# Embedding Theorems in the Reducibility Ordering of Partial Degrees

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# Abstract

Embedding theorems in the reducibility ordering of the partial degrees.

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The main result in this paper is:

<u>Theorem.</u> Given partial degrees $\underline{a} <_e \underline{b}$ such that [$\underline{b}$ is a total degree or $\underline{a}$ is total] and $(\exists f)(\exists B \in \underline{b})$ ( $f$ is a total function $\& A = \{<x, f(x)> | x \in N\} \in \underline{a}$ $B \leq_T A'$) then the partial ordering $\mathcal{E}$ can be embedded in the partial degrees between $\underline{a}$ and $\underline{b}$. That is, there exist partial degrees $\underline{c}_i$ such that $\underline{a} < \underline{c}_i < \underline{b}$ and $i <_F j \Leftrightarrow \underline{c}_i <_e \underline{c}_j$. Where $\mathcal{E}$ is the partial ordering of the recursively enumerable sets under inclusion.

A proof is given using this theorem of Gutteridge's result that there are no minimal partial degrees.

Two conjectures of John Case are proved. The first is that the measure of the sets enumeration incomparable with their complements is one. The second is that the measure of the quasi-minimal sets (those sets with no nonrecursive total function enumeration reducible to them)is one.

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## Introduction

Until quite recently, not much work has been done on enumeration reducibility. Perhaps this is because the most obvious question : is there a minimal partial degree? has proved so hard to solve. This has been disconcerting because over fifteen years ago, Spector [1956] had proved that in the closely related Turing reducibility there do exist minimal degrees.

Finally, Cooper [1971] announced that there exists a minimal partial degree. Three months later, Gutteridge [1971] announced that the partial degrees are dense. The author was able to obtain a copy of Gutteridge's paper and convince himself that there is no minimal partial degree. He started working on various embedding theorems and seemed to be getting results when Gutteridge withdrew his paper. After several sleepless nights, the author realized that the error in the Gutteridge paper was in the relativization. Thus there are no minimal partial degrees but the density question remains open. A proof that there is no minimal partial degree is given later in this paper.

The embedding theorems given in this paper might be strengthened in three ways. First, the type of partial ordering that can be embedded might be strengthened to include all countable partial orderings. This cannot be done using the methods employed in the paper and the author does not think it can be done. Second, the restrictions on the degree $\underline{b}$ might be removed. The author conjectures that this can be done. Third, the result might be relativised for all degrees $\underline{a} < \underline{b}$ without the restriction that either $\underline{a}$ or $\underline{b}$ be a total degree. This implies density. The author conjectures that this cannot be done, i.e. that the partial degrees are not dense.

The embedding theorems are an extension of work done by Lance Gutterridge. Their proofs also utilize many of the methods developed by him. For example, the use of recursive approximations when dealing with sets $B <_T 0'$ , and the construction of enumeration operators which depend only on the value of singletons.

# Chapter 1

## Preliminaries

The notation will follow that of Rogers [1967] with some additions given below. Subsets of N will be denoted by upper case letters with D and F being finite subsets, members of N will be denoted by lower case letters except that f and g will be reserved for total functions. Lower case underlined letters will denote partial degrees, e.g. $\underline{a}$ , $\underline{b}$ . $<x,E>$ will be used to represent $<x,u>$ where $D_u = E$ .

Definition:

i) $W(A) = \{x \mid <x,u> \in W \;\&\; D_u \subset A\}$ . This notation, which differs from that of Rogers, is very helpful in that it identifies a given set W with an operator. If the set W is r.e. then the associated operator will be an enumeration operator.

ii) A is <u>enumeration reducible</u> to B ( $A \leq B$) if there exists some r.e. set W such that $A = W(B)$. Usually this is written $\leq_e$ as not to confuse it with the more common Turing reducibility $\leq_T$ .

iii) $A \equiv_e B$ if $A \leq B$ and $B \leq A$ .

iv) $A < B$ if $A \leq B$ and $B \not\leq A$ .

v) $A \mid_e B$ if $A \not\leq B$ and $B \not\leq A$ .

$\equiv_e$ is an equivalence relation on the power set of $N$. The equivalence classes are called the partial or enumeration degrees. For the basic facts about the enumeration reducibility see Rogers [1967]. $\underline{0}$ will denote smallest partial degree, that of the r. e. sets. A degree $\underline{b}$ is said to be minimal if $\underline{0} < \underline{b}$ and if $\underline{a} < \underline{b}$ then $\underline{a} = \underline{0}$. A degree $\underline{a}$ is a total degree if there exists a total function $f$ such that $\underline{a}$ is the degree of the set $\{<x,f(x)> | x \in N\}$.

$W(T(B))$ will be denoted by $WT(B)$. Where $W \subseteq N$ and $T \subseteq N$.

$\bigvee_{i \in N} X_i$ is the infinite join of $X_0, X_1, \ldots$ i. e.
$x \in X_i \Leftrightarrow x \cdot 2^{i+1} + \sum_{k \leq i} k \in \vee X_i$. $\bigvee_{j \neq i_0} X_j$ and $\bigvee_{n \in N} X_{f(n)}$ have the obvious definitions if $i_0$ is a fixed integer and $f$ is a recursive total function. The following properties of the join can easily be verified

$$j_0 \neq i \Rightarrow X_{j_0} \leq \bigvee_{j \neq i} X_j$$

$$i \notin \text{Range } f \Rightarrow \left(\bigvee_{n \in N} X_{f(n)} \leq \bigvee_{j \neq i} X_j\right)$$

$$(\forall i)\left(X_i \leq \bigvee_{j \in N} X_j\right)$$

$W^s$ where $W$ is r. e. is the amount of $W$ enumerated in the first $s$ steps of some fixed enumeration. Enumeration operators have the following monotonicity type properties

i) $A \subset B \Rightarrow W(A) \subseteq W(B)$

ii) $W' \subset W \Rightarrow W'(A) \subseteq W(A)$

iii) $x \in W(A) \Rightarrow \exists s\ (x \in W^s(A))$

iv) $x \in W(A) \Rightarrow \exists D($ $D$ is finite $\&\ x \in W(D))$

$B^s$ is a <u>recursive approximation</u> to $B$ if there exists a recursive function $f$ such that $D_{f(s)} = B^s$ and the $B^s$ have the property that

$$\forall n \exists s' \forall s (s' > s \Rightarrow (n \in B^{s'} \Leftrightarrow n \in B))$$

It is easy to show that a set $B$ will have a recursive approximation $B^s$ if and only if $B \leq_T 0'$. This result can be relativised to : there exists an approximation $B^s$ to $B$ which is recursive in $A$ if and only if $B \leq_T A'$ .

# Chapter 2

## Embedding Theorems for Partial Degrees

Definition. $\leq_F \subset N \times N$ is a partial ordering if

$$\forall x \forall y \forall z ((\langle x,y\rangle \in \leq_F \& \langle y,z\rangle \in \leq_F) \Rightarrow \langle x,z\rangle \in \leq_F)$$

and if

$$\forall x \forall y ((\langle x,y\rangle \in \leq_F \& \langle y,x\rangle \in \leq_F) \Rightarrow x = y)$$

$\langle x,y\rangle \in \leq_F$ is abbreviated by $x \leq_F y$.

$x <_F y$ if $x \leq_F y$ and $y \not\leq_F x$.

$i$ is in the domain of $\leq_F$ if $\langle i,i\rangle \in \leq_F$.

Definition:

A partial ordering $\leq_F$ can be <u>embedded between two partial degrees</u> $\underline{a}$ and $\underline{b}$ where $\underline{a} < \underline{b}$ if for each $i$ and $j$ in the domain of $\leq_F$ there exist partial degrees $\underline{c}_i$, $\underline{c}_j$ such that

$$i <_F j \Leftrightarrow \underline{c}_i < \underline{c}_j$$

$$\underline{a} < \underline{c}_i < \underline{b}$$

THEOREM (Finite Embedding Theorem) *Given a degree* $\underline{b}$ *any finite partial ordering* $\leq_F$ *can be embedded below* $\underline{b}$, *if there exists a set* $B \in \underline{b}$ *such that* $B \leq_T 0'$.

*Proof*. Let $B \in b$ and $B^s$ be a recursive approximation to B.

If the partial ordering F has m elements then m enumeration operators $T_0, T_1, \ldots, T_{m-1}$ are constructed with the property that for all $i < m$

$$T_i(B) \not\leq \bigvee_{j \neq i} T_j(B)$$

Then m enumeration operators $S_0, \ldots, S_{m-1}$ are defined by

$$S_i(B) = \bigvee_{j \leq_F i} T_j(B)$$

Clearly if $i \leq_F j$ then by projection $S_i(B) \leq S_j(B)$. Also $T_i(B) \leq S_i(B)$ and if $i \not\leq_F j$ then again by projection

$$S_j(B) \leq \bigvee_{k \neq i} T_k(B)$$

Hence

$$i \not\leq_F j \Rightarrow S_i(B) \not\leq S_j(B)$$

otherwise by transitivity

$$T_i(B) \leq \bigvee_{j \neq i} T_j(B)$$

which would be a contradiction.

The $T_i$ are constructed using a priority argument. In the construction of the $T_i$ ordered pairs $\langle p,n \rangle$ will be given

$\surd$'s and x's. If an ordered pair $<p,n>$ in $T_i$ is given an x, then that ordered pair will always be given as an output, i.e. $<<p,n>,\phi> \in T_i$ and $<p,n> \in T_i(\phi)$. If $<p,n>$ has a $\surd$ in $T_i$, then $<p,n>$ is given as output whenever n is given as input, i.e. $<<p,n>,\{n\}> \in T_i$ and $<p,n> \in T_i(\{n\})$. An ordered pair can have both a $\surd$ and an x in which case the x will dominate. If $<p,n>$ has a $\surd$ but no x then

$$\forall X[<p,n> \in T_i(X) \Leftrightarrow n \in X]$$

Requirement $R_p$ where $p = mq + i$, is satisfied at stage s if there exists a $<p,n>$ which has a $\surd$ but no x in $T_i^s$ and

$$<p,n> \in T_i^s(B^s) \;\&\; <p,n> \notin W_q^s(\vee_{j \neq i}(D(j,p,s) \cup Z_p)) \qquad (*)$$

or

$$<p,n> \notin T_i^s(B^s) \;\&\; <p,n> \in W_q^s(\vee_{j \neq i}(D(j,p,s) \cup T_j^s(\emptyset))) \qquad (**)$$

Such a $<p,n>$ is said to satisfy $R_p$ at stage s. Where $D(j,p,s)$ is

$$D(j,p,s) \text{ is } \{<t,r> | t < p \;\&\; <t,r> \in T_j^s(B^s)\} =$$

$$\{<t,r> | t<p \;\&(<t,r> \text{ has an } x \text{ in } T_j^s) \text{ or}$$

$$(<t,r> \text{ has a } \surd \text{ and } r \in B^s)$$

and

$$Z_p = \{<t,r> | \; t \geq p\}$$

The $T_i$ are constructed in stages.

Stage 0 : $T_i = \phi$ , $i = 1, \ldots, m$ .

Stage s : Find the least unsatisfied requirement $R_p$ , where $p = mq + i$ .

Then in $T_i$ , $\surd$ $\langle p,r \rangle$, for all $r < s$ .

In each $T_j$ , where $j \neq i$ , x $\langle t,r \rangle$ for all $p \leq t < s$ and $r < s$ .

The motivation behind this method is that we want $R_p$ where $p = mq + i$ to insure that $W_q(\bigvee_{j \neq i} T_j(B)) \neq T_i(B)$ . To make sure that equality does not hold we want to find some $\langle p,n \rangle$ such that $\langle p,n \rangle \in T_i(B) \Leftrightarrow \langle p,n \rangle \notin W_q(\bigvee_{j \neq i} T_j(B))$. Of course we cannot hope to do this effectively since B is not r. e. and in general we do not even have an oracle for $W_q$ .

But since $B \leq_T 0'$ we do have a recursive approximation to B . The important property of a recursive approximation is that for any finite subset of B there exists an $s_0$ such that $s > s_0$ implies that $B^s$ coincides with B on that subset.

In the construction of the $T_i$ note that at stage s $\langle t,r \rangle$ can receive a $\surd$ or x only if $r < s$ and $p \leq t < s$ where $R_p$ is the least unsatisfied requirement. Thus, if after stage $s_0$ all $R_{p'}$, $p' < p$ are always satisfied then $\langle p',r \rangle$ will have a $\surd$ or x in any $T_j$ only if $r < s_0$ , and $\langle p',r \rangle$ already has that $\surd$ or x by stage $s_0$ . This means that for $s > s_0$ , $D(j,p,s)$ will depend only on the members of $B^s$ which are $< s_0$ . Thus if $s_1$ is large enough such that $s_1 > s_0$ and for $s \geq s_1$, $B^s$ coincides with B for all $r < s_0$ , then if $s > s_1$ $D(j,p,s) = D(j,p,s_1)$.

We say $D(j,p,s)$ has <u>stabilized by</u> $s_1$. When $D(j,p,s)$ has stabilized then $D(j,p,s) = D(j,p,s_1) = \{<p',r> | p'<p \& <p',r> \in T_j(B)\}$ where it should be noted that in the above equation neither $T_j$ nor B has an s. This also implies that for $s \geq s_1$

$$D(j,p,s) \cup T_j^s(\phi) \subseteq T_j(B) \subseteq D(j,p,s) \cup Z_p$$

Thus if $<p,n> \in W_q^s(\bigvee_{j \neq i}(D(j,p,s) \cup T_j^s(\phi)))$ and $s > s_1$ then $<p,n> \in W_q(\bigvee_{j\neq i} T_j(B))$ and if $\forall s > s_1 <p,n> \notin W_q^s(\bigvee_{j\neq i}(D(j,p,s) \cup Z_p))$ then $<p,n> \notin W_q(\bigvee_{j\neq i} T_j(B))$.

Similarly if $<p,n>$ has a $\surd$ and no x in $T_i^s$ and $s > s_0$ then $<p,n>$ will never receive an x because the only way $<p,n>$ can receive an x in $T_i$ is if some $R_{p'}$, $p'< p$ becomes unsatisfied. Thus if $s_2$ is large enough such that $s_2 > s_0$ and $s \geq s_2 \Rightarrow (n \in B^s \Leftrightarrow n \in B)$ then

$$<p,n> \in T_i^s(B^s) \Leftrightarrow <p,n> \in T_i(B)$$

Together, the last several equations imply that if $R_p$ is satisifed by $<p,n>$ for all s greater than some fixed integer then in fact

$$<p,n> \in T_i(B) \Leftrightarrow <p,n> \notin W_q(\bigvee_{j\neq i} T_j(B))$$

which insures that $T_i(B) \neq W_q(\bigvee_{j\neq i} T_i(B))$.

Thus to complete the proof that the method works we must prove the following two lemmas.

LEMMA 1 . Each requirement $R_p$ is unsatisfied at only finitely many stages.

LEMMA 2. For each requirement $R_p$ there exists a $<p,n>$ and an $s_0$ such that $\forall s > s_0$ $<p,n>$ satisfies $R_p$ at stage $s$ .

Of course anyone standing where the three roads meet could, and probably would, tell you (Sacks [1971]) that Lemma 1 follows from Lemma 2. Lemma 1, however, is used in the proof of Lemma 2 .

Proof of Lemma 1. Suppose all $R_{p'}$ $p' < p = qm + i$ are satisfied at all stages $s > s_0$ . Let $s_1 \geq s_0$ be large enough such that all $D(j,p,s)$ have stabilized by stage $s_1$ . Consider the following two sets :

$$B^* = \{<p,n> | n > s_0 \ \& \ n \in B\}$$

$$C^* = \{<p,n> | n > s_0 \ \& <p,n> \in W_q(\bigvee_{j \neq i} D(j,p,s_1) \cup Z_p))\}$$

The two sets must be unequal since the first is enumeration equivalent to B which is not r. e. and the second is r. e.. Thus

$$\exists <p,n>[(<p,n> \in B^* \& <p,n> \notin C^*) \vee (<p,n> \notin B^* \& <p,n> \in C^*)]$$

Let $s_2$ be such that $\forall s > s_2$ $(n \in B^s \Leftrightarrow n \in B) \ \& (s_2 > s_1)$.
Suppose

$$(<p,n> \in B^* \ \& <p,n> \notin C^*) \quad .$$

If $R_p$ is unsatisfied at any stage $s_3 > \max(s_2,n)$ then for all stages $s > s_3$ $<p,n>$ will have a $\checkmark$ and no x in $T_i^s$ .

Thus $<p,n> \in T_i^s(B^s)$ since $<p,n>$ has a $\checkmark$ in $T_i$ and $n \in B^s$, and $<p,n> \notin W_q^s(\bigvee_{j \neq 1}(D(j,p,s) \cup Z_p))$ since all the $D(j,p,s)$ have stabilized. Thus $<p,n>$ will satisfy $R_p$ for all stages $s > s_3$.

Suppose, however, that $<p,n> \notin B^* \& <p,n> \in C^*$. Since $<p,n> \in C^*$

$$(\exists <<p,n>,u> \in W_q)(D_u \subset (\bigvee_{j \neq i}(D(j,p,s_1) \cup Z_p)))$$

Let $s_3$ be such that $<<p,n>,u> \in W_p^{s_3}$.

If $x \in D_u$ then $x$ is **formed from** $<t_j,r_j>$, $j \neq i$ such that

$$<t_j,r_j> \in D(j,p,s_1) \cup Z_p \quad .$$

By the coding $t_j < u$ and $r_j < u$. Thus if $R_p$ is unsatisfied at some stage $s' > u$ and if $t_j \geq p$, then $<t_j,r_j>$ will receive an $x$ at stage $s'$. Thus if $s > \max(s', s_1)$

$$<t_j,r_j> \in D(j,p,s) \cup T_j^s(\phi) \quad .$$

It follows that if $R_p$ is unsatisfied at any stage $s_4$ such that $s_4 > \max(u,n,s_2,s_3)$ then $\forall s > s_4$

$$D_u \subset \bigvee_{j \neq i}(D(j,p,s) \cup T_j^s(\phi))$$

hence

$$<p,n> \in W_q^s(\bigvee_{j \neq 1}(D(j,p,s) \cup T_j^s(\phi)))$$

also

$$<p,n> \notin T_i^s(B^s)$$

Thus $R_p$ would be satisfied by $<\rho,n>$ for all $s > s_4$ which completes the proof of Lemma 1. □

<u>Proof of Lemma 2</u>. Suppose $p = mq+i$ and that $R_0, R_1, \ldots, R_p$ are all satisfied for all stages $s \geq s_0$. Then $<p,n>$ can satisfy $R_p$ only if $n < s_0$, since if $n \geq s_0$, $<p,n>$ can't have a $\surd$. Let $s_1 > s_0$ be large enough such that $\forall s \geq s_1$ all the $D(j,p,s)$ are stable and

$$(n < s_0 \Rightarrow (n \in B^s \Leftrightarrow n \in B)) \quad .$$

If at any stage $s_2 > s_1$ some $<p,n>$ satisfies $R_p$ by ** then $\forall s > s_2$ $<p,n>$ will satisfy $R_p$ by ** since

$$W_q^s \Big( \bigvee_{j \neq i} (D(j,p,s) \cup T_j^s(\phi)\Big)$$

is strictly increasing in $s$ if $s > s_1$.

So suppose at no stage $s > s_1$ does any $<p,n>$ satisfy $R_p$ by **. Then for all stages $s > s_1$ $R_p$ is satisfied by * . But if such a $<p,n>$ ever ceases to satisfy $R_p$ at some stage $s > s_1$ then it can never again satisfy $R_p$. Thus, since only a finite number of $<p,n>$ can possibly satisfy $R_p$ (namely those with $n < s_0$), at least one $<p,n>$ must satisfy $R_p$ for all $s > s_1$.

This completes the proof of Lemma 2 and hence of the main theorem. □

THEOREM. (Countable Embedding Theorem) Let $\leq_F \subseteq N \times N$ be any countable partial ordering with the property that for each $i$ in the domain of $F$ $\{j \mid j \leq_F i\}$ is r.e. Then if there exists a set $B \in \underline{b}$ and $\underline{b} > \underline{0}$ such that $B \leq_T \underline{0}'$ then $F$ can be embedded below $\underline{b}$.

Proof. This proof closely parallels the proof of the Finite Embedding Theorem. We shall construct enumeration operators $T_0, T_1, T_2, \ldots$ which have the property that for all $i$,

$$T_i(B) \not\leq \bigvee_{j \neq i} T_j(B)$$

where the join on the right is now infinite.

Requirement $R_p$ where $p = \langle q,i \rangle$ will insure that

$$T_i(B) \neq W_q\left( \bigvee_{j \neq i} T_j(B)\right) .$$

The construction of the $T_i$ and the proof that they have the desired properties is the same in the finite case except that $p = mq + i$ is everywhere replaced by $p = \langle q,i \rangle$.

We are given that if $i$ is in the domain of $F$ then $\{j \mid j \leq_F i\}$ is r.e. For such $i$ let $f_i$ be a recursive function which enumerates $\{j \mid j \leq_F i\}$, and define

$$S_i(X) = \bigvee_{n \in N} T_{f_i(n)}(X)$$

Then $S_i$ is r. e. By projection and the fact that $f_i$ is recursive (and hence we can effectively find some n such that $f_i(n) = i$) we will have that for all X

$$T_i(X) \leq S_i(X)$$

and if $i \not\leq_F j$ then, again because $f_i$ is recursive,

$$S_j(X) \leq \bigvee_{j \neq i} T_j(X)$$

thus

$$\phi < S_i(B) < B$$

and

$$i <_F j \Leftrightarrow S_i(B) < S_j(B)$$

which completes the proof. □

COROLLARY 1. <u>Any recursively enumerable partial ordering can be embedded below any degree</u> $\underline{b} > \underline{0}$ <u>if</u> $(\exists B \in \underline{b})(B \leq_T 0')$ .

COROLLARY 2 . <u>There exist countably many incomparable degrees below any degree</u> $\underline{b} > \underline{0}$ if $(\exists B \in \underline{b})(B \leq_T 0')$.

The partial ordering F does not have to be r. e. in which case the domain of F will be a non r. e. set. The following corollary gives a good example.

COROLLARY 3. <u>The partial ordering</u> $<_O$ <u>can be embedded below any degree</u> $\underline{b} > \underline{0}$ <u>if</u> $\exists B \in \underline{b}\ (B \leq 0')$. <u>Where</u> O <u>is the usual universal system of notation for the constructive ordinals, see</u>

Rogers [1967] p. 208.

The question naturally arises as to whether the previous theorem can be relativised, i.e. can a partial ordering be embedded between two degrees $\underline{a}$ and $\underline{b}$ where $\underline{a} < \underline{b}$ and $(\exists A \in \underline{a})(\exists B \in \underline{b})(B \leq_T A')$. The answer is yes if either $\underline{a}$ or $\underline{b}$ is a total degree. The case where $\underline{b}$ is a total degree will be treated later, for if $\underline{b}$ is total then the restriction that $B \leq_T 0'$ is no longer necessary.

THEOREM (Relativised Countable Partial Ordering Embedding Theorem). Let $F$ be any countable partial ordering with the property that if $i$ is in the domain of $F$ then $\{j \mid j \leq_F i\}$ is r.e. If $\underline{a}$ is a total degree, $\underline{a} < \underline{b}$, and $(\exists A \in \underline{a})(\exists B \in b)$ ($A$ is the diagram of a total function & $B \leq_T A'$) then $F$ can be embedded between $\underline{a}$ and $\underline{b}$.

Proof. Suppose $A \in \underline{a}$ is the diagram of the total function $f$ i.e. $A = \{<x,f(x)> \mid x \in N\}$ then define $A^s = \{<x,f(x)> \mid x < s\}$. Given any enumeration of $A$ we can produce the $A^s$. Also note that for the given $B \in \underline{b}$ such that $B \leq_T A'$ there exists an approximation $B^s$ to $B$ which will be recursive in $A$.

In this proof the $T_i$ will be constructed to have the property that the requirement $R_p$ where $p = <q,i>$ will insure that

$$T_i(B) \neq W_q(A \vee (\bigvee_{j \neq i} T_j(B))) \quad .$$

$R_p$, where $p = <q,i>$, is satisfied at stage $s$ if there exists a $<p,n>$ which has a $\surd$ but no $x$ in $T_i^s$ and

$$<p,n> \in T_i^s(B^s) \ \& <p,n> \notin W_q^s(A^s \vee (\bigvee_{j\neq i} (D(j,p,s) \cup Z_p)))$$

or

$$<p,n> \notin T_i^s(B^s) \ \& <p,n> \in W_q^s(A^s \vee (\bigvee_{j\neq i} (D(j,p,s) \cup T_j^s(\phi)))) .$$

The construction of the $T_i$ then proceeds in exactly the same way as in the previous proof.

The proofs of Lemma 1 and Lemma 2 are basically the same. The only difference is that now

$$C^* = \{<p,n> | \ n > s_0 \ \& <p,n> \in W_q(A \vee (\bigvee_{j\neq i} (D(j,p,s) \cup Z_p)))\} .$$

We will have $C^* \neq B^*$ since $C^* \leq A < B \equiv B^*$. The rest of the proof of the lemmas is essentially unchanged.

The $T_i$'s will no longer be r.e. However, given any enumeration of $A$ we can produce the $A^s$ and hence we can enumerate the $T_i$. Thus the $T_i$ are enumeration reducible to $A$ and $T_i = U_i(A)$ where $U_i$ is r.e. Also, $A < B$ so $T_i = U_i'(B)$ and hence $T_i(B) = (U_i'(B))(B) = U_i''(B)$ where

$$U_i'' = \{<x,u> | \ <<x,v>,v'> \in U_i' \ \& D_u = D_v \cup D_{v'}\}$$

For each $i$ in the domain of $F$, $S_i$ is now defined by

$$S_i(X) = U(X) \vee (\bigvee_{n \in N} U''_{f_i(n)}(X))$$

where $U$ is such that $U(B) = A$.

Then for all $i$ and $j$

$$A < S_i(B) < B$$

$$i <_F j \Leftrightarrow S_i(B) < S_j(B)$$

which completes the theorem. □

The question now immediately arises as to why $\underline{a}$ is restricted to being a total degree. If $(\exists A \in \underline{a})(A \leq_T 0')$ why cannot any recursive approximation $A^s$ to $A$ be used. The reason is that we cannot have $\forall s\,(A^s \subset A)$ since this would imply that $A$ is r.e. and hence there is no reason to suppose that

$$W_q^s(A^s \vee (\bigvee_{j \neq i} (D(j,p,s) \cup T_j^s(\phi))) \subset W_q(A \vee (\bigvee_{j \neq i} T_j(B)))$$

which is an essential assumption in the proof that the method works.

One way to get around this problem would be to use any given enumeration of $A$ letting $A^s$ be the first $s$ elements enumerated. Then the proof that

$$T_j(B) \neq W_q(A \vee (\bigvee_{j \neq i} T_i(B)))$$

will go through. The trouble in this case is that the $T_j$ will depend on the order in which $A$ is enumerated. But an essential property of enumeration operators is that the set enumerated as output does not depend on the order in which the input is enumerated. Thus there is no reason why $T_j \leq A$ and hence the proof does not work. This is the mistake that Gutteridge made in his proof that the enumeration degrees are dense.

These comments and the following theorem help to illustrate the strong difference between total and non-total degrees.

THEOREM (Countable Embedding Theorem for Total Degrees). *Let $\leq_F \subset N \times N$ be any countable partial ordering such that each $i$ in the domain of $F$ $\{j \mid j \leq_F i\}$ is r.e. If $\mathbf{b}$ is a total degree and $\mathbf{a} < \mathbf{b}$ then $F$ can be embedded between $\mathbf{a}$ and $\mathbf{b}$.*

*Proof.* Choose $B \in \mathbf{b}$ such that $B$ is the diagram of a total function $f$. Let $B^s = \{<x, f(x)> \mid x < s\}$ and let $A^s = U^s(B^s)$, where $U(B) = A$.

The construction of the $T_i$ is exactly the same as in the previous proof. The only difference is that now each $T_i$ is not reducible to $A$. The $T_i$ are, however, still reducible to $B$ and this is the only fact that we used before. Thus the $S_i$ are constructed as before and the same proof goes through. □

THEOREM. There is no minimal enumeration degree.

Proof. The first proof of this theorem was given by Lance Gutteridge [1971]. His proof uses game theoretic techniques. The proof given here is essentially based on that proof but is a more conventional priority method proof.

We shall construct a r.e. set $T$ such that

$$\forall X(\phi < T(X) < X \vee (X \leq_T 0')) .$$

This will complete the proof since we know, by the Finite Embedding Theorem that if $X \leq_T 0'$ then $X$ cannot be minimal.

The intuitive idea behind the proof is that $T$ will be constructed to have the two properties: 1) if $T(X) = W_p$, where $W_p$ is r.e., then $X \leq_T 0'$. (This will insure that $\phi < T(X)$ or $X \leq_T 0'$); 2) for all $q$ the only solution to $W_q T(X) = X$ will be r.e. (this will insure that if $X > \phi$ then $T(X) < X$).

The first property is achieved by having for each $n$ some $p$ such that $<p,n>$ has a $\checkmark$ and no x in $T$. If this is the case and $T(X) = W_q$ then to decide if $n \in X$ we first find some $<p,n>$ in $T$ with a $\checkmark$ and no x (this can be done effectively given an oracle for $0'$ since $T \leq_T 0'$). Next we see if $<p,n> \in W_q$. This tells us if $n \in x$ since $<p,n>$ has a $\checkmark$ but no x and thus $n \in x \Leftrightarrow <p,n> \in T(X) = W_q$. (again this can be done given an oracle for $0'$ since $W_q \leq_T 0'$.)

The second property is acheived by insuring that if $W_q T(X) = X$ then for all $n$

$$n \in X \Leftrightarrow n \in (W_q T)^{n+1}(X_q) \qquad (***)$$

where

$$X_q = \{x \in X \mid x < q\}$$

and

$$(W_q T)^0(Y) = Y$$

$$(W_q T)^1(Y) = W_q T(Y) \cup Y$$

$$(W_q T)^{n+1}(Y) = W_q T((W_q T)^n(Y)) \cup (W_q T)^n(Y)$$

$$(W_q T)^{\infty}(Y) = \bigcup_{n \in N} (W_q T)^n(Y)$$

If $Y$ is r.e. then for all $n$ $(W_q T)^n(Y)$ is also r.e. as is $(W_q T)^{\infty}(Y)$. Also $X = W_q T(X)$ and *** implies $X = (W_q T)^{\infty}(X_q)$. Hence if $X = W_p T(X)$ then $X$ is r.e.

The construction of $T$ proceeds in stages.

Stage 0 : $T^0 = \{\langle\langle 0,0\rangle, \{0\}\rangle\}$, i.e. $\langle 0,0\rangle$ has a √ .

Stage s : Give $\langle 0,s\rangle$ a √ . For each $\langle n,u\rangle \in W_q^s$, where $q < s$, and each $F \subseteq q$ such that the following four conditions are satisfied

i) $D_u \subseteq T^s(N)$ i.e. all the members of $D_u$ have a √ or an x in $T^s$.

ii) $\langle p,m\rangle \in D_u$ & $m < q \Rightarrow (\langle p,m\rangle$ has an x in $T^s$ or $m \in F$ or $m$ is $\langle q,F_m\rangle$ labelled where $F_m \subseteq F)$

iii) $\langle p, m\rangle \in D_u \,\&\, q \leq m < n \Rightarrow$ (m is $\langle q, F_m\rangle$ labelled where $F_m \subseteq F$).

iv) n is not already $\langle q, F\rangle$ labelled.

Do the following: $\langle q, F\rangle$ label n, and for each $\langle p, m\rangle \in D_u$ such that $m \geq \max(q, n)$ and $\langle p, m\rangle$ does not have an x in $T^s$, give $\langle p, m\rangle$ an x in $T^s$, and give $\langle p+1, m\rangle$ a ✓.

Claim: For each m only finitely many $\langle p, m\rangle$ are given an x and hence there will exist a $\langle p, m\rangle$ in T with a ✓ but no x. This insures that T will have the first property.

Proof of claim. $\langle p, m\rangle$ can receive an x only if some n receives a $\langle q, F\rangle$ label where $q \leq m$ and $n \leq m$. But for each q there are only finitely many different $\langle q, F\rangle$ labels since $F \subseteq q$. Once n is $\langle q, F\rangle$ labelled, it is never again $\langle q, F\rangle$ labelled with the same F, thus for a fixed m, only finitely many $\langle p, m\rangle$ are given an x.

We prove that T has the second desired property in two steps. First we will show that [$n \in F$ or n is $\langle q, F\rangle$ labelled] if and only if $n \in (W_q T)^{n+q}(F)$. Then we show that if $W_q T(X) = X$ then n is $\langle q, X_q\rangle$ labelled if and only if $n \in X$. Together this implies that if $W_q T(X) = X$ then $X = (W_q T)^{\infty}(X_q)$ and so X is r.e.

Given q we want to show that for all $F \subset q$ [$n \in F$ or n is $\langle q, F\rangle$ labelled] $\Leftrightarrow n \in (W_q T)^{n+q}(F)$.

Define the rank of a $\langle q, F\rangle$ label on $n$ as follows. If when $n$ was given a $\langle q, F\rangle$ label condition iii) or the second part of condition ii) was not used then the rank of the $\langle q, F\rangle$ label on $n$ is zero. If condition iii) or the second part of condition ii) was used then the rank of the $\langle q, F\rangle$ label on $n$ is one plus the maximum rank of the $\langle q, F_m\rangle$ label on $m$ where $\langle p, m\rangle \in D_u$ satisfies condition iii) or the second part of condition ii). Note that the rank of a $\langle q, F\rangle$ label on $n$ is $\leq \max(q, n) \leq n + q$.

Fix q. $n \in F \Leftrightarrow n \in (W_q T)^0(F)$. Suppose that for all $F \subset q$ $[n \in F$ or $(n'$ is $\langle q, F\rangle$ labelled and rank of $\langle q, F\rangle$ label on $n'$ is $< r)] \Rightarrow n' \in (W_q T)^r(F)$.

Suppose $n$ is $\langle p, F\rangle$ labelled and the rank of the $\langle p, F\rangle$ label on $n$ is $= r$. If $n$ was $\langle p, F\rangle$ labelled using $\langle n, u\rangle \in W_q^s$ then if $\langle p, m\rangle \in D_u$

we have four cases

i) $<p,m>$ has an $x$ in $T^s$ then $<p,m> \in T(\phi)$ so $<p,m> \in T(F)$

ii) $<p,m>$ has $m < q$ and $m \in F$ then $<p,m> \in T(F)$

iii) $<p,m>$ has $m < \max(q,n)$ and $m$ is $<q,F_m>$ labelled and $F_m \subset F$. Then the rank of the $<q, F_m>$ label on $m$ is is $<r$ so by the induction hypothesis $m \in (W_qT)^r(F_m)$ and hence $<p, m> \in T(W_qT)^r(F)$.

iv) $<p,m>$ has $m \geq \max(q,n)$ then $<p,m>$ was given an $x$ at stage $s$ and hence $<p,m> \in T(\phi)$ so $<p,m> \in T(F)$.

For all $m < r$ $(W_qT)\ (F) \subseteq (W_qT)^r(F)$ and thus $D_u \subseteq T((W_qT^n(F))$ which implies that $n \in (W_qT)^{r+1}(F)$.

Thus $n <q, F>$ labelled $\Rightarrow n \in (W_q T)^{n+q}(F)$.

If $n \in (W_qT)^{n+1}(F)$ then $n \in F$ or there exists $<n,u> \in W_q$ such that $D_u \subseteq T((W_qT)^n(F))$. Choose an $s$ large enough such that $<n,u> \in W_q^s$, $D_u \subseteq T^s(N)$ and for all $<p,m> \in D_u$ where $m < \max(q,n)$, either $<p,m>$ has an $x$ in $T^s$, or $m \in F$ or $m$ is $<q,F_m>$ labelled at stage $s$, where $F_m \subseteq F$. Then either $n$ is already $<q,F>$ labelled at stage $s$ or else $n$ will be $<q,F>$ labelled at stage $s$, and thus $n$ is $<q,F>$ labelled at some stage.

Now suppose $W_qT(X) = X$ and $n$ is $<q,X_q>$ labelled and $\forall n' < n$ we have $n' \in X \Leftrightarrow n'$ is $<q,X_q>$ labelled. If $n$ was caused to be $X_q$ labelled by $<n,u> \in W_q^s$ at stage $s$, then for all $<p,m> \in D_u$ we will have $<p,m>$ has an $x$ or $m \in X_p$ or $m < \max(q,n)$ and $m$ is $<q,X_q>$ labelled. This means that $D_u \subset T(X)$ so $n \in W_qT(X)$. Similarly if $n \in X$ and

$W_q T(X) = X$ then by induction $n$ must be $<q, X_q>$ labelled.

Therefore, if $W_q T(X) = X$ then $X = (W_q T)^{\infty} (X_q)$ and $X$ is r. e. This completes the proof that $T$ has the second desired property which completes the proof of the theorem.□

The previous result can be relativised to the following extent.

THEOREM <u>If $\underline{a}$ is a total degree then there is no minimal degree above $\underline{a}$</u>.

<u>Sketch of Proof</u>. Let $A \in \underline{a}$ where $A$ is the diagram of a total function $f$, i. e. $A = \{<x,f(x)> | x \in N\}$. If $A < B$ and $B \leq_T A'$ then by the Relativised Countable Embedding Theorem $B$ cannot be minimal above $A$.

A set $T$ is constructed as in the previous proof which will have the property that if $A < B$ and $B \not\leq_T A'$ then $A < A \vee T(B) < B$. As before $T$ will have two properties. First, for all $m$ there is a $<p,m>$ which has a $\surd$ but no $x$ in $T$. Second if $W_p(A)T(Y) = Y$ then $Y \leq A$.

The construction of the $T$ is the same as in the previous proof except that $W_q^s$ is everywhere replaced by $W_q^s(A^s)$ where $A^s = \{ <x,f(x)> | x < s\}$. $T$ is thus r. e. in any enumeration of

$A$ so $T = U(A) = U'(B)$ and $T(B) = U''(B)$ where $U, U'$ and $U''$ are r.e. If $U''(B) \leq A$ then since $T \leq A$ and $T$ has the first desired property we must have that $B \leq_T A'$. Therefore if $A < B$ and $B \not\leq_T A'$ then $A < A \vee U''(B) < B$. We will have $A \vee U''(B) < B$ since for each $p$ there exists a $p'$ and for each $p'$ there exists a $p$ such that $W_p(A \vee U''(B)) = W_p(A \vee T(B)) = W_{p'}(A)\ T(B)$.

# Chapter 3

## Some Measure Results in the Enumeration Reducibility

John Case, in his Ph. D. thesis, Case [1971] conjectured that the measure of the sets which are incomparable with their complements is one. The author devised the following proof which was his first result in recursion theory.

Notation: Throughout this chapter f and g will denote characteristic functions. $\hat{f}$ and $\hat{g}$ will denote initial segments of characteristic functions. $\ell(\hat{f})$ will denote the length of $\hat{f}$. C will denote the set of all characteristic functions. In a slight abuse of notation, sets $A \subset N$ will be used where strictly speaking we are talking about their associated characteristic functions. If any confusion might arise, $A_f$ or $f_A$ will be used where

$$A_f = \{x \mid f(x) = 1\} \quad \text{and} \quad f_A(x) = 1 \Leftrightarrow x \in A \quad .$$

C is made into a measure space by associating the equal-probable measure $\mu$, i.e. $\mu(\{0\}) = 1/2$ and $\mu(\{1\}) = 1/2$, with $\{0,1\}$ and then taking the product measure on C which we also call $\mu$. This measure is often called the probability measure on C because $\mu(\mathcal{A})$ is the probability that a random sequence of 0's and 1 is in $\mathcal{A}$. Thus we use the notation

$$Pr[f \in \mathcal{A}] = \mu(\mathcal{A})$$

and if $\mu(\mathcal{B}) \neq 0$ we define the conditional probability of $\mathcal{A}$ given $\mathcal{B}$

$$\Pr[f \in \mathfrak{G} \mid f \in \mathfrak{B}] = \mu(\mathfrak{G} \cap \mathfrak{B})/\mu(\mathfrak{B}) \quad .$$

THEOREM (Case's Conjecture I) <u>The measure of the sets incomparable with their complements is equal to one, i. e.</u> $\Pr[A\,|_e\,\overline{A}] = 1$ .

<u>Proof.</u> Suppose not. Then by the countable additivity of the measure there must exist some recursively enumerable set $W$ such that

$$\Pr[\,\overline{A} = W(A)\,] = \epsilon > 0 \quad .$$

Define

$$P_n(A) \equiv (\exists m)(\forall x)(m > n \,\&(x < n \Rightarrow (x \in \overline{A} \Leftrightarrow x \in W^m(A^m)))) \quad .$$

Then $\overline{A} = W(A) \Leftrightarrow \forall n\; P_n(A)$ for if $(\exists x)(x \in \overline{A} \;\& x \notin W(A))$ then for all $n > x$ $P_n(A)$ cannot hold and if $\exists x(x \notin \overline{A} \;\& x \in W(A))$ then $\exists m(x \in W^m(A^m))$ and thus for all $n \geq m$ $P_n(A)$ cannot hold. Thus for all $n$

$$\Pr[\overline{A} = W(A)] = \Pr[P_n(A)] \cdot \Pr[\overline{A} = W(A) \mid P_n(A)]$$

and $\bigcap_{n=0}^{\infty} \{A \mid P_n(A)\} = \{A \mid \overline{A} = W(A)\}$ . This implies

$$\lim_{n \to \infty} \Pr[P_n(A)] = \epsilon$$

and hence

$$\lim_{n \to \infty} \Pr[\overline{A} = W(A) \mid P_n(A)] = 1$$

Choose $n_0$ such that $n \geq n_0$ implies

$$Pr[\bar{A} = W(A) \mid P_n(A)] > 7/8$$

Let $\mathfrak{S}$ be the set of initial segments $\hat{f}$ such that

$$\forall x < n_0 \, (x \in \overline{A_{\hat{f}}} \Leftrightarrow x \in W^{\ell(\hat{f})}(A_{\hat{f}}^{\ell(\hat{f})}))$$

and for no proper initial segment of $\hat{f}$ does the above equation hold. Then

$$P_n(A) \Leftrightarrow (\exists \hat{f})(\hat{f} \in \mathfrak{S} \,\&\, \hat{f} \subset f_A)$$

Define $P_1(\hat{f}) = Pr[f \supset \hat{f}]$ and $P_2(\hat{f}) = Pr[\bar{A} = W(A) \mid f_A \supset \hat{f}]$ .
Then

$$Pr[P_n(A)] = \sum_{\hat{f} \in \mathfrak{S}} P_1(\hat{f})$$

$$Pr[\bar{A} = W(A)] = \sum_{\hat{f} \in \mathfrak{S}} P_1(\hat{f}) \cdot P_2(\hat{f})$$

and

$$Pr[\bar{A} = W(A) \mid P_{n_0}(A)] = (\sum_{\hat{f} \in \mathfrak{S}} P_1(\hat{f}) \cdot P_2(\hat{f})) / \sum_{\hat{f} \in \mathfrak{S}} P_1(\hat{f}) > 7/8$$

This last expression is a weighted average so there must exist an $f^* \in \mathfrak{S}$ such that

$$P_2[f^*] = Pr[\bar{A} = W(A) \mid f_A \supset f^*] > 7/8$$

Choose any $x > \ell(f^*)$ , then

$$Pr[x \in A \mid f_A \supset f^*] = 1/2$$

so

$$\Pr[\bar{A} = W(A) \,\&\, x \in A \mid f_A \supset f^*] \cdot 1/2 + \Pr[\bar{A} = W(A) \,\&\, x \notin A \mid f_A \supset f^*] \cdot 1/2 > 7/8$$

hence

$$\Pr[\bar{A} = W(A) \,\&\, x \in A \mid f_A \supset f^*] > 3/4 \quad .$$

Therefore

$$\Pr[x \notin W(A) \mid f_A \supset f^*] > 3/4$$

but also

$$\Pr[\bar{A} = W(A) \,\&\, x \notin A \mid f_A \supset f^*] > 3/4 \quad .$$

Therefore

$$\Pr[x \in W(A) \mid f_A \supset f^*] > 3/4$$

is a contradiction. Thus we must have $\Pr[\bar{A} = W(A)] = 0$ and hence $\Pr[A \mid_e \bar{A}] = 1$ . ◻

After the author had communicated this result to Dr. Case, Dr. Case devised a new and more difficult conjecture. Once again Case has conjectured correctly. It will be interesting to see what his next conjecture will be.

THEOREM (Case's Conjecture II) <u>The measure of the quasi-minimal sets is one. Where a set</u> $A$ <u>is quasi-minimal if whenever</u> $g \leq A$ , <u>where</u> $g$ <u>is a total function, then</u> $g$ <u>is recursive.</u>

<u>Proof.</u> A set is said to be a total function if it is the diagram of

a total function.

Suppose the measure of the quasiminimal sets is not equal to one. Then there exists a recursively enumerable set $W$ such that

$$\Pr[W(A) \text{ is a t.n.r.f.}] = \varepsilon > 0$$

where "t.n.r.f." and "t.n.r.0-1f." shall be abbreviations for "total non-recursive function" and "total nonrecursive 0-1 function" respectively. Let $W'$ be the recursively enumerable set which as an enumeration operator when given $<n,m>$ as input gives output $<<n,m>,1>$ and $<<n,m'>0>$ for all $m' \neq m$. Then clearly $W'(B)$ is a t.n.r.0-1f. $\Leftrightarrow$ $B$ is a t.n.r.f. .

Let $T$ be such that for all $X$

$$T(X) = W'W(X) \quad .$$

Then

$$\Pr[T(A) \text{ is a t.n.r.0-1f.}] = \varepsilon > 0$$

By the countable additivity of the measure there exist recursive functions $f_0, \ldots, f_{i'-1}$ such that

$$\Pr[T(A) \text{ is a recursive total function and } (\forall i < i')(T(A) \neq f_i)] < \varepsilon/20$$

Define $S_n$ to be the set of initial segments $\hat{f}$ such that

i) $T^{\ell(\hat{f})}(A_{\hat{f}})$ is a total function up to $n$, i.e. $\forall x < n\, \exists!\, y\, (<x,y> \in T^{\ell(\hat{f})}(A_{\hat{f}}))$ .

ii) $T^{\ell(\hat{f})}(A_{\hat{f}}) \not\subseteq f_i$ all $i < i'$

iii) i) and ii) do not hold for any proper initial segment of $\hat{f}$ .

We say a set $A$ has property $P_n$ , i.e. $P_n(A)$ , if $\exists \hat{f} \in S_n(\hat{f} \subset f_A)$ .

Then

$$P_n(A) \Rightarrow \forall n' < n \quad P_{n'}(A)$$

$$\sim P_n(A) \Rightarrow \forall n' > n \quad \sim P_{n'}(A)$$

and

$$Q(A) \equiv \forall n\ P_n(A) \Leftrightarrow ((T(A) \text{ is a t. n. r. f.}) \vee (T(A) \text{ is a total recursive function } \& \ \forall i < i' \ (T(A) \neq f_i)))$$

Therefore $Pr[Q(A)] = Pr[Q(A) | P_n(A)] \cdot Pr[P_n(A)]$ and since $Pr[Q(A)] \geq \epsilon > 0$ and

$$\lim_{n \to \infty} Pr[P_n(A)] = Pr[Q(A)]$$

we have

$$\lim_{n \to \infty} Pr[Q(A) | P_n(A)] = 1 \ .$$

Choose $n_0$ such that $Pr[Q(A) | P_{n_0}(A)] \geq .9$ . Now

$$Pr[T(A) \text{ is a t. n. r. f. } | Q(A)] \geq 20/21$$

and thus since $Q(A) \Rightarrow P_{n_0}(A)$

$$Pr[T(A) \text{ is a t. n. r. f. } | P_{n_0}(A)] > .8 \ .$$

By the same reasoning as in the previous proof, there must exist some particular $f^* \in S_{n_0}$ such that

$$\Pr[T(A) \text{ is a t.n.r.f.} \mid f_A \supset f^*] > .8 \quad .$$

Let $p_i, q_i$ and $\delta_i$ be such that

$$p_i = \Pr[<i, \delta_i > \in T(A) \mid f_A \supset f^*] \geq$$

$$q_i = \Pr[<i, (1-\delta_i) > \in T(A) \mid f_A \supset f^*]$$

Note that by the choice of $T$ $\delta_i$ will always be $0$ or $1$.

The $p_i, q_i$ and $\delta_i$ are well-defined for all $i$ and can be found recursively since

$$.9 \leq \Pr[T(A) \text{ is a well-defined total function } \mid f_A \supset f^*]$$

$$\leq \Pr[T(A) \text{ is well-defined at } i \mid f_A \supset f^*]$$

$$\leq \Pr[<i, 0> \in T(A) \;\&\; <i, 1> \notin T(A) \mid f_A \supset f^*]$$
$$+ \Pr[<i, 0> \notin T(A) \;\&\; <i, 1> \in T(A) \mid f_A \supset f^*]$$

$$\leq p_i (1-q_i) + q_i (1-p_i) = p_i + q_i - 2p_i q_i \quad .$$

The last inequality holds because

$$\Pr[<i, \delta_i> \in T(A) \wedge <i, 1-\delta_i > \notin T(A) \mid f_A \supset f^*]$$

$$= \Pr[<i, \delta_i > \in T(A) \mid f_A \supset f^*] \; .$$

$$\Pr[<i, 1-\delta_i > \notin T(A) \quad \mid <i, \delta_i > \in T(A) \;\&\; f_A \supset f^*]$$

$$=\Pr[\langle i, \delta_i\rangle \in T(A) \mid f_A \supset f^*]\cdot(1-\Pr[\langle i, 1-\delta_i\rangle \in T(A) \mid \langle i, \delta_i\rangle \in T(A) \;\&\; f_A \supset f^*]$$

and

$$\Pr[\langle i, 1-\delta_i\rangle \in T(A) \mid \langle i, \delta_i\rangle \in T(A) \;\&\; f_A \supset f^*]$$

$$\geq \Pr[\langle i, 1-\delta_i\rangle \in T(A) \mid f_A \supset f^*] = q_i \quad .$$

**This final** inequality holds because $T$ is an enumeration operator. The condition $\langle i, \delta_i\rangle \in T(A)$ only forces $A$ to contain more members which by the monotonicity of enumeration operators makes it more likely that $\langle i, 1-\delta_i\rangle \in T(A)$.

So, by elementary calculus, $p_i \geq .9$ and $q_i \leq .1$. Define $g = \{\langle i, \delta_i\rangle \mid i \in N\}$, then $g$ is a total recursive function.

Let $p_n^* = \Pr[\, i \leq n \Rightarrow \langle i, \delta_i\rangle \in T(A) \mid f_A \supset f^*]$. Then

$$.9 \leq \Pr[T(A) \text{ is a well-defined total function } \mid f_A \supset f^*]$$

$$\leq \Pr[T(A) \text{ is well-defined for all } i \leq n \mid f_A \supset f^*]$$

$$\leq \Pr[i \leq n \Rightarrow (\langle i, \delta_i\rangle \in T(A) \;\&\; \langle i, 1-\delta_i\rangle \notin T(A) \mid f_A \supset f^*]$$
$$+ \sum_{i\leq n} \Pr[\langle i, 1-\delta_i\rangle \in T(A) \;\&\; \langle i, \delta_i\rangle \notin T(A) \mid f_A \supset f^*]$$

$$\leq \Pr[i \leq n \Rightarrow \langle i, \delta_i\rangle \in T(A) \mid f_A \supset f^*]\cdot$$
$$\Pr[i \leq n \Rightarrow \langle i, \delta_i\rangle \notin T(A) \mid i \leq n \Rightarrow \langle i, \delta_i\rangle \in T(A) \;\&\; f_A \supset f^*]$$

$$+ \sum_{i\leq n}(\Pr[\langle i, 1-\delta_i\rangle \in T(A) \mid f_A \supset f^*]\cdot \Pr[\langle i, \delta_i\rangle \notin T(A) \mid \langle i, 1-\delta_i\rangle \in T(A) \;\&\; f_A \supset f^*])$$

$$\leq p_n^* (1 - \sum_{i \leq n} q_i) + \sum_{i \leq n} (q_i \cdot (1 - p_n^*))$$

$$\leq p_n^* + \sum_{1 \leq n} q_i - 2 \cdot p_n^* \cdot \sum_{i \leq n} q_i \quad .$$

So as before either $[p_n^* \geq .9$ and $\sum_{i<n} q_i \leq .1]$ or $[p_n^* \leq .1$ and $\sum_{i<n} q_i \geq .9]$. But $p_0^* = p_0 \geq .9$ and for all $n$ $p_n^* \geq p_{n-1}^* \cdot p_n$ and thus by induction $p_n^* \geq .9$.

Thus

$Pr[T(A) = g | f_A \supset f^*] \geq Pr[\forall i <i, \delta_i> \in T(A) | f_A \supset f^*] \cdot$

$Pr[T(A)$ is a well-defined total function $| f_A \supset f^*] \geq .9 \cdot .9 > .8$ .

Since $g$ is recursive this implies

$Pr[T(A)$ is a t.n.r.f. $| f_A \supset f^* ] < .2$ .

Therefore

$$.8 < \Pr[T(A) \text{ is a t.n.r.f.} \mid f_A \supset f^*] \leq .2$$

which is a contradiction, and it follows that the measure of the quasi-minimal degrees is one. ⊓

A different way to see that "most" sets are quasi-minimal is through the concept of category. See Rogers [1967] p. 271. C is made into a complete metric space by giving it the usual Cantor set topology.

THEOREM. <u>The quasi-minimal sets, and hence degrees, are of second category in Cantor space.</u>

<u>Proof.</u> We want to show that

$$\{A \mid \exists e(W_e(A) \text{ is a t.n.r.f})\}$$

is of first category in C. It is sufficient to show that for any given r.e. set W, $\mathfrak{B} = \{A \mid W(A) \text{ is a t.n.r.f.}\}$ is nondense. Suppose not, then there is some spherical neighborhood $\mathfrak{S} = \{f \mid f \supset \hat{f}_0\}$ which is contained in the closure of $\mathfrak{B}$. Every neighborhood contained in $\mathfrak{S}$ must therefore intersect $\mathfrak{B}$. Thus for any $\hat{f} \supset \hat{f}_0$, $W(A_{\hat{f}})$ must be a well-defined partial function, otherwise if $f \supset \hat{f}$ then $W(A_f)$ can <u>not</u> be a well defined total function and the neighborhood $\{f \mid f \supset \hat{f}\} \subset \mathfrak{S}$ cannot intersect $\mathfrak{B}$.

Suppose $W(A_{f_1}) \neq W(A_{f_2})$ where $f_1 \supset \hat{f}_0$ $f_2 \supset \hat{f}_0$ and $f_1, f_2 \in \mathfrak{B}$. Then there exist $\hat{f}_0 \subset \hat{f}_1 \subset f_1$ and $\hat{f}_0 \subset \hat{f}_2 \subset f_2$

such that

$$(\exists n)(\exists m)(\exists m')(m \neq m' \,\&\, \langle n,m\rangle \in W(A_{\hat{f}_1}) \,\&\, \langle n,m'\rangle \in W(A_{\hat{f}_2}))$$

Let $\hat{f}$ be defined as follows

$$\hat{f}(x) = \begin{cases} 1 & \text{if } \hat{f}_1(x) = 1 \text{ or } \hat{f}_2(x) = 1 \\ 0 & \text{otherwise if } x < \max(\ell(\hat{f}_1), \ell(\hat{f}_2)) \end{cases} .$$

Then $\hat{f} \supset \hat{f}_0$ and $W(A_{\hat{f}})$ is not a well defined partial function since it contains both $\langle n,m\rangle$ and $\langle n,m'\rangle$ .

Therefore, $W(A_f)$ can have only one value if $f \in \mathfrak{S}$ and $f \in \mathfrak{B}$ , but the only value it can have is

$$W(A_{f'})$$

where

$$f'(x) = \begin{cases} \hat{f}_0(x) & \text{if } x < \ell(\hat{f}_0) \\ 1 & \text{if } x \geq \ell(\hat{f}_0) \end{cases}$$

and $W(A_{f'})$ is an r. e. set so if $W(A_f)$ is a total function then it is a recursive total function which is a contradiction. Therefore $\mathfrak{B}$ is nondense and the quasiminimal degrees are of second category. □

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## Biography

Jay John Tuthill Lagemann was born in New York on June 14, 1944. He attended the Riverdale Neighborhood House Nursery School and completed his undergradeschool education in two more years at the Riverdale School. He then attended the Fieldstone School for the next twelve years, graduating in 1962. The next four years were spent attending Princeton University, being on the Dean's List most of the time. He graduated in 1966, receiving an A. B. Magna Cum Laude and was elected to the Phi Beta Kappa Society and to associate membership in the society of the Sigma Xi. After leaving Princeton he moved to Cambridge and has attended Massachusetts Institute of Technology.

for the last five years. He expects to receive his Ph. D. in the fall of 1971. While at M. I. T. he held an N. S. F. Traineeship for four years, was a teaching assistant for one year and was elected to full membership in the Society of the Sigma Xi.

During the summers after his first two years of college, Mr. Lagemann worked at I. B. M.'s Thomas J. Watson Research Center at Yorktown Heights, New York. This work resulted in two papers. The first "Some Improvements to the VAMFO Technique for Measuring Thickness of Thin Films" was published as a Research Note #NC-315, 9/30/1963, Thomas J. Watson Research Center, Yorktown Heights. The second paper, "A Method for Solving the Transportation Problem" was published in the Naval Research Logistics Quarterly in Vol.14 [1967] pp. 89-99.

The next summer he received a National Science Foundation Grant. During the following two summers he worked in Washington, D. C. for I. B. M. 's Federal Systems Division working on project T. R. A. M. P. S. which was to devise more efficient ways to get men and materials to Vietnam, Laos, and Cambodia so that we could keep the natives from becoming communists, "Better dead than Red, if they're Yellow."

After receiving his Ph. D. Mr. Lagemann would like to: ride a twenty foot wave on a surfboard, become a rock star, solve Fermat's last theorem, be a benevolent dictator on a small island in the Caribbean, and eventually retire to the family estate on Martha's Vineyard with two dogs, two women and a pet frog.