POSITIVE DEFINITE DISTRIBUTIONS ON SEMI-SIMPLE LIE GROUPS

by

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ABSTRACT

Let G be a connected unimodular Lie group, and T a positive definite distribution on G. Then the first main result of this thesis is that T may be written as a finite sum of invariant derivatives applied to bounded functions. For G connected, semi-simple and with finite center, and K a maximal compact subgroup, we use this theorem to show that the domain of definition of positive definite distributions can be extended to include Harish-Chandra's L^1-type Schwartz space $\mathcal{C}^1(G)$.

Restricting these distributions to the space $I^1(G)$ of K-biinvariant elements in $\mathcal{C}^1(G)$ we are able to use the recent harmonic analysis results of Trombi and Varadarajan to prove a spherical Bochner theorem for positive definite distributions on G. This uses and extends certain partial spherical Bochner theorems derived by Godement.

Since $I^1(G)$ is only one member in a series of Schwartz type spaces $I^p(G)$, $0 < p \leq 2$, where $I^p(G) \subset I^{p'}(G)$ when $p < p'$, our final result characterizes which $I^p(G)$ spaces a given positive definite distribution can be extended to by examining the support of its spherical Bochner measure. This theorem includes the Spherical Bochner theorem for tempered distributions originally proved by Muta.

Thesis Supervisor: Sigurdur Helgason
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TO

BETSY

AND MY PARENTS

WILBUR AND ANNABELLE BARKER

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Introduction

Suppose T is a positive definite distribution on $\mathbb{R}^n$. Then Schwartz [14] proves the following sequence of facts about T:

(1) T is a "bounded" distribution,

(2) T is a tempered distribution, and

(3) there is a unique tempered measure on $\mathbb{R}^n$ such that $T[\phi] = \int \ldots \int_{\mathbb{R}^n} \hat{\phi}(x) d\mu(x)$ for all test functions ϕ, where $\hat{\phi}$ is the fourier transform of ϕ (the Bochner theorem).

We generalize the above situation in the following way. Suppose G is a connected, unimodular Lie group. In Chapter I we prove that any positive definite distribution T on G may be written as a finite sum of invariant derivatives applied to bounded functions. This is essentially (1) for our more general situation.

To obtain (2) is more difficult since the notion of a "rapidly decreasing function" on an arbitrary Lie group is not well defined. Hence from this point on we restrict ourselves to the case where G is connected semi-simple and has a finite center, for then Schwartz spaces do exist. The complication is the existence of more than one Schwartz space - for each $0 < p \leq 2$ there is a space $\mathcal{C}^p(G) \subset L^p(G)$, where $\mathcal{C}^p(G) \subset \mathcal{C}^{p'}(G)$ when $p \leq p'$, and where $\mathcal{C}^2(G)$ roughly "corresponds" to the usual euclidean Schwartz space $\mathcal{S}(\mathbb{R}^n)$. These spaces, and their K-biinvariant subspaces $I^p(G)$, are described in detail in Chapter III, and from these descriptions it becomes clear that each positive definite distribution on G extends to a continuous linear functional on $\mathcal{C}^1(G)$. This generalizes (2).

In the $\mathbb{R}^n$ case proving (3) from (2) involves two main facts: (a) the fourier transform $f \to \hat{f}$ gives an isomorphism of $\mathcal{S}(\mathbb{R}^n)$ onto itself, and (b) any positive continuous linear functional on $\mathcal{S}(\mathbb{R}^n)$ must be a tempered measure on $\mathbb{R}^n$. For suppose that T is positive definite on $\mathbb{R}^n$. It is not hard to show that the continuous functional $\hat{T}$ defined on $\mathcal{S}(\mathbb{R}^n)$ by $\hat{T}[\hat{f}] = T[f]$ for all $f \in \mathcal{S}(\mathbb{R}^n)$ is positive, and hence (b) gives the desired result. The first problem which arises in trying to carry this argument over to the semi-simple Lie group case lies in (a), for although it is possible to define a Fourier transform on $\mathcal{C}^p(G)$, it is as yet unknown what the image space is, and whether the mapping is injective. The only known result in this area at the moment is [20], where positive results for $\mathcal{C}^2(G)$, G of real rank one, are obtained.

Strong results of the form of (a) have, however, been obtained recently by Trombi and Varadarajan [17], for the spaces $I^p(G)$, $0 < p \leq 2$, these results generalizing earlier theorems for $p = 1$ and 2 obtained by Harish-Chandra and Helgason. Trombi and Varadarajan define spaces $\bar{Z}(\mathfrak{F}^\varepsilon)$, $\varepsilon \geq 0$, consisting of functions defined on certain sets $\mathfrak{F}^\varepsilon$ of spherical functions such that the spherical Fourier transform is a topological isomorphism of $I^p(G)$ onto $\bar{Z}(\mathfrak{F}^\varepsilon)$ for $0 < p \leq 2$ and $\varepsilon \equiv (2/p) - 1$. Thus, considering the restrictions to $I^1(G)$ of positive definite distributions, we find that (a) holds true. Of course for a general positive definite distribution T this procedure can only be hoped to yield an integral formulation for $T[\phi]$ where ϕ is K-biinvariant. It does, however, yield the full formulation when T is K-biinvariant. The $\overline{Z}(\mathfrak{F}^{\varepsilon})$ spaces are fully described, and a certain amount of geometry is performed on them, in Chapter IV.

We have not tried to show (b) for the $\overline{Z}(\mathfrak{F}^{\varepsilon})$ spaces, and indeed there is reason to believe that all positive continuous linear functionals on $\overline{Z}(\mathfrak{F}^{\varepsilon})$, $\varepsilon > 0$, are not measures (see the discussion at the end of Chapter VI). In place of this we use the partial Bochner theorem of Godement [7(b)] which, for T a positive definite distribution on G, gives the existence of a unique measure μ supported in $\mathfrak{F}^1$ such that

$$T[\phi * \psi] = \int_{\mathfrak{F}^1} \hat{\phi}\hat{\psi}\, d\mu$$

for all K-biinvariant test functions ϕ and ψ. There is no property of "temperedness" given in this theorem, and the heart of proving an analogue of (3) is in defining a notion of a measure on $\mathfrak{F}^1$ being of "polynomial growth", and then showing that the above μ

is such an object. The Godement-Bochner theorems are proved in Chapter II with somewhat different proofs than the originals and Chapter V applies them as indicated above to arrive at the following generalization of (3):

The Spherical Bochner Theorem. Suppose T is a positive definite distribution on G. Then there exists a unique W-invariant positive regular Borel measure μ of polynomial growth on $\mathcal{P}$, the set of positive definite spherical functions, such that

$$T[\phi] = \int_{\mathcal{P}} \hat{\phi} d\mu, \quad \phi \in I^1(G).$$

This correspondence is bijective when restricted to K-biinvariant distributions, in which case the above formula holds for all $\phi \in \mathcal{C}^1(G)$.

In Chapter VI we prove a result relating the highest $I^p(G)$ space that a positive definite distribution T can be extended to with the support of its spherical Bochner measure μ. The result is that $T \in (I^p(G))'$ (the continuous dual of $I^p(G)$) if and only if $\text{supp}\, \mu \subset \mathfrak{F}^\varepsilon$ where $\varepsilon = (2/p) - 1$. While this seems to be a very natural theorem in light of the Trombi-Varadarajan result, the details are surprisingly complicated. It should be noted that for $p = 2$ this result was obtained by Muta [13] in much the same way as the euclidean Bochner theorem is proved.

For the convenience of the reader we have included a rather lengthy section on notation and preliminaries, this being essentially an elaboration of the same section in [10(c)].

Notation and Preliminaries

§1, General Notation. The standard notation $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ shall be used for the ring of integers, the field of real numbers and the field of complex numbers, respectively; $\mathbb{Z}^+$ is the set of nonnegative integers, $\mathbb{R}^+$ the set of nonnegative real numbers. If S is a set, T a subset and f a function on S, the restriction of f to T is denoted $f|_T$.

If S is a topological space, then $\mathcal{Cl}(T)$ denotes the closure of T in S, Int T the interior of T, and bdry T the boundary of T. The space of continuous functions from S to $\mathbb{C}$ is denoted by C(S), $C_c(S)$ the set of those of compact support. The support of any $f \in C_c(S)$ is denoted by supp f. Let Φ be a homeomorphism of S onto itself, and let $f : S \to S$, $T : C(S) \to \mathbb{C}$ and $A : C(S) \to C(S)$. We put

$$f^{\Phi}(s) = f(\Phi^{-1}(s)) \qquad s \in S$$

$$T^{\Phi}f = Tf^{\Phi^{-1}} \qquad f \in C(S)$$

$$A^{\Phi}f = (Af^{\Phi^{-1}})^{\Phi} \qquad f \in C(S)$$

and say that f is invariant under Φ if $f^{\Phi} = f$, and similarly for T and A. If Ψ is another homeomorphism of S onto itself, then $f^{\Phi\Psi} = (f^{\Psi})^{\Phi}$, $T^{\Phi\Psi} = (T^{\Psi})^{\Phi}$ and $A^{\Phi\Psi} = (A^{\Psi})^{\Phi}$. Similar notation will be used for spaces other than $C(S)$ when the terms are definable (as in §5).

If S is a locally compact hausdorff space, we say that $f \in C(S)$ "vanishes at infinity" if, for each $\varepsilon > 0$, there exists a compact set C such that $|f(s)| < \varepsilon$ for all $s \notin C$. The space of all continuous functions on S which vanish at infinity is denoted by $C_0(S)$. Following Halmos [8] we call the σ-ring generated by the compact subsets of S the Borel sets of S, and any measure defined on these sets, and finite on compacts, a Borel measure. A Borel measure μ is called "regular" if

$$\inf \{\mu(U) : E \subset U, U \text{ open Borel set}\} = \sup \{\mu(C) : C \subset E, C \text{ compact}\}$$

for all Borel sets E. Then the regular Borel measures correspond in a one-to-one fashion with the positive linear functionals on $C_c(S)$ [8, Theorem E, p. 248]. Finally, for E a topological vector space, let E' be its continuous dual.

Convex Sets. Let C be a convex subset of a real or complex linear space E. Then a point $x \in C$ is an "extreme" point of C if given any y and z in C and $0 < \alpha < 1$ such that $x = \alpha y + (1-\alpha)z$, then $x = y = z$. Then the following version of the Krein-Milman Theorem holds true: Let C be a compact, convex subset of a locally convex topological vector space (real or complex). Then C equals the convex hull of its extreme points [4, p. 440].

§3. Representations. Let G be a locally compact group which is countable at infinity, and E a locally convex, complete, Hausdorff topological vector space over $\mathbb{C}$. Then a (continuous) representation π of G on E is a homomorphism of G into Aut E such that $(g,v) \to \pi(g)v$ of $G \times E \to E$ is continuous. Note that for E barreled the continuity condition is equivalent to having $g \to \pi(g)v$ of $G \to E$ continuous for each fixed $v \in E$ [18(a), p. 219]. A representation π lifts to a homomorphism of the algebra of Radon measures on G with compact support into the continuous endomorphisms of E by $\pi(\mu)v = \int_G \pi(g)v\,d\mu(g)$ (Bochner integral), i.e., for each $T \in E'$ (the topological dual of E) we have $T[\pi(\mu)v] = \int_G T[\pi(g)v]d\mu(g)$ [18(a), p. 221].

A representation π of G on a Hilbert space $\mathcal{H}$ is said to be unitary if $\pi(g)$ is a unitary operator for each $g \in G$. Such a representation is called irreducible if there are no proper closed subspaces of E which are invariant under all the operators $\{\pi(g) \mid g \in G\}$. It is easily seen that a unitary representation is irreducible if and only if any projection operator commuting with all the operators $\{\pi(g) \mid g \in G\}$ is either zero or the identity.

Finally, for K a closed subgroup of G, we define a unitary representation π of G on $\mathcal{H}$ to be of class 1 if it is irreducible and there exists a vector $e \neq 0$ such that $\pi(k)e = e$ for all $k \in K$ (i.e., e is a K-fixed vector).

§4. Positive Definite Functions. Let G be an arbitrary group with identity e, not necessarily topological. A complex-valued function f on G is said to be positive-definite (written $f >> 0$) if the inequality

$$(1) \qquad \sum_{j=1}^{m} \sum_{k=1}^{m} \alpha_j \bar{\alpha}_k f(x_j^{-1} x_k) \geq 0$$

holds for all subsets $\{x_1, \ldots, x_m\}$ of elements of G and all sequences $\{\alpha_1, \ldots, \alpha_m\}$ of complex numbers. For such functions the following properties hold: [12(b), Theorem 32.4 p. 255 and Theorem 32.9 p. 259]

(2) $\quad f(e) \geq 0$

(3) $\quad |f(x)| \leq f(e) \qquad x \in G$

(4) $\quad f(x^{-1}) = \overline{f(x)} \qquad x \in G$

(5) $\quad \overline{f} \gg 0$

(6) $\quad \alpha_1 f_1 + \alpha_2 f_2 \gg 0 \quad \text{for} \quad \alpha_1, \alpha_2 \geq 0$

(7) $\quad f_1 f_2 \gg 0$

Let G be a unimodular locally compact group with Haar measure dx. Then for $g, h \in L^1(G)$ and f a locally integrable function on G we define

(8) $\quad g*h(y) = \int_G g(x)h(x^{-1}y)dx \qquad y \in G$

(9) $\quad g^*(x) = \overline{g(x^{-1})} \qquad x \in G$

(10) $\quad f[\phi] = \int_G f(x)\phi(x)dx \qquad \phi \in C_c(G)$

We now say that a locally integrable function f is integrally positive definite if

(11) $f[\phi * \phi^*] \geq 0$ for all $\phi \in C_c(G)$.

If f is essentially bounded then (11) holds true for all $\phi \in L^1(G)$ and thus by [12(b), Theorem 32.36, p. 275] we have that f is integrally positive definite and essentially bounded if and only if $f = g$ a.e. (almost everywhere) for some continuous $g \gg 0$. Moreover, the continuity condition on g may be dropped in view of [12(b), Theorem 32.12, p. 260]. It is important to note that if f is continuous, then (11) implies (1), so that these two notions of positive definiteness are equivalent in the continuous case.

Now each positive definite $f \not\equiv 0$ gives rise to a unitary representation of G as follows: Let V denote the set of all complex linear combinations of left translates $f^{L(x)}$ $(x \in G)$ of f. We define a scalar product on V by the formula

$$(12)\quad \left(\sum_j \alpha_j f^{L(x_j)}, \sum_k \beta_k f^{L(y_k)}\right) = \sum_{j,k} \alpha_j \bar{\beta}_k f(x_j^{-1} y_k).$$

Now if $F \in V$ such that $(F,F) = 0$, then $(F,G) = 0$ for all G V. Hence for each $x \in G$ we have

$F(x) = (F, f^{L(x)}) = 0$ so that $(F,F) = 0 \iff F \equiv 0$. Hence V is a pre-hilbert space, and we let $\mathcal{H}$ be the completion. Each $x \in G$ gives rise to an endomorphism. $F \to F^{L(x)}$ of V; this endomorphism preserves the inner product $(\, , \,)$ and extends uniquely to a unitary operator $\pi(x)$ of $\mathcal{H}$. Moreover π is a unitary representation of G on $\mathcal{H}$, and if $e_0 \in \mathcal{H}$ corresponds to $f \in V$ we have $f(x) = (e_0, \pi(x)e_0)$, for all $x \in G$. Clearly the complex linear combinations of $\pi(x)e_0$ $(x \in G)$ are dense in $\mathcal{H}$ [10(a), Theorem X.4.4 p. 414].

§5. Manifolds. Let M be a C^∞ manifold satisfying the second countability axiom. The space $\mathcal{E}(M)$ denotes the space of all C^∞ functions on M, topologized by means of uniform convergence on compacts of functions along with their derivatives. Let $\mathcal{D}(M)$ be the space of all C^∞ functions of compact support on M, and for each compact subset H of M, $\mathcal{D}_H(M)$ the subspace of $\mathcal{D}(M)$ of functions with support in H. Then each $\mathcal{D}_H(M)$ is given the topology induced by $\mathcal{E}(M)$, and $\mathcal{D}(M)$ is given the inductive limit topology of the

$\mathcal{D}_H(M)$ spaces. All the above spaces are locally convex, complete, Hausdorff, topological vector spaces; in particular, $\mathcal{E}(M)$ and $\mathcal{D}_H(M)$ are Frechet spaces. For each $m \geq 0$ we also define $\mathcal{D}^m(M)$ as the space of m-times continuously differentiable functions of compact support on M, topologized in a manner similar to $\mathcal{D}(M)$.

$\mathcal{D}'(M)$ denotes the dual space of $\mathcal{D}(M)$, called the space of distributions on M. $\mathcal{E}'(M)$ denotes the dual space of $\mathcal{E}(M)$, and can be identified with the distributions on M of compact support.

Let τ be a diffeomorphism of M onto itself, and take $f \in \mathcal{E}(M)$, $T \in \mathcal{D}'(M)$ and D a differential operator on M. Then $f^\tau \in \mathcal{E}(M)$, $T^\tau \in \mathcal{D}'(M)$ and D^τ is another differential operator on M (see §1).

Suppose M has dimension m and ω is an m-form on M of maximal rank, that is, $\omega_p(X_1,\dots,X_m) \neq 0$ if $X_1,\dots,X_m$ are arbitrary linearly independent tangent vectors at an arbitrary point $p \in M$. Let D be a differential operator on M. Then there exists a unique differential operator ${}^tD$ on M, called the adjoint of D, such that

$$(1) \qquad \int_M (Df)g\omega = \int_M f({}^tDg)\omega$$

whenever f and g are two C^∞ functions on M, at least one of which has compact support [10(a), p. 450].

Let M be a pseudo-Riemannian manifold with Riemannian measure dx and Laplace-Beltrami operator Δ. Then Δ is symmetric with respect to dx, that is,

$$(2) \qquad \int_M (\Delta f)(x)g(x)dx = \int_M f(x)(\Delta g)(x)dx$$

whenever f and g are two C^∞ functions on M, at least one of which has compact support [10(a), Prop. X.2.1, p. 387]. For a relationship between (1) and (2) see §7 (5).

If V is a vector space over $\mathbb{R}$, $\mathcal{S}(V)$ denotes the space of rapidly decreasing functions on V with the usual topology [14, p. 233]. Notice that if $\Phi \geq 0$ in $\mathcal{S}(V)$, then there exists a sequence $\{\Phi_n\}$ in $\mathcal{S}(V)$ such that $|\Phi_n^2| \to \Phi$ in $\mathcal{S}(V)$. This is a consequence of the proof of [14, Theorem XVIII, p. 276] when noting that $\mathcal{S}(V)$ is a Frechet space.

§6. Lie Groups. If A is a group and $a \in A$, $L(a)$ denotes the left translation $g \to ag$ and $R(a)$ denotes the right translation $g \to ga^{-1}$ on A. If B is a subgroup of A then A/B denotes the set of left cosets aB, $a \in A$. Lie groups will be denoted by Latin capital letters and their Lie algebras by corresponding lower case German letters. If G is a Lie group and $\mathfrak{g}$ its Lie algebra the adjoint representation of G is denoted by Ad (or Ad_G) and the adjoint representation of $\mathfrak{g}$ by ad (or $ad_{\mathfrak{g}}$). The identity of G is denoted by e.

Let G be a connected semi-simple Lie group with finite center, $\mathfrak{g}$ the Lie algebra of G, and $\langle \, , \rangle$ the Killing form of $\mathfrak{g}$. Let θ be a Cartan involution of $\mathfrak{g}$. This is an involutive automorphism such that the form $(X,Y) \to -\langle X, \theta Y\rangle$ is strictly positive definite on $\mathfrak{g} \times \mathfrak{g}$. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the decomposition of $\mathfrak{g}$ into eigenspaces of θ (a Cartan decomposition) and K the analytic subgroup of G with Lie algebra $\mathfrak{k}$. Notice that any maximal compact subgroup K_1 of G is associated with some Cartan decomposition of $\mathfrak{g}$ [10(a), p. 218].

Let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal abelian subspace, $\mathfrak{a}^*$ its dual, $\mathfrak{a}^*_{\mathbb{C}}$ the complexification of $\mathfrak{a}^*$, i.e. the space of $\mathbb{R}$-linear maps of $\mathfrak{a}$ into $\mathbb{C}$. Let $A = \exp \mathfrak{a}$ and log the inverse of the map $\exp : \mathfrak{a} \to A$. For $\lambda \in \mathfrak{a}^*$ put

$$\mathfrak{g}_\lambda = \{X \in \mathfrak{g} \mid [H,X] = \lambda(H)X, \text{ for all } H \in \mathfrak{a}\}.$$

If $\lambda \neq 0$ and $\mathfrak{g}_\lambda \neq \{0\}$ then λ is called a (restricted) root and $m_\lambda = \dim(\mathfrak{g}_\lambda)$ is called its multiplicity. Let $\mathfrak{g}_{\mathbb{C}}$ denote the complexification of $\mathfrak{g}$. If $\lambda, \mu \in \mathfrak{a}^*_{\mathbb{C}}$ let $H_\lambda \in \mathfrak{a}_{\mathbb{C}}$ be determined by $\lambda(H) = \langle H_\lambda, H\rangle$ for $H \in \mathfrak{a}$ and put $\langle\lambda,\mu\rangle = \langle H_\lambda, H_\mu\rangle$. Since $\langle\ ,\ \rangle$ is positive definite on $\mathfrak{p}$ we put $|\lambda| = \langle\lambda,\lambda\rangle^{1/2}$ for $\lambda \in \mathfrak{a}^*$ and $|X| = \langle X,X\rangle^{1/2}$ for $X \in \mathfrak{p}$. Let $\mathfrak{a}'$ be the open subset of $\mathfrak{a}$ where all restricted roots are $\neq 0$. The components of $\mathfrak{a}'$ are called Weyl chambers. Fix a Weyl chamber $\mathfrak{a}^+$ and call a (restricted) root α positive if it is positive on $\mathfrak{a}^+$. Let Σ denote the set of restricted roots and Σ^+ the set of positive roots. A root $\alpha \in \Sigma^+$ is called simple if it is not a sum of two positive roots. Let ρ denote half the sum of the positive roots with multiplicity, i.e. $\rho = \frac{1}{2} \sum_{\alpha \in \Sigma^+} m_\alpha \alpha$. Let $\mathfrak{n} = \sum_{\alpha > 0} \mathfrak{g}_\alpha$, $\bar{\mathfrak{n}} = \theta\mathfrak{n}$

and let N and $\bar{N}$ denote the corresponding analytic subgroups of G. Let M denote the centralizer of A in K, M' the normalizer of A in K, W the (finite) factor group M'/M, the Weyl group. The group W acts as a group of linear transformations of $\mathfrak{a}$ and also on $\mathfrak{a}_c^*$ by $(s\lambda)(H) = \lambda(s^{-1}H)$ for $H \in \mathfrak{a}$, $\lambda \in \mathfrak{a}_c^*$ and $s \in W$. Let w denote the order of W, $A^+ = \exp \mathfrak{a}^+$; then we have the decompositions

(1) $G = K\,\mathrm{Cl}(A^+)K$ (Cartan decomposition),

(2) $G = KAN$ (Iwasawa decomposition).

Here (1) means that each $g \in G$ can be written $g = k_1 A(g) k_2$ where $k_1, k_2 \in K$ and $A(g) \in \mathrm{Cl}(A^+)$; here $A(g)$ is actually unique. In (2) each $g \in G$ can be uniquely written $g = k(g)\exp H(g)n(g)$, $k(g) \in K$, $H(g) \in \mathfrak{a}$, $n(g) \in N$.

Let $m^* \in M'$ satisfy $\mathrm{Ad}(m^*)\mathfrak{a}^+ = -\mathfrak{a}^+$, and let s^* denote the coset m^*M in W. Then since $\alpha^s \in \Sigma$

whenever $\alpha \in \Sigma$ and $s \in W$, it is easily seen that $s^*(\Sigma^+) = -\Sigma^+$ and hence

$$(3) \qquad \rho^{s^*} = -\rho.$$

The number ℓ which equals dim $\mathfrak{a}$ is called the real rank of G and the rank of the symmetric space G/K.

§7. Convolutions and Normalization of Measures.

With G as in §6 it is convenient to make some conventions concerning the normalization of certain invariant measures. Let $\ell = \dim \mathfrak{a}$. The Killing form induces Euclidean measures on A, $\mathfrak{a}$ and $\mathfrak{a}^*$; multiplying these by the factor $(2\pi)^{-\ell/2}$ we obtain invariant measures da, dH and $d\lambda$, and the inversion formula for the Fourier transform

$$(1) \qquad \tilde{f}(\lambda) = \int_A f(a) e^{-i\lambda(\log a)} da \qquad \lambda \in \mathfrak{a}^*$$

holds without any multiplicative constant,

$$(2) \qquad f(a) = \int_{\mathfrak{a}^*} \tilde{f}(\lambda) e^{i\lambda(\log a)} d\lambda \qquad f \in \mathcal{S}(A).$$

We normalized the Haar measure dk on the compact group

K such that the total measure is one. The Haar measures of the nilpotent groups N, $\bar{N}$ are normalized such that

$$(3) \qquad \theta(dn) = d\bar{n}, \quad \int_{\bar{N}} e^{-2\rho(H(\bar{n}))} d\bar{n} = 1$$

The Haar measure dg on G can be normalized such that

$$(4) \quad \int_G f(g)dg = \int\int\int_{K\times A\times N} f(kan)e^{2\rho(\log a)}dkdadn \quad f \in \mathcal{D}(G).$$

On G/K there exists G-invariant measure dx (up to a constant factor) such that

$$(5) \quad \int_G f(g)dg = \int_{G/K} (\int_K f(xk)dk)dx \qquad f \in C_c(G)$$

when dx is suitably normalized. But on G/K we have a unique G-invariant Riemannian structure induced by $\langle\ ,\ \rangle$ [10(a), Chap. V, §5]; thus the Riemannian measure induced on G/K (being G-invariant) is proportional to dx.

If S is a locally compact space with a measure μ and $p \geq 1$, then $L^p(S)$ denotes the set of measurable functions f such that $|f|^p$ is μ-integrable. Moreover we let $C^\natural(G)$, $L^\natural(G)$ and $I_c(G)$ be the subspaces of K-biinvariant functions in $C_c(G)$, $L^1(G)$ and $\mathfrak{D}(G)$ respectively. These subspaces are all commutative under convolution.

For f locally integrable on G we define

(6) $$f^\natural(g) = \int\int_{K\times K} f(k_1 g k_2)\,dk_1\,dk_2 \qquad g \in G$$

(7) $$f^v(g) = f(g^{-1}) \qquad g \in G.$$

We easily see that for $f, g \in L^1(G)$

(8) $$(f^v * g^v) = (g * f)^v$$

§8. Differential operators. If A is a Lie group $D(A)$ denotes the algebra of all left invariant differential operators on A. If $B \subset A$ is a closed subgroup, $D(A/B)$ denotes the algebra of A-invariant differential operators on A/B.

The notation being as in §6 let $D_0(G)$ denote the set of $D \in D(G)$ which are invariant under all right translations from K. There is a homomorphism μ of $D_0(G)$ onto $D(G/K)$ such that

$$(\mu(D)f)\circ \pi = D(f\circ \pi) \quad \text{for} \quad D \in D_0(G),\ f \in \mathcal{E}(G/K),$$

π denoting the natural mapping of G onto G/K. Let ω be the Casimir operator on G [10(a), p. 451]. Then $\mu(\omega)$ is the Laplace-Beltrami operator on G/K when using the Riemannian structure on G/K induced by $\langle\ ,\ \rangle$.

Let $X_1,\ldots,X_n$ be any basis of $\mathfrak{g}$. Then $\{X_1^{e_1}\ldots X_n^{e_n} \mid e_j \geq 0\}$ is a basis of $D(G)$ when each X_j is considered as a left invariant vector field on G.

§9. Spherical Functions. The notation being as in §6 let a non-zero function $\phi \in C(G)$ be a (zonal) spherical function if it satisfies any one of the following equivalent conditions:

(1) $\int_K \phi(xky)dk = \phi(x)\phi(y) \qquad x,y \in G$

(2) ϕ is K-biinvariant, $\phi(e) = 1$ and

$f*\phi = (\int_G f(g)\phi(g^{-1})dg)\phi \qquad f \in C^{\natural}(G)$

(3) ϕ is K-biinvariant and $L : f \to \int_G f(g)\phi(g)dg$ is a homomorphism of $C^{\natural}(G)$ onto $\mathbb{C}$.

(4) ϕ is K-viinvariant, C^∞, $\phi(e) = 1$ and

$$D\phi = (D\phi(e))\phi \qquad D \in D_0(G).$$

Notice that (1) and (4) are equivalent by [10(a), Prop. X.3.2, p. 399], (3) and (4) are equivalent by [10(a), Lemma X.4.2, p. 409], and (1) implies (2) implies (3) using §7 (4).

Let $\mathcal{F}^1$ be the set of all bounded spherical functions and $\mathcal{P}$ the subset of all positive definite spherical functions. Then giving $\mathcal{P}$ the Godement topology, that is, the weak* topology as a subset of $L^\infty(G)$, makes $\mathcal{P}$ into a locally compact Hausdorff space [7(b), p. 7]. Now suppose $\phi >> 0$ with the associated unitary representation π on $\mathcal{H}$, and $e_0 \in \mathcal{H}$ such that $\phi(x) = (e_0, \pi(g)e_0)$ for all $g \in G$. Then

$\phi \in \mathcal{P} \iff \pi$ is class one with e_0 a K-fixed vector [10(a), Theorem X.4.5, p. 414].

For any measurable function f on G we define is spherical Fourier transform $\hat{f}$ by

(5) $$\hat{f}[\phi] = \int_G f(g)\phi(g^{-1})dg$$

for all spherical functions ϕ for which this integral makes sense. In particular, if $f \in L^1(G)$, then $\hat{f}$ is defined on $\mathfrak{F}^1$, hence on $\mathfrak{P}$, and (using obviously abusive notation) we have $\hat{f} \in C_0(\mathfrak{P})$ [7(b), p. 7]. Moreover, the following properties are quickly verified for $f, g \in L^1(G)$:

(6) $\widehat{f^\natural} = \hat{f}$ on $\mathfrak{F}^1$

(7) $\widehat{f^*}[\phi] = \overline{\hat{f}[\phi^*]}$ for $\phi \in \mathfrak{F}^1$; hence $\widehat{f^*} = \overline{\hat{f}}$ on $\mathfrak{P}$

(8) $(f*g)^\wedge = \hat{f} \cdot \hat{g}$ on $\mathfrak{F}^1$ if f is right K-invariant or if g is left K-invariant.

Hence $f \in L^1(G)$ and right K-invariant implies

(9) $(f*f^*)^\wedge = |\hat{f}|^2$ on $\mathfrak{P}$.

There exists a basic parametrization and formula for the spherical functions given by Harish-Chandra: the spherical functions are precisely the functions

(10) $\phi_\lambda(g) = \int_K \exp\,(i\lambda-\rho)(H(gk))dk, \quad g \in G$

where $\lambda \in \mathfrak{a}_c^*$ is arbitrary; moreover, $\phi_\lambda = \phi_\mu$ if and only if $\lambda = s\mu$ for some $s \in W$. Hence $\mathfrak{P}$ and $\mathfrak{F}^1$ can be viewed as subsets of $W\backslash\mathfrak{a}_c^*$, or by an obvious abuse of notation, as subsets of $\mathfrak{a}_c^*$. Certain properties are:

(11) $\phi_{-\lambda}(g^{-1}) = \phi_\lambda(g) \qquad \lambda \in \mathfrak{a}_{\mathbb{C}}^*, \quad g \in G$

(12) $\phi_{-\lambda}(g) = \overline{\phi_{\bar{\lambda}}(g)} \qquad \lambda \in \mathfrak{a}_{\mathbb{C}}^*, \quad g \in G$

(13) $\lambda \in \mathfrak{P}$ implies λ and $\bar{\lambda}$ are Weyl group conjugate

(14) If ω is the Casimir operator on G, then

$$\omega\phi_\lambda = -(\langle\lambda,\lambda\rangle + \langle\rho,\rho\rangle)\phi_\lambda \qquad \lambda \in \mathfrak{a}_{\mathbb{C}}^*$$

(15) Helgason-Johnson Theorem: Let C_ρ be the convex hull of $\{s\rho \mid s \in W\}$ in $\mathfrak{a}^*$. Then $\mathfrak{F}^1 = \mathfrak{a}^* + iC_\rho$.

Remarks: (11) is [9(a), Lemma 45, p. 294]; (12) follows by easy computation; (13) follows from (11), (12) and §4(4); (14) is [9(a), Cor. 2, p. 271]; and (15) is [11, Theorem 2.1, p. 587].

In addition to the Godement topology $\mathfrak{P}$ also has a topology induced by the Euclidean topology of $\mathfrak{a}_{\mathbb{C}}^*$. That these are the same is proved as follows: since both topologies are hausdorff locally compact, then in each case the weak topology generated by the appropriate $C_0(\mathfrak{P})$ space equals the given topology. Hence we have only to show that $C_0(\mathfrak{P})$ is the same space in both $\mathfrak{P}$-topologies. But this will follow when we show that

(16) $\mathcal{R} = \{\hat{f} \mid f \in I_c(G)\}$

satisfies the conditions of Stone's theorem on $\mathcal{P}$ using either $\mathcal{P}$-topology. For the Godement topology the assertion is easily seen as true since $\mathcal{R}$ contains all conjugates from (7). For the Euclidean topology we need only demonstrate that for each $f \in I_c(G)$ we have $\hat{f}$ is continuous and vanishing at infinity. The continuity is easily verified using the compactness of the support of f. Now let ω be the Casimir operator on G. Then

$$\begin{aligned}
(\omega f)^{\wedge}(\phi_\lambda) &= \int_G (\omega f)(g)\phi_{-\lambda}(g)dg \quad \text{from (5) and (11)} \\
&= \int_{G/K} (\omega f)(xK)\phi_{-\lambda}(xK)dx \quad \text{from §7 (5)} \\
&= \int_{G/K} f(xK)(\omega\phi_{-\lambda})(xK)dx \quad \text{from §5 (2) and §8} \\
&= -(\langle\lambda,\lambda\rangle + \langle\rho,\rho\rangle)\hat{f}(\phi_\lambda) \quad \text{from (14).}
\end{aligned}$$

But $\omega f \in I_c(G)$ gives $(\omega f)^{\wedge}$ is a bounded function on $\mathcal{P}$. Hence the above equation implies that $\hat{f}$ vanishes at infinity on $\mathcal{P}$ in the Euclidean induced topology. Hence the two topologies on $\mathcal{P}$ are the same.

From (11) we may write

(17) $\hat{f}(\lambda) = \int_G f(g)\phi_{-\lambda}(g)dg.$

Then for $f \in I_c(G)$ we have

(18) $f(g) = \frac{1}{w}\int_{\mathfrak{a}^*} \hat{f}(\lambda)\phi_\lambda(g) \mid c(\lambda)|^{-2}d\lambda \qquad g \in G,$

where $|c(\lambda)|^{-2} = c(\lambda)c(-\lambda)$ for $\lambda \in \mathfrak{a}^*$ and

(19) $c(\lambda) = \int_{\bar{N}} \exp\ (-(i\lambda+\rho)(H(\bar{n})))d\bar{n} \qquad \lambda \in \mathfrak{a}^*.$

We now define two functions which will be very important later on.

(20) $\Xi(g) = \int_K \exp\ (-\rho(H(gk)))dk, \quad g \in G$

(21) $\sigma(g) = |X|$ where $g = k \exp X,\ k \in K,\ X \in \mathfrak{p}$.

Notice that $\Xi = \phi_\lambda$ for $\lambda = 0$. Certain properties are:

(22) Ξ and σ are K-biinvariant.

(23) $\sigma(xy) \leq \sigma(x) + \sigma(y)\ , \quad x,y \in G.$

(24) $1 \leq \Xi(a)\exp\ (\rho(\log a)) \leq c(1+\sigma(a))^d$ for some $d \geq 0$, $c > 0$ and for all $a \in \mathfrak{a}(A^+)$.

(25) There exists $r \geq 0$ such that

$\int_G \Xi(g)^2(1+\sigma(g))^{-r}dg < \infty$.

(26) On any compact $C \subset G$ there exists $c > 0$ such that

$\Xi(y_1 x y_2) \leq c\Xi(x)$ for all $y_1, y_2 \in C$, $x \in G$.

(27) On any compact $C \in G$ there exists $d_1, d_2 > 0$ such

that $d_1(1+\sigma(xy)) \leq 1 + \sigma(x) \leq d_2(1+\sigma(xy))$ $y \in C$, $x \in G$.

Remarks: (22) for σ is [18(b), p. 66]; (23) is [9(d), Lemma 10, p. 15]; (24) is [18(b), p. 154]; (25) is [9(d), Lemma 11, p. 16]; (26) is [9(c), Lemma 32, p. 108]; and (27) is [18(b), Cor. 8.1.2.2, p. 67].

§10. Distributions on Lie Groups. Suppose G is a separable Lie group. Then the topologies on $\mathcal{D}(G)$ and $\mathcal{E}(G)$ can be described by means of the left (or right) invariant differential operators on G [5, Prop. 2, p. 593]. Moreover, both $\mathcal{D}(G)$ and $\mathcal{E}(G)$ are reflexive topological spaces [5, Prop. 3, p. 593]. Let $f, g \in \mathcal{E}(G)$, one of which has compact support, and let D (resp. E) be a left (resp. right) invariant differential operator on G. Then

(1) $D(f*g) = f*(Dg)$ and $E(f*g) = (Ef)*g$

both follow from the mean value and dominated convergence theorems. It follows that if $\phi_n \to \phi$ and $\psi_n \to \psi$ in $\mathcal{D}(G)$, then $\phi_n * \psi_n \to \phi * \psi$ in $\mathcal{D}(G)$.

Since $\mathcal{D}(G)$ and $\mathcal{E}(G)$ are reflexive, then the strong duals $\mathcal{D}'(G)$ and $\mathcal{E}'(G)$ are both barreled [16, Prop. 36.4, p. 373]. Furthermore, if dx is a left haar measure on G, then a function f, locally summable with respect to dx, can be identified with the distribution $T_f \in \mathcal{D}'(G)$ by

$$(2) \qquad T_f[\phi] = \int_G \phi(x)f(x)dx \qquad \phi \in \mathcal{D}(G).$$

Suppose $T, S \in \mathcal{D}'(G)$, one of which has compact support. We may define the convolution $T*S \in \mathcal{D}'(G)$ by

$$(3) \qquad T*S[\phi] = T_{[x]}(S_{[y]}[\phi(xy)]) = S_{[y]}(T_{[x]}[\phi(xy)]) \qquad \varphi \in \mathcal{D}(G)$$

[18(a), p. 489]. Convolution is associative but not necessarily commutative, and clearly

$$(4) \quad \text{support } (S*T) \subset (\text{support } S)\cdot(\text{support } T).$$

For convenience assume G to be unimodular. Then if $T \in \mathcal{D}'(G)$ (resp. $\mathcal{E}'(G)$) and $f \in \mathcal{D}(G)$ (resp. $\mathcal{E}(G)$), then $f*T$ and $T*f$ are C^∞ functions (as in (2)) given by

$$
(6) \qquad \begin{aligned} T*\alpha[\phi] &= T[\phi*\check{\alpha}] \\ \alpha*T[\phi] &= T[\check{\alpha}*\phi], \quad \alpha,\phi \in \mathcal{D}(G), \quad T \in \mathcal{D}'(G). \end{aligned}
$$

For each differential operator D on G we let ${}^tD$ be the adjoint with respect to Haar measure on G. Then given $T \in \mathcal{D}'(G)$ we define $DT \in \mathcal{D}'(G)$ by

$$
(7) \qquad DT[\phi] = T[{}^tD\phi], \quad \phi \in \mathcal{D}(G).
$$

In particular, ${}^tX = -X$ for X either a left or right invariant vector field. If $\delta \in \mathcal{D}'(G)$ is defined by $\delta[\phi] = \phi(e)$ for $\phi \in \mathcal{D}(G)$, and if X (resp. Y) is a left (resp. right) invariant vector field on G, we have $XT = T*(X\delta)$ and $YT = (Y\delta)*T$. Associativity then gives $X(T*S) = T*(XS)$ and $Y(T*S) = (YT)*S$ for $S, T \in \mathcal{D}'(G)$, one of which has compact support. Hence D a left invariant differential operator, and E a right invariant differential operator, imply

$$
(8) \qquad \begin{aligned} D(T*S) &= T*(DS) \\ E(T*S) &= (ET)*S, \text{ T and S as above.} \end{aligned}
$$

Given $T \in \mathcal{D}'(G)$ we also define $\check{T} \in \mathcal{D}'(G)$ by

(9) $\check{T}[\phi] \equiv T[\check{\phi}] \qquad \phi \in \mathcal{D}(G)$.

Let G be a connected, semisimple Lie group with finite center, and K a maximal compact subgroup. Then we say T is K-biinvariant if $T^{L(k_1)R(k_2)} = T$ for all $k_1, k_2 \in K$. If T and S are both K-biinvariant then $T*S = S*T$, with the same proof as in [10(a), Theorem X.4.1, p. 408]. Moreover,

$$(10) \qquad (T*S)^{L(k_1)R(k_2)} = T^{L(k_1)}*S^{R(k_2)}$$

so that $T*S$ is K-biinvariant if and only if T is left K-invariant and S is right K-invariant. Also, T K-biinvariant gives

$$(11) \qquad T[\phi^{\natural}] = T[\phi] \qquad \phi \in \mathcal{D}(G).$$

Chapter I.

Positive Definite Distributions

Let G be a connected unimodular Lie group. Then a distribution T is said to be positive definite (written $T >> 0$) if and only if

$$(1) \qquad T[\phi * \phi^*] \geq 0, \qquad \phi \in \mathcal{D}(G).$$

In this chapter we will show that such distributions can be written as sums of left and right invariant differential operators applied to bounded functions (Theorem 1.6). We start by proving some rather general results on $\mathcal{D}'$ and $\mathcal{E}'$.

Lemma 1.1. Suppose $T \in \mathcal{D}'(G)$ and $S \in \mathcal{E}'(G)$. Then the mappings

$u_T : \mathcal{D} \to \mathcal{E}$ by $\phi \to \phi * \check{T}$

$v_S : \mathcal{D} \to \mathcal{D}$ by $\phi \to \phi * \check{S}$

$w_T : \mathcal{D} \to \mathcal{E}$ by $\phi \to \check{T} * \phi$ are all continuous.

Proof: First consider $u_T : \mathcal{D} \to \mathcal{E}$ by $\phi \to \phi * \check{T}$. It suffices to show that $\phi \to \phi * \check{T}$ is continuous from $\mathcal{D}_H$ into $\mathcal{E}$ for each compact set H in G. Hence for each right invariant differential operator E and each compact set C in G we have to show that there exists a continuous semi-norm p on $\mathcal{D}_H$ and $M \geq 0$ such that

$$(2) \qquad \sup_{x \; C} |E(\phi * \check{T})(x)| \leq Mp(\phi), \qquad \phi \in \mathcal{D}_H.$$

But we have $E(\phi * \check{T})(x) = (E\phi) * \check{T}(x) = \check{T}_{[y]}[E\phi(xy^{-1})] = T_{[y]}[E\phi(xy)] = T[(E\phi)^{L(x^{-1})}]$. Now if $\phi \in \mathcal{D}_H$ and $x \in C$ we have $(E\phi)^{L(x^{-1})} \in \mathcal{D}_{C^{-1}H}$. But T restricted to $\mathcal{D}_{C^{-1}H}$ is continuous, and hence there exists $M \geq 0$ and D_j, $j = 1, \ldots, m$, left invariant differential operators such that

$$|T[\psi]| \leq M \sup_{\substack{y \in G \\ j=1,\ldots,m}} |D_j \psi| , \quad \psi \in \mathcal{D}_{C^{-1}H} .$$

Now applying this formula for $\psi = (E\phi)^{L(x^{-1})}$ gives

$$\sup_{x \in C} |E(\phi * \check{T})(x)| = \sup_{x \in C} |T[(E\phi)^{L(x^{-1})}]|$$

$$\leq \sup_{x \in C} |M \sup_{\substack{y \in G \\ j=1,\ldots,m}} |D_j((E\phi)^{L(x^{-1})})||$$

$$= M \sup_{\substack{y \in G \\ j=1,\ldots,m}} |D_j E\phi(y)|, \qquad \phi \in \mathcal{D}_H$$

which is the desired form of (2). Hence u_T is proved continuous.

Now consider $v_S : \mathcal{D} \to \mathcal{D}$ by $\phi \to \phi * \check{S}$. Again we may restrict ourselves to $\mathcal{D}_H \to \mathcal{D}$. By what we have proved above, $\phi \to \phi * S$ is continuous from $\mathcal{D}_H$ to $\mathcal{E}$. But $\text{supp}(\phi * \check{S}) \subset (\text{supp}\ \phi)(\text{supp}\ \check{S})$. Hence v_S is continuous into $\mathcal{D}$.

Finally consider $w_T \cdot \mathcal{D} \to \mathcal{E}$ by $\phi \to \check{T} * \phi$. The proof here is precisely the same as for u_T except for reversing the roles of left and right invariant differential operators. □

<u>Lemma 1.2</u>. Suppose $T \in \mathcal{D}'(G)$ and $S \in \mathcal{E}'(G)$. Then the mappings

$U : \mathcal{E}' \to \mathcal{D}'$ by $S \to S*T$

$V : \mathcal{D}' \to \mathcal{D}'$ by $T \to T*S$

$W : \mathcal{E}' \to \mathcal{D}'$ by $S \to T*S$

are all continuous in the strong topologies of $\mathcal{E}'$ and $\mathcal{D}'$.

Proof. If ${}^t u$ denotes the transpose of a mapping u, then in the notation of Lemma 1.1 we have that $U = {}^t u_T$, $V = {}^t v_S$ and $W = {}^t w_T$. Hence the result follows from Prop. 1.1 and [16, Cor. to Prop. 19.5, p. 199], i.e., the transpose of a continuous linear map is continuous in the strong dual topology. □

Prop. 1.3. Suppose $T \in \mathcal{D}'(G)$ and $S_1, S_2 \in \mathcal{E}'(G)$. Then the mapping of $\mathcal{E}' \times \mathcal{E}' \to \mathcal{D}'$ given by $(S_1,S_2) \to S_1*T*S_2$ is hypocontinuous in the strong dual topologies.

Proof Separate continuity is a consequence of Lemma 1.2 when we write $S_1*T*S_2 = (S_1*T)*S_2$. Hence, from [16, Theorem 41.2, p. 424] this mapping is hypocontinuous since $\mathcal{E}'$ is barreled (Notation §10). □

Prop. 1.4. For any relatively compact, open coordinate neighborhood ω of e in G and any $m \in \mathbb{Z}^+$ there exists a differential operator D, a function $f \in \mathcal{D}^m(\omega)$ and a function $\zeta \in \mathcal{D}(\omega)$ such that $Df - \zeta = \delta$ in the sense of distributions.

Proof. On $\mathbb{R}^n$ Schwartz shows that with

$\Delta \equiv \frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial x_n^2}$, then given any open neighborhood

W of zero and any $m > 0$, then there exists $k > 0$, $F \in \mathcal{D}^m(W)$ and $Z \in \mathcal{D}(W)$ such that $\Delta^k F - Z = \delta_0$ in the sense of distributions, where $\delta_0[\phi] \equiv \phi(0)$ for all $\varphi \in \mathcal{D}(\mathbb{R}^n)$ [14, Eq. II.3.19, p. 47 and Eq. VI.6.22, p. 191]. Pick ω_0 to be an open, relatively compact, canonical coordinate neighborhood about e in G, with coordinates $\phi_0(g) = (x_1(g),\ldots,x_n(g))$, and pick ω_1 to be an open neighborhood about e such that $\mathcal{Cl}(\omega_1) \subset \omega_0$. Then take W_0, W_1 contained in $\mathbb{R}^n$ by $W_0 \equiv \phi_0(\omega_0)$ and $W_1 \equiv \phi_0(\omega_1)$. Then Haar measure on ω_0 can be expressed as $G(x_1,\ldots,x_n)dx_1 \wedge \ldots \wedge dx_n$ for G a strictly positive C^∞ function on W_0 [10(a), p. 364].

Let $W = W_1$ and pick F and Z as above. Then define $f \in \mathcal{D}^m(\omega_1)$ and $\zeta \in \mathcal{D}(\omega_1)$ by

$$f(g) = \begin{cases} (F/G)(\phi_0(g)) , & g \in \omega_0 \\ 0 & \text{otherwise} \end{cases}$$

$$\zeta(g) = \begin{cases} (Z/G)(\phi_0(g)) , & g \in \omega_0 \\ 0 & \text{otherwise} \end{cases}$$

We may regard Δ^k as a differential operator on $\mathcal{D}(\omega_0)$ by $\Delta^k f \equiv \Delta^k(f \circ \phi_0^{-1}) \circ \phi_0$, $f \in \mathcal{D}(\omega_0)$. But by the definitions of ω_0 and ω_1 there exists $\gamma \in \mathcal{D}(\omega_0)$ such that $\gamma|_{\omega_1} \equiv 1$. Hence considering γ itself as a differential operator on $\mathcal{D}(G)$ gives that $\Delta^k \cdot \gamma$ is also a differential operator on $\mathcal{D}(G)$. Now let $D \equiv {}^t(\Delta^k \cdot \gamma)$, that is, let D be the adjoint of $\Delta^k \cdot \gamma$ with respect to Haar measure (Notation, §5(1), §10). Then for any $\phi \in \mathcal{D}(G)$ we have

$$(Df - \mathcal{S})[\phi] = f[\Delta^k(\gamma\phi)] - \mathcal{S}[\phi]$$

$$= \int_{\omega_0} f(g)\Delta^k(\gamma\phi)(g)dg - \int_{\omega_0} \mathcal{S}(g)\phi(g)dg$$

$$= \int \cdots \int_{W_0} F(x)\Delta^k((\gamma\phi)\circ\phi_0^{-1})(x)dx - \int\cdots\int_{W_0} Z(x)\phi(\phi_0^{-1}(x))dx$$

where $dx = dx_1 \ldots dx_n$

$$= (\Delta^k F - Z)[(\gamma\phi)\circ\phi_0^{-1}] \quad \text{since } \Delta^k \text{ is symmetric}$$

$$= \delta_0[(\gamma\phi)\circ\phi_0^{-1}]$$

$$= \delta[\phi]$$

Thus $Df - \mathcal{S} = \delta$ in the sense of distributions. □

<u>Lemma 1.5</u>. Suppose $\psi \in \mathcal{D}(G)$, $f \in \mathcal{D}^m(\omega)$ for ω an open subset of G, D a left invariant differential operator, and E a right invariant differential operator. Then

(a) $\psi Df = \sum_j D^j f_j$ for D^j left invariant, $f_j \in \mathcal{D}^m(\omega)$

(b) $\psi Ef = \sum_j E^j g_j$ for E^j right invariant, $g_j \in \mathcal{D}^m(\omega)$.

These sums are finite and are in the sense of distributions.

Proof. We prove only (a) as the proof for (b) is identical. Take $\{X_1, \ldots, X_n\}$ as a basis of $\mathfrak{g}$ and let $\tilde{X}_i$ be the left invariant vector field generated by X_i. Then without loss of generality we may take D to be of the form $\tilde{X}_{i_1} \ldots \tilde{X}_{i_k}$. We induct on k (i.e., the order of D):

1. Let $D = \tilde{X}$. Then for $\phi \in \mathcal{D}(G)$ we have

$$\psi\tilde{X}f[\phi] = \tilde{X}f[\psi\phi] = f[-\tilde{X}(\psi\phi)]$$

$$= f[-(\tilde{X}\psi)\phi - (\tilde{X}\phi)\psi] = \psi f[-\tilde{X}\phi] - (\tilde{X}\psi)f[\phi]$$

$$= (\tilde{X}(\psi f) - (\tilde{X}\psi)f)[\phi]$$

Notice we have used ${}^t\tilde{X} = -\tilde{X}$ for a unimodular Lie group. Hence $\psi\tilde{X}f = \tilde{X}(\psi f) - (\tilde{X}\psi)f$. But ψf and $(\tilde{X}\psi)f$ are both in $\mathcal{D}^m(\omega)$, so that case $k = 1$ is done.

2. Assume formula (a) is true for all left invariant differential operators of degree $\leq k - 1$ and for all $\psi \in \mathcal{D}(G)$ and $f \in \mathcal{D}^m(\omega)$. Then let $D = \tilde{X}_{i_1} \ldots \tilde{X}_{i_k}$.

$$\psi Df = \psi\tilde{X}_{i_1}(\tilde{X}_{i_2}\cdots\tilde{X}_{i_k}f)$$

$$= \tilde{X}_{i_1}(\psi\tilde{X}_{i_2}\cdots\tilde{X}_{i_k}f) - (\tilde{X}_{i_1}\psi)\tilde{X}_{i_2}\cdots\tilde{X}_{i_k}f$$

as shown in case $k = 1$

$$= \tilde{X}_{i_1}(\sum_j D_1^j f_j) - \sum_j D_2^j g_j \quad \text{by the induction assumption.}$$

This is, however, the desired form. □

<u>Theorem 1.6</u>. Suppose $T \in \mathcal{D}'(G)$, $T >> 0$. Then T can be expressed as a finite sum

(3) $$T = \sum_j D_j E_j f_j$$

where, for each j, $f_j \in L^\infty(G)$, D^j is a left invariant differential operator, and E^j is a right invariant differential operator.

<u>Proof</u>. For each $\alpha \in \mathcal{D}(G)$ we have that $\alpha * T * \alpha^*$ is a C^∞ function such that $\alpha * T * \alpha^*[\phi * \phi^*] = T[(\check{\alpha} * \phi) * (\check{\alpha} * \phi)^*] \geq 0$ for each $\phi \in \mathcal{D}(G)$. Hence (Notation §5(11)) gives that $\alpha * T * \alpha^* >> 0$, and in particular that

(4) $|\alpha * T * \alpha^*(g)| \leq \alpha * T * \alpha^*(e) = T[\check{\alpha} * \bar{\alpha}]$ for all $g \in G$, $\alpha \in \mathcal{D}(G)$.

Moreover, from

(5) $$4\alpha * T * \beta = (\alpha+\beta^*)*T*(\alpha+\beta^*)^* - (\alpha-\beta^*)*T*(\alpha-\beta^*)^*$$
$$+ i(\alpha-(i\beta)^*)*T*(\alpha-(i\beta)^*)^* - i(\alpha+(i\beta)^*)*T*(\alpha+(i\beta)^*)^*$$

we see that $\alpha * T * \beta \in L^\infty(G)$ for all $\alpha, \beta \in \mathcal{D}(G)$.

Let $B \equiv \{\phi \in \mathcal{D}(G) \mid ||\phi||_1 \leq 1\}$ where $||\phi||_1 = \int_G |\phi(x)|dx$ and for each $\phi \in B$ define $T_\phi : \mathcal{D} \times \mathcal{D} \to \mathbb{C}$ by

(6) $$T_\phi[(\alpha,\beta)] = \alpha * T * \beta[\phi].$$

From (Notation §10(6)) we see that each T_ϕ is a separately continuous bilinear mapping. Thus, if $\mathcal{D} \otimes \mathcal{D}$ is the algebraic tensor product of $\mathcal{D}$ with itself, equipped with the inductive topology [18(a), p. 483], we see that each T_ϕ defines a continuous mapping (also called T_ϕ) from $\mathcal{D} \otimes \mathcal{D} \to \mathbb{C}$ by $T_\phi[\alpha \otimes \beta] = T_\phi[(\alpha,\beta)]$ [18(a), Thm. A.2.2.4, p. 483]. Therefore each T_ϕ

extends to a continuous linear mapping $\mathcal{D} \bar{\otimes} \mathcal{D} \to \mathbb{C}$, where $\mathcal{D} \bar{\otimes} \mathcal{D}$ is the completion of $\mathcal{D} \otimes \mathcal{D}$ [16, Thm. 5.2, p. 41]. But $\mathcal{D}(G) \bar{\otimes} \mathcal{D}(G) = \mathcal{D}(G \times G)$ [18(a), p. 485] so that we have $\{T_\phi \mid \phi \in B\}$ as a collection of distributions on $G \times G$.

We claim that $\{T_\phi \mid \phi \in B\}$ is actually a bounded set of distributions on $G \times G$. To see this first note that for each fixed pair $(\alpha,\beta) \in \mathcal{D} \times \mathcal{D}$, $\{T_\phi[\alpha \otimes \beta] \mid \phi \in B\}$ is a bounded set of numbers in $\mathbb{C}$ since $||\alpha * T * \beta||_\infty < \infty$ and $||\phi||_1 \leq 1$. Now take Ψ arbitrary in $\mathcal{D}(G \times G)$. Since $\mathcal{D}(G \times G)$ is the inductive limit of the collection of spaces $\mathcal{D}_H(G) \bar{\otimes} \mathcal{D}_H(G)$, H compact in G [18(a), Thm. A.2.2.5, p. 484], then $\Psi \in \mathcal{D}_H(G) \bar{\otimes} \mathcal{D}_H(G)$ for some particular H. But $\mathcal{D}_H(G)$ is Frechet which implies that the inductive and projective topologies on $\mathcal{D}_H(G) \bar{\otimes} \mathcal{D}_H(G)$ are the same [18(a), p. 483]; hence we can apply [16, Th. 45.1, p. 459] to prove that Ψ is the sum of an absolutely convergent series $\sum_{k=0}^{\infty} \lambda_k \alpha_k \otimes \beta_k$, where $\{\lambda_k\}$ is a sequence of complex numbers such that $\sum_{k=1}^{\infty} |\lambda_n| < 1$ and $\{\alpha_k\}$ and $\{\beta_k\}$

are sequences converging to zero in $\mathcal{D}_H(G)$. Thus

$$|T_\phi[\Psi]| = |T_\phi[\sum_{k=0}^{\infty} \lambda_k \alpha_k \otimes \beta_k]| \leq \sum_{k=0}^{\infty} |\lambda_k| \; |T_\phi(\alpha_k \otimes \beta_k)|.$$

Applying (4) and (5) shows that $\lim_{k \to \infty} |T_\phi(\alpha_k \otimes \beta_k)| = 0$ uniformly for $\phi \in B$, so that $\{T_\phi[\Psi] \mid \phi \in B\}$ is a bounded set in $\mathbb{C}$. Thus [14, Th. IX(b), p. 72] gives that $\{T_\phi \mid \phi \in B\}$ is a bounded set of distributions (see Remark following this proof).

Let ω be an open, relatively compact, coordinate neighborhood of e in G. Then since $\{T_\phi \mid \phi \in B\}$ is bounded, there exists a differential operator D on $G \times G$, and a collection $\{f_\phi \mid \phi \in B\}$ of continuous <u>uniformly</u> bounded functions on $G \times G$ such that

$$T_\phi|_{\omega\times\omega} = Df_\phi \qquad \phi \in B$$

[14, Thm. XXII, p. 86]. Thus we have

(7) $\alpha * T * \beta[\phi] = f_\phi[{}^tD(\alpha \otimes \beta)] \qquad \alpha, \beta \in \mathcal{D}(\omega), \; \phi \in B.$

Let m be the order of D on $\omega \times \omega$, and take

$\alpha, \beta \in \mathfrak{D}^m(\omega)$ and $\{\alpha_j\}, \{\beta_j\} \subset \mathfrak{D}(\omega)$ such that $\alpha_j \to \alpha$ and $\beta_j \to \beta$ in $\mathfrak{D}^m(\omega)$. Since ${}^tD(\alpha_j \otimes \beta_j) \to {}^tD(\alpha \otimes \beta)$ in $L^1(G \times G)$, then (6) allows us to conclude that

1. there exists $M < \infty$ such that $|\alpha_j * T * \beta_j[\phi]| \leq M$ for all ϕ B and $j = 1,2,\ldots;$ and

2. $\lim_{j \to \infty} \alpha_j * T * \beta_j[\phi]$ exists for each $\phi \in B$. From the first statement it is easy to conclude that $||\alpha_j * T * \beta_j||_\infty \leq M$ for all j, from which a routine calculation shows that the second statement holds for all $\phi \in L^1(G)$. We are thus able to define a linear operator $F : L^1(G) \to \mathbb{C}$ by $F[\phi] = \lim_{j \to \infty} \alpha_j * T * \beta_j[\phi]$, $\phi \in L^1(G)$. But $|F[\phi]| \leq M||\phi||_1$, so that F must be some bounded function $f \in L^\infty(G)$. It is clear that this f depends only on α and β, and not on the choice of $\{\alpha_j\}$ and $\{\beta_j\}$. We have therefore established the following:

Given $T >> 0$ and ω any open, relatively compact coordinate neighborhood of e in G, then there exists an integer m such that given any $\alpha, \beta \in \mathcal{D}^m(\omega)$ there exists a function $f \in L^\infty(G)$ with the property that

$$(8) \quad \lim_{j \to \infty} \alpha_j * T * \beta_j[\phi] = f[\phi], \quad \phi \in L^1(G)$$

for any sequences, $\{\alpha_j\}, \{\beta_j\} \subset \mathcal{D}(\omega)$ where $\alpha_j \to \alpha$ and $\beta_j \to \beta$ in $\mathcal{D}^m(\omega)$.

We will now proceed to show that this function f is actually the distribution $\alpha * T * \beta$. By Prop. 1.3 we have that the mapping $(S_1, S_2) \mapsto S_1 * T * S_2$ of $\mathcal{E}'(G) \times \mathcal{E}'(G) \to \mathcal{D}'(G)$ is hypocontinuous. But thus, since $\alpha_j \to \alpha$ and $\beta_j \to \beta$ in $\mathcal{E}'(G)$, then $\alpha_j * T * \beta_j \to \alpha * T * \beta$ in $\mathcal{D}'(G)$. Hence this along with (8), gives $\alpha * T * \beta = f$.

With ω and m as above, take D, f and ζ as in Prop. 1.4, so that $Df - \zeta = \delta$. Since supp $f \subset \omega$ we may multiply Df by any function $\gamma \in \mathcal{D}(G)$ such that $\gamma|_\omega \equiv 1$ without changing its properties; hence we expand D to be either of the two finite sums

$$D = \sum_j \phi_j D_j \quad \text{and} \quad D = \sum_k \psi_k E_k \ ,$$

where ϕ_j, $\psi_k \in \mathcal{D}(G)$ and D_j (resp. E_k) is a left invariant (resp. right invariant) differential operator for all j, k. We then have

$$T = \delta * T * \delta$$

$$= (Df - \zeta) * T * (Df - \zeta)$$

$$= (\sum_k \psi_k E_k f - \zeta) * T * (\sum_j \phi_j D_j f - \zeta)$$

Now apply Lemma 1.5 to all the $\psi_k E_k f$ and $\phi_j D_j f$ terms. Combining the summations we obtain $T = (\sum_k E^k g_k) * T * (\sum_j D^j f_j)$ for g_k, $f_j \in \mathcal{D}^m(\omega)$ and D^j (resp. E^k) a left invariant (resp. right invariant) differential operator for all j, k. Hence

$$T = \sum_{j,k} D^j E^k (g_k * T * f_j)$$

from (Notation §10(8)), where each $g_k * T * f_j$ is a bounded function from what we previously showed. □

Remark. While [14, Thm. IX(b), p. 72] is stated only for distributions on $\mathbb{R}^n$, our use of it on $\mathcal{D}'(G \times G)$ is justified for the following reason: the theorem in question can be viewed as a corollary to [19, Thm. V.6, p. 123], which states that sequential convergence in $\mathcal{D}'(\Omega)$ is the same in both the weak* and the strong topologies, where Ω is any open set in $\mathbb{R}^n$. The proof of this theorem, upon examination, does carry over exactly to the separable manifold case.

It should be noted that motivation for Theorem 1.6 came from [14, Thm. XVII, p. 275] while the outline of its proof is from [14, Thm. XXV, p. 201]. Its usefulness comes from allowing us to extend the domain of definition of a positive definite distribution from $\mathcal{D}(G)$ to certain subspaces of $L^1(G)$. In $\mathbb{R}^n$, for example, Schwartz uses this to show that all positive definite distributions are tempered. For G a connected, semi simple Lie Group with finite center, we will exhibit a similar result in Chapter V.

Chapter II.

The Godement-Bochner Theorems

Let G be a connected, semi-simple Lie group with finite center, and K a maximal compact subgroup of G. Then define

$$\mathcal{P}_0 = \{f : G \to \mathbb{C} \mid f \text{ is continuous, K-biinvariant, } f \gg 0,\ f(e) \leq 1\}$$

$$\mathcal{P} = \{f \in \mathcal{P}_0 \mid f \text{ is (zonal) spherical}\}.$$

Both $\mathcal{P}_0$ and $\mathcal{P}$ are given the weak* topology as subsets of $L^\infty(G)$. Also define

$$\mathcal{M} = \{\text{positive regular Borel measures } \mu \text{ on } \mathcal{P} \text{ such that } \mu(\mathcal{P}) \leq 1\}$$

and give $\mathcal{M}$ the vague topology, that is, the weak* topology as a subset of $(C_0(\mathcal{P}))'$. For $\mu \in \mathcal{M}$ define $\mathfrak{F}_\mu : G \to \mathbb{C}$ by $\mathfrak{F}_\mu(x) = \int_{\mathcal{P}} \phi(x^{-1})d\mu(\phi)$ and let $\mathfrak{F}(\mathcal{M}) = \{\mathfrak{F}_\mu \mid \mu \in \mathcal{M}\}$. All other symbols are from (Notation §9).

Lemma 2.1. $\mathcal{P}_0$ is a weak*-compact, convex subset of $L^\infty(G)$.

Proof. Convexity is trivial. To show $\mathcal{P}_0$ is weak* compact we have only to show $\mathcal{P}_0$ is norm-bounded and weak* closed [19, p. 137]. Norm-boundedness is trivial since $f \in \mathcal{P}_0$ implies $||f||_\infty \leq f(e) \leq 1$. Hence consider a net $\{f_\alpha\} \subset \mathcal{P}_0$ such that $f_\alpha \to f_0$ in the weak* topology of $L^\infty(G)$. Then $||f_0||_\infty < \infty$ since $f_0 \in L^\infty(G)$, and f_0 is integrally positive definite since each f_α is. Hence we may pick f_0 to be a continuous positive definite function bounded by 1 from the discussion following (Notation §4(11)). We have only to show that f_0 is K-biinvariant. But for each $\phi \in L^1(G)$ we have $f_0[\phi] = \lim_\alpha f_\alpha[\phi] = \lim_\alpha f_\alpha[\phi^\natural]$ $= f_0[\phi^\natural] = f_0^\natural[\phi]$. Thus $f_0 = f_0^\natural$ since both are continuous, proving that f_0 is K-biinvariant. □

Lemma 2.2 $\mathcal{F}(\mathfrak{M})$ is a weak* compact, convex subset of $L^\infty(G)$, and $\mathcal{P} \subset \mathcal{F}(\mathfrak{M}) \subset \mathcal{P}_0$.

Proof. We first note that $\mu \to \mathfrak{F}_\mu$ is continuous from $\mathcal{M}$ to $L^\infty(G)$ when $L^\infty(G)$ is given the weak* topology. For suppose $\{\mu_\alpha\} \to \mu_0$ vaguely in $\mathcal{M}$, and $f \in L^1(G)$. Then $\hat{f} \in C_0(P)$ so that we have $\lim_\alpha \mathfrak{F}_{\mu_\alpha}(f)$

$$= \lim_\alpha \int_{\mathcal{P}} \hat{f}(\phi)d\mu_\alpha(\phi) = \int_{\mathcal{P}} \hat{f}(\phi)d\mu_0(\phi) = \mathfrak{F}_{\mu_0}(f) ,$$

proving the desired continuity.

Now by (Notation §4(4), (5)), for each $\phi \in \mathcal{P}$ we can define $\mu_\phi \in \mathcal{M}$ by $\mu_\phi(\bar{\phi}) = 1$ and $\mu_\phi(\psi) \equiv 0$ otherwise, and we find that $\mathfrak{F}_{\mu_\phi} = \phi$. Hence $\mathcal{P} \subset \mathfrak{F}(\mathcal{M})$. To show $\mathfrak{F}(\mathcal{M}) \subset \mathcal{P}_0$ consider μ any measure in $\mathcal{M}$. Then μ is the vague limit of a net of finite, convex combinations with sum ≤ 1 of measures μ_ϕ, $\phi \in \mathcal{P}$ (generalization of Riemann sums). Hence the continuity of $\mu \mapsto \mathfrak{F}_\mu$ shows that all the $\mathfrak{F}_\mu$, $\mu \in \mathcal{M}$, are weak* limits of functions in $\mathcal{P}_0$, so that Lemma 2.1 gives $\mathfrak{F}(\mathcal{M}) \subset \mathcal{P}_0$.

It is clear that $\mathfrak{F}(\mathcal{M})$ is convex. To show it is weak* compact we note that it is the continuous image of $\mathcal{M}$, a vaguely closed subset of the unit ball in

$(C_0(\mathcal{P}))'$, which is therefore vaguely compact [19, p. 137]. ◻

Lemma 2.3. The extreme points of $\mathcal{P}_0$ are contained in $\mathcal{P} \cup \{0\}$.

Remark. It can be proved by the methods of [2, Thm. A, p. 85] that the extreme points of $\mathcal{P}_0$ actually equal $\mathcal{P} \cup \{0\}$. We never need this fact, however.

Proof. For each $\phi \in \mathcal{P}_0$, $\phi \not\equiv 0$, let π be the unitary representation of G on $\mathcal{H}_\phi$ which is associated to ϕ (Notation §4) and let $e_0 \in \mathcal{H}_\phi$ be such that $\phi(g) = (e_0,\pi(g)e_0)$ for all $g \in G$, where $(\ ,\ )$ is the inner product of $\mathcal{H}_\phi$. Note that ϕ K-biinvariant implies that e_0 is a K-fixed vector. We will now show that if ϕ is extremal, then π is irreducible. This will complete the proof since then ϕ must be spherical (Notation §9).

Suppose ϕ is extremal, and let A be any projection operator commuting with all $\pi(g)$. Then π is irreducible if and only if A = zero or the identity operator I. Now $\phi(g) = (e_0,\pi(g)e_0)$ $= (Ae_0,\pi(g)e_0) + (e_0-Ae_0,\pi(g)e_0) = (Ae_0,\pi(g)Ae_0)$ $+ (e_0-Ae_0,\pi(g)(e_0-Ae_0))$ where we have applied the facts that $A^2 = A$, $A^* = A$ and $(e_0 - Ae_0, \pi(g)Ae_0) = 0$. But both Ae_0 and $e_0 - Ae_0$ are vectors of norm ≤ 1 which are invariant under $\pi(k)$ for all $k \in K$. Hence the functions $\phi_1(g) = (Ae_0, \pi(g)Ae_0)$ and $\phi_2(g) = (e_0 - Ae_0, \pi(g)(e_0 - Ae_0))$ both lie in $\mathcal{P}_0$, and $\phi = \phi_1 + \phi_2$. But ϕ extremal implies $\phi_1 = \lambda\phi$ for some $0 \leq \lambda \leq 1$, so that $(Ae_0, \pi(g)e_0) = \lambda(e_0, \pi(g)e_0)$ for all $g \in G$. Then using the facts that π is unitary and $\pi(xy) = \pi(x)\pi(y)$ gives us $(A - \lambda I)\pi(x)e_0 \perp \pi(y)e_0$ for all $x, y \in G$. But $\{\sum_j \alpha_j \pi(x_j)e_0 \mid \alpha_j \in \mathbb{C}, x_j \in G\}$ is dense in $\mathcal{H}_\phi$, and hence $A = \lambda I$ on $\mathcal{H}_\phi$. Therefore $\lambda = 0$ or 1 since $A^2 = A$, proving that π is irreducible. $\square$

Prop. 2.4. Suppose μ is a finite, complex-valued regular Borel measure on $\mathcal{P}$. Then $\mu \equiv 0$ if and only if $\mathcal{F}_\mu \equiv 0$.

Proof. From (Notation §9(16)) we see that $\mathcal{R} \equiv \{\hat{f} \mid f \in I_c(G)\}$ is dense in $C_0(\mathcal{P})$. But for any μ as specified, $\mu[\hat{f}] = \mathcal{F}_\mu[f] = 0$ for all $f \in I_c(G)$.

Hence $\mathcal{F}_\mu \equiv 0$ implies $\mu \equiv 0$, and the converse is obvious. $\square$

<u>Theorem 2.5</u> (Godement-Bochner) $\mathcal{P}_0$ and $\mathcal{M}$ are in a one-to-one correspondence given by $f = \mathcal{F}_\mu$, or alternately, $f[\psi] = \mu[\hat{\psi}]$ for all $\psi \in L^1(G)$.

<u>Proof</u>. The first three lemmas show that the Krein-Milman theorem applied to $\mathcal{P} \subset \mathcal{F}(\mathcal{M}) \subset \mathcal{P}_0$ gives $\mathcal{F}(\mathcal{M}) = \mathcal{P}_0$. That this correspondence is one-to-one follows directly from Prop. 2.4. $\square$

<u>Remark</u>. Theorem 2.5 was first proved for $G = \mathbb{R}^n$ by Bochner [1, Thm. 19, p. 326], and was later extended to the above form by Godement [7(b), p. 7]. Godement's theorem actually deals with positive definite measures, and its proof relies on a certain amount of operator theory. We now derive this theorem (in its form for distributions) directly from Theorem 2.5.

The proofs up to this point are generalizations of those given by Cartan and Godement for the abelian group case [2], some of which trace further back to Gelfand and Raikov.

Theorem 2.6. (Godement) Let $T \in \mathcal{D}'(G)$, $\dot{T} >> 0$. Then there exists a unique positive regular Borel measure μ on $\mathcal{P}$ such that

(i) $\hat{\phi} \in L^2(\mu)$, $\phi \in \mathcal{D}(G)$

(ii) $T[\phi * \psi^*] = \int_{\mathcal{P}} \hat{\phi}\overline{\hat{\psi}} d\mu$, $\phi, \psi \in I_c(G)$.

Proof. $\phi \in I_c(G)$ implies $\overline{\phi}^* * T * \overline{\phi} >> 0$ (from proof of Theorem 1.6) and is K-biinvariant, even if T is not K-biinvariant itself (Notation §10(10)). Hence Theorem 2.5 gives that $\overline{\phi}^* * T * \overline{\phi} = \mathcal{F}_{\mu_\phi}$ for some finite, positive, regular Borel measure μ_ϕ. Now take any other $\psi \in I_c(G)$ and $f \in L^1(G)$. Then

(1) $\overline{\psi}^* * \overline{\phi}^* * T * \overline{\phi} * \overline{\psi}[f] = \overline{\phi}^* * T * \overline{\phi}[\psi * f * \psi^*]$ (Notation §10(6))

$= \mu_\phi[(\psi * f * \psi^*)\hat{}\,]$ (Theorem 2.5)

$= \int_{\mathcal{P}} \hat{f}(|\hat{\psi}|^2 d\mu_\phi)$ (Notation §9(7)(8))

Since $I_c(G)$ is commutative under convolution, then (1) gives that $\int_{\mathcal{P}} \hat{f}(|\hat{\psi}|^2 d\mu_\phi) = \int_{\mathcal{P}} \hat{f}(|\hat{\phi}|^2 d\mu_\psi)$

for all $f \in L^1(G)$, and thus $|\hat{\psi}|^2 d\mu_\phi = |\hat{\phi}|^2 d\mu_\psi$ for all $\phi,\psi \in I_c(G)$ by Prop. 2.4 and its proof.

For each pair $\phi,\psi \in I_c(G)$ define a measure $d\mu_{\phi,\psi}$ by $4d\mu_{\phi,\psi} = d\mu_{\phi+\psi} - d\mu_{\phi-\psi} + id\mu_{\phi+i\psi} - id\mu_{\phi-i\psi}$. Then using the breakdown of $\phi * \psi^*$ similar to (Ch. I, (5)) we obtain

$$(2) \qquad |\hat{\rho}|^2 d\mu_{\phi,\psi} = \hat{\phi}\bar{\hat{\psi}} d\mu_\rho \quad \text{for all } \rho \in I_c(G).$$

We claim this is enough to show the existence of a positive regular Borel measure μ on $\mathcal{P}$ such that $d\mu_{\phi,\psi} = \hat{\phi}\bar{\hat{\psi}} d\mu$. The argument is a strict generalization of a similar one used by Cartan and Godement in the abelian group case [2, pp. 91-2]. We define

$\mathcal{B} \equiv \{\Phi : \mathcal{P} \to \mathbb{C} \mid \Phi$ is continuous and bounded, and there exists on $\mathcal{P}$ a finite, $\mathbb{C}$-valued, regular Borel measure ν^Φ such that $\Phi d\mu_\rho = |\hat{\rho}|^2 d\nu^\Phi$ for all $\rho \in I_c(G)\}$

In particular, (2) shows that $\hat{\phi}\bar{\hat{\psi}} \in \mathcal{B}$ for all $\phi,\psi \in I_c(G)$. We now establish certain properties of $\mathcal{B}$:

(a) $\Phi \in \mathfrak{B} \Rightarrow \nu^{\Phi}$ is unique. For take ψ_{ε} an approximation of the identity on G such that $||\psi_{\varepsilon}||_1 = 1$ for each $\varepsilon > 0$. Then with $\phi_{\varepsilon} = \psi_{\varepsilon}^{\flat}$ we have $|\hat{\phi}_{\varepsilon}(x)| \leq 1$ for all $\varepsilon > 0$ and $x \in G$, and $|\hat{\phi}_{\varepsilon}|^2 \to 1$ pointwise. Hence $|\hat{\phi}_{\varepsilon}|^2 d\nu^{\Phi} \to d\nu^{\Phi}$ vaguely. But $|\hat{\phi}_{\varepsilon}|^2 d\nu^{\Phi} = \Phi d\mu_{\phi_{\varepsilon}}$ so that $d\nu^{\Phi}$ is unique.

(b) If $\Phi \in \mathfrak{B}$ such that $\Phi \geq 0$, then $\nu^{\Phi} \geq 0$. This follows from (a) by taking $\phi_{\varepsilon} * \phi_{\varepsilon}^{*}$ in place of ϕ_{ε}, i.e., we make $\phi_{\varepsilon} >> 0$. For then we have $\Phi d\mu_{\phi_{\varepsilon}} \to d\nu^{\Phi}$ vaguely, where $d\mu_{\phi_{\varepsilon}} \geq 0$.

(c) If $\Phi = \hat{\phi}\overline{\hat{\psi}}$ for $\phi, \psi \in I_c(G)$, then $\nu^{\Phi} = \mu_{\phi,\psi}$. This is clear from (2) and (a).

(d) If $\Phi, \Psi \in \mathfrak{B}$, then $\Phi + \Psi \in \mathfrak{B}$ and $\nu^{(\Phi+\Psi)} = \nu^{\Phi} + \nu^{\Psi}$.

(e) If $\Phi \in \mathfrak{B}$ and $A : \mathfrak{P} \to \mathbb{C}$ is continuous and bounded, then $A\Phi \in \mathfrak{B}$ and $\nu^{A\Phi} = A\nu^{\Phi}$. Both (d) and (e) follow from (a).

We now claim that $C_c(\mathfrak{P}) \subset \mathfrak{B}$. For take any $\Psi \in C_c(\mathfrak{P})$.

Then there exists $\Phi \in \mathcal{G}$ such that Φ is bounded away from zero on the support of Ψ by the following argument: for each $\varepsilon > 0$ let ψ_ε be as in (a), and define $\mathcal{U}_\varepsilon \subset \mathcal{P}$ by $\mathcal{U}_\varepsilon \equiv \{\phi \in \mathcal{P} \mid |1-\hat{\psi}_\varepsilon[\phi]| < 1/2\}$. Then since $\hat{\psi}_\varepsilon$ goes pointwise to 1 on $\mathcal{P}$ we have that $\{\mathcal{U}_\varepsilon\}_{\varepsilon>0}$ covers $\mathcal{P}$. Moreover, by definition of the topology on $\mathcal{P}$, each $\mathcal{U}_\varepsilon$ is open. Now let $\mathcal{C} \equiv \operatorname{supp} \Psi$, so that $\mathcal{C} \subset \bigcup_{j=1}^{m} \mathcal{U}_{\varepsilon_j}$ for some ε_j, $j=1,\dots,m$.

Then since $\mathcal{P}$ is Hausdorff locally compact, applying [15, Thm. A, p. 164] and [3, Thm. 6.2(3), p. 238] gives the existence of continuous bounded functions α_j, $j=1,\dots,m$, on $\mathcal{P}$ such that $\operatorname{supp} \alpha_j \subset \mathcal{U}_{\varepsilon_j}$, $1 \geq \alpha_j \geq 0$, and $(\Sigma\alpha_j)|_{\mathcal{C}} \equiv 1$. Define $\Phi = \sum_{j=1}^{m} \alpha_j \hat{\psi}_{\varepsilon_j}$. Then (d) and (e) show that $\Phi \in \mathcal{G}$. Moreover, for $\phi \in \mathcal{C}$ we have

$$|1-\Phi(\phi)| = \Big|\sum_{j=1}^{m} \alpha_j(\phi)(1-\hat{\phi}_{\varepsilon_j}(\phi))\Big| \leq \sum_{j=1}^{m} \alpha_j(\phi)|1-\hat{\phi}_{\varepsilon_j}(\phi)|$$

$\leq \frac{1}{2} \sum_{j=1}^{m} \alpha_j(\phi) = \frac{1}{2}$. Thus Φ is bounded away from zero on $\operatorname{supp} \Psi$. It is then clear that $\Psi = A\Phi$ for some A continuous and bounded on $\mathcal{P}$, so that (e) implies $\Psi \in \mathcal{G}$.

Hence $C_c(\mathcal{P}) \subset \mathcal{Y}$ as claimed.

We now define a function from $C_c(\mathcal{P})$ to $\mathbb{C}$ by $\Psi \to \int_{\mathcal{P}} d\nu^{\Phi}$, which is linear from (d) and positive from (b). Thus there exists a unique positive regular Borel measure μ on $\mathcal{P}$ such that $\int_{\mathcal{P}} d\nu^{\Psi} = \int_{\mathcal{P}} \Psi d\mu$ by the Riesz Theorem. Now for each $\Phi \in \mathcal{Y}$ and $A \in C_c(\mathcal{P})$ we have $\int_{\mathcal{P}} A d\nu^{\Phi} = \int_{\mathcal{P}} d\nu^{A\Phi} = \int_{\mathcal{P}} A\Phi d\mu$ and hence $d\nu^{\Phi} = \Phi d\mu$ for all $\Phi \in \mathcal{Y}$. In particular $d\mu_{\phi,\psi} = \hat{\phi}\hat{\bar{\psi}} d\mu$ for all $\phi,\psi \in I_c(G)$, and hence $T[\phi*\psi^*] = \bar{\phi}^* * T * \bar{\psi}(e) = \int_{\mathcal{P}} d\mu_{\phi,\psi}$ $= \int_{\mathcal{P}} \hat{\phi}\hat{\bar{\psi}} d\mu$, proving (i) and (ii).

We now have only to show that μ is unique. Suppose ν is any other regular Borel measure satisfying (i) and (ii). Then for any $\rho \in I_c(G)$ we have $\int_{\mathcal{P}} \hat{\rho}\hat{\phi}\hat{\bar{\psi}} d\nu = T[(\rho*\phi)*\psi^*]$ $= \int_{\mathcal{P}} \hat{\rho}\hat{\phi}\hat{\bar{\psi}} d\mu$ and thus by Prop. 2.4 and its proof we have that $\hat{\phi}\hat{\bar{\psi}} d\nu = \hat{\phi}\hat{\bar{\psi}} d\mu$ for all $\phi,\psi \in I_c(G)$. Hence take any $\Psi \in C_c(\mathcal{P})$. Then as done above, there exist, for each $j = 1,\ldots,m$, functions $\psi_j \in I_c(G)$ and α_j continuous bounded on $\mathcal{P}$, such that, with $\Phi \equiv \sum_{j=1}^{m} \alpha_j |\hat{\psi}_j|^2$, we have

$\Psi = A\Phi$ for some A continuous bounded on $\mathcal{P}$. Therefore $\Psi d\mu = A(\sum_{j=1}^{m} \alpha_j |\hat{\psi}_j|^2) d\mu = A(\sum_{j=1}^{m} \alpha_j |\hat{\psi}_j|^2 d\nu = \Psi d\nu$. Hence $\mu = \nu$. □

Remark. Theorem 2.6 was proved by Godement for measures, the proof just given being similar to the original [7(b)] except that Theorem 2.5 is used to replace certain facts from Hilbert space operator theory.

Notice that we have not used the results of Chapter I for the above proofs. The Chapter I results, combined with the properties of the $I^p(G)$ and $\overline{Z}(\mathfrak{F}^{\varepsilon})$ spaces to be defined in the next two chapters, will allow us to characterize the measures in Theorem 2.6, and also to refine the result into the usual Bochner Theorem form.

Chapter III.

The $\mathcal{C}^p(G)$ and $I^p(G)$ Spaces

Let G be a connected, semi-simple Lie group with finite center, and K a maximal compact subgroup. Recalling the functions Ξ and σ (Notation §9(20)(21)) we define the following K-biinvariant "Schwartz-like" spaces: for each $0 < p \leq 2$ let $I^p(G)$ be the set of infinitely differentiable, K-biinvariant functions f such that for all $m \in \mathbb{Z}^+$ and all left invariant differential operators D we have

$$(1) \quad \sup_G (1+\sigma)^m \Xi^{-2/p} |Df| < \infty.$$

Topologized in the obvious manner makes each $I^p(G)$ into a Frechet space.

Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be a Cartan decomposition for (G,K), and let the function $\delta : \mathfrak{p} \to \mathbb{R}$ be determined by the formula $\int_{G/K} f(x)dx = \int_{\mathfrak{p}} f((\exp X)K)\delta(X)dX$, where dX is the Euclidean measure on $\mathfrak{p}$, normalized so that $\delta(0) = 1$. Using the notation $g = k \exp X$ for $k \in K$, $X \in \mathfrak{p}$, then for each $m \in \mathbb{Z}^+$ we define $\delta_m(g) = \delta(X)(1+|X|^m)$. We claim that the definition of each $I^p(G)$ space is unaltered by replacing (1) with

$$(2) \qquad \sup_G |\delta_m^{1/p} Df| < \infty.$$

For we have that each δ_m is K-biinvariant and

$$\delta(H) = \prod_{\alpha \in \Sigma^+} \left(\frac{\sinh \alpha(H)}{\alpha(H)}\right)^{m_\alpha} \quad \text{for } H \in \mathfrak{a} \text{ [10(c), pp. 33-4]},$$

implying that $\delta(H)e^{-2\rho(H)}$ remains bounded on $\mathfrak{a}^+$. Our claim then follows easily from (Notation §9(24)).

<u>Notation</u>. [18(b), p. 104] For $X \in \mathfrak{g}$ we define $f(x;X) \equiv \frac{d}{dt} f(x \exp tX)|_{t=0}$ for all $x \in G$. Then in the usual way this extends to a representation of the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$ on the algebra of all left invariant differential operators on G. This we also write as $f(x;D)$, $D \in \mathfrak{U}(\mathfrak{g})$. On the other hand, defining $f(X;x) \equiv \frac{d}{dt} f(\exp tX \cdot x)|_{t=0}$, we have that this extends to a representation of $\widetilde{\mathfrak{U}(\mathfrak{g})}$ on the algebra of all right invariant differential operators on G

Chapter III.

The $\mathcal{C}^p(G)$ and $I^p(G)$ Spaces

Let G be a connected, semi-simple Lie group with finite center, and K a maximal compact subgroup. Recalling the functions Ξ and σ (Notation §9(20)(21)) we define the following K-biinvariant "Schwartz-like" spaces: for each $0 < p \leq 2$ let $I^p(G)$ be the set of infinitely differentiable, K-biinvariant functions f such that for all $m \in \mathbb{Z}^+$ and all left invariant differential operators D we have

(1) $\sup_G (1+\sigma)^m \Xi^{-2/p} |Df| < \infty.$

Topologized in the obvious manner makes each $I^p(G)$ into a Frechet space.

Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be a Cartan decomposition for (G,K), and let the function $\delta : \mathfrak{p} \to \mathbb{R}$ be determined by the formula $\int_{G/K} f(x)dx = \int_{\mathfrak{p}} f((\exp X)K)\delta(X)dX$, where dX is the Euclidean measure on $\mathfrak{p}$, normalized so that $\delta(0) = 1$. Using the notation $g = k \exp X$ for $k \in K$, $X \in \mathfrak{p}$, then for each $m \in \mathbb{Z}^+$ we define $\delta_m(g) = \delta(X)(1+|X|^m)$. We claim that the definition of each $I^p(G)$ space is unaltered by replacing (1) with

$$(2) \qquad \sup_G |\delta_m^{1/p} Df| < \infty.$$

For we have that each δ_m is K-biinvariant and

$$\delta(H) = \prod_{\alpha \in \Sigma^+} \left(\frac{\sinh \alpha(H)}{\alpha(H)}\right)^{m_\alpha} \quad \text{for } H \in \mathfrak{a} \quad [10(c), \text{ pp. } 33\text{-}4],$$

implying that $\delta(H)e^{-2\rho(H)}$ remains bounded on $\mathfrak{a}^+$. Our claim then follows easily from (Notation §9(24)).

<u>Notation</u>. [18(b), p. 104] For $X \in \mathfrak{g}$ we define $f(x;X) \equiv \frac{d}{dt} f(x \exp tX)\big|_{t=0}$ for all $x \in G$. Then in the usual way this extends to a representation of the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$ on the algebra of all left invariant differential operators on G. This we also write as $f(x;D)$, $D \in \mathfrak{U}(\mathfrak{g})$. On the other hand, defining $f(X;x) \equiv \frac{d}{dt} f(\exp tX \cdot x)\big|_{t=0}$, we have that this extends to a representation of $\widetilde{\mathfrak{U}(\mathfrak{g})}$ on the algebra of all right invariant differential operators on G

($\widetilde{\mathfrak{U}(\mathfrak{g})}$ is the <u>reverse</u> algebra of $\mathfrak{U}(\mathfrak{g})$; i.e., they are the same linear spaces but multiplication is reversed: $(D_1 \cdot D_2)_{\tilde{\mathfrak{U}}} \equiv (D_2 \cdot D_1)_{\mathfrak{U}}$). hence we have $f(X_1 \ldots X_k; x; Y_1 \ldots Y_\ell) = \tilde{X}_k \ldots \tilde{X}_1 \tilde{Y}_1 \ldots \tilde{Y}_\ell f(x)$, where the $\tilde{X}_j$ are right invariant, the $\tilde{Y}_j$ left invariant differential operators.

Let $L(g)x \equiv gx$, $R(g)x \equiv xg^{-1}$. We can then extend $Ad(g)$ to $\mathfrak{U}(\mathfrak{g})$ by $Ad(g)D \equiv D^{R(g)}$ where $\mathfrak{U}(\mathfrak{g})$ here is identified with the left invariant differential operators. The same extension is obtained by definint $Ad(g)$ on $\widetilde{\mathfrak{U}(\mathfrak{g})}$ by $Ad(g)D = D^{L(g)}$, where $\widetilde{\mathfrak{U}(\mathfrak{g})}$ is identified with the right invariant differential operators as above. We use the notation $Ad(g)D \equiv gD$.

Now straightforward calculations show that for $f \in C^\infty(G)$ and $D, \tilde{D} \in \mathfrak{U}(\mathfrak{g})$ we have

(3) $$f(\tilde{D}; x_1 x x_2; D) = f(x_1^{-1}\tilde{D}; x; x_2 D), \text{ and}$$

(4) $$f(\tilde{D}; x; D) = f(x; (x^{-1}\tilde{D})D).$$

Using this we prove the following lemma.

Lemma 3.1. (Harish-Chandra) Given $D, \tilde{D} \in \mathfrak{U}(\mathfrak{g})$ there exists a finite number of $D_j \in \mathfrak{U}(\mathfrak{g})$ $(1 \leq j \leq p)$ such that if $f \in C^\infty(G)$, K-biinvariant, then

$$|f(\tilde{D};x;D)| \leq \sum_{j=1}^{p} |f(x;D_j)| \quad \text{and} \quad |f(\tilde{D};x;D)| \leq \sum_{j=1}^{p} |f(D_j;x)|$$

for all x G.

Proof. Fix a basis $\{X_1,\ldots,X_n\}$ of $\mathfrak{g}$ and for each $m \in \mathbb{Z}^+$ let $\mathfrak{U}_m(\mathfrak{g})$ be the finite dimensional subspace of $\mathfrak{U}(\mathfrak{g})$ spanned by $\{X_{i_1}\ldots X_{i_k} \mid k \leq n\}$. Now pick m such that $D, \tilde{D} \in \mathfrak{U}_m(\mathfrak{g})$ and choose a basis B for $\mathfrak{U}_m(\mathfrak{g})$ with the following properties:

(a) Each $D_b \in B$ has the form $D_b = D_{\mathfrak{k}} D_{\mathfrak{a}} D_{\mathfrak{n}}$ where $D_{\mathfrak{k}} \in \mathfrak{U}(\mathfrak{k})$, $D_{\mathfrak{a}} \in \mathfrak{U}(\mathfrak{a})$ and $D_{\mathfrak{n}} \in \mathfrak{U}(\mathfrak{n})$; and

(b) For each $D_b \in B$ we have $m_j \in \mathbb{Z}^+$ $(1 \leq j \leq p)$ such that for all $a \in \mathfrak{Cl}(A^+)$ we have

$aD_{\mathfrak{n}} = \exp(\sum_j m_j\lambda_j(\log a))D_{\mathfrak{n}}$, where the $\{\lambda_j\}$ are a fundamental positive root system. Notice that (b) is possible by taking a basis of $\mathfrak{n}$ comprised of root vectors: i.e., $\mathfrak{n} = \sum_{\alpha>0} \mathfrak{g}_\alpha$, and $X \in \mathfrak{g}_\alpha$ implies

$aX = \mathrm{Ad}(\exp(\log a))X = \exp(\mathrm{ad}(\log a))X = \exp(\alpha(\log a))X$

$= \exp(\sum_j m_j\lambda_j(\log a))X$ where $\alpha = \sum_j m_j\lambda_j$.

We now have $kD = \sum_B a_b(k)D_b$ and $k\tilde{D} = \sum_B \tilde{a}_b(k)D_b$, where a_b and $\tilde{a}_b$ are continuous functions on K for each $b \in B$. Hence, letting $M_0 = \sup_{k\in K}\max_{b\in B} \{|a_b(k)|, |\tilde{a}_b(k)|\}$ and $x = k_1ak_2$, we obtain $|f(\tilde{D};x;D)| = |f(k_1^{-1}\tilde{D};a;k_2D)|$

$\leq \sup_{k_1,k_2\in K} |f(k_1^{-1}\tilde{D};a;k_2D)| \leq M_0^2 \sum_{B\times B} |f(\tilde{D}_b;a;D_b)|$.

Now write $\tilde{D}_b = \tilde{D}_{\mathfrak{k}}\tilde{D}_{\mathfrak{a}}\tilde{D}_{\mathfrak{n}}$. Since f is K-biinvariant we may assume $\tilde{D}_{\mathfrak{k}} \equiv 1$, for otherwise $f(\tilde{D}_b;x;D_b) = 0$. Then we have $f(\tilde{D}_b;x;D_b) = f(a;(a^{-1}\tilde{D}_b)D_b) = f(a;\tilde{D}_{\mathfrak{a}}(a^{-1}\tilde{D}_{\mathfrak{n}})D_b)$

$= f(a;\tilde{D}_{\mathfrak{a}}(\prod_j \exp(-m_j\lambda_j(\log a))\tilde{D}_{\mathfrak{n}})D_b)$

$= \prod_j \exp(-m_j\lambda_j(\log a))f(a;\tilde{D}_{\mathfrak{a}}\tilde{D}_{\mathfrak{n}}D_b)$. Thus we have

$|f(\tilde{D}_b;a;D_b)| \leq |f(a;\tilde{D}_{\mathfrak{a}}\tilde{D}_{\mathfrak{n}}D_b)|$ since

$|\exp(-m_j\lambda_j(\log a))| \leq 1$ for all $a \in A^+$.

Now let $\{D_j\}_{j=1}^p$ be a basis of the subspace of $\mathcal{H}(\mathfrak{g})$ spanned by $\{k(\tilde{D}_{\mathfrak{a}}\tilde{D}_{\mathfrak{n}}D_b) \mid k \in K, \tilde{D}_b \in B, D_b \in B\}$. then $k(\tilde{D}_{\mathfrak{a}}\tilde{D}_{\mathfrak{n}}D_b) = \sum_{j=1}^{p} c_j(k,\tilde{D}_b,D_b)D_j$ for $c_j : K \times B \times B \to \mathbb{R}$, where each c_j is continuous with respect to k when D_b and $\tilde{D}_b$ are held constant. Thus $M_2 = \sup_{j,k,D_b,\tilde{D}_b} |c_j| < \infty$ and we obtain

$$|f(a;\tilde{D}_{\mathfrak{a}}\tilde{D}_{\mathfrak{n}}D_b)| = |f(x;k_2^{-1}(\tilde{D}_{\mathfrak{a}}\tilde{D}_{\mathfrak{n}}D_b)| = |f(x;\sum_j c_jD_j)|$$

$$\leq M_2 \sum_j |f(x;D_j)| .$$

Putting everything together we have $|f(\tilde{D};x;D)|$

$$\leq M_0^2 \sum_{B\times B} |f(\tilde{D}_b;a;D_b)| \leq M_0^2 \sum_{B\times B} |f(a;\tilde{D}_{\mathfrak{a}}\tilde{D}_{\mathfrak{n}}D_b)|$$

$\leq cM_0^2M_2 \sum_{j=1}^{p} |f(x;D_j)|$ where c = the number of elements in B. Hence the first half of Lemma 3.1 is proved, and the second follows in an identical manner. $\square$

<u>Remark</u>. The proof of Lemma 3.1 is merely a rewriting of [18(b), Lemma 2, p. 164] for the simpler, K-biinvariant

case.

Now for each $0 < p \leq 2$ let $\mathcal{C}^p(G)$ be the set of infinitely differentiable functions f such that for all $m \in \mathbb{Z}^+$ and all $D, \tilde{D} \in \mathfrak{U}(\mathfrak{g})$ we have

$$(5) \qquad \sup_{x \in G} (1+\sigma(x))^m \Xi(x)^{-2/p} |f(\tilde{D};x;D)| < \infty$$

Topologized in the obvious manner makes each $\mathcal{C}^p(G)$ into a Frechet space. Since $|\Xi| \leq 1$ it is clear that $\mathcal{C}^{p_1}(G) \subset \mathcal{C}^{p_2}(G)$ whenever $p_1 \leq p_2$, and (Notation §9 (25)) shows that if $f \in \mathcal{C}^p(G)$ then $Df \in L^p(G)$ for any left or right invariant differential operator D.

<u>Prop. 3.2</u>. $I^p(G) = \{f \in \mathcal{C}^p(G) \mid f$ is K-biinvariant$\}$, and the topology on $I^p(G)$ is the relative topology as a subset of $\mathcal{C}^p(G)$.

<u>Proof</u>. From Lemma 3.1 it is clear that if $f \in I^p(G)$, then (1) is true for D a <u>right</u> invariant differential operator. Hence this proposition follows trivially from the definition of $\mathcal{C}^p(G)$. ◻

Remark. For $p = 2$, $I^p(G)$ is the space $I(G)$ from [9(b), §12, p. 585], $\mathcal{C}^p(G)$ is the space $\mathcal{C}(G)$ from [9(d), §9, p. 19], and Prop. 3.2 is contained in [9(d), §20, p. 46].

A trivial but useful result of Prop. 3.2 is that

6) $f^* \in I^p(G)$ if and only if $f \in I^p(G)$,

where $f^*(x) = \overline{f(x^{-1})}$.

Prop. 3.3. Suppose $f \in I^p(G)$ for all $p > p_0$ (or $p \geq p_0$) for some fixed $p_0 > 0$. Then there exists a sequence $\{\phi_j\} \subset I_c(G)$ such that $\phi_j \to f$ in $I^p(G)$ for all $p > p_0$. In particular, $I_c(G)$ is dense in all the $I^p(G)$ spaces.

Proof. As noted in [10(b), p. 571] there exists a sequence $\{\psi_j\}$ in $I_c(G)$ and U_j open sets in G, increasing up to G, such that (a) $\psi_j|_{U_j} \equiv 1$, and (b) for each left invariant differential operator D on G we have $||D\psi_j||_\infty \leq c(D)$, where $c(D)$ is a constant depending only on D. The proof that $\phi_j \equiv f\psi_j$ satisfies the desired conditions for all $p > p_0$ (or $p \geq p_0$) is then the same as the $p = 2$ case which is given in [10(b), p. 571]. □

Prop. 3.4. Suppose S is a continuous linear functional on $I^p(G)$. Then the restriction of S to $I_c(G)$ extends uniquely, and in a one-to-one fashion, to a K-biinvariant distribution T on G.

Proof. For each $\phi \in \mathcal{D}(G)$ define $T[\phi] \equiv S[\phi^\natural]$, where $\phi^\natural$ is defined as in (Notation §7(6)). In order to show that $T \in \mathcal{D}'(G)$ we have only to show that if $\{\phi_\alpha\}$ is a net in $\mathcal{D}(G)$ converging to zero, then $\{\phi_\alpha^\natural\}$ converges to zero in $I^p(G)$. To do so take D any left invariant differential operator, and note that we may write $D = \sum_{j=1}^{m} \alpha_j D^j$ where $\alpha_j \in C^\infty(G)$ and the D^j are right invariant. We may also assume without loss of generality that the supports of all the ϕ_α's are contained in some compact set C. We then have

$$\sup_{G} (1+\sigma)^m \Xi^{-2/p} |D\phi_\alpha^\natural| \leq \sup_{\substack{g\in G \\ k\in K}} (1+\sigma(g))^m \Xi^{-2/p}(g) |D\phi_\alpha^{R(k)}(g)|$$

$$\leq C \sum_{j=1}^{m} \sup_{\substack{g\in G \\ k\in K}} |D^j \phi_\alpha(gk)|$$

where we set $c \equiv \sup_{\substack{g\in \mathcal{C} \\ j=1,\dots,m}} |(1+\sigma(g))^m \Xi^{-2/p}(g)\alpha_j(g)|$

Hence $\{\phi_\alpha^\natural\}$ converges to zero in $I^p(G)$, proving that $T \in \mathcal{D}'(G)$. The uniqueness of T is obvious from (Notation §10(11)), and the one-to-one nature of this correspondence is a consequence of Prop. 3.3. □

<u>Remark</u>. Prop. 3.4 allows us to consider $(I^p(G))'$ as a subset of K-biinvariant elements in $\mathcal{D}'(G)$. This is not, however, an onto mapping. What we will now proceed to prove is that $(I^p(G))'$ is in a one-to-one correspondence with the K-biinvariant elements of $(\mathcal{C}^p(G))'$.

We define two maps L and R from G into $\operatorname{Aut} C^\infty(G)$ by $L(g)f \equiv f^{L(g)}$ and $R(g)f \equiv f^{R(g)}$. We then have

Prop. 3.5. For each $0 < p \leq 2$ we have

(a) both L and R are differentiable representations of G on $\mathcal{C}^p(G)$,

(b) $\mathcal{C}^p(G)$ is a convolution algebra.

Proof. For $p = 2$, (a) is [18(b), Prop. 8.3.7.8, p. 158] and (b) is [18(b), Prop. 8.3.7.14, p. 164]. Both these proofs carry over to the general $0 < p \leq 2$ case for the following reasons: for (a) the only property of Ξ used in the proof is (Notation §9(26)), which is clearly also true for $\Xi^{2/p}$. The proof of (b) starts by showing (Notation §10(1)) is true for arbitrary f and $g \in \mathcal{C}^2(G)$. This obviously proves it for the $\mathcal{C}^p(G)$ case since $\mathcal{C}^p(G) \subset \mathcal{C}^2(G)$ for all $0 < p \leq 2$. For the rest of the proof we need only substitute $\Xi^{2/p}$ wherever Ξ appears, and note that $\Xi^2 f \in L^1(G)$ implies $\Xi^{4/p} f \in L^1(G)$ for any function f since Ξ is bounded. □

Theorem 3.6. For each $0 < p \leq 2$ we have

(a) $T \in (\mathcal{C}^p(G))'$ implies that the restriction $\overline{T}$ of T to $I^p(G)$ is in $(I^p(G))'$; and

(b) $S \in (I^p(G))'$ implies there exists a unique K-biinvariant $T \in (\mathcal{C}^p(G))'$ such that $S = \bar{T}$.

Remark. Thus $(I^p(G))'$ may be considered as the space of K-biinvariant elements of $(\mathcal{C}^p(G))'$, which form a subspace of the K-biinvariant distributions by Prop. 3.4.

Proof. We obtain (a) directly from Prop. 3.2. In order to prove (b) we first show that $f \mapsto f^{\natural}$ defines a continuous endomorphism of $\mathcal{C}^p(G)$ to $\mathcal{C}^p(G)$.

By Prop. 3.5 (a) and (Notation §3) we see, for each radon measure μ on G with compact support, that $f \mapsto L(\mu)f = \int_G L(x)f d\mu(x)$ (Bochner integral) and $f \mapsto R(\mu)f = \int_G R(x)f d\mu(x)$ are continuous endomorphisms of $\mathcal{C}^p(G)$. Moreover, for each $T \in (\mathcal{C}^p(G))'$ we have $T(L(\mu)f) = \int_G T(L(x)f)d\mu(x)$ and $T(R(\mu)f) = \int_G T(R(x)f)d\mu(x)$.

Now if dk indicates the Haar measure on K such that $\int_K dk = 1$, and for each $x \in G$ if $S_x \in (\mathcal{C}^p(G))'$ is defined by $S_x[\phi] = \phi(x)$, then we find $f^{\natural} = R(dk)L(dk)f$, which proves our claim.

Now take $S \in (I^p(G))'$, and define $T[f] = S[f^\natural]$ for all $f \in \mathcal{C}^p(G)$. Then since $f \mapsto f^\natural$ is continuous from $\mathcal{C}^p(G)$ into $\mathcal{C}^p(G)$ (hence also into $I^p(G)$) we see that $T \in (\mathcal{C}^p(G))'$, and $\bar{T} = S$.

Suppose $T_1 \in (\mathcal{C}^p(G))'$ is K-biinvariant. Then for any $f \in \mathcal{C}^p(G)$ we have $T_1[f^\natural] = T_1[R(dk)L(dk)f]$ $= \int\int_{K\times K} T_1[R(k_1)L(k_2)f]dk_1dk_2 = T_1[f]$. Thus the T defined above must be unique. □

Chapter IV.

The $\overline{Z}(\mathfrak{F}^{\varepsilon})$ Spaces

In this chapter we define the $\overline{Z}(\mathfrak{F}^{\varepsilon})$ spaces, which turn out to be images under the fourier transform of the $I^p(G)$ spaces. We also construct certain special elements in $\overline{Z}(\mathfrak{F}^{\varepsilon})$ which will be needed in the last two chapters.

Let C_ρ be the closed, convex hull in $\mathfrak{a}^*$ of the finite set $\{s\rho \mid s \in W\}$, and for each $\varepsilon \geq 0$ let $\mathfrak{F}^{\varepsilon} \equiv \mathfrak{a}^* + i\varepsilon C_\rho$ in $\mathfrak{a}_{\mathbb{C}}^*$. This notation is consistent with out previous definition of $\mathfrak{F}^1$ in (Notation §9) because of (Notation §9(15)). Then define $Z(\mathfrak{F}^0) = \mathcal{S}(\mathfrak{a}^*)$ and for each $\varepsilon > 0$ define

$Z(\mathfrak{F}^{\varepsilon}) \equiv \{\Phi : \text{Int } \mathfrak{F}^{\varepsilon} \to \mathbb{C}$ such that

(1) Φ is holomorphic on Int $\mathfrak{F}^{\varepsilon}$

(2) For each holomorphic differential operator D with polynomial coefficients we have $\sup_{\text{Int } \mathfrak{F}^{\varepsilon}} |D\Phi| < \infty\}$

For all $\varepsilon \geq 0$ we let $\overline{Z}(\mathfrak{F}^{\varepsilon})$ be the subspace of W-invariant elements in $Z(\mathfrak{F}^{\varepsilon})$.

Since C_ρ is W-invariant, then (Notation §6(3)) shows that $-C_\rho = C_\rho$, and hence $-\mathfrak{F}^{\varepsilon} = \mathfrak{F}^{\varepsilon}$ for each $\varepsilon \geq 0$. An important property for us will be

$$(3) \qquad \text{Int } \mathfrak{F}^{\varepsilon} = \bigcup_{0<\varepsilon'<\varepsilon} \mathfrak{F}^{\varepsilon'}$$

which is [17, Lemma 3.2.2, p. 275].

We have that each $Z(\mathfrak{F}^{\varepsilon})$ is an algebra, and when topologized in the obvious manner, becomes a Frechet space where multiplication is jointly continuous [17, §3.4, p. 278]. It is clear that $\overline{Z}(\mathfrak{F}^{\varepsilon})$ is a closed subalgebra of $Z(\mathfrak{F}^{\varepsilon})$. Another useful fact is that for each D as in (2), and each $f \in Z(\mathfrak{F}^{\varepsilon})$, Df extends to a continuous function on all of $\mathfrak{F}^{\varepsilon}$ [17, §3.4, p. 278]. We will now construct certain special elements in $\overline{Z}(\mathfrak{F}^{\varepsilon})$.

<u>Prop. 4.1</u>. For each $\varepsilon > 0$ there exists a non-constant W-invariant holomorphic polynomial on $\mathfrak{a}_{\mathbb{C}}^{*}$ which is uniformly bounded away from zero on $\mathfrak{F}^{\varepsilon}$. Moreover,

for P any such polynomial and $\Phi \in \bar{Z}(\mathfrak{F}^{\varepsilon})$, then $\Psi \in \bar{Z}(\mathfrak{F}^{\varepsilon})$ for $\Psi = \Phi/P$.

Proof. Let $\varepsilon_1, \ldots, \varepsilon_\ell$ be any basis of $\mathfrak{a}^*$. Then if $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ let $\lambda = \sum_{j=1}^{\ell} \lambda_j \varepsilon_j$, $\lambda_j \in \mathbb{C}$, $\lambda = \xi + i\eta$ where $\xi, \eta \in \mathfrak{a}^*$, and for each $c > 0$ let $P_c(\lambda) = c + \sum_{j=1}^{\ell} \lambda_j^2$.
We then have $P_c(\lambda) = c + \Sigma\, \xi_j^2 - \Sigma\, \eta_j^2 + 2i\, \Sigma\, \xi_j \eta_j$.
But as a function on $\mathfrak{F}^{\varepsilon}$ we have that $|\Sigma\, \eta_j^2|$ is uniformly bounded by some finite constant, say c_0.
Taking $c = 2c_0$ we obtain $|P_c(\lambda)| \geq |2c_0 + \sum_j \xi_j^2 - \sum_j \eta_j^2|$ $\geq 2c_0 - c_0 = c_0 > 0$ for all $\lambda \in \mathfrak{F}^{\varepsilon}$. Then $P(\lambda) = \prod_{s \in W} P_c(\lambda^s)$ satisfies the desired conditions since $\mathfrak{F}^{\varepsilon}$ is W-invariant.

Suppose $\Phi \in \bar{Z}(\mathfrak{F}^{\varepsilon})$. Then $\Psi = \Phi/P$ on $\operatorname{Int} \mathfrak{F}^{\varepsilon}$ is clearly well-defined, holomorphic and W-invariant. We have then only to show that for each holomorphic differential operator D with polynomial coefficients we have $\sup_{\operatorname{Int} \mathfrak{F}^{\varepsilon}} |D\Psi| < \infty$. We proceed by induction on the order of D.

(a) Suppose order $D = 0$. Then $D = Q$ for some holomorphic polynomial Q on $\mathfrak{a}^*_{\mathbb{C}}$. Then

$$\sup_{\mathrm{Int}\,\mathfrak{F}^{\varepsilon}} |D\Psi| \leq \frac{1}{c} \sup_{\mathrm{Int}\,\mathfrak{F}^{\varepsilon}} |Q\Phi| \quad \text{where} \quad |P| \geq c \text{ on } \mathfrak{F}^{\varepsilon}.$$

But this last quantity is finite since $\Phi \in \bar{Z}(\mathfrak{F}^{\varepsilon})$.

(b) Suppose that $\sup_{\mathrm{Int}\,\mathfrak{F}^{\varepsilon}} |D\Psi| < \infty$ for all D of order $\leq k - 1$ and for all $\Psi = \Phi/P$ where $\Phi \in \bar{Z}(\mathfrak{F}^{\varepsilon})$ and P is as above.

(c) Take D of order k. Using the coordinate system given above we may assume without loss of generality that $D = E \frac{\partial}{\partial \lambda_1}$ where E is a differential operator as in (b). Hence $D\Psi = E\left(\frac{\frac{\partial}{\partial \lambda_1}\Phi}{P}\right) + E\left(\frac{(\frac{\partial P}{\partial \lambda_1})\Phi}{P^2}\right)$.

But both $\frac{\partial}{\partial \lambda_1}\Phi$ and $(\frac{\partial P}{\partial \lambda_1})\Phi$ are in $\bar{Z}(\mathfrak{F}^{\varepsilon})$, and both P and P^2 satisfy the conditions of (b). Hence the induction assumption proves that $\sup_{\mathrm{Int}\,\mathfrak{F}^{\varepsilon}} |D\Psi| < \infty$. □

Remark. While Prop. 4.1 will be sufficient in its present form for completing the Spherical Bochner Theorem, a more refined version, which we now derive, will be needed in the last chapter.

Lemma 4.2. For each $\alpha \in \mathfrak{a}^*_{\mathbb{C}}$ there exists a W-invariant holomorphic polynomial P_α on $\mathfrak{a}^*_{\mathbb{C}}$ such that degree P_α is uniformly bounded in α, and

(i) $P_\alpha(\alpha) = 0$ for all $\alpha \in \mathfrak{a}^*_{\mathbb{C}}$

(ii) $P_{\alpha^n} \to P_\alpha$ uniformly on compacts as $\alpha^n \to \alpha$ in $\mathfrak{a}^*_{\mathbb{C}}$

(iii) Given U a compact set in $\mathfrak{a}^*_{\mathbb{C}}$ and $\varepsilon > 0$ such that $\mathfrak{F}^\varepsilon \cap U = \emptyset$, then there exists a $c > 0$ such that $|P_\alpha(\lambda)| \geq c$ for all $\alpha \in U$ and $\lambda \in \mathfrak{F}^\varepsilon$.

Proof. We let L be any hyperplane in $\mathfrak{a}^*$ which lies on a face of C_ρ, and take a basis $\varepsilon_1,\ldots,\varepsilon_\ell$ of $\mathfrak{a}^*$ such that $\varepsilon_1 \in L$ and $\varepsilon_2,\ldots,\varepsilon_\ell$ span a hyperplane parallel to L. This is possible since zero is an interior point of the convex set C_ρ. Then coordinating $\mathfrak{a}^*$ by $\eta = \Sigma\, \eta_j \varepsilon_j$ gives that L is the solution set to the equation $\eta_1 = 1$. Also notice that the set of solutions to $\eta_1 = -1$ must also lie on a face of C_ρ since $C_\rho = -C_\rho$. Hence C_ρ lies between the hyperplanes determined by $\eta_1{}^2 = 1$.

Since $\mathfrak{a}_{\mathbb{C}}^* = \bigcup_{\varepsilon>0} \mathfrak{F}^{\varepsilon}$ and $\operatorname{Int} \mathfrak{F}^{\varepsilon} = \bigcup_{\varepsilon'<\varepsilon} \mathfrak{F}^{\varepsilon'}$, we have that each $\alpha \in \mathfrak{a}_{\mathbb{C}}^*$ must lie on the boundary of some unique $\mathfrak{F}^{\varepsilon}$, $\varepsilon \geq 0$. Hence, let $\varepsilon(\alpha)$ be defined as that particular ε. Notice that $\alpha \mapsto \varepsilon(\alpha)$ is continuous.

Now parametrize $\mathfrak{a}_{\mathbb{C}}^*$ by the $\varepsilon_1,\ldots,\varepsilon_\ell$ from above by $\lambda = \Sigma\lambda_j\varepsilon_j$, $\lambda_j = \xi_j + \eta_j$ and $\alpha = \Sigma\alpha_j\varepsilon_j$, $\alpha_j = \beta_j + i\gamma_j$ for all $\lambda,\alpha \in \mathfrak{a}_{\mathbb{C}}^*$. Then let Q_α be the holomorphic polynomial in λ given by

$$Q_\alpha(\lambda) = \varepsilon(\alpha)^2 + (\lambda_1-\beta_1)^2.$$

Suppose U is a compact set in $\mathfrak{a}_{\mathbb{C}}^*$ and $\varepsilon > 0$ such that $\mathfrak{F}^{\varepsilon} \cap U = \emptyset$. Then $\alpha \in U$ implies $\varepsilon(\alpha) > \varepsilon$ (since $\mathfrak{F}^{\varepsilon} = \bigcap_{\varepsilon<\varepsilon'} \mathfrak{F}^{\varepsilon'}$) and taking $\lambda \in \mathfrak{F}^{\varepsilon}$ arbitrary we obtain $|Q_\alpha(\lambda)| = |\varepsilon(\alpha)^2+(\xi_1-\beta_1)^2-\eta_1{}^2+2i(\xi_1-\beta_1)\eta_1|$ $\geq \varepsilon(\alpha)^2-\eta_1{}^2$. But $\lambda \in \mathfrak{F}^{\varepsilon}$ implies $\eta \in \varepsilon C_\rho$, which in turn gives $|\eta_1| \leq \varepsilon$. Hence, letting $c = \inf_{\alpha \in U} \varepsilon(\alpha)^2 - \varepsilon^2$ we have $|Q_\alpha(\lambda)| \geq C$ for all $\alpha \in U$ and $\lambda \in \mathcal{F}^\varepsilon$, where $c > 0$ since U is compact. Therefore Q_α satisfies (iii), and it also follows trivially from the definition of Q_α that it satisfies (ii), at least pointwise, since $\alpha \mapsto \varepsilon(\alpha)$ and $\alpha \mapsto \beta_1$ are continuous. The uniform convergence on compacts then comes because degree $Q_\alpha = 2$ for all $\alpha \in \mathfrak{a}_{\mathbb{C}}^*$.

Let $\{L_1,\ldots,L_s\}$ be all the hyperplanes of $\mathfrak{a}^*$ which lie on faces of C_ρ. Then for each L_j we have a set of polynomials $\{P_\alpha^j\}$ as defined above. Then for each $\alpha \in \mathfrak{a}_{\mathbb{C}}^*$ we let $R_\alpha = \prod_{j=1}^{s} P_\alpha^j$. Then (ii) and (iii) still hold for R_α, but we also claim (i) holds from the following: by definition of $\varepsilon(\alpha)$ we have $\alpha \in \text{bdry } \mathcal{F}^{\varepsilon(\alpha)}$. Thus, with $\alpha = \beta + i\gamma$ we have that γ lies on at least one face $\varepsilon(\alpha)L_j$ of $\varepsilon(\alpha)C_\rho$. Hence, in the coordinate system associated with L_j we have $\gamma_1 = \varepsilon(\alpha)$, and if P_α^j is the polynomial associated with L_j we have $P_\alpha^j(\alpha) = \varepsilon(\alpha)^2 + (\beta_1 + i\gamma_1 - \beta_1)^2 = 0$. Hence

$R_\alpha(\alpha) = 0$.

Finally, since $\mathfrak{F}^\varepsilon$ is W-invariant we have that $P_\alpha = \prod_{s \in W} R_\alpha^s$ satisfies all the desired conditions. □

Prop. 4.3. Suppose $\{P_\alpha \mid \alpha \in \mathfrak{a}_{\mathbb{C}}^*\}$ as in Lemma 4.2, and fix both $\varepsilon > 0$ and $\Phi \in \bar{Z}(\mathfrak{F}^\varepsilon)$. Then if $\Psi_\alpha = \Phi/P_\alpha$ on $\mathfrak{F}^\varepsilon$ for each $\alpha \in \mathfrak{a}_{\mathbb{C}}^* - \mathfrak{F}^\varepsilon$, we have that $\alpha \mapsto \Psi_\alpha$ is continuous from $\mathfrak{a}_{\mathbb{C}}^* - \mathfrak{F}^\varepsilon$ into $\bar{Z}(\mathfrak{F}^\varepsilon)$.

Proof. First note that $\Psi_\alpha \in \bar{Z}(\mathfrak{F}^\varepsilon)$ for all $\alpha \notin \mathfrak{F}^\varepsilon$ from Prop. 4.1. Let $\alpha^n \to \alpha$ in $\mathfrak{a}_{\mathbb{C}}^* - \mathfrak{F}^\varepsilon$. We have only to show that, if D is a holomorphic differential operator on $\mathfrak{a}_{\mathbb{C}}^*$ with polynomial coefficients, $\sup_{\mathrm{Int}\, \mathfrak{F}^\varepsilon} |D(\Psi_\alpha - \Psi_{\alpha^n})| \to 0$. We first notice that

$\Psi_\alpha - \Psi_{\alpha^n} = ((P_{\alpha^n} - P_\alpha)/P_\alpha P_{\alpha^n})\Phi$ which is of the form

$(Q_n/R_n)\Phi$ where Q_n and R_n are W-invariant holomorphic polynomials on $\mathfrak{a}_{\mathbb{C}}^*$ such that (a) $|R_n| \geq c > 0$ on $\mathfrak{F}^\varepsilon$ for all n, (b) Q_n and R_n converge uniformly on compact sets, $Q_n \to 0$ and $R_n \to R$, where R is some

W-invariant holomorphic polynomial on $\mathfrak{a}_{\mathbb{C}}^*$, and (c) the degrees of Q_n and R_n are uniformly bounded in n by some $N < \infty$. We thus set $\Psi_n = (Q_n/R_n)\Phi$ and will show

$$\sup_{\operatorname{Int} \mathfrak{F}^\varepsilon} |D\Psi_n| \to 0$$ by inducting on the order of D.

(i) Suppose order $D = 0$. Then $D = P$ for some holomorphic polynomial P on $\mathfrak{a}_{\mathbb{C}}^*$, and we have

$$\sup_{\operatorname{Int} \mathfrak{F}^\varepsilon} |D\Psi_m| \leq \frac{1}{c} \sup_{\operatorname{Int} \mathfrak{F}^\varepsilon} |Q_n P\Phi|.$$ But we may express

$Q_n(\lambda)$ as $\sum_{|I|\leq n} c_I^n \lambda^I$ where $I = (i_1, \ldots, i_\ell)$, $|I| = i_1 + \ldots + i_\ell$ and $\lambda^I = \lambda_1^{i_1} \ldots \lambda_\ell^{i_\ell}$ for $\lambda_1, \ldots, \lambda_\ell$ some complex coordinate system on $\mathfrak{a}_{\mathbb{C}}^*$. Moreover, we have that for each fixed I, $c_I^n \to 0$ as $n \to \infty$. Thus we obtain $|D\Psi_n(\lambda)| \leq \frac{1}{c} |Q_n(\lambda)P(\lambda)\Phi(\lambda)| \leq \frac{1}{c}(\sum_{|I|\leq N} \sup_n |c_I^n||\lambda^I|)|P(\lambda)\Phi(\lambda)|$

for $\lambda \in \mathfrak{F}^\varepsilon$, where $\sup_n |c_I^n| < \infty$ for each I since $c_I^n \to 0$ as $n \to \infty$. Now take any $\delta > 0$. Since $\Phi \in \overline{Z}(\mathfrak{F}^\varepsilon)$ we have that there exists a compact set $C \subset \mathfrak{F}^\varepsilon$ such that $(\sum_I \sup_n |c_I^n||\lambda^I|)|P(\lambda)\Phi(\lambda)| < (\frac{c}{2})\delta$

for all $\lambda \in \mathfrak{F}^{\varepsilon} - C$. But on C we have $Q_n \to 0$ uniformly so that we can take n_0 such that $\sup_C |Q_n| < (\frac{c}{2})\delta \sup_C |P\Phi|$ for all $n \geq n_0$. Hence $|D\Psi_n(\lambda)| < \delta$ for all $n \geq n_0$ and $\lambda \in \operatorname{Int} \mathfrak{F}^{\varepsilon}$, and therefore $D\Psi_n \to 0$ uniformly on $\operatorname{Int} \mathfrak{F}^{\varepsilon}$ when D is of order zero.

(ii) Suppose $D\Psi_n \to 0$ uniformly on $\operatorname{Int} \mathfrak{F}^{\varepsilon}$ for all D of order less than k and for all Ψ_n of the specified form.

(iii) Take D of order k. Using some complex coordinate system $\lambda_1, \ldots, \lambda_\ell$ on $\mathfrak{a}^*_{\mathbb{C}}$ we may assume without loss of generality that $D = E \frac{\partial}{\partial \lambda_1}$ where E is a differential operator as in (ii). We then find

$$\sup_{\operatorname{Int} \mathfrak{F}^{\varepsilon}} |D\Psi_n| \leq \sup_{\operatorname{Int} \mathfrak{F}^{\varepsilon}} \left| E\left(\frac{(\frac{\partial Q_n}{\partial \lambda_1})\Phi}{R_n}\right) + E\left(\frac{-Q_n(\frac{\partial R_n}{\partial \lambda_1})\Phi}{R_n^{\,2}}\right) + E\left(\frac{Q_n}{R_n}\frac{\partial \Phi}{\partial \lambda_1}\right) \right|$$

But each of these terms satisfies the induction hypothesis. □

<u>Remark</u>. We now prove two very technical results, the first

to be used in the Spherical Bochner Theorem proof, the other to be used in the last chapter.

Prop. 4.4. Suppose $\{\mathfrak{g}_j\}_{j=1}^{\infty}$ is a sequence of functions in $C_c(G)$ such that $\cup_j \operatorname{supp} \mathfrak{g}_j$ is relatively compact and $\int_G |\mathfrak{g}_j(g)|dg$ is uniformly bounded in j. Then for D any constant coefficient holomorphic differential operator on $\mathfrak{a}_{\mathbb{C}}^*$, $\{D\hat{\mathfrak{g}}_j\}_{j=1}^{\infty}$ is a uniformly bounded set of functions on $\mathfrak{F}^{\varepsilon}$ for each $\varepsilon > 0$.

Proof. Let $C = \mathcal{C}\ell(\bigcup_j \operatorname{supp} \mathfrak{g}_j)$. Then $D\hat{\mathfrak{g}}_j(\lambda)$ $= \int_C \mathfrak{g}_j(g)D_\lambda(\phi_{-\lambda}(g))dg$ since C is compact. Hence $|D\hat{\mathfrak{g}}_j(\lambda)| \leq c_1 \sup_{g\in C} |D_\lambda(\phi_{-\lambda}(g))|$ for all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ and $j \in \mathbb{Z}^+$, where $c_1 \equiv \sup_j \int_G |\mathfrak{g}_j(g)|dg$. We thus have only to show that $\sup_{g\in C} |D_\lambda(\phi_{-\lambda}(g))|$ is uniformly bounded on $\mathfrak{F}^{\varepsilon}$ for each $\varepsilon > 0$.

Take $\varepsilon_1,\ldots,\varepsilon_\ell$ to be some basis of $\mathfrak{a}^*$, and if $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ let $\lambda = \Sigma_j\lambda_j\varepsilon_j$ where $\lambda_j \in \mathbb{C}$ and

$\lambda_j = \xi_j + i\eta_j$. Now let $H_1,\ldots,H_\ell$ be a dual basis of $\mathfrak{a}$, so that $\varepsilon_i(H_j) = \delta_{ij}$. We then write $H(g) = \Sigma_{j=1}^{\ell} c_j(g)H_j$ with each c_j a continuous function from G to $\mathbb{R}$. Hence $\lambda(H(g)) = \sum_{j=1}^{\ell} \lambda_j c_j(g)$ so that $D_\lambda e^{-i\lambda(H(g))} = P_D(-ic(g))e^{-i\lambda(H(g))}$ for P_D some polynomial in ℓ-variables and $c(g) = (c_1(g),\ldots,c_\ell(g))$.

Thus $|D_\lambda(\phi_{-\lambda}(g))| = |D_\lambda \int_K e^{-(i\lambda+\rho)H(gk)}dk|$

$$\leq \sup_{k\in K} |P_D(-ic(gk))e^{-(i\lambda+\rho)H(gk)}|$$

$$= \sup_{k\in K} |P_D(-ic(gk))e^{(\eta-\rho)H(gk)}|$$

where $\lambda = \xi + i\eta$. Consider $g \in C$ and $\lambda \in \mathfrak{F}^\varepsilon$ for some fixed $\varepsilon > 0$. Then $gk \in CK$ which is compact in G, and hence $\sup_{k\in K} |P_D(-ic(gk))|$ is uniformly bounded for $g \in C$. But now $\lambda \in \mathfrak{F}^\varepsilon$ if and only if $\lambda = \xi + i\eta$ where $\eta \in \varepsilon C_\rho$. Hence $\{\eta-\rho \mid \lambda \in \mathfrak{F}^\varepsilon\}$ is compact in $\mathfrak{a}^*$ which gives that $\{(\eta-\rho)H(gk) \mid \lambda \in \mathfrak{F}^\varepsilon, g \in C, k \in K\}$ is bounded in $\mathbb{R}$. Hence we have proved that $|D_\lambda(\phi_{-\lambda}(g))|$

is uniformly bounded for $\lambda \in \mathfrak{F}^{\varepsilon}$ and $g \in C$, which proves the lemma from the first paragraph. □

Prop. 4.5. Suppose for each $\alpha \in \mathfrak{a}_{\mathbb{C}}^{*}$ we have a W-invariant holomorphic polynomial P_α on $\mathfrak{a}_{\mathbb{C}}^{*}$ with degree $\leq N < \infty$ for all α such that

(i) $P_\alpha(\alpha) = 0$ for all $\alpha \in \mathfrak{a}_{\mathbb{C}}^{*}$

(ii) $P_{\alpha^n} \to P_\alpha$ uniformly on compact sets as $\alpha^n \to \alpha$ in $\mathfrak{a}_{\mathbb{C}}^{*}$.

Then the function $F : \mathfrak{a}_{\mathbb{C}}^{*} \times \mathfrak{a}_{\mathbb{C}}^{*} \to \mathbb{R}$ defined by

$$F(\alpha,\lambda) = \begin{cases} |P_\alpha(\lambda)|/||\lambda-\alpha|| & \text{if} \quad \lambda \neq \alpha \\ 0 & \text{if} \quad \lambda = \alpha \end{cases}$$

is jointly continuous.

Remark. The norm $||\cdot||$ on $\mathfrak{a}_{\mathbb{C}}^{*}$ is defined by $||\xi + i\eta||^2 = |\xi|^2 + |\eta|^2$ for $\xi, \eta \in \mathfrak{a}^{*}$, where $|\cdot|$ is as in (Notation §6).

Proof. Pick $\varepsilon_1,\ldots,\varepsilon_\ell$ as an orthonormal basis of $\mathfrak{a}^{*}$ with respect to $||\cdot||$ and let $\lambda = \Sigma\lambda_j\varepsilon_j$. Then

$P_\alpha(\lambda) = \sum_{0<|I|\leq N} a_I(\alpha)(\lambda-\alpha)^I$ (notation as in proof of Prop. 4.3.) where $a_I : \mathfrak{a}_{\mathbb{C}}^* \to \mathbb{C}$ is continuous for each I. Hence continuity of F at $(\alpha,\lambda) \neq (0,0)$ is clear, and to conclude the proof we need only show that $F(\alpha,\lambda) \to 0$ as $(\alpha,\lambda) \to (0,0)$.

Since $\varepsilon_1,\ldots,\varepsilon_\ell$ was picked orthonormal with respect to $||\cdot||$ we have that $|(\lambda-\alpha)^I| \leq ||\lambda-\alpha||^{|I|}$. Hence we obtain $|F(\alpha,\lambda)| \leq (\sum_{0<|I|\leq N} |a_I(\alpha)||(\lambda-\alpha)^I|)/||\lambda-\alpha||$ $\leq \sum_{0<|I|\leq N} |a_I(\alpha)|\ ||\lambda-\alpha||^{|I|-1}$. But from this it is clear that $F(\alpha,\lambda) \to 0$ as $(\alpha,\lambda) \to (0,0)$. □

<u>Remark</u>. We now state the main theorem concerning the $\overline{Z}(\mathfrak{F}^\varepsilon)$ spaces.

<u>Theorem 4.6</u>. Let $0 < p \leq 2$ and $\varepsilon \equiv (2/p) - 1$. Then the spherical transform map $f \mapsto \hat{f}$ is a linear topological isomorphism of $I^p(G)$ onto $\overline{Z}(\mathfrak{F}^\varepsilon)$ which preserves the algebraic structure.

<u>Remark</u>. For $p = 2$ the theorem is a combination of the results of Harish-Chandra [9(d), p. 48] and

Helgason [10(b), p. 572 and p. 576]; for $p = 1$ and G either of real rank one or complex, the theorem is due to Helgason [10(c), Thm. 2.1, p. 28]; in the full generality the theorem is due to Trombi and Varadarajan [17, Thm. 3.10.1, p. 298]. Notice that proving surjectivity is the non-trivial part of the theorem.

Chapter V.

The Spherical Bochner Theorem

Given G and K as in the previous three chapters, we shall now refine Theorem 2.6, putting it into the standard Bochner theorem form for distributions. The main tools for this procedure will be Theorem 1.6 and Theorem 4.6.

Theorem 5.1. Suppose T is a positive definite distribution on G. Then T restricted to $I_c(G)$ extends uniquely to an element in $(I^1(G))'$, which we also label T.

Proof. By Theorem 1.6 we have that $T = \sum_j D^j E^j f_j$ for D^j left invariant differential operators, E^j right invariant differential operators, and $f_j \in L^\infty(G)$. Hence $S[f] = \sum_j f_j[{}^tE^j \; {}^tD^j \; f]$ is in $(I^1(G))'$ by Prop. 3.2, and it is the unique extension of $T|_{I_c(G)}$ since $I_c(G)$ is dense in $I^1(G)$. □

Lemma 5.2. Suppose T is a positive definite distribution on G with spherical Bochner measure μ. Further suppose $\psi \in I^1(G)$ and $\phi \in I_c(G)$ such that $\hat{\phi} \in L^1(\mu)$. Then

$$T[\phi * \psi] = \int_{\mathfrak{P}} \hat{\phi}\hat{\psi}\,d\mu.$$

<u>Proof</u>. Pick $\{\psi_n\}_{n=1}^{\infty} \subset I_c(G)$ such that $\psi_n \to \psi$ in $I^1(G)$. Then Theorem 2.6 gives that

$$T[\phi * \psi_n] = \int_{\mathfrak{P}} \hat{\phi}\hat{\psi}_n\,d\mu,$$

and since $\phi * \psi_n \to \phi * \psi$ in $I^1(G)$, then Theorem 5.1 gives that $T[\phi * \psi_n] \to T[\phi * \psi]$. Hence we are left with showing $\int_{\mathfrak{P}} \hat{\phi}\hat{\psi}_n\,d\mu \to \int_{\mathfrak{P}} \hat{\phi}\hat{\psi}\,d\mu$ as $n \to \infty$.

Note that the Helgason-Johnson theorem (Notation §9(15)) tells us that $\mathfrak{P} \subset \mathfrak{F}^1$, and hence $\int_{\mathfrak{P}} \hat{\phi}\hat{\psi}\,d\mu$ is well defined since $\hat{\psi}$ is extendable to all of $\mathfrak{F}^1$ as a continuous function. Now Theorem 4.6 gives that $\hat{\psi}_n \to \hat{\psi}$ in $\overline{Z}(\mathfrak{F}^1)$, hence in particular, $\{\hat{\psi}_n\}_{n=1}^{\infty}$ is uniformly bounded on $\mathfrak{F}^1$. Thus dominated convergence gives $\int_{\mathfrak{P}} \hat{\phi}\hat{\psi}_n\,d\mu \to \int_{\mathfrak{P}} \hat{\phi}\hat{\psi}\,d\mu$. □

Definition A positive measure μ on $\mathfrak{P}$ is said to be of polynomial growth if there exists a holomorphic polynomial Q on $\mathfrak{a}_{\mathbb{C}}^*$ such that $\int_{\mathfrak{P}} \frac{d\mu}{|Q|} < \infty$.

Lemma 5.3. Suppose T is a positive definite distribution with spherical Bochner measure μ. Then μ is of polynomial growth.

Proof. By Theorem 5.1 we have $T \in (I^1(G))'$. Hence Theorem 4.6 gives that $\hat{T} \in (\overline{Z}(\mathfrak{F}^1))'$, where $\hat{T}[\hat{\psi}] = T[\psi]$ for all $\psi \in I^1(G)$. Now take $\varepsilon_1,\ldots,\varepsilon_\ell$ to be any basis of $\mathfrak{a}^*$, and for $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ set $\lambda = \Sigma\lambda_j\varepsilon_j$ where $\lambda_j \in \mathbb{C}$. For each $m, t \in \mathbb{Z}^+$ define the continuous semi-norm σ_m^t on $Z(\mathfrak{F}^1)$ by

$$\sigma_m^t(\Phi) = \sup_{\substack{\lambda \in \operatorname{Int} \mathfrak{F}^1 \\ |M| \le m}} (1 + ||\lambda||^2)^t \left|\frac{d^M}{d\lambda^M} \Phi(\lambda)\right|$$

where $M = (m_1.,,,.m_\ell)$, $|M| = m_1+\ldots+m_\ell$, $\frac{d^M}{d\lambda^M} = (\frac{\partial}{\partial\lambda_1})^{m_1}\ldots(\frac{\partial}{\partial\lambda_\ell})^{m_\ell}$ and $||\xi + i\eta||^2 = |\xi|^2 + |\eta|^2$ for all $\xi, \eta \in \mathfrak{a}^*$.

Since $\hat{T}$ is continuous on $\overline{Z}(\mathfrak{F}^1)$, there exist $\eta > 0$ and $m, t \in \mathbb{Z}^+$ such that $|\hat{T}[\phi]| \le 1$ for all

$\phi \in \overline{Z}(\mathfrak{F}^1)$ satisfying $\sigma_m^t(\phi) < \eta$. Thus Lemma 5.2 gives

(1) $$|\int_{\mathfrak{P}} \hat{\phi}\hat{\psi}d\mu| \leq 1$$

for all $\phi \in I_c(G)$, $\psi \in I^1(G)$ such that $\hat{\phi} \in L^1(\mu)$ and $\sigma_m^t(\hat{\phi}\hat{\psi}) < \eta$.

Let $P(\lambda)$ be as in Prop. 4.1 for $\varepsilon = 1$, and take $Q(\lambda) = (P(\lambda)\overline{P(\bar{\lambda})})^t$, with t from the previous paragraph. Then $Q(\lambda)$ also satisfies Prop. 4.1, and from (Notation §9(13)) we see that $Q(\lambda) = |P(\lambda)|^{2t}$ for $\lambda \in \mathfrak{P}$.

Take δ_j to be an approximation of 1 in $\mathfrak{D}(G)$, where $||\delta_j||_1 = 1$ for all j. Then on $\mathfrak{a}_{\mathbb{C}}^*$ we see that $\hat{\delta}_j \to 1$ pointwise since $\phi_\lambda(e) = 1$ for all $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$, and on $\mathfrak{P}$ we see that $|\hat{\delta}_j| \leq 1$ since $|\phi_\lambda| \leq 1$ for all $\lambda \in \mathfrak{P}$. Defining $\phi_j = \delta_j^{\natural} * (\delta_j^*)^{\natural}$ we see that for $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$,

$$\hat{\phi}_j(\lambda) = \hat{\delta}_j(\lambda)\widehat{\delta_j^*}(\lambda) \qquad \text{(Notation §9(6)(8))}$$

$$= \hat{\delta}_j(\lambda)\overline{\hat{\delta}_j(\bar{\lambda})} \qquad \text{(Notation §9(7)(11)(12))}$$

Hence $\hat{\phi}_j \to 1$ pointwise on $\mathfrak{X}^*_{\mathbb{C}}$, and by (Notation §9(14)) we have $\hat{\phi}_j = |\hat{\delta}_j|^2 \leq 1$ on $\mathcal{P}$. Hence $\hat{\phi}_j \in L^1(\mu)$ since $\hat{\delta}_j \in L^2(\mu)$ by Theorem 2.6.

For each j define $\Psi_j = \hat{\phi}_j/Q$. Then Prop. 4.1 gives that $\Psi_j \in \bar{Z}(\mathfrak{F}^1)$, and hence there exists a unique $\psi_j \in I^1(G)$ such that $\Psi_j = \hat{\psi}_j$ by Theorem 4.6. We now claim that for m and t as specified above we have $\{\sigma^t_m(\hat{\phi}_j\hat{\psi}_j)\}^\infty_{j=1}$ uniformly bounded with respect to j. For let $\mathcal{G}_j = \phi_j * \phi_j$ so that $\hat{\phi}_j\hat{\psi}_j = \mathcal{G}_j/Q$ on $\mathfrak{F}^1$. Then

$$\sigma^t_m(\hat{\phi}_j\hat{\psi}_j) = \sup_{\substack{\lambda \in \text{Int } \mathfrak{F}^1 \\ |M| \leq m}} (1 + ||\lambda||^2)^t \left|\frac{d^M}{d\lambda^M} (\hat{\mathcal{G}}_j/Q)(\lambda)\right|$$

and we can expand out each $\frac{d^M}{d\lambda^M}(\hat{\mathcal{G}}_j/Q)$ term into the form $\sum_{N+R=M} C_N \frac{d^N}{d\lambda^N}(\frac{1}{Q})\frac{d^R}{d\lambda^R}(\hat{\mathcal{G}}_j)$. Hence

$$\sigma^t_m(\hat{\phi}_j\hat{\psi}_j) \leq C \sum_{|N+R| \leq m} (\sup_{\lambda \in \text{Int } \mathfrak{F}^1} (1+||\lambda||^2)^t |\frac{d^N}{d\lambda^N}(\frac{1}{Q})|)(\sup_{\lambda \in \text{Int } \mathfrak{F}^1} |\frac{d^R}{d\lambda^R}(\hat{\mathcal{G}}_j)|)$$

Now first consider $\left(\sup_{\lambda \in \text{Int}\, \mathfrak{Z}^1} (1+||\lambda||^2)^t \left|\frac{d^N}{d\lambda^N}\left(\frac{1}{Q}\right)\right|\right)$. We claim this is bounded (obviously independently of j) from the following: $\frac{d^N}{d\lambda^N}\left(\frac{1}{Q}\right)$ is a rational function on $\mathfrak{Z}^1$ with the order of the denominator minus order of the numerator being $\geq$ order $Q \geq 2t$. Hence, since Q is strictly bounded away from zero on $\mathfrak{Z}^1$ we must have that $(1+||\lambda||^2)^t \left|\frac{d^N}{d\lambda^N}\left(\frac{1}{Q}\right)\right|$ is a bounded function on $\mathfrak{Z}^1$.

Hence, to show $\sigma_m^t(\hat{\phi}_j\hat{\psi}_j)$ uniformly bounded in j we have only to show $\sup_{\lambda \in \text{Int}\, \mathfrak{Z}^1} \left|\frac{d^R}{d\lambda^R}(\hat{\mathfrak{g}}_j)\right|$ is uniformly bounded in j for each R. By Prop. 4.4 we have only to show that $\{\mathfrak{g}_j\}_{j=1}^{\infty}$ is such that $\bigcup_{j=1}^{\infty} \text{supp}\, \mathfrak{g}_j$ is relatively compact and that $||\mathfrak{g}_j||_1$ is uniformly bounded in j. Both of these follow trivially from $\mathfrak{g}_j = \delta_j^{\natural} * (\delta_j^*)^{\natural} * \delta_j^{\natural} * (\delta_j^*)^{\natural}$, where δ_j is an approximation of 1 in $\mathfrak{D}(G)$ such that $||\delta_j||_1 = 1$.

Hence there exists $c > 0$ such that $\sigma_m^{\ell}(c\hat{\phi}_j\hat{\psi}_j) < \eta$ for all $j \in \mathbb{Z}^+$, which proves $\left|\int_{\mathfrak{P}} \hat{\phi}_j\hat{\psi}_j d\mu\right| \leq \frac{1}{c}$ for all

$j \in \mathbb{Z}^+$ by (1). Therefore

$$\left|\int_{\mathcal{P}} \hat{\phi}_j^2/Q d\mu\right| \leq \frac{1}{c} \text{ for all } j \in \mathbb{Z}^+$$

But on $\mathcal{P}$ we have that Q is strictly positive and bounded away from zero, $0 \leq \hat{\phi}_j \leq 1$ and $\hat{\phi}_j \to 1$ pointwise. Thus applying monotone convergence to $\int_{\mathcal{P}} \inf_{k \geq j} \hat{\phi}_k^2/Q d\mu$ gives $\int_{\mathcal{P}} \frac{d\mu}{Q} \leq \frac{1}{c} < \infty$, or that μ is of polynomial growth. □

Theorem 5.4. The Spherical Bochner Theorem

Suppose $T \in \mathcal{D}'(G)$, $T >> 0$. Then there exists a unique W-invariant positive regular Borel measure μ of polynomial growth on $\mathcal{P}$ such that

$$(2) \qquad T[\phi] = \int_{\mathcal{P}} \hat{\phi} d\mu, \quad \phi \in I^1(G).$$

The correspondence is bijective when restricted to the K-biinvariant distributions. This set also corresponds bijectively to the set of positive definite K-biinvariant $\mathcal{C}^1$ distributions, the mapping given by (2) for all

$\phi \in \mathcal{C}^1(G)$.

Proof. Suppose we are given μ a positive regular Borel measure of polynomial growth on $\mathfrak{P}$. We then claim that $\bar{Z}(\mathfrak{F}^1) \subset L^1(\mu)$. For take $\Phi \in \bar{Z}(\mathfrak{F}^1)$. Since $\mathfrak{P} \subset \mathfrak{F}^1$ by the Helgason-Johnson theorem, then Φ is defined on the support of μ. Moreover, there exists some holomorphic polynomial P on $\mathfrak{a}^*_{\mathbb{C}}$ such that $\int_{\mathfrak{P}} \frac{d\mu}{|P|} < \infty$, and thus $\int_{\mathfrak{P}} |\Phi| \mu \leq \sup_{\mathfrak{F}^1} |P\Phi| \int_{\mathfrak{P}} \frac{d\mu}{|P|} < \infty$, proving the claim.

Hence the linear functional $\hat{T}$ on $\bar{Z}(\mathfrak{F}^1)$ given by $\hat{T}[\Phi] = \int_{\mathfrak{P}} \Phi d\mu$ is well-defined. Suppose $\Phi_n \to \Phi$ in $\bar{Z}(\mathfrak{F}^1)$. Then $|\hat{T}[\Phi] - \hat{T}[\Phi_n]| \leq \sup_{\mathfrak{F}^1} |P(\Phi - \Phi_n)| \int_{\mathfrak{P}} \frac{d\mu}{|P|}$ which converges to zero, proving $\hat{T} \in (\bar{Z}(\mathfrak{F}^1))'$.

We can then define $T \in (I^1(G))'$ by $T[\psi] = \hat{T}[\hat{\psi}]$ for all $\psi \in I^1(G)$. But by Prop. 3.4 there exists a unique K-biinvariant extension of T in $\mathfrak{D}'(G)$ given by $T[\phi] = T[\phi^{\natural}]$ for all $\phi \in \mathfrak{D}(G)$. Hence $T[\phi] = \int_{\mathfrak{P}} \hat{\phi} d\mu$ and we have only to show $T >> 0$. Take any $\phi \in \mathfrak{D}(G)$ and consider $(\phi * \phi^*)^{\wedge}(\lambda) = \phi_{-\lambda}[\phi * \phi^*] \geq 0$ for all $\lambda \in \mathfrak{P}$ since $\phi_{-\lambda} >> 0$. Hence $T[\phi * \phi^*] \geq 0$ for all $\phi \in \mathfrak{D}(G)$.

Conversely, suppose $T \in \mathcal{D}'(G)$, $T >> 0$. Then by Theorem 2.6 there exists a unique (W-invariant) positive regular Borel measure μ on $\mathcal{P}$ such that $T[\phi * \psi] = \int_{\mathcal{P}} \hat{\phi}\hat{\psi} d\mu$ for all $\phi, \psi \in I_c(G)$. But by Lemma 5.3 the measure μ is of polynomial growth on $\mathcal{P}$, and our above proof shows that we can define $T_0 \in \mathcal{D}'(G)$ by $T_0[\phi] \equiv \int_{\mathcal{P}} \hat{\phi} d\mu$. But clearly $\{\phi * \psi \mid \phi, \psi \in I_c(G)\}$ is dense in $I_c(G)$, so that T and T_0 must agree on $I_c(G)$, and hence also on $I^1(G)$. If T is K-biinvariant, then they must agree on all of $\mathcal{D}(G)$, proving the first asserted bijection.

The second bijection arises from the first by applying Theorems 3.6(b) and 5.1. ◻

<u>Remark</u>. The proof of Lemma 5.3 is based on the proof for the euclidean case in [14, Thm. VII, p. 242].

Chapter VI

A Characterizing Theorem

Using the spherical Bochner theorem and the Trombi-Varadarajan theorem we can deduce which $(I^p(G))'$ space a given positive definite distribution lies in by examining the support of its spherical Bochner measure. The relationship turns out to be $T \in (I^p(G))'$ if and only if $\operatorname{supp} \mu \subset \mathfrak{F}^{\varepsilon}$, where $1 \leq p \leq 2$ and $\varepsilon = \frac{2}{p} - 1$. This is a very natural occurrence in light of the Trombi-Varadarajan theorem, but while the underlying idea of the proof is straightforward, the details are surprisingly complicated. The desired result will be a consequence of the first lemma after some rather unpleasant measure theory and geometry on $\mathfrak{a}_{\mathbf{c}}^*$.

Lemma 6.1. Suppose $T \in \mathcal{D}'(G)$, $T >> 0$ such that $T \in (I^{p_0}(G))'$ for some $p_0 \geq 1$, let μ be the spherical Bochner measure of T, and let α_0 to be any point in $\mathcal{P}$ outside of $\mathfrak{F}^{\varepsilon_0}$, where $\varepsilon_0 = \frac{2}{p_0} - 1$. Then for each non-zero integer m there exists a compact neighborhood

U of α_0, $R > 0$ and $M < \infty$ such that

$$\int \frac{d\mu(\lambda)}{||\lambda-\alpha||^{2m}} \leq M \quad \text{for all} \quad \alpha \in U,$$

$$\mathcal{Cl}(B_R(\alpha)) \cap \text{Int}\, \mathfrak{F}^{\varepsilon(\alpha)}$$

where $B_R(\alpha)$ is the open ball of radius R about α, and $\varepsilon(\alpha)$ is defined by $\alpha \in \text{bdry}\ \mathfrak{F}^{\varepsilon(\alpha)}$

Proof. Let $\phi \in I_c(G)$ be such that $|\hat{\phi}(\alpha_0)| > 0$ and let V be a compact neighborhood of α_0 such that (i) $V \cap \mathfrak{F}^{\varepsilon_0} = \phi$, and (ii) $|\hat{\phi}(\alpha)| \geq c > 0$ for all $\alpha \in V$. Taking $\{P_\alpha\}_{\alpha \in \mathfrak{a}_{\mathbb{C}}^*}$ a collection of holomorphic polynomials as constructed in Lemma 4.2, for each $\alpha \in \mathfrak{a}_{\mathbb{C}}^* - \mathfrak{F}^{\varepsilon_0}$ define

(1) $\qquad \Psi_\alpha \equiv \hat{\phi}/P_\alpha^m$ on $\text{Int}\ \mathfrak{F}^{\varepsilon(\alpha)}$

Then Prop. 4.3 gives that $\Psi_\alpha \in \overline{Z}(\mathfrak{F}^{\varepsilon})$ for all $\varepsilon < \varepsilon(\alpha)$, and

(2) $\qquad \alpha \mapsto \Psi_\alpha$ is continuous from V into $\overline{Z}(\mathfrak{F}^{\varepsilon_0})$.

The Trombi-Varadarajan Theorem now gives the existence of functions ψ_α which are in $I^p(G)$ for all $p > p(\alpha)$, where $p(\alpha) = 2/(\varepsilon(\alpha)-1)$, and are such that $\Psi_\alpha = \hat{\psi}_\alpha$ on Int $\mathfrak{F}^{\varepsilon(\alpha)}$. Thus from (2) we have

(3) $\alpha \mapsto \psi_\alpha$ is continuous from V into $I^{p_0}(G)$,

and by assumption on T,

(4) $\alpha \mapsto T[\psi_\alpha * \psi_\alpha^*]$ is continuous from V into $\mathbb{C}$.

For each $\alpha \in V$ take $\{\psi_{\alpha,n}\}_{n=1}^{\infty} \subset I_c(G)$ such that $\psi_{\alpha,n} \to \psi_\alpha$ in $I^p(G)$ for all $p > p(\alpha)$ (Prop. 3.3). Then Theorem 5.4 gives that

(5) $T[\psi_{\alpha,n} * \psi_{\alpha,n}^*] = \int_{\mathfrak{P}} |\hat{\psi}_{\alpha,n}|^2 d\mu$

But the right hand side of (5) is greater than or equal to $\int_{\mathfrak{P} \cap \mathfrak{F}^\varepsilon} |\hat{\psi}_{\alpha,n}|^2 d\mu$ for each $\varepsilon < \varepsilon(\alpha)$, and these latter quantities tend toward $\int_{\mathfrak{P} \cap \mathfrak{F}^\varepsilon} |\hat{\psi}_\alpha|^2 d\mu$ as $n \to \infty$ since the measure $\nu_\varepsilon(E) = \mu(E \cap \mathcal{F}^\varepsilon)$, $\varepsilon < \varepsilon(\alpha)$ defines a continuous linear functional on $\mathcal{Z}(\mathcal{F}^\varepsilon)$ (see the proof of Theorem 5.4). Hence (5) gives

$$T[\psi_\alpha * \psi_\alpha^*] \geq \int_{\mathcal{P} \cap \mathcal{F}^\varepsilon} |\hat{\psi}_\alpha|^2 d\mu, \quad \alpha \in V, \ \varepsilon < \varepsilon(\alpha),$$

and then monotone convergence implies

$$(6) \qquad T[\psi_\alpha * \psi_\alpha^*] \geq \int_{\mathcal{P} \cap \operatorname{Int} \mathcal{F}^{\varepsilon(\alpha)}} |\hat{\psi}_\alpha|^2 d\mu, \qquad \alpha \in V.$$

Now pick $R > 0$ and U a compact neighborhood of α_0 such that $\mathcal{Cl}(\bigcup_{\alpha \in U} B_R(\alpha)) \subset V$. Then by Prop. 4.5 there exists $M_0 < \infty$ such that $|P_\alpha(\lambda)| \leq M_0 ||\lambda - \alpha||$ for all $\alpha \in U$ and $\lambda \in \mathcal{Cl}(B_R(\alpha))$. Hence from (6) we have

$$T[\psi_\alpha * \psi_\alpha^*] \geq \int_{\operatorname{Int} \mathcal{F}^{\varepsilon(\alpha)}} |\hat{\phi}|^2 / |P_\alpha|^{2m} \, d\mu$$

$$\geq \int_{\mathcal{Cl}(B_R(\alpha)) \cap \operatorname{Int} \mathcal{F}^{\varepsilon(\alpha)}} |\hat{\phi}(\lambda)|^2 / M_0^{2m} ||\lambda - \alpha||^{2m} \, d\mu(\lambda).$$

Taking $M = (M_0^{2m}/c^2)(\sup_{\alpha \in U} T[\psi_\alpha * \psi_\alpha^*])$, which is finite from (4), proves the Lemma. □

Remark. We actually wish to conclude in the situation of Lemma 6.1 that $\mu \equiv 0$ in a neighborhood of the point α_0, and hence that $\mu \equiv 0$ outside of $\mathfrak{F}^{\varepsilon_0}$. To do so we have to carefully examine the sets $B_r(\alpha) \cap \operatorname{Int} \mathfrak{F}^{\varepsilon(\alpha)}$ for all $\alpha \in U$ and r small. There are two problems to be dealt with: (1) the sets $B_r(\alpha) \cap \operatorname{Int} \mathfrak{F}^{\varepsilon(\alpha)}$ are "irregular" in shape, and (2) $\alpha \notin B_r(\alpha) \cap \operatorname{Int} \mathfrak{F}^{\varepsilon(\alpha)}$. These are circumvented by means of the following proposition.

Prop. 6.2. For each non-zero $\alpha \in \mathfrak{A}_{\mathbb{C}}^*$ and $r \geq 0$ define $\alpha_r \in \mathfrak{A}_{\mathbb{C}}^*$ by $\alpha_r \equiv (1 - (r/2||\alpha||))\alpha$. Then for each compact set U in $\mathfrak{A}_{\mathbb{C}}^*$ disjoint from zero there exists $0 < c < 1$ such that

$$B_{cr}(\alpha_r) \subset B_r(\alpha) \cap \operatorname{Int} \mathfrak{F}^{\varepsilon(\alpha)}$$

for all $\alpha \in U$ and $r \leq R_0 = \inf_{\alpha \in U} ||\alpha||$.

Proof. α_r is simply translation of α by a distance of $r/2$ towards the origin, and hence α_r is trivially in $B_r(\alpha)$ for $r > 0$. Also, since α_r is a convex combination of α and zero for $r \leq R_0$, with α a boundary point of the convex set $\mathcal{F}^{\varepsilon(\alpha)}$ and zero an interior point, then α_r is also clearly an interior point. Hence, for each such α and $r \leq R_0$ there exists a constant $c(r,\alpha)$ such that $B_{cr}(\alpha_r) \subset B_r(\alpha) \cap \text{Int } \mathcal{F}^{\varepsilon(\alpha)}$. The non-trivial part of the proposition is that C can be taken independently of both α and r for $\alpha \in U$ and $r \leq R_0$.

Let $S(r,\alpha) = \sup \{s \mid B_s(\alpha_r) \subset B_r(\alpha) \cap \text{Int } \mathcal{F}^{\varepsilon(\alpha)}\}$.

Then $S(r,\alpha) > 0$ for each $\alpha \in U$ and $0 < r \leq R_0$ from the above. We thus have only to show the existence of $0 < c < 1$ such that $c < S(r,\alpha)/r$ for all $\alpha \in U$ and $0 < r \leq R_0$.

(i) First we claim that $S(r,\alpha)/r$ is non-decreasing for each fixed α as r goes down to zero. For suppose $0 < r_1 < r$. Then

$$S'(r_1,\alpha)/r_1 = \sup_s \{s/r_1 \mid B_s(\alpha_{r_1}) \subset B_{r_1}(\alpha) \cap \text{Int } \mathcal{F}^{\varepsilon(\alpha)}\}$$

We now alter the coordinate system on σ_c^* by shifting α to zero. Then Int $\mathcal{F}^{\varepsilon(\alpha)}$ becomes some open convex set C with zero on the boundary, and we have

$$S(r_1,\alpha)/r_1 = \sup_s \{s/r_1 \mid B_s(\alpha_{r_1}-\alpha) \subset B_{r_1}(0) \cap C\}.$$

But now $(\alpha_{r_1}-\alpha) = r_1/r(\alpha_r-\alpha)$ and $B_s(t\alpha) = tB_{s/t}(\alpha)$ so that

$$S(r_1,\alpha)/r_1 = \sup_s \{s/r_1 \mid (r_1/r)B_{rs/r_1}(\alpha_r-\alpha) \subset (r_1/r)B_r(0) \cap C\}$$

$$= \sup_s \{s/r_1 \mid B_{rs/r_1}(\alpha_r-\alpha) \subset B_r(0) \cap (r/r_1)C\}$$

$$= \sup_s \{s/r \mid B_s(\alpha_r-\alpha) \subset B_r(0) \cap (r/r_1)C\}$$

But since $(r/r_1) > 1$ and C is an open convex set with zero on the boundary we get that $C \subset (r/r_1)C$, hence

$$S(r_1,\alpha)/r_1 \geq \sup_s \{s/r \mid B_s(\alpha_r-\alpha) \subset B_r(0) \cap C\}$$

$$= S(r,\alpha)/r.$$

This of course proves (i).

(ii) Our second claim is that $\alpha \mapsto S(r,\alpha)/r$ is continuous from U to $\mathbb{R}$ for each fixed $0 < r \leq R_0$. For suppose $||\alpha-\beta|| \leq \delta$ for $\delta > 0$. Then take any s such that $B_s(\alpha_r) \subset B_r(\alpha) \cap \mathrm{Int}\, \mathcal{F}^{\varepsilon(\alpha)}$. First note that $B_{s-\delta}(\alpha_r) \subset B_r(\beta) \cap \mathrm{Int}\, \mathcal{F}^{\varepsilon(\alpha)}$ for δ small. But $||\alpha-\beta|| < \delta \Rightarrow ||\alpha_r-\beta_r|| < \delta$ when $r \leq R_0$ by simple verification, and hence

$$B_{s-2\delta}(B_r) \subset B_r(\beta) \cap \mathrm{Int}\, \mathcal{F}^{\varepsilon(\alpha)} \tag{7}$$

for δ small. Now define

$$\Delta = \sup_{\lambda} \{d(\lambda, \mathrm{Int}\, \mathcal{F}^{\varepsilon(\beta)}) \mid \lambda \in \mathrm{Int}\, \mathcal{F}^{\varepsilon(\alpha)}\}$$

$$+ \sup_{\lambda} \{d(\lambda, \mathrm{Int}\, \mathcal{F}^{\varepsilon(\alpha)}) \mid \lambda \in \mathrm{Int}\, \mathcal{F}^{\varepsilon(\beta)}\}$$

where $d(\lambda,E)$ is the distance of λ from the set E given by $\inf\{||\lambda-\lambda'|| \mid \lambda' \in E\}$. (Notice that one of the two terms defining Δ always has to be zero.) Then from (7) we find that

$$B_{s-2\delta-\Delta}(\beta_r) \subset B_r(\beta) \cap \text{Int } \mathfrak{F}^{\varepsilon(\beta)}$$

for small δ and Δ, and since everything we did was symmetric with respect to α and β, we find that

$$|S(r,\alpha) - S(r,\beta)| \leq 2\delta + \Delta$$

for $||\alpha-\beta|| \leq \delta$, δ small. Our desired continuity will then be established once we show $\Delta \to 0$ as $\delta \to 0$ for α fixed.

Without loss of generality suppose $\varepsilon(\alpha) > \varepsilon(\beta)$. If $\lambda \in \text{Int } \mathfrak{F}^{\varepsilon(\alpha)}$ we have $d(\lambda, \text{Int } \mathfrak{F}^{\varepsilon(\beta)})$

$$= \inf_{\lambda'} \{ ||\lambda-\lambda'|| \mid \lambda' \in \text{Int } \mathfrak{F}^{\varepsilon(\beta)}\} \leq \inf_{\eta'} \{ ||\eta-\eta'|| \mid \eta' \in \varepsilon(\beta) \text{ Int } C_\rho\}$$

where $\lambda = \xi + i\eta$, $\eta \in \varepsilon(\alpha) \text{ Int } C_\rho$. But C_ρ is norm bounded in $\mathfrak{A}^*$, say by M_2. Hence $||\eta-(\varepsilon(\beta)/\varepsilon(\alpha))\eta||$ $\leq \left|\frac{\varepsilon(\alpha)-\varepsilon(\beta)}{\varepsilon(\alpha)}\right| M_2$, and since $(\varepsilon(\beta)/\varepsilon(\alpha))\eta \in \varepsilon(\beta) \text{ Int } C_\rho$ we find that $\Delta \leq \left|\frac{\varepsilon(\alpha)-\varepsilon(\beta)}{\varepsilon(\alpha)}\right| M_2 \to 0$ as $\delta \to 0$ since $\alpha \mapsto \varepsilon(\alpha)$ is continuous. Hence (ii) is proved.

(iii) Now (i) and (ii) prove the proposition, since if $0 < r \leq R_0$ and $\alpha \in U$ we find that

$$S(r,\alpha)/r \geq S(R_0,\alpha)/R_0 \quad \text{from (i)}$$

$$\geq \inf_{\alpha \in U} S(R_0,\alpha)/R_0 > 0 \quad \text{from (ii).} \quad \square$$

Lemma 6.3. Suppose T, μ, p_0, ε_0 and α_0 as in Lemma 6.1, and let Λ be lebesgue measure on $\mathfrak{a}_{\mathbb{C}}^*$. Then there exists a compact neighborhood U_1 of α_0 and $R_1 > 0$ such that $\mu(B_r(\alpha)) \leq \Lambda(B_r(\alpha))r$ for all $r \leq R_1$ and $\alpha \in U_1$.

Proof. Let $\ell = \dim \mathfrak{a}^*$ and apply Lemma 6.1 with $m = \ell + 1$. Then there exists a compact neighborhood U of α_0, $R > 0$ and $M < \infty$ such that

$$\int_{B_r(\alpha) \cap \operatorname{Int} \mathcal{F}^{\varepsilon(\alpha)}} \frac{d\mu(\lambda)}{||\lambda-\alpha||^{2\ell+2}} \leq M \text{ for all } \alpha \in U \text{ and all } r \leq R.$$

Hence

$$(8) \qquad \mu(B_r(\alpha) \cap \operatorname{Int} \mathcal{F}^{\varepsilon(\alpha)}) \leq Mr^{2\ell+2}, \ \alpha \in U, \ r \leq R.$$

But from Prop. 6.2 we find, with

$\alpha_r = (1 - (r/2||\alpha||))\alpha$ for all $\alpha \in U$ and $0 < r$ small, that there exists $0 < c < 1$ such that

(9) $B_{cr}(\alpha_r) \subset B_r(\alpha) \cap \text{Int}\, \mathcal{F}^{\varepsilon(\alpha)}$, $\alpha \in U$, $0 < r$ small.

Hence (8) and (9) combined yield

$$\mu(B_r(\alpha_{(r/c)})) \leq (M/C^{2\ell+2})r^{2\ell+2},\ \alpha \in U,\ 0 < r \text{ small.}$$

Since $\Lambda(B_r(\alpha))$ is proportional to $r^{2\ell}$, then there exists $M_0 > 0$ such that $\mu(B_r(\alpha_{(r/c)})) \leq M_0\Lambda(B_r(\alpha_{(r/c)}))r^2$, or

(10) $\mu(B_r(\alpha_{(r/c)})) \leq \Lambda(B_r(\alpha_{(r/c)}))r$, $\alpha \in U$, $0 < r$ small.

To show (10) implies the lemma we have only to show that there exists a compact subneighborhood U_1 of α_0 and some sufficiently small R_1 such that for each $\alpha \in U_1$ and $r \leq R_1$ we can find a $\beta \in U$ for which $\alpha = \beta_{(r/c)}$. To do so simply define
$U_1 = \mathcal{C}\ell(B_{R_2}(\alpha_0))$ for some $R_2 > 0$ such that
$B_{2R_2}(\alpha_0) \subset U$. Then for each $\alpha \in U_1$ and $r > 0$ the

equation $\alpha = \beta(r/c)$ uniquely determines β to be $(1 + (r/2c||\alpha||))\alpha$ by simple verification. But again, simple verification yields that we have only to restrict r so that $r \leq 2cR_2$ to have $\beta \in U$. Hence the desired R_1 equals $2cR_2$. □

Remark. Our theorem will now result from the above lemma.and the following general covering theorem, a variant of material found in Federer [6, Thms. 2.8.4 and 2.8.7].

Prop. 6.4. Suppose C is a compact set in $\mathbb{R}^n$, Λ lebesgue measure on $\mathbb{R}^n$, and μ a Borel measure on $\mathbb{R}^n$ with the property that for every $\varepsilon > 0$ there exists $r_\varepsilon > 0$ such that $\mu(B_r(x)) \leq \varepsilon\Lambda(B_r(x))$ for all $x \in C$ and all $r \leq r_\varepsilon$. Then $\mu \equiv 0$ on C.

Proof. If $S = B_r(x)$ then define $\hat{S} = B_{3r}(x)$ and for each $\varepsilon > 0$ define $\mathcal{S}_\varepsilon = \{B_r(x) \mid x \in C \text{ and } r \leq r_\varepsilon/3\}$. We claim that $\mathcal{S}_\varepsilon$ has a disjoint subfamily $\mathcal{G}_\varepsilon$ with the property that for each $T \in \mathcal{S}_\varepsilon$ there exists $S \in \mathcal{G}_\varepsilon$ such that $T \cap S \neq \emptyset$ and $T \subset \hat{S}$. To prove this let

$\Omega_\varepsilon = \{ \mathcal{H} \mid \mathcal{H}$ is a disjoint subfamily of $\mathcal{S}_\varepsilon$ such that

for each $T \in \mathcal{S}_\varepsilon$ either

(i) for all $S \in \mathcal{H}$, $T \cap S = \emptyset$, or

(ii) for some $S \in \mathcal{H}$, $T \cap S \neq \emptyset$ and $T \subset \hat{S}\}$

Notice that (i) implies Ω_ε is non-empty since $\{\emptyset\} \in \Omega_\varepsilon$. Now partially order Ω_ε by inclusion -- then every chain in Ω_ε has an upper bound which is also in Ω_ε. Hence Zorn's lemma gives the existence of a maximal subfamily $\mathcal{G}_\varepsilon$ in Ω_ε, and to show $\mathcal{G}_\varepsilon$ is the collection desired we have only to show that each $T \in \mathcal{S}_\varepsilon$ satisfies (ii) relative to $\mathcal{G}_\varepsilon$, not (i). Hence we must show $\mathcal{K} = \emptyset$ where

$\mathcal{K} = \{T \in \mathcal{S}_\varepsilon \mid T \cap S = \emptyset$ for all $S \in \mathcal{G}_\varepsilon\}$.

Suppose $\mathcal{K} \neq \emptyset$. Then there exists $W \in \mathcal{K}$ such that $2\Lambda(W) \geq \sup_{T \in \mathcal{K}} \Lambda(T)$ since this supremum must be finite. But we then claim that $\mathcal{G}_\varepsilon \cup \{W\} \in \Omega_\varepsilon$. For take any $T \in \mathcal{S}_\varepsilon$. Then if (ii) holds for T relative to $\mathcal{G}_\varepsilon$ it also clearly holds for T relative to $\mathcal{G}_\varepsilon \cup \{W\}$.

Thus suppose (i) holds for T relative to $\mathcal{G}_\varepsilon$, i.e., $T \in \mathcal{K}$. Then $2\Lambda(W) \geq \Lambda(T)$ by definition of W, which then implies

(11) $$2 \text{ radius } (W) \geq \text{radius } (T).$$

There are only two cases to consider here: $T \cap W$ empty or non-empty. If $T \cap W$ is empty, then (i) holds for T relative to $\mathcal{G}_\varepsilon \cup \{W\}$. If $T \cap W$ is non-empty, then $T \subset \hat{W}$ from (11) and hence (ii) holds for T relative to $\mathcal{G}_\varepsilon \cup \{W\}$. We have thus verified that $\mathcal{G}_\varepsilon \cup \{W\} \in \Omega_\varepsilon$ which is a contradiction to the maximality of $\mathcal{G}_\varepsilon$. Hence $\mathcal{K} = \emptyset$ and our claim is proved.

We use the subfamily $\mathcal{G}_\varepsilon$ to show that $\mu \equiv 0$ on C. Let $C_0 = \{x \in \mathbb{R}^n \mid d(x,C) \leq r_\varepsilon \text{ where } \varepsilon = 1\}$. We then see that $C \subset \bigcup_{T \in \mathcal{S}_\varepsilon} T \subset \bigcup_{S \in \mathcal{G}_\varepsilon} \hat{S} \subset C_0$ for $\varepsilon < 1$ where without loss of generality we assume $r_\varepsilon \leq r_1$ when $\varepsilon \leq 1$. Hence $\sum_{S \in \mathcal{G}_\varepsilon} \Lambda(S) \leq \Lambda(C_0) < \infty$, and therefore $\mathcal{G}_\varepsilon$ is countable since $\Lambda(S) > 0$ for all

$S \in \mathcal{B}_\varepsilon$. Thus for each $\varepsilon > 0$ we have $\mu(C) \leq \mu(\bigcup_{S \in \mathcal{B}_\varepsilon} \hat{S})$

$\leq \sum_{S \in \mathcal{B}_\varepsilon} \mu(\hat{S}) \leq \sum_{S \in \mathcal{B}_\varepsilon} \varepsilon\Lambda(\hat{S}) = 3^n\varepsilon \sum_{S \in \mathcal{B}_\varepsilon} \Lambda(S) \leq 3^n\varepsilon\Lambda(C_0)$.

But therefore $\mu(C) = 0$ since $\Lambda(C_0) < \infty$. □

<u>Theorem 6.5</u>. Suppose T is a positive definite distribution with spherical Bochner measure μ. Then $T \in (I^p(G))'$ if and only if $\operatorname{supp} \mu \subset \mathcal{F}^\varepsilon$, where $1 \leq p \leq 2$ and $\varepsilon \equiv 2/p - 1$. In such a case $T[\phi] = \int_{\mathcal{P}} \hat{\phi} d\mu$ for all $\phi \in I^p(G)$.

<u>Proof</u>. Suppose $\operatorname{supp} \mu \subseteq \mathcal{F}^\varepsilon$ for some $0 \leq \varepsilon \leq 1$. Then we easily see that the linear functional $\hat{T}$ on $\overline{Z}(\mathcal{F}^\varepsilon)$ defined by $\hat{T}[\Phi] \equiv \int_{\mathcal{P}} \Phi d\mu$ is continuous (same procedure as in the proof of Theorem 5.4 for the $\varepsilon = 1$ case). Hence we can extend T to $I^p(G)$ by $T[\phi] = \hat{T}[\hat{\phi}] = \int_{\mathcal{P}} \hat{\phi} d\mu$.

Now suppose $T \in (I^p(G))'$ for some $1 \leq p \leq 2$. Then by Lemma 6.3 and Prop. 6.4 we have that $\operatorname{supp} \mu \subset \mathcal{F}^\varepsilon$. □

<u>Remark</u>. When considered for $p = 2$ Theorem 6.5 becomes simply the Bochner theorem for tempered (i.e., distributions which lie in $(\mathcal{C}^2(G))'$) K-biinvariant positive definite distributions, which was first proved by Muta [13] in much the same fashion as the euclidean Bochner theorem is proved in Schwartz [14, Thm. XVII, p. 275]. It is of interest to note that Muta's definition of a tempered K-biinvariant positive distribution differs from ours, while the spherical Bochner theorem indirectly proves them equal. Muta defines a tempered K-biinvariant distribution T to be positive definite if $T[\phi * \phi^*] \geq 0$ for all $\phi \in I_c(G)$, which is on the surface a less restrictive definition than ours. We can prove the equivalence directly in the following manner: suppose T is positive definite in Muta's sense, and take $\phi \in \mathcal{D}(G)$. Then $(\phi * \phi^*)^\wedge(\lambda) \geq 0$ for all $\lambda \in \mathcal{P}$ since $\phi_{-\lambda} >> 0$ for all $\lambda \in \mathcal{P}$. But by (Notation §5) we see that there exists a sequence $\{\Phi_n\}_{n=1}^{\infty} \subset \mathcal{S}(\mathfrak{a}^*) = Z(\mathcal{F}^0)$ such that $|\Phi_n|^2 \to (\phi * \phi^*)$ in $\mathcal{S}(\mathfrak{a}^*)$. Noting Schwartz's construction we see that these Φ_n may be taken as W-invariant, and

thus $|\Phi_n|^2 \to (\phi * \phi^*)^\wedge$ in $\mathcal{Z}(\mathcal{F}^0)$. But Theorem 4.6 gives $\phi_n \in I^2(G)$ such that $\hat{\phi}_n = \Phi_n$, and hence $\phi_n * \phi_n^* \to (\phi * \phi^*)^\natural$ in $I^2(G)$. Therefore $T[\phi_n * \phi_n^*] \to T[\phi * \phi^*]$ in $\mathbb{C}$, but it is easy to see that $T[\psi * \psi^*] \geq 0$ for all $\psi \in I^2(G)$. Hence T is positive definite in our sense.

It is by no means obvious that this equivalence of definitions holds for non-tempered K-biinvariant positive definite distributions. This would seem to depend on the truth of the following conjecture: given $\Psi \in \mathcal{Z}(\mathcal{F}^\varepsilon)$, $\Psi \geq 0$ on $\mathcal{P} \cap \mathcal{F}^\varepsilon$, then there exists a sequence Φ_n in $\mathcal{Z}(\mathcal{F}^\varepsilon)$ such that, with $\Psi_n(\lambda) \equiv \Phi_n(\lambda)\overline{\Phi_n(\bar{\lambda})}$ for each $\lambda \in \mathcal{F}^\varepsilon$, we have $\Psi_n \to \Psi$ in $\mathcal{Z}(\mathcal{F}^\varepsilon)$. This seems like a very unobvious statement except for the case $\varepsilon = 0$.

<u>Corollary 6.6</u>. Consider $T \in \mathcal{D}'(G)$, $T \gg 0$, K-biinvariant. Then if the spherical Bochner measure μ is supported in $\mathfrak{a}^* \cup C$ where C is a compact subset of $\mathfrak{a}_{\mathbb{C}}^*$, then $T = S + f$, where S is a tempered

K-biinvariant positive definite distribution and f is a continuous K-biinvariant positive definite function.

In particular, this is always true in the real rank one case.

Proof. Let $\mu_1(E) \equiv \mu(E \cap \mathfrak{a}^*)$ and $\mu_2(E) \equiv \mu(E \cap (\mathfrak{a}_{\mathbb{C}}^* - \mathfrak{a}^*))$ for all Borel sets E in $\mathfrak{a}_{\mathbb{C}}^*$. Then by Theorem 6.5 we have that μ_1 gives rise to a tempered K-biinvariant positive definite distribution S. Now by Lemma 5.3 we have that μ_2 must be finite since it is supported in the compact set C. Hence μ_2 gives rise to a continuous K-biinvariant function f by Theorem 2.5. Clearly $T = S + f$.

Suppose that G is real rank one. Then $\mathcal{P}$ is contained in $\mathfrak{a}^* \cup (i\mathfrak{a}^*)$ since λ and $\bar{\lambda}$ are W-conjugate for all $\lambda \in \mathcal{P}$, and $W = \{s_1, s_2\}$, where $s_1\lambda = \lambda$, $s_2\lambda = -\lambda$ for all $\lambda \in \mathfrak{a}^*$. Then the Helgason-Johnson theorem gives that $\mathcal{P} \subset \mathfrak{a}^* \cup iC_\rho$, which is the desired form. □

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Biographical Note

William H. Barker was born on April 20, 1946, in Brooklyn, New York, which iS an alright place to be born in, but not such a hot place to live. Luckily he and his parents agreed early on this point, and seven years later they all moved out to Long Island.

In 1964 he graduated from high school and entered Harpur College (known now to the masses as SUNY Binghamton), where he spent four very pleasant years and received his B.A. in 1968. At this point his meteoric career was sidetracked by the Asian Red Menace, and he spent four months in Missouri defending the country by learning to stick bayonets into old rubber tires.

It was thus as a trained killer that he entered M.I.T. in September 1969 on an N.S.F. Graduate Fellowship, which he was to hold for two years. The last two years he spent as a half-time teaching assistant while trying to learn all about analysis on Lie groups. In September he'll continue trying to learn all about analysis on Lie groups while an Instructor at Dartmouth College.