A Finite Difference Formulation of Biot's Equations for Vertically Heterogeneous Full Waveform Acoustic Logging Problems

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ABSTRACT

In this paper we present a finite difference formulation for Biot's equations for wave propagation in saturated porous media which vary in range and depth. One objective of well logging petroleum exploration holes is to determine the permeability of a section. There are indications that the Stoneley wave in the full waveform acoustic logging tool is sensitive to permeability, but we need modeling techniques to fully understand the mechanism. One question to be addressed is how narrow horizontal fissures of varying permeability affect Stoneley or tube wave propagation in the borehole. Another question is the trade-off between attenuation due to viscous losses in the pore fluid and due to scattering. A technique for modeling acoustic logs in two-dimensionally varying Biot solids will give insight into these problems.

THE WAVE EQUATION FOR A HETEROGENEOUS, ISOTROPIC BIOT SOLID

Biot (1956a,b; 1962) developed a theory to study wave propagation in porous media saturated with a viscous fluid. Analysis of his equations has been carried out previously for homogeneous media and for vertically homogeneous acoustic logging problems. We present here a finite difference formulation of the Biot equations which can handle two-dimensional structure. We refer throughout to Schmitt (1986), which is an excellent review of Biot's theory and discusses applications to acoustic logging. Equation numbers in Schmitt (1986) will be identified by square brackets.

We will first derive the Biot wave equations for isotropic, heterogeneous material. The stress-strain relation for a Biot solid is [15]:

$$\begin{aligned} \sigma_{ij} &= 2Ne_{ij} + (Ae + Q\varepsilon)\delta_{ij} \\ s\delta_{ij} &= (Qe + \tilde{R}\varepsilon)\delta_{ij} \end{aligned} \tag{1}$$

where σ_{ij} is the stress tensor for the solid, s is the elastic component of the stress tensor for the fluid filling pores, e_{ij} is the strain tensor of the solid, and ε_{ij} is the strain in the fluid. A and N are analogous to Lamé's coefficients.

The appropriate equations of motion are [24]:

$$\begin{aligned} \partial_j\sigma_{ij} &= \rho_{11}\ddot{u}_i + \rho_{12}\ddot{U}_i + b(\dot{u}_i - \dot{U}_i) \\ \partial_i s &= \rho_{12}\ddot{u}_i + \rho_{22}\ddot{U}_i - b(\dot{u}_i - \dot{U}_i) \end{aligned} \tag{2}$$

where u_i are the displacements of the solid, U_i are the displacements of the fluid in the pores, and the coefficients b, ρ_{11}, ρ_{12}, and ρ_{22} are in general frequency dependent. Frequency dependent coefficients cannot be handled in a time domain finite difference formulation as discussed here because frequency is not generally known specifically (except in the case of a continuous wave source). So for the remainder of this discussion we will assume that the coefficients b, ρ_{11}, ρ_{12}, and ρ_{22} are frequency independent, at least over the band width of the pulse source.

Substituting equations (1) into (2), and maintaining gradients in parameters yields the wave equations for a heterogeneous, isotropic Biot solid:

$$\begin{gathered} (A+N)\nabla(\nabla\cdot\vec{u}) + N\nabla^2\vec{u} + Q\nabla(\nabla\cdot\vec{U}) \\ + \nabla A(\nabla\cdot\vec{u}) + \nabla N\times(\nabla\times\vec{u}) + 2(\nabla N\cdot\nabla)\vec{u} + \nabla Q(\nabla\cdot\vec{U}) \\ = \rho_{11}\ddot{\vec{u}} + \rho_{12}\ddot{\vec{U}} + b(\dot{\vec{u}} - \dot{\vec{U}}) \\ Q\nabla(\nabla\cdot\vec{u}) + \tilde{R}\nabla(\nabla\cdot\vec{U}) + \nabla Q(\nabla\cdot\vec{u}) + \nabla\tilde{R}(\nabla\cdot\vec{U}) \\ = \rho_{12}\ddot{\vec{u}} + \rho_{22}\ddot{\vec{U}} - b(\dot{\vec{u}} - \dot{\vec{U}}) \ . \end{gathered} \tag{3}$$

If parameters are constant in space, these equations reduce to equations [47] in Schmitt (1986). In turn, if porosity is zero, equations [47] reduce to the elastic wave equation which reduces further to the acoustic wave equation if N equals 0. This hierarchy of equations is applicable to the well-logging problem which will contain media of all types.

THE FINITE DIFFERENCE SCHEME

We solve equation (3) using an explicit, second order finite difference scheme analogous to the formulations of Bhasavanija (1983) and Nicoletis (1981), as discussed in Stephen (1985). The equations will be solved in cylindrical coordinates (r, z, θ) in which there is no θ dependence in the parameters, but only r, z dependence. Figure 1 summarizes the grid configuration used. The left-hand side is the axis of symmetry in cylindrical coordinates. A point source is introduced in the upper, left-hand corner, and the top edge is also an axis of symmetry. Axes of symmetry are easy to formulate and are more stable than absorbing boundaries. The inside of the grid is separated into three vertical regions. The left region, into which the source is introduced, is a homogeneous fluid and represents the borehole. The right region is a homogeneous elastic solid which acts as a buffer zone along the right absorbing boundary. The center region is a transition region in which arbitrary, two-dimensional variation of fluids, elastic solids, or Biot solids can be specified. The bottom part of the transition zone should not contain Biot solid parameters, since it would complicate the absorbing boundary formulations. All of the regions except the upper transition region, have been described by Stephen *et al.* (1985) and Stephen (1985). The formulation for the upper transition region is given below.

The finite difference solution can be simplified by considering repeated use of the elastic wave equation finite difference operator defined by:

$$\vec{\Theta}(\vec{u}; A, N) = F \cdot D \cdot \{(A+N)\nabla(\nabla \cdot \vec{u}) + N\nabla^2\vec{u}$$
$$+ \nabla A(\nabla \cdot \vec{u}) + \nabla N \times (\nabla \times \vec{u}) + 2(\nabla N \cdot \nabla)\vec{u}\}. \quad (4)$$

where $F \cdot D \cdot \{\}$ means a finite difference representation of the quantity inside the brace brackets. For example, the finite difference formulation of the elastic wave equation used by Stephen (1985) and Hunt and Stephen (1987) can be written:

$$F \cdot D \cdot \{\ddot{\vec{u}}\} = \vec{\Theta}(\vec{u}; \lambda, \mu) \quad (5)$$

where λ and μ are Lamé's parameters. Equations (3) can be reduced to

$$\begin{aligned} F \cdot D \cdot \{\rho_{11}\ddot{\vec{u}} + \rho_{12}\ddot{\vec{U}} + b(\dot{\vec{u}} - \dot{\vec{U}}\} &= \vec{\Theta}(\vec{u}; A, N) + \vec{\Theta}(\vec{U}; Q, 0) = \vec{I}^l \\ F \cdot D \cdot \{\rho_{12}\ddot{\vec{u}} + \rho_{22}\ddot{\vec{U}} - b(\dot{\vec{u}} - \dot{\vec{U}})\} &= \vec{\Theta}(\vec{u}; Q, 0) + \vec{\Theta}(\vec{U}; \tilde{R}, 0) = \vec{II}^l \end{aligned} \quad (6)$$

So the right-hand sides of the above equations, designated $\vec{I}^l$ and $\vec{II}^l$, contain only spatial derivatives, and in the context of a time marching algorithm can be evaluated at present time $l\Delta t$.

Taking centered finite differences in time of the left-hand side of the first equation in (6) gives:

$$\begin{aligned}\rho_{11}\,\frac{[\vec{u}^{l+1}-2\vec{u}^{l}+\vec{u}^{l-1}]}{\Delta t^2} &+ \rho_{12}\,\frac{[\vec{U}^{l+1}-2\vec{U}^{l}+\vec{U}^{l-1}]}{\Delta t^2}\\ +b\left[\frac{\vec{u}^{l+1}-\vec{u}^{l-1}}{2\Delta t}\right] &- b\left[\frac{\vec{U}^{l+1}-\vec{U}^{l-1}}{2\Delta t}\right]=\vec{I}^{l}\,,\end{aligned} \tag{7}$$

and rearranging yields

$$a_1\vec{u}^{l+1}-\vec{b}_1+a_2\vec{U}^{l+1}-\vec{b}_2=\vec{I}^{l} \tag{8}$$

where

$$\begin{aligned} a_1 &= \frac{\rho_{11}}{\Delta t^2}+\frac{b}{2\Delta t}\\ a_2 &= \frac{\rho_{12}}{\Delta t^2}-\frac{b}{2\Delta t}\\ \vec{b}_1 &= -\frac{2\rho_{11}}{\Delta t^2}\,\vec{u}^{l}+\left(\frac{\rho_{11}}{\Delta t^2}-\frac{b}{2\Delta t}\right)\vec{u}^{l-1}\\ \vec{b}_2 &= -\frac{2\rho_{12}}{\Delta t^2}\,\vec{U}^{l}+\left(\frac{\rho_{12}}{\Delta t^2}+\frac{b}{2\Delta t}\right)\vec{U}^{l-1}\,. \end{aligned} \tag{9}$$

Similarly the second equation in (6) becomes:

$$c_1\vec{u}^{l+1}-\vec{d}_1+c_2\vec{U}^{l+1}-\vec{d}_2=\vec{II}^{l} \tag{10}$$

where

$$\begin{aligned} c_1 &= \frac{\rho_{12}}{\Delta t^2}-\frac{b}{2\Delta t}\\ c_2 &= \frac{\rho_{22}}{\Delta t^2}+\frac{b}{2\Delta t}\\ \vec{d}_1 &= -\frac{2\rho_{12}}{\Delta t^2}\,\vec{u}^{l}+\left(\frac{\rho_{12}}{\Delta t^2}+\frac{b}{2\Delta t}\right)\vec{u}^{l-1}\\ \vec{d}_2 &= -\frac{2\rho_{22}}{\Delta t^2}\,\vec{U}^{l}+\left(\frac{\rho_{22}}{\Delta t^2}-\frac{b}{2\Delta t}\right)\vec{U}^{l-1}\,. \end{aligned} \tag{11}$$

We want to solve for $\vec{u}^{l+1}$ and $\vec{U}^{l+1}$ knowing $\vec{u}^{l}$, $\vec{u}^{l-1}$, $\vec{U}^{l}$, and $\vec{U}^{l-1}$. Note that $\vec{I}^{l}$ and $\vec{II}^{l}$ contain only terms in $\vec{u}^{l}$ and $\vec{U}^{l}$.

Solving (8) and (10) for $\vec{u}^{l+1}$ and $\vec{U}^{l+1}$ then gives:

$$\begin{aligned} \vec{u}^{l+1} &= [c_2\vec{J}^{l}-a_2\vec{JJ}^{l}]/(a_1c_2-a_2c_1)\\ \vec{U}^{l+1} &= [a_1\vec{JJ}^{l}-c_1\vec{J}^{l}]/(a_1c_2-a_2c_1) \end{aligned} \tag{12}$$

where

$$\begin{aligned} \vec{J}^l &= \vec{I}^l - \vec{b}_1 - \vec{b}_2 \\ \vec{JJ}^l &= \vec{II}^l - \vec{d}_1 - \vec{d}_2 \end{aligned} \tag{13}$$

In summary, the displacements in the solid and in the fluid at future time, $\vec{u}^{l+1}$ and $\vec{U}^{l+1}$, can be computed from (12), using (13) for $\vec{J}^l$ and $\vec{JJ}^l$. The a's, b's, c's and d's are given by equations (9) and (11) and $\vec{I}^l$ and $\vec{II}^l$ are given by (6). The necessary coefficients are A, N, Q, $\tilde{R}$, b, ρ_{11}, ρ_{12} and ρ_{22}.

DISCUSSION OF THE COEFFICIENTS

At best the eight new coefficients which replace V_p, V_s and ρ for the elastic wave equation are difficult to visualize. At worst some are frequency dependent, which makes solutions difficult. In this section we discuss the new coefficients in more detail.

The coefficients A, N, Q and $\tilde{R}$ can be expressed in terms of the bulk moduli of the solid matrix (K_s), the skeleton (i.e., the dry, porous solid) (K_b) and the fluid (K_f), the shear modules of the skeleton (μ_b) and the porosity, $\tilde{\phi}$ [16]:

$$\begin{aligned} A &= \frac{(1-\tilde{\phi})\left(1-\tilde{\phi}-\dfrac{K_b}{K_s}\right) K_s + \tilde{\phi}\dfrac{K_s}{K_f}K_b}{1-\tilde{\phi}-\dfrac{K_b}{K_s}+\tilde{\phi}\dfrac{K_s}{K_f}} - \frac{2}{3}\mu_b \\ Q &= \frac{\left(1-\tilde{\phi}-\dfrac{K_b}{K_s}\right)\tilde{\phi}K_s}{1-\tilde{\phi}-\dfrac{K_b}{K_s}+\tilde{\phi}\dfrac{K_s}{K_f}} \\ \tilde{R} &= \frac{\tilde{\phi}^2 K_s}{1-\tilde{\phi}-\dfrac{K_b}{K_s}+\tilde{\phi}\dfrac{K_s}{K_f}} \\ N &= \mu_b \end{aligned} \tag{14}$$

If porosity is set to zero, it can be shown that A and N reduce to λ and μ for the dry solid matrix, and Q and $\tilde{R}$ reduce to zero. It is convenient to express the coefficients K_b, N and K_f in terms of the compressional and shear velocities of the dry but still porous rock, α_m and β_m, and the fluid velocity, α_f [17]:

$$\begin{aligned} K_b &= (1-\tilde{\phi})\rho_s(\alpha_m^2 - \tfrac{4}{3}\beta_m^2) \\ N &= (1-\tilde{\phi})\rho_s\beta_m^2 \\ K_f &= \rho_f\alpha_f^2 \,. \end{aligned} \tag{15}$$

In general, b, ρ_{11}, ρ_{12}, and ρ_{22} are frequency dependent, but in certain cases frequency independence can be justified. For example, at low frequencies the viscous coupling coefficient is given by ([22] and [Figures 2a–6a]):

$$b = \frac{\eta\tilde{\phi}^2}{\tilde{k}} \tag{16}$$

where η is the dynamic viscosity of the fluid and $\tilde{k}$ is the intrinsic permeability. Similarly the mass coupling coefficient is ([82] and [Figures 2a–6a]):

$$\rho_{22} = \frac{4}{3}\tilde{\phi}\rho_f \,. \tag{17}$$

Once ρ_{22} is established, the other coupling coefficients are obtained by [41]:

$$\begin{aligned} \rho_{12} &= \rho_2 - \rho_{22} \\ \rho_{11} &= \rho_1 - \rho_{12} \,, \end{aligned} \tag{18}$$

where ρ_1 and ρ_2 are the liquid and solid phase densities per unit volume [1]:

$$\begin{aligned} \rho_1 &= (1-\tilde{\phi})\rho_s \\ \rho_2 &= \tilde{\phi}\rho_f \,. \end{aligned} \tag{19}$$

With these relationships, equations (18) become:

$$\begin{aligned} \rho_{12} &= \tilde{\phi}\rho_f - \tfrac{4}{3}\tilde{\phi}\rho_f = -\tfrac{1}{3}\tilde{\phi}\rho_f = -\tfrac{1}{4}\rho_{22} \\ \rho_{11} &= (1-\tilde{\phi})\,\rho_s + \tfrac{1}{3}\tilde{\phi}\rho_f \,. \end{aligned} \tag{20}$$

Under these assumptions, ρ_{12} is not an independent coefficient.

In summary, the coefficients A, Q, $\tilde{R}$, and N can be obtained from K_s, α_m, β_m, α_f, ρ_s, ρ_f, and $\tilde{\phi}$ using equations (14) and (15). The coefficients ρ_{11}, ρ_{22} and b (since ρ_{12} is not independent) can be obtained from η, $\tilde{k}$, $\tilde{\phi}$, ρ_s and ρ_f using equations (16), (17) and (20).

PROGRESS REPORT

A finite difference code based on the above equations has been drafted and it is being tested for stability and accuracy. As for the elastic solid case, there may be problems at the liquid-solid boundary. In the elastic case we were able to use the wave equation for heterogeneous media without specifically coding the liquid-solid boundary conditions. Stephen (1985) showed excellent agreement between the discrete wavenumber method and the finite difference method using this approach. In this work we will compare our finite difference solutions with the discrete wavenumber results of Schmitt (1986) to check if the same will be true for Biot solids.

Ultimately we will apply the code to problems which contain vertical variations in porosity and permeability such as horizontal, porous fissures.

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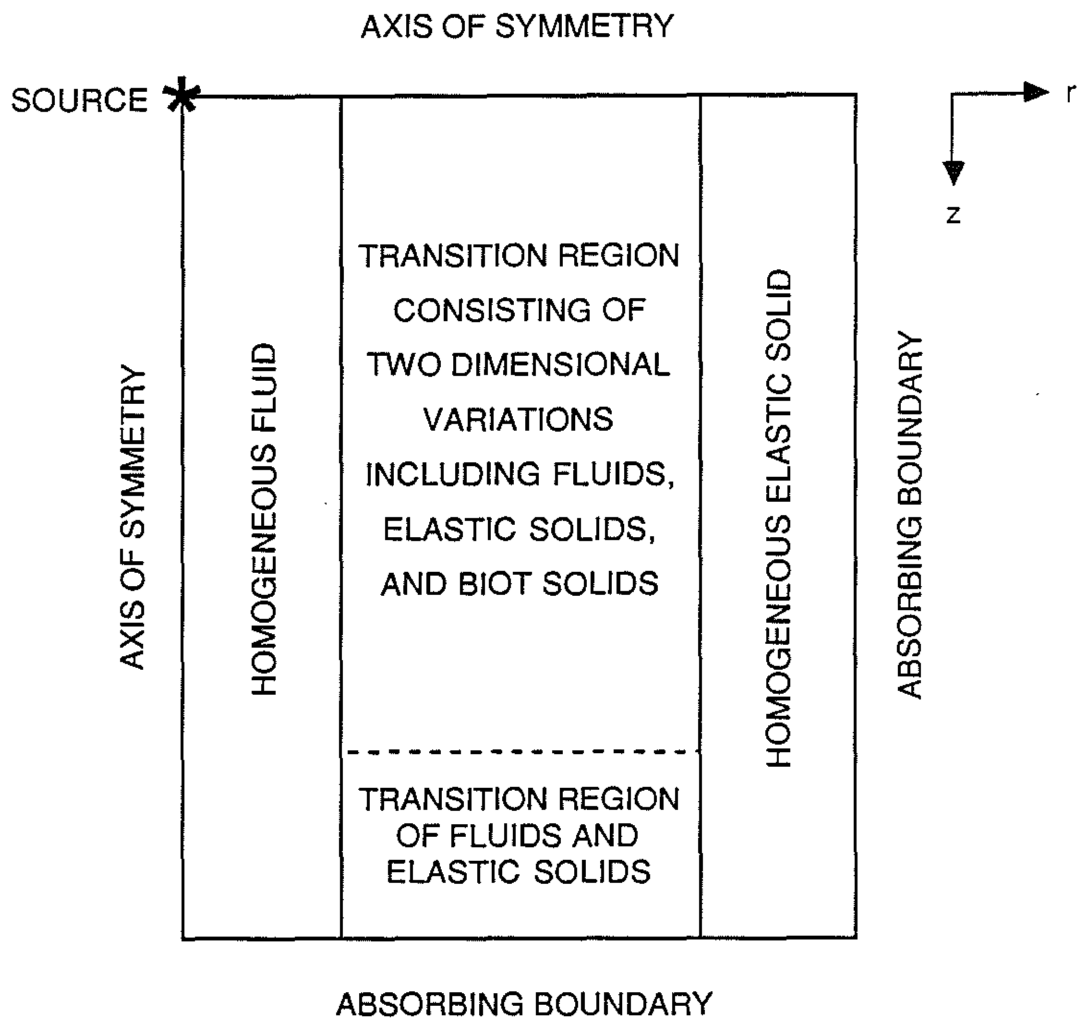

Figure 1: The grid configuration used for finite difference synthetic acoustic logs in Biot solids.