STABILITY OF LINEAR FEEDBACK SYSTEMS

by

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ABSTRACT

This thesis is concerned with bounded input - bounded output stability of feedback control systems. The problem is treated in the framework of the spectral theory of linear operators on Banach spaces. It is shown that the determination of stability is equivalent to the calculation of the spectrum of the open loop operator. Several well-known sufficient conditions for stability follow directly from this result and spectral estimates for the operators involved. The general theorem is applied to obtain new results concerning the establishment of stability results by positive operator arguments. In particular, it is shown that systems which are stable for all positive feedback gains may be characterized by the fact that the open loop operator may be factored as the square of an operator which is similar to a positive operator. Necessary and sufficient stability conditions are derived for multiple input - multiple output time invariant systems, and for discrete systems with a periodic feedback gain. These criteria involve the locus of the eigenvalues of a certain matrix of functions. By applying eigenvalue estimation results a number of sufficient conditions for either stability or instability are obtained.

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CONTENTS

CONTENTS (continued)

LIST OF FIGURES

I. INTRODUCTION

A. GENERALITIES

There are two basic approaches to the problem of the stability of feedback control systems. One is to employ a differential (or difference) equation description of the system and to use the ideas of Lyapunov to establish stability.[7,8] The second approach is to employ a so-called input-output description of the system. This leads to the consideration of integral equations in place of the corresponding differential equations and ideas having their roots in functional analysis are used to establish stability of the system. This second point of view is the one taken in this thesis.

The systematic application of functional analysis to feedback system stability seems to have begun with the work of G. Zames[9] and I. W. Sandberg,[15] and since the appearance of their work considerable research has been done in this direction. Treating feedback system stability by means of functional analysis requires care in the area of the definitions of the systems to be considered. The definitions employed in this thesis are strongly motivated by the work of Zames, although the emphasis is somewhat different in several respects.

The main concept required is that of separating the problem of existence of "solutions" of the system equations from the problem of stability. The key is that the "solution" of an unstable system will not in general belong to the Banach space being used for stability analysis. Hence it is impossible to require a priori that "solutions" of the system equations lie in the Banach space chosen for the stability analysis.

In the work of Zames (and others) this problem is circumvented by the introduction of so-called "extended spaces". The corresponding device in this treatment is the definition of the concept of a "regular system" in Chapter II. Here the concept of input-output stability is defined only for "regular systems", with the result that the basic theorem of Chapter II, to the effect that the stability problem is equivalent to an operator spectrum calculation, follows readily from the definitions.

This one result is the focal point of the thesis; it serves to identify the basic mathematical problem of feedback system stability. It provides a theoretical framework from which to attack concrete problems, as well as it generates results of a more abstract nature.

B. ORGANIZATION

The second Chapter of this thesis is devoted to results of a general nature. After some preliminary definitions the basic theorem mentioned above is proved, giving necessary and sufficient conditions for the stability of a certain class of feedback systems. This theorem immediately leads to a class of sufficient conditions for stability through the idea of a spectral estimate. These conditions have been previously obtained by Zames, essentially through the technique of manipulating the sytem equations with the use of the "triangle" and Cauchy-Schwartz inequalities.

The theorem is also applied to derive original results of an abstract nature. One of these is a "converse theorem" related to the establishment of stability conditions by positive operator arguments, and another is an abstract generalization of a theorem characterizing feedback systems which are stable for all positive constant feedback gains. The original form of this result is due to Brockett and Willems,[58] and a slightly more general case is treated in Zames and Freedman.[56]

The third chapter considers the stability of linear time invariant systems, i.e. systems described by convolution equations. Such systems are of great practical interest, and have been extensively studied.[57] Here results are derived from the main theorem of Chapter II by applying some results on the spectrum of convolution operators due to Gokhberg and Krein.[11,33] The stability of such a system is determined essentially by the locus of the eigenvalues of a certain matrix of functions. By applying an eigenvalue estimation result for matrices due to Bauer and Ficke,[39] sufficient conditions for stability or instability may be obtained. This condition generalizes one due to Rosenbrock,[40] and is useful in connection with the problems of Chapter IV.

The general topic of the fourth chapter is the stability of linear discrete time systems with a periodically time varying feedback gain. Such systems are used to illustrate the application of the results and ideas of the previous two chapters. It turns out that it is possible to actually carry out a spectral calculation for the operators which arise in this context, so that the main theorem of Chapter II may be applied to obtain necessary and sufficient stability conditions. These conditions are related to those obtained by use of discrete Fourier series techniques for the special case where the latter are applicable.

Matrix eigenvalue estimation results are applied to obtain a sufficient condition for the validity of the "frozen time Nyquist condition", and a result which turns out to be a discrete-time version of one previously obtained by Willems[44] for the continuous time case. Finally, a necessary and sufficient condition for the positivity of an operator which is the composition of a discrete convolution and a periodic gain is obtained.

C. NOTATION AND DEFINITIONS

Free use of certain basic notions of functional analysis is made, and most of the standard manipulations (e.g., that a convolution with an L^1 kernel defines a bounded operator in certain Banach spaces) which are readily available in standard references have been omitted. While a few definitions of concepts of functional analysis are included for the sake of continuity of reading, no attempt to include a short course on functional analysis is made.

Upper case script letters (e.g., $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{E}$, etc.) are used to denote Banach spaces, and lower case letters denote elements of such a Banach space. Operators are denoted by capital letters. (e.g., A, K, etc). Finite dimensional vectors and matrices are indicated with a single underline and double underline respectively.

An attempt has been made to make all the notations and definitions of terms not explicitly defined in the text coincide with those of Reference 1.

II. THEORETICAL ASPECTS

A. INTRODUCTION

In this chapter a theoretical framework for the problem of input-output stability for a class of feedback systems is presented. The consideration of feedback systems from an input-output point of view leads naturally to the use of functional analysis for the solution of the stability problem. By suitably defining the class of systems to be considered, it is possible to obtain a complete theoretical solution to the problem, in the sense of obtaining a necessary and sufficient condition for stability. This condition involves the spectrum of a certain operator, and as may be expected, it is difficult to verify in most concrete cases of interest. This difficulty leads to the search for more easily verified conditions which are only sufficient to guarantee stability (or instability). For this purpose the general result is valuable, as it provides a unifying point of view for the generation of such sufficient conditions in the form of spectral estimates.

The results of this chapter are of an abstract character; their application to concrete problems will be given in later chapters.

B. PHYSICAL MOTIVATION

This chapter considers the stability of a certain class of linear feedback systems governed by the functional equations

$$y = Ge, \quad e = u - Ky \tag{2.1}$$

The physical interpretation of system (2.1) is that the original input to a system G has been replaced by a linear combination of a new input u and the result of passing the output y through a system K. (See Fig. 1). Such configurations arise naturally in the context of regulation schemes (Ref. 3), and differential equations may be readily given such an interpretation. (Ref. 4). The intuitive notion of stability which is sought is that the total system (1.1) should behave in a non-explosive manner; small inputs u should generate small respenses e and y throughout the loop.

C. BASIC DEFINITIONS

There are two basic approaches to the problem of feedback system stability. One is based on an internal or "state variable" description of the systems involved and a Lyapunov method approach to stability.[7, 8] The second is to consider a system strictly from an input-output point of view, to consider a system as a mapping of a class of inputs into a (possibly different) class of outputs. It is this second point of view that is taken here.

It is first necessary to select classes of admissable inputs and outputs for the systems considered. From an input-output point of view the stability properties of a feedback system are determined by the "size" of its response to the application of an input from the admissable class. The concept of applying an input to a closed loop system may be considered as implicitly defining an origin of time after which the input is applied. This consideration, together with the necessity of measuring the "size" of responses in some sense, requires the use of Banach spaces defined on a half-axis in time as input and output classes.

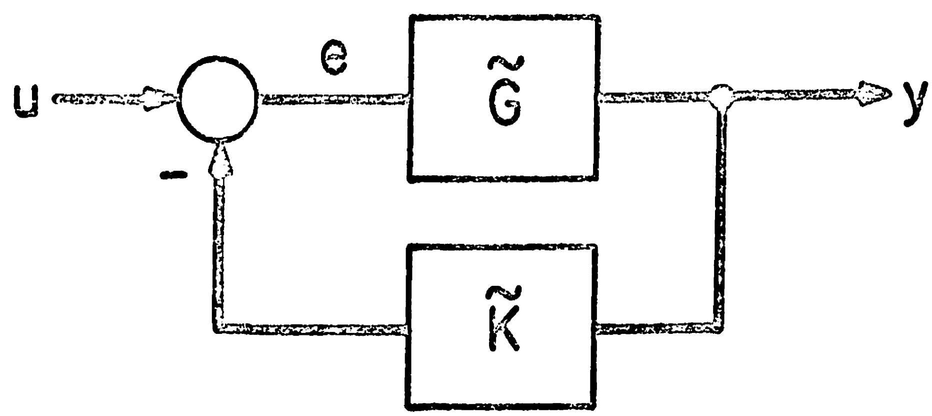


Fig. 1 A Feedback System

Notation: Denote by T^+ either of the sets R^+ (non-negative real numbers) or Z^+ (non-negative integers), and by $\mathcal{X}(T^+)$, the class of admissable inputs, any Banach space of functions defined on T^+. The norm on $\mathcal{X}(T^+)$ will be denoted by $\| \ \|_{\mathcal{X}}$. Similarly, the class of admissable outputs will be denoted by $\mathcal{Y}(T^+)$, and the corresponding norm by $\| \ \|_{\mathcal{Y}}$.

Typical examples of such Banach spaces are the usual $Lp(0, \infty)$ spaces, or $C^+(0, \infty)$, the space of continuous functions on $[0, \infty]$ for which the limit as $t \to \infty$ exists, equipped with the sup. norm.

Before formally defining a concept of stability, it is necessary to restrict the class of systems to be considered. These restrictions are motivated by the consideration of formulating the stability problem in a manner that is both physically reasonable and mathematically tractable. It is assumed that the elementary systems from which the feedback systems are constructed are represented as linear operators from the space of admissable inputs $\mathcal{X}(T^+)$ to the space of admissable outputs $\mathcal{Y}(T^+)$. From the norm structure on the input-output spaces, the space of linear operators inherits a natural norm structure.[1]

Definition 2.1: $\mathcal{B}(\mathcal{X}, \mathcal{Y})$ is the space of bounded linear operators from $\mathcal{X}$ to $\mathcal{Y}$, that is, the class of all those operators G for which

$$\|G\| \triangleq \sup_{\|x\|_{\mathcal{X}} = 1} \left\{ \|Gx\|_{\mathcal{Y}} \right\} < \infty \tag{2.2}$$

As is well known, the space $\mathcal{B}(\mathcal{X}, \mathcal{Y})$ forms a Banach Algebra using the usual operator composition as product and the obvious definitions for addition and scalar multiplication. A central problem in the theory of bounded linear operators is that of the existence of inverses in the algebra.

Definition 2.2: The spectrum of a bounded linear operator $A: \mathcal{X} \to \mathcal{X}$ is the set of all complex scalars λ such that the operator $(\lambda I - A)$ does not have a bounded inverse on $\mathcal{X}$. The spectrum of A is denoted by $\sigma(A)$.

$\sigma(A)$ is a non-empty compact subset of the complex plane.[1]

If $\lambda \in \sigma(A)$, then one of three things must happen: Ref. 2, p. 54.

1. λ is an eigenvalue, i.e., there exists an element $x_o \in \mathcal{X}$ such that $(\lambda I - A)x_o = 0$.

2. λ is "almost" an eigenvalue, i.e., there exists a sequence $\{x_n\}_{n=1}^{\infty}$ in $\mathcal{X}$ such that $\|x_n\|_{\mathcal{X}} = 1$, while $\lim_{n \to \infty} \|(\lambda I - A)x_n\|_{\mathcal{X}} = 0$

3. The range of $(\lambda I - A)$ is not dense in $\mathcal{X}$.

While there is considerable literature on the classification and properties of operator spectra, the above facts are sufficient background for what follows.

One of the assumptions made below essentially concerns the finite time interval behavior of the class of feedback systems considered here. The following formalism is useful in formulating the assumptions.

Definition 2.3: Define the truncation operator $P_T : \mathcal{X}(T^+) \to \mathcal{X}_T(T^+)$ by the condition

$$P_\tau x(t) = \begin{cases} x(t) & 0 \leq t \leq \tau \\ 0 & t > \tau \end{cases} \qquad (2.3)$$

For notational convenience $P_T x$ is denoted by x_T, and the subspace $P_T \mathcal{X}(T^+)$ by $\mathcal{X}_T$. $\mathcal{X}_T$ for each fixed T may be easily interpreted as a Banach space under the natural norm it inherits from $\mathcal{X}(T^+)$, provided that P_T is closed, as will usually be the case.

Definition 2.4: The functional equation $\{(I+KG)e = u\}$ is said to define a regular linear feedback system (relative to the input space $\mathcal{X}(T^+)$ and output space $\mathcal{Y}(T^+)$) if the following two conditions are satisfied:

1. G defines a bounded linear operator from $\mathcal{X}(T^+)$ to $\mathcal{Y}(T^+)$ and K defined a bounded linear operator from $\mathcal{Y}(T^+)$ to $\mathcal{X}(T^+)$;

2. For each $T > 0$ and each T-truncated input $u_T \in \mathcal{X}_T$ the equation

$$(I+KG)\, e_T = u_T \quad ; \quad 0 \leq t \leq T \tag{2.4}$$

has a unique solution $e_T \in \mathcal{X}_T$.

That it is necessary that both G and K separately define bounded linear operators, and not just that their composition have such an interpretation, may be seen from simple examples using convolution operators with transfer function pole-zero cancellation. Also the assumption of the boundedness of the operators is not as restrictive in practice as it may at first seem. A common case is that where G represents a system described by a system of linear time invariant differential equations of the form

$$\dot{\underline{x}} = \underline{\underline{A}}\, \underline{x} + \underline{b}\, e \; ; \quad y = \underline{c}'\underline{x} \tag{2.5}$$

If the matrix $\underline{\underline{A}}$ has eigenvalues in the closure of the right half plane, then two problems occur. First, when the differential equation is turned into an integral equation in order to obtain an input-output description of the system, the kernel of the integral equation will contain non-decreasing exponential terms. This means that the integral operator will not be bounded on any of the usual spaces $\mathcal{X}(T^+)$. At the same time, the initial condition response may also contain growing exponential terms, and so cannot be lumped in with the closed-loop system inputs. One way to avoid both problems is to rewrite the differential equation in the form

$$\dot{\underline{x}} = \underline{\underline{A}}_s\underline{x} + (\underline{\underline{A}} - \underline{\underline{A}}_s)\underline{x} + \underline{b}e \tag{2.6}$$

where $\underline{\underline{A}}_s$ is a stable matrix, and to consider the term $(\underline{\underline{A}} - \underline{\underline{A}}_s)\underline{x}$ as an additional feedback.

The second assumption about a regular system is essentially that the finite time version of the loop equations has a unique solution. The fact that is is necessary to somehow separate the problem of the existence of solutions from the stability problem seems first to be recognized in the work of Zames,[9] who introduced so called "extended spaces" for this purpose. This assumption is usually not very troublesome in practice. For instance if $\mathcal{X}(T^+)$ is a space of continuous functions, and KG represents a cascade of non-anticipative linear filters, then the equation

$$(I + KG)e_T = u_T \quad , \quad 0 \leq t \leq T \tag{2.7}$$

is just a Volterra integral equation of the second kind. However systems which involve feedthroughs are potentially more troublesome in this regard, and may cause problems in the area of "well-posedness" of the stability problem as well.[59]

The concept of bounded input – bounded output stability will be defined below only for regular feedback systems. Part of the reason for this is that the resulting theory is mathematically tractable. The remainder, however, is based on the physically motivated idea that a system which is not regular according to the above definition is not suitably characterized by the concept of input-output stability. For example, a system for which finite time solutions do not exist for all truncation times exhibits a finite escape time of some sort. (One could go so far as to define the supremum of the truncation times for which a unique solution exists as an escape time.) Such a system should be considered so unstable as to be outside the scope of the following definition.

Definition 2.5: The regular feedback system described by the functional equation $\{(I + KG)e = u\}$ is said to be bounded input – bounded output stable in the $\mathcal{X}(T^+)$ sense, if there exists a constant M such that

$$\|e\|_{\mathcal{X}} \leq M \|u\|_{\mathcal{X}} \tag{2.8}$$

for all possible inputs $u \in \mathcal{X}(T^+)$. If no such M exists, then the system is called unstable.

D. STABILITY OF REGULAR SYSTEMS

Theorem 2.1: The regular linear feedback system described by the functional equation $\{(I + KG)e = u\}$ is bounded input – bounded output stable in the $\mathcal{X}(T^+)$ sense if and only if the point -1 does not belong to the spectrum of the bounded linear operator KG acting on $\mathcal{X}(T^+)$.

Proof: The sufficiency portion of the theorem is straight-forward. For if -1 does not belong to the spectrum of KG, then by Definition 1.2,

$(I + KG)^{-1}$ exists as a bounded linear operator mapping $\mathcal{X}(T^+)$ onto itself. From the functional equation

$$(I + KG)e = u \tag{2.9}$$

one immediately obtains

$$e = (I + KG)^{-1}u \tag{2.10}$$

and since the inverse is bounded on $\mathcal{X}(T^+)$

$$\|e\|_{\mathcal{X}} = \|(I + KG)^{-1}u\|_{\mathcal{X}} \leq M \|u\|_{\mathcal{X}} \tag{2.11}$$

To show the necessity of the condition assume that $-1 \in \sigma(KG)$. By the remarks following Definition 1.2, this implies that either the range of $(I + KG)$ is not dense or that there exists a sequence in $\mathcal{X}$, $\{e_n\}_1^\infty$, $\|e_n\|_{\mathcal{X}} = 1$, such that $\|(I + KG)e_n\|_{\mathcal{X}} \to 0$.

In the second case choose a sequence of inputs $\{u_n\}_1^\infty$ defined by

$$(I + KG)e_n = u_n \tag{2.12}$$

Then by the above $\|u_n\|_{\mathcal{X}} \to 0$, while the return difference e_n has norm equal to unity for all n. Hence there exists no M such that $\|e\|_{\mathcal{X}} \leq M \|u\|_{\mathcal{X}} \quad \forall\, u \in \mathcal{X}(T^+)$.

If -1 belongs to the spectrum, but no such sequence exists, then the range of $(I + KG)$ is not dense in $\mathcal{X}$. This means that we may select an input u in $\mathcal{X}(T^+)$ but not in the closure of the range of $(I + KG)$. For such a u, generate a sequence of inputs $\{u_n\}_1^\infty$ defined by $u_n = u_{T_n}$, where $\{T_n\}_1^\infty$ is a sequence of truncation times increasing monotonically without bound. By the assumption that the system is regular, this sequence of inputs generates a sequence of finite time solutions $\{e_{T_n}\}_1^\infty$. Since the range of (I + KG) is not dense, this sequence cannot converge to an element of $\mathcal{X}(T^+)$, and hence

$$\| e_{T_n} \|_{\mathcal{X}} \rightarrow \infty \qquad (2.13)$$

which establishes the theorem.

It is necessary to emphasize the point that the Banach space $\mathcal{X}(T^+)$ is defined on a half-axis in time, and that the spectrum of the operator KG must be computed relative to its action on the space $\mathcal{X}(T^+)$. The crucial difference that this makes may be seen from the examples of the convolution operators treated in Chapter III.

The concept of applying an input to a feedback system implicitly establishes an origin of time; stability or instability is determined by the behavior of the closed loop system response in the time interval following the application of an input. This makes appropriate the consideration of Banach spaces defined on a half-axis in time as a natural setting for the input-output stability problem.

In the present formulation, the notion of causality of operators plays an implicit role, in that it is intimately connected with the second requirement for a regular feedback system, the condition that the finite time loop equations have a unique solution. For if the responses in the time interval [0, T) were to depend on the inputs for times greater than T, then there obviously can be no unique solution to the finite time truncated loop equations. By making explicit use of the concept of causality, it is possible to state stability conditions in terms of spaces defined on the whole time axis.[10]

The main usefulness of Theorem 2.1 is self evident: it provides a necessary and sufficient condition for input-output stability. Another benefit is that the spectrum of the operator KG may turn out to be the same relative to a number of different half-axis Banach spaces. In such a case a single spectral calculation yields input-output stability for a wide class of inputs. This is illustrated by the convolution operators in Chapter III.

As may be expected, any necessary and sufficient condition for stability will in general be difficult to verify. Essentially the only classes of operators for which there are relatively complete spectral theories are the so-called compact operators, and classes of commuting normal operators. Typical of compact operators are integral operators with square integrable kernel acting on $L_2(0,1)$. The compactness property (and hence the available general results) disappear in the case of a semi-infinite time interval.[11,12,35] Scalar convolution operators arise in practice from the consideration of constant coefficient linear differential equations, and such operators are in fact normal on $L_2(0,\infty)$. In a common case of interest, where the operator K represents multiplication by a time varying gain, K and G do not commute unless K is a constant. Hence the general results are again of little use.

Perhaps the most useful way to view Theorem 2.1 is as an identification of the mathematical problem involved, rather than as the solution. The problem of spectral calculations for various classes of operators is an active area of research in mathematical analysis, and any result obtained in this area has a potential interpretation as a feedback stability result. Examples of this type are given for convolution equations in Chapter III, and for a class of discrete systems with periodic feedback gain in Chapter IV.

While explicit spectral calculations are rather rare, it is possible to obtain estimates of the spectra of operators. As may be seen from Theorem 2.1, any condition sufficient to guarantee that the point -1 does not belong to the spectrum of KG is a sufficient condition for the stability of the feedback system. Similarly, a condition that guarantees that -1 belongs to the spectrum of KG is a sufficient condition for the instability of the corresponding feedback system. Thus Theorem 2.1 provides a rational means of generating sufficient stability (instability) conditions through spectral estimates.

E. STANDARD LOOP MANIPULATIONS

The process of generating a spectral estimate generally involves establishing an inequality of some sort. To get the maximum use out of such inequalities, it is useful to establish classes of feedback systems whose stability properties are equivalent. This will allow one to conclude the stability of one system on the basis of a spectral estimate for an equivalent system.

Theorem 2.2: Let the operators K and G of the regular feedback system $\{(I + KG)e = u\}$ map the space $\mathcal{X}(T^+)$ into $\mathcal{X}(T^+)$. (That is, the input space is the same as the output space.) Then the system $\{(I + KG)e_1 = u_1\}$ is bounded input – bounded output stable in the $\mathcal{X}(T^+)$ sense if and only if the system $\{(I + GK)e_2 = u_2\}$ is also stable.

Proof: If K and G belong to $\mathcal{B}(\mathcal{X}, \mathcal{X})$ and $(I + KG)$ is invertible, then it may be directly verified that the expression

$$I - G(I + KG)^{-1} K$$

is an inverse for the operator $(I + GK)$. By symmetry, $-1 \in \sigma(KG)$ if and only if $-1 \in \sigma(GK)$. The conclusion now follows from Theorem 2.1.

The following is closely related to the above, and finds application in stability conditions of the positive operator type.

Theorem 2.3: Let M_1 and M_2 be invertible bounded linear operators mapping $\mathcal{X}(T^+)$ onto $\mathcal{X}(T^+)$ and $\mathcal{Y}(T^+)$ onto $\mathcal{Y}(T^+)$ respectively. Then the regular feedback system $\{(I + KG)e_1 = u_1\}$ is bounded input – bounded output stable in the $\mathcal{X}(T^+)$ sense if and only if the system $\{(I + M_1^{-1} K M_2^{-1} M_2 G M_1) e_2 = u_2\}$ is also stable.

Proof: The spectrum is invariant under similarity transformations.

A common transformation which arises in the theory of functions of a complex variable is the bilinear or homographic transformation defined by

$$z \rightarrow \frac{az + b}{cz + d} \qquad \text{Ref. 13, 14}$$

This class of transformations occurs naturally in the theory of feedback systems when equivalent systems are constructed by the device of adding a pair of self-cancelling feedback paths inside the original feedback loop. See Fig. 2.

Definition 2.6: Two feedback systems $S_1 : \{(I + K_1 G_1) e_1 = u_1\}$ and $S_2 : \{(I + K_2 G_2) e_2 = u_2\}$ are said to be bilinearly related by parameter set (a, b, c, d) if the following conditions are satisfied:

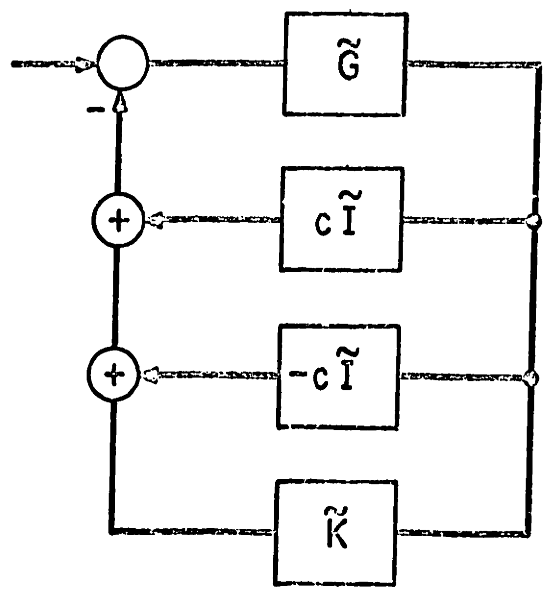


Fig. 2 Standard Loop Manipulation

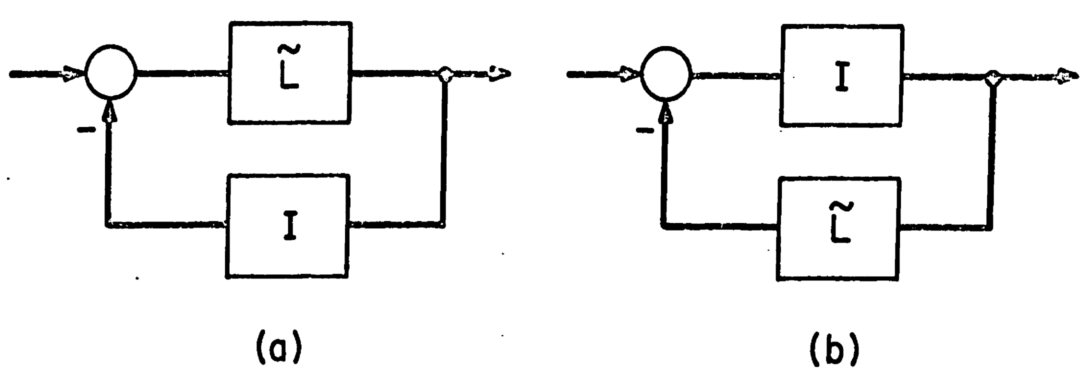


Fig. 3 Equivalent Systems

$$\left.\begin{array}{l} 1.\ \ K_2 = (dK_1 - cI)(aI - bK_1)^{-1} \\ 2.\ \ G_2 = (aG_1 + bI)(cG_1 + dI)^{-1} \\ 3.\ \ d(ad - bc) \neq 0 \end{array}\right\} \qquad (2.14)$$

Two bilinearly related feedback systems S_1 and S_2 are called bilinearly equivalent if S_1 is stable if S_2 is stable.

Theorem 2.4: Two bilinearly related feedback systems are bilinearly equivalent provided that $(aI - bK_1)$ and $(cG_1 + dI)$ both have bounded inverses.

Proof: Let S_1 be given by

$$(I + K_1 G_1)e_1 = u_1 \qquad (2.15)$$

then

$$(I + cG_1 + (K_1 - cI)G_1)e_1 = u_1 \qquad (2.16)$$

Let

$$e_1' = (I + cG_1)e_1 \qquad (2.17)$$

Then if $(I + cG_1)^{-1}$ exists as a bounded operator, the stability of S_2 defined by

$$\{(I + (K_1 - cI)G_1(I + cG_1)^{-1})e_1' = u_1\} \qquad (2.18)$$

imply that of the original system S_1. The above proof covers the parameter set $(1, 0, c, 1)$. The general case is handled by a sequence of the same manipulations, requiring only more algebra to keep track of the parameters.

F. THE SPECTRAL RADIUS

From the previous section it is clear that there exist a large number of linear feedback systems whose stability properties are equivalent. In view of the large number of rearrangements, factorizations, etc., which are possible, it is convenient at this time to drop the distinction between the operators K and G, and to describe a feedback system by the equation

$$(I + L)e = u \tag{2.19}$$

where the operator L is called the loop transfer operator. (See Fig. 3 for interpretations.)

As discussed above, a natural way to generate conditions sufficient for the stability of a feedback system is to derive bounds on the location of the spectrum of the loop transfer operator, L.

<u>Definition 2.7</u>: For each $L \in \mathcal{B}(\mathcal{X}, \mathcal{X})$, define the <u>spectral radius</u> $\nu(L) = \sup_{\lambda \in \sigma(L)} |\lambda|$.

The following relation between $\nu(L)$ and $\|L\|$ is well known.

<u>Theorem 2.5</u>: The spectral radius of a bounded linear operator from $\mathcal{X}$ to $\mathcal{X}$ satisfies

$$\nu(L) \leq \|L\|$$

<u>Proof</u>: Write $\lambda I - A = \lambda(I - A/\lambda)$ $(\lambda \neq 0)$. Since $\left\|\frac{A}{\lambda}\right\| = \frac{\|A\|}{|\lambda|} < 1$ for $|\lambda| > \|A\|$, the expression $\sum_{i=0}^{\infty} \frac{A}{\lambda}^{i}$ converges in the uniform norm topology to the inverse of $(I - \frac{A}{\lambda})$. Hence for $|\lambda| > \|A\|$, λ cannot be in the spectrum of A, so $\nu(A) \leq \|A\|$.

From Theorem 2.1, a feedback system is stable if -1 does not belong to the spectrum of the loop transfer operator. Combining this with the above estimate gives the following theorem, due originally to Zames.[9]

Theorem 2.6: (Small Gain Theorem) A sufficient condition that the regular feedback system described by the functional equation $\{(I+L)e = u\}$ be bounded input – bounded output stable in the $\mathcal{X}(T^+)$ sense is that:

$$\|L\| = \sup_{\|x\|_{\mathcal{X}} = 1} \|Lx\|_{\mathcal{X}} < 1 \qquad (2.20)$$

Proof: Since $\|L\| < 1$, $\nu(L) < 1$ by Theorem 2.5. Hence $-1 \notin \sigma(L)$, and stability follows from Theorem 2.1.

The intuitive basis of the "small gain" result may be seen from Fig. 3b. If the operator L is in some sense small, then the output e is very nearly equal to the input u. The result shows that any L with norm less than unity is "small enough" to give stability.

It is relatively unusual for a feedback system to arise naturally with a loop transfer operator L which satisfies the small gain theorem condition. However, it is possible to modify the original system using the standard manipulations of the previous section and to obtain a system to which the small gain theorem is applicable.

Another use of the small gain theorem is to establish results which might be considered in the nature of a perturbation estimate. Suppose it is known that the system $S : \{(I + L)e = u\}$ is stable, while the system to be investigated has the form $S_\Delta : \{(I + L + \Delta)e = u\}$. Since S is stable, the stability of S_Δ is equivalent to that of $S_L : \{(I + \Delta(I+L)^{-1})e' = u\}$. The small gain theorem guarantees stability of S_L if

$$\|\Delta(I+L)^{-1}\| \leq \|\Delta\| \cdot \|(I+L)^{-1}\| < 1 \qquad (2.21)$$

so that S_Δ is stable if the perturbation Δ is small enough. Note that the allowable size of Δ is inversely proportional to the quantity $\|(I+L)^{-1}\|$, which may be considered as a measure of sensitivity of the closed loop system S.

Example: (The Circle Theorem)

The above results may be used to prove a version of the so-called Circle Theorem, originally due to Sandberg[15] and Zames.[9] The system considered is given by

$$e(t) + k(t) \int_0^t g(t-s)\, e(s)\, ds = u(t) \qquad 0 \leq t < \infty \tag{1.22}$$

$\mathcal{X}(T^+)$ is taken as $L_2(0,\infty)$, and it will be assumed that $g(\cdot) \in L_1(0,\infty)$, while the measurable function $k(\cdot)$ satisfies $|k(t)| \leq \beta$. Clearly the operator $K : x(t) \to k(t)x(t)$ defines a bounded linear operator on $L_2(0,\infty)$, and $\|K\| = \beta$.

Let the operator G be defined by

$$G : x(t) \to \int_0^t g(t-s)\, x(s)\, ds \quad , \quad x(\cdot) \in L_2(0,\infty)$$

Then as is well known, G also defines a bounded linear operator. Defining the transfer function of G as

$$\hat{\mathcal{G}}(i\omega) = \int_0^\infty g(t)\, e^{-i\omega t}\, dt \tag{2.22}$$

it may be seen by using the fact that the Fourier transform of a convolution is the product of the Fourier transforms of the two functions, together with Plancherel's Theorem that

$$\|G\| = \sup_{\omega \in R} |\hat{\mathcal{G}}(i\omega)| \tag{2.23}$$

Combining the two estimates, the small gain theorem establishes stability provided that

$$\sup_{\omega \in R} |\hat{\mathcal{G}}(i\omega)| < \frac{1}{\beta} \tag{2.24}$$

The geometric interpretation of the condition (1.24) is that the locus of points $\{\hat{\mathcal{G}}(i\omega) \mid \omega \in R\}$, known as the Nyquist locus, lies inside a circle in the complex plane of radius $1/\beta$.

The case where the time varying gain satisfies $0 < \alpha \leq k(t) \leq \beta$ instead of the symmetric constraint is handled by making a standard transformation of the type of Theorem 2.4 in order to make the interval symmetric. This change replaces the k(t) constraint with $|k_1(t)| \leq \frac{\beta - \alpha}{2}$, and replaces $\hat{\mathcal{G}}(i\omega)$ with $\hat{\mathcal{G}}(i\omega)/(1 + \frac{(\alpha+\beta)}{2}\hat{\mathcal{G}}(i\omega))$. The geometric interpretation on the Nyquist locus of $\hat{\mathcal{G}}$ then becomes that it neither encircle nor intersect a disc in the complex plane of radius $r = \frac{1}{2}(\frac{1}{\alpha} - \frac{1}{\beta})$, and center $c = -\frac{1}{2}(\frac{1}{\alpha} + \frac{1}{\beta})$. See Fig. 4.

The small gain theorem is, as seen above, a consequence of a simple bound on the spectral radius. The implicit use of the spectral radius result to provide a means to solve an integral equation is not a recent development.[16] In fact, the well-known method of successive approximations (the Neumann series) is closely related to the convergence of a "candidate" for the inverse of the operator $(\lambda I - A)$ in the form of a geometric series.

In fact, any method for obtaining a solution to an equation in a Banach space of the form

$$(\lambda I - A) f = g \tag{2.25}$$

has a possible interpretation as a feedback stability result. Any condition

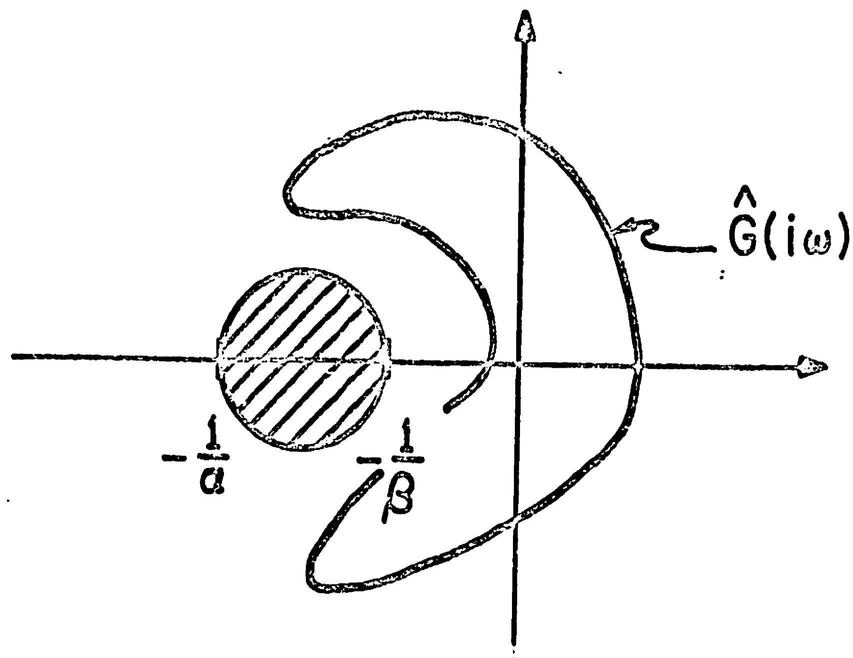


Fig. 4 The Circle Criterion

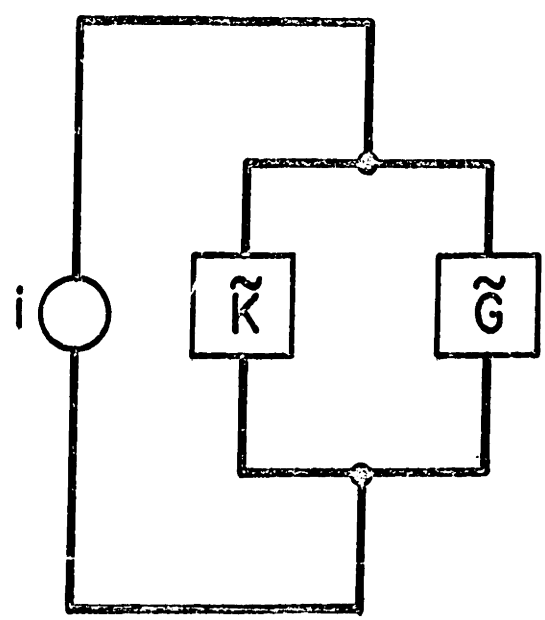


Fig. 5 A Network Interpretation

on λ and the operator A that guarantees a unique solution f of the equation (1.25) for any "forcing term" g, effectively generates an inverse for $(\lambda I - A)$. Hence $\lambda \notin \sigma(A)$ whenever such a method converges.

A whole class of methods which may be interpreted as generalizing the method of successive approximations is given in the monograph by Luchka.[17] In principle, one should be able to derive various sufficient conditions for the stability of feedback systems with these methods. As a practical matter, however, it seems to be difficult to derive conditions which (like the Circle Theorem) involve characteristics of the system operators which are relatively easy to compute.

G. HILBERT SPACE INPUTS

The additional structure of an inner product obtained by specializing the Banach space $\mathcal{X}(T^+)$ of the previous sections to a Hilbert space (notation: $\mathcal{H}(T^+)$) proves useful in the derivation of stability results. This brings in the ideas of power, energy, and passivity which have their roots in the analysis of electrical network systems,[18] and leads to a possible interpretation of a feedback system from a network point of view. (See References 9, 19 and Fig. 5.) In this interpretation the operator G of Fig. 1 represents an impedance, K an admittance, and the input u appears as a current source driving the parallel combination of the two.

A large number of stability results have been obtained from this interpretation in recent years;[20-24] the intuitive motivation is the idea that the interconnection of two systems which dissipate energy should be stable.

In what follows $\mathcal{H}(T^+)$ will be a Hilbert space defined on a half-axis in time, and the standard inner product between two elements $x, y \in \mathcal{H}(T^+)$

will be denoted by $\langle x, y \rangle$. The usual method of establishing stability by passivity arguments is to write down the system equation and to then manipulate inner products using various inequalities. Here a derivation is given directly on the basis of Theorem 2.1, so that the passivity argument appears as simply another method of estimating the spectrum.

Definition 2.8: For a bounded linear operator $A: \mathcal{H}(T^+) \to \mathcal{H}(T^+)$ define the numerical range $W(A) = \{\lambda \mid \lambda = \langle Ax, x \rangle \text{ for some } \|x\| = 1\}$. $W(A)$ is a bounded, not necessarily closed, and convex subset of the complex planes.

Definition 2.9: A is said to be positive (strictly positive) if $\operatorname{Re} W(A) \geq 0$ $(\operatorname{Re} W(A) \geq \epsilon > 0)$.

In the mathematical literature such operators are called accretive; the term positive is reserved for self-adjoint operators. However, the term positive is standard in the control literature and that usage is followed here.

Examples: 1. Consider the convolution operator G defined on $L_2(0,\infty)$ by

$$G: x(t) \to \int_0^t g(t-s)\, x(s)\, ds, \quad \text{where } g(\cdot) \in L_1(0,\infty)$$

Then the Plancherel Theorem gives

$$\langle Gx, x \rangle = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{\mathcal{G}}(i\omega)\, |\hat{x}(i\omega)|^2\, d\omega \tag{2.26}$$

where $\hat{x}(i\omega)$ is the Fourier-Plancherel transform of $x \in L_2(0,\infty)$ and $\hat{\mathcal{G}}(i\omega)$ is the transfer function corresponding to the L_1 function $g(\cdot)$.

Clearly, positivity of the operator G is equivalent to the condition

$$\mathrm{Re}\,\hat{\mathcal{G}}(i\omega) \geq 0 \quad (\text{a.e.}), \quad \omega \in R \tag{2.27}$$

2. Similarly in the case of a discrete convolution acting on ℓ_2^+ and defined by

$$G : x(n) \rightarrow \sum_{j=0}^{n} g(n-j)\, x(j)$$

where the sequence $\{g_i\}_0^\infty \in \ell_1$, the condition becomes

$$\mathrm{Re}\,\hat{\mathcal{G}}(e^{i\theta}) \geq 0 \ \ (\text{or } \mathrm{Re}\,\hat{\mathcal{G}}(e^{i\theta}) \geq \epsilon > 0, \text{ for strict positivity})\ (\text{a.e.})\ -\pi \leq \theta \leq \pi \tag{2.28}$$

where $\hat{\mathcal{G}}(\cdot)$ is the pulse transfer function (z - transform) of the sequence $\{g_i\}_0^\infty$ defined by

$$\hat{\mathcal{G}}(z) = \sum_{0}^{\infty} g_i\, z^{-i} \tag{2.29}$$

which converges absolutely, $|z| \geq 1$.

3. The "time varying gain" operator defined by

$$K : x(t) \rightarrow k(t)\, x(t) \quad (\text{a.e.})$$

where $k(\cdot)$ is a measurable and essentially bounded function, is positive (strictly positive) if and only if

$$\mathrm{Re}\,(k(t)) \geq 0 \quad (\mathrm{Re}\,k(t) \geq \epsilon > 0) \quad (\text{a.e.}) \tag{2.30}$$

The following well known result explains the connection between the spectrum of a bounded linear operator and its numerical range.

<u>Theorem 2.7</u>: Let $A : \mathcal{H}(T^+) \rightarrow \mathcal{H}(T^+)$. Then $\sigma(A) \subseteq \overline{W(A)}$

<u>Proof</u>: See Reference 26 or 27.

The following generalization of the above is due to Williams.[28] Here the set $\overline{W(B)}/\overline{W(A)}$ is interpreted as the set of all possible quotients b/a, with $b \in \overline{W(B)}$, $a \in \overline{W(A)}$.

<u>Theorem 2.8</u> (Williams) : Let A and B be bounded linear operators on $\mathcal{H}(T^{+})$. Then if $0 \notin \overline{W(A)}$, the spectrum of the operator $A^{-1}B$ satisfies

$$\sigma(A^{-1}B) \subseteq \overline{W(B)}/\overline{W(A)} \tag{2.31}$$

<u>Proof</u>: If $0 \notin \overline{W(A)}$, then by Theorem 2.7, A is invertible. Hence the identity

$$(\lambda I - A^{-1}B) = A^{-1}(\lambda A - E) \tag{2.32}$$

shows that $\lambda \in \sigma(A^{-1}B)$ if and only if $0 \in \sigma(\lambda A - B)$. Using Theorem 2.7 again

$$0 \in \overline{W(\lambda A - B)} \subseteq \lambda\, \overline{W(A)} - \overline{W(B)}$$

from which

$$\lambda \subseteq \frac{\overline{W(B)}}{\overline{W(A)}}$$

The basic idea in establishing stability by positivity arguments is to use the standard manipulations of Section E above to obtain an equivalent system in which the loop transfer operator may be factored as the product of two positive operators, one of which is strictly positive.

<u>Theorem 2.9</u>: Suppose that the regular feedback system $S_1 : \{(I + KG)e_1 = u_1\}$ is such that it is equivalent to a second system $S_2 : \{(I + L)e_2 = u_2\}$, where L has the property:

$L = P_1^{-1} P_2$, where P_1 is strictly positive; and P_2 is positive. Then the system S_1 is bounded input — bounded output stable in the $\mathcal{H}(T^{+})$ sense.

Proof: Since $L = P_1^{-1}P_2$ and P_1 is strictly positive, Theorem 2.8 implies

$$\sigma(L) \subseteq \frac{\overline{W}(P_2)}{\overline{W}(P_1)}$$

$\overline{W}(P_2)$ is contained in the closed right half plane, and $\overline{W}(P_1)$ is contained in the open right half plane, so that the above set of quotients does not contain any part of the negative real axis. Specifically, it does not contain the point -1, so S_2 is stable. By the hypothesis of equivalence, so is S_1.

H. FUNCTIONS OF OPERATORS

A problem which has been avoided so far is that of the spectra of the individual factors which occur in the loop transfer operator after one of the standard loop manipulations. This is one aspect of the general problem of the definition and properties of functions of an operator. In the case where the operator is a finite dimensional matrix, the problem may be resolved directly. (See Ref. 28, for example.) For more general operators the problem of the class of functions to be admitted, and of a suitable definition of a function of an operator is not so simple. The whole theory is sometimes referred to as an "Operational Calculus", and a feature of its various occurrences is a tendency to widen the class of admissable mapping functions at the cost of restrictions on the class of operators to be mapped. A version suitable for the present purposes is given in Ref. 1, p. 568, and the basic facts are reproduced here for convenience. Throughout T is a bounded linear operator on a Banach space $\mathcal{X}$.

Definition 1.10: The complement of $\sigma(T)$ is called the resolvent set for T, denoted by $\rho(T)$.

Since $\sigma(T)$ is closed, $\rho(T)$ is open.

Lemma: For $\lambda \in \rho(T)$, define $R(\lambda : T) = (\lambda I - T)^{-1}$, called the resolvent of T. Then the resolvent is a (vector valued) analytic function of T.

Definition 1.11: Denote by $\mathcal{F}(T)$ the class of (scalar valued) functions analytic on some neighborhood of $\sigma(T)$. Let $f \in \mathcal{F}(T)$ and let U be an open set whose boundary B consists of a finite set of rectifiable Jordan curves, positively oriented. Suppose $U \supseteq \sigma(T)$, and that $U \quad B$ is contained in the domain of analyticity of f. Then $f(T)$ is defined as

$$f(T) = \frac{1}{2\pi i} \int_B f(\lambda)\, R(\lambda : T)\, d\lambda \tag{2.33}$$

From Cauchy's integral theorem the above is well defined, and turns out to have all the expected properties. The result needed below is the following, known as the spectral mapping theorem.

Theorem 2.10 (Spectral Mapping Theorem):

If $f \in \mathcal{F}(T)$, then $\sigma(f(T)) = f(\sigma(T))$ (2.34)

In words, the spectrum of the function $f(T)$ of the operator T is the image under the function f of the points belonging to the spectrum of T.

I. A POSITIVE OPERATOR CONVERSE THEOREM

There exists a vast literature on the general topic of stability theorems for ordinary differential equations derived by the so-called second method of Lyapunov. (See for example Refs. 7, 8, 16, and 31.) Certainly a comforting fact to anyone engaged in the pursuit of Lyapunov functions is the existence of a theorem to the effect that for a stable differential equation there exists a suitable Lyapunov function which establishes its stability. The problem, of course, is to be sufficiently clever to find one. As mentioned above, recently there has been a large amount of work done in the area of establishing stability by positive operator arguments. Roughly the situation is that every newly discovered positive operator yields a new sufficient condition for stability.

A natural question to occur is now whether or not anything analogous to the Lyapunov theory case occurs, that is, whether the positive operator methods are sufficient to prove the stability of a system which is actually stable. It turns out that this is true, and that it may be proved by use of the spectral mapping theorem in combination with a recent result due to Williams. The suggestion for the method of the proof stems from a positive operator derivation of the Circle Theorem.[19]

Example: (Circle Theorem re-examined)

As previously mentioned in connection with Theorem 2.4, bilinear transformations occur naturally in the context of feedback systems. They occur through manipulations of the loop which may be interpreted as the addition of self-cancelling additional loops within the original system, (Fig. 2). The distinguishing feature of bilinear transformations from the point of view of complex variables is that they map circles into circles (with the usual interpretation of a straight line as a limiting case).

Consider a system described by

$$e(t) + k(t) \int_0^t g(t-s)\, e(s)\, ds = u(t) \qquad t \geq 0 \quad (\text{a.e.}) \tag{2.35}$$

under the assumptions that $0 < \alpha + \epsilon \leq k(t) \leq \beta - \epsilon$, and the $L_1(0,\infty)$ functions $g(\cdot)$ is such that the circle theorem conditions are satisfied, i.e., the Nyquist locus of $\hat{\mathcal{G}}$, the Fourier transform of the function $g(\cdot)$, neither encircles nor intersects the critical disc in the complex plane having the interval $\left[-\frac{1}{\alpha}, -\frac{1}{\beta}\right]$ as diameter.

It is easily seen that the bilinear transformation $z \to \frac{\beta z + 1}{\alpha z + 1}$ has the property that it maps the exterior of the critical disc onto the right half plane.

The next crucial step is to note that a bilinear tranformation with parameter set $(\beta, 1, \alpha, 1)$ (cf. Definition 2.6) results in an equivalent system. This is true because $\beta I - K$ is invertible by virtue of the bound $k(t) \leq \beta + \epsilon$. $I + \alpha G$ is invertible by the Nyquist criterion (Chapter III).

In the equivalent (by virtue of Theorem 2.5) system K_2 is again a multiplication by a time varying gain,

$$k_2(t) = \frac{k(t) - \alpha}{\beta - k(t)} \tag{2.36}$$

which is strictly positive by virtue of the bounds on $k(t)$. G_2 is an element of the convolution algebra $I \oplus L_1$, and its Fourier transform (frequency domain) representation is simply

$$\hat{\mathcal{G}}_2(i\omega) = \frac{\beta\hat{\mathcal{G}}(i\omega) + 1}{\alpha\hat{\mathcal{G}}(i\omega) + 1} \tag{2.37}$$

By virtue of the fact that the circle criterion conditions are satisfied, the above quantity has a positive real part for all real ω, and so G_2 is a positive operator.

Hence stability is concluded from Theorem 2.9.

It will be shown in Chapter III, that the spectrum of a convolution operator (acting on a half-axis) consists of the image of the right half plane under the Laplace transform of the convolution kernel. Thus the expression (2.37) above is essentially an example of the spectral mapping theorem in the particular case where the mapping function is a simple rational function. Another useful fact about scalar convolutions is that such operators are of the <u>normal</u> type. Normal operators have the property (among others) that the numerical range is the convex hull of the spectrum; hence the fact that the spectrum of the operator G_2 above lies in the right half plane is sufficient to insure its positivity. That this is not in general the case may be seen from the two dimensional matrix:

$$\underline{\underline{A}} = \begin{bmatrix} \lambda & a \\ 0 & \lambda \end{bmatrix} \tag{2.38}$$

for which

$$W(\underline{\underline{A}}) = \{z \mid |z - \lambda| \leq \tfrac{1}{2} |a| \} \tag{2.39}$$

Clearly for $|a|$ sufficiently large compared to $|\mathrm{Re}\,\lambda|$, the numerical range intersects the left half plane although $\mathrm{Re}\,\lambda > 0$.

The following result is true, however. Here H^+ denotes the right half plane.

<u>Theorem 2.11</u>: (Williams) Let A be a bounded linear operator on a Hilbert space $\mathcal{H}$ such that $\sigma(A) \subset H^+$. Then the following two conditions hold:

1. There exists a self-adjoint, positive operator P such that

$$\overline{W(AP)} \subset H^+$$

2. There exists an invertible operator S such that

$$\overline{W(S^{-1}AS)} \subset H^{+}$$

In other words, if the spectrum of an operator lies in the right half of the complex plane, then it may be made positive either by postmultiplication by a suitable positive and self-adjoint P, or by a similarity transformation by a suitable invertible operator S.

Proof: See Ref. 32.

Theorem 2.12: (Converse Positive Operator Theorem)
Suppose that the regular system $S_1 : \{(I+L_1)e_1 = u_1\}$ is bounded input—bounded output stable in the $\mathcal{H}(T^{+})$ sense. Then there exists an equivalent system (in fact a bilinearly equivalent system) $S_2 : \{(I+L_2)e_2 = u_2\}$ with the property that

$$L_2 = P_1 P_2^{-1} \tag{2.40}$$

with P_1 and P_2 each positive.

Proof: Since S_1 is stable, from Theorem 2.1 it follows that $-1 \notin \sigma(L_1)$. Since the resolvent set is open, there exists a disc of radius $\rho > 0$ about -1 which does not intersect $\sigma(L_1)$. Choose any ϵ satisfying $0 < \epsilon < \rho$.

The bilinear transformation T

$$z \rightarrow \frac{z-(-1+\epsilon)}{z-(-1-\epsilon)}$$

maps the exterior of the region $|z+1| < \epsilon$ onto the right half plane, and since the point $(-1-\epsilon)$ belongs to the resolvent set of L_1 the system

$$S_2 : \{(I + (L_1 - (-1+\epsilon)I)(L_1 - (-1-\epsilon)I)^{-1})e_2 = u_2\}$$

is bilinearly equivalent to S_1. Let

$$L_2 = (L_1 - (-1+\epsilon)I)(L_1 - (-1-\epsilon)I)^{-1} \tag{2.41}$$

By the spectral mapping theorem and the mapping properties of T, $\sigma(L_2)$ is a closed subset of the open right half plane. By Theorem 2.11 there exists a positive invertible P such that L_2P is positive. Hence

$$L_2 = (L_2P)P^{-1}$$

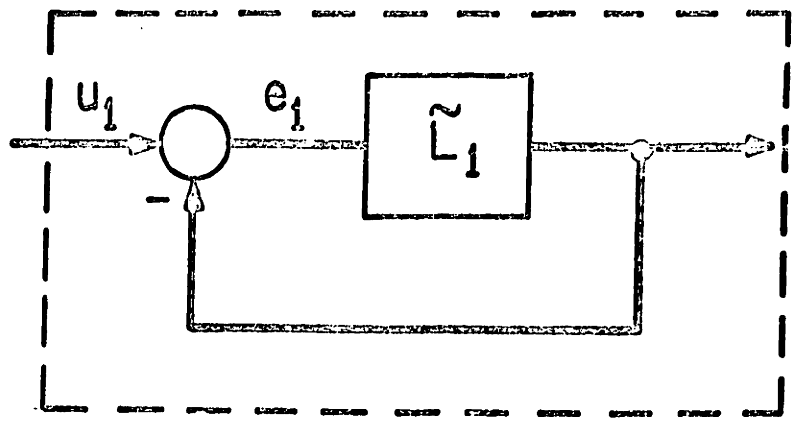
is the required factorization of L_2 into the product of two strictly positive operators.

Remarks: It is interesting to give a "block diagram" interpretation to the above result. The process of constructing equivalent systems through the use of bilinear transformations has an interpretation in terms of factoring certain operators out of the loop. If the original system is written in the form

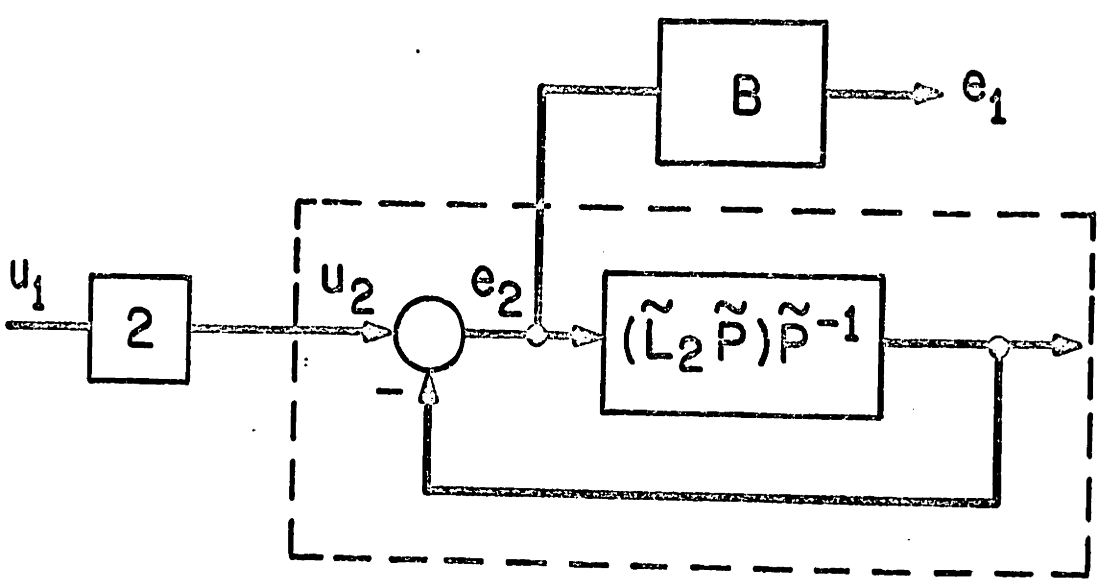
$$(I(1-\epsilon) + L_1 + I(1+\epsilon) + L_1)e = 2u \tag{2.42}$$

then the block diagram interpretation of Figure 6 appears. The bounded operator B is just $(I(1+\epsilon)+L_1)^{-1}$, and the system S_2 in which the loop transfer operator may be factored appears as the only feedback loop imbedded in the overall system.

As may be easily seen, the proof follows the same idea as the example immediately preceding the theorem. A loop transformation is made to put the spectrum of the forward loop operator of the new equivalent system in the right half plane, and at the same time to preserve the positivity of the feedback operator. In the case of Theorem 2.13 the feedback operator is just the identity, and the transformation T has been chosen to leave it invariant.



(a)



(b)

Fig. 6 System S_2 embedded in S_1

It is quite easy to interpret Theorem 2.13 in the case that the original system is of the form

$$e_1(t) + k\int_0^t g(t-s)\,e_1(s)\,ds = u_1(s)\ ,\ t \geq 0 \tag{2.43}$$

with $g(\cdot) \in L_1(0,\infty)$. Taking Laplace transforms leads to the relation

$$(1 + k\hat{\mathcal{G}}(s))\hat{e}_1(s) = \hat{u}_1(s) \tag{2.44}$$

The Laplace transform representation for S_2 becomes

$$\left(1 + \frac{(1-\epsilon) + k\hat{\mathcal{G}}(s)}{(1+\epsilon) + k\hat{\mathcal{G}}(s)}\right)\hat{e}_2(s) = \hat{u}_2(s) \tag{2.45}$$

Letting

$$\frac{(1-\epsilon) + k\hat{\mathcal{G}}(s)}{(1+\epsilon) + k\hat{\mathcal{G}}(s)} = \hat{\mathcal{G}}_2(s) \tag{2.46}$$

it follows that since the Nyquist criterion is satisfied for a stable system of the form (1.43), the transfer function $\hat{\mathcal{G}}_2$ is a positive real function. Hence L_2 already represents a positive operator, and in this instance the operator P of Theorem 2.13 need not be introduced.

J. SYSTEMS STABLE FOR ALL POSITIVE FEEDBACK GAINS

In this section a characterization of systems which are stable for all positive constant feedback gains is given. Consider a regular system of the form $S_k : \{(I+kL)e = u\}$, where k is a positive constant. Since the spectrum of L is compact, it follows that S_k is stable for all sufficiently small values of k. If one considers the stability of the class of systems S_k, it is typically found that for some sufficiently large value of k, an instability may develop.

For some classes of systems, however, such an instability cannot occur. For instance, if L is such that it may be factored as a product of two positive operators, then the addition of a positive constant multiplier to one of the factors does not disturb the positivity, and so Theorem 2.10 continues to guarantee stability irrespective of the magnitude of k.

In the case where L corresponds to a scalar convolution operator, then something like a converse is true. Brockett and Willems have shown in the rational transform case that, for any $\epsilon > 0$, $L + \epsilon I$ may be factored into a product of two positive operators. Freedman and Zames have extended this result to the case of transfer functions which are not necessarily rational functions.

The result below may be interpreted as a generalization of the above.

Theorem 2.13: Suppose that the regular system $S_k : \{(I + kL)e = u\}$ is bounded input – bounded output stable in the $\mathcal{H}(T^+)$ sense for each fixed k in the range $0 < k < \infty$. Then for any $\epsilon > 0$, there exists an invertible operator S such that $S^{-1}(L + \epsilon I)S = P^2$, with P strictly positive.

Proof: By Theorem 2.1, the condition that S_k be stable for all $0 < k < \infty$ is equivalent to the statement that $\sigma(L)$ does not meet the negative real axis, with the possible exception of the point $\{0\}$. Hence for any $\epsilon > 0$, the operator

$$(L + \epsilon I)$$

has a spectrum which does not meet $(-\infty, 0]$.

The function $f(z) = z^{1/2}$ is analytic in the z-plane cut along the negative real axis: hence $z^{1/2}$ belongs to the class $\mathcal{F}(L + \epsilon I)$.

The operator $(L + \epsilon I)^{1/2}$ is well defined, and by the spectral mapping theorem has its spectrum in the (open) right half plane. By Theorem 2.12, there exists an invertible bounded linear operator S such that

$$P = S^{-1}(L + \epsilon I)^{1/2} S \tag{2.47}$$

is positive. Since

$$P^2 = S^{-1}(L + \epsilon I)S$$

the theorem is proved.

<u>Remarks</u>: The 2 X 2 matrix example preceding Theorem 2.12 strongly suggests that it is necessary to introduce the similarity transformation S in order even to extend the original scalar result to the case of matrix convolution operators. That the similarity S may be needed is further indicated by the fact that the hypothesis of the theorem is invariant under similarity transformations of the loop transfer operator. The reason that the similarity is not needed in the case of a scalar convolution is again the fact that scalar convolutions are normal operators. In that case, the fact that the spectrum lies in the right half plane is sufficient to give positivity.

III. TIME-INVARIANT SYSTEMS

A. INTRODUCTION

In this chapter a treatment is given of feedback systems which may be described by the vector equations

$$\underline{e}(t) + \underline{\underline{K}} \int_0^t \underline{\underline{G}}(t-s)\,\underline{e}(s)\,ds = \underline{u}(t), \quad t \geq 0 \tag{3.1}$$

(in the case of a continuous time variable) or by

$$\underline{e}(n) + \underline{\underline{K}} \sum_{j=0}^{n-1} \underline{\underline{G}}(n-j)\,\underline{e}(j) = \underline{u}(n) \quad n \geq 0 \tag{3.2}$$

(in the case of discrete time).

Following conventional notation, the convolution integal of (3.1) and the convolution sum of (3.2) will be denoted by an asterisk, so that the system equations take the form

$$\underline{e}(\tau) + \underline{\underline{K}} \; (\underline{\underline{G}} * e)(\tau) = \underline{u}(\tau), \qquad \tau \geq 0, \tag{3.3}$$

where τ may take integer values corresponding to (3.2) are the continuous values corresponding to (3.1).

Such systems arise naturally from the consideration of ordinary differential equations with constant coefficients. A dynamical system governed by the vector differential equation

$$\dot{\underline{x}} = \underline{\underline{A}}\,\underline{x} + \underline{\underline{B}}\,\underline{e}, \tag{3.4}$$

with "input" $\underline{e}$ and "output" $\underline{y} = \underline{\underline{C}}\,\underline{x}$ can by use of the "variation of constants" formula be put in the form of an input-output relation

$$y(t) = \int_0^t \underline{C}\, e^{\underline{\underline{A}}(t-\sigma)} \underline{\underline{B}}\, e(\sigma)\, d\sigma \tag{3.5}$$

(assuming zero initial conditions). Similarly, a system governed by a vector difference equation of the form

$$\underline{x}(k+1) = \underline{\underline{A}}\, x(k) + \underline{\underline{B}}\, \underline{e}\,(k) \tag{3.6}$$

with "output" $y(k) = \underline{\underline{C}}\, x(k)$ may be described from an input-output point of view by

$$y(n) = \sum_{j=0}^{n-1} \underline{C}\, \underline{\underline{A}}^{n-j} \underline{\underline{B}}\, \underline{e}(j) \tag{3.7}$$

(again assuming zero initial conditions).

If in (3.4) one considers the input e to consist of the sum of a linear combination of the outputs $-\underline{\underline{K}}\,\underline{y}$ and a new input $\underline{u}$, then it is easy to see that the vector $\underline{e}$ satisfies the integral equation

$$e(t) + \underline{\underline{K}} \int_0^t \underline{\underline{c}}\, e^{\underline{\underline{A}}(t-s)} \underline{\underline{B}}\, \underline{e}(s)\, ds = u(t) - \underline{\underline{K}}\,\underline{\underline{c}}\, e^{\underline{\underline{A}}t} \underline{x}_o \tag{3.8}$$

where $\underline{x}_o$ is the initial condition for the differential equation (3.4). In order for (3.8) to fall into the class of systems to which the theory of Chapter II is applicable, it is necessary that the integral operator in (3.8) be bounded on some input space $X(T^+)$. For the usual $L_p(0, \infty)$ spaces, etc., this requires that the eigenvalues of the matrix $\underline{\underline{A}}$ satisfy $\text{Re}\, \lambda_i(\underline{\underline{A}}) < 0$. This condition also means that the initial condition response $\underline{\underline{K}}\,\underline{\underline{c}}\, e^{\underline{\underline{A}}t} x_o$ will belong to the space $X(T^+)$ for most spaces of practical interest, and hence may be lumped with the input $\underline{u}$.

The above remarks apply equally well to the case of the difference equation (3.7) in which case the restriction on the matrix $\underline{\underline{A}}$ becomes $|\lambda_i(\underline{\underline{A}})| < 1$.

As mentioned in Section II-C, it is simple to manipulate equations of the types (3.4) and (3.6) into an equivalent form so that these conditions are satisfied:

<u>Notation</u>: It turns out that the stability properties of the feedback systems (3.1) and (3.2) are the same relative to a number of possible input spaces $X(T^+)$. For this reason it is convenient to let E^+ stand for anyone of the following continuous time spaces:

a. $L_p(0,\infty)$, for $1 \leq p \leq \infty$

b. $M_c(0,\infty)$, the continuous $L_\infty(0\ \infty)$ functions

c. $M_u(0,\infty)$, the uniformly continuous $L_\infty(0,\infty)$ functions

d. $C(0,\infty)$, continuous functions for which a limit as $t \to \infty$ exists

c. $C_o(0,\infty)$, the subspace of $C(0,\infty)$ for which the limit is zero.

By E_n^+ is meant the space of n-vector functions, each component of which belongs to E^+.

In the case of a discrete time parameter, let $\mathcal{E}^+$ stand for any one of the following:

a. l_p^+, for $1 \leq p \leq \infty$

b. C^+, the convergent sequences on Z^+

c. C_o^+, sequences convergent to zero on Z^+.

Analogously, $\mathcal{E}_n^+$ denotes the space of n-vector sequences, each component of which belongs to $\mathcal{E}^+$.

B. REGULARITY OF STATIONARY SYSTEMS

In order to apply the theory of Chapter II, it is necessary to determine a class of regular systems defined by (3.1) and (3.2). The first problem is that of the boundedness of the operators.

Lemma 3.1 a: Let $\underline{\underline{G}}(\cdot)$ be an nxn matrix of functions belonging to $L_1(0, \infty)$. Then the operator G defined by

$$G: \quad \underline{x}(t) \rightarrow \int_0^t \underline{\underline{G}}(t-s)\underline{x}(s)\,ds, \qquad t \geq 0,$$

is a bounded linear operator $E_n^+ \rightarrow E_n^+$.

Lemma 3.1 b: Let $\{\underline{\underline{G}}(i)\}_1^\infty$ be a sequence of nxn matrices, each element of which belongs to l_1^+. Then the operator G defined by

$$G: \quad \underline{x}(n) \rightarrow \sum_{y=0}^{n-1} \underline{\underline{G}}(n-j)\underline{x}(j) \quad n = 0, 1, ..$$

is a bounded linear operator $\mathcal{E}_n^+ \rightarrow \mathcal{E}_n^+$.

Proofs: The above are standard results; [11] has a sketch of the proof, together with further references.

The next problem is to verify that the loop equations have a unique finite time solution. As mentioned in Chapter II, this follows from the fact that the equations involved are of the Voltera type.

Lemma 3.2 a: Let $\underline{\underline{G}}(\cdot)$ be an nxn matrix of functions belonging to $L_1(0, \infty)$. Then for each finite $T > 0$, the equation

$$\underline{e}_T(t) + \int_0^t \underline{\underline{G}}(t-s)\underline{e}_T(s)\,ds = \underline{u}_T(t), \quad 0 \le t \le T \tag{3.10}$$

has a unique solution $\underline{e}_T \in (E_n^+)_T$ corresponding to each $u_T \in (E_n^+)_T$.

Proof: In the case of $L_p^{(n)}(0, T)$, $1 \le p \le \infty$, the result follows from Ringrose [34]. (Strictly speaking 34 covers the case n=1; however the present case follows by replacing absolute values with the appropriate vector and matrix norms throughout 34). The case of continuous functions follows from the L_∞ case, and the fact that the integral operator defining the inverse is of the same type as the original, i.e. it maps continuous functions into continuous ones.

Lemma 3.2 b: Let $\{G(i)\}_1^\infty$ be a sequence of matrices, each element of which belongs to l_1^+. Then for each finite $T > 0$ the equation

$$\underline{e}_T(n) + \sum_{j=0}^{n-1} \underline{\underline{G}}(n-j)\underline{e}_T(j) = \underline{u}_T(n), \quad 0 \le n \le T \tag{3.11}$$

has a unique solution $\underline{e}_T \in (\mathcal{E}_n^+)_T$ for each $\underline{u}_T \in (\mathcal{E}_n^+)_T$.

Proof: This is trivial, since the equation (3.4) defines the sequence $\{e_T\}_0^T$ recursively.

C. THE NYQUIST CONDITION

Since Section B above shows that equations (3.1) and (3.2) define regular systems in each of the spaces E_n^+ (or $\mathcal{E}_n^+$ as appropriate), Theorem 2.1 may be applied to give necessary and sufficient conditions for stability.

Definition 3.1: For each matrix of $L_1(0,\infty)$ functions; define the transfer function matrix

$$\hat{\underline{\underline{G}}}(s) = \int_0^\infty \underline{\underline{G}}(t)\, e^{-st}\, dt, \quad \mathrm{Re}\, s \geq 0 \tag{3.12}$$

Similarly, for each sequence of matrices $\{\underline{\underline{G}}_i\}_0^\infty$, each element of which belongs to l_1^+, define the pulse transfer function matrix

$$\hat{\underline{\underline{G}}}(z) = \sum_{i=0}^{\infty} \underline{\underline{G}}(i)\, z^{-i}, \quad |z| \geq 1, \tag{3.13}$$

The problem of spectrum calculations for convolution equations on a half-axis is a highly non-trivial one. It has been solved only comparatively recently, and its complete solution hinges on certain nations from the theory of generalized Fredholm operators. (Φ-operators in the Russian literature) 11, 36, 37. For the sake of keeping the proof of Theorem 3.1 below to a reasonable length, it is necessary to introduce the following definitions.

Definition 3.2 [36]: A closed linear operator A is called a generalized Fredholm operator if the following conditions are satisfied:

a. α_A, the dimension of the null space, is finite

b. β_A, the codimension of the range, is finite

c. the equation $Ax = y$ is solvable if and only if y is orthagonal to the null space of the adjoint of A.

The integer $k = \alpha_A - \beta_A$ is called the index of A.

Theorem 3.1 a: Let $\underline{\underline{G}}(\cdot)$ be a matrix of $L_1(0,\infty)$ functions. Then the feedback system

$$\underline{e}(t) + \underline{\underline{K}} \int_0^t \underline{\underline{G}}(t-s)\underline{e}(s)\,ds = \underline{u}(t), \quad t \geq 0 \tag{3.1}$$

is bounded input - bounded output stable in the E_n^+ sense if and only if the following two conditions are satisfied:

$$\begin{aligned} &1.\ \det\,(\underline{\underline{I}} + \underline{\underline{K}}\,\underline{\underline{G}}(i\omega)) \neq 0 \qquad -\infty \leq \omega \leq \infty \\ &2.\ k = \frac{1}{2\pi} \int_{-\infty}^{\infty} d_\omega \arg\,(\det\,(\underline{\underline{I}} + \underline{\underline{K}}\,\underline{\underline{G}}(i\omega))) = 0 \end{aligned} \tag{3.14}$$

Proof: By Lemmas 3.1 and 3.2 the equation 3.1 defines a regular feedback system relative to each E_n^+, and so the system is stable if and only if $-1 \notin \sigma(KG)$ on E_n^+.

If the first condition is violated, then the operator $I+KG$ is not even invertible on the banach space E_n obtained by extening E_n^+ to the corresponding class of functions defined on the whole real axis, and hence cannot be invertible on E_n^+ [33].

If condition 1 is satisfied, then Gokhberg and Krein show that $I+KG$ is a generalized Fredholm operator on E_n^+. Further $I+KG$ has a bounded inverse on E_n^+ if and only if each of the numbers α_A, β_A are zero.

In Ref. 33,, the integers α_A and β_A are found by the process of spectral factorization of a matrix of functions. However, in the present case this is not required, since α_A is identically zero. This follows from the observation that an element of the null space of $I+KG$ would generate a solution to the homogeneous equation

$$\underline{e}_T + \underline{\underline{K}}(\underline{\underline{G}} * \underline{e}_T) = \mathbf{0}, \quad 0 \leq t \leq T, \qquad (3.15)$$

contradicting the regularity of the system.

It is further shown that the integer k in condition 2 is actually the index of the generalized Fredholm operator I+KG. Since α_A is zero, β_A is zero if and only if $k = 0$. Hence I+KG is invertible on E_n^+ if and only if $k = 0$.

Theorem 3.1b: Let $\{G(i)\}_1^{\infty}$ be a sequence of matrices each element of which belongs to l_1^+. Then the feedback system

$$\underline{e}(n) + \underline{\underline{K}} \sum_{j=0}^{n-1} \underline{\underline{G}}(n-j)\underline{e}(j) = \underline{u}(n), \quad n \geq 0, \qquad (3.2)$$

is bounded input - bounded output stable in the $\mathcal{E}_n^+$ sense if and only if the following two conditions are satisfied:

$$1. \det (I + \underline{\underline{K}}\, \underline{\underline{G}}(e^{i\theta})) \neq 0, \quad -\pi \leq \theta \leq \pi$$

$$2. k = \frac{1}{2\pi} \int_{-\pi}^{\pi} d_\theta \arg(\det(\underline{\underline{I}} + \underline{\underline{K}}\, \underline{\underline{G}}(e^{i\theta}))) = 0 \qquad (3.16)$$

Proof: The proof is exactly parallel to that of Theorem 3.1a, merely exchanging the boundary of the unit disc $|z| = 1$ for the imaginary axis.

D. A CLASS OF SUFFICIENT CONDITIONS

If the matrices occurring in Theorem 3.1 alone are of large dimension, then the evaluation of the determinants involved becomes a computationally difficult problem. It may occur that the matrix $\underline{\underline{K}}\,\underline{\underline{G}}(s)$ has a form which is nearly block diagonal, i.e. it has the form

$$\underline{\underline{K}}\,\underline{\underline{G}}(s) = \left[\begin{array}{c|c|c} \underline{\underline{M}}_1(s) & \underline{\underline{0}} \;\cdots & \underline{\underline{0}} \\ \hline \underline{\underline{0}} & \underline{\underline{M}}_2(s) & \\ \hline \underline{\underline{0}} & & \underline{\underline{M}}_r(s) \end{array}\right] + \underline{\underline{R}}(s) \tag{3.17}$$

where the $\underline{\underline{M}}_i(s)$ are square matrices of functions of appropriate dimension, and the remainder $\underline{\underline{R}}(s)$ is small in some sense. If $\underline{\underline{R}}(s)$ were identically zero, then the stability of the original system could be determined from the appropriate quantities

$$f_r(s) = \det\,(\underline{\underline{I}} + \underline{\underline{M}}_i(s)), \quad i = 1, 2, r, \tag{3.18}$$

since the encirclements of the origin by $\det(\underline{\underline{I}} + \underline{\underline{K}}\,\underline{\underline{G}}(i\omega))$, would then be just the sum of the encirclements of the individual functions (3.18).

Theorem 3.1 shows that the spectrum of the operator KG (acting on E^+) is just the locus of the eigenvalues of the matrix of functions $\underline{\underline{M}}(s) = -\underline{\underline{K}}\,\underline{\underline{G}}(s)$ for s is the region $\mathrm{Re}\, s \geq 0$. One way to obtain sufficient conditions for stability (or instability) is to find bounds on the eigenvalues of $-\underline{\underline{K}}\,\underline{\underline{G}}(s)$, $\mathrm{Re}\, s \geq 0$, which will insure that the point -1 is not (or is) included.

The problem of obtaining bounds on the location of the eigenvalues of finite dimensional matrices is a widely studied topic, with a large amount of literature. See Ref. 38 for both a treatment of some aspects of the problem, as well as a large bibliography on the topic.

A large number of the results in the literature are concerned with special classes of matrices, and hence are not useful for the present application. The following Theorem, due to Bauer and Fike [39], is one example of a general result which includes a number of special cases previously derived.

Theorem 3.2: (Bauer and Fike). Let $\| \underline{\underline{M}} \|$ be induced norm for the nxn matrix $\underline{\underline{M}}$. Then the set $\{ z \mid \| (\underline{\underline{I}} z - \underline{\underline{B}})^{-1} \|^{-1} \leq \| \underline{\underline{A}} - \underline{\underline{B}} \|$ or $\det (\underline{\underline{I}} z - \underline{\underline{B}}) = 0 \}$ contains all the characteristic roots of $\underline{\underline{A}}$.

The above is an inclusion theorem for the spectrum of the matrix $\underline{\underline{A}}$. The contrapositive of the above gives an exclusion theorem, which is essentially a matrix version of the "small gain theorem" of Section II-F.

Corollary: If $\det (\underline{\underline{I}} z - \underline{\underline{B}}) \neq 0$, and $\| (\underline{\underline{I}} z - \underline{\underline{B}})^{-1} \| \cdot \| \underline{\underline{A}} - \underline{\underline{B}} \| < 1$, then z does not belong to $\sigma(\underline{\underline{A}})$.

By taking just for a norm on C^n the sum of the absolute values of the components (the l_1 norm), and then for a norm the maximum absolute value of the components, the classical Gershgorin results [41] may be obtained by choosing $\underline{\underline{B}}$ to be the diagonal of $\underline{\underline{A}}$ in Theorem 3.2. The Gershgorin result has been used by Rosenbrock [40] to obtain a sufficient condition for stability based on essentially the diagonal entries of the matrix of functions $\underline{\underline{K}}\,\underline{\underline{G}}(s)$. By using Theorem 3.2 to estimate eigenvalue locations rather than Gershgorin's theorem, the following generalization is obtained.

Theorem 3.3: Let $\underline{\underline{G}}(\cdot)$ be a matrix of $L^1(0,\infty)$ functions. Then suppose that the feedback system

$$\underline{e}(t) + \underline{\underline{K}} \int_0^t \underline{\underline{G}}(t-s)\underline{e}(s)\,ds = \underline{u}(t), \quad t \geq 0 \tag{3.19}$$

is such that there exists a decomposition

$$\underline{\underline{K}}\,\underline{\underline{G}}(s) = \underline{\underline{D}}(s) + \underline{\underline{R}}(s) \tag{3.20}$$

where

$$\begin{aligned} &1.\ \det\,(\underline{\underline{I}} + \underline{\underline{D}}(i\omega)) \neq 0 \quad -\infty < \omega < \infty \\ &2.\ \| (\underline{\underline{I}} + \underline{\underline{D}}(i\omega))^{-1} \| \ \| \underline{\underline{R}}(i\omega) \| < 1, \quad -\infty < \omega < \infty \end{aligned} \tag{3.21}$$

for some induced matrix norm $\| \ \|$.

Then the system (3.19) is bounded input - bounded output stable in the E_n^+ sense if

$$K_D = \frac{1}{2\pi} \int_{-\infty}^{\infty} d_\omega \arg\,(\det\,(\underline{\underline{I}} + \underline{\underline{D}}(i\omega))) = 0,$$

and (3.19) is unstable if $K_D \neq 0$.

Proof: Consider $\underline{\underline{M}}_\epsilon(i\omega) = \underline{\underline{D}}(i\omega) + \epsilon\,\underline{\underline{R}}(i\omega)$, $0 \leq \epsilon \leq 1$. Clearly $\underline{\underline{M}}_0(i\omega) = \underline{\underline{D}}(i\omega)$, while $\underline{\underline{M}}_1(i\omega) = \underline{\underline{K}}\,\underline{\underline{G}}(i\omega)$.

First conditions 1 and 2 together with Theorem 3.2 imply that $\det\,(\underline{\underline{I}} + \underline{\underline{K}}\,\underline{\underline{G}}(i\omega)) \neq 0$.

By the Riemann-Lebesgue Lemma, for fixed ϵ the locus of points $\det\,(\underline{\underline{I}} + \underline{\underline{M}}_\epsilon(i\omega))$, $-\infty \leq \omega \leq \infty$ forms a continuous closed curve in the complex plane. If K_D, the number of encirclements of the origin by $\det\,(\underline{\underline{I}} + \underline{\underline{D}}(i\omega)$ is not the same as the number of encirclements of the quantity $\det(\underline{\underline{I}} + \underline{\underline{K}}\,\underline{\underline{G}}(i\omega))$, then for some ϵ_o, $0<\epsilon<1$, and some ω_o, $-\infty<\omega_o<\infty$

$$\det(\underline{\underline{I}} + \underline{\underline{M}}_{\epsilon_o}(i\omega_o)) = 0 \qquad (3.23)$$

But this is contradicted by condition 2 and Theorem 3.2, so the encirclements are the same, and the result follows from Theorem 3.1.

Obviously, a result completely parallel to the above holds in the case of discrete time systems.

E. EVENTUALLY TIME-INVARIANT SYSTEMS.

One aspect of the behavior of what has been defined as regular systems is that their stability depends only on their long term response to the application of an input. A consequence of this is that in a sense their stability is determined by the "asymptotic behavior" of the systems. This is illustrated by the following example of a convolution operator followed by a time-varying feedback gain which may be considered as the sum of a constant and a transient term.

<u>Theorem 3.4:</u> Suppose that the regular feedback system S is described by the equation

$$\underline{e}(t) + \underline{\underline{K}}(t)\int_0^t \underline{\underline{G}}(t-s)\underline{e}(s)\,ds = \underline{u}(t), \quad t \geq 0, \qquad (3.24)$$

where $\underline{\underline{K}}(\cdot)$ is a matrix of continuous functions, uniformly bounded on $(0,\infty)$. Suppose also that $\lim_{t\to\infty} \underline{\underline{K}}(t) = \underline{\underline{K}}_\infty$, and that the system $S_\infty : \{(\underline{\underline{I}} + \underline{\underline{K}}_\infty \underline{\underline{G}}*)\underline{e} = \underline{u}\}$ is bounded input - bounded output stable in the E_n^+ sense. Then the system S is also bounded input - bounded output stable in the E_n^+ sense.

Proof: Since the system S_∞ is stable, there exists a constant M such that

$$\|G(I + K_\infty G)^{-1}\| < M \tag{3.25}$$

Since $\lim_{t \to \infty} \underline{\underline{K}}(t) = \underline{\underline{K}}_\infty$, there exists a $T < \infty$ such that

$$\|\underline{\underline{K}}(t) - \underline{\underline{K}}_\infty\| < \frac{1}{M}, \quad t \geq T. \tag{3.26}$$

The system equations may be rewritten in the form

$$(I + KG)\, e_{1T} = u_{1T} \quad 0 \leq t \leq T$$

$$\text{and} \qquad \underline{e}_2(t) + \underline{\underline{K}}(t+T) \int_0^t \underline{\underline{G}}(t-s)\underline{e}_2(s) = \underline{u}_2(t), \quad t \geq 0 \tag{3.27}$$

where $\underline{e}_2(t) = \underline{e}(t+T)$, $u_2(t) = u(t+T)$.

Since the system is assumed regular

$$\|e_T\| \leq M_1 \|u_T\| \tag{3.28}$$

Rewrite (3.27) in the form

$$(I + K_\infty G + (K - K_\infty)G)\, e_2 = u_2$$

which is equivalent to

$$(I + (K-K_\infty)\, G\, (I+K_\infty G)^{-1})\, e_2^{1} = u_2,$$

which is stable according to the small gain theorem using inequalities (3.24) and (3.25). Hence

$$\|e_2\| \leq M_2 \|u_2\|,$$

from which together with (3.28) it follows that S is stable.

F. SOME REMARKS ON FREDHOLM OPERATORS

One of the most useful aspects of Theorem 3.1 is that it shows that the stability of a time-invariant system may be determined entirely from the Nyquist locus, the set of points $\{\det(\underline{I} + \underline{K}\,\underline{G}(i\omega)), -\infty \leq \omega \leq \infty\}$. As shown in the course of the proof, the number of encirclements of the origin of the Nyquist locus is just the so-called index of a certain generalized Fredholm operator. Because of the definition of a regular system, the feedback equations in the general case

$$(I + KG)\, e = u \tag{3.29}$$

will be such that the operator $(I+KG)$ has no null space. Otherwise, just as argued in the proof of Theorem 3.1, a vector e_o annihilated by $(I+KG)$ would generate a non-trivial solution to the finite-time truncated equations with zero input,

$$(I+KG)\, e_{o_T} = 0, \qquad 0 \leq t \leq T. \tag{3.30}$$

This contradicts the regularity assumption that the finite-time truncated equation has a uniqe solution.

If the operator $(I+KG)$ is a generalized Fredholm operator, then again, just as in Theorem 3.1, its index $K = -\beta_{I+KG}$ and the operator $I+KG$ will again be invertible if and only if $K = 0$. This suggests the possibility of generalizing Theorem 3.1 to the case when the system is not necessarily described by a convolution equation, and obtaining a "Nyquist condition" for time varying systems. All that is required is an analytic method of computing the index for the integral operator involved.

In fact, it is possible to interpret Theorem 3.4 in this manner. Sahbagjan (Ref.42) shows that a certain class of integral equations on a half-axis whose kernels are "eventually stationary" indeed do define generalized Fredholm operators. Moreover, their index is the same as the index of the stationary limit kernel. From this point of view, Theorem 3.4 is simply computing the index of $I+KG$ in a case where the system is not time invariant.

The problem of obtaining solutions of a convolution equation defined on a half-axis is commonly referred to as a Wiener-Hopf problem. It is well known that by taking the Fourier transform of the original equation, the problem is transformed into one involving certain functions which are analytic in certain regions of the plane, and whose limiting values at the boundary of the regions satisfy certain linear relationships. This approach is fully exploited in[11].

If one considers equations of the form

$$e(t) + \int_0^t g(t-s)\cos(s)\, e(s)\, ds = u(t) \quad t \geq 0, \qquad (3.31)$$

which may be interpreted as a time-invariant system with a cosine function as a time-varying feedback gain, then again by use of the Fourier transform the problem may be transformed into one involving analytic functions; however in this case the linear relationships at the boundaries involved <u>shifted</u> values of the functions. Problems of this type also have an index theory associated with them, and have been widely studied. Ref. 43 is a survey paper containing a large bibliography. While this is still an active area of research in complex analysis the theory is relatively complete only in the case where the shift satisfies a so-called generalized Carleman condition, which requires that a finite number of applications of the shift bring the boundary back to the original. This is not the case for the problem corresponding to equation (3.31), so the theory at present seems inapplicable. The discrete analog of (3.31), however, does lead to a problem which may be handled by these methods. However, this particular problem may also be solved by more direct approaches. (See Chapter IV).

IV. PERIODIC DISCRETE SYSTEMS

A. INTRODUCTION

This chapter is concerned with feedback systems which have the form of a convolution operator in the forward loop, coupled with a periodically time-varying gain in the feedback path. (Fig. 7.) Such systems are governed by equations of the form

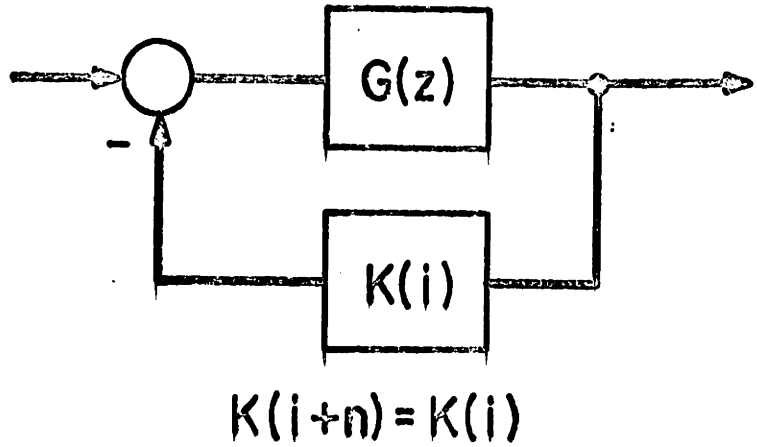
$$\underline{e}(\tau) + \underline{\underline{K}}(\tau)\,(\underline{\underline{G}} * \underline{e})\,(\tau) = \underline{u}(\tau) \qquad \tau \geq 0 \tag{4.1}$$

where the "gain function" $K(\cdot)$ satisfies $\underline{\underline{K}}(i+n) = \underline{\underline{K}}(i)$, $i \geq 0$, in the discrete time case, or $\underline{\underline{K}}(t+T) = \underline{\underline{K}}(t)$, $t \geq 0$ for the case of a continuous time variable.

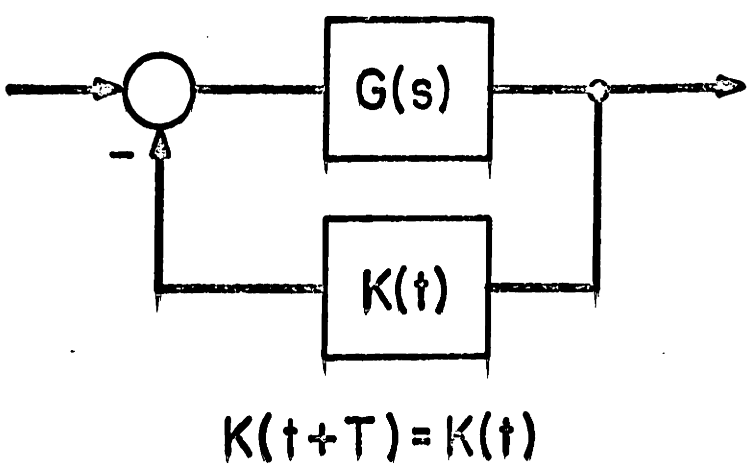
Such equations arise in an engineering context in connection with the analysis of parametric amplifiers; the stability of a periodic solution of certain classes of non-linear differential equations is also determined from systems of the form (4.1). As a result, a considerable amount of work has been done on systems of this type (eg. 44-48).

Most of the previous work on system (4.1) has been concerned with the continuous time variable case, and generally speaking, sufficient conditions for stability are what has been obtained in an analytically tractable form. However, for the discrete time version of (4.1), it is possible to obtain <u>necessary and sufficient</u> conditions for stability in closed form.

By applying the several spectral estimation results of Chapters II and III to the conditions derived, sufficient conditions for the stability of (4.1) may be derived.



(a)



(b)

Fig. 7 Systems with Periodic Feedback

B. A CLASS OF EQUIVALENT DISCRETE TIME SYSTEMS

The results of Chapter III are apparently the only spectral calculations which are presently known and directly applicable to the feedback stability problem. These results apply, however, only to convolution equations (i.e. time-invariant feedback systems). The system (4.1) appears to be a time-varying one, but it is possible to re-interpret (4.1) in such a way that the resulting system appears stationary. This makes it possible to apply the results of Chapter III to the equivalent system in order to obtain necessary and sufficient conditions for stability.

The key to the problem is to focus attention on the action of the periodically time-varying gain, $\underline{\underline{K}}(\cdot)$. To illustrate the idea involved, consider a scalar input—scalar output system described by the equations

$$x(i+1) = \underline{\underline{A}}\, x(i) + \underline{b}\, e(i)\ , \qquad y(i) = \underline{c}'\underline{x}(i)\ ,$$
$$e(i) = v(i) - k(i)\ y(i) \qquad i \geq 0 \tag{4.2}$$

where $\underline{x}$ is an r vector, e and y are scalars, and the gain satisfies $k(i+n) = k(i)$. This corresponds to an input-output description of the form

$$e(i) + k(i) \sum_{j=0}^{i-1} \underline{c}'\, \underline{\underline{A}}^{i-j}\underline{b}\ e(j) = u(i)\ , \quad i \geq 0 \tag{4.3}$$

Consider the sequences which correspond to the output y , and the feedback term, Ky .

$$y : \{y_0, y_1, \ldots, y_{n-1}, y_n, \ldots, y_{2n-1}, \ldots\}$$

$$Ky : \{k_0 y_0, k_1 y_1, \ldots, k_{n-1} y_{n-1}, k_0 y_n, \ldots, k_{n-1} y_{2n-1}, \ldots\} \tag{4.4}$$

Due to the periodicity of the gain $k(\cdot)$, every n^{th} term in the y sequence is multiplied by the same value assumed by the gain. Hence, if the elements of the y sequence are grouped together as sequences of column vectors of dimension n:

$$y: \left\{ \begin{bmatrix} y_0 \\ \vdots \\ y_{n-1} \end{bmatrix}, \begin{bmatrix} y_n \\ \\ y_{2n-1} \end{bmatrix}, \begin{bmatrix} y_{2n} \\ \\ y_{3n-1} \end{bmatrix}, \ldots \begin{bmatrix} y_{\ell n} \\ \\ y_{(\ell+1)n-1} \end{bmatrix}, \ldots \right\} \tag{4.5}$$

then the feedback term Ky has a corresponding representation

$$Ky: \left\{ \begin{bmatrix} k_0 & & \underline{\underline{0}} \\ & \ddots & \\ \underline{\underline{0}} & & k_{n-1} \end{bmatrix} \begin{bmatrix} y_0 \\ \\ y_{n-1} \end{bmatrix}, \begin{bmatrix} k_0 & & \underline{\underline{0}} \\ & \ddots & \\ \underline{\underline{0}} & & k_{n-1} \end{bmatrix} \begin{bmatrix} y_n \\ \\ y_{2n-1} \end{bmatrix}, \ldots \begin{bmatrix} k_0 & & \underline{\underline{0}} \\ & \ddots & \\ \underline{\underline{0}} & & k_{n-1} \end{bmatrix} \begin{bmatrix} y_{\ell n} \\ \\ y_{(\ell+1)n-1} \end{bmatrix} \right.$$

$$= \begin{bmatrix} k_0 & & \underline{\underline{0}} \\ & \ddots & \\ \underline{\underline{0}} & & k_{n-1} \end{bmatrix} \left\{ \begin{bmatrix} y_0 \\ \\ y_{n-1} \end{bmatrix}, \begin{bmatrix} y_n \\ \\ y_{2n-1} \end{bmatrix}, \ldots \begin{bmatrix} y_{\ell n} \\ \\ y_{(\ell+1)n-1} \end{bmatrix} \ldots \right\} \tag{4.6}$$

If the ℓ^{th} column vector in (4.5) is denoted by $\mathcal{Y}_\ell$, then (4.5) and (4.6) have the form

$$y: \{\mathcal{Y}_0, \mathcal{Y}_1, \mathcal{Y}_2, \ldots, \mathcal{Y}_\ell, \ldots\}$$

$$\begin{bmatrix} k_0 & & \underline{\underline{0}} \\ & \ddots & \\ \underline{\underline{0}} & & k_{n-1} \end{bmatrix} \{\mathcal{Y}_0, \mathcal{Y}_1, \ldots, \mathcal{Y}_\ell, \ldots\}$$

so that according to the "time" index of the n-vector sequences the action of K is time-invariant.

Consider now Equation (4.2), and with a view toward grouping sequences as above, write it in the form

$$\begin{cases} \underline{x}_0 = \underline{x}_0 \\ \underline{x}_1 = \underline{\underline{A}}\,\underline{x}_0 + \underline{b}\,e_0 \\ \underline{x}_2 = \underline{\underline{A}}^2\underline{x}_0 + \underline{\underline{A}}\,\underline{b}\,e_0 + \underline{b}\,e_1 \\ \vdots \\ \underline{x}_{n-1} = \underline{\underline{A}}^{n-1}\underline{x}_0 + \underline{\underline{A}}^{n-2}\underline{b}\,e_0 + \dots + \underline{b}\,e_{n-2} \end{cases}$$

$$\begin{cases} x_n = \underline{\underline{A}}^n\underline{x}_0 + \underline{\underline{A}}^{n-1}\underline{b}\,e_0 + \dots + \underline{b}\,e_{n-1} \\ x_{n+1} = \underline{\underline{A}}\,\underline{x}_n + \underline{b}\,e_n \\ \vdots \\ x_{2n-1} = \underline{\underline{A}}^{n-1}\underline{x}_n + \underline{\underline{A}}^{n-2}\underline{b}\,e_n + \dots + \underline{b}\,e_{2n-2} \end{cases} \tag{4.7}$$

$$\vdots$$

Following the notation above, let

$$\begin{bmatrix} e_{\ell n} \\ \vdots \\ e_{(\ell+1)n-1} \end{bmatrix} = \underline{\mathcal{E}}_\ell$$

Then since $y(i) = c'x(i)$ from (4.2), the input-output behavior of (4.2) is duplicated by the system

$$\underline{\omega}(\ell+1) = \underline{\underline{A}}^n\,\underline{\omega}(\ell) + [\underline{\underline{A}}^{n-1}\underline{b},\ \underline{\underline{A}}^{n-2}\underline{b}, \dots, \underline{b}]\ \underline{\mathcal{E}}(\ell)$$

$$\mathcal{Y}(\ell) = \begin{bmatrix} -\underline{c}'- \\ \vdots \\ -\underline{c}'\underline{\underline{A}}^{n-1}- \end{bmatrix} \omega(\ell) + \begin{bmatrix} 0 & & & \\ \underline{c}'\underline{b} & \ddots & 0 & \\ \vdots & \ddots\ \underline{c}'\underline{b} & \ddots & \\ \underline{c}'\underline{\underline{A}}^{n-2}\underline{b} & \dots & \underline{c}'\underline{b} & 0 \end{bmatrix} \underline{\mathcal{E}}(\ell) \tag{4.8}$$

which is again a linear time-invariant difference equation.

The condition that the operator G be bounded requires that the eigenvalues of the matrix $\underline{\underline{A}}$ lie strictly inside the unit disc. Assuming the (and zero initial conditions for (4.8)), the relationship between the z-transform of the input sequence $\mathcal{E}$ and the z-transform of the output sequence $\mathcal{Y}$ is given by

$$\hat{\underline{\mathcal{Y}}}(z) = \hat{\underline{\underline{\mathcal{G}}}}(z)\ \hat{\underline{\mathcal{E}}}(z)$$

where

$$\hat{\underline{\underline{\mathcal{G}}}}(z) = \begin{bmatrix} \underline{c}'(\underline{I}z - \underline{\underline{A}}^n)^{-1}\underline{\underline{A}}^{n-1}\underline{b} & \underline{c}'(\underline{I}z - \underline{\underline{A}}^n)^{-1}\underline{\underline{A}}^{n-2}\underline{b} \ \cdots & \underline{c}'(\underline{I}z - \underline{\underline{A}}^n)^{-1}\underline{b} \\ z\,\underline{c}'(\underline{I}z - \underline{\underline{A}}^n)^{-1}\underline{b} & & \\ z\,\underline{c}'(\underline{I}z - \underline{\underline{A}}^n)^{-1}\underline{\underline{A}}\underline{b} & & \\ \vdots & & \\ z\,\underline{c}'(\underline{I}z - \underline{\underline{A}}^n)^{-1}\underline{\underline{A}}^{n-2}\underline{b} & & \underline{c}'(\underline{I}z - \underline{\underline{A}}^n)^{-1}\underline{\underline{A}}^{n-1}\underline{b} \end{bmatrix} \qquad (4.9)$$

$\hat{\underline{\underline{\mathcal{G}}}}(z)$ is an $n \times n$ matrix of functions, where n is the period of the periodic gain, and has the form of a finite Toeplitz matrix.

A result parallel to (4.9) may be derived without the assumption that the original convolution operator arises from a finite dimensional system such as (4.2). Let $\{\underline{\underline{G}}(i)\}_1^{\infty}$ be a sequence of $r \times s$ dimensional matrices such that each entry of the sequence belongs to ℓ_1^+. Such a sequence defines a convolution operator according to

$$\underline{\mathcal{Y}}_i = \sum_{j=0}^{i-1} \underline{\underline{G}}(i-j)\,\underline{e}(i) \qquad i \geq 0 \qquad (4.10)$$

which may be rewritten in the form

$$\begin{bmatrix} \underline{y}_0 \\ \underline{y}_1 \\ \underline{y}_2 \\ \vdots \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 & 0 & \cdots & & \\ \underline{\underline{G}}_1 & 0 & 0 & \ddots & \\ \underline{\underline{G}}_2 & \underline{\underline{G}}_1 & 0 & \ddots & \\ \underline{\underline{G}}_3 & \underline{\underline{G}}_2 & \underline{\underline{G}}_1 & 0 & \ddots \\ \vdots & \underline{\underline{G}}_3 & \ddots & \ddots & \ddots \\ & & \ddots & & \end{bmatrix} \begin{bmatrix} \underline{e}_0 \\ \underline{e}_1 \\ \underline{e}_2 \\ \vdots \\ \vdots \end{bmatrix} \tag{4.11}$$

The semi-infinite matrix has a "block Toeplitz" form, and is lower triangular since the operator G is of the Volterra type. In the equation (4.11) the entries in the $\underline{e}$ and $\underline{y}$ sequences may be grouped in sections of length n as before:

$$\begin{bmatrix} \left\{\begin{matrix} \underline{y}_0 \\ \vdots \\ \underline{y}_{n-1} \end{matrix}\right\} \\ \left\{\begin{matrix} \underline{y}_n \\ \vdots \\ \underline{y}_{2n-1} \end{matrix}\right\} \\ \left\{\begin{matrix} \vdots \\ \\ \end{matrix}\right\} \\ \vdots \end{bmatrix} = \begin{bmatrix} \left\{\begin{matrix} 0 & \ddots & \\ \underline{\underline{G}}_1 & \ddots & \\ \vdots & \ddots & \\ \underline{\underline{G}}_{n-1} & \underline{\underline{G}}_1 & 0 \end{matrix}\right\} & \underline{\underline{0}} \ddots & \\ \left\{\begin{matrix} \underline{\underline{G}}_n & & \underline{\underline{G}}_1 \\ \vdots & \ddots & \\ \underline{\underline{G}}_{2n-1} & & \underline{\underline{G}}_n \end{matrix}\right\} & \left\{\begin{matrix} 0 & \ddots & \\ \underline{\underline{G}}_1 & \ddots & \\ \underline{\underline{G}}_{n-1} & \underline{\underline{G}}_1 & 0 \end{matrix}\right\} & \\ \left\{\begin{matrix} \underline{\underline{G}}_{2n} & & \underline{\underline{G}}_{n+1} \\ & \ddots & \\ \underline{\underline{G}}_{3n-1} & & \underline{\underline{G}}_{2n} \end{matrix}\right\} & \left\{\begin{matrix} \underline{\underline{G}}_n & & \underline{\underline{G}}_1 \\ & \ddots & \\ \underline{\underline{G}}_{2n-1} & & \underline{\underline{G}}_n \end{matrix}\right\} & \ddots \\ & \ddots & \end{bmatrix} \begin{bmatrix} \left\{\begin{matrix} \underline{e}_0 \\ \\ \underline{e}_{n-1} \end{matrix}\right\} \\ \left\{\begin{matrix} \underline{e}_n \\ \\ \underline{e}_{2n-1} \end{matrix}\right\} \\ \left\{\begin{matrix} \ddots \\ \\ \ddots \end{matrix}\right\} \end{bmatrix} \tag{4.12}$$

By defining

$$\begin{Bmatrix} \underline{\underline{G}}_{\ell n} & \cdots & \underline{\underline{G}}_{(\ell-1)n+1} \\ \vdots & \ddots & \vdots \\ \underline{\underline{G}}_{(\ell+1)n-1} & \cdots & \underline{\underline{G}}_{\ell n} \end{Bmatrix} = \underline{\underline{\mathcal{G}}}_{\ell}$$

and using the same notation as above for the grouped $\underline{e}$ and $\underline{y}$ vectors (4.12) becomes

$$\begin{bmatrix} \underline{\mathcal{Y}}_0 \\ \underline{\mathcal{Y}}_1 \\ \underline{\mathcal{Y}}_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} \underline{\underline{\mathcal{G}}}_0 & 0 & \cdots & 0 & \cdots \\ \underline{\underline{\mathcal{G}}}_1 & \underline{\underline{\mathcal{G}}}_0 & \cdots & 0 & \cdots \\ \underline{\underline{\mathcal{G}}}_2 & \underline{\underline{\mathcal{G}}}_1 & \cdots & & \cdots \\ \underline{\underline{\mathcal{G}}}_3 & \underline{\underline{\mathcal{G}}}_2 & \cdots & & \cdots \end{bmatrix} \begin{bmatrix} \underline{\mathcal{E}}_0 \\ \underline{\mathcal{E}}_1 \\ \underline{\mathcal{E}}_2 \\ \vdots \end{bmatrix} \tag{4.13}$$

which again represents a discrete convolution,

$$\underline{\mathcal{Y}}_i = \sum_{j=0}^{i} \underline{\underline{\mathcal{G}}}_{i-j} \, \underline{\mathcal{E}}_j \tag{4.14}$$

with a corresponding z-transform relationship

$$\underline{\mathcal{Y}}(z) = \underline{\underline{\mathcal{G}}}(z) \, \underline{\mathcal{E}}(z)$$

where

$$\underline{\underline{\mathcal{G}}}(z) = \sum_{\ell=0}^{\infty} \underline{\underline{\mathcal{G}}}_{\ell} \, z^{-\ell} \tag{4.15}$$

It is convenient to have some terminology for the equivalent system (4.14) obtained from an original discrete convolution (4.10). The equation (4.14) has been obtained by a process of compressing the time scale of the sequences, while expanding the dimension of the vectors involved. This suggests the following:

Definition: Consider the discrete convolution operator G determined by the sequence of $r \times s$ matrices $\{\underline{\underline{G}}_i\}_1^\infty$. The discrete convolution operator G_n determined by the sequence of $rn \times sn$ matrices

$$\left\{ \begin{bmatrix} \underline{\underline{G}}_{\ell n} & \underline{\underline{G}}_{\ell n-1} & \cdots & \underline{\underline{G}}_{(\ell-1)n+1} \\ \underline{\underline{G}}_{\ell n+1} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \\ \underline{\underline{G}}_{(\ell+1)n-1} & \cdots & & \underline{\underline{G}}_{\ell n} \end{bmatrix} \right\}_{\ell=0}^{\infty}$$

will be referred to as the convolution operator G companded by n. Similarly the pulse transfer function corresponding to the operator G_n

$$\hat{\underline{\underline{\mathcal{G}}}}(z) = \sum_{\ell=0}^{\infty} \begin{bmatrix} \underline{\underline{G}}_{\ell n} & & \underline{\underline{G}}_{(\ell-1)n+1} \\ \vdots & & \\ \underline{\underline{G}}_{(\ell+1)n-1} & \cdots & \underline{\underline{G}}_{\ell n} \end{bmatrix} z^{-\ell} \tag{4.16}$$

will be called the companded pulse transfer function.

The companded pulse transfer function may be obtained directly from the pulse transfer function of the original convolution operator. If the original pulse transfer function

$$\hat{\underline{\underline{G}}}(z) = \sum_{i=1}^{\infty} \underline{\underline{G}}_i\, z^{-i}$$

is written in the form

$$\hat{\underline{\underline{G}}}(z) = \hat{\underline{\underline{G}}}_0(z^n) + z^{-1}\hat{\underline{\underline{G}}}_1(z^n) + \ldots + z^{-(n-1)}\hat{\underline{\underline{G}}}_{n-1}(z^n) \tag{4.17}$$

then from (4.16) it follows that

$$\underline{\underline{\hat{\mathcal{G}}}}(z) = \begin{bmatrix} \hat{\underline{\underline{G}}}_0(z) & \frac{1}{z}\hat{\underline{\underline{G}}}_{n-1}(z) & \cdots & \frac{1}{z}\hat{\underline{\underline{G}}}_1(z) \\ \hat{\underline{\underline{G}}}_1(z) & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \frac{1}{z}\hat{\underline{\underline{G}}}_{n-1}(z) \\ \hat{\underline{\underline{G}}}_{n-1}(z) & \cdots & \hat{\underline{\underline{G}}}_1(z) & \hat{\underline{\underline{G}}}_0(z) \end{bmatrix} \tag{4.18}$$

The functions $\hat{\underline{\underline{G}}}_i(z)$ $i = 0, \ldots n-1$ may be calculated from the original pulse transfer function by means of contour integrals.

Theorem 4.1: Let the pulse transfer function of the ℓ_1^+ r x s matrix sequence $\{\underline{\underline{G}}_i\}_1^\infty$ be written in the form

$$\hat{\underline{\underline{G}}}(z) = \hat{\underline{\underline{G}}}_0(z^n) + z^{-1}\hat{\underline{\underline{G}}}_1(z^n) + \ldots + z^{-(n-1)}\hat{\underline{\underline{G}}}_{n-1}(z^n)$$

Then the functions $\hat{\underline{\underline{G}}}_a(z)$ have an integral representation of the form

$$\hat{\underline{\underline{G}}}_a(z) = \frac{1}{2\pi i} \oint \hat{\underline{\underline{G}}}(\xi)\, \xi^{a-1} \frac{z}{\xi^n - z}\, d\xi \tag{4.19}$$

The above representation is valid for $|z| \geq 1$, is to be interpreted as a principal value for $|z| = 1$, and the contour integral is taken about $|\xi| = 1$ in the clockwise sense.

Proof: By Cauchy's residue theorem

$$\underline{\underline{G}}_{\ell n + a} = \frac{1}{2\pi i} \oint \hat{\underline{\underline{G}}}(\xi) \left(\frac{1}{\xi}\right)^{-(\ell n + a + 1)} d\left(\frac{1}{\xi}\right)$$

where the contour is clockwise about $|\xi| = 1$. Therefore $\hat{\underline{\underline{G}}}_a(z)$ may be expressed as

$$\hat{\underline{\underline{G}}}_a(z) = \sum_{\ell=0}^{\infty} z^{-\ell} \oint \frac{1}{2\pi i} \hat{\underline{\underline{G}}}(\xi) \left(\frac{1}{\xi}\right)^{-(\ell n + a + 1)} d\left(\frac{1}{\xi}\right)$$

In the region $|z| \geq 1 + \epsilon$, $\epsilon > 0$, the expression

$$\hat{\underline{\underline{G}}}_a(z) = \frac{1}{2\pi i} \oint \hat{\underline{\underline{G}}}(\xi)\, \xi^{a-1} \frac{1}{\xi^n - z}\, d\xi \tag{4.20}$$

is obtained by appealing to the dominated convergence theorem. Since $\epsilon > 0$ is arbitrary, the above is valid for all $|z| > 1$.

To show that the representation may be extended to the boundary $|z| = 1$, (4.20) may be expressed as

$$\hat{\underline{\underline{G}}}_a(z) = \frac{1}{2\pi} \int_0^{-2\pi} \hat{\underline{\underline{G}}}(e^{i\theta})\, e^{ia\theta} \frac{z}{e^{in\theta} - z}\, d\theta$$

Making the change of variables $\theta = \phi/n$ gives

$$\hat{\underline{\underline{G}}}_a(z) = \sum_{j=1}^{w} \frac{1}{2\pi n} \int_0^{-2\pi} \hat{\underline{\underline{G}}}\left(e^{i\phi/n}\, e^{-\frac{2\pi i j}{n}}\right) e^{ia\left(\phi/n - \frac{2\pi j}{n}\right)} \frac{z}{e^{i\phi} - z}\, d\phi$$

The above may be regarded as a sum of Cauchy-type integrals, each of whose densities is continuous except for a jump discontinuity at $\phi = 0$,

$$\hat{\underline{\underline{G}}}_a(z) = \sum_{j=1}^{n} \frac{1}{2\pi} \int_0^{-2\pi} \underline{\underline{h}}j(\phi) \frac{1}{e^{i\phi} - z}\, d\phi \tag{4.21}$$

Let $\underline{\underline{a}}j = \underline{\underline{h}}j(2\pi) - \underline{\underline{h}}j(0)$. Then if $\Delta(\cdot)$ is <u>any</u> function continuous on $[0, 2\pi]$ and such that $\Delta(2\pi) - \Delta(0) = 1$, the function

$$\underline{\underline{h}}j(\cdot) - \underline{\underline{a}}j\, \Delta(\cdot)$$

is continuous on the circle, and its Cauchy integral may be extended as a principal value continuously to the boundary $|z| = 1$.[49] Since $\hat{\underline{\underline{G}}}(e^{i\cdot})e^{ia\cdot}$

is continuous on the circle, $\sum_{j=1}^{n} \underline{\underline{a}}j = 0$ so that $\sum_{j=1}^{n} \underline{\underline{a}}j\, \Delta(\phi)$ may be added under the integral in (4.21) with no effect on $\hat{\underline{\underline{G}}}_a(z)$. Hence $\hat{\underline{\underline{G}}}_a(z)$ may be extended to the boundary as claimed.

C. BASIC STABILITY THEOREM

The use of the "companding" manipulations of the previous section gives the following result on systems with a periodic feedback gain.

<u>Theorem 4.2</u>: Let $\{\underline{\underline{G}}_i\}_1^{\infty}$ be a sequence of matrices of dimension $r \times s$, each entry of which belongs to ℓ_1^+. Then the regular feedback system defined by

$$\underline{e}(i) + \underline{\underline{K}}(i) \sum_{j=0}^{i-1} \underline{\underline{G}}_{i-j}\, e(j) = \underline{u}(i), \quad i \geq 0 \tag{4.22}$$

where the $s \times r$ matrices $\underline{\underline{K}}(i)$ satisfy $\underline{\underline{K}}(i+n) = \underline{\underline{K}}(i)$, is bounded input — bounded output stable in the sense of $\ell_p^{+(s)}$, $p \geq 1$, $c^{+(s)}$ and $c_0^{+(s)}$ if and only if the following conditions hold:

1. $\det(\underline{\underline{I}} + \underline{\underline{\mathcal{K}}}\,\hat{\underline{\underline{\mathcal{G}}}}(e^{i\theta})) \neq 0, \quad -\pi \leq \theta \leq \pi$

2. $$\frac{1}{2\pi} \int_{-\pi}^{\pi} d_\theta \arg(\det(\underline{\underline{I}} + \underline{\underline{\mathcal{K}}}\,\hat{\underline{\underline{\mathcal{G}}}}(e^{i\theta})) = 0 \tag{4.23}$$

Here the $sn \times rn$ matrix $\underline{\underline{\mathcal{K}}}$ is given by

$$\underline{\underline{\mathcal{K}}} = \begin{bmatrix} \underline{\underline{K}}(0) & & \\ & \underline{\underline{K}}(1) & \\ & & \ddots & \\ & & & \underline{\underline{K}}(n-1) \end{bmatrix}$$

and $\hat{\mathcal{G}}(\cdot)$ is the companded pulse transfer function determined by the sequence $\{G_i\}_1^{\infty}$.

Proof: By the structure of the companded convolution operator, the system

$$\mathcal{E}(i) + \mathcal{K} \sum_{j=0}^{i} \mathcal{G}_{i-j} \mathcal{E}(j) = \mathcal{Y}(i), \quad i \geq 0 \qquad (4.24)$$

is entirely equivalent to the original one, (4.22). In fact, the systems generate identical trajectories, and by choosing the natural norm on c^{sn} suggested by any of the possible input spaces $\ell_p^{(s)}$, $c^{(s)}$ or $c_0^{(s)}$ it is easily seen that norms of sequences are preserved by the transformation from (4.22) to (4.24).

Applying Theorem 3.16 to (4.24) gives the desired result.

It should be remarked that the conditions (4.23) are equivalent to the requirement that the analytic function $\det(I + \mathcal{K}\hat{\mathcal{G}}(z))$ has no zero in the region $|z| \geq 1$.

Theorem 4.2 has an analog for the case of continuous time systems in the case where the convolution operator arises from a finite dimensional linear system. This case is the differential equation analog of Equations (4.2).

$$\dot{x}(t) = A x(t) + b e(t), \; y(t) = c' x(t), \; e(t) = u(t) - k(t) y(t) \qquad (4.25)$$

where x is an r-vector, e and y are scalars, and the gain satisfies $k(t+T) = k(t)$. It is also necessary to assume enough regularity on the function $k(\cdot)$, to guarantee existence of solutions of (4.25). It is sufficient to assume that $k(\cdot)$ is bounded and measurable on $[0, T]$.

In the discrete time case, the system (4.2) was shown to be equivalent to (4.8)

$$\underline{\omega}(\ell+1) = \underline{\underline{A}}^n \underline{\omega}(\ell) + [\underline{\underline{A}}^{n-1}\underline{b}, \underline{\underline{A}}^{n-2}\underline{b}, \ldots, \underline{b}]\underline{\mathcal{E}}(\ell) \tag{4.26a}$$

$$\underline{\mathcal{Y}}(\ell) = \begin{bmatrix} -\underline{c}'- \\ \vdots \\ -\underline{c}'\underline{\underline{A}}^{n-1}- \end{bmatrix} \omega(\ell) + \begin{bmatrix} 0 & & & \underline{0} \\ \underline{c}'\underline{b} & \ddots & & \\ \vdots & & \ddots & 0 \\ \underline{c}'\underline{\underline{A}}^{n-2}\underline{b} & \cdots & \underline{c}'\underline{b} & 0 \end{bmatrix} \underline{\mathcal{E}}(\ell) \tag{4.26b}$$

$$\underline{\mathcal{E}}(\ell) = \underline{\mathcal{U}}(\ell) - \begin{bmatrix} k(0) & & \\ & \ddots & \\ & & k(n-1) \end{bmatrix} \underline{\mathcal{Y}}(\ell) \tag{4.26c}$$

Combining (4.26b) and (4.26c) gives

$$\underline{\mathcal{E}}(\ell) = \left(I + \begin{bmatrix} k(0) & & \\ & \ddots & \\ & & k(n-1) \end{bmatrix} \begin{bmatrix} 0 & & & \\ \underline{c}'\underline{b} & & & \\ \vdots & & & \\ \underline{c}'\underline{\underline{A}}^{n-2}\underline{b} & \cdots & \underline{c}'\underline{b} & 0 \end{bmatrix} \right)^{-1} \left(\begin{bmatrix} k(0) & & \\ & \ddots & \\ & & k(n-1) \end{bmatrix} \begin{bmatrix} \underline{c}' \\ \vdots \\ -\underline{c}'\underline{\underline{A}}^{n-1}- \end{bmatrix} \underline{\omega}(\ell) + \underline{\mathcal{U}}(\ell) \right)$$

and substituting in (4.26a)

$$\omega(\ell+1) = \left[\underline{\underline{A}}^n + \left(\underline{\underline{I}} + \begin{bmatrix} k(0) & & \\ & \ddots & \\ & & k(n-1) \end{bmatrix} \begin{bmatrix} 0 & & \\ \underline{c}'\underline{b} & & \\ \underline{c}'\underline{\underline{A}}^{n-2}\underline{b} & \underline{c}'\underline{b} & 0 \end{bmatrix} \right)^{-1} \right.$$

$$\left. \begin{bmatrix} k(0) & & \\ & \ddots & \\ & & k(n-1) \end{bmatrix} \begin{bmatrix} -\underline{c}'- \\ \\ -\underline{c}'\underline{\underline{A}}^{n-1}\underline{b} \end{bmatrix} \right] \underline{\omega}(\ell) + \underline{\underline{M}}\, \underline{\mathcal{U}}(\ell)$$

$$= \underline{\mathcal{A}}\, \omega(\ell) + \underline{\underline{M}}\, \underline{\mathcal{U}}(\ell) \tag{4.27}$$

The matrix $\mathcal{A}$ in (4.27) is in fact just an expression for the transition matrix $\underline{\underline{\Phi}}(n, 0)$ of the original system (4.2). This follows from the fact that the system (4.8) is constructed in such a way that it propagates the state of (4.2) in time steps equal to the period of the gain $k(\cdot)$. The system (4.2) is asymptotically stable if and only if all of the eigenvalues of the matrix $\underline{\underline{\Phi}}(n, 0)$ lie inside the unit disc. Although it is far from immediately apparent, the condition for stability,

$$\det\left(\underline{\underline{I}}z - \underline{\underline{\Phi}}(n, 0)\right) = \det\left(\underline{\underline{I}}z - \underline{\underline{\mathcal{A}}}\right) \neq 0, \quad |z| \geq 1$$

may be algebraically manipulated to yield the conditions of Theorem 4.2 above. The actual manipulations involved are tiresome; but the point of interest here is the observation that the z-transform variable which appears in connection with the companded system corresponds to the indeterminate in the characteristic equation of the closed-loop transition matrix.

Applying the variation of constants formula to (4.25) gives an integral equation

$$\underline{x}(\tau) = e^{\underline{\underline{A}}\tau}\underline{x}_0 + \int_0^{\tau} e^{\underline{\underline{A}}(\tau - s)}\underline{b}\,\underline{c}'\,k(s)\,\underline{x}(s)\,ds \qquad (4.28)$$

so that

$$\underline{x}(T) = e^{\underline{\underline{A}}T}\underline{x}_0 + \int_0^{T} e^{\underline{\underline{A}}(T - s)}\underline{b}\,\underline{c}'\,k(s)\,\underline{x}(s)\,ds \qquad (4.29)$$

Now if z is an eigenvalue of the closed loop transition matrix $\Phi(T, 0)$

$$\underline{x}(T) = \underline{\underline{\Phi}}(T, 0)\,\underline{x}_0 = z\,\underline{x}_0, \text{ for some } \underline{x}_0 \qquad (4.30)$$

Solving for $\underline{x}_0$ from (4.29) and (4.30) and substituting back in (4.28) gives

$$\underline{x}(\tau) = \int_0^T (\underline{\underline{I}} z - e^{\underline{\underline{A}} T})^{-1} e^{\underline{\underline{A}}(\tau - s + T)} \underline{b}\,\underline{c}' k(s)\,\underline{x}(s)\,ds$$

$$+ \int_0^\tau e^{\underline{\underline{A}}(\tau - s)} \underline{b}\,\underline{c}' k(s)\,\underline{x}(s)\,ds \tag{4.31}$$

The closed loop transition matrix $\underline{\underline{\Phi}}(T, 0)$ has an eigenvalue z if and only if the Fredholm integral equation of the second kind (4.31) has a solution. This statement is also true of the integral equation obtained by taking the inner product of (4.31) with the vector $\underline{c}'$, provided that the original system is observable. (See Ref. 4 for definitions of observability.) These observations lead to the following:

Lemma 4.1: The differential equation

$$\dot{\underline{x}} = (\underline{\underline{A}} - \underline{b}\,\underline{c}' k(t))\underline{x}(t) \tag{4.32}$$

where $\text{Re}\,\lambda i\,(A) < 0$, $k(t)$ is a bounded measurable function periodic of period T, and the pair $[\underline{c}', \underline{\underline{A}}]$ is observable in exponentially stable (i.e., all solutions approach zero at an exponential rate) if and only if the following condition holds: The scalar integral equation

$$\lambda\,\alpha(t) = f(t) + \int_0^T \underline{c}' (\underline{I} z - e^{\underline{\underline{A}} T})^{-1} e^{\underline{\underline{A}}(t - s + T)} \underline{b}\,k(s)\,\alpha(s)\,ds$$

$$+ \int_0^t \underline{c}' e^{\underline{\underline{A}}(t - s)} \underline{b}\,k(s)\,\alpha(s)\,ds\,, \quad 0 \leq t \leq T \tag{4.33}$$

does not have the point $\lambda = 1$ in its spectrum for any value of the parameter z in the region $|z| \geq 1$.

Proof: (4.33) is a Fredholm equation of the second kind, and by the above remarks it has a homogeneous solution if and only if the transition matrix of (4.32) satisfies

$$\det(\underline{\underline{I}} z - \underline{\underline{\Phi}}(T, 0)) = 0$$

Since (4.32) is exponentially stable if and only if all the eigenvalues of $\Phi(T, 0)$ lie inside the unit disc, the theorem follows.

Theorem 4.3: Suppose that the eigenvalues of the matrix $\underline{\underline{A}}$ satisfy $\mathrm{Re}\ \lambda i(\underline{\underline{A}}) < 0$, and that $[\underline{\underline{A}}, \underline{b}, \underline{c}']$ is a minimal system, i.e. the pair $[\underline{\underline{A}}, \underline{b}]$ is controllable, while the pair $[\underline{\underline{A}}, \underline{c}']$ is observable. Suppose further that $k(\cdot)$ is a bounded measurable function which is periodic of period T. Then the regular feedback system

$$e(t) + k(t) \int_0^t \underline{c}' e^{\underline{\underline{A}}(t-s)} \underline{b}\ e(s) = u(t) \qquad t \geq 0 \tag{4.34}$$

is bounded input — bounded output stable in the Lp $(0, \infty)$ sense, $1 \leq p < \infty$, if and only if the closure of the locus of the eigenvalues of the Fredholm integral equation

$$-\lambda\, a(t) = f(t) + \int_0^T \underline{c}' (\underline{\underline{I}} z - e^{\underline{\underline{A}} T})^{-1} e^{\underline{\underline{A}}(t-s+T)} \underline{b}\ k(s)\, a(s)\, ds$$

$$+ \int_0^t \underline{c}' e^{\underline{\underline{A}}(t-s)} \underline{b}\ k(s)\, a(s)\, ds \tag{4.35}$$

for values of the parameter z in the region $|z| \geq 1$ does not include the point -1.

Proof: Bounded input – bounded output stability of (4.34) in the spaces $Lp(0,\infty)$ is equivalent under the hypothesis of minimality to the exponential stability of the corresponding differential equation (4.32). The result then follows directly from Lemma 4.3.

Theorem 4.3 has an interpretation as an identification of the spectrum of the operator KG defined by

$$KG : x(t) \rightarrow k(t) \int_0^t g(t-s)\, x(s)\, ds \qquad (4.36)$$

for the case where the $L'(0,\infty)$ function $g(\cdot)$ in (4.36) has a rational Fourier transform, and $k(t+T) = k(t)$. This is due to Theorem 2.1, which establishes the equivalence of input-output stability of (4.34) to the absence of the point -1 from the spectrum of the operator KG in (4.36).

The analogy between Theorems 4.2 and 4.3 becomes apparent with the introduction of an eigenvector ν with eigenvalue -1 for the matrix $\underline{\underline{\mathcal{K}}}\,\underline{\underline{\hat{\mathcal{G}}}}(z)$ of Theorem 4.2. Then the condition that $\underline{\underline{\mathcal{K}}}\,\underline{\underline{\hat{\mathcal{G}}}}(z)$ have an eigenvalue -1 for some z_0 in the region $|z_0| \geq 1$ becomes

$$(\underline{\underline{I}} + \underline{\underline{\hat{\mathcal{G}}}}(z_0)\underline{\underline{\mathcal{K}}})\nu = 0 \qquad |z_0| \geq 1 \qquad (4.37)$$

In the homogeneous version of the integral equation (4.35), the eigenfunction $\alpha(\cdot)$ corresponds to the vector ν of (4.37), the integral kernel corresponds to the matrix $\underline{\underline{\hat{\mathcal{G}}}}(z_0)$, and the function $k(\cdot)$ corresponds to the matrix $\underline{\underline{\mathcal{K}}}$. Essentially, the discrete points of number equal to the period of the gain are replaced by a continuous interval of length equal to the period of the gain. In the process, the matrices of the discrete problem become the integral kernels of the continuous one. Using these ideas it is possible to heuristically derive condition (4.35) directly from (4.37) by use of a difference equation approximation to the corresponding differential equation. As the step size goes to zero, the various terms of (4.37) may readily be identified with the expressions in (4.35).

D. DIAGONALIZATION AND A DUAL RESULT

It is possible to consider Section C as a "time domain" approach to the problem of the stability of systems with periodic feedback, since the results have an interpretation as a determination procedure for the eigenvalues of a certain state transition matrix (at least in the case that the underlying system has a finite dimensional realization). An alternative approach to the stability of systems with periodic coefficients is to employ Fourier series expansions of the various time functions involved, and so carry out a "frequency domain" analysis of the problem. The results of this section have such an interpretation. Only the case of scalar input—scalar output systems is considered; the motivation is that such systems are both more easy to handle algebraically, and yield more concrete results.

The following lemma is concerned with the structure of the companded pulse transfer function matrices introduced in Section III-B.

<u>Lemma 4.2</u>: Let $\underline{\underline{\hat{\mathcal{G}}}}(z)$ denote the $n \times n$ companded pulse transfer function matrix obtained from the ℓ_1^+ sequence $\{g_i\}_1^\infty$. Then for each fixed value of the complex parameter z, the matrix $\underline{\underline{\hat{\mathcal{G}}}}(z)$ has an eigenvector

col $(1, z^{1/n}, z^{2/n}, \ldots, z^{(n-1)/n})$ with corresponding eigenvalue equal to the original pulse transfer function $\hat{G}(\cdot)$ evaluated at the point $z^{1/n}$. Here $z^{1/n}$ is any of the (n possible) values of the n^{th} root of the complex variable z.

Proof: The easiest proof is by a direct verification. Recall that (4.17) if the original pulse transfer function is written in the form

$$\hat{G}(z) = \hat{G}_0(z^n) + z^{-1}\hat{G}_1(z^n) r \ldots + z^{-(n-1)}\hat{G}_{n-1}(z^n) \qquad (4.38)$$

then the companded pulse transfer function matrix may be written as

$$\hat{\mathcal{G}}(z) = \begin{bmatrix} \hat{G}_0(z) & \frac{1}{z}\hat{G}_{n-1}(z) & \cdots & \frac{1}{z}\hat{G}_1(z) \\ \hat{G}_1(z) & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \\ \hat{G}_{n-1}(z) & \cdots & \cdots & \hat{G}_0(z) \end{bmatrix} \qquad (4.39)$$

Evaluating (4.38) at the point $z^{1/n}$ shows that

$$\hat{G}(z^{1/n}) = \hat{G}_0(z) + z^{-\frac{1}{n}}\hat{G}_1(z) + \ldots + z^{-\frac{n-1}{n}}\hat{G}_{n-1}(z)$$

and this formula is the key to the verification of the lemma.

Notation: Let ωj denote the j^{th} of the $n - n^{th}$ roots of unity, i.e. $\omega j = e^{\frac{2\pi i j}{n}}$. Also let $\underline{\underline{\Omega}}$ denote a normalized Vandermonde matrix constructed from the $\{\omega j\}_0^{n-1}$ i.e.

$$\underline{\underline{\Omega}} = \begin{bmatrix} 1 & 1 & 1 \\ \omega_0 & \omega_1 & \omega_{n-1} \\ \omega_0^2 & & \vdots \\ \vdots & & \\ \omega_0^{n-1} & \omega_1^{n-1} & \omega_{n-1}^{n-1} \end{bmatrix} \cdot \frac{1}{\sqrt{n}}$$

$\underline{\underline{\Omega}}$ is a unitary matrix, as is easy to verify.

Lemma 4.3: The nxn companded pulse transfer function matrix can be written as

$$\hat{\mathcal{G}}(z) = \begin{bmatrix} 1 & & & \\ & z^{\frac{1}{n}} & & \\ & & \ddots & \\ & & & z^{\frac{n-1}{n}} \end{bmatrix} \underline{\underline{\Omega}} \begin{bmatrix} \hat{G}(\omega_0 z^{\frac{1}{n}}) & & \\ & \ddots & \\ & & \hat{G}(\omega_{n-1} z^{\frac{1}{n}}) \end{bmatrix} \underline{\underline{\Omega}}^* \begin{bmatrix} 1 & & & \\ & z^{-\frac{1}{n}} & & \\ & & \ddots & \\ & & & z^{-\frac{(n-1)}{n}} \end{bmatrix} \tag{4.40}$$

where the principal value of the n^{th} root of z is taken in the above expression.

Proof: This follows immediately from the previous lemma by diagonalizing the matrix $\hat{\mathcal{G}}(z)$.

The above expression for the companded pulse transfer function matrix can be used to derive an alternative form of Theorem 4.2.

Theorem 4.4: Let $\{g_i\}_1^\infty$ be a sequence belonging to ℓ_1^+. Then the regular feedback system S

$$e(i) + k(i) \sum_{j=0}^{i-1} g_{i-j}\, e(j) = u(i)\,, \quad i \geq 0 \tag{4.41}$$

where the time varying feedback gain satisfies $k(i+n) = k(i)$, is bounded input–bounded output stable in the sense of ℓ_p^+, $p \geq 1$, c^+, or c_0^+ if and only if

$$\det\left(\underline{\underline{I}} + \underline{\underline{\Omega}}^* \begin{bmatrix} k(0) & & \\ & \ddots & \\ & & k(n-1) \end{bmatrix} \underline{\underline{\Omega}} \begin{bmatrix} \hat{G}(\omega_0 z^{\frac{1}{n}}) & & \\ & \ddots & \\ & & \hat{G}(\omega_{n-1} z^{\frac{1}{n}}) \end{bmatrix} \right) \neq 0 \tag{4.42}$$

for all z such that $|z| \geq 1$. Here $\hat{G}(\xi) = \sum_{i=1}^{\infty} g_i \xi^{-i}$ is the pulse transfer function corresponding to the convolution G determined by $\{g_i\}_1^{\infty}$.

Proof: From Theorem 4.2, the system is stable if and only if

$$\det\left(\underline{\underline{I}} + \begin{bmatrix} k(0) & & \\ & \ddots & \\ & & k(n-1) \end{bmatrix} \hat{\underline{\underline{\mathcal{G}}}}(z) \right) \neq 0, \quad |z| \geq 1$$

Substituting the expression (4.40) for $\hat{\underline{\underline{\mathcal{G}}}}(z)$ in the above and using the matrix identity

$$\det(\underline{\underline{I}} + \underline{\underline{A}}\,\underline{\underline{B}}) = \det(\underline{\underline{I}} + \underline{\underline{B}}\,\underline{\underline{A}})$$

(which incidentally is just the matrix version of the general result $\sigma(fg) = \sigma(gf) \pm \{0\}$) leads directly to (4.42).

The quantities $\hat{G}(\omega_j z^{\frac{1}{n}})$ which occur in (4.42) may be interpreted as one n^{th} part of the spectrum of the convolution operator G. The spectrum of the convolution G consists exactly of the set of points $\{\hat{G}(\xi) \;\; |\xi| \geq 1\}$. As z ranges over the exterior of the unit disc, the quantity $\omega_j z^{\frac{1}{n}}$ ranges over an angular region of the plane of angle $2\pi/n$. (See Fig. 8.) Hence the range of the function $\hat{G}(\omega_j z^{\frac{1}{n}})$ consists of just a portion of the spectrum of the convolution operator G; however, the union of the ranges over the n possible values of j consists of the entire spectrum.

The matrix $\underline{\underline{\Omega}}^* (\mathrm{diag}\; k(0) \ldots k(n-1))\underline{\underline{\Omega}}$ in (4.42) also has a ready interpretation as a matrix of the (discrete) Fourier coefficients of the periodic gain function $k(\cdot)$. As is well known, the n^{th} roots of unity have the property that

$$\frac{1}{n} \sum_{\ell=0}^{n-1} (\omega_i)^{\ell} (\bar{\omega}_j)^{\ell} = \delta_{ij} \tag{4.43}$$

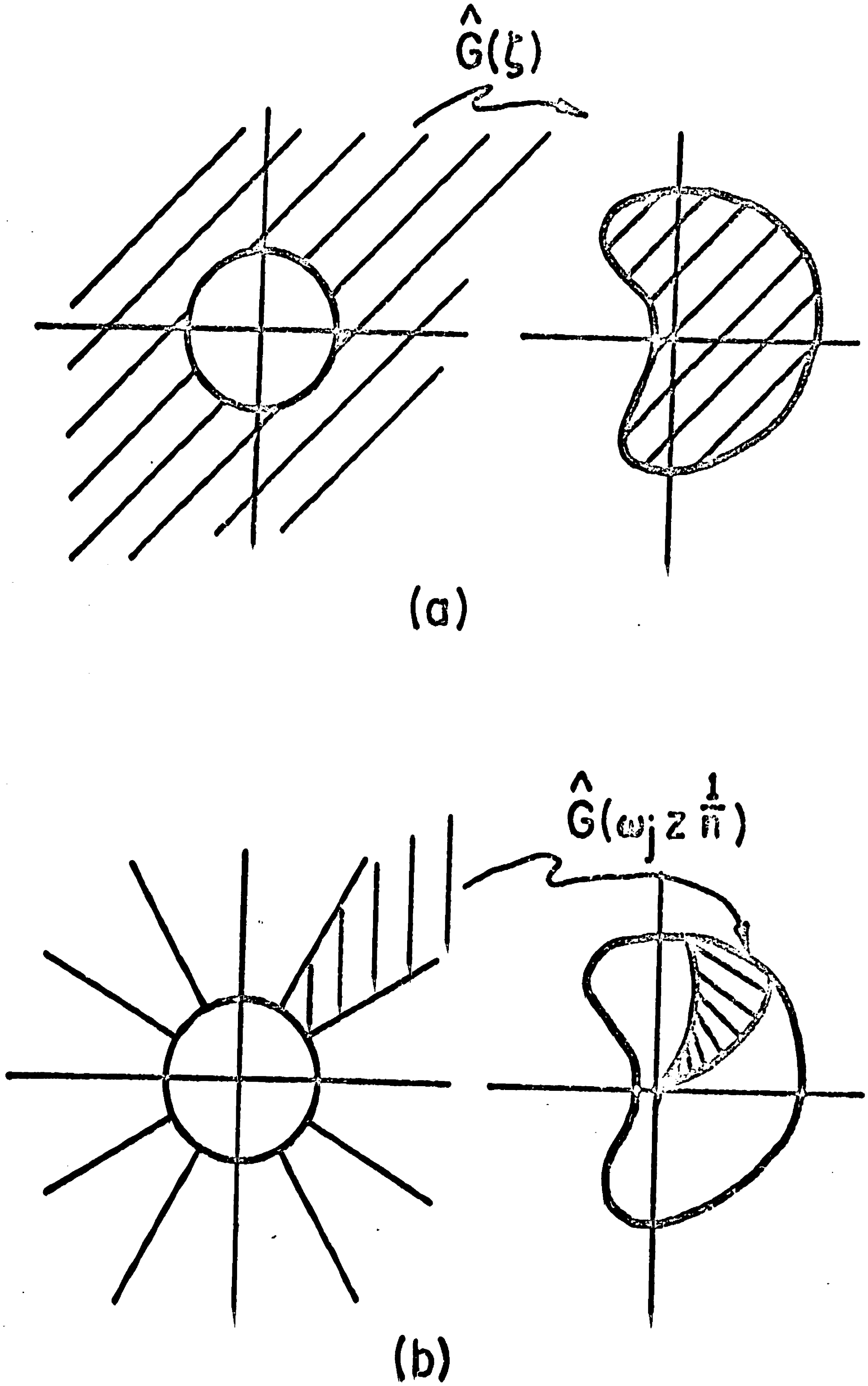


Fig. 8 Partitioning of the Spectrum of G

so that the powers of the $\{\omega_i\}_0^{n-1}$ form a complete orthagonal set of functions over the integers $(0, \ldots n-1)$. This is analogous to the use of the functions $\{e^{in\frac{2\pi t}{T}}\}_{-\infty}^{\infty}$ on the continuous interval $(0, T)$. Using the discrete orthagonality relation (4.43), the periodic feedback gain function may be expressed as a discrete Fourier series,

$$\left.\begin{aligned} k(\ell) &= \sum_{j=0}^{n-1} (\omega_j)^{\ell} k_j \\ \text{where} \quad k_j &= \frac{1}{n} \sum_{\ell=0}^{n-1} k(\ell)\, (\overline{\omega_j})^{\ell} \end{aligned}\right\} \tag{4.44}$$

With this notation it may be seen that

$$\underline{\underline{\Omega}}^* \begin{bmatrix} k(0) & & \\ & \ddots & \\ & & k(n-1) \end{bmatrix} \underline{\underline{\Omega}} = \begin{bmatrix} k_0 & k_1 & \cdots & k_{n-1} \\ k_{n-1} & \ddots & \ddots & k_1 \\ \vdots & & \ddots & \\ k_1 & \cdots & k_{n-1} & k_0 \end{bmatrix}$$

For a certain restricted class of systems it is possible to derive (4.42) using discrete Fourier series arguments, similar to those employed in the case of certain differential equations with periodic coefficients.

Consider a difference equation of the form

$$\begin{aligned} &x(j+r) + p_{r-1}\, x(j+r-1) + \ldots + p_0\, x(j) + \\ &\quad k(j)\, [q_{r-1}\, x(j+r-1) + \ldots + q_0\, x(j)] = 0, \quad j \geq 0 \end{aligned} \tag{4.45}$$

with $k(j+N) = k(j)$ and initial conditions $x(0) \ldots x(r-1)$ given. The above corresponds to a feedback system with a convolution operator with pulse transfer function

$$\hat{G}(z) = \frac{q_{r-1} z^{r-1} + \ldots + q_0}{z^r + p_{r-1} z^{r-1} + \ldots + p_0} = \frac{q(z)}{p(z)}$$

and a periodic feedback gain $k(\cdot)$. In order to carry out a Fourier series analysis of solutions of (4.45) it is necessary to assume that a Floquet representation of the solutions may be obtained in analogy with the continuous case.[52] This is true because the vector difference equation version of (4.45)

$$\underline{\omega}(n+1) = (\underline{\underline{A}} - \underline{b}\,\underline{c}'\, k(n))\underline{\omega}(n) \tag{4.46}$$

$$\underline{\omega}(0) = \mathrm{col}(x(0), \ldots x(r-1))$$

need not have a non-singular transition matrix. However, if in (4.46) the matrices $\underline{\underline{A}} - \underline{b}\,\underline{c}'\, k(n)$ are non-singular for $n=0, \ldots N-1$, then an argument parallel to that used in the differential equation case[52] shows that the transition matrix $\underline{\underline{\Phi}}(n, 0)$ has a representation of the form

$$\underline{\underline{\Phi}}(n, 0) = \underline{\underline{P}}(n)\, \underline{\underline{R}}^n \tag{4.47}$$

with the matrix $\underline{\underline{P}}$ periodic of period N.

If a representation such as (4.47) holds, then the necessary and sufficient condition that all solutions of (4.45) decay to zero is that (4.45) admit no solution of the form

$$x(n) = p(n)\rho^n \tag{4.48}$$

with $|\rho| \geq 1$, and $p(n+N) = p(n)$.

Assume that (4.45) has a solution of the form (4.48), and expand p in a discrete Fourier series,

$$p(n) = \sum_{j=0}^{N-1} a_j (\omega_j)^n$$

Along with the periodic gain

$$k(n) = \sum_{j=0}^{N-1} k_j (\omega_j)^n$$

Substitution of these assumed forms in the difference equation (4.45) gives

$$\sum_{m=0}^{N-1} \sum_{i=0}^{r} a_m p_i (\rho\omega_m)^n (\rho\omega_m)^i$$

$$+ \sum_{m=0}^{N-1} \sum_{j=0}^{N-1} \sum_{i=0}^{r-1} k_{m \ominus j} a_j q_i (\rho\omega_m)^n (\rho\omega_j)^i = 0 \qquad (4.49)$$

where the symbol $\ominus$ is defined by $m \ominus j = m-j \pmod N$. Setting the coefficients of $(\rho\omega_m)^n$ equal to zero and writing the resulting equations for the unknown coefficients a_m in the form of a vector-matrix equation gives

$$\left\{ \begin{bmatrix} p(\rho\omega_0) & & \\ & \ddots & \\ & & p(\rho\omega_{N-1}) \end{bmatrix} + \begin{bmatrix} k_0 & k_1 & \cdots & k_{N-1} \\ k_{N-1} & & & \\ \vdots & & & \\ k_1 & & & k_0 \end{bmatrix} \begin{bmatrix} q(\rho\omega_0) & & \\ & \ddots & \\ & & q(\rho\omega_{N-1}) \end{bmatrix} \right\} \begin{bmatrix} a_0 \\ \vdots \\ a_{N-1} \end{bmatrix} = 0 \qquad (4.50)$$

The system (4.46) will be stable if and only if

$$\det \left(\operatorname{diag}(p(\rho\omega_j)) + \begin{bmatrix} k_0 & k_1 & \cdots & k_{N-1} \\ k_{N-1} & \ddots & \ddots & \\ \vdots & & \ddots & \\ k_1 & & & k_0 \end{bmatrix} \operatorname{diag}(q(\rho\omega_j)) \right) \neq 0 \qquad (4.51)$$

for $|\rho| \geq 1$. The usual requirement that the convolution operator G be bounded requires in this context that the polynomial p(x) has no zero in the region $|x| \geq 1$. If this is the case, then diag $(p(\rho\omega_j))$ is a non-singular matrix for $|\rho| \geq 1$, and multiplying by its inverse in (4.51) gives exactly the condition of Theorem 4.4 above.

The above example makes clear the Fourier series ("frequency domain") interpretation of Theorem 4.4. Of course Theorem 4.4 covers a wider class of systems than the above example.

E. A "SHIFTING CIRCLE" CONDITION

The circle criterion for stability (Section I-F) may of course be applied to systems with a periodically time varying feedback gain. However, this criterion applies to a much wider class of systems. In fact, the feedback need not even be linear. The fact that the circle criterion applies to such a wide class of systems leaves open the possibility of obtaining improved conditions for stability by exploiting specific characteristics of the feedback element. Examples of this are systems with a "slope-restricted" nonlinearity, or systems with a "slowly varying" time varying feedback gain. (Cf. 53 - 56.) One of the characteristics of the circle criterion is that it involves only the location of the Nyquist locus of the convolution part of the system, and the "range" of the feedback gain. The example of Section D above illustrates the fact that instability in periodic systems where stability might otherwise be expected arises from a balance of the "harmonics" in the feedback loop; this process involves the relationship of points of the Nyquist locus which are essentially spaced at multiples of the feedback frequency apart. This indicates that it should be possible to derive conditions for stability involving a "parameterization" along the Nyquist locus in the case that the feedback gain is periodic. The following Lemma, first derived for the case of matrices by Wielandt, is the basic tool. In fact, it is simply a case of the circle criterion.

<u>Lemma 4.4</u>: Let A and B be normal operators on a Hilbert Space H. If there exists a circular region in the complex plane which strictly separates $\sigma(-A)$ from $\sigma(B)$, then $A + B$ is invertible on H.

Proof: Denote the center of the separating circle by μ, the radius by R and suppose $\sigma(B)$ is exterior to the circle (see Fig. 9)

$$A + B = -(-A + \mu I) + (B + \mu I),$$

and $B + \mu I$ is invertible, so that

$$(A + B) = (B + \mu I)(I - (B + \mu I)^{-1}(-A + \mu I)) \tag{4.52}$$

Since the induced norm of a normal operator is equal to its spectral radius,

$$\| -A + \mu I \| < R, \text{ and}$$

$$\| (B + \mu I)^{-1} \| < \frac{1}{R}$$

so that

$$\| (B + \mu I)^{-1}(-A + \mu I) \| < 1,$$

which shows from (4.52) that $A + B$ is invertible.

The connection of the above lemma with the linear version of the original circle theorem is immediate, since both a scalar convolution and a time varying gain correspond to normal operators on $L_2(0,\infty)$. The set points encircled by the Nyquist locus is the spectrum of the convolution, and the interval $[\frac{1}{\beta}, \frac{1}{\alpha}]$ contains the spectrum of the operator K^{-1} when K is multiplication by a gain limited by $0 < \alpha \leq k(t) \leq \beta$. Hence the circle with the segment $[-\frac{1}{\alpha}, -\frac{1}{\beta}]$ as diameter separates $\sigma(G + \frac{1}{2}(\frac{1}{\alpha} + \frac{1}{\beta})I)$ from $\sigma(-K^{-1} + \frac{1}{2}(\frac{1}{\alpha} + \frac{1}{\beta})I)$ if the circle criterion condition is satisfied.

Theorem 4.4 of the previous section provides the basis for an improvement of the circle condition in the case of a periodic feedback gain.

Proof: Denote the center of the separating circle by μ, the radius by R and suppose $\sigma(B)$ is exterior to the circle (see Fig. 9)

$$A + B = -(-A + \mu I) + (B + \mu I),$$

and $B + \mu I$ is invertible, so that

$$(A + B) = (B + \mu I)(I - (B + \mu I)^{-1}(-A + \mu I)) \qquad (4.52)$$

Since the induced norm of a normal operator is equal to its spectral radius,

$$\| -A + \mu I \| < R, \text{ and}$$

$$\| (B + \mu I)^{-1} \| < \frac{1}{R}$$

so that

$$\| (B + \mu I)^{-1}(-A + \mu I) \| < 1,$$

which shows from (4.52) that $A + B$ is invertible.

The connection of the above lemma with the linear version of the original circle theorem is immediate, since both a scalar convolution and a time varying gain correspond to normal operators on $L_2(0,\infty)$. The set points encircled by the Nyquist locus is the spectrum of the convolution, and the interval $[\frac{1}{\beta}, \frac{1}{\alpha}]$ contains the spectrum of the operator K^{-1} when K is multiplication by a gain limited by $0 < \alpha \leq k(t) \leq \beta$. Hence the circle with the segment $[-\frac{1}{\alpha}, -\frac{1}{\beta}]$ as diameter separates $\sigma(G + \frac{1}{2}(\frac{1}{\alpha} + \frac{1}{\beta})I)$ from $\sigma(-K^{-1} + \frac{1}{2}(\frac{1}{\alpha} + \frac{1}{\beta})I)$ if the circle criterion condition is satisfied.

Theorem 4.4 of the previous section provides the basis for an improvement of the circle condition in the case of a periodic feedback gain.

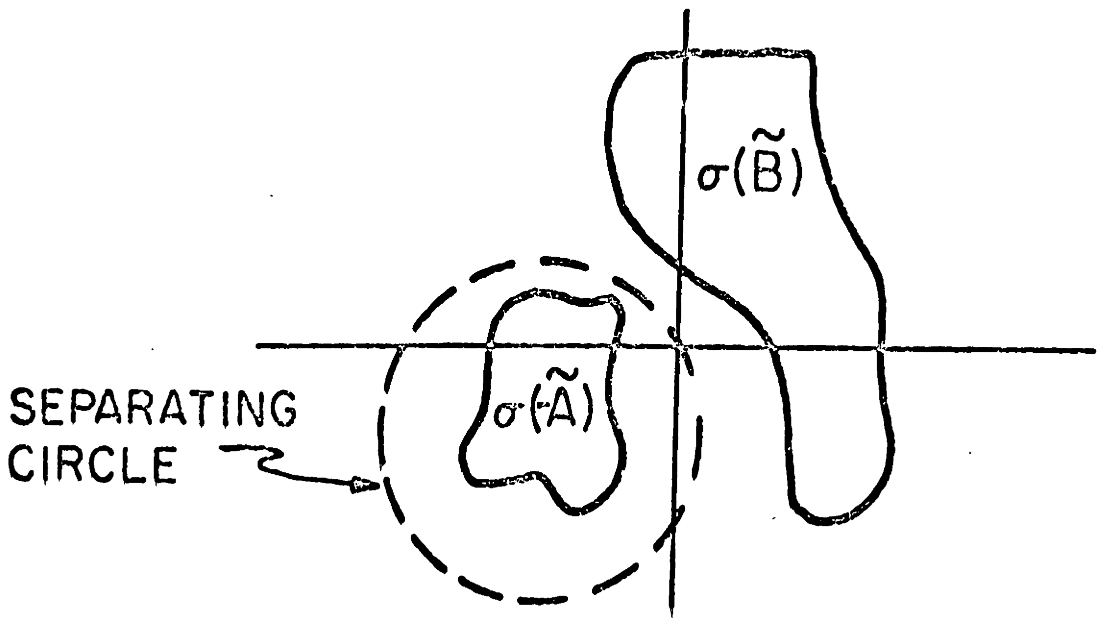


Fig. 9 Spectra of A and B

The use of the companded system for analysis essentially results in a partition of the spectrum of the convolution operator into separate pieces, as was noted above. This partitioning may be exploited to obtain a sufficient condition for stability.

Theorem 4.5: Let $\{g_i\}^{\infty}$ be a sequence belonging to ℓ_1^+, and let $\hat{g}(z)$ be the corresponding pulse transfer function. Suppose that the function $k(\cdot)$ is periodic of period n, and satisfies the bounds $0 < \alpha + \epsilon \leq k(i) \leq \beta - \epsilon$, $i \geq 0$. Then the regular feedback system

$$e(i) + k(i) \sum_{j=0}^{i-1} g_{i-j} e(j) = u(i) \quad i \geq 0 \tag{4.53}$$

is bounded input-bounded output stable in the sense of ℓ_p^+, $p \geq 1$, c^+, or c_0^+ if the following holds: For each complex z in the region $|z| \geq 1$, there exists a circle separating the points $\{\hat{g}(\omega_i z)\}_{i=0}^{n-1}$ from the segment $[-\frac{1}{\alpha}, -\frac{1}{\beta}]$.

Proof: This follows directly from Theorem 4.4 and the previous Lemma.

The geometrical interpretation of the above is that even if the Nyquist locus intersects the critical disc of the circle criterion, stability may still be guaranteed if it is possible to "shift the circle" so that it avoids the points $\{\hat{g}(\omega_i z)\}_{i=0}^{n-1}$ for each z. This is illustrated for the case $n = 2$ in Fig. 10. Even though the point $g(\omega_i z)$ originally may lie within the critical circle, it may be possible to move the circle upward to avoid the point $\hat{g}(\omega_1 z)$ and still not include the point $\hat{g}(\omega_0 z)$ within the circle.

Theorem 4.5 as it stands suffers from two limitations. It is hard to analytically characterize the parameters of the "shifting circle" involved, and the condition apparently involves more than just the boundary of the Nyquist plot.

These limitations may be overcome by recasting the problem in the form corresponding to the positive operator version of the circle criterion. In this form, the shifted circle corresponds to a straight line through the origin.

<u>Corollary 4.5</u>: A sufficient condition that the system (4.53), satisfying the conditions of Theorem 4.5 be stable is that for $i = 0, 1, \ldots n-1$

$$\left| \arg \frac{\beta \hat{g}(e^{i\theta}) + 1}{\alpha \hat{g}(e^{i\theta}) + 1} - \arg \frac{\beta \hat{g}(\omega_i e^{i\theta}) + 1}{\alpha \hat{g}(\omega_i e^{i\theta}) + 1} \right| \leq \Pi \tag{4.54}$$

for all θ that

$$\left| \arg \frac{\beta \hat{g}(e^{i\theta}) + 1}{\alpha \hat{g}(e^{i\theta}) + 1} \right| \geq \Pi/2 \tag{4.55}$$

Geometrically (see Fig. 11), the condition is that for all values of θ such that the circle criterion condition is violated, the points on the Nyquist plot corresponding to θ shifted by a multiple of the frequency of the periodic gain lie to the right of a straight line drawn through the origin and the θ-point of the Nyquist locus.

<u>Proof of Corollary:</u> Just as in Section II-I, use the bilinear transformation

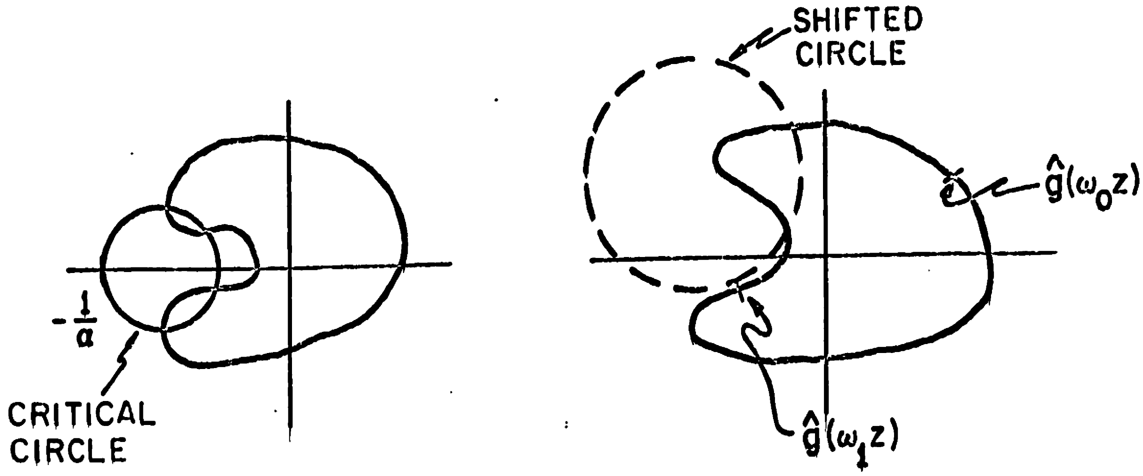


Fig. 10 The Shifting Circle Condition

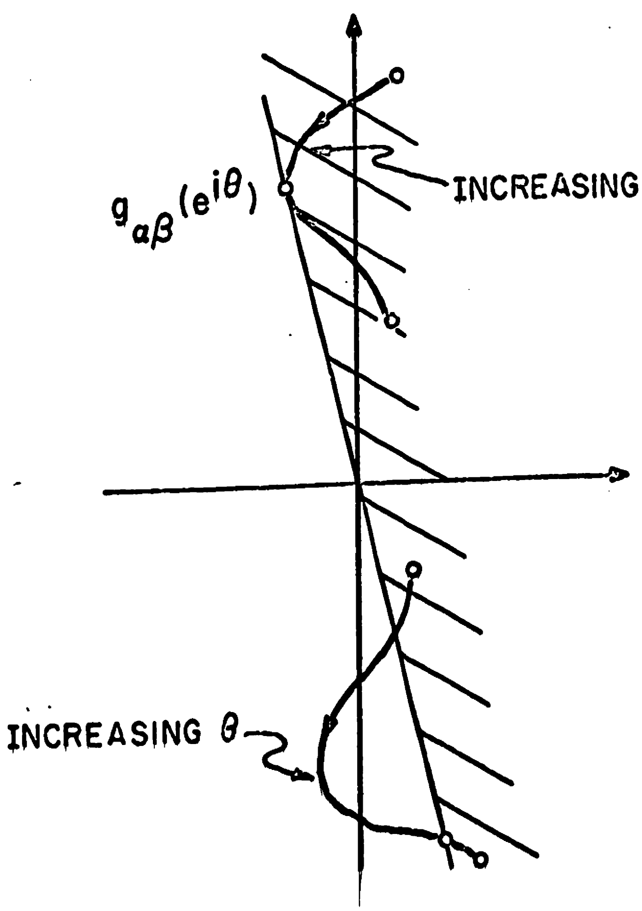


Fig. 11 Shifted Circle after Transformation

$$\omega \rightarrow \frac{\beta\omega + 1}{\alpha\omega + 1}$$

to obtain an equivalent system with pulse transfer function

$$\hat{g}_{\alpha\beta}(z) = \frac{\beta\hat{g}(z) + 1}{\alpha\hat{g}(z) + 1}$$

and a periodic feedback gain

$$k_{\alpha\beta}(i) = \frac{k(i) - \alpha}{\beta - k(i)}$$

The separating circle of Theorem 4.5 becomes a straight line through the origin and the point $\hat{g}_{\alpha\beta}(z)$.

The fact that the condition need be verified only for $|z| = 1$ rather than $|z| \geq 1$ follows from the maximum modulus principle applied to the harmonic function

$$h(z) = \arg\left\{\frac{\beta\hat{g}(z) + 1}{\alpha\hat{g}(z) + 1} \bigg/ \frac{\beta\hat{g}(\omega_i z) + 1}{\alpha\hat{g}(\omega_i z) + 1}\right\}$$

The above result is a discrete time version of a result obtained for the continuous time case by Willems.[44] The continuous time result was originally derived by Willems using a positive operator argument, and it is shown in Section IV-G that the corresponding positive operator arises rather naturally in the discrete time case. It is also true that the continuous time result may be derived by exactly the same arguments as were employed above. The appropriate method is to use a continuous time Fourier series to "diagonalize" the integral operator occurring in (4.35), just as the discrete Fourier series was employed to diagonalize the companded pulse transfer function matrix in Section IV-D. The details are omitted, since the continuous time version is previously known.

F. FROZEN TIME NYQUIST CONDITIONS

It is natural to ask what conclusions may be drawn about the stability of a system with a time-varying feedback gain on the basis of the application of the Nyquist criterion to the class of systems obtained by replacing the time-varying gain with a constant equal to a value assumed by the gain. The conjecture that such a system is stable provided that it is stable for each fixed gain value is referred to as the "frozen time Nyquist condition". Similarly, if the system were unstable for each fixed value of the gain, one might conjecture that the time-varying system is unstable. In fact, both of the above conjectures are untrue without additional assumptions on the original system. Moreover, the above conjectures have nothing to say about systems stable for some of the values assumed by the gain, but unstable for others.

The theorem below identifies a class of periodic discrete systems for which the stability properties are determined completely by the values assumed by the gain.

<u>Theorem 4.8:</u> Consider the discrete system

$$e(i) + k(i) \sum_{j=0}^{i-1} g_{i-j}\, e(j) = u(i) \quad , \quad i > 0$$

where $\{g_i\} \in \ell_1^+$, and $k(i+n) = k(i)$. Then if the pulse transfer function has the form

$$\hat{g}(z) = \hat{g}_0(z^n)$$

the following conditions hold.

a. The system is bounded input-bounded output stable in the sense of ℓ_p^+, c, or c_0 provided that it satisfies the Nyquist condition for each fixed value of the feedback gain.

b. The system is unstable if the Nyquist condition fails to be satisfied for <u>any one</u> of the fixed values assumed by the feedback gain.

<u>Proof</u>: The theorem follows from the observation that the companded pulse transfer function matrix is diagonal only in the case that $\hat{g}(z) = \hat{g}_0(z^n)$. If the companded pulse transfer function matrix is diagonal, the determinant satisfies

$$\det(\underline{\underline{I}} + \underline{\underline{\mathcal{K}}}\,\hat{\underline{\underline{\mathcal{G}}}}(z)) = \prod_{i=0}^{n-1} (1 + k(i)\hat{g}_0(z))$$

which shows that the system is stable if and only if $(1 + k(i)g_0(z)) \neq 0$, $|z| \geq 1$, which is simply the condition that $g_0(z)$ (equivalently $g_0(z^n)$) satisfy the Nyquist condition for each assumed value of the feedback gain.

It is relatively unusual for a pulse transfer function to be a function of z^n alone. In the case where the system originally arises from a difference equation of the form

$$\begin{aligned} &p_r\, x(j+r) + p_{r-1}\, x(j+r-1) + \ldots + p_0\, x(j) + \\ &k(j)\left[q_{r-1}\, x(j+r-1) + \ldots + q_0\, x(j)\right] = 0 \end{aligned} \qquad (4.59)$$

this corresponds to the vanishing of all the coefficients in (4.59) whose subscript is not an integral multiple of the feedback period, n. When this occurs, it is easy to see that the equation (4.59) may be replaced by n separate equations with constant coefficients.

If a pulse transfer function is written in the form

$$\hat{g}(z) = \hat{g}_0(z^n) + \hat{r}(z) \tag{4.60}$$

and if the residual $\hat{r}(z)$ is "small" in some sense, then it would be expected that stability could be concluded on the basis of the frozen time Nyquist condition. Such a result may be derived applying the "Gershgorin type" estimates of Theorem 3.3 to the companded pulse transfer function matrices corresponding to (4.60).

<u>Theorem 4.9</u>: Consider the regular feedback system

$$e(i) + k(i) \sum_{j=0}^{i-1} g_{i-j}\, e(j) = n(i) \qquad i \geq 0 \tag{4.61}$$

with $\{g_i\}_1^{\infty} \in \ell_1^+$, $k(i+n) = k(i)$, $i \geq 0$, and $k(i) \neq 0$. Let the pulse transfer function be given by

$$\hat{g}(z) = \sum_{i=1}^{\infty} \hat{g}_i z^{-i} = g_0(z^n) + r(z) \tag{4.60}$$

and suppose that

1. $1 + k(i)\,\hat{g}_0(e^{in\Theta}) \neq 0$, $-\Pi \leq \Theta \leq \Pi$, $i = 0, \ldots n-1$, and

2. $\left| \frac{1}{k(i)} + \hat{g}_0(e^{in\Theta}) \right| > \left| \hat{r}(e^{i\Theta}) \right|$, $\quad -\Pi \leq \Theta \leq \Pi$, $\quad i = 0, \ldots n-1$

Then the system (4.61) is bounded input-bounded output stable in the sense of ℓ_p^+, $p \geq 1$, c^+, or c_0^+ if

$$3. \quad k_0 = \sum_{i=0}^{n-1} \frac{1}{2\Pi} \int_{-\Pi}^{\Pi} d_\Theta \arg(1 + k(i)\hat{g}_0(e^{in\Theta})) = 0$$

and (4.61) is unstable if $k_0 \neq 0$.

<u>Proof</u>: The necessary and sufficient condition for stability is that the function

$$\det\left(\underline{\underline{I}} + \begin{bmatrix} k(0) & & \\ & \ddots & \\ & & k(n-1) \end{bmatrix} \hat{\underline{\underline{\mathcal{G}}}}(e^{i\Theta})\right)$$

where $\hat{\underline{\underline{\mathcal{G}}}}(e^{i\Theta})$ is the companded pulse transfer function matrix, neither vanish nor have any encirclements of the origin as Θ varies over the range $[0, 2\Pi]$. Since by hypothesis none of the $k(i)$ vanish, the above may be replaced by

$$\det\left(\begin{bmatrix} \frac{1}{k(0)} + \hat{g}_0(e^{i\Theta}) & & \\ & \ddots & \\ & & \frac{1}{k(n-1)} + \hat{g}_0(e^{i\Theta}) \end{bmatrix} + \hat{\underline{\underline{\mathcal{R}}}}(e^{i\Theta})\right) \qquad (4.62)$$

Here $\hat{g}_0(e^{i\Theta})$ is as in 4.60, and $\hat{\underline{\underline{\mathcal{R}}}}(e^{i\Theta})$ is the companded pulse transfer function matrix corresponding to $\hat{r}(z)$.

By Lemma 4.3, $\hat{\underline{\underline{\mathcal{R}}}}$ may be written in the form

$$\hat{\mathcal{R}}(e^{i\Theta}) = \begin{bmatrix} 1 & & & \\ & e^{i\Theta/n} & & \\ & & \ddots & \\ & & & e^{i\frac{n-1}{n}\Theta} \end{bmatrix} \underline{\underline{\Omega}} \begin{bmatrix} \hat{r}(\omega_0 e^{i\Theta/n}) & & \\ & \ddots & \\ & & r(\omega_{n-1} e^{i\Theta/n}) \end{bmatrix} \cdot$$

$$\underline{\underline{\Omega}}^* \begin{bmatrix} 1 & & & \\ & e^{-i\Theta/n} & & \\ & & \ddots & \\ & & & e^{-i\frac{n-1}{n}\Theta} \end{bmatrix},$$

or

$$\hat{\mathcal{R}}(e^{i\Theta}) = \underline{\underline{\mathcal{U}}} \begin{bmatrix} \hat{r}(\omega_0 e^{i\Theta/n}) & & \\ & \ddots & \\ & & \hat{r}(\omega_{n-1} e^{i\Theta/n}) \end{bmatrix} \underline{\underline{\mathcal{U}}}^*$$

where the matrix $\underline{\underline{\mathcal{U}}}$ is unitary. Since $\underline{\underline{\mathcal{U}}}$ is unitary, the Euclidian induced norm of the matrix $\hat{\mathcal{R}}(e^{i\Theta})$ is just

$$\max_j \left| \hat{r}(\omega_j e^{i\Theta/n}) \right| \tag{4.62}$$

Similarly, the induced norm of the inverse of

$$\begin{bmatrix} \frac{1}{k(0)} + \hat{g}_0(e^{i\Theta}) & \\ & \frac{1}{k(n-1)} + \hat{g}_0(e^{i\Theta}) \end{bmatrix}$$

is

$$\frac{1}{\min_j \left| \frac{1}{k(j)} + \hat{g}_0(e^{i\Theta}) \right|}$$

Applying Theorem 3.3 (more precisely, the obvious discrete time version) to (4.61) gives the result, after replacing Θ by $n \cdot \Theta$ in order to remove the max over j in (4.62).

In the case of stability the conditions of the theorem have a possible geometric interpretation. If the system with pulse transfer function $\hat{g}_0(z^n)$ is stable for each mixed value of the feedback gain $k(i)$, then conditions 1 and 3 are satisfied. This may of course easily be checked by constructing the Nyquist locus for $\hat{g}_0(z^n)$. If the Nyquist locus for $r(z)$ is plotted on the same set of axes, then condition 2 is just that the distance from the Nyquist locus of $\hat{g}_0(z^n)$ to the points $\{-\frac{1}{k(i)}\}_0^{n-1}$ should exceed the magnitude of the Nyquist locus for $\hat{r}(z)$ for the corresponding values of the "frequency" Θ.

It should be mentioned that the restriction that the feedback gain not vanish is not necessary. It may be removed by making a standard loop transformation to replace the gain value zero with one of magnitude ϵ. Specifically, for sufficiently small $\epsilon < 0$, a system with pulse transfer function

$$\hat{g}_\epsilon(z) = \frac{\hat{g}(z)}{1 + \epsilon\, \hat{g}(z)}$$

and periodic gain

$$k_\epsilon(i) = k(i) + \epsilon$$

is equivalent to the original one.

For the "ϵ-system" the feedback gain does not vanish. The conditions 1, 2, and 3 of the theorem are automatically satisfied because $\frac{1}{\epsilon}$ has large magnitude, and for sufficiently small ϵ the conditions

1, 2, and 3 corresponding to the other values of the gain are the same as for the original system due to the continuity of the conditions with respect to ϵ. Hence in case the feedback gain vanishes, conditions 1, 2, and 3 of Theorem 4.9 need only be verified for non-vanishing values of the gain.

G. PERIODIC DISCRETE POSITIVE OPERATORS

As mentioned previously, positive operators play a very useful role in establishing sufficient conditions for the stability of feedback systems. This is especially true in the case of systems which involve a convolution operator, since the Plancherel theorem provides a ready means to test positivity in such a case. As was shown in Section II-G, the necessary and sufficient condition for a scalar convolution operator to be positive is that the corresponding Nyquist locus lie in the right half of the complex plane.

According to Theorem 2.10, a feedback system is stable (relative to inputs in the Hilbert space $\mathcal{H}(T^+)$), provided that the operator KG may be factored as a product of two positive operators. If the operator G represents a convolution, then it is natural to factor KG in the form

$$KG = (KH^{-1})(HG)$$

where H is also a convolution, since in this case the positivity of HG may easily be checked. In actual practice, of course, H is restricted by the geometry of the Nyquist locus of the convolution G. The condition that KH^{-1} be positive imposes additional restrictions on H, so that in order for such an H to exist some combination of conditions involving the Nyquist locus of G and various properties of the operator K will have to be satisfied.

In the case of the periodic discrete systems considered in this chapter, the operator K represents a multiplication by a periodically time-varying gain. This means that the problem of whether the operator KH^{-1} is positive comes to the problem of the positivity of a composition of a convolution operator with a periodic gain. As might be expected, the device of constructing an equivalent companded system may be employed to solve the problem.

Theorem 4.10: Consider the operator KH defined on ℓ_2^+ by the relation

$$KH: \quad u(i) \to k(i) \sum_{j=0}^{i} h_{i-j} u(j) \qquad i > 0$$

where $k(i+n) = k(i)$, and the sequence $\{h_i\}_0^\infty \in \ell_1^+$.

Then a necessary and sufficient condition that the operator KH be strictly positive on ℓ_2^+, i.e., that $\mathrm{Re}\langle KHu, u\rangle \geq \epsilon \langle u,u\rangle$, $\forall u \in \ell_2^+$, and some $\epsilon > 0$ is that the Hermetian matrix

$$\begin{bmatrix} k(0) & & \\ & \ddots & \\ & & k(n-1) \end{bmatrix} \hat{\underline{\underline{\mathcal{H}}}}(e^{i\theta}) + \hat{\underline{\underline{\mathcal{H}}}}^*(e^{i\theta}) \begin{bmatrix} \bar{k}(0) & & \\ & \ddots & \\ & & \bar{k}(n-1) \end{bmatrix} \tag{4.63}$$

be strictly positive definite, $0 \leq \theta \leq 2\Pi$. Here $\hat{\underline{\underline{\mathcal{H}}}}(e^{i\theta})$ is the companded transfer function matrix corresponding to $\{h_i\}_0^\infty$.

Proof: According to the results of Section IV-B above the input-output behavior of the operator KH may be equivalently represented by a product of a diagonal gain matrix and a companded pulse transfer matrix,

$$KH \sim \begin{bmatrix} k(0) & & \\ & \ddots & \\ & & k(n-1) \end{bmatrix} \begin{bmatrix} h_0(z) & \frac{1}{z} & \cdots & \frac{1}{z} h_1(z) \\ h_1(z) & & & \\ \vdots & & & \\ h_{n-1}(z) & & & h_0(z) \end{bmatrix} \quad (4.63)$$

which corresponds to a (matrix) convolution acting on $\ell_2^{+(n)}$.

The result then follows from the Plancherel Theorem, since

$$<KHu,u> = \frac{1}{2\Pi} \int_{-\Pi}^{\Pi} \hat{\underline{\mathcal{U}}}^*(e^{i\Theta}) \begin{bmatrix} k(0) & & \\ & \ddots & \\ & & k(n-1) \end{bmatrix} \hat{\underline{\underline{\mathcal{H}}}}(e^{i\Theta}) \hat{\underline{\mathcal{U}}}(e^{i\Theta}) d\Theta \quad (4.64)$$

where $\hat{\underline{\mathcal{U}}}(e^{i\Theta})$ denotes the ℓ.i.m. z transform of the companded version of the sequence u.

Corrollary: A set of sufficient conditions that KH :

$u(i) \rightarrow k(i) \sum_{g=0}^{i} h_{i-j} u(j)$ be a positive operator are that

1. k(i) be real, and $k(i) \geq \epsilon > 0$
2. $\hat{h}(z) = \hat{h}_0(z^n)$, with Re $\hat{h}_0(e^{i\Theta}) \geq \delta > 0$

Proof: If $\hat{h}(z) = \hat{h}_0(z^n)$ then the companded pulse transfer function matrix is diagonal, and positivity is obvious.

The above corrollary may be applied to derive a stability criterion for discrete systems with periodic feedback gain.

Theorem 4.11: A sufficient condition that the feedback system

$$e(i) + k(i) \sum_{j=0}^{i-1} g_{i-j}\, e(j) = u(i) \qquad i \geq 0$$

with $k(i+n) = k(i)$, $k(i)$ real, and $\{g_i\}_1^{\infty} \in \ell_1^+$, be bounded input-bounded output stable in the ℓ_2^+ sense is that:

1. $\alpha + \epsilon \leq k(i) \leq \beta - \epsilon$
2. There exist an analytic function $h(\cdot)$ such that
   a. $\mathrm{Re}\,(\hat{h}(e^{i\Theta})) > 0$

   b. $\mathrm{Re}\left[\hat{h}(e^{in\Theta})\, \dfrac{\beta g(e^{i\Theta}) + 1}{\alpha g(e^{i\Theta}) + 1}\right] \geq 0 \qquad -\Pi \leq \Theta \leq \Pi$

Proof: Consider first the case where the periodic gain is bounded by $0 < \epsilon \leq k(i) < \infty$.

The positive real function $h(\cdot)$ determines a convolution with pulse transfer function $\hat{h}(z^n)$.

Since $\mathrm{Re}\,\hat{h}(e^{i\Theta}) > 0$, $\hat{h}(z^n)$ does not vanish for $|z| \geq 1$, and so has an analytic inverse, again representing a convolution operator on ℓ_2^+.

By the previous corrollary,

$$KH^{-1} \sim k(i) \cdot 1/\hat{h}(z^n)$$

is a positive operator. If the pulse transfer function $\hat{g}(z)$ is such that

$$\mathrm{Re}\left[\hat{h}(e^{in\Theta})\, \hat{g}(e^{i\Theta})\right] \geq 0 \qquad (4.65)$$

then the operator KG has been factored as a product of a positive and a non-negative operator, and stability follows from Theorem 2.10.

In the case that the gain is restricted to an interval $[\alpha + \epsilon, \beta - \epsilon,]$ then the use of a bilinear transformation as in Theorem 4.5 reduces the problem to the above case and (4.65) becomes condition b. of the theorem statement.

Theorem 4.11 above is a discrete version of the multiplier result obtained in Ref. 44. Just as is shown in that reference, the condition for such a multiplier to exist is just the geometrical condition on the Nyquist locus given in Theorem 4.5.

V. CONCLUSIONS AND SUGGESTIONS FOR FURTHER RESEARCH

The results included in this thesis show that the spectral theory of linear operators may be systematically applied to derive conditions for the input - output stability of a class of feedback control systems. The main benefits of this point of view are two. First, it identifies the central mathematical problem as that of the invertibility of a certain operator and so connects the stability problem with a currently active research area in mathematical analysis. In the second place it provides an unified interpretation of the origin of sufficient conditions for stability as estimates of the spectrum of the operator involved.

Certainly the most striking aspect of this point of view is the fact that it is capable of generating a wide range of results from quite general ones to some of an extremely concrete and specific nature. This is illustrated by the contrast between the theorems concerned with positive operator products in Chapter II, and the content of Chapter IV, devoted entirely to periodic discrete systems.

The relationship between feedback system stability and operator spectral calculations has been exploited almost entirely in one direction in this thesis. With the exception of Theorem 4.3, spectral calculations from the mathematical literature (and spectral estimates as well) have been employed to obtain conditions for the stability of feedback systems. Theorem 4.3, on the other hand, provides a characterization of the spectrum of a Volterra type integral operator acting in $L_p(0,\infty)$, $1 \leq p < \infty$, for the case of a kernel of the form $h(t,s) = k(t)g(t-s)$ with $k(\cdot)$ periodic, and $g(\cdot) \in L_1(0,\infty)$ having a rational Fourier transform.

It is possible to obtain spectral estimates for certain integral operators on a half-axis by exploiting sufficient conditions for stability which already occur in the control literature. For example, the results of[45] may be applied to a problem of a singular integral equation with shifts[44] by exploiting the connection mentioned in Section III-F. Similarly, results of the "restricted rate of variation" type[57] may be applied to integral operators on a half-axis with kernel $h(t, s) = k(t) g(t-s)$ under suitable restrictions.

Several results above have made use of eigenvalue estimates for matrices to derive stability conditions for vector input - vector output time-invariant systems. There is a large amount of literature on the problem of eigenvalue estimation which may be potentially useful in stability problems. Of particular interest may be results of the type which require "block partitioning" of the matrices involved, which strongly suggest application to "weakly coupled" systems of large dimension.

Another possible topic for future research is the application of such devices as positive operator arguments to the problem of matrix eigenvalue location. The classical Lyapunov theorem connecting matrices $\underline{\underline{A}}$ having eigenvalues in the region $\mathrm{Re}\,\lambda_i(\underline{\underline{A}}) < 0$ and the linear matrix equation

$$\underline{\underline{A}}\,\underline{\underline{K}} + \underline{\underline{K}}^*\underline{\underline{A}}^* = -Q, \quad Q = Q^* \text{ positive definite} \tag{5.1}$$

has a clear connection with the Theorem of Williams (Theorem 2.9):

$$\sigma(\underline{\underline{A}}) = \sigma(\underline{\underline{K}}^{-1}\underline{\underline{K}}\,\underline{\underline{A}}) \subseteq \frac{W(\underline{\underline{K}}\,\underline{\underline{A}})}{W(\underline{\underline{K}})} \tag{5.2}$$

If equation (5.1) is satisfied for some positive definite $\underline{\underline{Q}}$ and $\underline{\underline{K}}$, then (5.2) shows that the eigenvalues of A all have negative real parts, because (5.1) guarantees that $Re(W(\underline{\underline{K}}\underline{\underline{A}})) < 0$. It would be of interest to consider equation (5.1) under the assumption that K is unitary with spectrum restricted to a sector of the plane. Results along these lines potentially lead to wedge shaped regions containing the eigenvalues of the matrix $\underline{\underline{A}}$.

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