AN INVESTIGATION OF THE FEASIBILITY

OF A METHOD OF OBTAINING DATA

FROM WHICH

THE STABILITY DERIVATIVES $C_{m_{\dot{\alpha}}}$ AND C_{m_q}

CAN BE SEPARATED

by

DAVID ARTHUR CLUNIES

and

H. WILLIAM STILES, JR. '58

Submitted in Partial Fulfillment

of the Requirements for the

Degree of Bachelor of Science

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

June, 1957

Signature of Authors

....
Department of Aeronautical Engineering
May 20, 1957

Certified by

....
Thesis Supervisor

Accepted by

....
Chairman, Departmental Committee on Theses

TABLE OF CONTENTS

NOTATION

α	Angle of attack
θ	Angle between the flight path and the horizontal
δ or δ_e	Angle of the elevator, measured from the neutral position
M	Mass of the Aircraft
u	Forward velocity
τ	Time Constant of the system
c	Mean Aerodynamic chord of the wing
M_p	Moment about the center of gravity
Z_p	Force in Z direction of the standard right handed coordinate system
C_m	Pitching moment coefficient
C_z	Z force coefficient

DERIVATIVE NOTATION:

$$A_{N_R} \equiv \frac{dA_N}{dR}$$

DOT NOTATION:

$$\dot{A} \equiv \frac{dA}{dt} \qquad \ddot{A} \equiv \frac{d^2A}{dt^2}$$

$$C_{m_q} \equiv C_{m_{\dot{\theta}}}$$

SUMMARY

This paper is an investigation of a proposed method for obtaining flight data from which the stability derivatives $C_{m_{\dot{\alpha}}}$ and C_{m_q} can be separated. The investigation covers the use of a rocket pulse, in the longitudinal mode, in conjunction with a step elevator input. Data resulting from this type of excitation was used in attempting the separation. The pitching moment derivatives were extracted for three different forcing functions: elevator alone, pulse alone, and the combination of elevator and pulse. Responses were determined analytically by LaPlace transforms, using known constants. Sufficient test points were determined to assure an accurate representation of the response and these test points were used in the separative procedure. The Derivative Method and Shinbrot Method were used to obtain the equations to be solved. The equations were reduced in number by the Least Squares procedure, and the coefficients were determined by the Gauss Method. No adequate separation was achieved, but trends were established which would indicate that separation may be possible with future refinement of method.

INTRODUCTION

The separation of stability derivatives from response data has been taken up in detail by Klein and Sedney. [1] This paper will make use of the three conditions which they consider necessary for separation:

1. A fortunate choice of time increments is made.
2. The order of the true characteristic equation is equal to the order of the assumed characteristic equation.
3. Initial conditions are such that all modes are excited in the motion.

In purely longitudinal responses, caused by elevator disturbance, it is impossible to extract all the moment derivatives. Due to the linear dependence of the system, two of the derivatives can not be determined. [2] They are linearly dependent according to the lift equation:

$$C_{L_\alpha}(\alpha) + M U_0(\dot{\alpha}) + C_{L_\delta}(\delta) - (q) M U_0 = 0$$

Therefore, the simultaneous equations formed to determine the moment derivatives cannot be solved uniquely for the unknowns unless the value of one of them is known from other sources.

The sum of $C_{m_{\dot{\alpha}}}$ and C_{m_q} as separated agrees with the sum of the true values. This indeterminancy only prevents

1. Klein and Sedney; Journal of Aeronautical Sciences

2. TN 2340

the separation of the derivatives $C_{m_{\dot{\alpha}}}$ and C_{m_q} because they are of the same order of magnitude.

Taking these factors into account, the following procedure was used. A linear system was assumed, allowing two degrees of freedom. The forward velocity was to remain constant during the excitation of the system, to allow the order of the equations of motion to be reduced. The flight path was level initially. The rocket pulse was applied at the center of gravity of the aircraft producing a Z force, but no moment about the center of gravity.

PROCEDURE

Responses

The longitudinal equations of motion used are as follows:

Z Force

$$\dot{\alpha} = \dot{\theta} + g/U_0 + \frac{C_{Z\alpha}}{2\tau}(\alpha) + \frac{C_{Z\delta}}{2\tau}(\delta) - \frac{C_T}{2\tau}(\alpha + \alpha_T) \pm \frac{Z_p}{MU}$$

Moment

$$\ddot{\theta} = V_y C_{m_\alpha}(\alpha) + V_y\left(\frac{c}{2U}\right)C_{m_{\dot{\alpha}}}(\dot{\alpha}) + V_y\left(\frac{c}{2U}\right)C_{m_{\dot{\theta}}}(\dot{\theta}) + V_y C_{m_\delta}(\delta_e) \pm \frac{M_p}{I_y}$$

In this presentation α and δ_e are total angles, not increments. If α and δ_e are incremental changes from trim, these equations become:

$$\dot{\alpha} = \dot{\theta} + \frac{C_{Z\alpha}}{2\tau}(\alpha) + \frac{C_{Z\delta}}{2\tau}(\delta_e) \pm \frac{Z_p}{MU}$$

$$\ddot{\theta} = V_y C_{m_\alpha}(\alpha) + V_y\left(\frac{c}{2U}\right)C_{m_{\dot{\alpha}}}(\dot{\alpha}) + V_y\left(\frac{c}{2U}\right)C_{m_{\dot{\theta}}}(\dot{\theta}) + V_y C_{m_\delta}(\delta_e) \pm \frac{M_p}{I_y}$$

To obtain the responses, representative constants for an aircraft were obtained and applied to the equations. These figures are not necessarily for any one model.

$$V_y C_{m_\alpha} = -2.60 \qquad V_y\left(\frac{c}{2U}\right)C_{m_{\dot{\alpha}}} = -.107$$

$$V_y\left(\frac{c}{2U}\right)C_{m_{\dot{\theta}}} = -.473 \qquad \frac{C_{Z\delta}}{2\tau} = .053$$

$$\frac{C_{Z\alpha}}{2\tau} = -.863 \qquad V_y C_{m_{\delta_e}} = 5.511$$

Substituting these values and placing in LaPlace form, the equations become:

$$(p + .863)\alpha - (p)\,\theta = (.053)\delta \pm \frac{Z_p}{MU}$$

$$(.107\,p + 2.6)\alpha + (p^2 + .473p)\,\theta = (5.511)\delta \pm \frac{M_p}{I_y}$$

From this form, the equations are solved for α and θ by use of determinants.

I. For Elevator Input

α Determinant:

$$\bar{\alpha} = \frac{\begin{vmatrix} .053\delta & -p \\ 5.511\delta & (p^2 + .473p) \end{vmatrix}}{\begin{vmatrix} p + .863 & -p \\ (.107p + 2.6) & (p^2 + .473p) \end{vmatrix}}$$

α Determinant Expanded:

$$\bar{\alpha} = \frac{(.053p)\delta + (5.536069)\delta}{p^2 + 1.443p + 3.008199}$$

Six place accuracy is introduced here to in~~s~~sure accuracy in later work, as will become apparent. Such accuracy would not be attainable in the real case, however.

θ Determinant:

$$\bar{\theta} = \frac{\begin{vmatrix} (p+.863) & .053\,\partial \\ (.107p+2.6) & 5.511\,\partial \end{vmatrix}}{\begin{vmatrix} (p+.863) & -p \\ (.107p+2.6) & p(p+.473) \end{vmatrix}}$$

θ Determinant Expanded:

$$\bar{\theta} = \frac{(5.505329\,p)\partial + 4.618193\,\partial}{p(p^2+1.443p+3.008199)}$$

A unit step, $\partial = \frac{1}{p}$ was applied, giving:

$$\bar{\alpha} = \frac{.053}{p^2+1.443p+3.008199} + \frac{5.536069}{p(p^2+1.443+3.008199)}$$

$$\bar{\theta} = \frac{5.505329}{p(p^2+1.443p+3.008199)} + \frac{4.618193}{p^2\,(p^2+1.443p+3.008199)}$$

This is a standard form for which the following solution is obtained.[1] $\omega_N = 1.734416 \qquad \zeta = .415990$

$$\alpha(t) = \frac{.053}{3.008199}\left[\frac{\omega_N}{\sqrt{1-\zeta^2}}\, e^{-t/\tau} \sin(\omega_N\sqrt{1-\zeta^2}\, t)\right] + \frac{5.536069}{3.008199}\left[1 + \frac{e^{-t/\tau}}{\sqrt{1-\zeta^2}} \sin(\omega_N\sqrt{1-\zeta^2}\, t + \psi_1)\right]$$

WHERE $\frac{1}{\tau} = \zeta\omega_N \quad \psi_1 = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{-\zeta}$

$$\theta(t) = \frac{5.505329}{3.008199}\left[1 + \frac{e^{-t/\tau}}{\sqrt{1-\zeta^2}} \sin(\omega_N\sqrt{1-\zeta^2}\, t + \psi_1)\right] + \frac{4.618193}{3.008199}\left[t - \frac{2\zeta}{\omega_N} + \frac{e^{-t/\tau}}{\omega_N\sqrt{1-\zeta^2}} \sin(\omega_N\sqrt{1-\zeta^2}\, t + \psi_2)\right] \quad \psi_2 = 2\psi_1$$

---

1. Nixon, _Principles of Automatic Control_

Through the use of a zero check:

ψ_1 = - 1.999827 radians

ψ_2 = - 3.999655 radians

These expressions are evaluated for a δ magnitude of .1 radians. See Figure I (page 24) and Table I (page 26).

These expressions are now differentiated with respect to time and similarly e valuated and tabulated. In order to gain an accurate picture of the responses, the expressions were evaluated at one/tenth second intervals. The θ expression was differentiated twice to obtain the $\ddot{\theta}$ expression, and this was tabulated for use in the separation.

In order that this evaluation be commensurate with the 6-place accuracy of the rest of the work, it was necessary to interpolate in the sine, cosine, and exponential tables, as 6-place tables for these functions are not available. The tables interpolated are stated in the bibliography.

The interpolation was accomplished by the use of the Taylor Series Expansion.

The expansions were performed as follows:

Exponential

$$f(x_0+h) = [f(x_0) + \frac{h\,f'(x_0)}{1!} + \frac{h^2 f''(x_0)}{2!} \ldots$$

$$f(x) = e^{-x} \qquad f(x_0) = e^{-x_0}$$

$$f'(x) = -e^{-x} \qquad f'(x_0) = -e^{-x_0}$$

$$f''(x) = e^{-x} \qquad f''(x_0) = e^{-x_0}$$

$$x = x_0 + h$$

EXAMPLE: $x = .111111$, $x_0 = .1111$, $h = .000011$

$$e^{-x_0+h} = e^{-x_0}[1 - \frac{h}{1!} + \frac{h^2}{2!} \ldots$$

Sine

$$f(x) = \sin x \qquad f(x_0) = \sin x_0$$
$$f'(x) = \cos x \qquad f'(x_0) = \cos x_0$$
$$f''(x) = -\sin x \qquad f''(x_0) = -\sin x_0$$

$$f(x_0+h) = \sin(x_0+h) = \sin x_0 \left[1 + \frac{h \cot x_0}{1!} - \frac{h^2}{2} \right. \cdots$$

Cosine (Similar to Sine)

$$\cos(x_0+h) = \cos x_0 \left[1 - \frac{h \tan x_0}{1!} \right. \cdots$$

II. For Rocket Pulse Input

α Determinant:

$$\bar{\alpha} = \frac{\begin{vmatrix} \pm Z_p/MU & -p \\ \pm M_p/I_y & p(p+.473) \end{vmatrix}}{\begin{vmatrix} (p+.863) & -p \\ (.107p+2.6) & p(p+.473) \end{vmatrix}}$$

α Determinant Expanded:

(Consider pulse applied at center of gravity of the aircraft. $M_p = 0$)

$$= \frac{\pm Z_p/MU \ (p+.473)}{p^2 + 1.443p + 3.008199}$$

Θ Determinant:

$$\bar{\Theta} = \frac{\begin{vmatrix} (p+.863) & \pm Z_p/MU \\ (.107p+2.6) & \pm M_p/I_y \end{vmatrix}}{\begin{vmatrix} (p+.P63) & -p \\ (.107p+2.6) & p(p+.473) \end{vmatrix}}$$

<u>Θ Determinant Expanded</u>:

$$\bar{\Theta} = \frac{-(.107p + 2.6)(\pm\ Z_p/MU)}{p(p^2 + 1.443p + 3.008199)}$$

Apply a unit pulse in the negative Z direction. Since the duration of the pulse is to be two seconds, it may be considered analagous to a step, over this time interval. Taking the Inverse LaPlace Transform as before, the expressions for α and Θ as functions of time are:

$$\alpha(t) = \frac{-1}{3.008199}\left[\frac{\omega_N}{\sqrt{1-\rho^2}} e^{-t/\tau} \sin(\omega_N\sqrt{1-\rho^2}\,t)\right] - \frac{.473}{3.008199}\left[1 + \frac{e^{-t/\tau}}{\sqrt{1-\rho^2}} \sin(\omega_N\sqrt{1-\rho^2}\,t + \psi_1)\right]$$

$$\Theta(t) = \frac{.107}{3.008199}\left[1 + \frac{e^{-t/\tau}}{\sqrt{1-\rho^2}} \sin(\omega_N\sqrt{1-\rho^2}\,t + \psi_1)\right] + \frac{2.6}{3.008199}\left[t - \frac{2\rho}{\omega_N} + \frac{e^{-t/\tau}}{\omega_N\sqrt{1-\rho^2}} \sin(\omega_N\sqrt{1-\rho^2}\,t + \psi_2)\right]$$

τ, ω_N, ρ, ψ_1, ψ_2 AS FOR STEP

These functions were evaluated and plotted for a pulse of six thousand pounds. (See Figure II on page 25 and Table II on page 27.) The derivatives were also taken and evaluated. The responses were computed at one/tenth second intervals.

The results were checked for accuracy in two ways. First, the Initial Value Theorem was applied and secondly, the responses were obtained from a computer. In all cases, the accuracy was correct to at least the fifth-place decimal.

A rocket thrust of six thousand pounds, applied at the center of gravity, was used for the pulse response. This was found to be easily practicable with solid propellant or liquid propellant rockets at a minimum of additional weight to the aircraft. Either a single rocket or several smaller rockets could be used. The firing of several rockets can be electrically synchronized to within five to ten milliseconds. A nearly ideal square wave is obtainable with such rockets.

The intent of this work was to try a combination of step and pulse to determine the possibility of the combination providing separable results. Although the step and pulse were determined separately, the system is linear and the combination can be obtained by addition.

Derivative Method

In order to maintain an optimum degree of accuracy in fitting the response curve,[1] and to minimize the error of individual inaccuracies, twelve stations of data were selected from the twenty points tabulated.

The values of α, $\dot{\alpha}$, θ, $\dot{\theta}$, $\ddot{\theta}$ at these points were used in the equations of motion with the unknowns being the desired coefficients.

The twelve equations were reduced by the Method of Least Squares [2.] particularly adapted to the aeronautical equation. [3.]

Using the "Operator" notation, the equation of interest for the step, in symbol form, is as follows:

$$D^2(\theta) = A(\alpha) + B(D\alpha) + C(D\theta) + E(\delta_e)$$

1. According to Klein and Sedney.
2. Scarborough, Numerical Mathematical Analysis.
3. TN 2340.

The four equations obtained by Least Squares are:

$$\text{I]}\ \sum_{1-12} D^2(\theta)[\alpha] = A\sum_{1-12}(\alpha)^2 + B\sum_{1-12}\alpha(D\alpha) + C\sum_{1-12}\alpha(D\theta) + E\sum_{1-12}\delta_e(\alpha)$$

$$\text{II]}\ \sum_{1-12} D^2(\theta)[D\alpha] = A\sum_{1-12}(\alpha)(D\alpha) + B\sum_{1-12}(D\alpha)^2 + C\sum_{1-12}(D\alpha)(D\theta) + E\sum_{1-12}\delta_e(D\alpha)$$

$$\text{III]}\ \sum_{1-12} D^2(\theta)[D\theta] = A\sum_{1-12}\alpha(D\theta) + B\sum_{1-12}(D\alpha)(D\theta) + C\sum_{1-12}(D\theta)^2 + E\sum_{1-12}\delta_e(D\theta)$$

$$\text{IV}\ \sum_{1-12} D^2(\theta)\,\delta_e = A\sum_{1-12}\alpha(\delta_e) + B\sum_{1-12}D\alpha(\delta_e) + C\sum_{1-12}D\theta(\delta_e) + E\sum_{1-12}(\delta_e)^2$$

A sample table of the values of the responses at each point and the summation is included in Table III (page .) The computation was accomplished in this manner to make checking figures and procedure simple and direct.

In the case of the pulse, δ was equal to zero and the equation for solution was reduced in complexity to:

$$D^2(\theta) = A(\alpha) + B(D\alpha) + C(D\theta)$$

Reduced by Least Squares:

$$\text{I]}\ \sum_{1-12} D^2(\theta)(\alpha) = A\sum_{1-12}(\alpha)^2 + B\sum_{1-12}(D\alpha)\alpha + C\sum_{1-12}(D\theta)\alpha$$

$$\text{II]}\ \sum_{1-12} D^2(\theta)(D\alpha) = A\sum_{1-12}\alpha(D\alpha) + B\sum_{1-12}(D\alpha)^2 + C\sum_{1-12}(D\theta)(D\alpha)$$

$$\text{III}\ \sum_{1-12} D^2(\theta)(D\theta) = A\sum_{1-12}\alpha(D\theta) + B\sum_{1-12}(D\alpha)(D\theta) + C\sum_{1-12}(D\theta)^2$$

Determination of Unknowns

Since the number of unknowns equals the number of valid equations, the values of the stability derivatives can be obtained. One convenient method of determining the derivatives is the Gauss Method. [1] This method was used. For a fuller development of this method see Appendix A (page 33).

1. Hildebrand, *Introduction to Numerical Analysis*.

When separated, the alphabetical coefficients are analagous to the following:

$$A = \gamma_y C_{m_\alpha}$$
$$B = \gamma_y \left(\frac{c}{2u}\right) C_{m_{\dot\alpha}}$$
$$C = \gamma_y \left(\frac{c}{2u}\right) C_{m_{\dot\theta}}$$
$$E = \gamma_y C_{m_{\delta_e}}$$

The values of the above, as initiated in the response computation are:

A = -2.6 C = -.473

B = -.107 E = 5.511

The results of the separations are tabulated below.

TABLE IV

	A	B	C	E
Step	-2.504599	.000420	-.577367	5.495026
Pulse(6000 lbs.)	-3.188991	.206212	-.733333	
Pulse(60,000 lbs.)*	-2.223690	-.359600	-.237113	
Combination	-2.498285	-.006755	-.623609	5.627733

---

* Upon inspection of the data, it was observed that most of the six-place accuracy had been lost in the process of separation of the six-thousand pound pulse response. To determine whether this loss of accuracy was due merely to the size of the pulse used, a solution was obtained assuming a pulse of sixty-thousand pounds. This was easily managed by moving the decimal points of the responses to the six-thousand pound pulse.

DISCUSSION OF ERROR

It may be assumed that the response data is very accurate since the only source of error in the LaPlace usage would come from the user, and the responses check very well with computer results. [1.]

One must then examine possible sources of error other than calculative. The most damaging possibility would be that of violating condition one. [2.] How well do the twelve points of data used represent the curve? In Tables I and II, the points used are numbered in the left hand column. The α and θ values, being fairly regular, seem adequately represented by the test points. The derivatives are more irregular; the test point representation of the response here may not be accurate. It may be assumed that the degree of misrepresentation is not great, however, and will not significantly alter the results.

1. Tape 317-22-24	Kavanagh	2004.8	3-17-57
Tape 317-22-24	Kavanagh	1846.0	4-17-57

2. See page 3 .

CONCLUSION

Due to the limited scope of this paper, it is impossible to reach any definite conclusion or to sufficiently evaluate the effectiveness of this separation method. However, considerable evidence has been obtained to support the hypothesis upon which this paper is based.

As was expected for the step response separation, the desired coefficients B and C are not in agreement with the inputs. The sum B + C does compare favorably as expected. This indicates that the calculations were accurate. Therefore, the results of the other separations can be assumed precise enough to determine the success or failure of the separation.

The figures for the small pulse show that no separation has been accomplished. In addition, the sum B + C is not in agreement with the input sum. The figures for the large pulse show that the accuracy is improved, and a trend toward separation is demonstrated, due to the increase in size of the pulse. This suggests that a greater degree of excitation is necessary for the extraction, due to the loss of accuracy in the extraction process.

For the combination of the step elevator and the small pulse, the results are encouraging. The correct signs are evidenced, which was not true for the step alone. This indicates a decided trend toward separation, since the small pulse (although inaccurate itself) increased the accuracy of the combination.

From the indications of the two pulse response separations and the separations for the step alone and the combination, it may be theorized that the following is deserving of further investigation and refinement.

The separation of data compiled from the combination of a rocket pulse of value Z_P/MU approximately equal to .05, and a step elevator input of reasonable value, may yield acceptable values for the stability derivatives $C_{m_{\dot{\alpha}}}$ and C_{m_q} for any aircraft.

One other possibility which should be investigated would be that of a rocket pulse applied eccentrically with the center of gravity. It seems entirely possible that an eccentrically placed pulse (alone or combined) might provide responses which would be effective for the separation.

Shinbrot Method

In this "equations-of-motion" method developed by Shinbrot,[1] the equation to be solved is multiplied by N "method functions" $y_\gamma(t)$. The resulting equations are then integrated from zero to T, this being the length of the data run. Successive integration by parts eliminates the explicit dependence of the equations on derivatives of the data. The resulting N linear simultaneous equations contain the desired coefficients as integrals involving the recorded data which must be evaluated. Then the equations can be solved by least squares for the desired coefficients.

The method function chosen is $y_\gamma(t) = \sin^m \omega_\gamma t \qquad \gamma = 1, \ldots, N$ where m is the highest derivative occurring in the equation and $\omega_\gamma = \frac{\gamma \pi}{2T}$. Suitable frequencies are chosen to meet zero initial and final conditions. It will be noted that the above choice of method function results in the last quarter cycle of the odd number frequencies being considered identically zero to satisfy final conditions, but this seems to have negligible effect on the results.

Finally, the following approximation is used to evaluate the integrals occurring in the equations:

$$\int_a^b x(t)y(t)dt = \Delta t \sum_{n=0}^{2h} x(t_n)\Gamma_n(y)$$

$$\int_a^b \dot{x}(t)y(t)dt = -\Delta t \sum_{n=0}^{2h} x(t_n)\Gamma_n(\dot{y})$$

where the interval (a,b) is divided into 2h equal parts by points $t_0 = a < t_1 < \ldots < t_{2h} = b$, where $t_{n-1} - t_n = \Delta t =$ constant and x(t) is numerical data.

1. TN 3288.

The moment equation under consideration is

$$\dot{q} = A\alpha + B\dot{\alpha} + Cq + E\delta_e$$

where the coefficients are defined on page 13 and $q = \dot{\theta}$.

Introducing the method function produces

$$\int_0^T \dot{q}\sin\omega_\nu t\,dt = A\int_0^T \alpha\sin\omega_\nu t\,dt + B\int_0^T \dot{\alpha}\sin\omega_\nu t\,dt + C\int_0^T q\sin\omega_\nu t\,dt + E\int_0^T \delta_e\sin\omega_\nu t\,dt$$

For this problem T=2 seconds, $\Delta t = 0.1$ seconds and these frequencies

ν	1	2	3	4	5	6
ω_ν	$\frac{\pi}{2}$	π	$\frac{10}{9}\pi$	$\frac{3}{2}\pi$	$\frac{5}{3}\pi$	2π

Substitution produces six equations of the form

$$-\sum_{n=0}^{20} q\Gamma_n(\dot{y}_n) = A\sum_{n=0}^{20}\alpha\Gamma_n(y_n) - B\sum_{n=0}^{20}\alpha\Gamma_n(\dot{y}_n) + C\sum_{n=0}^{20} q\Gamma_n(y_n) + E\sum_{n=0}^{20}\delta_e\Gamma_n(y_n)$$

Defining the A coefficient summation to be I, the B coefficient summation to be II, the C coefficient summation to be III, the D coefficient summation to be IV, and the remaining summation to be V, the four equations defined by least squares are:

$$-\sum_1^6 (V)(I) = A\sum_1^6 (I)^2 - B\sum_1^6 (I)(II) + C\sum_1^6 (I)(III) + E\sum_1^6 (I)(IV)$$

$$-\sum_1^6 (V)(II) = A\sum_1^6 (I)(II) - B\sum_1^6 (II)^2 + C\sum_1^6 (II)(III) + E\sum_1^6 (II)(IV)$$

$$-\sum_1^6 (V)(III) = A\sum_1^6 (I)(III) - B\sum_1^6 (II)(III) + C\sum_1^6 (III)^2 + E\sum_1^6 (III)(IV)$$

$$-\sum_1^6 (V)(IV) = A\sum_1^6 (I)(IV) - B\sum_1^6 (II)(IV) + C\sum_1^6 (III)(IV) + E\sum_1^6 (IV)^2$$

The values of $\Gamma_n(y_n)$ and $\frac{1}{\omega_\nu}\Gamma_n(\dot{y}_n)$ for the six frequencies used are found in Table V.

The step response values of the indicated summations I, II, III, IV, V, $(I)^2$, (I)(II), (I)(III), (I)(IV), (I)(V), $(II)^2$, (II)(III), (II)(IV), (II)(V), $(III)^2$, (III)(IV), (III)(V), $(IV)^2$ and (IV)(V) are found in Table VI.

The 6000 pound pulse response values of the indicated summations appear in Table VII.

The step plus 6000 pound pulse response values of the indicated summations appear in Table VIII.

These values are plugged into the equations defined by least squares on page 18. The equations have been solved using the Gauss Method (see Appendix A) to yield the desired coefficients.

Results

The results of the analysis by the Shinbrot Method are:

True	-2.600	-.107	-.473	5.511	-.580
Step	-2.370466	.127199	-.698975	5.45342	-.571776
Pulse(6000)	-2.656194	-.039893	-.513229	0	-.553122
Combination	-2.455278	.042421	-.592018	5.425398	-.54966

Discussion of Error

It has already been determined that the response data is accurate (see page 14).

Errors inherent in the method of analysis might lay in the fact that only 6 frequencies were used to evaluate the data although Shinbrot regards 16 as adequate. Another source of error here would be the accuracy of the approximation used to evaluate the integrals with Gamma Functions.

The largest source of error present involves the large number of calculations necessary to evaluate all the summations. The use of a digital computer would eliminate the human error. The extreme accuracy needed and the necessary use of six significant figures do not lend themselves manual computation.

Considering the whole picture the error resolved into the results seems to been kept within reasonable limits.

Conclusion

Within the limited range of this analysis there can be no definite conclusions drawn but some interesting comparisons and trends may be noted.

In the step response separation the coefficients B and C compare unfavorably with the true values, although the sum B+C compares favorably. This was as expected.

The 6000 pound pulse response resulted in reasonable accuracy in all coefficients including the correct sign for B. B and C have tended to separate here although the sum B+C has become more in error.

The results of the combination response fell inbetween the first two cases as might be expected. Again the price for the trend towards separation was more error in the sum B+C.

It is interesting to note that the best results come from the small pulse condition which in the Derivative Method analysis were the poorest. This seems to indicate that the Shinbrot Method which uses more data points is sensitive enough to handle such a small excitation.

On the basis of the above results it seems plausible to extend investigation into the areas mentioned on page 16 and also into the use of Shinbrot's Method (preferably programmed on a digital computer) for extraction of the stability derivatives.

BIBLIOGRAPHY

Greenberg; *A Survey of Methods for Determining Stability Parameters of an Aircraft from Dynamic Flight Measurements*; NACA TN 2340.

Hildebrand; *Introduction to Numerical Analysis*; McGraw-Hill, Inc., New York, 1956.

Klein & Sedney; *Some Basic Concepts for Analysing Dynamic Flight-Test Data*; Journal of Aeronautical Sciences; Volume 20, 1953.

Nixon; *Principles of Automatic Control*; Prentice Hall Inc. Englewood Cliffs, New Jersey; 1955.

Scarborough; *Numerical Mathematical Analysis*; Johns Hopkins Press, Baltimore, 1955.

Shinbrot; *On the Analysis of Linear and Nonlinear Dynamical Systems from Transient-Response Data*, NACA TN 3288.

* * * *

Tables

1. Circular and Hyperbolic Sines and Cosines for Radian Arguments; WPA for New York City; N.B.S., 1939.
2. Circular and Hyperbolic Tangents and Cotangents for Radian Arguments; WPA for New York City; N.B.S., 1943.
3. Tables of Function e^x ; N.B.S. Computation Library, 1951.

FIGURE 1

STEP ELEVATOR RESPONSE

α, Θ vs t

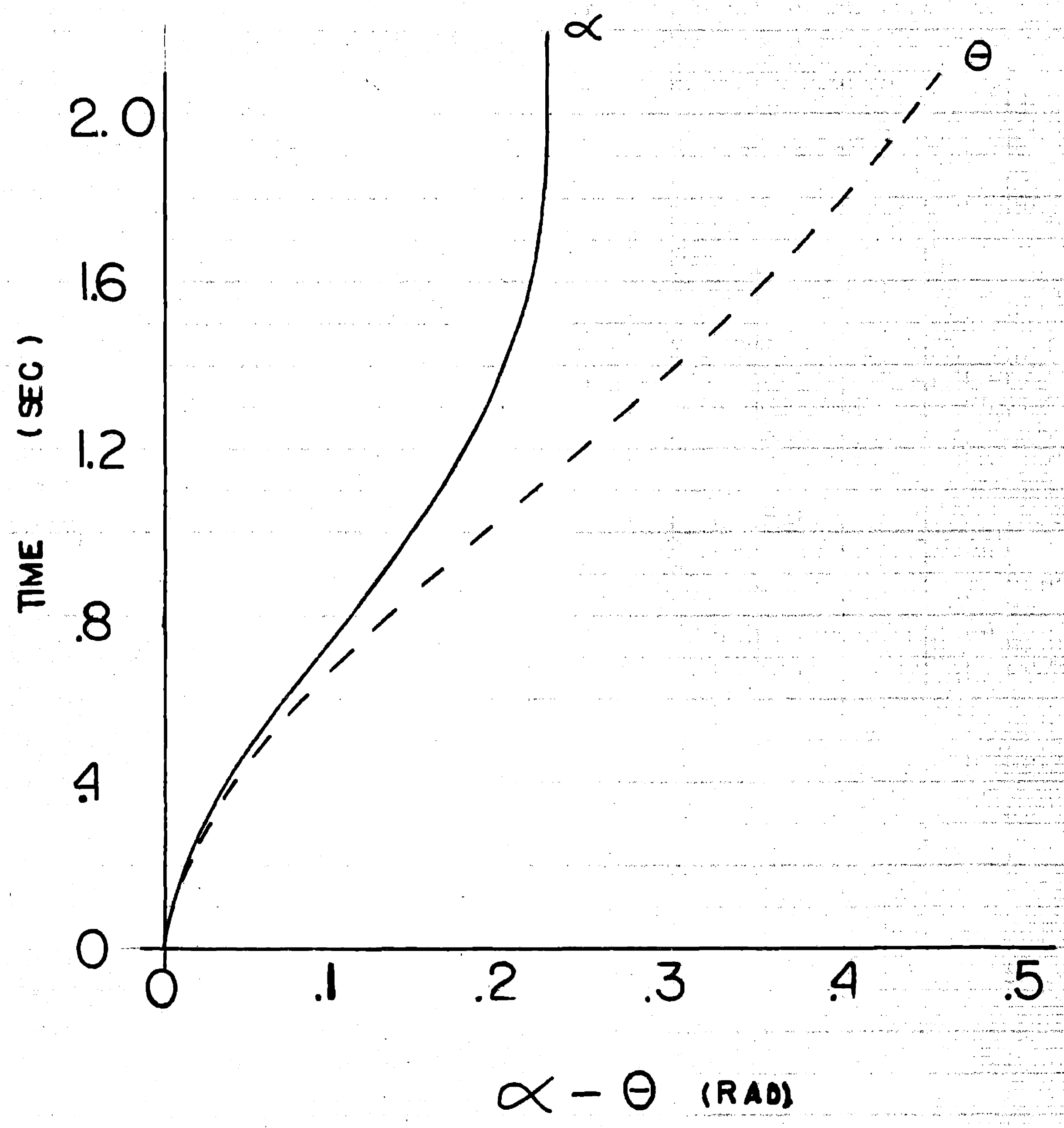

FIGURE 2

ROCKET PULSE RESPONSE

α, Θ vs t

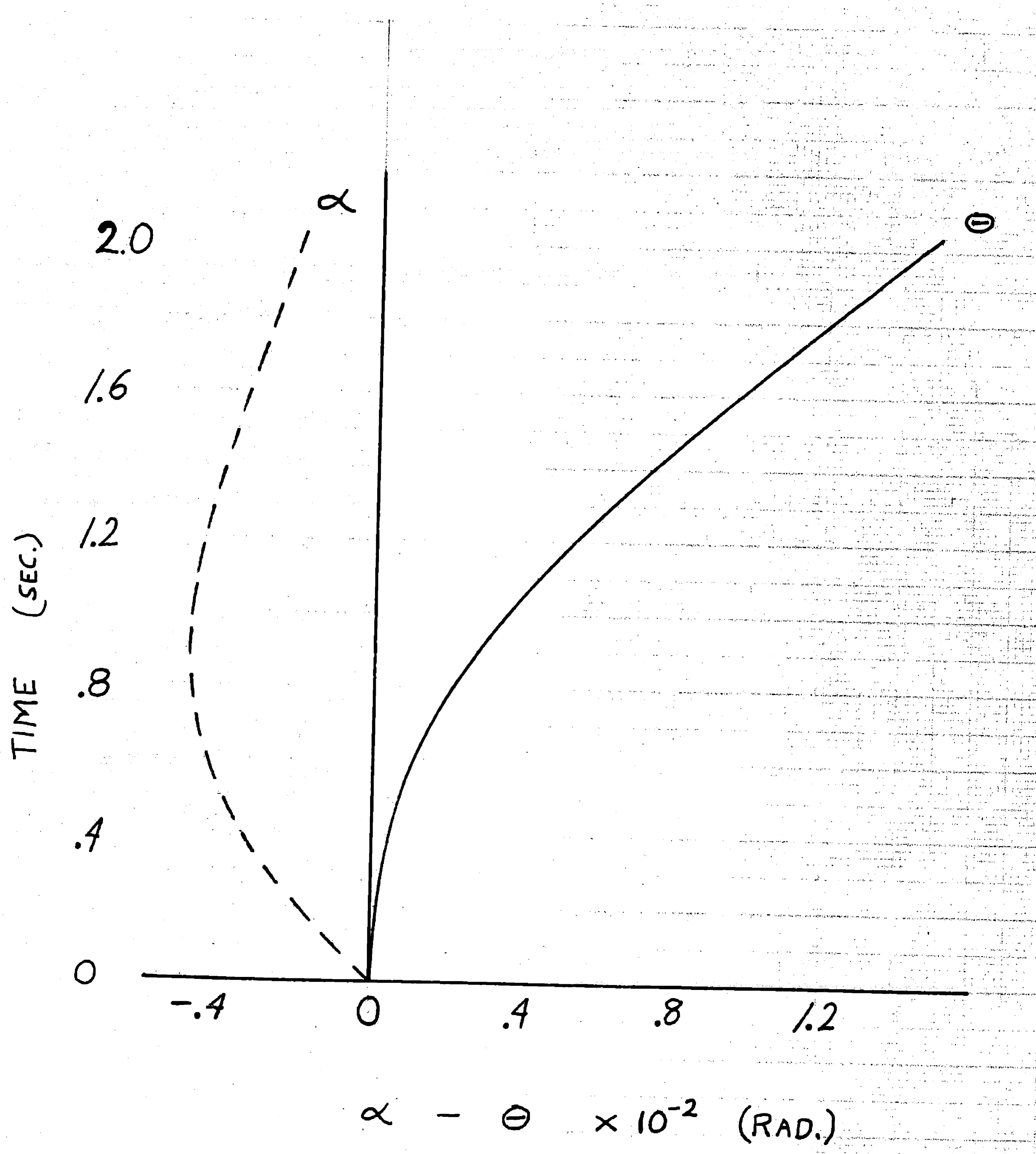

TABLE I

STEP ELEVATOR RESPONSE

	t	$\alpha(t)$	$\theta(t)$	$\dot{\theta}(t)$
	0	0	0	0
1	.1	.003125	.002693	.055232
2	.2	.010899	.010511	.102057
	.3	.022396	.022892	.145835
3	.4	.036958	.039468	.183872
	.5	.053632	.059511	.215982
4	.6	.071712	.082461	.242033
	.7	.090515	.107733	.262129
5	.8	.109377	.134692	.276456
6	.9	.127803	.162830	.285439
7	1.0	.145300	.191592	.289548
	1.1	.161574	.220591	.289256
8	1.2	.176267	.249346	.285256
	1.3	.200440	.277538	.278099
9	1.4	.200256	.304882	.268423
	1.5	.209371	.331159	.256839
10	1.6	.216553	.356207	.234770
	1.7	.221861	.379917	.230219
11	1.8	.225423	.402249	.216251
	1.9	.223050	.423164	.202339
12	2.0	.227746	.442724	.188965

	t	$\ddot{\theta}(t)$	$\dot{\alpha}(t)$
	0	.5505296	0
1	.1	.508924	.056664
2	.2	.464026	.097961
	.3	.409799	.131767
3	.4	.351236	.157235
	.5	.290798	.174990
4	.6	.230356	.185436
	.7	.171526	.189317
5	.8	.115941	.187351
6	.9	.004523	.180428
7	1.0	.018261	.169414
	1.1	-.022385	.155114
8	1.2	-.056902	.125275
	1.3	-.085192	.120093
9	1.4	-.107299	.090308
	1.5	-.123438	.081424
10	1.6	-.127353	.062311
	1.7	-.139321	.044024
11	1.8	-.140156	.026981
	1.9	-.136807	.011445
12	2.0	-.130159	-.002310

TABLE II

ROCKET PULSE RESPONSE

$Z_p = 6000$ lbs.

	t	$\alpha(t)$	$\theta(t)$	$\dot{\theta}(t)$
	0	0	0	0
1	.1	-.001094	.000011	.000334
2	.2	-.002060	.000059	.000749
	.3	-.002889	.000167	.001433
3	.4	-.003578	.000351	.002262
	.5	-.004120	.000623	.003195
4	.6	-.004528	.000992	.004191
	.7	-.004805	.001462	.005215
5	.8	-.004962	.002034	.006232
6	.9	-.005008	.002707	.007217
7	1.0	-.004959	.003476	.008146
	1.1	-.004829	.004334	.009001
8	1.2	-.004632	.005273	.009767
	1.3	-.004383	.006284	.010435
9	1.4	-.004095	.007357	.011000
	1.5	-.003782	.008481	.011459
10	1.6	-.003456	.009645	.011814
	1.7	-.003128	.010840	.012070
11	1.8	-.002807	.012055	.012233
	1.9	-.002501	.013284	.012308
12	2.0	-.002217	.014515	.012308

	t	$\ddot{\theta}(t)$	$\dot{\alpha}(t)$
	0	-.000840	-.011509
1	.1	.002068	-.010318
2	.2	.004733	-.008984
	.3	.006811	-.007586
3	.4	.008420	-.006165
	.5	.009577	-.004759
4	.6	.010311	-.003412
	.7	.010658	-.002151
5	.8	.010660	-.001000
6	.9	.009214	.000030
7	1.0	.009820	.000916
	1.1	.009073	.001657
8	1.2	.008173	.002252
	1.3	.007166	.002708
9	1.4	.006482	.003025
	1.5	.004996	.003214
10	1.6	.003905	.003288
	1.7	.002853	.002951
11	1.8	.001862	.003147
	1.9	.000951	.002958
12	2.0	.000137	.002713

TABLE III

POINT SUMMATION FOR PURPOSES OF LEAST SQUARE REDUCTION

	t	α^2	$\alpha(\dot{\alpha})$	$\alpha(\dot{\theta})$	$(\dot{\alpha})^2$
1	.1	.000010	.000177	.000173	.003211
2	.2	.000119	.001068	.001112	.009596
3	.4	.001366	.005811	.006795	.024723
4	.6	.005143	.013298	.017357	.034387
5	.8	.011963	.020492	.030238	.035100
6	.9	.016334	.023959	.036480	.032554
7	1.0	.021112	.024616	.042071	.028701
8	1.2	.031070	.022082	.050281	.015694
9	1.4	.040102	.018085	.053753	.008156
10	1.6	.046895	.013494	.050840	.003883
11	1.8	.050816	.006082	.048747	.000728
12	2.0	.051868	-.000526	.043036	.000005

	t	$\dot{\alpha}(\ddot{\theta})$	$(\dot{\theta})^2$	$\dot{\alpha}(\dot{\theta})$	$\alpha(\ddot{\theta})$
1	.1	.028838	.003051	.003130	.001590
2	.2	.045456	.010416	.009998	.005057
3	.4	.055227	.033809	.028911	.012981
4	.6	.042716	.058590	.044882	.016519
5	.8	.021722	.076428	.051794	.012681
6	.9	.011642	.081475	.051501	.008246
7	1.0	.003094	.083838	.049053	.002653
8	1.2	-.007128	.081371	.035735	-.010030
9	1.4	-.009690	.072051	.024241	-.021487
10	1.6	-.007935	.055117	.014629	-.027579
11	1.8	-.003782	.046764	.005835	-.031594
12	2.0	-.000301	.035708	-.000437	-.029643

	t	$\ddot{\theta}(\dot{\theta})$
1	.1	.028109
2	.2	.047357
3	.4	.064582
4	.6	.055754
5	.8	.032053
6	.9	.018417
7	1.0	.005287
8	1.2	-.016232
9	1.4	-.028802
10	1.6	-.029899
11	1.8	-.030309
12	2.0	-.024595

$\Sigma \alpha^2 = .276798$

$\Sigma \alpha(\dot{\alpha}) = .147738$

$\Sigma \alpha(\dot{\theta}) = .380883$

$\Sigma (\dot{\alpha})^2 = .196738$

$\Sigma (\ddot{\theta})(\dot{\alpha}) = .180461$

$\Sigma (\dot{\theta})^2 = .638608$

$\Sigma (\dot{\alpha})(\dot{\theta}) = .319272$

$\Sigma \alpha(\ddot{\theta}) = -.060606$

$\Sigma (\dot{\theta})(\ddot{\theta}) = .121722$

$\Sigma \ddot{\theta}(\partial_e) = .119140$

$\Sigma \alpha(\partial_e) = .155142$

$\Sigma \dot{\alpha}(\partial_e) = .133705$

$\Sigma \dot{\theta}(\partial_e) = .262830$

$\Sigma (\partial_e)^2 = .12$

TABLE V

n	t	$\Gamma_n(y_1)$	$\Gamma_n(y_2)$	$\Gamma_n(y_3)$	$\Gamma_n(y_4)$	$\Gamma_n(y_5)$	$\Gamma_n(y_6)$
0	0	.0002	.0014	.0019	.0045	.0061	.0104
1	.10	.2081	.4080	.4505	.5920	.6486	.7532
2	.20	.2070	.3994	.4386	.5618	.6066	.6787
3	.30	.6038	1.0681	1.1407	1.2879	1.2971	1.2187
4	.40	.3938	.6462	.6720	.6604	.6066	.4194
5	.50	.9405	1.3202	1.2971	.9220	.6486	0
6	.60	.5420	.6462	.5909	.2146	0	-.4194
7	.70	1.1851	1.0681	.8467	-.2040	-.6486	-1.2187
8	.80	.6371	.3994	.2334	-.4082	-.6066	-.6787
9	.90	1.3137	.4080	0	-1.1618	-1.2971	-.7532
10	1.00	.6699	0	-.2334	-.6944	-.6066	0
11	1.10	1.3137	-.4080	-.8467	-1.1618	-.6486	.7532
12	1.20	.6371	-.3994	-.5909	-.4082	0	.6787
13	1.30	1.1851	-1.0681	-1.2971	-.2040	.6486	1.2187
14	1.40	.5420	-.6462	-.6720	.2146	.6066	.4194
15	1.50	.9405	-1.3202	-1.1407	.9220	1.2971	0
16	1.60	.3938	-.6462	-.4386	.6604	.6066	-.4194
17	1.70	.6038	-1.0681	-.4505	1.2879	.6486	-1.2187
18	1.80	.2070	-.3994	-.0019	.5618	.0061	-.6787
19	1.90	.2081	-.4080	0	.5920	0	-.7532
20	2.00	.0002	-.0014	0	.0045	0	-.0104

n	t	$\frac{1}{\omega_1}\Gamma_n(\dot{y}_1)$	$\frac{1}{\omega_2}\Gamma_n(\dot{y}_2)$	$\frac{1}{\omega_3}\Gamma_n(\dot{y}_3)$	$\frac{1}{\omega_4}\Gamma_n(\dot{y}_4)$	$\frac{1}{\omega_5}\Gamma_n(\dot{y}_5)$	$\frac{1}{\omega_6}\Gamma_n(\dot{y}_6)$
0	0	.3350	.3397	.3412	.3477	.3502	.3568
1	.10	1.3137	1.2556	1.2377	1.1618	1.1234	1.0367
2	.20	.6371	.5497	.5227	.4082	.3502	.2205
3	.30	1.1851	.7760	.6586	.2040	0	-.3960
4	.40	.5420	.2100	.1185	-.2146	-.3502	-.5773
5	.50	.9405	0	-.2287	-.9220	-1.1234	-1.2814
6	.60	.3938	-.2100	-.3412	-.6604	-.7004	-.5773
7	.70	.6038	-.7760	-1.0090	-1.2879	-1.1234	-.3960
8	.80	.2070	-.5497	-.6412	-.5618	-.3502	.2205
9	.90	.2081	-1.2556	-1.3172	-.5920	0	1.0367
10	1.00	0	-.6795	-.6412	0	.3502	.7136
11	1.10	-.2081	-1.2556	-1.0090	.5920	1.1234	1.0367
12	1.20	-.2070	-.5497	-.3412	.5618	.7004	.2205
13	1.30	-.6038	-.7760	-.2287	1.2879	1.1234	-.3960
14	1.40	-.3938	-.2100	.1185	.6604	.3502	-.5773
15	1.50	-.9405	0	.6586	.9220	0	-1.2814
16	1.60	-.5420	.2100	.5227	.2146	-.3502	-.5773
17	1.70	-1.1851	.7760	1.2377	-.2040	-1.1234	-.3960
18	1.80	-.6371	.5497	.3412	-.4082	-.3502	.2205
19	1.90	-1.3137	1.2556	0	-1.1618	0	1.0367
20	2.00	-.3350	.3397	0	-.3472	0	.3568

TABLE VI

γ	ω_γ	$\frac{1}{\bar{\omega}_\gamma}$ I	$\frac{1}{\bar{\omega}_\gamma}$ II	$\bar{\text{I}}$	II
1	1.570796	-1.125538	-.517982	-1.76799	-.813644
2	3.14159	-.162608	-.811238	-.510848	-2.548577
3	3.49066	-.0735971	-.666089	-.256902	-2.325090
4	4.71239	-.00161917	-.206542	-.00763016	-.973306
5	5.23599	-.0089412	-.1540426	-.046816	-.806566
6	6.283185	-.0133903	-.174173	-.0841337	-1.094361

γ	I	II	III	IV	V
1	1.743279	-1.76799	3.179862	1.27325	-.813644
2	-.953154	-.510848	-.322045	0	-2.548577
3	-.832943	-.256902	-.453043	0	-2.325090
4	.440045	-.00763016	.430174	.4244	-.973306
5	.410830	-.046816	.436693	.38197	-.806566
6	-.376516	-.0841337	-.275064	0	-1.094361

$\sum_1^6 (\text{I})^2$	4.944677	$\sum_1^6 (\text{II})^2$	3.373770
$\sum_1^6 (\text{I})(\text{II})$	-2.351246	$\sum_1^6 (\text{II})(\text{III})$	-5.493826
$\sum_1^6 (\text{I})(\text{III})$	6.788813	$\sum_1^6 (\text{II})(\text{IV})$	-2.252537
$\sum_1^6 (\text{I})(\text{IV})$	2.563310	$\sum_1^6 (\text{II})(\text{V})$	3.475030
$\sum_1^6 (\text{I})(\text{V})$	2.599830		
$\sum_1^6 (\text{III})^2$	10.871894	$\sum_1^6 (\text{IV})^2$	1.947182
$\sum_1^6 (\text{III})(\text{IV})$	4.398129	$\sum_1^6 (\text{IV})(\text{V})$	-1.757127
$\sum_1^6 (\text{III})(\text{V})$	-1.183047		

TABLE VII

r	ω_r	$\frac{1}{\omega_r}$ I	$\frac{1}{\omega_r}$ II	I	II
1	1.570796	.00243662	-.0596622	.003827433	-.09371745
2	3.14159	.0158469	-.00940184	.04978446	-.0295367
3	3.49066	.0133778	-.00476286	.04669735	-.0166255
4	4.71239	.00421871	-.00111286	.01987978	-.00524423
5	5.23599	.00318337	-.00132185	.01666809	-.00692119
6	6.283185	.00374321	-.000740425	.02351928	-.004652227

r	I	II	III	IV	V
1	-.0535136	.003827433	.0965369	0.	-.093717145
2	-.000329563	.04978446	-.0500486	0	-.0295367
3	.00332199	.04669735	-.0438098	0	-.0166255
4	-.00574183	.01987978	.0240694	0	-.00524423
5	-.00616997	.01666809	.00220449	0	-.00692119
6	.00288631	.02351928	-.0209521	0	-.004652227

$\sum_1^6 (I)^2$	.00295422	$\sum_1^6 (II)^2$	.00589997
$\sum_1^6 (I)(II)$	-.000215203	$\sum_1^6 (II)(III)$	-.00414549
$\sum_1^6 (I)(III)$	-.00550736	$\sum_1^6 (II)(V)$	-.00293457
$\sum_1^6 (I)(V)$	.00502903		
$\sum_1^6 (III)^2$	.0147667		
$\sum_1^6 (III)(V)$	-.00688454		

TABLE VIII

ν	I	II	III	IV	V
1	1.689765	-1.764163	3.276399	1.27325	-.907361
2	-.953484	-.461064	-.372094	0	-2.578114
3	-.829621	-.210205	-.496853	0	-2.34172
4	.434303	.0122496	.454243	.4244	-.97855
5	.404660	-.0301479	.438897	.38197	-.813487
6	-.373630	-.0606144	-.296016	0	-1.099013

$\sum_1^6 (I)^2$	4.944677	$\sum_1^6 (II)^2$	3.373770
$\sum_1^6 (I)(II)$	-2.351246	$\sum_1^6 (II)(III)$	-5.493826
$\sum_1^6 (I)(III)$	6.788813	$\sum_1^6 (II)(IV)$	-2.252537
$\sum_1^6 (I)(IV)$	2.490379	$\sum_1^6 (II)(V)$	3.360804
$\sum_1^6 (I)(V)$	2.524155		
$\sum_1^6 (III)^2$	11.606700	$\sum_1^6 (IV)^2$	1.947182
$\sum_1^6 (III)(IV)$	4.532101	$\sum_1^6 (IV)(V)$	-1.881322
$\sum_1^6 (III)(V)$	-1.326296		

APPENDIX A

Separation of coefficients for step elevator response by the use of the Gauss' Method.The equations used are as follows:

Equation 1

$$\Sigma D^2\theta(\alpha) = A\Sigma\alpha^2 + B\Sigma\alpha(D\alpha) + C\Sigma\alpha(D\theta) + E\Sigma\delta_e(\alpha)$$

Equation 2

$$\Sigma D^2\theta(D\alpha) = A\Sigma\alpha(D\alpha) + B\Sigma(D\alpha)^2 + C\Sigma(D\alpha)(D\theta) + E\Sigma(D\alpha)\delta_e$$

Equation 3

$$\Sigma D^2\theta(D\theta) = A\Sigma\alpha(D\theta) + B\Sigma(D\alpha)(D\theta) + C\Sigma(D\theta)^2 + E\Sigma\delta_e(D\theta)$$

Equation 4

$$\Sigma D^2\theta(\delta_e) = A\Sigma\alpha(\delta_e) + B\Sigma(D\alpha)\delta_e + C\Sigma(D\theta)\delta_e + E\Sigma(\delta_e)^2$$

Substituting the values for the summations as obtained from Table III the equations become:

1) $-.060606 = A(.276798) + B(.147738) + C(.380883) + E(.155142)$

2) $.180461 = A(.147738) + B(.196738) + C(.319272) + E(.133705)$

3) $.121722 = A(.380883) + B(.319272) + C(.638608) + E(.262830)$

4) $.119140 = A(.155142) + B(.133705) + C(.262830) + E(.120000)$

Reduction

Equation 1 is divided by the coefficient of A. This yields an expression for A. This expression for A is substituted for A in the remaining three equations. The equations resulting from this are as follows:

$A = -.218954 -.533739(B) -1.376032(C) -.560488(E)$

2) $.212809 = B(.177884) + C(.115980) + E(.050900)$

3) $.205118 = B(.115980) + C(.114501) + E(.049350)$

4) $.153109 = B(.050900) + C(.049350) + E(.033045)$

Equation 2 is now divided by the coefficient of B. This yields an expression for B .This expression for B is substituted into the remaining two equations . Then in similar manner the resulting third equation is divided by the coefficient of C . Finally the resultant expression for C is substituted into the fourth equation yielding a numerical value for E . The equations of each of the coefficients in terms of the others are as follows:

$B = 1.196336 - .651998\ (C) - .286142\ (E)$

$C = 1.706882 - .415694\ (E)$

$E = +5.495026$

Now each value is substituted working backwards until A is finally evaluated.The subsequent values are:

$C = - .577367$

$B = +.000420$

$A = - 2.504599$