CONSTRUCTING POINTS THROUGH FOLDING AND INTERSECTION

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Constructing points through folding and intersection

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Abstract

Fix an $n \geq 3$. Consider the following two operations: given a line with a specified point on the line we can construct a new line through the point which forms an angle with the new line which is a multiple of π/n (folding); and given two lines we can construct the point where they cross (intersection). Starting with the line $y = 0$ and the points $(0, 0)$ and $(1, 0)$ we determine which points in the plane can be constructed using only these two operations for $n = 3, 4, 5, 6, 8, 10, 12, 24$ and also consider the problem of the minimum number of steps it takes to construct such a point.

1 Introduction

If an origami model is laid flat the piece of paper will retain a memory of the folds that went into the construction of the model as creases (or lines) in the paper. These creases will sometimes be reflected as places in the final model where the paper is bent and sometimes will be left over artifacts from early in the construction process. These creases can also be used in the construction of reference points, which play a useful role in the design of complicated origami models (see [6]). As such, tools to help efficiently construct reference points have been developed, i.e., ReferenceFinder [7].

The problem of finding which points can be constructed using origami has been extensively studied. In particular, using the Huzita-Hatori axioms it has been shown that all quartic polynomials can be solved using origami (see [4, pp. 285–291] for more information). So origami is more powerful than the use of ruler and compass, which can solve quadratics. This of course assumes that we put no limitation on the type of folds that we make, so one might ask what happens if we limit the folds. For example, the crease patterns for many origami models are designed with an angular grid system of π/n for some $n \geq 3$ (in practice, n is taken to be even). The crease pattern is formed by starting with two reference points $(0, 0)$ and $(1, 0)$ and a crease containing both (or it might be that the two reference points are two corners of the paper). New creases are formed by taking folds through an existing point with an angle of $i\pi/n$ for some i, and new points are formed by taking the intersection of two creases.

In particular, the grid system based on $\pi/8 = 22.5°$ has been used for centuries — one of the oldest is the classic origami crane — and the system keeps producing complex but organized origami expressions such as the *Devil* (1980) by Jun Maekawa [8] and the *Wolf* (2006) by Hideo Komatsu

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[5]. Toshikazu Kawasaki calls this system "Maekawa-gami". Figure 1 shows a square filled with several creases in the 22.5° grid system. The set of points which are possible to construct in this system have been examined [9].

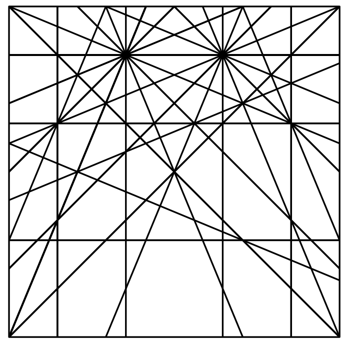


Figure 1: An example of several creases in the 22.5° grid system.

In this note we will consider the problem of what points are constructible in the plane for specific values of n generalizing the idea of folding and intersecting on origami paper. Our main result is the following.

Theorem 1. *Fix $n \geq 3$. Starting with the line $y = 0$ and the points $(0,0)$ and $(1,0)$ construct new lines and points by using the following two rules. To construct a new line take an existing point and introduce a new line forming an angle of $i\pi/n$ with another line through the point. To construct a new point take the intersection of two lines.*

The set of points that can be constructed for $n = 3, 4, 5, 6, 8, 10, 12, 24$ is given by the following table (where $a, \ldots, \ell$ are all arbitrary integers).

n	*Form for constructed points*
3	$a(1,0) + b(\frac{1}{2}, \frac{1}{2}\sqrt{3})$
4	$\left(\dfrac{a}{2^k}, \dfrac{b}{2^k}\right)$
5	$\left(\dfrac{a+b\sqrt{5}}{2}\right)(1,0) + \left(\dfrac{c+d\sqrt{5}}{2}\right)\left(\dfrac{1}{2}, \dfrac{1}{2}\sqrt{5-2\sqrt{5}}\right)$, *with* $a+b \equiv c+d \equiv 0 \pmod 2$
6	$\left(\dfrac{a}{2^k3^\ell}, \dfrac{b}{2^k3^\ell}\sqrt{3}\right)$
8	$\left(\dfrac{a+b\sqrt{2}}{2^k}, \dfrac{c+d\sqrt{2}}{2^k}\right)$
10	$\left(\dfrac{a+b\sqrt{5}}{2^k5^\ell}, \dfrac{c+d\sqrt{5}}{2^k5^\ell}\sqrt{5-2\sqrt{5}}\right)$
12	$\left(\dfrac{a+b\sqrt{3}}{2^k3^\ell}, \dfrac{c+d\sqrt{3}}{2^k3^\ell}\right)$
24	$\left(\dfrac{a+b\sqrt{2}+c\sqrt{3}+d\sqrt{6}}{2^k3^\ell}, \dfrac{e+f\sqrt{2}+g\sqrt{3}+h\sqrt{6}}{2^k3^\ell}\right)$

The proof of the theorem will be done in two parts. First, in Section 2 we will show that all of these points *can* be constructed and also consider the number of steps needed to construct a given point. Second, in Section 3 we will show that these are the *only* points that can be constructed. Finally, we will give some concluding comments in Section 4.

2 Constructing our points

We want to show that the points given in Theorem 1 can be constructed. The most useful tools to help do this is to show that we can add points together and scale; the problem then reduces to constructing several simple points.

Lemma 2. *The following holds for all $n \geq 3$.*

(a) Given constructed points (x_1, y_1) and (x_2, y_2) we can construct $a(x_1, y_1) + b(x_2, y_2)$ for arbitrary integers a and b.

(b) If we can construct (x_1, y_1) and $(\gamma, 0)$ (for any real number γ) then we can construct $(\gamma^k x_1, \gamma^k y_1)$ for $k \geq 0$.

Proof. For part (a) it suffices to show that we can construct $(2, 0)$, since if we can construct $(2, 0)$ then repeating the same steps used to produce (x_1, y_1) but with the points $(1, 0)$ and $(2, 0)$ we can construct $(x_1 + 1, y_1)$. Now starting with the points (x_1, y_1) and $(x_1 + 1, y_1)$ follow the same steps used to produce (x_2, y_2) and the result will be $(x_1, y_1) + (x_2, y_2)$. Similarly, if we can construct $(2, 0)$ then by symmetry of the process we can construct $(-1, 0)$. So given a construction for (x_1, y_1) we can by symmetry construct $(-x_1, -y_1)$. Combining these two we can now construct any integer combination of the two points.

We now give a construction of $(2, 0)$. Starting with the points $A = (0, 0)$ and $B = (1, 0)$, form line 1 out of point A with angle π/n and form lines 2 and 3 out of point B with angle $(n-1)\pi/n$ and π/n. The intersection of lines 1 and 2 is a new point $C = (\frac{1}{2}, \frac{1}{2}\tan\frac{\pi}{n})$. Through point C form a fold with an angle of 0 to form line 4 which intersects line 3 at a new point $D = (\frac{3}{2}, \frac{1}{2}\tan\frac{\pi}{n})$. Finally through point D form a fold with angle $(n-1)\pi/n$ to make line 5 which intersects line 0 at point $E = (2, 0)$.

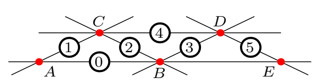


Figure 2: Constructing $(2, 0)$.

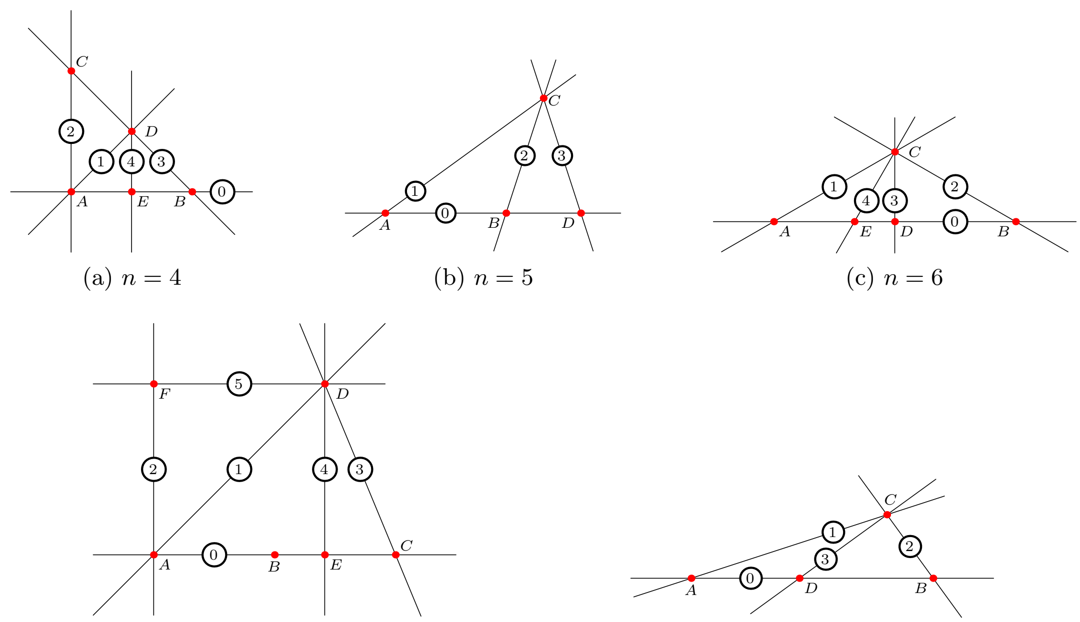
For part (b) we first note that if we can construct $(\gamma, 0)$ then by repeating the same steps but with $(0, 0)$ and $(\gamma, 0)$ we can construct $(\gamma^2, 0)$ and then by induction we can construct $(\gamma^k, 0)$. Finally, given the construction for (x_1, y_1) we now do the same construction but using the points $(0, 0)$ and $(\gamma^k, 0)$ instead of $(0, 0)$ and $(1, 0)$ which will produce the point $(\gamma^k x_1, \gamma^k y_1)$. □

An immediate consequence of this lemma is that points on the x-axis that can be constructed by folding form a ring. This is because we can add and multiply any two elements, and we have the additive and multiplicative identities.

We now work through the various cases of n and show how to construct the basic atoms and use the lemma to show that we can construct all points as given in Theorem 1.

For $n = 3$ the construction shown in Figure 2 shows that we can construct $(\frac{1}{2}, \frac{1}{2}\sqrt{3})$ and so by Lemma 2 we can construct $a(1,0) + b(\frac{1}{2}, \frac{1}{2}\sqrt{3})$ finishing this case.

For $n = 4$, consider Figure 3a where we start with points $A = (0,0)$ and $B = (1,0)$. Form lines 1 and 2 out of point A at angles of $\pi/4$ and $\pi/2$ and form line 3 out of B at an angle of $3\pi/4$. The intersection of lines 2 and 3 gives the point $C = (0,1)$, the intersection of lines 1 and 3 gives the point $D = (\frac{1}{2}, \frac{1}{2})$. Through D we construct line 4 with an angle of $\pi/2$. The intersection of lines 0 and 4 gives the point $E = (\frac{1}{2}, 0)$.



(a) $n = 4$ (b) $n = 5$ (c) $n = 6$ (d) $n = 8$ (e) $n = 10$

Figure 3: Construction of several points for various n.

By Lemma 2 we can get $a(1,0) + b(0,1) = (a,b)$. Since we also constructed $(\frac{1}{2}, 0)$ then by the second part of Lemma 2 we can scale (a,b) by any power of $\frac{1}{2}$ and so we can construct any point of the form $(a/2^k, b/2^k)$ finishing this case.

We used two features which will come in handy in future construction. One is that when we can form an angle of $\pi/2$ we can project onto the axes, and so, in particular, if n is divisible by 2 then we can form a point (x,y) if we can form $(x,0)$ and $(0,y)$. Conversely, if we can form $(x,0)$ and $(0,y)$ then by folding an angle of 0 through $(y,0)$ and an angle of $\pi/2$ through $(x,0)$ we can form (x,y). The other is that when we can form an angle of $\pi/4$ we can construct the point $(0,1)$, and so by symmetry we can construct the point (x,y) if and only if we can construct the point (y,x); in particular, we only need to know what points of the form $(x,0)$ can be constructed to find all points that can be constructed.

For $n = 5$ the construction in Figure 2 shows that we can construct $(\frac{1}{2}, \frac{1}{2}\sqrt{5 - 2\sqrt{5}})$. It suffices to show how to construct points of the form $\big((p + q\sqrt{5})/2, 0\big)$ with $p + q \equiv 0 \pmod 2$ (since we can, by scaling and adding the points $(\frac{1}{2}, \frac{1}{2}\sqrt{5 - 2\sqrt{5}})$ and $(1,0)$, construct all points of the desired form). To see this consider Figure 3b where we start with points $A = (0,0)$ and $B = (1,0)$, then form line 1 out of point A at an angle $\pi/5$ and form line 2 out of point B at an angle of

$2\pi/5$. The intersection of lines 1 and 2 gives point $C = \big(\frac{3}{4}+\frac{1}{4}\sqrt{5}, (\frac{3}{4}+\frac{1}{4}\sqrt{5})\sqrt{5-2\sqrt{5}}\big)$. Through C we construct line 3 with an angle of $3\pi/5$. The intersection of lines 0 and 3 gives the point $D = (\frac{1}{2}+\frac{1}{2}\sqrt{5},0)$. So by Lemma 2 we can form any integer combination $a(1,0)+b(\frac{1}{2}+\frac{1}{2}\sqrt{5},0)$ which are points of the form $\big((p+q\sqrt{5})/2,0\big)$ with $p+q\equiv 0 \pmod 2$.

For $n=6$, we can construct all points for the $n=3$ case, in particular we can construct $(a,b\sqrt{3})$ for a,b integer. It suffices to show that we can now scale by $1/2$ and $1/3$. To see this consider Figure 3c where we start with points $A=(0,0)$ and $B=(1,0)$. Form line 1 through A with an angle of $\pi/3$ and line 2 through B with an angle of $5\pi/3$. The intersection of lines 1 and 2 is the point $C=(\frac{1}{2},\frac{1}{6}\sqrt{3})$. Through C we construct lines 3 and 4 at angles of $\pi/2$ and $2\pi/3$. The intersection of lines 0 and 3 is the point $D=(\frac{1}{2},0)$ and the intersection of lines 0 and 4 is the point $E=(\frac{1}{3},0)$. So by the second part of Lemma 2 we can scale by arbitrary powers of $1/2$ and $1/3$ showing we can construct all points of the form $(a/2^k3^\ell, b\sqrt{3}/2^k3^\ell)$ finishing this case.

For $n=8$, we can construct all points for the $n=4$ case, including $(0,1)$ and $(\frac{1}{2},0)$ (so we can scale by $1/2$). The important part about this case is to show that we can construct $(\sqrt{2},0)$ and $(0,\sqrt{2})$. To see this consider Figure 3d where we start with points $A=(0,0)$ and $B=(1,0)$. Following the steps in Figure 2 we construct $C=(2,0)$ (steps are suppressed in the drawing). Through A we form lines 1 and 2 with angles of $\pi/4$ and $\pi/2$ and through C we form line 3 with an angle of $5\pi/8$. The intersection of lines 1 and 3 is the point $D=(\sqrt{2},\sqrt{2})$. Through D we construct lines 4 and 5 with angles of 0 and $\pi/2$. Then the intersection of lines 0 and 4 is $E=(\sqrt{2},0)$ and the intersection of lines 2 and 5 is $F=(0,\sqrt{2})$. Using Lemma 2 we can take all linear combinations of the points $(1,0)$, $(0,1)$, $(\sqrt{2},0)$ and $(0,\sqrt{2})$ and also scale by arbitrary powers of $1/2$. In particular, we can construct all points of the form $\big((a+b\sqrt{2})/2^k,(c+d\sqrt{2})/2^k\big)$ finishing this case.

For $n=10$, we can construct all points for the $n=5$ case, i.e., $\big(a+b\sqrt{5},(c+d\sqrt{5})\sqrt{5-2\sqrt{5}}\big)$ with a,b,c,d integer. By the same method used in the $n=4$ and $n=6$ case we can construct $(1/2,0)$ showing we can scale by $1/2$. So it remains to show that we can scale by $1/5$. To see this consider Figure 3e where we start with points $A=(0,0)$ and $B=(1,0)$. Through A we form line 1 with an angle of $\pi/10$ and through B we form line 2 with an angle of $7\pi/10$. Then the intersection of lines 1 and 2 is the point $C=\big((1+\sqrt{5})/4,(5+\sqrt{5})\sqrt{5-2\sqrt{5}}/20\big)$ Through C we construct line 3 with an angle of $\pi/5$. Then the intersection of lines 0 and 3 is the point $D=(1/\sqrt{5},0)$. By Lemma 2, since we can construct $(\gamma,0)=(1/\sqrt{5},0)$ then we can construct $(\gamma^2,0)=(1/5,0)$ and, in particular, we can scale by $1/5$ finishing this case.

For $n=12$ we note that by the $n=3$ case we can construct points of the form $(a/2^k3^\ell, d\sqrt{3}/2^k3^\ell)$ which combined with the comments following the $n=4$ case means that we can also construct points of the form $(b\sqrt{3}/2^k3^\ell, c/2^k3^\ell)$. Finally we can combine these points and therefore we can construct points of the form $\big((a+b\sqrt{3})/2^k3^\ell,(c+d\sqrt{3})/2^k3^\ell\big)$ finishing this case.

For the $n=24$ case we note that we can construct all points in the $n=3,4,6,8,12$ cases. In particular we can construct $(1,0)$, $(\sqrt{2},0)$ and $(\sqrt{3},0)$ and also scale by $1/2$ and by $1/3$. Using Lemma 2 we can scale $(\sqrt{2},0)$ by $\sqrt{3}$ showing we can construct $(\sqrt{6},0)$. So it follows that in this case we can construct any point of the form

$$\left(\frac{a+b\sqrt{2}+c\sqrt{3}+d\sqrt{6}}{2^k3^\ell},\frac{e+f\sqrt{2}+g\sqrt{3}+h\sqrt{6}}{2^k3^\ell}\right)$$

finishing this case.

2.1 Short construction of points

The above constructions show not only how to construct points but indicate how we can construct points efficiently (see [2] for more information about the efficiency of folding). In practice this is important since pieces of paper have physical limitations to the number of folds that can be made and still be viable for origami. In this section we will focus on the case $n = 8$, although the same analysis applies to the other values of n we have considered.

Theorem 3. *For $n = 8$ the point $\big((a + b\sqrt{2})/2^k, (c + d\sqrt{2})/2^k\big)$ with a, b, c, d, k integers and $k \geq 0$ can be constructed using at most*

$$A\big(\lg(1 + |a|) + \lg(1 + |b|) + \lg(1 + |c|) + \lg(1 + |d|)\big) + Bk + C$$

folds for some fixed constants A, B, C and where lg *denotes the base-*2 *logarithm.*

Before we begin the proof we note that by "fold" we generally mean constructing a line with some angle through a given point. In origami practice to form an angle for $n = 3, 4, 5, 6, 8, 10, 12, 24$ we might need to use several folds to form a given line; in either case the number of origami folds is bounded by an integer multiple of the idealized folds we have been using.

Proof. First we note if we can construct $(x, 0)$ and $(y, 0)$ then using at most four more lines we can construct (x, y). Namely, through the points $(x, 0)$ and $(0, 0)$ we fold lines at angles of $\pi/2$ and through the point $(y, 0)$ we fold a line at angle $\pi/4$ which intersects the line through $(0, 0)$ at $(0, y)$. Through $(0, y)$ we fold a line at an angle of 0 and this will intersect the line through $(x, 0)$ at (x, y). So it suffices to show that the result is true in the special case $\big((a + b\sqrt{2})/2^k, 0\big)$.

Next note that if we have constructed the point $(a + b\sqrt{2}, 0)$ then using at most $2k + 1$ more folds we can construct $\big((a + b\sqrt{2})/2^k, 0\big)$. Namely we fold a line through the origin at an angle of $\pi/4$ and then iteratively apply the following two fold step: fold a line through the current point at an angle of $3\pi/4$ take the point of intersection with the line through the origin and make a fold at an angle of $\pi/2$; where the new line intersect the x-axis is exactly the point we started with scaled by 1/2. So after applying this k times the point $(a + b\sqrt{2}, 0)$ yields $\big((a + b\sqrt{2})/2^k, 0\big)$. So it suffices to show that the results holds for $(a + b\sqrt{2}, 0)$. By a similar argument we may also assume that $a, b \geq 0$.

We now construct $(a, 0)$, to do this we write a in binary form, i.e., $a = \alpha_k \ldots \alpha_1\alpha_0$ where $\alpha_k = 1$ and $k \leq \lfloor \lg(1 + |a|)\rfloor$ and start with the points $(a_{k+1}, 0) = (0, 0)$ and $(a_{k+1} + 1, 0) = (1, 0)$. Given $(a_{i+1}, 0)$ and $(a_{i+1} + 1, 0)$ we from the points $(a_i, 0) = (2a_{i+1} + \alpha_i, 0)$ and $(a_i + 1, 0) = (2a_{i+1} + \alpha_i + 1, 0)$. By this construction it is easy to see that the binary expansion of a_i is $\alpha_k \ldots \alpha_i$ so that the point $(a_0, 0) = (a, 0)$ and $(a_0 + 1, 0) = (a + 1, 0)$, as desired. (This idea is similar to Horner's method for polynomial evaluation.)

To carry this out we will fold lines through $(0, 0)$ and $(1, 0)$ at $\pi/4$. To send the points $(x, 0)$ and $(x+1, 0)$ to $(2x, 0)$ and $(2x+1, 0)$ takes four folds, namely fold lines through $(x, 0)$ and $(x+1, 0)$ at angles of $\pi/2$. This will form intersections at (x, x) and $(x + 1, x)$ which we then fold lines through these points an angles of $3\pi/4$ which will intersect the axis at $(2x, 0)$ and $(2x + 1, 0)$. To send the points $(b, 0)$ and $(b + 1, 0)$ to $(2b + 1, 0)$ and $(2b + 2, 0)$ we first construct $(2b, 0)$ and $(2b + 1, 0)$ and then using the construction given in Lemma 2 we form $(2b + 2, 0)$ with four more folds.

In particular, it will take no more than $8\lfloor \lg(1 + |a|)\rfloor + 2$ folds to construct $(a, 0)$ and $(a + 1, 0)$. Similarly, it will take no more than $8\lfloor \lg(1 + |b|)\rfloor + 7$ folds to construct $(b\sqrt{2}, 0)$ (the 5 extra folds are used to go from $(b,0)$ to $(b\sqrt{2},0)$). Therefore we can construct $(a+b\sqrt{2},0)$ using at most

$$8\lfloor \lg(1+|a|)\rfloor + 8\lfloor \lg(1+|b|)\rfloor + 9$$

folds. The result now follows. $\square$

3 Restricting the points we can construct

In the previous section we showed how to construct points for $n = 3, 4, 5, 6, 8, 10, 12, 24$. In this section we turn to the problem of showing that these are the *only* such points that can be constructed.

3.1 A lattice for $n = 3$

We already saw that we can construct all points of the form $a(1,0)+b(\frac{1}{2},\frac{1}{2}\sqrt{3})$. If we plot all these points and then form all lines through these points with angles of $0, \pi/3, 2\pi/3$ we get Figure 4. In particular, we see there are no points of intersection other than the ones we started with and so these are the only points that are constructible.

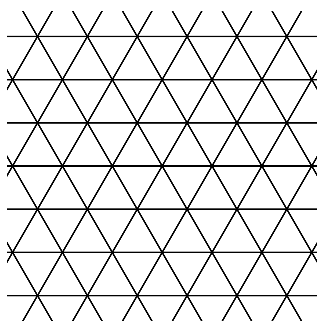


Figure 4: The lattice generated by all constructible points for $n = 3$.

3.2 Finding invariant subsets for $n = 4, 6, 8, 10, 12, 24$

We now turn to the cases $n = 4, 6, 8, 10, 12, 24$. We first begin by noting that every point (x, y) other than $(0, 0)$ and $(1, 0)$ are found by starting with two already constructed points (x_1, y_1) and (x_2, y_2) and two angles $\theta_1 \neq \theta_2$ and finding the points of intersection of the line through the first point with angle θ_1 and the line through the second point with angle θ_2. So that the newly constructed point is a solution to the following 2×2 system.

$$\begin{aligned}(x-x_1)\sin\theta_1 - (y-y_1)\cos\theta_1 &= 0\\(x-x_2)\sin\theta_2 - (y-y_2)\cos\theta_2 &= 0\end{aligned}$$

Solving this system, i.e., by Cramer's rule, we find the new point of intersection will be

$$\begin{aligned}x &= \frac{\cos\theta_1\cos\theta_2(y_1-y_2)+\cos\theta_1\sin\theta_2 x_2 - \sin\theta_1\cos\theta_2 x_1}{\sin(\theta_2-\theta_1)},\\ y &= \frac{\sin\theta_1\sin\theta_2(x_2-x_1)+\cos\theta_1\sin\theta_2 y_1 - \sin\theta_1\cos\theta_2 y_2}{\sin(\theta_2-\theta_1)}.\end{aligned}$$

In particular, one way to understand what form the points can take is to understand the following quantities (where $\theta_1 = i\pi/n$ and $\theta_2 = j\pi/n$ with $i \neq j$):

$$\begin{aligned} CC_n(i,j) &= \frac{\cos\left(\frac{i}{n}\pi\right)\cos\left(\frac{j}{n}\pi\right)}{\sin\left(\frac{j-i}{n}\pi\right)}, \\ SS_n(i,j) &= \frac{\sin\left(\frac{i}{n}\pi\right)\sin\left(\frac{j}{n}\pi\right)}{\sin\left(\frac{j-i}{n}\pi\right)}, \\ CS_n(i,j) &= \frac{\cos\left(\frac{i}{n}\pi\right)\sin\left(\frac{j}{n}\pi\right)}{\sin\left(\frac{j-i}{n}\pi\right)}. \end{aligned}$$

Theorem 4. *For n fixed, let $\mathcal{X}_n$ and $\mathcal{Y}_n$ be subsets of the real numbers closed under addition such that $0,1 \in \mathcal{X}_n$ and $0 \in \mathcal{Y}_n$. If for every $x \in \mathcal{X}_n$ and $y \in \mathcal{Y}_n$ and $i \neq j$ we have $CC_n(i,j)\mathcal{Y}_n \subseteq \mathcal{X}_n$, $CS_n(i,j)\mathcal{X}_n \subseteq \mathcal{X}_n$, $SS_n(i,j)\mathcal{X}_n \subseteq \mathcal{Y}_n$ and $CS_n(i,j)\mathcal{Y}_n \subseteq \mathcal{Y}_n$ then any constructible point (a,b) in the construction problem for angles with π/n satisfies $a \in \mathcal{X}$ and $b \in \mathcal{Y}$.*

The proof follows by simply examining the above solution to the system of linear equations and seeing that these sets contain the initial points and are closed under finding intersection. We now apply this theorem to get restrictions on which points can be constructed. The goal of course is to find $\mathcal{X}_n$ and $\mathcal{Y}_n$ as small as possible. We will see that our choice for $\mathcal{X}_n$ and $\mathcal{Y}_n$ are driven by the form of the numbers $CC_n(i,j)$, $SS_n(i,j)$ and $CS_n(i,j)$.

For $n = 4$ we have the values shown in Table 1. Based on the form of CC_4, CS_4 and SS_4, it is

$CC_4(i,j)$	$j=0$	$j=1$	$j=2$	$j=3$
$i=0$		1	0	-1
$i=1$	-1		0	$-\frac{1}{2}$
$i=2$	0	0		0
$i=3$	1	$\frac{1}{2}$	0	

$SS_4(i,j)$	$j=0$	$j=1$	$j=2$	$j=3$
$i=0$		0	0	0
$i=1$	0		1	$\frac{1}{2}$
$i=2$	0	-1		1
$i=3$	0	$-\frac{1}{2}$	-1	

$CS_4(i,j)$	$j=0$	$j=1$	$j=2$	$j=3$
$i=0$		1	1	1
$i=1$	0		1	$\frac{1}{2}$
$i=2$	0	0		0
$i=3$	0	$\frac{1}{2}$	1	

Table 1: Values of CC_4, SS_4 and CS_4.

easy to see that $\mathcal{X}_4 = \mathcal{Y}_4 = \{a/2^k : a,k \in \mathbb{Z}\}$ satisfies Theorem 4 showing that the only points that can be constructed have the form $(a/2^k, b/2^k)$ finishing this case.

For $n = 6$ we have values shown in Table 3. Examining them, it is easy to see that $\mathcal{X}_6 =$

$CC_6(i,j)$	$j=0$	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$
$i=0$		$\sqrt{3}$	$\frac{1}{3}\sqrt{3}$	0	$-\frac{1}{3}\sqrt{3}$	$-\sqrt{3}$
$i=1$	$-\sqrt{3}$		$\frac{1}{2}\sqrt{3}$	0	$-\frac{1}{4}\sqrt{3}$	$-\frac{1}{2}\sqrt{3}$
$i=2$	$-\frac{1}{3}\sqrt{3}$	$-\frac{1}{2}\sqrt{3}$		0	$-\frac{1}{6}\sqrt{3}$	$-\frac{1}{4}\sqrt{3}$
$i=3$	0	0	0		0	0
$i=4$	$\frac{1}{3}\sqrt{3}$	$\frac{1}{4}\sqrt{3}$	$\frac{1}{6}\sqrt{3}$	0		$\frac{1}{2}\sqrt{3}$
$i=5$	$\sqrt{3}$	$\frac{1}{2}\sqrt{3}$	$\frac{1}{4}\sqrt{3}$	0	$-\frac{1}{2}\sqrt{3}$	

$SS_6(i,j)$	$j=0$	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$
$i=0$		0	0	0	0	0
$i=1$	0		$\frac{1}{2}\sqrt{3}$	$\frac{1}{3}\sqrt{3}$	$\frac{1}{4}\sqrt{3}$	$\frac{1}{6}\sqrt{3}$
$i=2$	0	$-\frac{1}{2}\sqrt{3}$		$\sqrt{3}$	$\frac{1}{2}\sqrt{3}$	$\frac{1}{4}\sqrt{3}$
$i=3$	0	$-\frac{1}{3}\sqrt{3}$	$-\sqrt{3}$		$\sqrt{3}$	$\frac{1}{3}\sqrt{3}$
$i=4$	0	$-\frac{1}{4}\sqrt{3}$	$-\frac{1}{2}\sqrt{3}$	$-\sqrt{3}$		$\frac{1}{2}\sqrt{3}$
$i=5$	0	$-\frac{1}{6}\sqrt{3}$	$-\frac{1}{4}\sqrt{3}$	$-\frac{1}{3}\sqrt{3}$	$-\frac{1}{2}\sqrt{3}$	

$CS_6(i,j)$	$j=0$	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$
$i=0$		1	1	1	1	1
$i=1$	0		$\frac{3}{2}$	1	$\frac{3}{4}$	$\frac{1}{2}$
$i=2$	0	$-\frac{1}{2}$		1	$\frac{1}{2}$	$\frac{1}{4}$
$i=3$	0	0	0		0	0
$i=4$	0	$\frac{1}{4}$	$\frac{1}{2}$	1		$-\frac{1}{2}$
$i=5$	0	$\frac{1}{2}$	$\frac{3}{4}$	1	$\frac{3}{2}$	

Table 2: Values of CC_6, SS_6 and CS_6.

$\{a/2^k3^\ell : a,k,\ell \in \mathbb{Z}\}$ and $\mathcal{Y}_6 = \{b\sqrt{3}/2^k3^\ell : b,k,\ell \in \mathbb{Z}\}$ satisfies Theorem 4 showing that the only points that can be constructed are of the form $(a/2^k3^\ell, b\sqrt{3}/2^k3^\ell)$ finishing this case.

For $n=8$ we have values shown in Table 3. Examining them, it is easy to see that $CC_8(i,j)$, $CS_8(i,j)$ and $SS_8(i,j)$ are of the form $(\alpha+\beta\sqrt{2})/4$ where $\alpha,\beta \in \mathbb{Z}$. From this it is easy to construct sets satisfying Theorem 4, namely $\mathcal{X}_8 = \mathcal{Y}_8 = \{(a+b\sqrt{2})/2^k : a,b,k \in \mathbb{Z}\}$ showing that the only points that can be constructed are of the form $\big((a+b\sqrt{2})/2^k, (c+d\sqrt{2})/2^k\big)$ finishing this case.

For $n=10$ we do not produce the tables for space consideration but note that

$$CC_{10}(i,j), SS_{10}(i,j) \in \left\{\frac{a+b\sqrt{5}}{40}\sqrt{5-2\sqrt{5}} : a,b \in \mathbb{Z}\right\} \text{ and } CS_{10}(i,j) \in \left\{\frac{a+b\sqrt{5}}{8} : a,b \in \mathbb{Z}\right\}.$$

From this it is easy to construct sets satisfying Theorem 4, namely

$$\mathcal{X}_{10} = \left\{\frac{a+b\sqrt{5}}{2^k5^\ell} : a,b,k,\ell \in \mathbb{Z}\right\} \text{ and } \mathcal{Y}_{10} = \left\{\frac{a+b\sqrt{5}}{2^k5^\ell}\sqrt{5-2\sqrt{5}} : a,b,k,\ell \in \mathbb{Z}\right\}$$

showing that the only points that can be constructed are of the form

$$\left(\frac{a+b\sqrt{5}}{2^k5^\ell}, \frac{c+d\sqrt{5}}{2^k5^\ell}\sqrt{5-2\sqrt{5}}\right)$$

finishing this case.

For $n = 12$ we do not produce the tables but note that $CC_{12}(i,j)$, $CS_{12}(i,j)$ and $SS_{12}(i,j)$ are of the form $(\alpha+\beta\sqrt{3})/12$ where $\alpha,\beta\in\mathbb{Z}$. From this it is easy to construct sets satisfying Theorem 4, namely $\mathcal{X}_{12}=\mathcal{Y}_{12}=\{(a+b\sqrt{3})/2^k3^\ell : a,b,k,\ell\in\mathbb{Z}\}$ showing that the only points that can be constructed are of the form $\big((a+b\sqrt{3})/2^k3^\ell,(c+d\sqrt{3})/2^k3^\ell\big)$ finishing this case.

For $n = 24$ we do not produce the tables but note that $CC_{24}(i,j)$, $CS_{24}(i,j)$ and $SS_{24}(i,j)$ are of the form $(\alpha+\beta\sqrt{2}+\gamma\sqrt{3})/24$ where $\alpha,\beta,\gamma\in\mathbb{Z}$. From this it is easy to construct sets satisfying Theorem 4, namely $\mathcal{X}_{24}=\mathcal{Y}_{24}=\{(a+b\sqrt{2}+c\sqrt{3}+d\sqrt{6})/2^k3^\ell : a,b,c,d,k,\ell\in\mathbb{Z}\}$ showing that the only points that can be constructed are of the form

$$\left(\frac{a+b\sqrt{2}+c\sqrt{3}+d\sqrt{6}}{2^k3^\ell},\frac{e+f\sqrt{2}+g\sqrt{3}+h\sqrt{6}}{2^k3^\ell}\right)$$

finishing this case.

3.3 Case analysis for $n = 5$

Theorem 4 has some limitations. Namely, when calculating the intersection of two points the x and y coordinate is found by adding two terms. Theorem 4 says that when each individual term is contained in $\mathcal{X}$ (or $\mathcal{Y}$) then the combination of the terms is also contained; but this does not allow for the possibility that the two individual terms might not be in $\mathcal{X}$ (or $\mathcal{Y}$) but the combination is.

This is the case, for example, when $n = 5$ where we have $CS_5(1,4)=\frac{1}{2}$ and so the set $\mathcal{X}$ in Theorem 4 would need to contain $1/2^k$ for arbitrarily large k. But such points are not constructible for $n = 5$. For this case we will need a different method to limit what points are constructible. We will do this by showing that the set of points which we have already shown can be constructed is closed under taking intersection of two allowable lines through two points which are in the set. So suppose that we have two points

$$(x_1,y_1) = \left(\frac{p_1+q_1\sqrt{5}}{2}\right)(1,0)+\left(\frac{r_1+s_1\sqrt{5}}{2}\right)\left(\frac{1}{2},\frac{1}{2}\sqrt{5-2\sqrt{5}}\right) \quad (1)$$

$$(x_2,y_2) = \left(\frac{p_2+q_2\sqrt{5}}{2}\right)(1,0)+\left(\frac{r_2+s_2\sqrt{5}}{2}\right)\left(\frac{1}{2},\frac{1}{2}\sqrt{5-2\sqrt{5}}\right) \quad (2)$$

where $p_i+q_i\equiv r_i+s_i\equiv 0 \pmod 2$ for $i=1,2$. Let us form a new point (x^*,y^*) found by taking a fold with angle θ_1 through (x_1,y_1) and a fold with angle θ_2 through (x_2,y_2), where

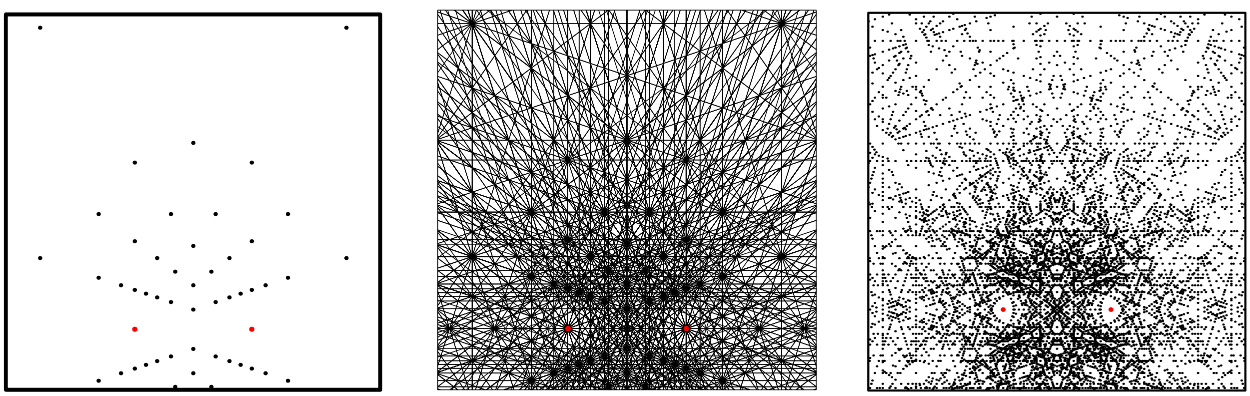
$$(x^*,y^*)=\left(\frac{p^*+q^*\sqrt{5}}{2}\right)(1,0)+\left(\frac{r^*+s^*\sqrt{5}}{2}\right)\left(\frac{1}{2},\frac{1}{2}\sqrt{5-2\sqrt{5}}\right).$$

We need to verify that p^*,q^*,r^*,s^* are integer with $p^*+q^*\equiv r^*+s^*\equiv 0 \pmod 2$. In Tables 4 and 5 we have listed the values of p^*,q^*,r^*,s^* for the various possible pairs of angles. Using the modular conditions of the p_i,q_i,r_i,s_i simple computations show that the modular conditions of p^*,q^*,r^*,s^* also hold. This shows that these points are closed under taking intersections of folds finishing this case.

4 Concluding comments

We have focused on some simple values of n, mainly those for which the form of the points were easy to describe and for which a practical implementation in origami is possible. Of course, one can ask the same question for arbitrary n, and in this direction a complete answer is known using some existence arguments expressing points as numbers in a subring of the complex plane (see [1]). Our approach was somewhat different by giving constructive arguments. We have also seen that each point can be constructed using relatively few steps.

One open question is how many points can be constructed which need at most m lines to construct. For instance in Figure 5 we have started with the two reference points and then took the intersection of all allowable lines that pass through these points for $n = 10$. The result are 74 points, some of which are indicated in Figure 5a (plus some more that are outside the region). All allowable lines through these points are shown in Figure 5b and some of the 18195 points of intersection are marked in Figure 5c.



(a) Points after initial folds (b) With all lines drawn in (c) All resulting intersections

Figure 5: Points that can be constructed for $n = 10$ using six or fewer lines.

In particular, every point in Figure 5c can be constructed using 6 or fewer lines. The number of points that can be constructed using m lines appears to grow quickly and might grow super exponentially. For example, if we repeat the above process for k generations then we get the following number of points that can be constructed within k generations (i.e., at most $2k$ lines).

	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$	$k=7$	$k=8$
$n=3$	2	4	8	20	60	204	748	2860	11180
$n=4$	2	8	39	277	2685	29321			
$n=5$	2	14	176	4188					
$n=6$	2	22	529	35035					
$n=8$	2	44	3239						
$n=10$	2	74	18195						

In general if $a_n(k)$ is the number of points that can be constructed in k generations for a given n

then we have that $a_n(0) = 2$ and that

$$a_n(k+1) \le \binom{a_n(k)}{2} n(n-1) < \left(\frac{na_n(k)}{2}\right)^2.$$

Since we have to choose two points that were already constructed and two distinct angles to fold through those points. Of course we can construct points in multiple ways giving us the inequality. Iteratively applying this inequality we can conclude that

$$a_n(k) \le \frac{n^{2^{k+1}-2}}{2^{2^k-1}}$$

showing that we have at most double exponential growth in the number of points that are formed (a similar iterated fold-intersect problem was recently shown to have double exponential growth [3]).

Another variation is to start with a larger collection of initial points and lines to work with. For example, in origami we have a square piece of paper and so from an origami perspective it is more natural to model this by using the four corners $(0,0)$, $(1,0)$, $(0,1)$, $(1,1)$ along with the lines $y = 0$, $y = 1$, $x = 0$, $x = 1$ instead of the two points $(0,0)$, $(1,0)$ and the line $y = 0$. When n is divisible by 4 we were able to construct these additional points and lines anyway and so there is no difference. On the other hand for $n = 6$ in the expanded case we would be able to construct the same points as we would be able to for $n = 12$.

There remain many interesting problems to explore about how to fold a piece of paper.

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CC	$j=0$	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$	$j=6$	$j=7$
$i=0$		$1+\sqrt{2}$	1	$-1+\sqrt{2}$	0	$1-\sqrt{2}$	-1	$-1-\sqrt{2}$
$i=1$	$-1-\sqrt{2}$		$1+\frac{1}{2}\sqrt{2}$	$\frac{1}{2}$	0	$-\frac{1}{4}\sqrt{2}$	$-\frac{1}{4}\sqrt{2}$	$-\frac{1}{2}-\frac{1}{2}\sqrt{2}$
$i=2$	-1	$-1-\frac{1}{2}\sqrt{2}$		$\frac{1}{2}\sqrt{2}$	0	$-1+\frac{1}{2}\sqrt{2}$	$-\frac{1}{2}$	$-\frac{1}{2}\sqrt{2}$
$i=3$	$1-\sqrt{2}$	$-\frac{1}{2}$	$-\frac{1}{2}\sqrt{2}$		0	$\frac{1}{2}-\frac{1}{2}\sqrt{2}$	$-1+\frac{1}{2}\sqrt{2}$	$-\frac{1}{4}\sqrt{2}$
$i=4$	0	0	0	0		0	0	0
$i=5$	$-1+\sqrt{2}$	$\frac{1}{4}\sqrt{2}$	$1-\frac{1}{2}\sqrt{2}$	$-\frac{1}{2}+\frac{1}{2}\sqrt{2}$	0		$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}$
$i=6$	1	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}$	$1-\frac{1}{2}\sqrt{2}$	0	$-\frac{1}{2}\sqrt{2}$		$1+\frac{1}{2}\sqrt{2}$
$i=7$	$1+\sqrt{2}$	$\frac{1}{2}+\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{4}\sqrt{2}$	0	$-\frac{1}{2}$	$-1-\frac{1}{2}\sqrt{2}$	

SS	$j=0$	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$	$j=6$	$j=7$
$i=0$		0	0	0	0	0	0	0
$i=1$	0		$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}$	$-1+\sqrt{2}$	$\frac{1}{4}\sqrt{2}$	$1-\frac{1}{2}\sqrt{2}$	$-\frac{1}{2}+\frac{1}{2}\sqrt{2}$
$i=2$	0	$-\frac{1}{2}\sqrt{2}$		$1+\frac{1}{2}\sqrt{2}$	1	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}$	$1-\frac{1}{2}\sqrt{2}$
$i=3$	0	$-\frac{1}{2}$	$-1-\frac{1}{2}\sqrt{2}$		$1+\sqrt{2}$	$\frac{1}{2}+\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{4}\sqrt{2}$
$i=4$	0	$1-\sqrt{2}$	-2	$-1-\sqrt{2}$		$1+\sqrt{2}$	1	$-1+\sqrt{2}$
$i=5$	0	$-\frac{1}{4}\sqrt{2}$	$-\frac{1}{2}\sqrt{2}$	$-\frac{1}{2}-\frac{1}{2}\sqrt{2}$	$-1-\sqrt{2}$		$1+\frac{1}{2}\sqrt{2}$	$\frac{1}{2}$
$i=6$	0	$-1+\frac{1}{2}\sqrt{2}$	$-\frac{1}{2}$	$-\frac{1}{2}\sqrt{2}$	-1	$-1-\frac{1}{2}\sqrt{2}$		$\frac{q}{2}\sqrt{2}$
$i=7$	0	$\frac{1}{2}-\frac{1}{2}\sqrt{2}$	$-1+\frac{1}{2}\sqrt{2}$	$-\frac{1}{4}\sqrt{2}$	$1-\sqrt{2}$	$-\frac{1}{2}$	$-\frac{1}{2}\sqrt{2}$	

CS	$j=0$	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$	$j=6$	$j=7$
$i=0$		1	1	1	1	1	1	1
$i=1$	0		$1+\frac{1}{2}\sqrt{2}$	$\frac{1}{2}+\frac{1}{2}\sqrt{2}$	1	$\frac{1}{2}+\frac{1}{4}\sqrt{2}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}$
$i=2$	0	$-\frac{1}{2}\sqrt{2}$		$1+\frac{1}{2}\sqrt{2}$	1	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}$	$1-\frac{1}{2}\sqrt{2}$
$i=3$	0	$\frac{1}{2}-\frac{1}{2}\sqrt{2}$	$-\frac{1}{2}\sqrt{2}$		1	$\frac{1}{2}$	$1-\frac{1}{2}\sqrt{2}$	$\frac{1}{2}-\frac{1}{4}\sqrt{2}$
$i=4$	0	0	0	0		0	0	0
$i=5$	0	$\frac{1}{2}-\frac{1}{4}\sqrt{2}$	$1-\frac{1}{2}\sqrt{2}$	$\frac{1}{2}$	1		$-\frac{1}{2}\sqrt{2}$	$\frac{1}{2}-\frac{1}{2}\sqrt{2}$
$i=6$	0	$1-\frac{1}{2}\sqrt{2}$	$\frac{1}{2}$	$\frac{1}{2}\sqrt{2}$	1	$1+\frac{1}{2}\sqrt{2}$		$-\frac{1}{2}\sqrt{2}$
$i=7$	0	$\frac{1}{2}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}+\frac{1}{4}\sqrt{2}$	1	$\frac{1}{2}+\frac{1}{2}\sqrt{2}$	$1+\frac{1}{2}\sqrt{2}$	

Table 3: Values of CC_8, SS_8 and CS_8.

θ_1	θ_2	p^*	q^*
0	$\frac{1}{5}\pi$	p_2	q_2
0	$\frac{2}{5}\pi$	$p_2 - \frac{3}{2}r_1 + \frac{3}{2}r_2 + \frac{5}{2}s_1 - \frac{5}{2}s_2$	$q_2 + \frac{1}{2}r_1 - \frac{1}{2}r_2 - \frac{3}{2}s_1 + \frac{3}{2}s_2$
0	$\frac{3}{5}\pi$	$p_2 + \frac{1}{2}r_1 - \frac{1}{2}r_2 - \frac{5}{2}s_1 + \frac{5}{2}s_2$	$q_2 - \frac{1}{2}r_1 + \frac{1}{2}r_2 + \frac{1}{2}s_1 - \frac{1}{2}s_2$
0	$\frac{4}{5}\pi$	$p_2 - r_1 + r_2$	$q_2 - s_1 + s_2$
$\frac{1}{5}\pi$	$\frac{2}{5}\pi$	p_1	q_1
$\frac{1}{5}\pi$	$\frac{3}{5}\pi$	p_1	q_1
$\frac{1}{5}\pi$	$\frac{4}{5}\pi$	p_1	q_1
$\frac{2}{5}\pi$	$\frac{3}{5}\pi$	$\frac{3}{2}p_1 - \frac{1}{2}p_2 + \frac{5}{2}q_1 - \frac{5}{2}q_2 + r_1 - r_2$	$\frac{1}{2}p_1 - \frac{1}{2}p_2 + \frac{3}{2}q_1 - \frac{1}{2}q_2 + s_1 - s_2$
$\frac{2}{5}\pi$	$\frac{4}{5}\pi$	$\frac{1}{2}p_1 + \frac{1}{2}p_2 + \frac{5}{2}q_1 - \frac{5}{2}q_2$ $-\frac{1}{2}r_1 + \frac{1}{2}r_2 + \frac{5}{2}s_1 - \frac{5}{2}s_2$	$\frac{1}{2}p_1 + \frac{1}{2}p_2 + \frac{1}{2}q_1 + \frac{1}{2}q_2$ $-\frac{1}{2}r_1 + \frac{1}{2}r_2 - \frac{1}{2}s_1 + \frac{1}{2}s_2$
$\frac{3}{5}\pi$	$\frac{4}{5}\pi$	$\frac{3}{2}p_1 - \frac{1}{2}p_2 + \frac{5}{2}q_1 - \frac{5}{2}q_2$ $+\frac{1}{2}r_1 - \frac{1}{2}r_2 + \frac{5}{2}s_1 - \frac{5}{2}s_2$	$\frac{1}{2}p_1 - \frac{1}{2}p_2 + \frac{3}{2}q_1 - \frac{1}{2}q_2$ $+\frac{1}{2}r_1 - \frac{1}{2}r_2 + \frac{1}{2}s_1 - \frac{1}{2}s_2$

Table 4: The values of p^* and q^* for the $n = 5$ case.

θ_1	θ_2	r^*	s^*
0	$\frac{1}{5}\pi$	r_1	s_1
0	$\frac{2}{5}\pi$	r_1	s_1
0	$\frac{3}{5}\pi$	r_1	s_1
0	$\frac{4}{5}\pi$	r_1	s_1
$\frac{1}{5}\pi$	$\frac{2}{5}\pi$	$-\frac{3}{2}p_1 + \frac{3}{2}p_2 - \frac{5}{2}q_1 + \frac{5}{2}q_2 + r_2$	$-\frac{1}{2}p_1 + \frac{1}{2}p_2 - \frac{3}{2}q_1 + \frac{3}{2}q_2 + s_2$
$\frac{1}{5}\pi$	$\frac{3}{5}\pi$	$-\frac{1}{2}p_1 + \frac{1}{2}p_2 - \frac{5}{2}q_1 + \frac{5}{2}q_2 + r_2$	$-\frac{1}{2}p_1 + \frac{1}{2}p_2 - \frac{1}{2}q_1 + \frac{1}{2}q_2 + s_2$
$\frac{1}{5}\pi$	$\frac{4}{5}\pi$	$-p_1 + p_2 + r_2$	$-q_1 + q_2 + s_2$
$\frac{2}{5}\pi$	$\frac{3}{5}\pi$	$-2p_1 + 2p_2 - 5q_1 + 5q_2$ $-\frac{1}{2}r_1 + \frac{3}{2}r_2 - \frac{5}{2}s_1 + \frac{5}{2}s_2$	$-p_1 + p_2 - 2q_1 + 2q_2$ $-\frac{1}{2}r_1 + \frac{1}{2}r_2 - \frac{1}{2}s_1 + \frac{3}{2}s_2$
$\frac{2}{5}\pi$	$\frac{4}{5}\pi$	$-\frac{1}{2}p_1 + \frac{1}{2}p_2 - \frac{5}{2}q_1 + \frac{5}{2}q_2$ $+\frac{1}{2}r_1 + \frac{1}{2}r_2 - \frac{5}{2}s_1 + \frac{5}{2}s_2$	$-\frac{1}{2}p_1 + \frac{1}{2}p_2 - \frac{1}{2}q_1 + \frac{1}{2}q_2$ $-\frac{1}{2}r_1 + \frac{1}{2}r_2 + \frac{1}{2}s_1 + \frac{1}{2}s_2$
$\frac{3}{5}\pi$	$\frac{4}{5}\pi$	$-\frac{3}{2}p_1 + \frac{3}{2}p_2 - \frac{5}{2}q_1 + \frac{5}{2}q_2$ $-\frac{1}{2}r_1 + \frac{3}{2}r_2 - \frac{5}{2}s_1 + \frac{5}{2}s_2$	$-\frac{1}{2}p_1 + \frac{1}{2}p_2 - \frac{3}{2}q_1 + \frac{3}{2}q_2$ $-\frac{1}{2}r_1 + \frac{1}{2}r_2 - \frac{1}{2}s_1 + \frac{3}{2}s_2$

Table 5: The values of r^* and s^* for the $n = 5$ case.