A MODAL EXPANSION TECHNIQUE FOR SPACE-TIME
REACTOR KINETICS

by

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by

LARRY R. FOULKE

Submitted to the Department of Nuclear Engineering in August, 1966, in partial fulfillment of the requirement for the degree of Doctor of Philosophy.

ABSTRACT

A space dependent reactor kinetics approximation called the Natural Mode Approximation (NMA) has been applied to the calculation of kinetic behavior and to the interpretation of kinetic experiments. The NMA is based on a modal expansion technique where the space and time dependent neutron density and other variables of a reactor system are approximated by a series of products of time dependent coefficients and space dependent expansion vectors. The expansion vectors are called the natural modes of the system; they are the eigenvectors of a linear operator derived from the complete set of equations describing the system at an initial, reference condition. A pair of computer codes, MUDMO-II and SYNSIG, are given. These codes are used to synthesize approximations to the natural modes in multidimensional systems without feedback.

An oscillation test is proposed which may be used to verify key parameters of the NMA. It is considered that the experimental verification of these parameters gives the NMA physical meaning. The experimental technique is described in detail. Results of applying the technique to both numerical and actual experiments are given.

The NMA is used (i) to calculate a large class of kinetics problems, and (ii) to interpret experiments when the experimental observations are functions of space and time. Calculations using the NMA are compared with independent calculations which are considered to be correct. It is concluded that the flux tilting following a localized perturbation is a sensitive function of the relative magnitudes of those parameters which may be experimentally verified. It is also concluded that the number of expansion modes which must be retained for good results depends on the magnitude and the degree of localization of the perturbation. The introduction of the novel idea of "correction modes" increases the accuracy of a low order NMA without an appreciable increase in computation time. A power excursion calculated with the NMA compares well with results obtained with a direct numerical integration of the space-time equations.

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CHAPTER 1

INTRODUCTION

1.1 Thesis Objective

The objective of this thesis is to develop a useful and physically meaningful approximation for space dependent reactor dynamics. An approximation is considered to be useful and physically meaningful when the approximation (i) is constructed in terms of experimentally verifiable parameters, (ii) can be used to answer questions concerning the kinetic behavior of the reactor, and (iii) can be used to interpret kinetic experiments when the observations are functions of space and time.

1.2 Space Dependent Kinetics Approximations

The simplest time dependent approximation to the complicated conservation relationships which describe the behavior of the neutron density in a nuclear reactor is a set of ordinary differential equations referred to as either the space independent kinetics approximation or the conventional kinetics equations [1-3]. This approximate description of nuclear reactor kinetics gives information about the time behavior of some spatial average of the neutron density, but gives no information about the time variation of the spatial shape of the neutron density. In situations (small cores, slow transients) where the shape of the neutron density distribution does not vary appreciably with time, this space independent approximation is often adequate either for calculating transient behavior or for interpreting kinetic experiments.
In order to understand the dynamic behavior of large nuclear reactor cores or small reactor cores in which large, rapid changes occur, one must consider the fact that the spatial shape of the neutron density may vary with time. Recognition of this fact has led to the development of a number of space dependent kinetics approximations which have been described in a review article by Kaplan et al.\(^4\), and which are briefly summarized here.

One approximation scheme\(^{[5-11]}\) is the straightforward numerical solution of the space and time dependent neutron balance equations. Differential operators are expressed in terms of finite difference approximations and the resulting equations are solved numerically.

A second approximation scheme has been labeled the "instantaneous tilt" method.\(^{[12,13]}\) In this scheme it is assumed that the shape of the neutron density at any instant of time corresponds to the nuclear properties of the reactor at that instant. A space and time dependent solution is developed by first solving a series of static problems which specify the neutron density shapes as a function of the nuclear properties to be encountered at various instants of time. Then time equations are solved which have the form of the space independent kinetics approximation. The input parameters to the time equations are functions of both the nuclear properties and the neutron density shapes which correspond to these nuclear properties.

A third approximation scheme\(^{[14]}\) treats the reactor as a coupled system of sub-reactors. The kinetic behavior of the neutron population within each sub-reactor is given by equations which have the form of
the space independent kinetics approximation. The coupling of one sub-reactor to another is expressed in terms of coupling parameters which are determined initially by means of detailed static calculations.

A fourth approximation scheme is the modal expansion technique. In this approximation, the space and time dependent neutron density is expressed as a series of products of unknown time dependent coefficients and known spatial distributions, sometimes referred to as modes. The space part of the problem consists of specifying the modes. A sequence of weighting and averaging operations leads to a set of coupled, ordinary differential equations which may be solved for the time dependent expansion coefficients. Because of the variety of choices which can be made for the modes, the expansion technique gives rise to a variety of space dependent kinetics approximations.

Common features of the different space dependent kinetics approximations summarized above are that they all represent an approximation of more complicated neutron balance relationships; they all involve the solution of ordinary differential equations in time, and they all may be used to answer questions concerning the kinetic behavior of nuclear reactors. However, to the author's knowledge, people who work with space dependent kinetics approximations have not emphasized the importance of being able (i) to use the approximations for interpreting kinetic experiments in which spatial effects are important, and (ii) to verify experimentally the parameters of the approximations in non-hazardous experiments. These two points are emphasized in this thesis.
It is considered here that in order for a space dependent kinetics approximation to be useful it must satisfy the following criteria:

1) the variables of the approximation can be related to experimental observations,
2) the parameters of the approximation can be calculated, and
3) the approximation can be used to answer questions concerning the kinetic behavior of the reactor.

In addition, in order for a space dependent kinetics approximation to be physically meaningful it must satisfy the following criterion:

4) parameters of the approximation can be experimentally verified.

These criteria will be referred to as the Approximation Criteria. An alternative statement of the thesis objective is to say that the objective of this thesis is to develop, verify experimentally, and use a space dependent kinetics approximation which satisfies the Approximation Criteria.

1.3 Method and Principal Contributions

The objective stated in Section 1.1 is pursued through the use of a particular modal expansion technique which is to be called the Natural Mode Approximation (NMA). The NMA is based on an expansion of the space and time dependent neutron density and other dependent variables of the reactor system into a series of products of unknown, time dependent coefficients and known, spatially dependent expansion vectors. These expansion vectors are called the natural modes of
the reactor; they are the eigenvectors of a linear operator derived from the complete set of equations describing the system at an initial, reference condition (See Chapters 2 and 3). The steady state parameters appearing in the NMA are the eigenvalues of the eigenvalue problem solved to generate the natural modes. The NMA is patterned after the work of Cohen[13], Kaplan[19] and Henry[20].

One feature of the NMA is that knowledge of the eigenvalue spectrum of the natural modes is often sufficient to answer questions concerning kinetic behavior. For example, it is known that questions concerning criticality[21] and stability[19,22] of the reference state may be answered by inspecting the eigenvalue spectrum of the natural modes. The major contributions of this thesis are (i) the demonstration of how certain natural mode eigenvalues may be used to indicate the tendency of the neutron density to undergo shape changes, and (ii) the demonstration of how these particular eigenvalues may be measured by a non-hazardous oscillation test. Others[23] have pointed out that the tendency to undergo shape changes is indicated by the spectrum of what is called the λ-eigenvalues, but these eigenvalues cannot be experimentally verified.

One of the major drawbacks of the NMA is that it is difficult to calculate the natural modes in multidimensional systems. Computer codes are in existence which can calculate the natural modes in one-dimensional systems both with[22,24] and without[21,25,26] xenon feedback. Apparently, numerical difficulties have prevented others from generating the natural modes in multidimensional systems. One contribution of this thesis is the demonstration that it is possible
to generate the natural modes in multidimensional systems without feedback by using a "synthesis" technique[^27] in conjunction with a "Stabilized March Technique"[^28].

Another contribution of this thesis is a series of illustrations which demonstrate how experiments may be interpreted in a meaningful manner in the presence of spatial effects by using a number of suitably located neutron detectors and interpreting these detector readings in terms of a natural mode expansion. Spatial effects can seriously affect the measurements made in standard kinetics experiments such as: (i) oscillation tests designed to measure subcriticality of the fundamental spatial mode, (ii) oscillation tests designed to measure feedback effects in a power reactor, and (iii) excursion experiments designed to measure the inherent self-shutdown mechanisms of a reactor.

A final contribution of this thesis is the presentation of results of transients calculated by means of the NMA. No matter how suitable a space dependent kinetics approximation may be for the interpretation of experiments, the utility of an approximation depends largely upon its ability to predict correct transient behavior with a reasonable expenditure of effort. Calculations are made which show how the results depend upon the number of natural modes retained. Calculations are made which show that the amount of shape change in the neutron density following a perturbation in a specified region depends upon the magnitude of the perturbation and the relative magnitudes of those natural mode eigenvalues which may be experimentally verified. Calculations are made for a one-dimensional
example which includes feedback. The results are compared with the results obtained by a direct numerical solution.

1.4 Organization of the Thesis

Chapter 2 reviews the simplifications involved in the construction of a large class of space dependent kinetics approximations. The formalism of one particular approximation, the Natural Mode Approximation, is developed in detail. Chapter 3 considers the problem of calculating the natural modes and their eigenvalues which are used in the NMA, and discusses the utility of the eigenvalue spectrum for characterizing the space dependent kinetic behavior. Chapter 4 considers the experimental verification of parameters (certain eigenvalues) of the NMA by means of small signal oscillation tests. Chapter 5 considers the use of the NMA for calculating space dependent kinetic behavior. Chapter 6 considers the use of the NMA for interpreting kinetic experiments in which spatial effects are important. Chapter 7 summarizes the results of the thesis and makes recommendations for further work.

The notation is defined when it is introduced. In addition, a glossary of principal symbols is included in Appendix A.
2.1 Introduction

The neutron population in a nuclear reactor is a complicated function of many variables such as spatial location, energy, direction and time. The behavior of the neutron population is described as a function of these variables by a set of integro-differential equations which cannot be solved. Thus it is necessary to make certain simplifications which lead to an approximate but tractable set of equations which can be solved.

The type of simplification involved depends upon the type of problem being studied. For example, the problem here is that of describing the neutron population as a function of space and time. Therefore the first simplification is to eliminate the energy and direction dependence by using multigroup diffusion theory. Even with this simplification, the resulting set of space and time dependent equations (referred to as the space dependent kinetics equations) cannot be solved and it is necessary to make additional simplifications.

These additional simplifications lead to a more tractable set of equations which is referred to as a space dependent kinetics approximation. The purpose of this chapter is to develop the formalism of a particular space dependent kinetics approximation called the Natural Mode Approximation (NMA).
The chapter is organized in the following way. Section 2.2 introduces the basic equations. These equations contain operators which depend implicitly upon the solution. Section 2.3 defines an approximation which allows this implicit dependence to be included explicitly by the introduction of addition variables called feedback variables. Section 2.4 considers the operations which are performed on the space dependent kinetics equations in order to get what is referred to as a space dependent kinetics approximation. Section 2.5 defines a particular approximation, the NMA, which satisfies the Approximation Criteria introduced in Chapter 1. The chapter ends with an example of how the Approximation Criteria may be used to decide about the usefulness and the physical meaning of a kinetics approximation.

### 2.2 The Basic Equations

The behavior of a nuclear reactor can always be described by the equation,

\[
\frac{\partial}{\partial t} \psi(x, E, \Omega, t) = \left[ H(\psi, x, E, \Omega, t) \right] \psi(x, E, \Omega, t) + S(x, E, \Omega, t),
\] (2.1)

where \( \psi \) is a \( K \)-vector containing the \( K \) dependent variables of the system as components and \( x, E, \) and \( \Omega \) represent the independent variables, space, energy and direction, respectively. The \( K \) dependent variables in this description are the \( I \) delayed neutron precursor.

*Column vectors will be indicated by an underscore, \( \mathbf{A} \); row vectors will be indicated by the transpose of a column vector, \( \mathbf{A}^T \); and matrices will be indicated by square brackets, \([\mathbf{A}]\). Superscripts in parentheses will index the components of a vector, and subscripts will index terms of an expansion.*
densities, \( c^{(i)}(x,t) \), and the density of neutrons, \( N(x,E,\Omega,t) \) in (\( x,E,\Omega \))-space. That is,

\[
\psi(x,E,\Omega,t) \equiv \text{col}[N(x,E,\Omega,t), c^{(1)}(x,t), \ldots, c^{(I)}(x,t)].
\]

The K by K matrix operator, \([H(\psi, x,E,\Omega,t)]\), comes from general transport theory and it governs the relationships between the dependent variables. This operator is not linear because of its implicit dependence upon \( \psi \). The vector, \( \mathbf{S}(x,E,\Omega,t) \), contains all external sources.

Equation (2.1) cannot be solved in general and it is necessary and customary to make certain simplifications. This work will concentrate upon the space and time dependent behavior of the neutron density. Hence the first simplification will be to eliminate the energy and angular dependence of \( N(x,E,\Omega,t) \) by using multigroup diffusion theory. In the multigroup approximation the entire energy range is broken up into \( G \) intervals and it is assumed that the energy dependence of \( N(x,E,\Omega,t) \) can be represented by a simple function of the neutron energy over each interval. Thus the component, \( N(x,E,\Omega,t) \), of the vector, \( \psi \), becomes a \( G \) vector with components, \( N^{(1)}(x,\Omega,t) \), \( N^{(2)}(x,\Omega,t) \), \ldots, \( N^{(G)}(x,\Omega,t) \). Diffusion theory is based upon the assumption that the angular distribution of neutron velocity vectors is nearly isotropic and hence it is sufficient to characterize the neutron density in terms of the total number density,

\[
N^{(g)}(x,t) = \int_\Omega d\Omega \, N^{(g)}(x,\Omega,t).
\]
A more complete discussion of the derivation of and the restrictions of multigroup diffusion theory may be found in Refs. 29 and 30.

In the framework of G group diffusion theory with I groups of delayed neutron precursors in a reactor with stationary fuel and one fissionable nuclide, the elements appearing in Eq. (2.1) are defined by:

\[ \mathbf{\Psi} = \text{col}[N^{(1)}, \ldots, N^{(G)}, C^{(1)}, \ldots, C^{(I)}], \]

\[ N^{(g)} = \text{neutron density in group } g, \]

\[ C^{(i)} = \text{concentration of delayed neutron precursors of type } i, \]

\[ [H] = \begin{bmatrix} [\Sigma][V] + \nabla \cdot [D][\nabla][V] + (1-\beta)[\mathbf{K} \cdot \mathbf{F}^T][V] & [\Gamma] \\ [\Phi \cdot \mathbf{F}^T][V] & -[\lambda] \end{bmatrix}, \tag{2.2} \]

\[ [\Sigma] = [\Sigma_a] + [\Sigma_r], \]

\[ [\Sigma_a] = \begin{bmatrix} \Sigma_{a1} & 0 & \ldots & 0 \\ 0 & \Sigma_{a2} \\ \vdots & \vdots \\ 0 & 0 & \ldots & \Sigma_{aG} \end{bmatrix}, \]

\[ [\Sigma_r] = \begin{bmatrix} \Sigma_{r1} & -\Sigma_{21} & \ldots & -\Sigma_{G1} \\ -\Sigma_{12} & \Sigma_{r2} & \ldots & -\Sigma_{G2} \\ \vdots & \vdots & \ddots & \vdots \\ -\Sigma_{1G} & -\Sigma_{2G} & \ldots & \Sigma_{rG} \end{bmatrix}. \]
\[
[v] = \begin{bmatrix}
v_1 & 0 & \cdots & 0 \\
0 & v_2 & \cdots & 0 \\
\vdots & \vdots & & \vdots \\
0 & 0 & \cdots & v_G \\
\end{bmatrix}
\]

\[
[D] = \begin{bmatrix}
D_1 & 0 & \cdots & 0 \\
0 & D_2 & \cdots & 0 \\
\vdots & \vdots & & \vdots \\
0 & 0 & \cdots & D_G \\
\end{bmatrix}
\]

\[
F = \text{col}[ v f_1, v f_2, \ldots, v f_G ]
\]

\[
v_g = \text{neutron velocity in group } g,
\]

\[
\Sigma_{gj} = \text{scattering cross section from group } g \text{ to group } j \ (\Sigma_{gg} = 0),
\]

\[
\Sigma_{rg} = \text{total scattering cross section for group } g \ (\Sigma_{rg} = \sum_j \Sigma_{gj}),
\]

\[
\Sigma_{ag} = \text{total absorption cross section for group } g,
\]

\[
D_g = \text{diffusion coefficient for group } g,
\]

\[
v \Sigma_{fg} = \text{neutrons per fission times fission cross section in group } g,
\]

\[
X_p = \text{col}[x_{p1}, x_{p2}, \ldots, x_{pG}] = \text{emission spectrum for prompt neutrons},
\]

\[
X_i = \text{col}[x_{i1}, x_{i2}, \ldots, x_{iG}] = \text{emission spectrum for delayed neutrons of type } i,
\]

\[
[\Gamma] = [\lambda_1 x_1, \lambda_2 x_2, \ldots, \lambda_1 x_1]
\]

\[
\lambda_i = \text{decay constant of precursors of type } i,
\]
\[
[\lambda] = \begin{bmatrix}
\lambda_1 & 0 & \ldots & 0 \\
0 & \lambda_2 & 0 \\
0 & 0 & \ldots & \lambda_L \\
\end{bmatrix}
\]

\[
\beta = \text{col} [\beta_1, \beta_2, \ldots, \beta_L],
\]

\[
\beta_i = \text{fraction of fission neutrons appearing as delayed type } i,
\]

\[
\beta = \sum_{i=1}^{L} \beta_i.
\]

2.3 The Inclusion of Feedback

As a first step in solving the system of multigroup diffusion equations represented by Eq. (2.1) some approximation must be made for the implicit dependence of \([H(\mathbf{x}, \mathbf{t})]\) upon \(\mathbf{f}\). This is conveniently done by the introduction of a number of additional variables usually referred to as feedback variables. Examples of these are xenon concentration, material temperatures, and void volume.

Let \(\mathcal{Q}(j)(\mathbf{x}, t)\) represent the departure of the \(j\)th feedback variable from its value, \(T_o(j)(\mathbf{x})\), at the reference condition. Let the feedback variables be related to \(\mathbf{f}\) by an equation,

\[
[Q(\mathbf{f}, x, t)] \psi(x, t) + [G(\mathbf{f}, x, t)] (T_o + \mathcal{Q}(x, t)) = \frac{\partial \mathcal{Q}}{\partial t}(x, t), \tag{2.3}
\]

where \(\mathcal{Q} = \text{col} [\mathcal{Q}(1), \ldots, \mathcal{Q}(J)]\), \([Q]\) is a \(J\) by \(K\) matrix operator and \([G]\) is a \(J\) by \(J\) matrix operator. The exact form of the matrix operators, \([V]\) and \([G]\), is specified by conservation relationships which describe the feedback variable. Now assume that \([H]\), \([Q]\), and \([G]\) may be represented as
\[ [H_0(x, t)] = [H_0(x)] + [h(x, t)] + \sum_{j=1}^{J} \frac{\partial [H_0]}{\partial \phi(j)} \phi(j)(x, t) \quad , \quad (2.4) \]

\[ [Q_0(x, t)] = [Q_0(x)] + [q(x, t)] + \sum_{j=1}^{J} \frac{\partial [Q_0]}{\partial \phi(j)} \phi(j)(x, t) \quad , \]
and

\[ [G_0(x, t)] = [G_0(x)] + [g(x, t)] + \sum_{j=1}^{J} \frac{\partial [G_0]}{\partial \phi(j)} \phi(j)(x, t) \quad . \]

\[ [H_0], [Q_0], \text{ and } [G_0] \text{ represent the matrix operators at a steady state,} \]
reference condition; \([h], [q] \text{ and } [g] \text{ represent externally controlled} \]
perturbations to these operators.

Substitution of these representations for the operators into
Eqs. (2.1) and (2.3) yields the following set of coupled equations:

\[ \frac{\partial \psi}{\partial t} = [H_0] \psi + [h] \psi + \sum_{j=1}^{J} \frac{\partial [H_0]}{\partial \phi(j)} \phi(j) \psi_0 + \sum_{j=1}^{J} \frac{\partial [H_0]}{\partial \phi(j)} \phi(j)(\psi_0 - \psi_0) + \psi \quad , \quad (2.5) \]

and

\[ \frac{\partial \phi}{\partial t} = [Q_0] \psi + [q] \psi + [G_0] \phi + [g] \phi + \sum_{j=1}^{J} \frac{\partial [Q_0]}{\partial \phi(j)} \phi(j) \phi_0 + \sum_{j=1}^{J} \frac{\partial [G_0]}{\partial \phi(j)} \phi(j) \phi_0 + \phi + \sum_{j=1}^{J} \frac{\partial [G_0]}{\partial \phi(j)} \phi(j)[H_0] \phi_0 + \phi \quad . \quad (2.6) \]

Equations (2.5) and (2.6) can be written in a more convenient form by
manipulating the terms which are linear in \( \phi(j) \) to give
\[
\sum_{j=1}^{J} \frac{\partial [H_0^\circ]}{\partial \psi_j} \psi_j(x) = H_1^\circ \psi_1 + H_2^\circ \psi_2 + \ldots + H_J^\circ \psi_J \\
= [H_1^\circ, H_2^\circ, \ldots, H_J^\circ] \begin{bmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_J \end{bmatrix} \\
= [H_0^\circ] \psi .
\]

The vectors, \( H_j^\circ \), and the matrix operator, \( [H_0^\circ] \), are defined by Eqs. (2.7). The superscript "\( \circ \)" of the operator, \( [H_0^\circ] \), is used to denote the dependence of this operator on \( \psi_0 \). If similar manipulations are used to define \( [Q_0^\circ] \) and \( [G_0^\circ] \) then Eqs. (2.5) and (2.6) may be written in matrix form as

\[
\frac{\partial}{\partial t} \begin{bmatrix} \psi \\ \varphi \end{bmatrix} = \begin{bmatrix} [H_0] & [H_0^\circ] \\ [Q_0 + Q_0^\circ] & [G_0 + G_0^\circ] \end{bmatrix} \begin{bmatrix} \psi \\ \varphi \end{bmatrix} + \begin{bmatrix} [h] & [0] \\ [q] & [g] \end{bmatrix} \begin{bmatrix} \psi \\ \varphi \end{bmatrix} \\
+ \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} + \begin{bmatrix} S(x, t) \\ [G_0^\circ T_0] \end{bmatrix} ,
\]

where the nonlinear terms,

\[
f_1 = \sum_{j=1}^{J} \frac{\partial [H_0^\circ]}{\partial \psi_j} \psi_j \left( \psi - \varphi \right),
\]

and

\[
f_2 = \sum_{j=1}^{J} \frac{\partial [Q_0^\circ]}{\partial \psi_j} \psi_j \left( \psi - \varphi \right) + \sum_{j=1}^{J} \frac{\partial [G_0^\circ]}{\partial \psi_j} \psi_j \varphi ,
\]
contain products of incremental quantities.

Equation (2.8) can be written more compactly as

\[
\frac{\partial}{\partial t} \varphi(x,t) = \left[ L_0^0 \right] \varphi(x,t) + \left[ \varphi_0 \right] \varphi(x,t) + f(x,\varphi(x,t)) + S'(x,t), \quad (2.11)
\]

where \( \varphi \) is a K-vector given by

\[
\varphi = \text{col}[N^{(1)}, \ldots, N^{(G)}, c^{(1)}, \ldots, c^{(I)}, q^{(1)}, \ldots, q^{(J)}]
\]

The other notation in Eq. (2.11) is defined by a comparison with Eq. (2.8).

In many cases of experimental interest the reference reactor system is at very low power and the dependence of Eq. (2.11) upon the feedback variables, \( \varphi \), is negligible. In this case, Eq. (2.11) reduces to

\[
\frac{\partial \psi(x,t)}{\partial t} = \left[ H_0 \right] \psi(x,t) + \left[ h \right] \psi(x,t) + S'(x,t), \quad (2.12)
\]

where \( \psi \) is a K-vector given by

\[
\psi = \text{col}[N^{(1)}, \ldots, N^{(G)}, c^{(1)}, \ldots, c^{(I)}]
\]

2.4 A Class of Approximation Techniques

Even with the simplifications of G-group diffusion theory and the assumption of linear dependence of the operators upon the feedback variables, the ensuing space and time dependent equations, Eqs. (2.11)
or (2.12), cannot be solved in the general case. As pointed out by Kaplan, present digital computers are not adequate to handle the complexity of three space dimensions plus one time dimension in a direct numerical solution. However, digital computers are adequate for calculating static distributions in the three dimensional $x$-space. Thus one simplification which is widely used is the technique of expanding the dependent variables into a finite series of products of known, space dependent vectors, $\varphi_{mk}(x)$, and unknown, time dependent expansion coefficients, $A_{mk}(t)$. That is,

$$
\varphi(x,t) = \left\{ \varphi(x,t) \right\} \approx \sum_{m=1}^{M} \sum_{k=1}^{K} A_{mk}(t) \varphi_{mk}(x) \quad (2.13)
$$

where the upper limit $K$, in the $k$ summation is meant to be identical to the number of dependent variables in $\varphi$, and the upper limit, $M$, is arbitrary. The upper limit, $M$, is unity in a space independent kinetics approximation. As $M$ increases it is assumed that the space dependent kinetics approximation improves. The $\varphi_{mk}(x)$'s are considered to satisfy the same boundary conditions as $\varphi(x,t)$.

Substitution of this expansion into the system of equations, Eq. (2.11), yields
\[
\sum_{m=1}^{M} \sum_{k=1}^{K} \frac{\partial A_{mk}}{\partial t} \psi_{mk} = \sum_{m=1}^{M} \sum_{k=1}^{K} A_{mk} [L_0] \psi_{mk} + \sum_{m=1}^{M} \sum_{k=1}^{K} A_{mk} [\mathcal{L}] \psi_{mk} + \\
+ \tilde{f}(A_{11}A_{12}A_{11}A_{13}, \ldots, A_{MK}A_{MK}) + S'. \tag{2.14}
\]

Weighting of this equation by \(MK\) linearly independent weighting vectors, \(\psi_{nj}\), and subsequent integration \(T\) over all space yield the following set of \(MK\), coupled nonlinear ordinary differential equations,

\[
\sum_{m=1}^{M} \sum_{k=1}^{K} \frac{dA_{mk}}{dt} \left\langle \psi_{nj} \psi_{mk} \right\rangle = \sum_{m=1}^{M} \sum_{k=1}^{K} A_{mk} \left\langle \psi_{nj} [L_0 \psi_{mk}] \right\rangle + \\
+ \sum_{m=1}^{M} \sum_{k=1}^{K} A_{mk} \left\langle \psi_{nj} [\mathcal{L}] \psi_{mk} \right\rangle + \\
+ \left\langle \psi_{nj} \tilde{f} \right\rangle + \left\langle \psi_{nj} S' \right\rangle, \tag{2.15}
\]

where \(n=1, \ldots, M\) and \(j=1, \ldots, K\).

These equations may be solved for the \(MK\) expansion coefficients, \(A_{mk}\).

The approximate solution is then given by Eq. (2.13).

The result of using this expansion technique is a set of equations, Eq. (2.15), which may be referred to as a space dependent kinetics approximation. The number of independent variables has been reduced from four (three coordinate variables and one time variable) to one (a time variable); but the number of equations has been increased from

---

*The Dirac bracket notation, \(\langle A, B \rangle\), is used to denote integration of the scalar product of the row vector, \(A^T\), and the column vector, \(B\), over the volume of the reactor. The notation, \(\langle AB \rangle\), is used to denote integration of the product of two scalars \(A\) and \(B\) over the volume of the reactor.*
K to MK. However, the equations of the approximation are more tractable and can be solved numerically. Instead of being constructed in terms of neutron densities and nuclear cross sections, the approximation is constructed in terms of new variables, $A_{mk}(t)$, and the averaged quantities, $\langle \omega_{nj}, [L^0_o] \phi_{nk} \rangle$, and $\langle \omega_{nj}, [\ell] \phi_{nk} \rangle$. These averaged quantities will be referred to as the parameters of the approximation. The exact meaning of the $A_{mk}$ and the $\langle \omega_{nj}, [L^0_o] \phi_{nk} \rangle$, and the differences in various space dependent kinetics approximations depend upon the particular choice of expansion vectors and weighting vectors. The fact that one space dependent kinetics approximation is better than another in a specific problem results from using vectors which are better suited to the specific problem. However, a specific set of expansion vectors will, in general, lead to a kinetics approximation which is useful for only a limited range of problems.

This thesis is particularly concerned with (i) the relationship between the $A_{mk}$ and experimentally observable quantities, and (ii) the expediency with which the parameters can be experimentally verified and calculated. These considerations lead to the particular choice of expansion vectors and weighting vectors which are described in the next section.

2.5 The Natural Mode Approximation

It has been suggested in Ref. 19 that a suitable set of expansion vectors, $\phi_{nk}(x)$, are the eigenvectors of the equation,

$$[L^0_o] \phi_{nk}(x) = \omega_{nk} \phi_{nk}(x),$$

(2.16)
and a convenient set of weighting vectors are the eigenvectors of the equation,

$$[L_o^0]^* \phi_{mk}^* (x) = w_{nk}^* \phi_{nk}^* (x) . \quad (2.17)$$

The linear operator, $[L_o^0]^*$, is the adjoint of the operator, $[L_o^0]$, and is defined by the equation,

$$\langle \phi_{nj}^* , [L_o^0] \phi_{nk} \rangle = \langle \phi_{nk}^* , [L_o^0]^* \phi_{nj} \rangle , \quad (2.18)$$

where it is considered that both $\phi_{nk}$ and $\phi_{nj}^*$ have homogeneous boundary conditions.

The set of eigenvectors defined by Eq. (2.16) may be considered to be the natural modes of vibration of the system at a linearized, reference condition. These modes have a spatial dependence which reflects the heterogeneities of the system. They will be subsequently referred to as the natural modes of the system.

It is assumed that the eigenvalues, $w_{nk}$, are distinct. This implies the following orthogonality relationship between the expansion vectors and weighting vectors:

$$\langle \phi_{nj}^* , \phi_{nk} \rangle = 0 , \quad \text{for } w_{nk} \neq w_{nj}^* . \quad (2.19)$$

It is further assumed that

$$\langle \phi_{nk}^* , \phi_{nk} \rangle \neq 0 , \quad (2.20)$$

which implies that the set of eigenvalues, $w_{nk}$, is the same as the set, $w_{nk}^*$. 
If these expansion vectors, weighting vectors and orthogonality relationships are used, the space dependent kinetics approximation, Eq. (2.15), may be written in the form,

$$\frac{d\mathbf{A}}{dt} = \text{diag}[\omega] \mathbf{A} + [\mathbf{P}] \mathbf{A} + \mathbf{f} + \mathbf{S}$$  \hspace{1cm} (2.21)

where $\mathbf{A} = \text{col}[A_{11}, A_{12}, \ldots, A_{MK}]$, and

$$\text{diag}[\omega] = \text{an MK by MK diagonal matrix with diagonal elements, } \omega_{mk}.$$  

The perturbation matrix, $[\mathbf{P}]$, has elements,

$$P_{\mu\nu} = \frac{\langle \phi_{mk}^{*}, [\phi_{n,l}] \rangle}{\langle \phi_{nk}^{*}, \phi_{nk} \rangle}$$  \hspace{1cm} (2.22)

where $\mu = (m-1)K + k$ for $m = 1, \ldots, M$ and $k = 1, \ldots, K$; and where $\nu = (n-1)K + j$ for $n = 1, \ldots, M$ and $j = 1, \ldots, K$. The vector of nonlinear terms, $\mathbf{f}$, has MK components given by

$$f_{mk} = \frac{1}{\langle \phi_{nk}^{*}, \phi_{nk} \rangle} \sum_{n=1}^{M} \sum_{j=1}^{K} \sum_{l=1}^{M} \sum_{i=1}^{K} \left[ \rho_{mk,nj,lij} A_{lij} (A_{lij} - A_{lij}(0)) \right]$$  \hspace{1cm} (2.23)

$$+ \rho_{mk,nj,lij} A_{nj} A_{lij}$$

where

$$\rho_{mk,nj,lij} = \langle \phi_{mk}^{*}, \sum_{\mu=1}^{J} \frac{\partial[H_{q}]}{\partial \psi_{m}(\mu)} \phi_{n,j}(\mu) \psi_{lij} \rangle +$$

$$+ \langle \phi_{mk}^{*}, \sum_{\mu=1}^{J} \frac{\partial[Q_{q}]}{\partial \phi_{n,j}(\mu)} \phi_{n,j}(\mu) \psi_{lij} \rangle \hspace{1cm} (2.24)$$
The source vector, $S$, has MK components given by

$$S_{mk} = \langle \Phi_{mk}^*, S \rangle \left( S_{ml} \right).$$  (2.26)

If the reactor is operating at very low power, feedback is negligible. The set of equations which describe the system behavior is Eq. (2.12) which is repeated here for convenience.

$$\frac{\partial \psi(x,t)}{\partial t} = [H_0] \psi(x,t) + [h] \psi(x,t) + S(x,t).$$  (2.27)

In this case $\psi$ is expanded in terms of the natural modes of the reactor without feedback. That is,

$$\psi(x,t) \approx \sum_{m=1}^{M} \sum_{k=1}^{K} A_{mk}(t) \psi_{mk},$$  (2.28)

where the $\psi_{mk}$'s are the solutions of

$$[H_0] \psi_{mk} = \omega_{mk} \psi_{mk}.$$  (2.29)

The weighting vectors are the eigenvectors of the adjoint equation,

$$[H_0]^* \psi^*_{mk} = \omega_{mk} \psi^*_{mk}.$$  (2.30)

The substitute-weight-and-integrate procedure now yields the set of equations,
\[
\frac{dA}{dt} = \text{diag}[^w]A + [P]A + S,
\] (2.31)

which is analogous to Eq. (2.21), but which does not contain the effects of feedback. The perturbation matrix, \([P]\), now has elements of the form,

\[
P_{\mu\gamma} = \frac{\langle \psi^*_{mk} , [h] \psi_{nk} \rangle}{\langle \psi^*_{mk} , \psi_{mk} \rangle},
\] (2.32)

where \(\mu = (m-1)K + k\) for \(m=1, \ldots, M\) and \(k=1, \ldots, K\); and where \(\gamma = (n-1)K + j\) for \(n=1, \ldots, M\) and \(j=1, \ldots, K\).

The source vector has \(MK\) components of the form,

\[
S_{mk} = \frac{\langle \psi^*_{mk} , S(x,t) \rangle}{\langle \psi^*_{mk} , \psi^*_{mk} \rangle}.
\] (2.33)

Equations (2.21) or (2.31) are the time dependent equations which will be referred to as the Natural Mode Approximation (NMA). It is constructed in terms of variables, \(A_{mk}\), which are the natural mode expansion coefficients and in terms of reference state parameters, \(\omega_{mk}\), which are the natural mode eigenvalues. The NMA, its variables, and its parameters are nothing more than mathematical constructs which approximate our understanding of a physical theory. Nevertheless, it is considered here that these constructs are useful and physically meaningful if they satisfy the Approximation Criteria which are repeated here for convenience:
A space dependent kinetics approximation is useful if

1) the variables of the approximation can be related to experimental observations,

2) the parameters of the approximation can be calculated, and

3) the approximation can be used to answer questions concerning the kinetic behavior of the reactor.

In addition, a space dependent kinetics approximation is physically meaningful if

4) parameters of the approximation can be experimentally verified.

It will be shown that the first criterion may be satisfied. At any instant of time, a detailed and continuous spatial distribution of any variable such as a neutron group density can be constructed by combining the $A_{nk}$ and the natural modes in the manner prescribed by Eq. (2.13) or Eq. (2.28). Thus experimental observations of the behavior of the neutron density can be constructed from the $A_{nk}$.

Conversely, it is often possible to infer experimentally the behavior of the N most predominant $A_{nk}$ by analyzing the results of N experimental observations. This is covered more fully in Chapters 4 and 6.

It will be shown that the second criterion may be satisfied to a certain extent. The steady state parameters of the NMA are the eigenvalues, $\omega_{mk}$, of the matrix operator which describes the unperturbed, reference condition. The perturbation parameters and the parameters appearing in $f$, the vector having nonlinear components, are simply integrals involving the natural modes and the changes in reactor properties. Therefore, if the natural modes can be calculated,
the parameters of the NMA can be calculated. The calculation of the natural modes in multidimensional systems is a difficult problem. Even though this thesis makes contributions to the solution of this problem, the solution is not completely satisfactory. The calculation of the natural modes and their eigenvalues is the subject of Chapter 3.

It will be shown that the third criterion may be satisfied. Answers to questions regarding kinetic behavior of the system can be found merely by solving the NMA. A particularly attractive feature of using the NMA is that knowledge of the spectrum of natural mode eigenvalues is sufficient to answer many questions about kinetic behavior, and hence it is unnecessary actually to solve the NMA. Such questions include questions about asymptotic stability, spatial stability, subcriticality, and the tendency for the neutron density distribution to undergo shape changes following a perturbation. The characterization of space dependent reactor kinetics by means of the eigenvalue spectrum is covered more fully in Chapters 3 and 5.

Finally, it will be shown in Chapter 4 that several eigenvalues of the NMA may be measured by means of small signal oscillation tests. Thus the fourth criterion may be satisfied.

2.6 Application of the Approximation Criteria in Determining the Usefulness and Physical Significance of a Kinetics Approximation

In order to elaborate on how the Approximation Criteria may be used to decide about the suitability of a kinetics approximation, an example is discussed which deals with the interpretation of oscillation tests in terms of the conventional kinetics equations.
The points to be made can be illustrated by using a simple, one group diffusion theory, one delayed precursor group model in which there is no feedback, and no external sources. The equation which describes this system is

\[ \frac{d}{dt} \psi = [H_o] \psi + [h] \psi \quad \text{(2.34)} \]

where

\[ \psi = \text{col} [N(x,t), C(x,t)] \]

\[ [H_o] = \begin{bmatrix} -\nu \Sigma_{a_0} + \nu \nabla \cdot D \nabla + (1-\beta) \nu \Sigma_f v & \lambda \\ \beta \nu \Sigma_f v & -\lambda \end{bmatrix} \]

\[ \psi(0,t) = \psi(a,t) = 0 \]

and

\[ [h] = \begin{bmatrix} -\nu \delta \Sigma_a(x,t) & 0 \\ 0 & 0 \end{bmatrix} \]

The solution vector, \( \psi \), contains two dependent variables, hence \( K = 2 \). The use of the two linearly independent weighting vectors,

\[ w_1 = \text{col} [w(x), 0] \]

and

\[ w_2 = \text{col} [0, w(x)] \]

in a weighted averaging of Eq. (2.34) leads to the ordinary differential equations,

\[ \frac{d}{dt} \langle w(x) N(x,t) \rangle = \langle w(x) (\nu \Sigma_f (1-\beta) - \nu \Sigma_a + \nu \nabla \cdot D \nabla) N(x,t) \rangle + \lambda \langle w(x) C(x,t) \rangle \quad \text{(2.35)} \]

and
\[
\frac{d}{dt} \langle w(x)C(x,t) \rangle = \beta \langle w(x)\nu \Sigma_f N(x,t) \rangle - \lambda \langle w(x)C(x,t) \rangle , \tag{2.36}
\]

where \( \Sigma_a = \Sigma_{ao} + \delta \Sigma_a(x,t) \).

These equations can be put into the form of the conventional kinetics equations \([3]\),

\[
\frac{dP}{dt} = \frac{\beta - \rho}{\Lambda} P(t) + \lambda C(t) , \tag{2.37a}
\]

and

\[
\frac{dC}{dt} = \frac{\beta}{\Lambda} P(t) - \lambda C(t) , \tag{2.37b}
\]

if the following definitions are made:

\[
P(t) \equiv \langle w(x)N(x,t) \rangle , \tag{2.38a}
\]

\[
C(t) \equiv \langle w(x)C(x,t) \rangle , \tag{2.38b}
\]

\[
\frac{\rho}{\Lambda} = \frac{\langle w(x)(\nu \Sigma_f - \nu \Sigma_a + \nu \nabla \cdot \nabla)N(x,t) \rangle}{\langle w(x)N(x,t) \rangle} , \tag{2.38c}
\]

and

\[
\frac{\beta}{\Lambda} = \frac{\langle w(x)\nu \Sigma_f N(x,t) \rangle}{\langle w(x)N(x,t) \rangle} \beta . \tag{2.38d}
\]

Equations (2.37) may be considered to be a kinetics approximation. Intelligent use of this approximation requires an understanding of the precise meaning of the parameters and variables involved. It will be shown that the precise meaning of these quantities depends upon the choice of weighting function, \( w(x) \).

An example is now given which illustrates that the results of a "numerical" oscillation test cannot be interpreted successfully in
terms of the kinetics approximation. The failure of the kinetics approximation is traced to the fact that the Approximation Criteria cannot be satisfied.

Consider the following numerical experiment. A small amount of localized, neutron absorber is oscillated with angular frequency, $\omega$, in a plane at $x = x_0$ in a uniform, critical, slab reactor of extrapolated width, $a$. The oscillating perturbation causes the neutron density to oscillate with the same angular frequency but with an amplitude and phase relationship to the perturbation which depends on the dynamic characteristics of the reactor. The oscillating perturbation can be represented as

$$\delta \Sigma_a(x,t) = \delta \Sigma_a(x-x_0)e^{jwt},$$

where $\delta \Sigma_a$ is the magnitude of the oscillating perturbation and where only the real part of the oscillating time function is to be taken.

For the simple system under consideration, the corresponding space and frequency dependent solution, $N(x,t)$, of Eq. (2.34) can be found explicitly. This solution is derived in Appendix B and is given by the relation,

$$N(x,t) = N_0(x) + \frac{2\delta \Sigma_a}{D} N_0(x_0) \frac{1}{\pi^2} \sum_{n=1}^{\infty} \frac{\sin \frac{\pi x}{a}}{n^2} \sin \frac{\pi x_0}{a} \frac{\sin \frac{\pi x}{a}}{n^2} e^{jwt},$$

where

$$B^2 = \left(\frac{n}{a}\right)^2 - \frac{j\omega}{\nu D} \left\{ \frac{j\omega + \lambda + \nu \Sigma_f \beta}{j\omega + \lambda} \right\}.$$ 

This expression may also be written as (Appendix B)
\[ N(x,t) = N_0(x) + \frac{\delta \sum_a N_a(x_0)}{D} G(x, x_0, j\omega) e^{j\omega t}, \quad (2.39) \]

where

\[ G(x, x_0, j\omega) = \frac{\sin Bx \sin B(a-x_0)}{B \sin Ba} \text{ for } x \leq x_0, \]

and

\[ G(x, x_0, j\omega) = \frac{\sin B\omega \sin B(a-x)}{B \sin Ba} \text{ for } x > x_0. \]

A detector at position, \( x = x_1 \), with a response function, \( v_0 \delta(x-x_1) \), would have an output, \( R_1(t) \), with an oscillating component, \( r_1(t) \), given by

\[ r_1(t) = \langle v_0 \delta(x-x_1) \{N(x,t) - N_0(x)\} \rangle = v_0 G(x_1, x_0, j\omega) e^{j\omega t}. \quad (2.40) \]

The phase of this oscillating output component, with respect to the phase of the perturbation, is

\[ \frac{r_1(t)}{\delta \sum_a} = G(x_1, x_0, j\omega). \quad (2.41) \]

Figure 2.1(b) illustrates the phase that would be measured by detectors at \( x = 10 \text{ cm} \) or \( x = 100 \text{ cm} \) with the oscillator at \( x_0 = a/2 = 100 \text{ cm} \).

Behavior of this type has been observed in actual oscillation tests (for example, see Fig. 4.16 of Chapter 4).
Figure 2.1 Interpretation of an Oscillation Test in Terms of the Space Independent Kinetics Approximation.
Suppose now that this experiment is to be interpreted in terms of the kinetics approximation given by Eqs. (2.37). The phase between $P(t)$ and $p/\Lambda(t)$, which is determined from Eqs. (2.37), is plotted in Fig. 2.1(b). Note that this phase is space independent and does not coincide with the phase indicated by either of the detectors. Hence the kinetics approximation is not useful for interpreting this experiment. This conclusion could also have been reached independently by noting that the kinetics approximation fails to satisfy the Approximation Criteria. This is shown below for two different choices of the weighting function, $w(x)$, which is used in the derivation of the kinetics approximation.

First, consider that $w(x)$ is chosen as the response function of the $i^{th}$ detector. In this case, it is seen from Eq. (2.38a) that the variable of the approximation, $P(t)$, is identical to the reading of the $i^{th}$ detector. That is,

$$P(t) = \langle w(x)N(x,t) \rangle = \langle v_d \delta(x-x_i)N(x,t) \rangle = v_d N(x_i,t) = v_d N_0(x_i) + r_i(t). \quad (2.42)$$

Hence Criterion 1 (which requires that the variable of the approximation can be related to experimental observations) is satisfied. However, the parameter, $p/\Lambda$, given by Eq. (2.38c) as
Cannot be calculated without knowledge of \( N(x,t) \). In general, \( N(x,t) \) is unknown and hence the parameters of the approximation cannot be calculated. Criterion 2 (which requires that the parameters can be calculated) cannot be satisfied and the conclusion is made that the kinetics approximation is not useful for interpreting this experiment.

In this simple numerical example for which \( N(x,t) \) is known, \( \rho/\Lambda \) can be evaluated. An evaluation of \( \rho/\Lambda \) from Eq. (2.43) reveals that there is a phase between \( \rho/\Lambda \) and \( \delta \Sigma_a(x,t) \), \( \left[ \frac{\rho(t)}{\Lambda} \right]/\delta \Sigma_a \), which is equal to the apparent discrepancy between \( \left[ P(t)/\delta \Sigma_a \right] \) and \( \left[ P(t)/\Lambda \right] \). This point is illustrated in Fig. 2.1(a).

Second, consider that \( w(x) \) is chosen as the adjoint neutron density, \( \bar{N}_o(x) \), of the steady state problem. In this case, \( \rho/\Lambda \) can be calculated in spite of the fact that its definition, Eq. (2.38c), contains the unknown solution, \( N(x,t) \). The reason for this is that for suitably small values of \( \delta \Sigma_a \) (see Appendix B), the result for \( \rho/\Lambda \) obtained by using \( N(x,t) \) is identical to the result for \( \rho/\Lambda \) obtained by replacing \( N(x,t) \) by \( \bar{N}_o(x) \). Since \( \bar{N}_o(x) \) is known, in general, it may be concluded that Criterion 2 can be satisfied. However, the variable of the approximation in this case is the amplitude of the adjoint weighted neutron density. That is,
\[ P(t) = A(t) = \langle N^*(x)N(x,t) \rangle . \]

There is no instrument which observes the adjoint weighted neutron density and hence Criterion 1 cannot be satisfied. The conclusion is made that the kinetics approximation is not useful for interpreting the experiment.

In this simple numerical example for which \( N(x,t) \) is known, the relationship between the detector reading and the adjoint weighted neutron density can be evaluated. An evaluation of \( P(t) = A(t) = \langle N^*(x)N(x,t) \rangle \) reveals that there is a phase between \( P(t) \) and the detector reading, \( R(t) \), \( \frac{R(t)}{A(t)} \), which is equal to the apparent discrepancy between \( \frac{R(t)}{A(t)} \) and \( \frac{A(t)}{A(t)} \).

This illustration of the interpretation of oscillation tests is another contribution to that body of works\([2, 21, 32-36]\) which has sought to understand the meaning and utility of the conventional kinetics equations. The purpose of this illustration has been to show how the usefulness and meaning of this approximation or any kinetics approximation may be judged in terms of the Approximation Criteria.
CHAPTER 3

THE NATURAL MODES AND THE EIGENVALUES OF THE NATURAL MODE APPROXIMATION

3.1 Introduction

The purposes of this chapter are (i) to treat the problem of calculating the natural modes, (ii) to note the properties of the natural modes and their eigenvalues, and (iii) to indicate that the eigenvalue spectrum may be used to answer questions about many aspects of kinetic behavior.

As pointed out in Chapter 2, the natural modes of a reactor without feedback are the eigenvector solutions of Eq. (2.29), while the natural modes with feedback are the eigenvector solutions of Eq. (2.16). There are computer codes in existence which calculate the natural modes in nonuniform, one-dimensional systems both with\textsuperscript{[22,24]} and without\textsuperscript{[21,25,26]} xenon feedback; but there are no codes at the present time which calculate the modes in multidimensional systems. An original contribution of this chapter is the demonstration that approximations to the natural modes in nonuniform, two-dimensional systems without feedback may be "synthesized"\textsuperscript{[27]} from the results of one-dimensional calculations. One-dimensional calculations of the natural modes are easily carried out by means of the code, MUDMO-II, which is described in Appendix C. The inclusion of feedback in MUDMO-II is possible in principle, but the effort in this work is directed at the problem of generating multidimensional modes. Therefore, all examples of this chapter will be restricted to systems which do not include feedback.
The major portion of the calculations in this chapter are for one-dimensional systems. The natural modes in one-dimensional systems are easy to find with MUDMO-II, and results of these calculations lead to an increased understanding of the properties of the natural modes and their eigenvalues. As pointed out in Chapter 2, knowledge of the eigenvalue spectrum of the natural modes is often sufficient to answer questions about kinetic behavior. It is shown in this chapter that the relative magnitudes of those eigenvalues which can be measured (See Chapter 4) are (i) a sensitive function of the reactor configuration, and (ii) a sensitive indicator of the tendency of the neutron density to undergo shape changes.

The chapter is organized in the following way. Section 3.2 briefly describes the equations which are solved numerically for the natural modes and their eigenvalues. Section 3.3 summarizes the properties of the natural modes in a reactor without feedback and introduces a terminology for discussing the natural modes. The practical importance of the eigenvalue spectrum is discussed in Section 3.4. The chapter concludes with the presentation of the results of two-dimensional calculations of the natural modes.

3.2 Calculation of the Natural Modes in Reactors without Feedback

The eigenvalue problem to be solved in this chapter is

\[ [H_0] \psi_{mk}(x) = w_{mk} \psi_{mk}(x) , \]  \hspace{1cm} (3.1)
where \( \psi_{mk}(x) \) has vanishing boundary conditions. By using the definition of \( [H_b] \), Eq. (2.2), the eigenvalue problem may be written in two equations as

\[
\begin{align*}
\left[ \nabla \cdot [D] \nabla - [I] + (1-\beta)[\lambda \mathbf{F}_p^T] \right] [V] N_{mk} + [\Gamma] C_{mk} = w_{mk} N_{mk},
\end{align*}
\]

and

\[
\begin{align*}
[\beta \mathbf{F}_p^T] [V] N_{mk} - [\lambda] C_{mk} = w_{mk} C_{mk}.
\end{align*}
\]

The latter equation may be solved to eliminate \( C_{mk} \). The result of this may be written

\[
\begin{align*}
\nabla \cdot [D(x)] N_{mk}(x) + [\mu(x,w_{mk})] N_{mk}(x) = 0 ,
\end{align*}
\]

(3.2)

where

\[
[\mu(x,w_{mk})] = \left[ [\Sigma] + (1-\beta)[\lambda \mathbf{F}_p^T] + [\Gamma][w_{mk}[I] + [\lambda]]^{-1}[\beta \mathbf{F}_p^T] [V] w_{mk}[I] \right],
\]

(3.3)

and \( [I] \) is the unit matrix.

In one-dimensional systems, the diffusion operator is expressed as

\[
\nabla \cdot [D(x)] \nabla = \nabla_x \cdot [D(x)] \nabla_x - [DB_T^2],
\]

where \( \nabla_x \) is an operator in only the \( x \)-direction and \( [DB_T^2] \) is a diagonal matrix whose elements account for neutron leakage in the directions transverse to the \( x \)-direction. Thus in one-dimensional systems, Eq. (3.2) reduces to

\[
\nabla_x \cdot [D(x)] \nabla_x N_{mk}(x) + [\mu(x,w_{mk})] N_{mk}(x) = 0 \quad (3.4)
\]

where

\[
[\mu(x,w_{mk})] = [\mu(x,w_{mk})] - [DB_T^2]. \quad (3.5)
\]
Equation (3.4) may be solved numerically by means of the computer code, MUDMO-II, which is described in Appendix C.

In nonuniform, two-dimensional systems, Eq. (3.2) is solved by a "synthesis" technique.\[^{27}\] A synthesized solution is an approximate solution which is constructed from an expansion of the form,

\[ N(x,z) = \sum_{i=1}^{L} [X_i(x)]Z_i(z) . \]  

(3.6)

The elements of the G by G matrix, \([X_i(x)]\), are specified trial functions which are considered to represent the x dependence of the G neutron groups in different z-regions of the reactor. The elements of the G-vectors, \(Z_i\), are unknown coefficients of combination which are to be determined. A set of differential equations for the unknown \(Z_i\) can be generated by the standard substitute-, weight- and integrate procedure. That is, Eq. (3.6) is substituted into Eq. (3.2); the result is weighted by L weighting matrices, \([X^*(x)]_j\), and then integrated over all x. This procedure yields a one-dimensional equation,

\[ \nabla_z [\tilde{D}(z)] \nabla_z Z(z) + [\tilde{\mu}(z,w)]Z(z) = 0 , \]  

(3.7)

which can be solved numerically by MUDMO-II. The vector \(Z\) is now an IG-vector,

\[ Z = \text{col}[Z_1, Z_2, \ldots, Z_G] , \]

and the matrices \(\tilde{D}(z)\) and \(\tilde{\mu}\) are now IG by IG matrices whose elements are integrals over x. For example, the \(j^{th}\) - \(i^{th}\) element of the matrix, \(\tilde{D}\), is a G by G matrix,
The computer code, SYNSIG, is used to prepare input for the synthesis of a natural mode in two dimensions. This code, and a more detailed description of the synthesis technique are described in Appendix D. Results of two-dimensional natural modes found by this synthesis procedure are given in Section 3.5. First, however, it is desirable to develop a terminology for discussing the modes.

3.3 Summary of the Properties of the Natural Modes and their Eigenvalues

In this section the properties of the solutions of Eq. (3.1) are summarized. This summary is based upon the results of this work and the work of others\[20,21,37\]. The purpose of this section is not only to summarize but also to develop a terminology for discussing the natural modes.

The first property of the solutions of Eq. (3.1) is summarized as follows:

Property 1. For a reactor described by \( I \) delayed precursor groups and \( G \) neutron energy groups, the eigenvectors (natural modes) of Eq. (3.1) appear to come in clusters of \( K \) where \( K = G + I \). Each of the \( K \) eigenvectors of the \( m \)th cluster has components of similar spatial shape; and, in general, these component shapes become more oscillatory in space as the index \( m \) increases. Thus the \( m \)th cluster of eigenvectors may be
considered as representing the $m^{th}$ spatial harmonic. The $k^{th}$ eigenvector of the $m^{th}$ cluster will often be referred to as the $k^{th}$ natural mode of the $m^{th}$ spatial harmonic.

In uniform reactors each of the $K$ eigenvectors of the $m^{th}$ cluster has components of identical shape; and all of these components become more oscillatory in space as the index $m$ increases. Thus there is no ambiguity associated with classifying the natural modes according to Property 1.

In nonuniform reactors the classification of the natural modes into clusters is not as straightforward as in the uniform reactor case. It still appears possible, however, to make the classification. Ambiguities arise because of appreciable differences in the spatial shapes of the eigenvectors of a cluster. Two examples for nonuniform reactors are given which illustrate the classification of the eigenvectors into clusters. The first example is for an almost uniform reactor whose natural modes may be classified without ambiguity. The second example is for a nonuniform reactor whose natural modes cannot be classified in a straightforward manner. In both examples, two neutron energy groups ($G = 2$) and one delayed precursor group ($I = 1$) are considered.

The properties of the reactor of the first example are tabulated in Table 3.1. This reactor has increased neutron absorption in the central region ($a/4 \leq x \leq 3a/4$ where $a$ = extrapolated width of 60 cm) which tends to flatten the neutron density distribution. Figure 3.1 shows the components, $N_{mk}^{(1)}$ and $N_{mk}^{(2)}$, of the eigenvector,
Table 3.1

Nuclear Parameters of the Slab Reactors
of the Illustrations Shown in Figure 3.1 and Figure 3.2

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Reactor of Figure 3.1</th>
<th>Reactor of Figure 3.2</th>
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<tr>
<td></td>
<td>Region 1</td>
<td>Region 2</td>
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<td>$\Sigma_{a1}$, cm$^{-1}$</td>
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<td>$\nu\Sigma_{f2}$, cm$^{-1}$</td>
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</tr>
<tr>
<td>$v_2$, cm/sec</td>
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</tr>
<tr>
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</tr>
<tr>
<td>$\lambda$, sec$^{-1}$</td>
<td>0.08</td>
<td></td>
</tr>
</tbody>
</table>
\( \psi_{mk} = \text{col}[N_{mk}^{(1)}, N_{mk}^{(2)}, C_{mk}] \) for the \( m = 1 \) cluster and the \( m = 2 \) cluster of eigenvectors. Note that the eigenvectors come in clusters of three \( (K = G + I = 3) \), where each eigenvector (mode) of the \( m \)th cluster (spatial harmonic) has components of similar shape. It is seen in this nonuniform reactor example that the natural modes may be classified into clusters without ambiguity.

The properties of the reactor of the second example are tabulated in Table 3.1. This reactor has a strongly absorbing core surrounded by a weakly absorbing reflector. Figure 3.2 shows the components, \( N_{mk}^{(1)} \) and \( N_{mk}^{(2)} \), for the \( m = 1 \) and the \( m = 3 \) cluster of eigenvectors. Note that even though the components of the eigenvectors associated with the fundamental spatial harmonic \( (m = 1) \) do not have identical shapes, they may all (with one exception) be associated with a fundamental spatial harmonic. Note the exceptional case where the thermal neutron component, \( N_{13}^{(2)} \), of the \( k = 3 \) mode, \( \psi_{13} \), bears little resemblance to the thermal neutron components of the other eigenvectors of the \( m = 1 \) cluster. However, the epithermal neutron component, \( N_{13}^{(1)} \), of \( \psi_{13} \) can be classified as belonging to the \( m = 1 \) cluster. Note also that the cluster of eigenvectors associated with the third spatial harmonic \( (m = 3) \) do not have identical shapes. Note in particular that the epithermal neutron component, \( N_{32}^{(1)} \), of the \( k = 2 \) mode has no zero crossing. However, the thermal neutron component, \( N_{32}^{(2)} \), of \( \psi_{32} \) can be classified as belonging to the \( m = 3 \) cluster.

The author is undecided as to whether the behavior of \( N_{13}^{(2)} \) and \( N_{33}^{(2)} \) of Fig. 3.2 is real or just a numerical difficulty. No calcu-
Figure 3.1 Natural Mode Eigenvector Components for an Almost Uniform Slab Reactor.
Figure 3.2 Natural Mode Eigenvector Components for a Nonuniform Slab Reactor.
lations of these eigenvectors have been published by others. Even though these shapes are contrary to intuition, they appear to be invariant to changes in such numerical solution parameters as number of mesh points and the frequency of conditioning.

It is the behavior of the natural modes in the second example which prevents a more precise statement of Property 1.

Three more properties of the solutions of Eq. (3.1) are summarized as follows:

**Property 2.** The K eigenvalues corresponding to the K eigenvectors of the m\textsuperscript{th} cluster fall into K distinct groups of different magnitudes. The I algebraically largest eigenvalues have magnitudes which are of the order of the precursor decay constants. The G remaining eigenvalues have larger magnitudes. The former group of eigenvalues will be referred to as delayed neutron eigenvalues of the m\textsuperscript{th} spatial harmonic and the corresponding eigenvectors will be referred to as the delayed neutron modes. The latter group of eigenvalues will be referred to as prompt neutron eigenvalues of the m\textsuperscript{th} spatial harmonic and the corresponding eigenvectors will be referred to as the prompt neutron modes. As the spatial harmonic index, m, increases the I delayed neutron eigenvalues asymptotically approach the I values, -\lambda_1, and the G prompt neutron eigenvalues asymptotically approach -\infty.
Property 3. For a critical reactor the algebraically largest delayed neutron eigenvalue is zero. The components of the corresponding eigenvector are the static distributions of the K dependent variables. For a non-critical reactor the algebraically largest eigenvalue is the reciprocal of the asymptotic period with which the neutron density grows or decays in time.

Property 4. Of the G prompt neutron eigenvalues of the $m^{th}$ spatial harmonic there is one of algebraically largest magnitude which is a sensitive indicator of the subcriticality of the corresponding natural mode. It is this eigenvalue which can be verified experimentally (See Chapter 4). In a thermal reactor this eigenvalue is sensitive to both the thermal group parameters and the geometrical configuration of the reactor. It will be referred to as the prompt thermal neutron eigenvalue and it may be associated with the relaxation time for the distribution of prompt thermal neutrons of the $m^{th}$ spatial harmonic. The corresponding eigenvector will be referred to as the prompt thermal neutron mode. As the prompt thermal neutron eigenvalue of the $m^{th}$ spatial harmonic increases, the I delayed neutron eigenvalues of the $m^{th}$ spatial harmonic also increase.

Property 2 is illustrated in Fig. 3.3 for the first example reactor of the preceding discussion. Note that the three eigenvalues for the $m^{th}$ cluster fall into distinct groups of different
Figure 3.3  Eigenvalue Spectrum of a Nonuniform, Slab Reactor in which Two Neutron Groups and One Precursor Group are Considered.
magnitudes. The delayed neutron eigenvalue, \( w_{ml} \), asymptotically approaches the precursor decay constant, \( -\lambda \), as \( m \) increases. The prompt thermal neutron eigenvalue, \( w_{m2} \), is the reciprocal relaxation time of the asymptotic energy distribution* of neutrons in the \( m \)th spatial harmonic. The epithermal neutron eigenvalue, \( w_{m3} \), effectively represents the decay of energy transients for the \( m \)th spatial harmonic. These neutron group eigenvalues asymptotically approach \(-\infty\) as \( m \) increases.

Property 3 may be examined in the preceding figures, Figs. 3.1 and 3.2. For the critical reactors the algebraically largest eigenvalue delayed neutron eigenvalue is zero. The components of the corresponding eigenvector are the static density distributions.

Property 4 will be illustrated in the next section.

3.4 The Practical Importance of the Eigenvalue Spectrum

The distribution of eigenvalues, like that shown in Fig. 3.3, will be referred to as the eigenvalue spectrum of the reactor system. Knowledge of the behavior of the eigenvalue spectrum is often sufficient to answer many questions concerning the space dependent kinetic behavior of the neutron density. To see why this is so, some simple solutions to the NMA are examined in this section.

First, consider the special case where the reactor is slightly perturbed by external sources in such a way that nonlinear contri-

*Even though the energy variable has been eliminated, the multigroup model still retains some representation of the energy transient which exists until the group density components attain a fixed ratio with each other. This fixed ratio crudely defines an asymptotic energy distribution.
butions are negligible. For this case Eq. (2.21) reduces to

$$\dot{\mathbf{A}} = \text{diag}[w] \mathbf{A} + \mathbf{S} \quad (3.8)$$

These equations are completely uncoupled and the behavior of each

$$A_{mk}(t)$$

may be found by solving

$$\dot{A}_{mk}(t) = w_{mk} A_{mk}(t) + S_{mk}(t) \quad (3.9)$$

The NMA has the desirable property of "finality" [19] for this example. This means that each coefficient, $$A_{mk}$$, may be found independently of the other coefficients and of the number, M, of natural modes retained in the expansion, Eq. (2.28). If the source perturbation is a step function in time (i.e., $$S_{mk}(t) = S_{mk}$$ for $$t \geq 0$$ and $$S_{mk}(t) = 0$$ for $$t < 0$$), the solution to Eq. (3.9) is simply

$$A_{mk}(t) = A_{mk}(0) e^{w_{mk} t} - \frac{S_{mk}}{w_{mk}} [1 - e^{w_{mk} t}] \quad (3.10)$$

Note that the time behavior of the $$m_k$$th natural mode is controlled by its associated eigenvalue, $$w_{mk}$$. If the eigenvalue is complex the $$m_k$$th natural mode will oscillate. The rate of divergence or convergence of this mode will be controlled by the real part of $$w_{mk}$$. Therefore, questions about asymptotic stability and spatial stability (oscillations of the higher order spatial harmonics) may be answered by inspecting the eigenvalue spectrum.

Now, consider the special case where a critical reactor at very low power is perturbed by a change in reactor properties through the perturbation operator, $$[h]$$. For this case Eq. (2.31) reduces to

$$\dot{\mathbf{A}} = \text{diag}[w] \mathbf{A} + [P] \mathbf{A} \quad (3.11)$$
The perturbation, \([P]\), couples all the equations; hence there is no "finality" and all coefficients must be found simultaneously. An iteration scheme can be used to find the coefficients when the elements of \([P]\) are independent of time. In this case all coefficients will eventually have a time behavior given by \(e^{t/T}\) where \(T\) is called the asymptotic period. Each of the \(MK\) equations of Eqs. (3.11) is then of the form,

\[
\frac{1}{T} A_{mk} = - |w_{mk}| A_{mk} + \sum_{n=1}^{M} \sum_{j=1}^{K} P_{mk,nj} A_{nj},
\]

(3.12)

where

\[
P_{mk,nj} = \frac{\langle \psi_{mk}^* [h] \psi_{nj} \rangle}{\langle \psi_{mk}^* \psi_{mk} \rangle}.
\]

(3.13)

One product, call it \(P_{\alpha\beta} A_{\alpha\beta}\), will predominate in Eq. (3.12). If each of the \(MK\) equations is divided by \(A_{\alpha\beta}\) there will be one equation of the form,

\[
\frac{1}{T} = - |w_{\alpha\beta}| + \sum_{m=1}^{M} \sum_{k=1}^{K} P_{nj, mk} \left( \frac{A_{mk}}{A_{\alpha\beta}} \right),
\]

(3.14)

and \(MK-1\) equations of the form,

\[
\frac{A_{mk}}{A_{\alpha\beta}} = \frac{1}{T-1 + |w_{mk}|} \sum_{n=1}^{M} \sum_{j=1}^{K} P_{mk, nj} \left( \frac{A_{nj}}{A_{\alpha\beta}} \right),
\]

(3.15)

where \(m \neq \alpha\) and \(k \neq \beta\).

These equations are of a form which suggest an iterative solution. The \((\mu+1)^{st}\) estimate of \(\frac{A_{mk}}{A_{\alpha\beta}}\) may be found from
\[
\frac{A_{mk}}{A_{\alpha\beta}}(\mu+1) = \frac{1}{|w_{mk}| + (\frac{1}{T})(\mu)} \sum_{n=1}^{M} \sum_{j=1}^{K} P_{njk,m} \left( \frac{A_{nj}}{A_{\alpha\beta}}(\mu) \right), \quad (3.16)
\]

where

\[
(\frac{1}{T})(\mu) = -|w_{\alpha\beta}| + \sum_{n=1}^{M} \sum_{k=1}^{K} P_{nj,mk} \left( \frac{A_{mk}}{A_{\alpha\beta}}(\mu) \right). \quad (3.17)
\]

When this iteration converges, the estimated coefficient ratios may be used in Eq. (2.28) to construct the asymptotic shape of the perturbed neutron distribution. Before the perturbation the only nonzero coefficient is \( A_{11} \). The amount of shape difference between the initial distribution and the asymptotic (perturbed) distribution depends upon the magnitude of the coefficients of higher order spatial harmonics. It will be argued that the magnitudes of the coefficients of the \( m \)th spatial harmonic are a sensitive function of the magnitude of the prompt thermal neutron eigenvalue of the \( m \)th spatial harmonic. Before this can be done, however, it is necessary to examine the behavior of the perturbation elements.

For a reactor described by two neutron groups and one delayed precursor group the perturbation elements may be written as

\[
P_{mk,nj} = \frac{\langle \psi_{mk}^{\star} [h] \psi_{nj} \rangle}{\sum_{g=1}^{2} \langle \psi_{mk}^{\star} (g) \psi_{mk} (g) \rangle} \frac{1}{1 + \Lambda_{mk}^{-1} \frac{\beta_{\lambda}}{(\lambda + |w_{mk}|)^2}}, \quad (3.18)
\]

where
\[
\Lambda_{mk}^{-1} = \sum_{g=1}^{2} \frac{N_{mk}^*(1)_{N_{mk}}(g) \psi_{ng} \psi g \Sigma g}{\sum_{g=1}^{2} \langle N_{mk}^*(g)_{N_{mk}}(g) \rangle}. \tag{3.19}
\]

Consider the perturbation elements in the equations, Eq. (3.15), for the coefficients of the prompt neutron modes \(2 \leq k \leq 3\). The prompt neutron eigenvalues are of such large magnitude that the term, \(\Lambda_{mk}^{-1} \lambda \beta/(\omega_{mk} + \lambda)^2\), is small compared with unity; hence \(P_{mk,nj}\) may be approximated by

\[
P_{mk,nj} \simeq \frac{\langle \psi_{mk}^*, h \psi_{nj} \rangle}{\sum_{g=1}^{2} \langle N_{mk}^*(g)_{N_{mk}}(g) \rangle} \tag{3.20}
\]

Now consider the perturbation elements in the equations, Eq. (3.15), for the coefficients of the delayed neutron modes \((k = 1)\). The delayed neutron eigenvalue, \(\omega_{ml}\), approaches its asymptote, \(-\lambda\), in a manner which may be approximated by \(\omega_{ml} \simeq -\lambda(1 - \omega_{m2}/\omega_{m2})\), where the subscript \(k = 2\) denotes the prompt thermal neutron eigenvalue. Thus as \(m\) increases, \(\omega_{m2}\) approaches \(-\infty\); \((\omega_{ml} + \lambda)^2\) approaches zero; and \(P_{ml,nj}\) may be approximated by the relation,

\[
P_{ml,nj} \simeq \frac{\langle \psi_{ml}^*, [h] \psi_{nj} \rangle}{\sum_{g=1}^{2} \langle N_{ml}^*(g)_{N_{ml}}(g) \rangle} \frac{\Lambda_{ml} \lambda}{\omega_{m2}^2}. \tag{3.21}
\]

Now an approximate expression for the coefficients of the delayed neutron modes is

\[
\frac{A_{ml}}{A_{\alpha\beta}} = \frac{1}{T^{-1} + |\omega_{ml}|} \frac{\Lambda_{ml} \lambda}{\omega_{m2}} \sum_{n=1}^{M} \sum_{j=1}^{K} \frac{\langle \psi_{ml}^*, [h] \psi_{nj} \rangle}{\sum_{g=1}^{2} \langle N_{ml}^*(g)_{N_{ml}}(g) \rangle}. \tag{3.22}
\]
Therefore the relative contribution of the delayed neutron mode of the $m^{th}$ spatial harmonic to the perturbed distribution is closely related to the magnitude of the quantity $(\omega_{12}/\omega_{m2})^2$.

An approximate expression for the coefficients of the prompt thermal neutron modes is

$$A_{m2} = \frac{1}{1 + |\omega_{m2}|} \sum_{n=1}^{M} \sum_{j=1}^{K} \frac{\langle \psi_{m2}^*, [h] \psi_{n,j} \rangle}{\sum_{g=1}^{2} \langle N_{m2}^*(g) N_{m2}(g) \rangle} A_{n,j}.$$ 

Except for extremely rapid transients, $|\omega_{m2}|$ is ordinarily much greater than $T^{-1}$. Therefore the relative contribution of the prompt thermal neutron mode of the $m^{th}$ spatial harmonic to the perturbed distribution is closely related to the magnitude of $\omega_{m2}$.

The relative contribution of the prompt epithermal neutron modes ($k = 3$) to the perturbed distribution is negligible because of the large magnitude of the prompt epithermal neutron eigenvalues, $\omega_{m3}$.

The conclusion to be drawn from the preceding development is that the relative magnitudes of the prompt thermal neutron eigenvalues of higher order spatial harmonics are a sensitive indicator of the tendency of the neutron density to undergo shape changes. If the magnitude of $\omega_{m2}$ is large with respect to $\omega_{12}$ the coefficients of the natural modes of the $m^{th}$ spatial harmonic will be small relative to the coefficients of the fundamental spatial harmonic. Numerical examples which illustrate this conclusion will be presented in Chapter 5.
The relative magnitudes of the prompt thermal neutron eigenvalues are a sensitive function of the reactor configuration. This is illustrated by the examples which follow.

Four of the reactors to be considered are Reactors I through IV whose properties are tabulated in Table 3.2. Two neutron energy groups and one delayed precursor group are used in the theoretical description. Reactor I has uniform properties and it is typical of a small, $H_2O$ moderated reactor. Reactor II has uniform properties and it is typical of a large, $H_2O$ moderated reactor. Reactor III is a large, $H_2O$ moderated reactor similar to Reactor II except that it has been "decoupled" by the addition of absorbing material in the central region and removal of absorbing material from the outer regions. Reactor IV has uniform properties and it is typical of a $D_2O$ moderated reactor. Figure 3.4 shows the geometrical configurations of these reactors. Figure 3.5 shows the eigenvalue spectra of these four reactors. Note that even though the prompt thermal neutron eigenvalue of the fundamental spatial harmonic is essentially the same for Reactors I through III, the prompt thermal neutron eigenvalues of the higher order spatial harmonics are considerably different. Note also that differences in the rate at which the delayed neutron eigenvalues of each reactor approach the asymptote of $-\lambda$ is reflected by the differences in behavior of the prompt thermal neutron eigenvalue spectrum. From the knowledge of the behavior of the solution (Eq. (3.15)) of Eq. (3.11) it may be concluded that for similar perturbations (similar $P_{nj, mk}'s$) to Reactors I, II and III, Reactor III will experience the greatest change in neutron density shape.
<table>
<thead>
<tr>
<th>Parameter</th>
<th>Reactor I</th>
<th>Reactor II</th>
<th>Reactor III</th>
<th>Reactor IV</th>
</tr>
</thead>
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<tr>
<td>width, cm</td>
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<td>240.</td>
<td></td>
<td>228.</td>
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<td>0.033515</td>
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</tr>
<tr>
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<td>0.0194962</td>
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<td>1.69531</td>
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<td>$v_1$, cm/sec</td>
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<tr>
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<td></td>
<td></td>
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</tr>
<tr>
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<td></td>
<td></td>
<td></td>
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</tr>
<tr>
<td>Fuel</td>
<td></td>
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<td></td>
<td></td>
</tr>
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</table>

<table>
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</tr>
<tr>
<td>Fuel plus Poison</td>
</tr>
<tr>
<td>Fuel</td>
</tr>
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</table>

<table>
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</thead>
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</tr>
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<table>
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</thead>
<tbody>
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</tr>
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</tr>
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<td>Fuel</td>
</tr>
<tr>
<td>M₁</td>
</tr>
<tr>
<td>M₂</td>
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<table>
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</thead>
<tbody>
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</tr>
<tr>
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</tr>
<tr>
<td>M₁</td>
</tr>
<tr>
<td>Fuel</td>
</tr>
<tr>
<td>M₂</td>
</tr>
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</table>

<table>
<thead>
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</thead>
<tbody>
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</tr>
<tr>
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</tr>
<tr>
<td>Fuel</td>
</tr>
<tr>
<td>M₁</td>
</tr>
<tr>
<td>Fuel</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Reactor VII, D₂O Moderated</th>
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</thead>
<tbody>
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</tr>
<tr>
<td>-50</td>
</tr>
<tr>
<td>0</td>
</tr>
<tr>
<td>50</td>
</tr>
<tr>
<td>100</td>
</tr>
</tbody>
</table>

Distance, Centimeters

M₁ = Moderator I, M₂ = Moderator II

Figure 3.4 Geometric Configuration of Slab Reactor Examples I through VII.
Figure 3.5 Eigenvalue Spectra of Reactors I, II, III and IV.
The space dependent kinetic characteristics of Reactor IV cannot be compared with the characteristics of Reactors I through III in Fig. 3.5 because the prompt, kinetic behavior of Reactor IV occurs on a different time scale. The magnitude of the prompt thermal neutron eigenvalue of the fundamental spatial harmonic is essentially the parameter, \((\beta - \rho) / \Lambda\), of the space independent kinetics approximation, where \(\rho\) is the reactivity, \(\beta\) is the total fraction of delayed neutrons, and \(\Lambda\) is the generation time. The parameter, \(\beta / \Lambda\), is often used to characterize the space independent kinetic behavior of a nuclear reactor. In a similar manner the prompt thermal neutron eigenvalue of the fundamental spatial harmonic may be used to characterize the kinetic behavior of an importance weighted, total neutron population. What is more relevant to the space dependent aspects of kinetic behavior is the normalized spectrum of the prompt thermal neutron eigenvalues, \(w_{m2} / w_{12}\). Normalized spectra for Reactors I through IV are shown in Fig. 3.6 along with the spectra for three additional reactors, Reactors V, VI and VII. (The subscript \(k = p\) is more generally used throughout the thesis to indicate the prompt thermal neutron eigenvalue. This notation is used in Fig. 3.6.) Before Reactors V through VII are described, note that the space dependent kinetic characteristics of the D_2O reactor, Reactor IV, can now be compared with the three H_2O reactors, Reactors I, II and III. It may be concluded that for similar perturbations (similar \(P_{n,j,m_k}\)’s) to Reactors I, II, III and IV, Reactor IV will experience the smallest change in the neutron density shape. The conclusion is based on the fact that the normalized eigenvalues are largest for Reactor IV.
Figure 3.6 Normalized Spectra of Prompt Thermal Eigenvalues for Reactors I through VII.
The geometrical configurations of the D\textsubscript{2}O reactors, Reactors IV through VII, are shown in Fig. 3.4 along with the geometrical configurations of the H\textsubscript{2}O reactors, Reactors I, II and III. The nuclear properties of Reactors V through VII are tabulated in Table 3.3.

Reactor V is a nonuniform, reflected reactor; Reactor VI is a nonuniform reactor which has been decoupled by a moderator region; and Reactor VII is a nonuniform reactor in which an additional fuel region has been inserted between decoupled fuel regions. Note that the normalized eigenvalue spectrum of Reactor V indicates that the neutron density in this reactor will have a very small tendency to experience shape changes. The normalized spectrum of Reactor VI indicates that the neutron density in this reactor will have a large tendency to experience shape changes; but this tendency is not as great as it is in the H\textsubscript{2}O reactors, Reactors II and III. The normalized spectrum of Reactor VII indicates that the addition of the central fuel region between decoupled regions has a large effect on reducing the tendency of the neutron density to experience shape changes.

### 3.5 Results of Two-Dimensional Calculations of the Natural Modes

The results of the preceding sections have been for one-dimensional reactor systems. Any computation of transients in real reactor systems requires multidimensional calculations of the natural modes. This is a difficult problem and it has been recognized\textsuperscript{[20]} as one of the serious drawbacks of the NMA. The author believes that this problem will eventually be overcome, and it will be possible to calculate directly the natural modes in two-dimensional systems either
Table 3.3
Nuclear Parameters of Reactors V, VI and VII

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<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
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</tr>
<tr>
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<table>
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<th>Parameter</th>
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<th>Moderator II</th>
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<td>0.0</td>
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<td>$\Sigma_{f}$, cm$^{-1}$</td>
<td>0.010471</td>
<td>0.010471</td>
<td>0.00262</td>
</tr>
<tr>
<td>$D_1$, cm</td>
<td>1.313</td>
<td>1.313</td>
<td>1.017</td>
</tr>
<tr>
<td>$D_2$, cm</td>
<td>0.804</td>
<td>0.837</td>
<td>0.917</td>
</tr>
<tr>
<td>$v_1$, cm/sec</td>
<td>4.06x10$^6$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$v_2$, cm/sec</td>
<td>2.48x10$^5$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\beta$</td>
<td>0.0074</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\lambda$, sec$^{-1}$</td>
<td>0.0133</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$B_T$, cm$^{-2}$</td>
<td>0.000561</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
by use of the Wielandt Iteration Technique[38] or the Stabilized March Technique[28]. The three-dimensional natural modes may then be constructed by "synthesizing" the results of two-dimensional calculations. The formalism of the synthesis technique was sketched briefly in Section 3.2. The formalism is discussed in more detail in Appendix D which describes SYNSIG, a computer code used to prepare synthesis input parameters for the code, MUDMC-II.

The results of a first attempt to synthesize two-dimensional natural modes from one-dimensional calculations are summarized here. The geometrical configuration of the two-dimensional reactor is shown in Fig. 3.7 and the properties of the reactor are tabulated in Table 3.4. Two neutron groups and one delayed precursor group are considered. Figures 3.8 and 3.9 illustrate the thermal neutron component of the eigenvectors \(|\psi_{11,p}(x,z)|\) and \(|\psi_{22,p}(x,z)|\), respectively. The first subscript indexes the order of the spatial harmonic in the x-direction and the second subscript indexes the order of the spatial harmonic in the z-direction. The subscript \(p\) indicates that these eigenvectors correspond to the prompt thermal neutron eigenvalues. The spectra of the delayed neutron eigenvalues and the prompt thermal neutron eigenvalues are tabulated in Table 3.5.

Tests of the orthogonality relationship,

\[
\langle \psi_{mn,p} \mid \psi_{rs,p} \rangle = \begin{cases} 
1.0 & (m = r \text{ and } n = s) \\
0.0 & (m \neq r \text{ and } n \neq s)
\end{cases}
\]
Figure 3.7 Geometrical Configuration of a Two-Dimensional Reactor.
Table 3.4
Nuclear Parameters of a Two-Dimensional Reactor

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Region 1</th>
<th>Region 2</th>
<th>Region 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Sigma_{a1, \text{cm}^{-1}}$</td>
<td>0.0322365</td>
<td>0.0340843</td>
<td>0.0322026</td>
</tr>
<tr>
<td>$\Sigma_{a2, \text{cm}^{-1}}$</td>
<td>0.2653829</td>
<td>0.266275</td>
<td>0.265820</td>
</tr>
<tr>
<td>$\nu\Sigma_{f1, \text{cm}^{-1}}$</td>
<td></td>
<td>0.0194903</td>
<td></td>
</tr>
<tr>
<td>$\nu\Sigma_{f2, \text{cm}^{-1}}$</td>
<td></td>
<td>0.497707</td>
<td></td>
</tr>
<tr>
<td>$\Sigma_{r, \text{cm}^{-1}}$</td>
<td>0.0164444</td>
<td></td>
<td></td>
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<tr>
<td>$D_1, \text{cm}$</td>
<td>1.69531</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$D_2, \text{cm}$</td>
<td></td>
<td>0.409718</td>
<td></td>
</tr>
<tr>
<td>$v_1, \text{cm/sec}$</td>
<td></td>
<td>4.06x10^6</td>
<td></td>
</tr>
<tr>
<td>$v_2, \text{cm/sec}$</td>
<td></td>
<td>2.2x10^5</td>
<td></td>
</tr>
<tr>
<td>$\beta$</td>
<td>0.0064</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\lambda, \text{sec}^{-1}$</td>
<td>0.08</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Figure 3.8 Thermal Neutron Component of the Two-Dimensional Natural Mode, \( \Psi_{11,B}(x,z) \).
Figure 3.9 Thermal Neutron Component of the Two-Dimensional Natural Mode, $\tilde{Y}_{22, \rho}(x, z)$. 
Table 3.5
Spectra of the Prompt Thermal Neutron Eigenvalues and the Delayed Neutron Eigenvalues for the Natural Modes of a Two-Dimensional Reactor

<table>
<thead>
<tr>
<th>Prompt Thermal Neutron Eigenvalues, $\omega_{mn,p}$</th>
<th>x index, m</th>
</tr>
</thead>
<tbody>
<tr>
<td>z index, n</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>-416.64</td>
</tr>
<tr>
<td>2</td>
<td>-1049.1</td>
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<tr>
<td>3</td>
<td>-2592.4</td>
</tr>
<tr>
<td>4</td>
<td>-5656.2</td>
</tr>
<tr>
<td>5</td>
<td>-8590.2</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Delayed Neutron Eigenvalues, $\omega_{mn,d}$</th>
<th>x index, m</th>
</tr>
</thead>
<tbody>
<tr>
<td>z index, n</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>0.0</td>
</tr>
<tr>
<td>2</td>
<td>-0.04845</td>
</tr>
<tr>
<td>3</td>
<td>-0.06747</td>
</tr>
<tr>
<td>4</td>
<td>-0.07446</td>
</tr>
</tbody>
</table>

*Coincided with extraneous eigenvalue, could not calculate eigenvector. Changed weighting vector from adjoint eigenvector to unit vector to break multiplicity; eigenvalue was then -3670.
are tabulated in Table 3.6. Since there are no exact calculations of the two-dimensional natural modes, the orthogonality relation is the only test by which the synthesized modes may be judged. Note that in some cases the orthogonality test fails. This means that one of the particular modes involved is unsatisfactory. This unsatisfactory behavior points out the major difficulty involved with synthesized solutions. The technique gives good results if the trial functions are good. The ability to choose good trial functions comes with experience and there is no experience with the natural modes in multidimensional systems.

Another difficulty involved with the synthesis of multidimensional natural modes of higher spatial harmonics is that the synthesis procedure introduces extraneous eigenvalues. It has been established in Section 3.3 that a system with $K$ dependent variables has $K$ eigenvalues associated with the cluster of $K$ natural modes of the $m^{th}$ spatial harmonic. The synthesis procedure transforms a two-dimensional problem with $K$ dependent variables into an approximate, one-dimensional problem with $LK$ dependent variables, where $L$ is the number of trial vectors used in the synthesis expression, Eq. (3.6). Therefore the approximate, one-dimensional problem has $LK$ eigenvalues and eigenvectors associated with the $m^{th}$ spatial harmonic, $(L-1)K$ of which are extraneous. This difficulty can be overcome, however. MUDMO-II can be used to scan the entire range of eigenvalue magnitudes so as to locate the eigenvalues roughly. After a few of the modes are determined, a pattern of relative behavior is soon recognized which allows the approximate location of the real eigenvalues to be determined.
<table>
<thead>
<tr>
<th>$m, n_s$</th>
<th>$n_s$</th>
<th>1,1</th>
<th>1,2</th>
<th>1,3</th>
<th>1,4</th>
<th>2,1</th>
<th>2,2</th>
<th>2,3</th>
<th>2,4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1,1</td>
<td>1.0</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
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<td>+</td>
</tr>
<tr>
<td>1,2</td>
<td>+</td>
<td>1.0</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>0.083</td>
</tr>
<tr>
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<td>+</td>
<td>0.011</td>
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<td>+</td>
<td></td>
</tr>
<tr>
<td>1,4</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>1.0</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
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</tr>
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<td>2,1</td>
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<td>0.217</td>
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<td>+</td>
<td></td>
</tr>
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<td>+</td>
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<td>+</td>
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<td>+</td>
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</tr>
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<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>1.0</td>
<td></td>
</tr>
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<td>-2.942</td>
<td>+</td>
<td>+</td>
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<td>0.125</td>
<td>0.953</td>
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<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
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<td>+</td>
<td></td>
</tr>
<tr>
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<td>+</td>
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<td>+</td>
<td>+</td>
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<td>+</td>
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<tr>
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<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
</tbody>
</table>

+ Denotes that the number is less than 0.01
Table 3.6(b)

Results of a Check on the Orthogonality of a Number of the Synthesized Natural Modes of a Two Dimensional Reactor

Normalized Value of $\langle \chi_{m,n}(x,z) \chi_{n,s}(x,z) \rangle$

<table>
<thead>
<tr>
<th>$m,n$</th>
<th>$n,s$</th>
<th>3,1</th>
<th>3,2</th>
<th>3,3</th>
<th>3,4</th>
<th>4,1</th>
<th>4,2</th>
<th>4,3</th>
<th>4,4</th>
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<tbody>
<tr>
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<td>0.039</td>
<td>+</td>
<td>-0.010</td>
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</tr>
<tr>
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</tr>
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<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>1,4</td>
<td></td>
<td>+</td>
<td>-0.072</td>
<td>+</td>
<td>-0.127</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>2,1</td>
<td></td>
<td>0.013</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
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<td>+</td>
<td>0.029</td>
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<td>0.017</td>
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<td>+</td>
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<td>-0.011</td>
</tr>
<tr>
<td>2,3</td>
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<td>0.020</td>
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<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
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<td>+</td>
<td>0.024</td>
<td>+</td>
<td>-0.020</td>
<td>-0.020</td>
<td>-0.60</td>
<td>0.014</td>
<td>+</td>
</tr>
<tr>
<td>3,1</td>
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<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
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<tr>
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<td>-0.113</td>
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<td>+</td>
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<td>+</td>
<td>1.0</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>4,1</td>
<td></td>
<td>0.014</td>
<td>+</td>
<td>+</td>
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<td>1.0</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
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<td>+</td>
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<td>1.0</td>
<td>-0.045</td>
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<tr>
<td>4,3</td>
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<td>0.011</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>-0.020</td>
<td>1.0</td>
<td>-0.014</td>
</tr>
<tr>
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<td>+</td>
<td>+</td>
<td>+</td>
<td>1.0</td>
</tr>
</tbody>
</table>

+ Denotes that the number is less than 0.01
A finer calculation is then done which actually calculates the eigenvalues and eigenvector. Difficulties caused by extraneous eigenvalues can be quite serious if an extraneous eigenvalue and an actual eigenvalue are so close in magnitude that they appear to the code as a multiple eigenvalue. MUDMO-II cannot calculate the eigenvectors for multiple eigenvalues. This situation occurred in the attempt to synthesize \( \chi^2_{p}(x,z) \). However, the multiplicity was broken by a change in the weighting function, \([x^m]_j\), which is used in the reduction of the two-dimensional equation, to the one-dimensional equation, Eq. (3.7).
4.1 Introduction

In Chapters 2 and 3 the Natural Mode Approximation for treating problems in space dependent kinetics was developed. It was pointed out that in thermal reactors certain natural mode eigenvalues, called the prompt thermal neutron eigenvalues, are sensitive indicators of the susceptibility of the neutron density to undergo shape changes and that these eigenvalues can be measured. The purpose of this chapter is to describe how small signal oscillation tests, performed on reactors at very low power, may be used to measure these eigenvalues.

The basic elements of the experimental procedure are as follows:

1. The natural modes of a small number, $M$ (say $M = 2$ to 5), of the lowest order spatial harmonics are calculated.

2. A localized, oscillating, thermal neutron absorber is located in the reactor in such a way to excite these spatial harmonics.

3. $M$ thermal neutron detectors are located at positions which are prescribed by a rule of thumb. This rule of thumb is based on the results of many numerical experiments.

4. The space and frequency dependent readings of these $M$ detectors are analyzed in terms of a natural mode expansion for $M$ expansion coefficients. This step uses the modes calculated in Step 1 above.
5) The prompt thermal neutron eigenvalues are inferred from the frequency dependent behavior of the expansion coefficients.

This procedure is summarized in greater detail in Section 4.4 after the theoretical background is developed in Sections 4.2 and 4.3. Numerical experiments are analyzed in Section 4.5 to show the feasibility of the procedure. In Section 4.6 the data of an actual experiment are analyzed.

Even though the oscillation test to be described here is not the only procedure which could conceivably be used to measure these eigenvalues, the author feels that it is the most suitable technique which can be used with large, critical reactors. The pulsed neutron technique\[39] is not suitable for critical systems because the time dependent portion of the detector reading governed by the prompt thermal neutron eigenvalues is masked by a background reading which increases with time. (It is of interest to note that Bliss\[40], Lopez and Beyster\[41] and Utzinger et al\[42] have measured the prompt thermal neutron eigenvalues of the natural modes of higher spatial harmonics in subcritical systems using the pulsed neutron technique. These authors, however, did not attach any practical significance to these measurements.) The results of measurements using noise techniques\[43] involve cross products of the various spatial harmonics; hence, this technique suffers from difficulties in isolating the behavior of particular spatial harmonics. On the other hand, the oscillation test can be used on critical reactors and the measurements can be resolved into natural mode expansion coefficients. In addition,
the oscillation test, when coupled with a cross-correlation\[33\] of the
detector reading and the oscillating perturbation, yields accurate
measurements even in the presence of high background noise.

### 4.2 Interpretation of Oscillation Tests in Terms of the Natural Mode Approximation

The measurement of the prompt thermal neutron eigenvalues of
higher order spatial harmonics is based on the following theoretical
interpretation of oscillation tests.

Consider a reactor at very low power which is described by
the equation,

\[ [H_o] \psi_o(x) + S_o(x) = 0 \quad (4.1) \]

The vector, \( \psi_o(x) \), is a K-vector containing the K dependent variables
of the system as components, \([H_o]\) is a K by K matrix operator
governing the relationships between variables, and \( S_o(x) \) is a vector
containing external sources. Let the reactor be excited by a localized,
thermal neutron absorber which oscillates sinusoidally with frequency, \( \omega \).
The kinetic behavior of the perturbed reactor is described by the
equation,

\[ [H_o] \psi(x,t) + [h] \psi(x,t)e^{j\omega t} + S_o(x) = \frac{\partial \psi(x,t)}{\partial t}, \quad (4.2) \]

where it is to be understood that only the real parts of complex numbers
will be considered. The perturbation matrix operator, \([h]\), contains
only one element, which gives the magnitude and location of the
oscillating thermal neutron absorber.
After the perturbation is introduced, the mean value, $\bar{\psi}_o(x)$, of $\psi(x,t)$ will be different from $\bar{\psi}_o(x)$ of Eq. (4.1). In addition, the operator $[H_o]$, will be slightly different from $[H_o^*]$ of Eq. (4.1).

One reason for the differences is that the introduction of the perturbation mechanism must be accompanied by a small change in reactor properties in order to keep the reactor critical. Another reason for the differences is that the product of $[h] \exp(jwt)$ and $(\psi(x,t) - \bar{\psi}_o(x))$ causes the power level to increase. This increase, which must be arrested by a change in $[H_o^*]$, will be neglected here.

It is assumed that the variation of $\psi(x,t)$ from its mean value, $\bar{\psi}_o(x)$, is so small that $[h] \psi(x,t)$ may be replaced by $[h] \bar{\psi}_o(x)$. Now if it is assumed that the solution vector may be expanded in terms of the natural modes of $[H_o]$ as

$$\psi(x,t) = \bar{\psi}_o(x) + \sum_{m=1}^{\infty} \sum_{k=1}^{K} A_{mk}(jw) e^{jwt} \psi_{mk}(x),$$

(4.3)

then the usual substitute-, weight- and integrate procedure yields the following expression for the expansion coefficients:

$$A_{mk}(jw) = c_{mk}(jw - w_{mk})^{-1},$$

(4.4)

where $m = 1, \ldots, \infty$, and $k = 1, \ldots, K$. It has been assumed that $[H_o] \bar{\psi}_o(x) + S_o(x) = 0$. The constant, $c_{mk}$, is given by

$$c_{mk} = \frac{\langle \psi_{mk}^*(x) [h] \bar{\psi}_o(x) \rangle}{\langle \psi_{mk}^*(x) \psi_{mk}(x) \rangle}.$$  

(4.5)
The expansion vector, $\psi_{mk}$, is that natural mode of $[H_0]$ which corresponds to the eigenvalue, $w_{mk}$. The weighting vector, $\psi^*_m$, is that natural mode of the adjoint operator, $[H_0]^*$, which corresponds to $w_{mk}$.

The reading, $R_i(t)$, of the $i^{th}$ thermal neutron detector is composed of an oscillating portion, $r_i(j\omega)e^{j\omega t}$, superimposed upon a time averaged detector reading, $R_i^0$. The time average of the detector reading is given by

$$R_i^0 = \left< v_{th i}^0 \right>`(x)N_{th}(x)`, (4.6)$$

and the oscillating portion is given by

$$r_i(j\omega)e^{j\omega t} = \left< v_{th i}^0(x) \right> \sum_{m=1}^{\infty} \sum_{k=1}^{K} A_{mk}(j\omega)e^{j\omega t} N_{mk}(x) \right> (4.7)$$

The function, $\sigma_i(x)$, is the response function of the $i^{th}$ thermal neutron detector. It depends upon the size and location of the detector. The reading of the $i^{th}$ detector is conveniently plotted in terms of two quantities: (i) the magnitude of the ratio, $r_i(j\omega)/R_i^0$, and (ii) the phase difference between $r_i(j\omega)$ and the oscillating perturbation. Figure 4.1 illustrates these quantities.
Figure 4.1 Illustration of the Quantities Measured by the $i^{th}$ Detector in an Oscillation Test.
4.3 Inference of the Prompt Thermal Neutron Eigenvalues from the Results of Oscillation Tests

If the expansion coefficients, $A_{mk}(jw)$, of the higher order spatial harmonics ($m > 1$) in Eq. (4.7) are appreciable, then the normalized reading, $r_i(jw)/R_i^{0}$, will depend upon the spatial location of the $i^{th}$ detector. The estimation of the prompt thermal neutron eigenvalues is based upon analyzing the readings of, say, $M$ detectors to infer the behavior of $M$ coefficients, $A_{mk}$. The eigenvalues are then estimated from the frequency dependent behavior of the $A_{mk}$ which goes as $(jw - w_{mk})^{-1}$. If the $A_{mk}$ are negligible for $m > 1$ then the differences in the $r_i(jw)/R_i^{0}$ are too small to analyze. However, in this case a space independent kinetics approximation is appropriate and there is little motivation for measuring the eigenvalues of higher order spatial harmonics.

In order to understand why it is feasible to infer the $A_{mk}$ from the $r_i(jw)$, consider the following development. Denote the prompt thermal neutron eigenvalue of the $m^{th}$ spatial harmonics by the subscript, $k = p$, and write Eq. (4.7) as

$$
 r_i(jw) = \sum_{m=1}^{\infty} \frac{c_{mp}}{jw - \omega_{mp}} \langle v_{th}^{(th)N_{mp}} \rangle + \sum_{m=1}^{\infty} \sum_{k=1}^{K} \frac{c_{mk}}{jw - \omega_{mk}} \langle v_{th}^{(th)N_{mk}} \rangle 
 + \sum_{m=1}^{\infty} \sum_{k=1}^{K} \frac{c_{mk}}{jw - \omega_{mk}} \langle v_{th}^{(th)N_{mk}} \rangle 
 = \sum_{m=1}^{\infty} \frac{c_{mp}}{jw - \omega_{mp}} \langle v_{th}^{(th)N_{mp}} \rangle + E_i^*(jw)$$

$$
 = \sum_{m=1}^{M} \frac{c_{mp}}{jw - \omega_{mp}} \langle v_{th}^{(th)N_{mp}} \rangle + E_i^*(jw) + E_M(jw) + E_i^*(jw) \quad (4.8)
$$
Consider that the location of the oscillator has been chosen in such a way to maximize the constants, \( c_{mp} \), for the first \( M \) spatial harmonics.

The term, \( E_i'(j\omega) \), is a sum over all delayed neutron modes and all prompt epithermal modes. For oscillation frequencies which are large compared with the magnitude of delayed neutron eigenvalues and small compared with the magnitude of prompt epithermal neutron eigenvalues, this term is negligible in a thermal reactor. This is illustrated in Fig. 4.2 for an oscillator at \( x_0 = a/6 \) in Reactor II of Fig. 3.4. Two neutron energy groups and one delayed precursor group are used in the theoretical description. Note that the coefficients of the prompt epithermal modes, indexed by \( k = 3 \), and of the delayed neutron modes, indexed by \( k = 1 \), are negligible (\(< 1\) percent) for frequencies, \( |\omega_m| \ll 10 < \omega < 10,000 \ll |\omega_3| \). The term, \( E_{M1}(j\omega) \), is a sum over all prompt thermal neutron modes associated with spatial harmonics of higher order than the \( M^{th} \). That is,

\[
E_{M1}(j\omega) = \sum_{m=M+1}^{\infty} A_{mp}(j\omega) \left\langle r_i \sigma_i^{(th)} N_{mp}^{(th)} \right\rangle.
\]

If the detectors are small so that \( \sigma_i^{(th)} \propto \delta(x-x_i) \), then \( E_{M1} \) is an oscillatory function of the location of the \( i^{th} \) detector and has at least \( M \) zero crossings. Hence if \( M \) detectors are placed at or near locations where the quantity, \( E_{M1} = E_i + E_i' \), is very small then the readings of these \( M \) detectors are given to a good approximation by

\[
r_i(j\omega) \approx \sum_{m=1}^{M} A_{mp}(j\omega) \left\langle r_i \sigma_i^{(th)} N_{mp}^{(th)} \right\rangle,
\]

\( (4.10) \)
Figure 4.2 Comparison of the Relative Magnitudes of Natural Mode Coefficients in an Oscillation Test on a Uniform, Slab Reactor.
where \( i = 1, \ldots, M \). In matrix form this set of equations may be written as

\[
\mathbf{r} = [\mathbf{U}]\mathbf{A}, \tag{4.11}
\]

where \( \mathbf{r} = \text{col}[r_1, \ldots, r_M] \), \( \mathbf{A} = \text{col}[A_{1p}, \ldots, A_{Mp}] \) and \( [\mathbf{U}] \) is a matrix with elements,

\[
U_{im} = \left< v_{th}^{(th)} i_{\text{th}} N_{mp} \right>. \tag{4.12}
\]

Presuming that the \( N_{mp}^{(th)} (m = 1, \ldots, M) \) are known so that the \( U_{im} \) may be calculated and presuming that \( [\mathbf{U}] \) is non-singular, estimates of the expansion coefficients are given by

\[
\mathbf{A} = [\mathbf{U}]^{-1} \mathbf{r}. \tag{4.13}
\]

The prompt thermal neutron eigenvalues may be estimated from the \( A_{mp} \)'s by plotting \( \log|A_{mp}| \) as a function of \( \log \omega \). A typical plot is illustrated in Fig. 4.3. At low frequencies (\( \omega \ll |w_{mp}| \)), \( |A_{mp}| \) has a value of \( |c_{mp}/w_{mp}| \). At high frequencies (\( \omega \gg |w_{mp}| \)), \( |A_{mp}| \) has a behavior given by \( |c_{mp}|/\omega \). As shown in the figure, the intersection of the extensions of lines drawn along the low frequency behavior and the high frequency behavior define that frequency for which \( \omega = |w_{mp}| \).

The maximum number of detectors that can be used depends upon the range of experimental frequencies available. It is known from the results of Chapter 3 that the prompt thermal neutron eigenvalues monotonically approach \(-\infty\) as \( m \) increases. Therefore there is some number, \( N \), for which \( w_{\text{max}} < |w_{N+1,p}| \) where \( w_{\text{max}} \) is the maximum oscillation frequency. This value of \( N \) is the upper limit on the number of detectors that can be used. Any number of detectors less
Figure 4.3 Typical Plot of $|A_{mp}|$ as a Function of Oscillation Frequency.
than $N$, say $M$, may be used to estimate only $M$ eigenvalues. In this case the maximum oscillation frequency required is correspondingly less; namely, $\omega_{\text{max}} < |\omega_{M+1,p}| < |\omega_{N+1,p}|$.

The best locations for the $M$ detectors are specified by a rule of thumb which has been formulated on the basis of considerable experience with numerical studies of this procedure. The best locations for $M$ detectors are at those locations where the quantity, $E_{M+1} = E_{M+1} + E_1$, is zero. These locations cannot be specified in the general case. However, the locations of $M$ of the zeros of the lowest order, non-negligible harmonic in $E_{M+1}(j\omega)$ can be specified merely by calculating the natural mode, $\overline{\lambda}_{MP}(x)$, for this harmonic. If $c_{M+1,p} \neq 0$ then a first estimate of the $\omega_{MP}$ may be obtained by placing the $M$ detectors at those $x_i$ where $N_{M+1,p}(\text{th})(x_i) = 0$. These first estimates of the $\omega_{MP}$ are often quite good but can be further improved. The results of many numerical experiments indicate that the zeros of the quantity, $\overline{E}_{M+1} = E_{M+1} + E_1$, are at locations which are displaced from the zeros of $N_{M+1,p}(\text{th})(x)$ by a small distance in the direction towards the location of the oscillator. In addition, results of these numerical experiments indicate that: (i) when the detectors are at the zeros of $N_{M+1,p}(\text{th})$, the phase of the inferred $A_{1p}(j\omega)$ falls below its proper high frequency asymptote of $-90^\circ$; (ii) as the detectors are simultaneously moved towards the oscillator (i.e., closer to the zeros of $\overline{E}_{M+1}(j\omega)$), the phase of $A_{1p}$ attains its proper high frequency asymptote (for $\omega \ll |\omega_{M+1,p}|$); and, (iii) when the detectors are moved too far, the phase of $A_{1p}$ falls above its proper high frequency asymptote.

These results have led to the following rule of thumb for locating the
detectors: (i) place the \( M \) detectors at the zeros of the lowest order, non-negligible \( (c_{mp} \neq 0) \) mode of \( E_{Mi}(j\omega) \); and (ii) move the \( M \) detectors a small distance towards the position of the oscillator until the phase of \( A_{lp} \) attains its proper high frequency asymptotic behavior. It is emphasized that this rule is empirical and that it is based on the results of numerical experiments. The results of one of these experiments is given in Section 4.5; but first, the experimental procedure will be summarized.

4.4 Summary of the Experimental Procedure

Suppose that the maximum oscillation frequency available is \( \omega_{\text{max}} \). Consider that the prompt thermal neutron eigenvalues and eigenvectors (modes) of a number of low order spatial harmonics are calculated and that \( N \) of the eigenvalues are smaller in magnitude than \( \omega_{\text{max}} \). To verify experimentally \( M \) of these \( N \) eigenvalues where \( M \leq N \), this procedure is followed:

1) The oscillator is located at a position which will excite the \( M \) prompt thermal neutron modes of interest. This requires that the oscillator be located far from the zeros of these modes.

2) \( M \) thermal neutron detectors are placed at the zeros of the lowest order, non-negligible mode in the summation, \( E_{Mi} \). For example, if \( M = 3 \) and \( c_{4p} \neq 0 \), then three detectors are placed at the zeros of \( N_{4p}^{(th)}(x) \).
3) The oscillator is oscillated at frequencies from \( w < |w_{lp}| \) to \( w > |w_{mp}| \) and the frequency dependent behavior of the M detector readings, \( r_1(jw) \), is analyzed according to Eq. (4.13) for M coefficients, \( A_{zp}(jw) \).

The analysis requires the computation of the matrix, \([U]\), whose elements (See Eq. (4.12)) are integrals of the M prompt thermal neutron modes.

4) The coefficients, \( A_{zp} \), depend upon \( w_{mp} \) in a manner given by Eq. (4.4), and the \( w_{mp} \) may be estimated from plots of \( |A_{zp}(jw)| \) as a function of \( w \).

5) The preliminary results for the \( w_{mp} \) may be improved by repeating steps 3 and 4 with the detectors at new positions which are determined by the rule of thumb. That is, the detectors are simultaneously moved from their original positions towards the location of the oscillator until the phase of \( A_{zp}(jw) \) attains its proper high frequency asymptote.

Figure 4.4 summarizes this procedure.

Results obtained by using this procedure on numerical experiments are given in the next section. Results obtained by using this procedure on an actual experiment are given in the final section of this chapter.
Estimation of $M$ Prompt Thermal Neutron Eigenvalues, $\omega_{mp}$

Figure 4.4 Schematic Summary of Experimental Procedure.
4.5 Results of Numerical Experiments

Some numerical experiments have been analyzed for the purpose of testing the feasibility of using oscillation tests as a means for measuring the prompt thermal neutron eigenvalues of the natural modes. The results which would be measured in an experiment are generated by an independent calculation which is considered to be correct. These results are analyzed for the eigenvalues by the procedure outlined in the preceding section. Comparison of the eigenvalues inferred from this numerical experiment with the actual eigenvalues leads to an evaluation of the feasibility of the procedure.

Three numerical experiments are described. The first two, Oscillation Tests I and II, occur in uniform, slab reactors. In these examples, a closed form solution can be derived and used to generate experimental observations. The third experiment, Oscillation Test III, occurs in a nonuniform, cylindrical reactor. The experimental observations for this case have been generated numerically by MacDonald and Johnson\textsuperscript{[45]}.

4.5.1 Uniform Slab Reactor Examples

The reactors in the first two examples are uniform, one-dimensional, critical slabs of extrapolated width, a. These reactors are perturbed by a plane, thermal neutron absorber which oscillates with angular frequency, \( \omega \), at the position, \( x = x_0 \). One group of delayed neutron precursors and two energy groups of neutrons are considered.

The equations which describe the dynamic behavior of this system are:
The boundary conditions to be imposed are \( \psi(0,t) = \psi(a,t) = 0 \).

The epithermal neutron group is indexed by the superscript 1 and the thermal group is indexed by the superscript 2. The perturbation approximation of replacing \( E_{h_1}(x,t) \) by \( \psi(x) \) has been made. The magnitude of the perturbation is \( \delta \Sigma_{a2} \). The other nomenclature is standard and is tabulated in Appendix A. The closed form solution to Eq. (4.14) is derived in Appendix F and is

\[
N^{(2)}(x,t) - N^{(2)}_0(x) = K(x,x_o, j\omega) e^{j\omega t},
\]
where \( K(x, x_0, j\omega) \) is given by Eq. (F.20) of Appendix F.

The behavior of the thermal neutron density is observed by means of plane detectors having response functions,

\[
v_{\text{th}1}(x) = v_{\text{th}1}(x)\delta(x-x_1),
\]

where \( x_1 \) is the location of the \( i^{th} \) detector. Thus the oscillating portion of the \( i^{th} \) detector reading is given by

\[
v_i(j\omega)e^{jut} = \langle v_{\text{th}1}(x)\{N_1(2)(x,t) - \overline{N_0(2)(x)}\} \rangle e^{jut} = v_{\text{th}1}(x)K(x_1, x_0, j\omega)e^{jut}.
\]

\[(4.17)\]

In a uniform reactor of extrapolated width, \( a \), the \( K \) natural modes of the \( m^{th} \) spatial harmonic have identical spatial shapes given by \( \sin \frac{m\pi x}{a} \). Therefore the derivative operators \( \left( \frac{\partial^2}{\partial x^2} \right) \) appearing in \( [H_0] \) can be easily applied to reduce the eigenvalue problem,

\[
[H_0] = \omega_{mn} \delta_{mn},
\]

to a system of \( K \) homogeneous, algebraic equations which may be solved for the \( K \) eigenvalues and the \( K \) natural modes of the \( m^{th} \) spatial harmonic. Hence for uniform reactors the natural modes, their eigenvalues, and the expansion coefficients, \( A_{nm} \), may be specified analytically. For example, the natural mode eigenvalues of the operator defined by Eq. (4.15) are the roots of the equation,

\[
w^3 + b_{2m}w^2 + b_{1m}w + b_{0m} = 0,
\]

\[(4.18)\]

where \( m = 1, \ldots, \infty \),
\[ b_{2m} = \lambda + \alpha_2 + \alpha_1 + (v_2 D_2 + v_1 D_1) \left( \frac{\mu_1}{a} \right)^2, \]

\[ b_{1m} = (\alpha_2 + v_2 D_2) \left( \frac{\mu_1}{a} \right)^2 (\alpha_1 + v_1 D_1) \left( \frac{\mu_1}{a} \right)^2 - v_1 v_2 \nu \Sigma_{f2} \Sigma_r (1-\beta), \]

and

\[ b_{0m}/\lambda = (\alpha_2 + v_2 D_2) \left( \frac{\mu_1}{a} \right)^2 (\alpha_1 + v_1 D_1) \left( \frac{\mu_1}{a} \right)^2 - v_1 v_2 \nu \Sigma_{f1} - v_1 v_2 \nu \Sigma_{f2} \Sigma_r. \]

Now consider Oscillation Test I which is for the purpose of illustrating the application of the procedure outlined in Section 4.4. The prompt thermal neutron eigenvalues of the first four spatial harmonics are found from Eq. (4.18) and are tabulated in Table 4.1. The properties of the uniform, slab reactor \( a = 200 \text{ cm} \) are tabulated in Table 4.2. Presume for the sake of illustration that the maximum experimental frequency, \( \omega_{\text{max}} \), is 1000 radians/second. Only the first three eigenvalues are smaller than \( \omega_{\text{max}} \); hence, only these three eigenvalues may be verified.

Table 4.1

<table>
<thead>
<tr>
<th>Spatial Harmonic Index, ( m )</th>
<th>( \omega_{mp}, \text{sec}^{-1} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>49.13</td>
</tr>
<tr>
<td>2</td>
<td>355</td>
</tr>
<tr>
<td>3</td>
<td>850</td>
</tr>
<tr>
<td>4</td>
<td>-1514</td>
</tr>
</tbody>
</table>

Prompt Thermal Neutron Eigenvalues of the First Four Spatial Harmonics of a Slab Reactor Example
Table 4.2

Nuclear Parameters of the Uniform Slab Reactor of Oscillation Test I

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>width, cm</td>
<td>200.0</td>
</tr>
<tr>
<td>$\Sigma_{a1}, \text{cm}^{-1}$</td>
<td>0.0</td>
</tr>
<tr>
<td>$\Sigma_{a2}, \text{cm}^{-1}$</td>
<td>0.02775</td>
</tr>
<tr>
<td>$\nu \Sigma_{f1}, \text{cm}^{-1}$</td>
<td>0.0</td>
</tr>
<tr>
<td>$\nu \Sigma_{f2}, \text{cm}^{-1}$</td>
<td>0.0282</td>
</tr>
<tr>
<td>$\Sigma_r, \text{cm}^{-1}$</td>
<td>0.03776</td>
</tr>
<tr>
<td>$D_1, \text{cm}$</td>
<td>1.313</td>
</tr>
<tr>
<td>$D_2, \text{cm}$</td>
<td>0.804</td>
</tr>
<tr>
<td>$v_1, \text{cm/sec}$</td>
<td>$4.06 \times 10^6$</td>
</tr>
<tr>
<td>$v_2, \text{cm/sec}$</td>
<td>$2.48 \times 10^5$</td>
</tr>
<tr>
<td>$\beta$</td>
<td>0.0074</td>
</tr>
<tr>
<td>$\lambda, \text{sec}^{-1}$</td>
<td>0.0138</td>
</tr>
</tbody>
</table>
Consider that the oscillator is located at the position, \( x_0 = a/4 \). At this location the oscillator excites the first three spatial harmonics but not the fourth spatial harmonic. According to the experimental procedure, three detectors are located at three of the zeros of the lowest order, non-negligible spatial harmonic appearing in \( E_{31} \). Since the fourth spatial harmonic is not excited (\( c_{4p} = 0 \)), this lowest order spatial harmonic is the fifth (\( \sin \frac{5\pi x}{a} \)) which has zeros at \( x_i = ia/5 \), where \( i = 1, \ldots, 4 \). Detector readings obtained by evaluating Eq. (4.17) for \( x_1 = 2a/5 \), \( x_2 = 3a/5 \) and \( x_3 = 4a/5 \) are illustrated in Fig. 4.5. Analyzing these detector readings by Eq. (4.13) leads to the inferred coefficients, \( A_{mp} \), illustrated in Fig. 4.6. Note that the inferred behavior of the \( A_{mp} \)'s is in fair agreement with the actual behavior. Note also that the phase of \( A_{1p} \) falls below its proper high frequency asymptote of \(-90^\circ\). The prompt thermal neutron eigenvalues inferred from the estimated coefficients (Detectors at Zeros of Fifth Harmonic) are compared with the actual eigenvalues in Table 4.3. Note that good results are obtained for the first two eigenvalues.

Figure 4.7 illustrates the results obtained for the expansion coefficients when all detectors are moved closer to the oscillator by an arbitrary distance of 10 cm. Note that the phase of \( A_{1p}(ju) \) more closely follows its proper high frequency asymptotic behavior. However, the phase still falls below its asymptote so it may be concluded that the detectors have not been moved too far. The prompt thermal neutron eigenvalues inferred from the estimated coefficients (Detectors Displaced towards Oscillator) are compared with the actual
Figure 4.5 Phase and Normalized Magnitude of the Oscillating Portion of the Reading of the ith Detector. Oscillation Test I.
Figure 4.6 Comparison of Actual Behavior of Expansion Coefficients with Inferred Behavior in Oscillation Test I. Detectors at Zeros of Lowest Order, Non-negligible, Neglected Mode.
eigenvalues in Table 4.3. Note that there is improvement over those results obtained with the detectors at the zeros of the fifth spatial harmonic.

<table>
<thead>
<tr>
<th>Spatial Harmonic Index, m</th>
<th>Actual Eigenvalue, sec⁻¹</th>
<th>Inferred Eigenvalues, sec⁻¹</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Detectors at Zeros of Fifth Harmonic</td>
</tr>
<tr>
<td>1</td>
<td>-49.1</td>
<td>-48.9</td>
</tr>
<tr>
<td>3</td>
<td>-350.</td>
<td>-360.</td>
</tr>
</tbody>
</table>

The second example, Oscillation Test II, is for the purpose of illustrating the type of results which led to the rule of thumb procedure for locating the neutron detectors.

Consider Reactor II (a = 240 cm) of Fig. 3.4 whose properties are tabulated in Table 3.2. This reactor is perturbed by an oscillating, thermal neutron absorber at the position, \( x_0 = 40 \) cm; and the response, \( r_i(jw) \), of the \( i^{th} \) detector is observed. The detector reading is given in terms of a natural mode expansion by Eq. (4.3), which is repeated here for convenience,

\[
x_i(jw) = \sum_{m=1}^{M} \lambda_{mp}(jw) \left< v_{th}^i \sigma_{th}^i \right|_{nmp} (th) + E_{M_i}(jw) + E_i^r(jw) \quad (4.19)
\]

It is apparent that if \( M \) neutron detectors (\( i = 1, \ldots, M \)) are located at positions where the quantity, \( E_{M_i}^r = E_{M_i} + E_i^r \), is zero, then the system of \( M \) equations formed by Eqs. (4.19) can be solved for
Figure 4.7 Comparison of Actual Behavior of Expansion Coefficients with Inferred Behavior in Oscillation Test I. Detectors Displaced towards Oscillator.
estimates of the $M$ expansion coefficients, $A_{mp}$. The locations of the zeros of $E_{M1}$ are unknown, in general; but, for the simple system of Oscillation Test II, the behavior of $E_{M1}$ may be examined. The quantity, $r_1(jw)$, may be calculated from Eq. (4.17). The coefficients, $A_{mp}(jw)$, may be calculated from Eq. (4.4) in this simple example. The eigenvalues are given by Eq. (4.18) and the shapes of the modes are merely $\sin \frac{\theta}{a}$. Therefore, $E_{M1}$, which represents an infinite number of neglected terms, may be calculated from the equation,

$$E_{M1}(jw) = r_1(jw) - \sum_{m=1}^{M} A_{mp}(jw) \left<v_{th_i}(th)N_{mp}(th)\right>.$$ 

Typical results of such calculations for Reactor II are shown in Fig. 4.8. In this figure the real part of $E_{M1}$ for $\omega = 30$ radians/sec is plotted as a function of detector position for $M = 1$ to $M = 5$. Note that for $M$ detectors there do exist $M$ locations, indicated by the circles, for which $\text{Re}(E_{M1}) = 0$. Note also that the zeros of $\text{Re}(E_{M1})$ are displaced towards the oscillator from the zeros of the $(M + 1)^{st}$ spatial harmonic, indicated by the crosses. The only exception to this last observation occurs when the $(M + 1)^{st}$ spatial harmonic is negligible. For example, the oscillator is located at $x_0 = a/6$ and the sixth spatial harmonic is not excited. Therefore when five detectors are used ($M = 5$) the detectors are displaced from the zeros of the lowest order, non-negligible spatial harmonic in $E_{51}$, i.e., the seventh.

---

It is considered sufficient to study the behavior of the real part of $E_{M1}$ because the imaginary part is very small. When $\omega \ll \left|M+1, p\right|$ the coefficients, $A_{mp}$, for $m > M$ are approximately equal to the real number, $a_{mp}/(\omega_{mp})$. 
All Curves for $\omega = 30$ Radians/sec.

- $M$ - Number of Coefficients to be Estimated
- $\bigcirc$ - Zeros of $E_{\text{TM}}$
- $\times$ $M+1$ - Zeros of $(M+1)$st Mode

Figure 4.8 Real Part of $E_{\text{TM}}(j\omega)$ as a Function of the Location of the $i$th Detector. Oscillation Test II.
Note also in Fig. 4.8 that the magnitude of \( \text{Re}(E_{M1}) \) is largest in the vicinity of the oscillator. This means that the results for the inferred coefficients, \( A_{mp} \), will be very sensitive to the location of the detector nearest the oscillator. Note also that the variation of the magnitude of \( \text{Re}(E_{M1}) \) gets smaller as the number of detectors is increased. This means that for a large number of detectors the results for the \( A_{m1} \) become less sensitive to the locations of those detectors far from the oscillator.

Figure 4.9 shows the excellent results obtained for the coefficients, \( A_{mp} \), when three detectors are placed very near those locations where \( E_{31} \) is zero. This figure also shows the behavior of the phase of \( A_{l1} \) as the detectors are moved from the zeros of the fourth spatial harmonic, and through the zeros of \( E_{31} \). Note that:

(i) the phase of \( A_{l1} \) is below its proper high frequency asymptote when the detectors are at the zeros of the fourth harmonic, and

(ii) the phase of \( A_{l1} \) is above its high frequency asymptote when the detectors are moved past the zeros of \( E_{31} \).

It is such observations of the results of many studies of the behavior of \( E_{M1} \) and \( A_{l1} \) which have led to the following rule of thumb for locating the detectors: To measure the coefficients of \( M \) non-negligible modes \( (c_{mp} \neq 0) \), (i) place \( M \) detectors at \( M \) of the zeros of the lowest order, non-negligible mode \( (c_{mp} \neq 0) \) of \( E_{M1} \); and (ii) move these \( M \) detectors a small distance in the direction of the oscillator until the phase of \( A_{l1} \) attains its proper high frequency asymptotic behavior. No general specification can be made as to the relative movements of each detector.
Figure 4.9 Comparison of Actual Behavior of Expansion Coefficients with Inferred Behavior in Oscillation Test II. Detectors at Zeros of Re($E_{21}(3CJ)$).
It is of interest to note that poorly placed detectors yield poor results which cannot be interpreted for the eigenvalues. Consider an example where the detectors are placed in close proximity to each other at \( x_1 = 60 \text{ cm}, x_2 = 80 \text{ cm}, \) and \( x_3 = 100 \text{ cm} \). Figure 4.10 shows that the estimated coefficients in this case are very far off from the actual coefficients. The reason for the poor estimation is that the quantities, \( E_{31} \), are not negligible and they combine in an unfavorable way in the process of solving for the estimated coefficients.

### 4.5.2 Nonuniform, Cylindrical Reactor Example

This numerical experiment, Oscillation Test III, is for the purpose of illustrating the feasibility of measuring the prompt thermal neutron eigenvalues in nonuniform reactors.

The reactor in this example is a one-dimensional representation of a particular fuel loading\(^{[46]}\) in NORA, a heavy water moderated, zero power reactor. The reactor is perturbed by an oscillating thermal absorber at the position, \( r_0 = 0 \). The resulting space and frequency dependent reading of the \( i \)th neutron detector is illustrated in Figs. 4.11 and 4.12 for oscillation frequencies between \( 1.0 \) and \( 1000.0 \) radians/second. These results were calculated by MacDonald and Johnson \(^{[45]}\) who used static techniques. Two groups of neutrons and fourteen groups of delayed neutron precursors were considered. The neutron group parameters and the effective delayed precursor parameters used in this calculation are tabulated in Table 4.4. Analysis of these results by the procedure summarized in Section 4.4 leads to estimates of the prompt thermal neutron eigenvalues.
Figure 4.10 Comparison of Actual Behavior of Expansion Coefficients with Inferred Behavior in Oscillation Test II. Poorly Placed Detectors.

Three Detectors at $x_1=60$, $x_2=30$ and $x_3=100$ cm.
Figure 4.11 Magnitude of the Oscillating Portion of the $i^{th}$ Thermal Neutron Detector Reading in a Nonuniform, Cylindrical Reactor. Oscillation Test III.
Figure 4.12 Phase of the Oscillating Portion of the \( i \)th Detector in a Nonuniform, Cylindrical Reactor. Oscillation Test III.
Table 4.4
Nuclear Parameters of the Nonuniform, Cylindrical Reactor of Oscillation Test III

<table>
<thead>
<tr>
<th>Neutron Group Parameter</th>
<th>Region 1 0 to 59.7 cm</th>
<th>Region 2 59.7 to 112.5 cm</th>
<th>Region 3 112.5 to 162.5 cm</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Sigma_{a1}, \text{cm}^{-1} )</td>
<td>0.000235</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>( \Sigma_{a2}, \text{cm}^{-1} )</td>
<td>0.00365</td>
<td>0.00011</td>
<td>0.00083</td>
</tr>
<tr>
<td>( v \Sigma_{f1}, \text{cm}^{-1} )</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>( v \Sigma_{f2}, \text{cm}^{-1} )</td>
<td>0.00556</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>( \Sigma_{p}, \text{cm}^{-1} )</td>
<td>0.00968</td>
<td>0.01090</td>
<td>0.00282</td>
</tr>
<tr>
<td>( D_1, \text{cm} )</td>
<td>1.29197</td>
<td>1.305</td>
<td>1.1920</td>
</tr>
<tr>
<td>( D_2, \text{cm} )</td>
<td>0.81796</td>
<td>0.82550</td>
<td>1.0500</td>
</tr>
<tr>
<td>( v_1, \text{cm/sec} )</td>
<td>2.0x10^6</td>
<td>2.0x10^6</td>
<td>2.0x10^6</td>
</tr>
<tr>
<td>( v_2, \text{cm/sec} )</td>
<td>2.2x10^5</td>
<td>2.2x10^5</td>
<td>2.2x10^5</td>
</tr>
<tr>
<td>( B_z^2, \text{cm}^{-2} )</td>
<td>0.0005536047</td>
<td>0.0005536047</td>
<td>0.0005536047</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Delayed Precursor Group Index, ( i )</th>
<th>Precursor Decay Constants ( \lambda_i, \text{sec}^{-1} )</th>
<th>Effective Delayed Neutron Fractions ( \beta_i, 10^{-5} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0127</td>
<td>24.8</td>
</tr>
<tr>
<td>2</td>
<td>0.0317</td>
<td>140.2</td>
</tr>
<tr>
<td>3</td>
<td>0.115</td>
<td>124.5</td>
</tr>
<tr>
<td>4</td>
<td>0.311</td>
<td>270.1</td>
</tr>
<tr>
<td>5</td>
<td>1.40</td>
<td>86.8</td>
</tr>
<tr>
<td>6</td>
<td>3.87</td>
<td>18.2</td>
</tr>
<tr>
<td>7</td>
<td>0.273</td>
<td>48.2</td>
</tr>
<tr>
<td>8</td>
<td>0.0169</td>
<td>15.1</td>
</tr>
<tr>
<td>9</td>
<td>0.0049</td>
<td>5.2</td>
</tr>
<tr>
<td>10</td>
<td>0.00152</td>
<td>2.5</td>
</tr>
<tr>
<td>11</td>
<td>0.000427</td>
<td>1.5</td>
</tr>
<tr>
<td>12</td>
<td>0.000116</td>
<td>1.7</td>
</tr>
<tr>
<td>13</td>
<td>0.000044</td>
<td>0.2</td>
</tr>
<tr>
<td>14</td>
<td>0.000037</td>
<td>0.07</td>
</tr>
</tbody>
</table>
There is one modification of the technique for this particular example because the prompt thermal neutron eigenvalue of the fundamental spatial harmonic is of the same order of magnitude as the precursor decay constants. Thus the frequency dependent behavior of the fundamental mode coefficient is influenced to some extent by those terms in the \( k \) summation which contain the delayed neutron eigenvalues. The expansion solution, Eq. (4.3), must be written in a form which does not neglect these terms. That is,

\[
\begin{align*}
    r_i(j\omega) &= \sum_{k=1}^{K-1} A_{1k}(j\omega) \left< \sigma_i(\text{th})(r) N_{1k}(\text{th})(r) \right> \\
    &+ \sum_{m=2}^{M} A_{1m}(j\omega) \left< \sigma_i(\text{th})(r) N_{1m}(\text{th})(r) \right> \\
    &+ E_{M1}(j\omega) ,
\end{align*}
\]

where

\[
\begin{align*}
    E_{M1}(j\omega) &= A_{1K}(j\omega) \left< \sigma_i(\text{th})(r) N_{1K}(\text{th})(r) \right> \\
    &+ \sum_{m=2}^{M} \sum_{k=1}^{K} A_{mk}(j\omega) \left< \sigma_i(\text{th})(r) N_{mk}(\text{th})(r) \right> \\
    &+ \sum_{m=1+1}^{\infty} A_{mp}(j\omega) \left< \sigma_i(\text{th})(r) N_{mp}(\text{th})(r) \right> .
\end{align*}
\]

The appearance of \( M + K - 2 \) unknown coefficients in Eq. (4.20) offers little difficulty because the \( K - 1 \) thermal neutron density components,
are the same function of \( r \) to a very good approximation for this reactor. Thus the summation of terms,

\[
\sum_{k=1}^{K-1} A_{1k}(j\omega) \left\langle \sigma_i^{(th)}(r) N_{1k}^{(th)}(r) \right\rangle,
\]

may be replaced by

\[
A_1(j\omega) \left\langle \sigma_i^{(th)}(r) N_{1p}^{(th)}(r) \right\rangle,
\]

where

\[
A_1(j\omega) = \sum_{k=1}^{K-1} A_{1k}(j\omega).
\]

The behavior of only that coefficient which contains the prompt thermal neutron eigenvalue may be found by subtracting the coefficients of the delayed neutron modes from the measured coefficients. That is,

\[
A_{1p}(j\omega) = A_1(j\omega) - \sum_{k=1}^{K-2} A_{1k}(j\omega).
\]

The coefficients, \( A_{1k} \), for \( k = 1, \ldots, K-2 \) must be calculated from their defining expression,

\[
A_{1k} = c_{mk} (j\omega - \omega_{mk})^{-1}.
\]

The \( c_{mk} \) are given by Eq. (4.5).

The natural modes which are used in the processing of the detector readings in this nonuniform reactor must be calculated numerically by means of the computer code described in Appendix C. The thermal neutron density components, \( N_{mp}^{(2)}(r) \), of the calculated modes are illustrated in Fig. 4.13. The theoretical values of the prompt thermal neutron eigenvalues of these modes are listed in Fig. 4.13. The theoretical values of the delayed neutron eigenvalues, \( \mu_{1k} \), and the coefficients, \( c_{1k} \), are listed in Table 4.5.
Figure 4.12 Thermal Neutron Density Components of Natural Modes of the Nonuniform, Cylindrical Reactor of Oscillation Test III.
Table 4.5

Delayed Neutron Eigenvalues of the Fundamental Spatial Harmonic of the Nonuniform, Cylindrical Reactor of Oscillation Test III

<table>
<thead>
<tr>
<th>Delayed Precursor Group Index, k</th>
<th>Delayed Neutron Eigenvalue, $\omega_{lk}$, sec$^{-1}$</th>
<th>Coefficient, $c_{lk}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0</td>
<td>0.3928</td>
</tr>
<tr>
<td>2</td>
<td>-5.6256x10$^{-6}$</td>
<td>0.1962</td>
</tr>
<tr>
<td>3</td>
<td>-4.8993x10$^{-5}$</td>
<td>0.0512</td>
</tr>
<tr>
<td>4</td>
<td>-2.1071x10$^{-4}$</td>
<td>0.4312</td>
</tr>
<tr>
<td>5</td>
<td>-5.8574x10$^{-4}$</td>
<td>0.5030</td>
</tr>
<tr>
<td>6</td>
<td>-1.7816x10$^{-3}$</td>
<td>0.3040</td>
</tr>
<tr>
<td>7</td>
<td>-5.3743x10$^{-3}$</td>
<td>0.1627</td>
</tr>
<tr>
<td>8</td>
<td>-1.4319x10$^{-2}$</td>
<td>0.1201</td>
</tr>
<tr>
<td>9</td>
<td>-1.8696x10$^{-2}$</td>
<td>0.1807</td>
</tr>
<tr>
<td>10</td>
<td>-7.2017x10$^{-2}$</td>
<td>1.6733</td>
</tr>
<tr>
<td>11</td>
<td>-1.8955x10$^{-1}$</td>
<td>2.0358</td>
</tr>
<tr>
<td>12</td>
<td>-2.8368x10$^{-1}$</td>
<td>0.0401</td>
</tr>
<tr>
<td>13</td>
<td>-1.2184</td>
<td>5.3475</td>
</tr>
<tr>
<td>14</td>
<td>-3.6595</td>
<td>12.051</td>
</tr>
</tbody>
</table>
Figures 4.14 and 4.15 show the results for the expansion coefficients obtained by using two ($M = 2$) and three ($M = 3$) detectors, respectively. The detectors were located according to the rule of thumb procedure. The prompt thermal neutron eigenvalues inferred from the behavior of these expansion coefficients are compared with the actual eigenvalues in Table 4.6. Note that the agreement is excellent.

4.6 The Analysis of an Actual Experiment

In this section the results of an oscillation test performed on the NORA reactor are analyzed by the procedure outlined in Section 4.4. Only the essential features of the reactor configuration and the experimental arrangement are summarized here. More detail about the experiment is given in Ref. 46. A detailed description of the NORA reactor is given in Ref. 47.

The NORA reactor is a critical assembly designed for the study of mixed moderator ($\text{H}_2\text{O}/\text{D}_2\text{O}$) systems. The reactor configuration and composition at the time of the experiment are summarized in Table 4.7.

The reactor was perturbed by an oscillator which consisted of three pairs of cadmium discs mounted on a vertical assembly. One disc of each pair was shaped in such a way that the reactor was perturbed sinusoidally when this disc was rotated past the other disc. The approximate elevations of these pairs of discs from the interface between the lower reflectors were 46 cm, 66 cm and 86 cm, respectively. The oscillator was located 7.1 cm from the vertical centerline of the reactor and had a diameter of 7 cm.
Figure 4.14 Comparison of Theoretical Behavior of the Expansion Coefficients with Inferred Behavior in Oscillation Test III. Two Detectors.
Figure 4.15 Comparison of Theoretical Coefficients with Inferred Behavior in Three Detectors.

Three Detectors at
\[ r_1 = 22.83 \text{ cm}, \]
\[ r_2 = 68.5 \text{ cm}, \]
\[ r_3 = 103.7 \text{ cm}. \]
Table 4.6

Comparison of Inferred Eigenvalues and Actual Eigenvalues of the Nonuniform, Cylindrical Reactor of Oscillation Test III

<table>
<thead>
<tr>
<th>Detector Locations, centimeters</th>
<th>Harmonic Index, m</th>
<th>Theoretical Eigenvalue, $\omega_{mp}$, seconds$^{-1}$</th>
<th>Inferred Eigenvalue, $\omega'_{mp}$, seconds$^{-1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>27.36 and 86.10</td>
<td>1</td>
<td>-6.088</td>
<td>(-6.088)*</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>-357.1</td>
<td>-361.†</td>
</tr>
<tr>
<td>23.88, 63.5, and 103.7</td>
<td>1</td>
<td>-6.088</td>
<td>(-6.088)*</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>-357.1</td>
<td>-350.†</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>-728.9</td>
<td>-716.†</td>
</tr>
</tbody>
</table>

* Data fit so well with theoretical eigenvalue that no effort was made to improve the fit.

† Determined by fitting

\[ K/(\omega_{mp})^2 = |A_{mp}|^2 [1+(\omega/\omega_{mp})^2] \text{ for } \omega<|\omega_{mp}| \]

and fitting

\[ K = |A_{mp}|^2 \omega^2 [1+(\omega_{mp}/\omega)^2] \text{ for } \omega>|\omega_{mp}| \]

then

\[ \omega_{mp} = \sqrt{\frac{K}{K/(\omega_{mp})^2}} \]
Table 4.7

Configuration and Composition of the NORA Reactor

<table>
<thead>
<tr>
<th>Component</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fuel</td>
<td>3 o/o enriched UO₂, 112 rods of 12 mm diameter on a square lattice spacing of 10 cm.</td>
</tr>
<tr>
<td>Moderator</td>
<td>99.605 o/o D₂O</td>
</tr>
<tr>
<td>Critical Core Height</td>
<td>115.25 cm (18.8°C)</td>
</tr>
<tr>
<td>Core Diameter</td>
<td>119.4 cm</td>
</tr>
<tr>
<td>Moderator/fuel Volume Ratio</td>
<td>98.8</td>
</tr>
<tr>
<td>Side Reflector Thickness (D₂O)</td>
<td>52.8 cm</td>
</tr>
<tr>
<td>Side Reflector Thickness (Graphite)</td>
<td>50 cm</td>
</tr>
<tr>
<td>Lower Reflector (D₂O)</td>
<td>8.8 cm</td>
</tr>
<tr>
<td>Lower Reflector (Graphite)</td>
<td>70 cm</td>
</tr>
</tbody>
</table>

The oscillations of the neutron density were measured with a fission chamber which was 26 cm in length and 5.6 cm in diameter. The center of the chamber was at a height of approximately 66 cm above the interface between the lower reflectors. The radial location of this chamber was varied during the course of the experiment.

The oscillating portion of the output signal from the chamber was Fourier analyzed for that portion of the signal which oscillated with the fundamental frequency of oscillation.

Figure 4.16 shows the experimental data for the magnitude and phase of the oscillating response of the neutron detector at five different radial locations. The magnitude data are reported as the normalized ratio, $|r_1(j\omega)]/R_1^{0}$, in decibel units (G in decibels =
Figure 4.16 Experimental Data for Oscillation Test on KORA Reactor
20 \log_{10} G). The mean errors for the experimental data are ± 0.8 degrees and ± 0.1 decibels for the phase and magnitude, respectively. The original data are tabulated in Table 4.8.

These data were smoothed by hand fitting a smooth curve to the results from each detector. The smoothed data were plotted as a function of detector position for various frequencies and these plots were used to interpolate between detector positions. By this procedure data were estimated for any desired detector location.

The natural modes which were used in the processing of the detector readings were calculated numerically by means of the computer code described in Appendix C. The nuclear parameters which were used in this calculation are tabulated in Tables 4.9 and 4.10. The thermal neutron density components, \( N_{mp}^{(th)}(r) \), of these calculated modes are illustrated in Fig. 4.17. The theoretical calculations of the prompt thermal neutron eigenvalues are listed in this figure.

Figures 4.18 and 4.19 show the inferred behavior of the expansion coefficients of the first three radial harmonics as determined by three detectors located at 17.7 cm, 62.5 cm and 114.2 cm. These detector locations were chosen by the rule of thumb procedure. Shown also in these figures is the theoretical behavior of the expansion coefficients.

The maximum experimental frequency was not high enough to allow an estimate of the prompt thermal neutron eigenvalues of the higher order spatial harmonics. However, the inferred behavior of the higher mode expansion coefficients is certainly reasonable when compared with the type of behavior observed in the numerical experiments.
**Table 4.8(a)**

Experimental Data for the Oscillation Test on the NCRA Reactor

<table>
<thead>
<tr>
<th>Location A</th>
<th>Location B</th>
<th>Location C</th>
</tr>
</thead>
<tbody>
<tr>
<td>Detector at r=7.9 cm</td>
<td>Detector at r=17.7 cm</td>
<td>Detector at r=37.6 cm</td>
</tr>
<tr>
<td>Frequency, cycles/sec</td>
<td>Amplitude, decibels</td>
<td>Phase, degrees</td>
</tr>
<tr>
<td>27.17</td>
<td>-23.10</td>
<td>-40.35</td>
</tr>
<tr>
<td>24.03</td>
<td>-22.30</td>
<td>-40.87</td>
</tr>
<tr>
<td>22.02</td>
<td>-22.53</td>
<td>-41.44</td>
</tr>
<tr>
<td>20.02</td>
<td>-22.15</td>
<td>-41.62</td>
</tr>
<tr>
<td>18.02</td>
<td>-21.92</td>
<td>-44.33</td>
</tr>
<tr>
<td>16.02</td>
<td>-21.43</td>
<td>-45.54</td>
</tr>
<tr>
<td>12.0</td>
<td>-20.21</td>
<td>-51.67</td>
</tr>
<tr>
<td>10.0</td>
<td>-19.23</td>
<td>-54.40</td>
</tr>
<tr>
<td>8.0</td>
<td>-17.80</td>
<td>-57.95</td>
</tr>
<tr>
<td>6.5</td>
<td>-16.43</td>
<td>-60.53</td>
</tr>
<tr>
<td>5.0</td>
<td>-14.47</td>
<td>-63.07</td>
</tr>
<tr>
<td>4.5</td>
<td>-13.66</td>
<td>-63.56</td>
</tr>
<tr>
<td>3.0</td>
<td>-10.61</td>
<td>-63.32</td>
</tr>
<tr>
<td>2.25</td>
<td>- 8.27</td>
<td>-61.06</td>
</tr>
<tr>
<td>1.067</td>
<td>- 3.55</td>
<td>-48.57</td>
</tr>
<tr>
<td>0.65</td>
<td>- 1.65</td>
<td>-38.57</td>
</tr>
<tr>
<td>0.40</td>
<td>- 0.41</td>
<td>-29.86</td>
</tr>
<tr>
<td>0.108</td>
<td>+ 1.46</td>
<td>-23.07</td>
</tr>
<tr>
<td>0.0433</td>
<td>+ 3.47</td>
<td>-29.78</td>
</tr>
</tbody>
</table>
### Table 4.8(b)
Experimental Data for the Oscillation Test on the Nima Reactor

<table>
<thead>
<tr>
<th>Location D</th>
<th>Location E</th>
</tr>
</thead>
<tbody>
<tr>
<td>Detector at r=62.5 cm</td>
<td>Detector at r=114.2 cm</td>
</tr>
<tr>
<td>Frequency, cycles/sec</td>
<td>Frequency, cycles/sec</td>
</tr>
<tr>
<td>Amplitude, decibels</td>
<td>Amplitude, decibels</td>
</tr>
<tr>
<td>Phase, degrees</td>
<td>Phase, degrees</td>
</tr>
<tr>
<td>27.08</td>
<td>27.17</td>
</tr>
<tr>
<td>24.02</td>
<td>24.0</td>
</tr>
<tr>
<td>22.0</td>
<td>22.0</td>
</tr>
<tr>
<td>20.0</td>
<td>20.0</td>
</tr>
<tr>
<td>18.0</td>
<td>18.0</td>
</tr>
<tr>
<td>16.0</td>
<td>16.0</td>
</tr>
<tr>
<td>14.0</td>
<td>14.0</td>
</tr>
<tr>
<td>12.0</td>
<td>12.0</td>
</tr>
<tr>
<td>10.0</td>
<td>10.0</td>
</tr>
<tr>
<td>8.0</td>
<td>8.0</td>
</tr>
<tr>
<td>6.5</td>
<td>6.5</td>
</tr>
<tr>
<td>5.0</td>
<td>5.0</td>
</tr>
<tr>
<td>3.98</td>
<td>4.0</td>
</tr>
<tr>
<td>3.0</td>
<td>3.0</td>
</tr>
<tr>
<td>1.067</td>
<td>1.0</td>
</tr>
<tr>
<td>0.65</td>
<td>0.65</td>
</tr>
<tr>
<td>0.40</td>
<td>0.40</td>
</tr>
<tr>
<td>0.20</td>
<td>0.20</td>
</tr>
<tr>
<td>0.10</td>
<td>0.10</td>
</tr>
<tr>
<td>0.0467</td>
<td>0.045</td>
</tr>
<tr>
<td>Parameters</td>
<td>Oscillator Region a</td>
</tr>
<tr>
<td>------------</td>
<td>---------------------</td>
</tr>
<tr>
<td>Inner Radius, cm</td>
<td>0.0</td>
</tr>
<tr>
<td>Outer Radius, cm</td>
<td>3.5</td>
</tr>
<tr>
<td>$\Sigma_{a1}, \text{cm}^{-1}$</td>
<td>0.00048647</td>
</tr>
<tr>
<td>$\Sigma_{a2}, \text{cm}^{-1}$</td>
<td>0.01286025</td>
</tr>
<tr>
<td>$v\Sigma_{f1}, \text{cm}^{-1}$</td>
<td>0.0</td>
</tr>
<tr>
<td>$v\Sigma_{f2}, \text{cm}^{-1}$</td>
<td>0.0</td>
</tr>
<tr>
<td>$\Sigma_{r}, \text{cm}^{-1}$</td>
<td>0.010471</td>
</tr>
<tr>
<td>$D_{1}, \text{cm}$</td>
<td>1.313</td>
</tr>
<tr>
<td>$D_{2}, \text{cm}$</td>
<td>0.804</td>
</tr>
<tr>
<td>$v_{1}, \text{cm/sec}$</td>
<td>4.06x10^6 (calculated from Eq. (4.24))</td>
</tr>
<tr>
<td>$v_{2}, \text{cm/sec}$</td>
<td>2.48x10^5 (\bar{v} of a Maxwellian distribution at 293^0K)</td>
</tr>
<tr>
<td>$B_{z}, \text{cm}^{-2}$</td>
<td>0.000514520 (corresponds to H_{ext} = 138.50 cm, the critical height with oscillator inserted)</td>
</tr>
</tbody>
</table>

(a) Same as core parameters with $\Sigma_{a2}$ adjusted for criticality when H_{ext} = 138.50 cm
(b) Moen and Haugest [49]
(c) Etherington [49]
Table 4.10

Delayed Precursor Parameters Used in Calculating the Natural Modes of the NORA Reactor [46]

<table>
<thead>
<tr>
<th>Precursor Group</th>
<th>Delayed Neutron Fraction</th>
<th>Precursor Decay Constant</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$24.8 \times 10^{-5}$</td>
<td>0.0127</td>
</tr>
<tr>
<td>2</td>
<td>$140.2$</td>
<td>0.0317</td>
</tr>
<tr>
<td>3</td>
<td>$124.5$</td>
<td>0.115</td>
</tr>
<tr>
<td>4</td>
<td>$270.1$</td>
<td>0.311</td>
</tr>
<tr>
<td>5</td>
<td>$86.8$</td>
<td>1.40</td>
</tr>
<tr>
<td>6</td>
<td>$18.2$</td>
<td>3.87</td>
</tr>
<tr>
<td>7</td>
<td>$48.2$</td>
<td>0.278</td>
</tr>
<tr>
<td>8</td>
<td>$15.1$</td>
<td>0.0169</td>
</tr>
<tr>
<td>9</td>
<td>$5.2$</td>
<td>0.0049</td>
</tr>
<tr>
<td>10</td>
<td>$2.5$</td>
<td>0.00152</td>
</tr>
<tr>
<td>11</td>
<td>$1.5$</td>
<td>0.000427</td>
</tr>
<tr>
<td>12</td>
<td>$1.7$</td>
<td>0.000116</td>
</tr>
<tr>
<td>13</td>
<td>$0.2$</td>
<td>0.000044</td>
</tr>
<tr>
<td>14</td>
<td>$0.07$</td>
<td>0.0000037</td>
</tr>
</tbody>
</table>

$\beta_{tot} = 0.007392$
Figure 4.17 Thermal Neutron Density Components of Natural Modes of the NCRA Reactor
Theoretical
Inferred

$|A_1(j\omega)|$

$|A_{2p}(j\omega)|$

$|A_{3p}(j\omega)|$

Oscillation Frequency, $\omega$ - Radians/sec

Three Detectors at $r_1=17.7$, $r_2=62.5$
and $r_3=114.2$ cm

Figure 4.18 Comparison of the Magnitude of the Theoretical Expansion Coefficients with Inferred Magnitude in the NORA Reactor.
Figure 4.19 Comparison of the Phase of the Theoretical Expansion Coefficients with the Inferred Phase in the NORA Reactor

Three Detectors at $r_1=17.7$, $r_2=62.5$ and $r_3=114.2$ cm.
From the inferred behavior of the fundamental spatial harmonic, the prompt thermal neutron eigenvalue, $\omega_{1p}$, has been estimated to be $-5.68 \pm 0.20$. This eigenvalue was estimated by making a least squares fit to $A_{1p}(j\omega)$ where

$$A_{1p}(j\omega) = A_1(j\omega) - \sum_{k=1}^{14} A_{1k}(j\omega),$$

and $A_1(j\omega)$ is the measured behavior of the fundamental spatial harmonic. The constants, $c_{1k}$, are defined by Eq. (4.5) and are listed in Table 4.11 along with the delayed neutron eigenvalues, $\omega_{1k}$ ($k = 1, 2, \ldots, 14$), for the fundamental spatial harmonic.

The quoted error was determined from the propagation of two errors. The low frequency behavior of $|A_{1p}(j\omega)|$ (for $\omega \ll |\omega_{1p}|$, $|A_{1p}|^2 = K/|\omega_{1p}|^2$) was empirically fit for the parameter,

$$\frac{K}{|\omega_{1p}|^2} = |A_{1p}|^2[1 + (\omega/|\omega_{1p}|)^2]. \quad (4.22)$$

This was determined by a least squares fit to be $0.547 \pm 0.017$ where $0.017$ is the standard deviation. The high frequency behavior of $|A_{1p}(j\omega)|$ (for $\omega \gg |\omega_{1p}|$, $|A_{1p}|^2 = K/\omega^2$) was empirically fit for the constant parameter,

$$K = \omega^2|A_{1p}|^2[1 + (\omega_{1p}/\omega)^2]. \quad (4.23)$$
Table 4.11

Delayed Neutron Eigenvalues
of the Fundamental Spatial Harmonic of the NORA Reactor

<table>
<thead>
<tr>
<th>Delayed Precursor Group Index, ( k )</th>
<th>Delayed Neutron Eigenvalue, ( \omega_{1k} ), sec(^{-1} )</th>
<th>Coefficient, ( c_{1k} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0.2472</td>
</tr>
<tr>
<td>2</td>
<td>-0.5626 \times 10^{-5}</td>
<td>0.1235</td>
</tr>
<tr>
<td>3</td>
<td>-0.4899 \times 10^{-4}</td>
<td>0.0322</td>
</tr>
<tr>
<td>4</td>
<td>-0.2107 \times 10^{-3}</td>
<td>0.2714</td>
</tr>
<tr>
<td>5</td>
<td>-0.5847 \times 10^{-3}</td>
<td>0.3167</td>
</tr>
<tr>
<td>6</td>
<td>-0.1782 \times 10^{-2}</td>
<td>0.1914</td>
</tr>
<tr>
<td>7</td>
<td>-0.5374 \times 10^{-2}</td>
<td>0.1024</td>
</tr>
<tr>
<td>8</td>
<td>-0.1432 \times 10^{-1}</td>
<td>0.0756</td>
</tr>
<tr>
<td>9</td>
<td>-0.1870 \times 10^{-1}</td>
<td>0.1137</td>
</tr>
<tr>
<td>10</td>
<td>-0.7056 \times 10^{-1}</td>
<td>1.0527</td>
</tr>
<tr>
<td>11</td>
<td>-0.1864 \times 10^{0}</td>
<td>1.3003</td>
</tr>
<tr>
<td>12</td>
<td>-0.2836 \times 10^{0}</td>
<td>0.0247</td>
</tr>
<tr>
<td>13</td>
<td>-0.1220 \times 10^{1}</td>
<td>3.2938</td>
</tr>
<tr>
<td>14</td>
<td>-0.3677 \times 10^{1}</td>
<td>6.4808</td>
</tr>
</tbody>
</table>

All calculations used the following group velocities:

\[ v_1 = 4.06 \times 10^6 \text{ cm/sec} \]
\[ v_2 = 2.48 \times 10^5 \text{ cm/sec} \]
This was determined by a least squares fit to be $17.61 \pm 0.672$ where $0.672$ is the standard deviation. The eigenvalue, $\omega_{lp}$, was determined as

$$\omega_{lp} = -\sqrt{\frac{K}{\left| w_{lp} \right|^2}} = -\sqrt{\frac{17.61 \pm 0.672}{0.547 \pm 0.017}} = -5.68 \pm 0.20$$

Thus the quoted error represents the propagation of standard deviations.

This experimental estimate is in disagreement with the theoretical value of $-6.473$. The source of this disagreement is unknown, but the theoretical eigenvalue is quite sensitive to the thermal group parameters used in its calculation.

The nuclear parameters which were used in this calculation are listed in Tables 4.9 and 4.10. For the most part they were estimated from information provided by Moen and Haugest[48]. This reference did not provide estimates of group velocities. However, it is known by the author that this reference used the computer program, BIGG[50], which calculates averaged parameters for the thermal group. Thus it was considered appropriate to use the average velocity[51] of a Maxwellian distribution at $293^\circ$K for the thermal group velocity. That is,

$$v_2 = v_0 \frac{2}{\sqrt{\pi}}$$

where $v_0 = 2.2 \times 10^5$ cm/sec, the most probable velocity of a Maxwellian distribution at $293^\circ$K. The fast velocity was determined arbitrarily from the following relationship:
\[
\frac{\int_{v_t}^{v_1} v n(v) dv}{\int_{v_t}^{v_0} n(v) dv} = 4.06 \times 10^6 \text{ cm/sec},
\]

where \( n(v) = n_0/v^2 \). The lower limit, \( v_t \), was chosen as the velocity which corresponds to an energy cutoff of 5kT and the upper limit, \( v_o \), was chosen as the velocity which corresponds to 2 Mev, the average energy of the fission spectrum.

Theory and experiment could be forced to agree by using smaller group velocities. For example, if \( v_1 \) were chosen as \( 2.0 \times 10^6 \) cm/sec and \( v_2 \) were chosen as \( v_2 = v_o = 2.2 \times 10^5 \) cm/sec, then the theoretical eigenvalue would be -5.778 which is in good agreement with the experimental value of -5.68 \pm 0.20. Table 4.12 shows the sensitivity of the calculated eigenvalue, \( \omega_{1p} \), to variations in the group velocities and to variations in the number of delayed neutron groups considered. Note that \( \omega_{1p} \) is very sensitive to the choice of \( v_2 \) and only slightly sensitive to the choice of \( v_1 \). Note also that delayed precursor contributions cannot be neglected in this case.
Table 4.12

Prompt Thermal Neutron Eigenvalue Calculations
for the NORA Reactor as a Function of the Neutron Group Velocities
and the Number of Delayed Precursor Groups

<table>
<thead>
<tr>
<th>$v_1$ cm/sec</th>
<th>$v_2$ cm/sec</th>
<th>Number of Delayed Precursor Groups</th>
<th>$\omega_{1p}$ sec$^{-1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Variation of $v_1$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$4.06 \times 10^6$</td>
<td>$2.48 \times 10^5$</td>
<td>14</td>
<td>$-6.473$</td>
</tr>
<tr>
<td>$2.0 \times 10^6$</td>
<td></td>
<td></td>
<td>$-6.380$</td>
</tr>
<tr>
<td>Variation of $v_1$ and $v_2$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$4.06 \times 10^6$</td>
<td>$2.48 \times 10^5$</td>
<td>14</td>
<td>$-6.473$</td>
</tr>
<tr>
<td>$2.0 \times 10^6$</td>
<td>$2.2 \times 10^5$</td>
<td></td>
<td>$-5.778$</td>
</tr>
<tr>
<td>Variation of Number of Precursor Groups</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$4.06 \times 10^6$</td>
<td>$2.48 \times 10^5$</td>
<td>14</td>
<td>$-6.473$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0</td>
<td>$-5.913$</td>
</tr>
</tbody>
</table>
CHAPTER 5

EXAMPLE CALCULATIONS USING THE NATURAL MODE APPROXIMATION

5.1 Introduction

The purpose of this chapter is to give results of space dependent kinetic calculations obtained with the aid of the Natural Mode Approximation. First, in Section 5.2, the asymptotic period and the asymptotic, spatial shape of the neutron density are calculated for localized, step perturbations applied to one-dimensional reactors without feedback. The results of this study lead to conclusions about the number of modes required for accurate representation of a transient and about the importance of the eigenvalue spectrum in characterizing space dependent kinetic behavior. Feedback is introduced in Section 5.3 where the Natural Mode Approximation is applied to the study of power excursion behavior.

5.2 Localized, Step Perturbations Applied to Reactors without Feedback

According to the Natural Mode Approximation, the space and time dependent neutron group densities are given by the expansion,

\[
\begin{align*}
\left\{ N^{(1)}(x,t) \right\} & = \sum_{m=1}^{M} \sum_{k=1}^{K} A_{mk}(t) \left\{ \frac{N^{(1)}_{mk}(x)}{N^{(G)}_{mk}(x)} \right\} , \\
\left\{ \frac{N^{(G)}(x,t)}{N^{(G)}(x)} \right\} & = \sum_{m=1}^{M} \sum_{k=1}^{K} A_{mk}(t) \left\{ \frac{N^{(1)}_{mk}(x)}{N^{(G)}_{mk}(x)} \right\}
\end{align*}
\]

(5.1)

where \( N^{(G)}_{mk}(x) \) is the \( g \)th group component of the \( mk \)th natural mode. For a source free reactor without feedback the \( A_{mk} \) are determined by solving Eq. (2.31) which is repeated here for convenience:
\[ A = \text{diag}[\omega]A + [P]A, \quad t \geq 0. \quad (5.2) \]

For \( t < 0 \) the perturbation matrix, \([P]\), is zero and the initial density distributions in the critical reactor are given by the components of
\[ \psi_{11} = \text{col}[N_{11}^{(1)}, \ldots, N_{11}^{(G)}, C_{11}^{(1)}, \ldots, C_{11}^{(I)}], \quad \text{where } \omega_{11} = 0. \]

By restricting \([P]\) to be independent of time for \( t > 0 \), the \( A_{nk}(t) \) eventually have a time behavior of \( e^{t/T} \) where \( T \) is called the asymptotic period. When this asymptotic behavior is achieved the shape of the neutron densities remains fixed in what is called the asymptotic shape. This shape may be calculated from Eq. (5.1) if the \( A_{nk} \) are known. The asymptotic period and the relative magnitudes of the \( A_{nk} \) may be calculated from Eq. (5.2) by means of the iteration scheme developed in Section 3.4. Such calculations have been made for a number of one-dimensional reactor examples. In all examples two neutron groups and one precursor group are considered.

The perturbation in each example is a removal of thermal neutron absorber from a localized region of a slab reactor of extrapolated width, \( a \). The magnitude of the perturbation is given in terms of a perturbation parameter, \( PP \), which is defined as

\[ PP = \frac{1}{\omega_{lp}} \frac{\left< N_{11}^{*(2)} \nu_2 \Sigma a_2(x) N_{11}^{(2)} \right>}{\left< N_{11}^{*(1)} N_{11}^{(1)} \right> + \left< N_{11}^{*(2)} N_{11}^{(2)} \right>}. \quad (5.3) \]

The neutron group densities, \( N_{11}^{(1)} \) and \( N_{11}^{(2)} \), and the adjoint densities, \( N_{11}^{*(1)} \) and \( N_{11}^{*(2)} \), are initial distributions. This perturbation parameter is equal to the reactivity (in dollar units) that would be calculated in a fundamental mode, space independent kinetics approximation.
The amount of tilting which occurs in each example is reported in terms of a tilt index, $\tau$, which is defined as

$$\tau = \frac{\sum_{k=1}^{K} A_{2k}(t)}{\sum_{k=1}^{K} A_{1k}(t)} ,$$  \hspace{1cm} (5.4)

with the condition that the thermal neutron components of $\psi_{2k}(x)$ and $\psi_{1k}(x)$ are normalized as follows: $N_{1k}(2)(a/2) = N_{2k}(2)(a/4) = 1.0$, for $k = 1, \ldots, K$. The tilt index is a measure of the ratio of the amplitude of the second spatial harmonic to the amplitude of the first (fundamental) spatial harmonic.

The first series of example calculations is for the purpose of illustrating that distortion or tilting of the neutron density shape following a localized perturbation is a sensitive function of the normalized spectrum of prompt thermal neutron eigenvalues and the magnitude of the perturbation. Reactors I, II, and III of Fig. 3.4 are made supercritical by a step removal of thermal neutron absorber in the region $0 < x < a/4$. Reactor I is uniform with $a = 60$ cm, Reactor II is uniform with $a = 240$ cm and Reactor III is a nonuniform, decoupled version of Reactor II. Figure 5.1 shows the periods, tilt indices and the normalized, asymptotic shapes of the thermal neutron density in each reactor for a case where $PP = 0.75$. The shapes and periods have been found by means of both direct computation and a Natural Mode Approximation which retained five ($M = 5$) spatial harmonics. Only the precursor mode ($k = 1$) and the prompt thermal neutron mode ($k = p = 2$) of each spatial harmonic were considered in
Figure 5.1 Asymptotic Thermal Neutron Density Shapes which Result from Step Perturbations in Three Different Slab Reactors with no Feedback.
the summation over \( k \) (See Eq. (5.1).); the prompt epithermal mode 
\( (k = 3) \) was neglected. Tabulated in Table 5.1 are the prompt thermal neutron eigenvalues of the first five spatial harmonics. Note that even though \( \omega_{lp} \) is approximately the same for all reactors, the \( \omega_{mp} 's \) for \( m > 1 \) are different. The normalized spectrum, \( \omega_{mp}/\omega_{lp} \), is tabulated in Table 5.1. Note that a large amount of tilting is associated with a small ratio, \( \omega_{2p}/\omega_{lp} \). This ratio is considered here to be a single number which is most representative of the difference in eigenvalue spectra.

Figure 5.2 illustrates the results of many calculations for different values of PP. The results are reported in terms of the tilt index, \( \tau \), which is plotted as a function of PP with \( \omega_{2p}/\omega_{lp} \) as a parameter. Note that the distortion or tilting of the neutron density is a sensitive function of both PP and the ratio, \( \omega_{2p}/\omega_{lp} \). Figure 5.3 illustrates the same results plotted as a function of the quantity, 
\[ \mathcal{E} = PP(\omega_{lp}/\omega_{2p}) \]. Note that the results for all three reactors correlate rather well in terms of \( \mathcal{E} \), which will be referred to as the tilt parameter.

The results of the preceding examples lead to the following conclusion which is valid for these examples: The tilt or distortion of the neutron density shape as measured by the tilt index, \( \tau \), resulting from a particular localized perturbation (in the region, \( 0 \leq x \leq a/4 \)) to different reactors may be correlated with the magnitude of the perturbation and with the eigenvalue ratio, \( \omega_{2p}/\omega_{lp} \) by means of the tilt parameter, \( \mathcal{E} = PP(\omega_{lp}/\omega_{2p}) \). In particular, for the specific examples discussed here, the tilt index, \( \tau \), is less than 0.4 when the tilt parameter, \( \mathcal{E} \), is less than 0.1.
Table 5.1
Spectra of Prompt Thermal Neutron Eigenvalues
for Reactors I, II and III

<table>
<thead>
<tr>
<th>Prompt Thermal Neutron Eigenvalue, sec(^{-1})</th>
<th>Reactor</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>I</td>
</tr>
<tr>
<td>(w_{1p})</td>
<td>-429.</td>
</tr>
<tr>
<td>(w_{2p})</td>
<td>-14960.</td>
</tr>
<tr>
<td>(w_{3p})</td>
<td>-29939.</td>
</tr>
<tr>
<td>(w_{4p})</td>
<td>-41502.</td>
</tr>
<tr>
<td>(w_{5p})</td>
<td>-50011.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Normalized Eigenvalue</th>
<th>Reactor</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>I</td>
</tr>
<tr>
<td>(w_{2p}/w_{1p})</td>
<td>34.9</td>
</tr>
<tr>
<td>(w_{3p}/w_{1p})</td>
<td>69.8</td>
</tr>
<tr>
<td>(w_{4p}/w_{1p})</td>
<td>96.9</td>
</tr>
<tr>
<td>(w_{5p}/w_{1p})</td>
<td>116.8</td>
</tr>
</tbody>
</table>
Reactor III, $\frac{\omega_{2p}}{\omega_{1p}} = 2.04$

Reactor II, $\frac{\omega_{2p}}{\omega_{1p}} = 3.73$

Reactor I, $\frac{\omega_{2p}}{\omega_{1p}} = 34.9$

All Calculations Based on Natural Mode Approximation with $\Omega = 5$

All Perturbations in the Region, $\frac{\alpha x}{\alpha} < \frac{\alpha}{4}$

Figure 5.2 Tilt Index as a Function of the Perturbation Parameter with $\frac{\omega_{2p}}{\omega_{1p}}$ as a Parameter
All Calculations Based on the Natural Mode Approximation with $M=5$

All Perturbations in the Region, $0 < x < a/4$

Figure 5.3 Tilt Index as a Function of the Tilt Parameter

Tilt Index, $\tau$

Tilt Parameter, $\epsilon = (P/P)_{l_p}/(P/P)_{l_p}$
The second set of example calculations is for the purpose of illustrating how the number of modes required for an accurate representation of a transient depends on (i) the magnitude of the perturbation parameter, PP, and (ii) the degree of localization of the perturbation. Perturbation I is in the region, \( 0 \leq x \leq a/4 \), and it is considered to have a low degree of localization. Perturbation II is in the region, \( a/8 \leq x \leq a/4 \), and it is considered to have a high degree of localization.

Figure 5.4 illustrates the error in the calculated reciprocal of the asymptotic period as a function of \( M \), the number of spatial harmonics retained when Perturbation I \((0 \leq x \leq a/4)\) is applied to Reactor II. The magnitude of Perturbation I is varied from \( PP = 0.25 \) to \( PP = 1.0 \). Figure 5.4(a) shows the error obtained when epithermal modes are neglected and Fig. 5.4(b) shows the error obtained when epithermal modes are considered. Note that the reciprocal period is calculated to within 10 percent in all cases when five spatial harmonics are retained. Note also that the consideration of epithermal modes decreases the error by about 2 percent.

It is interesting to note that the convergence to the correct period is best for the smallest perturbation, \( PP = 0.25 \); the convergence is worst for the intermediate perturbation, \( PP = 0.5 \); and the convergence is the same for the largest perturbations, \( PP = 0.75 \) and \( PP = 1.0 \). A possible explanation for this behavior may be understood by inspecting these results in another form. Figure 5.5 shows the value of the calculated period as a function
Table 5.4 Percent Error in the Asymptotic Period as a Function of the Number of Modes Retained. Variation of the Magnitude of Perturbation I Applied to Reactor II.

<table>
<thead>
<tr>
<th>Symbol</th>
<th>PP</th>
<th>Actual Period</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.25</td>
<td>13.46</td>
</tr>
<tr>
<td></td>
<td>0.50</td>
<td>0.00624</td>
</tr>
<tr>
<td></td>
<td>0.75</td>
<td>0.001159</td>
</tr>
<tr>
<td></td>
<td>1.0</td>
<td>0.0005705</td>
</tr>
</tbody>
</table>

Epithermal Modes Neglected

Epithermal Modes Considered

Figure 5.4: Percent Error in the Asymptotic Period as a Function of the Number of Modes Retained. Variation of the Magnitude of Perturbation I Applied to Reactor II.
Figure 5.5 Estimated Asymptotic Period as a Function of the Number of Spatial Harmonics Retained. Variation of the Magnitude of Perturbation I to Reactor II.
of $M$ as the perturbation magnitude is varied. First note the behavior of the calculated period as a function of $PP$ when only a fundamental mode description ($M = 1$) is retained. As $PP$ increases from 0.125 to 1.0, the period decreases in a regular fashion. For $PP > 1.0$ the calculated period is of a much different order of magnitude. In the terminology of a fundamental mode description of reactor kinetics, $PP > 1.0$ represents a prompt critical condition which means that the fundamental mode is critical without delayed neutron contributions and the reactor behaves as if there were no delayed neutrons. This is the reason for the transition from one scale of periods to another as a certain perturbation magnitude is exceeded.

Now by inspecting the behavior of the actual periods in Fig. 5.5 it can be seen that $PP \geq 0.5$ represents a prompt critical condition and $PP \leq 0.25$ represents a below prompt critical condition (critical with delayed neutron contributions). For $PP = 0.75$ and 1.0, a two harmonic ($M = 2$) description reveals that the reactor is prompt critical; whereas for $PP = 0.5$, a three harmonic description is necessary before it is recognized that the reactor is prompt critical. It appears that there is a certain range of $PP$ for which many spatial harmonics are required before it is recognized that the reactor is prompt critical; therefore, it is reasonable to expect that convergence to the correct answer may be quite slow in this range of perturbation magnitudes. This is a possible explanation for the poor results shown in Fig. 5.4 for the perturbation of intermediate magnitude, $PP = 0.5$. 
The above reasoning is consistent with the results shown in Fig. 5.6. Figure 5.6 illustrates the error in the calculated reciprocal of the asymptotic period as a function of $M$, the highest order spatial harmonic retained, for Perturbation I ($0 \leq x \leq a/4$) applied to Reactors I, II, and III. Recall that Reactor I is uniform ($a = 60$ cm, $w_2p/w_{lp} = 34.9$), Reactor II is uniform ($a = 240$ cm, $w_2p/w_{lp} = 3.73$), and Reactor III is a nonuniform, decoupled version of Reactor II ($a = 240$ cm, $w_2p/w_{lp} = 2.04$). Epithermal modes are neglected; and the magnitude of the perturbation is $PP = 0.75$.

This example is a further investigation of the transients illustrated in Fig. 5.1. Note that the convergence to the correct reciprocal period is poorest for Reactor I which is least susceptible to neutron density shape changes, and the convergence is best for Reactor III which is most susceptible to neutron density shape changes. Inspection of the actual values of the asymptotic periods tabulated in Fig. 5.6 reveals that a perturbation of $PP = 0.75$ leads to considerably different transients in each reactor; Reactors II and III are prompt critical and Reactor I is almost prompt critical. Thus $PP = 0.75$ is in that range of perturbation magnitudes in Reactor I for which slow convergence to the correct period is observed.

Figure 5.7 illustrates the error in the calculated reciprocal of the asymptotic period as the degree of localization of the perturbation is changed. Figure 5.7(a) shows results obtained for Perturbations I ($0 \leq x \leq a/4$) and II ($a/8 \leq x \leq a/4$) applied to Reactor I ($a = 60$ cm); and Fig. 5.7(b) shows results obtained for
Figure 5.6. Percent Error in the Estimated Asymptotic Period as a Function of the Number of Spatial Harmonics Retained. Perturbation I Applied to Reactors I, II and III.
Figure 5.7 Percent Error in the Estimated Asymptotic Period as a Function of the Number of Spatial Harmonics Retained. Perturbations I and II Applied to Reactors I and II.
Perturbations I and II applied to Reactor II \((a = 240 \text{ cm})\). Epithermal modes are included and the magnitude of the perturbation in all cases is \(PP = 0.75\). Note that for Perturbation II (high degree of localization) at least eight spatial harmonics must be retained in order to calculate the reciprocal period to within 10 percent, whereas for Perturbation I (lower degree of localization) only five spatial harmonics must be retained. Note also that the convergence to the correct period is better for Reactor II than it is for Reactor I. This is consistent with previous observations about the suitability of the NMA for very fast transients. The transient in Reactor II is above prompt critical; whereas the transient in Reactor I is on the order of prompt critical.

The preceding series of examples leads to the following conclusions which are valid for these examples: 

(i) the NMA works best for transients in the delayed critical regime or for very fast (super prompt critical) transients; the NMA does not work as well for transients which are on the order of prompt critical. 

(ii) For an accurate representation \((T \text{ to within } 10 \text{ percent})\) of a transient initiated by a perturbation of width \(\Delta x\), at least \((M + 1)\) modes must be retained where \(M\) is in the order of the spatial harmonic whose period of oscillation, \(\lambda_M\) (twice the average distance between zero crossings), is less than \(2\Delta x\).
5.3 Power Excursion Behavior in One-Dimensional Reactors

This section presents the results of a calculation of a power excursion in a reactor which has an inherent shutdown mechanism (negative feedback). The calculation has been performed by Radd[52], who used a version of WIGLE[8], a code which solves directly the time and space dependent, multigroup diffusion equations in one-dimensional systems. This same calculation is performed here with the NMA.

The calculational advantages of using the NMA are: (i) it involves less computer time if many problems are to be solved for the same reactor with different types of feedback, and (ii) it is extendable to multidimensional systems. The disadvantage of using the NMA is that it involves the additional problem of computing the natural modes.

In this example a nonuniform slab reactor of 60 cm thickness is perturbed by a stepwise increase in the fission cross sections, $\nu \Sigma_{f1}$ and $\nu \Sigma_{f2}$, in the region, $0 \leq x \leq 15$ cm. Two neutron groups and one delayed precursor group are considered. Shutdown is obtained by changing the thermal absorption cross section in each region in proportion to the time- and volume-integral of the thermal neutron density in each region; that is,

$$\Sigma_{a2}(\text{Region } i, t) = \Sigma_{a2o}(\text{Region } i) + \nu_{1} E_{1}(t) ,$$

where

$$E_{1}(t) \equiv \nu_{2} \int_{\text{Region } i} dx' \int_{0}^{t} dt' N^{(2)}(x', t') ,$$
and $\gamma_1$ is the feedback coefficient in the $i^{th}$ region. A description of the reactor considered in this example is tabulated in Table 5.2. The equations which describe this one-dimensional system are

$$\frac{\partial \xi(x,t)}{\partial t} = [H_0]\xi(x,t) + [h]\xi(x,t) + \xi,$$  \hspace{1cm} (5.5)

where

$$[H_0] = \begin{bmatrix}
  v_1 \frac{\partial}{\partial x} D_1 \frac{\partial}{\partial x} - \alpha_1 & v_2 \nu \Sigma f_2 (1-\beta) & \lambda \\
  v_1 \Sigma r & v_2 \frac{\partial}{\partial x} D_2 \frac{\partial}{\partial x} - \alpha_2 & 0 \\
  v_1 \nu \Sigma r f_1^\beta & v_2 \nu \Sigma r f_2^\beta & -\lambda
\end{bmatrix}, \hspace{1cm} (5.6)$$

$$\alpha_1 \equiv v_1 (\Sigma a_1 + \Sigma r - \nu \Sigma f_1 (1-\beta) + D_1 B_T^2),$$

$$\alpha_2 \equiv v_2 (\Sigma a_2 + D_2 B_T^2),$$

$$[h] \equiv \begin{bmatrix}
  v_1 \nu \delta \Sigma f_1 (1-\beta) & v_2 \nu \delta \Sigma f_2 (1-\beta) & 0 \\
  0 & 0 & 0 \\
  v_1 \nu \delta \Sigma r f_1^\beta & v_2 \nu \delta \Sigma r f_2^\beta & 0
\end{bmatrix} \hspace{1cm} (t \geq 0), \hspace{1cm} (5.7)$$

$$\xi(x,t) \equiv \text{col}[N^{(1)}(x,t), N^{(2)}(x,t), C(x,t)] \hspace{1cm} (5.8)$$

and

$$\xi \equiv \text{col}[0, -v_2 \sum_{i=1}^{3} \Delta_i \gamma_i E_i(t) N^{(2)}(x,t), 0]. \hspace{1cm} (5.9)$$

The quantity, $\Delta_1$, is introduced to denote the discontinuous nature of the feedback in this example. This quantity has the properties:
Table 5.2
Summary of the Reactor System Considered in the Power Excursion Calculation

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Region 1</th>
<th>Region 2</th>
<th>Region 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>width, cm</td>
<td>0 to 15 cm</td>
<td>15 to 45 cm</td>
<td>45 to 60 cm</td>
</tr>
<tr>
<td>(D_1, \text{cm})</td>
<td></td>
<td>1.69531</td>
<td></td>
</tr>
<tr>
<td>(D_2, \text{cm})</td>
<td></td>
<td>0.409718</td>
<td></td>
</tr>
<tr>
<td>(\Sigma_x, \text{cm}^{-1})</td>
<td></td>
<td>0.016444</td>
<td></td>
</tr>
<tr>
<td>(\Sigma_{a1}, \text{cm}^{-1})</td>
<td>0.0322302</td>
<td>0.0340950</td>
<td>0.0322302</td>
</tr>
<tr>
<td>(\Sigma_{a2}, \text{cm}^{-1})</td>
<td>0.265827</td>
<td>0.266278</td>
<td>0.265827</td>
</tr>
<tr>
<td>(\nu \Sigma_{f1}, \text{cm}^{-1}(t&lt;0))</td>
<td>0.0213296</td>
<td>0.0213296</td>
<td>0.0213296</td>
</tr>
<tr>
<td>(\nu \Sigma_{f1}, \text{cm}^{-1}(t&gt;0))</td>
<td>0.025733</td>
<td>0.0213296</td>
<td>0.0213296</td>
</tr>
<tr>
<td>(\nu \Sigma_{f2}, \text{cm}^{-1}(t&lt;0))</td>
<td>0.544678</td>
<td>0.544678</td>
<td>0.544678</td>
</tr>
<tr>
<td>(\nu \Sigma_{f2}, \text{cm}^{-1}(t&gt;0))</td>
<td>0.65712</td>
<td>0.544678</td>
<td>0.544678</td>
</tr>
<tr>
<td>(\gamma_1)</td>
<td>6.515x10^{-6}</td>
<td>2.7x10^{-6}</td>
<td>5.4x10^{-6}</td>
</tr>
</tbody>
</table>
\[ \Delta_i = 1 \text{ for } x \text{ in Region } i, \text{ and } \Delta_i = 0, \text{ otherwise.} \] There is no difficulty in treating a continuously varying feedback with the NMA. However, this example is meant to be identical to the example calculated by WIGLE.

The solution may be expanded in terms of the natural modes of the reference condition described by the operator, \([H_0]\). That is,

\[ \psi(x, t) = \sum_{m=1}^{M} \sum_{k=1}^{3} A_{mk}(t) \psi_{mk}(x), \quad (5.10) \]

where \([H_0]\psi_{mk} = \omega_{mk} \psi_{mk}\). The subscripts, \(k = 1, 2, \text{ and } 3\), denote the delayed neutron, prompt thermal neutron and prompt epithermal neutron modes, respectively. The usual procedure of substituting Eq. (5.10) into Eq. (5.5), weighting with \(3M\) vectors, \(\psi_{mk}^*\), and integrating leads to the equations

\[ \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} = \begin{bmatrix} \text{diag}[W_1] & [0] & [0] \\ [0] & \text{diag}[W_2] & [0] \\ [0] & [0] & \text{diag}[W_3] \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} + \begin{bmatrix} [p_{11}] [p_{12}] [p_{13}] \\ [p_{21}] [p_{22}] [p_{23}] \\ [p_{31}] [p_{32}] [p_{33}] \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}, \quad (5.11) \]

where

\[ A_k = \text{col} [A_{1k}, \ldots, A_{Mk}], \]

\[ \text{diag}[W_k] = \begin{bmatrix} \omega_{1k} & 0 & \ldots & 0 \\ 0 & \omega_{2k} & \ldots & \vdots \\ 0 & \ldots & \omega_{Mk} \end{bmatrix}; \]
and

$$[p_{rs}] = \text{an } M \text{ by } M \text{ matrix with elements, } P_{mr,ns},$$

with

$$P_{mr,ns} = \frac{\langle \varphi^*_{mr}, [h] \varphi_{ns} \rangle}{\langle \varphi^*_{mr}, \varphi_{mr} \rangle}, \text{ for } m = 1, \ldots, M; \text{ and } n = 1, \ldots, M.$$

The vector, $f_k,$ is an $M$ component vector with elements

$$f_{mk} = \sum_{n=1}^{M} \sum_{j=1}^{K} \sum_{\ell=1}^{M} \sum_{i=1}^{K} q_{mk,nj,\ell\mu} A_{n\ell} \int_{0}^{t} A_{\ell\mu}(t') dt', \quad (5.12)$$

where

$$q_{mk,nj,\ell\mu} = \frac{v_2^2 \sum_{i=1}^{3} \Delta_i \chi_i \langle N^{(2)}_{\mu}(x) \rangle_{\text{Region } i} \langle N^{* (2)}_{mk,nj} \rangle_{\text{Region } i}}{\langle \varphi^*_{mk}, \varphi_{mk} \rangle}. \quad (5.13)$$

The initial conditions for Eqs. (5.11) in an initially critical reactor are: $A_1(0) = \text{col}[1,0, \ldots, 0]$, $A_2(0) = \text{col}[0, \ldots, 0]$, and $A_3(0) = \text{col}[0, \ldots, 0]$. Equations (5.11) are a set of $3M$ nonlinear, ordinary differential equations which may be solved numerically with the aid of a digital computer. However, by considering the solutions in various time ranges, this system of $3M$ equations can be effectively reduced to three separate problems, each involving only $M$ equations\[63].

This reduction is possible because of the fact that the eigenvalues, $\omega_{mk}$, fall into three distinct clusters of different magnitudes. For example, in the time range where $t < 1/|\omega_{m1}|$, the variation of $A_1$ (which contains the expansion coefficients of the delayed neutron
modes) is small. Thus, in this time range, $A_1(t)$ may be approximated by $A_1(0)$. Similarly, in a time range where $t > 1/|\omega_m|$, the time derivatives, $A_3$, of the expansion coefficients of the epithermal modes are negligible in comparison with the terms, $\text{diag}[W_3]A_3$. With the preceding approximations, Eqs. (5.11) reduces to

$$
\begin{align*}
\frac{d}{dt} \begin{bmatrix} A_2 \\ 0 \\
\end{bmatrix} &= \begin{bmatrix} \text{diag}[W_2] & [0] \\
[0] & \text{diag}[W_3] \\
\end{bmatrix} \begin{bmatrix} A_2 \\ A_3 \\
\end{bmatrix} + \begin{bmatrix} [p_{21}]A_1(0) \\
[p_{31}]A_1(0) \\
\end{bmatrix} + \begin{bmatrix} f_2 \\
f_3 \\
\end{bmatrix} \\
&+ \begin{bmatrix} [p_{22}][p_{23}] \\
[p_{32}][p_{33}] \\
\end{bmatrix} \begin{bmatrix} A_2 \\ A_3 \\
\end{bmatrix}.
\end{align*}
$$

(5.14)

The only non-negligible terms of the summation, Eq. (5.12), which define the components of the vectors, $f_k$, are those terms which contain products of the expansion coefficients, $A_{m2}$. Thus the components of the $f_k$ may be written as

$$
f_{mk} \sim \sum_{n=1}^{M} \sum_{\ell=1}^{M} q_{m2,n2\ell2} A_{n2} \int_{0}^{t} A_{2\ell}(t') dt'.
$$

(5.14a)

Now, $A_3$ appears only in an algebraic manner in the second equation of Eqs. (5.14). Hence for cases where the $[p_{ij}]$ are independent of time, $A_3$ may be eliminated and Eqs. (5.14) may be written as

$$
\frac{d}{dt} \begin{bmatrix} A_2 \\
\end{bmatrix} = \begin{bmatrix} \text{diag}[W_2] + [p_{22}] - [u][p_{23}] \\
[p_{21}] - [u][p_{31}] \\
\end{bmatrix} A_1(0) + f_2 - uf_3,
$$

(5.15)

where

$$
[u] = [p_{23}]^{-1}\left[\text{diag}[W_3] + [p_{33}]\right].
$$
Equation 5.15 is a set of only $M$ nonlinear, ordinary differential equations which may be solved by any convenient numerical integration scheme. For this example calculation, Eq. (5.15) was integrated by means of the subprogram, DPNV[53], one of a library of subprograms built into the computer system at M.I.T. This subprogram obtains a numerical solution by using an Adams four point integration formula. A time increment of $10^{-6}$ seconds gave a numerically stable solution. This time increment is not considered to be optimum.

Results obtained by solving Eq. (5.15) for $M = 3$ and $M = 4$ are illustrated in Fig. 5.8. The result for $M = 4$ is in good agreement with the result obtained by using WIGLE. The result for $M = 3$ is quite poor. The fact that at least four spatial harmonics are necessary for a good representation is compatible with the conclusions of Section 5.2 concerning the number of spatial harmonics required in relation to the degree of localization of the perturbation.

The total computation time* required for calculating a single problem with the NMA is the sum of:

i) the time required for computing the natural modes with MUDMO-II [average time for the one-dimensional modes of this example = 0.25 minutes/mode],

ii) the time required for calculating the integrals of the natural modes which appear as matrix elements in Eq. (5.15) [not timed, but certainly less than 0.5 minutes], and

*All computations performed on an IBM 7094.
Figure 5.8 Volume Integral of the Time Dependent Thermal Neutron Density in a Power Excursion Calculation.
iii) the time required for numerically integrating Eq. (5.15)
[1.29 minutes for integrating the $M = 3$ example from 0 to 15 msec,
1.72 minutes for integrating the $M = 4$ example].

Thus the total time used for the $M = 3$ example was approximately

$$\text{prompt} \left( 1 - \frac{\text{thermal mode}}{\text{spatial harmonic}} + 1 - \frac{\text{epithermal mode}}{\text{spatial harmonic}} \right) \left( 3 \text{ harmonics} \right) \left( 0.25 \text{ minutes per mode} \right)$$

$$+ 0.5 \text{ minutes} + 1.29 \text{ minutes} = 3.29 \text{ minutes.}$$

The total time used for the $M = 4$ example was approximately

$$(2)(4)(0.25) \text{ minutes} +$$

$$+ 0.5 \text{ minutes} + 1.72 \text{ minutes} = 4.22 \text{ minutes.}$$

The total time used by WIGLE with six groups of delayed neutrons in
a calculation$^{[52]}$ from 0 to 18 msec was 4.8 minutes; scaled to a time
range of 0 to 15 msec, the time required would be approximately 4.17
minutes. (In fairness to WIGLE, it should be noted that a WIGLE
calculation without delayed neutrons would yield essentially the same
results for this example problem; a timing formula in Ref. 52
indicates that the WIGLE calculation without delayed neutrons would
required only 2.8 minutes). The conclusion to be drawn from this
comparison of computation times is that the NMA offers greatest
advantage when many calculations are being performed for the same
reactor and the natural modes need be calculated only once. Then
each problem after the first would require only 1.72 minutes of
computation time if the NMA with $M = 4$ were used.
An improvement in the results of the NMA can be obtained by using a novel idea called "correction modes." Correction modes are approximations to higher order spatial harmonics beyond the Mth which are included in an algebraic manner, i.e., in a manner which does not introduce additional differential equations. The motivation for the introduction of correction modes is based on the following observations:

1) A small error in the initial asymptotic period, T, leads to a large error in the power (which goes initially as $e^{t/T}$) if $t >> T$. In Table 5.3 the initial asymptotic periods found by various calculations are tabulated along with the percent error in the power found by these calculations. Note that 5.2 percent error in the initial asymptotic period gives a 55 percent error in the power after 10 msec.

ii) The NMA overestimates the period; it converges to the correct period from above as more spatial harmonics are retained.

Therefore, the poor result shown in Fig. 5.3 for $M = 3$ is due to the fact that the initial asymptotic period is overestimated by too great a percentage. A better estimate for the initial asymptotic period would be obtained if the expansion contained additional spatial harmonics up to, say, the $N^{th}$, where $N > M$. That is, the expansion is considered to be

$$\hat{y}(x,t) = \sum_{m=1}^{M} \sum_{k=1}^{3} A_{mk}(t)\psi_{mk}(x) + \sum_{m=M+1}^{N} \sum_{k=1}^{3} A_{mk}(t)\psi_{mk}(x), \quad (5.16)$$
Table 5.3
Comparison of Calculations of the Asymptotic Period and the Power Using Different Approximations

<table>
<thead>
<tr>
<th>Calculation</th>
<th>Asymptotic Period (T,\text{msec})</th>
<th>Percent Error from WIGLE</th>
<th>Power at (t=10) msec (Arbitrary Units)</th>
<th>Percent Error from WIGLE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Direct Calculation of the Largest Eigenvalue of the Perturbed Reactor by means of MUDMO-II WIGLE[52]</td>
<td>0.506</td>
<td></td>
<td>3.77</td>
<td></td>
</tr>
<tr>
<td>Natural Mode Approximation:</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Eq. (5.15), (M=3)</td>
<td>0.532</td>
<td>5.2 o/o</td>
<td>1.70</td>
<td>55.0/o</td>
</tr>
<tr>
<td>Eq. (5.15), (M=4)</td>
<td>0.509</td>
<td>1.0/o</td>
<td>3.15</td>
<td>16.5 o/o</td>
</tr>
<tr>
<td>Eq. (5.17), (M=3)</td>
<td>0.505</td>
<td>negligible</td>
<td>3.55</td>
<td>&lt; 6 o/o</td>
</tr>
<tr>
<td>Eq. (5.17), (M=4)</td>
<td>0.505</td>
<td>negligible</td>
<td>3.55</td>
<td>&lt; 6 o/o</td>
</tr>
</tbody>
</table>
instead of Eq. (5.10). If the additional $3(N-M)$ modes are included in the usual way, there is nothing novel about the procedure. It is the following simplifications which allow the additional modes to be included in an algebraic manner:

i) Consider that spatial harmonics beyond the $M^{th}$, because of their small magnitude, do not make appreciable nonlinear contributions; hence, additional terms beyond the $M^{th}$ introduced into Eq. (5.14a) are neglected.

ii) Consider that the $w_{m2}'s$ for $m > M$ are much greater in magnitude than $T^{-1}$; hence, time derivatives, $A_{m2}$, are neglected in comparison with $w_{m2}A_{m2}$ for $m > M$.

Now, if one proceeds as follows:

(i) substitute Eq. (5.16) into Eq. (5.5);

(ii) neglect the time behavior of delayed neutron modes;

(iii) weight by $2N$ vectors, $\tilde{\psi}_{mk}$, where $k = 2, 3$;

(iv) integrate all equations over the volume of the reactor;

(v) neglect all nonlinear terms for $m > M$, and $k \neq 2$;

(vi) set all $A_{mk} = 0$ for $m > M$, and $k \neq 2$;

(vii) eliminate all $A_{m2}$ for $m > M$, and all $A_{m3}$ by the same procedure that was used to obtain Eq. (5.15) from Eq. (5.14);

one derives the system of $M$ equations,

$$\dot{A}_2 = \left[\text{diag}[W_2] + [p_{22}] - [u'][\beta_{32}]A_2 + \left[[p_{21}] - [u'][\beta_{31}]\right]A_1(0)\right] (5.17)$$

$$+ \dot{f}_2 - [u]f_3.$$
Equation (5.17) is like Eq. (5.15) except that now the matrices, \([\mathbf{u}']\), \([\mathbf{\beta}_{31}]\) and \([\mathbf{\beta}_{32}]\), include elements which contain weighted integrals of higher order spatial harmonics.

The above procedure becomes particularly attractive if the natural modes are easy to find for spatial harmonics beyond the \(M\)th. For the reactor of this example it was observed that as \(m\) increased the spatial shapes of the natural modes of the nonuniform reactor approached the spatial shapes of the natural modes of a uniform reactor (i.e., sinusoids). In addition it was observed that the spectra of the prompt thermal neutron eigenvalues of the nonuniform reactor and of a uniform version of the reactor became quite similar as \(m\) increased. Hence the correction modes for this example are taken as the natural modes of a uniform version of the critical, nonuniform reactor whose properties are summarized in Table 5.2.

Results obtained by solving Eq. (5.17) for \(M = 3\) and \(M = 4\) are illustrated in the preceding figure, Fig. 5.8. In each case additional spatial harmonics up to and including the eighth were considered. Note that the results for \(M = 3\) and \(M = 4\) (which cannot be distinguished in Fig. 5.8) both agree quite well with the WIGLE result. Table 5.3 shows that the initial asymptotic periods obtained by using the idea of correction modes agree quite well with the WIGLE result. The computer time used to obtain the NMA results with correction modes was essentially the same as that used to obtain the NMA results without correction modes.

Figure 5.9 shows the space dependent thermal neutron density at \(t = 10\) msec as calculated by the various approximations discussed here.
Thermal Neutron Density Distribution at Time, t=10 milliseconds in a Power Excursion Calculation

Number of Spatial Harmonics Retained Indicated on Curve

WIGLE Result
- Prompt Thermal, Prompt Epithermal and "Correction" Modes
- Prompt Thermal and Prompt Epithermal Modes

Position, x - centimeters

Figure 5.9 Space Dependent Thermal Neutron Density in a Power Excursion Calculation.
Note that the results obtained by using correction modes are very good in comparison with the WIGLE result. Note that the result obtained without correction modes for $M = 4$ is quite good considering the fact that the average neutron density has increased by more than a factor of $3 \times 10^8$ during the course of the transient.
CHAPTER 6

INTERPRETATION OF EXPERIMENTS IN TERMS OF NATURAL MODE EXPANSIONS

6.1 Introduction

The purpose of this chapter is to illustrate how a natural mode expansion can be used for the interpretation of: (i) oscillation tests designed to infer subcriticality of the fundamental spatial harmonic, (ii) oscillation tests designed to infer asymptotic stability of the fundamental spatial harmonic, and (iii) reactor excursion tests designed to measure shutdown coefficients. When spatial dynamics effects are important the results of these tests may be grossly in error if only a single neutron detector is used. The error may be reduced through use of $M$ detectors at different locations and analysis of these detector readings in terms of $M$ appropriate modes of a natural mode expansion.

To make the preceding statements clearer consider that the reading, $R_i(t)$, of the $i$th detector may be given to a good approximation by

$$R_i(t) = \left< \sigma_i^T \psi(x,t) \right> \approx \sum_{m=1}^{M} A_{ms}(t) \left< \sigma_i^T \psi_{ms}(x) \right> , \quad (6.1)$$

where $\sigma_i$ is the response function of the $i$th detector, $A_{ms}(t)$ is an unknown expansion coefficient, and $\psi_{ms}$ is the predominant natural mode of the $K$ modes of the $m$th spatial harmonic. If $M$ detectors at different locations are observed, Eqs. (6.1) form a system of $M$ equations,
\[ R = [U][A] , \] (6.2)

where \( R = \text{col}[R_1, \ldots, R_M] \), \( A = \text{col}[A_{1r}, \ldots, A_{Mr}] \) and \([U]\) is a matrix with elements, \( U_{im} = \langle \xi_i^T, \psi_{ms} \rangle \). Assuming that \([U]\) is non-singular allows estimates of the expansion coefficients in terms of the detector readings; that is,

\[ A^{\text{est}} = [U]^{-1} R. \] (6.3)

The estimated expansion coefficients may be of interest themselves, or quantities which may be constructed from the expansion coefficients may be of interest. For example, an experimental estimate of the power distribution which is continuous in both space and time would be given by

\[
\text{Power} (x,t) \approx \sum_{m=1}^{M} \langle \xi_f, \psi_{ms} \rangle A_{ms}^{\text{est}}(t) =
\]

\[
= \xi^T A^{\text{est}} = \xi^T [U]^{-1} R ,
\] (6.4)

where

\[ \xi \equiv \text{col}[\langle \xi_f, \psi_{1s} \rangle, \ldots, \langle \xi_f, \psi_{Ms} \rangle] , \]

and

\[ \Sigma_f \equiv \text{col}[\Sigma_{f1}, \Sigma_{f2}, \ldots, \Sigma_{fG}] . \]

The number of detectors required and the best locations for these detectors cannot be specified in the general case. In the examples which follow, three detectors are placed at the zeros of the fourth spatial harmonic. The emphasis in the examples is not upon the
details of the experiment and the concepts involved but rather it is upon the proper consideration of spatial effects.

6.2 **Subcriticality Measurements**

Subcriticality of the fundamental spatial harmonic is related[^21] to the prompt thermal neutron eigenvalue, \( w_{lp} \), which is often inferred from an oscillation test. Consider the following experiment which is designed to measure this eigenvalue in a large, subcritical reactor.

Reactor II of Fig. 3.4 is made subcritical by a uniform adjustment of the thermal absorption cross section. Figure 6.1 shows the computed responses of three plane neutron detectors A, B and C located at \( x = a/4, a/2 \) and \( 3a/4 \), to oscillations induced by a plane, oscillating thermal absorber located at \( x_o = a/6 \). These responses are found by evaluating Eq. (4.16). Included in the figure are also the eigenvalues, \( w_{lp} \), derived by considering each response individually or analyzing all responses simultaneously for \( A_{lp}(j\omega) = c_{lp}/(j\omega - w_{lp}) \). The data clearly indicate the errors which may be introduced by spatial effects in subcriticality measurements and how these errors may be reduced through the use of a natural mode expansion.

6.3 **Stability Measurements in Power Reactors**

Small signal oscillation tests are often performed on reactors at high power for the purpose of estimating the power level at which the reactor will become asymptotically unstable[^54,55]. Consider such an oscillation test on a high power version of Reactor II (\( a = 240 \) cm) of Fig. 3.4. Let this reactor be described in terms of two neutron
Figure 6.1 Comparison of the Magnitude of Actual Fundamental Mode Coefficient with Inferred Behavior. Oscillation Test on Subcritical Reactor.
groups, one delayed precursor group and two temperatures, $T^{(1)}$ and $T^{(2)}$. It is assumed that $\Sigma_{a1}$ and $\Sigma_{a2}$ depend upon $T^{(1)}$ and $T^{(2)}$ in the following way:

\[
v_{2} \Sigma_{a2} N^{(2)}(x,t) = \Sigma_{a20} v_{2} N^{(2)}(x,t) + \gamma_{2} v_{2} N_{0}^{(2)}(x) \phi^{(2)}(x,t),
\]

\[
v_{1} \Sigma_{a1} N^{(1)}(x,t) = \Sigma_{a10} v_{1} N^{(1)}(x,t) + \gamma_{1} v_{1} N_{0}^{(1)}(x) \phi^{(1)}(x,t),
\]

where

\[
\phi^{(1)}(x,t) = T^{(1)}(x,t) - T_{0}^{(1)}(x),
\]

and

\[
\phi^{(2)}(x,t) = T^{(2)}(x,t) - T_{0}^{(2)}(x).
\]

The equation which describes this small signal oscillation test is considered to be

\[
\frac{\partial \phi(x,t)}{\partial t} = [I_{0} \phi(x,t)] + [L] \phi e^{\jmath \omega t},
\]

where

\[
\phi(x,t) \equiv \text{col}(N^{(1)} - N_{0}^{(1)}, N^{(2)} - N_{0}^{(2)}, (C - C_{0}), \phi^{(1)}, \phi^{(2)}),
\]
\[
\begin{bmatrix}
\nu_1 D_1 \frac{\partial^2}{\partial x^2} - \alpha_1 & \nu_2 \nu \Sigma_{f2} (1 - \beta) & \lambda & \nu_1 \nu N_0 (1) (x) & 0 \\
\nu_1 \Sigma_r & \nu_2 D_2 \frac{\partial^2}{\partial x^2} - \Sigma_{a2} v_2 & 0 & 0 & \gamma_2 \nu_2 N_0 (2) (x)
\end{bmatrix}
\]

\[
[L_0^0] = \begin{bmatrix}
\beta \nu_1 \nu \Sigma_{f1} & \beta \nu_2 \nu \Sigma_{f2} - \lambda & 0 & 0 \\
\nu_1 \Sigma_{f1} & \nu_2 \Sigma_{f2} & 0 & 1/\tau_1 & 1/\tau_1 \\
0 & 0 & 0 & 1/\tau_2 & (1/\tau_2 + 1/\tau_3)
\end{bmatrix}
\]

and \(\alpha_1 = \nu_1 \left\{ \Sigma_{a10} + \Sigma_r - \nu \Sigma_{f1} (1 - \beta) \right\} \). The symbol, \(\zeta\), denotes heat capacity in units of \(^\circ\text{C} \cdot \text{cm}^3 / \text{fission}\); the symbols, \(\tau_1\), \(\tau_2\) and \(\tau_3\), denote time constants of the heat removal processes.

Figure 6.2 shows the computed responses of three plane neutron detectors, A, B, and C, located at \(x = a/4\), \(a/2\), and \(3a/4\), respectively, to oscillations induced by a plane thermal neutron absorber oscillating at \(x_0 = a/6\). These responses were found by evaluating a special version of Eq. (4.16) in which two-temperature feedback was incorporated with \(\gamma_2 \nu_2 N_0 (2) (x) = 2.55 \times 10^{10} \text{ cm}^{-3} \text{ see}^{-1} \text{ Oc}^{-1}\), \(\gamma_1 \nu_1 N_0 (1) (x) = 5.25 \times 10^8 \text{ cm}^{-3} \text{ see}^{-1} \text{ Oc}^{-1}\), \(\zeta = 9.42 \times 10^{-12} \text{ cm}^3 / \text{Oc} / \text{fission}\), \(\tau_1 = 0.255 \text{ sec}\), \(\tau_2 = 0.73 \text{ see}\) and \(\tau_3 = 0.21 \text{ sec}\).

Note in Fig. 6.2 that the results obtained in this oscillation test depend upon the location of the neutron detector. The most meaningful
Figure 6.2 Phase and Normalized Magnitude of Inferred Fundamental Mode Behavior. Oscillation Test on a Reactor at Power.
behavior to observe in this case is the behavior of the fundamental spatial harmonic. Figure 6.2 shows this behavior which has been inferred by analyzing the detector readings according to Eq. (6.3).

6.4 Measurement of Self-Shutdown Mechanisms in Power Excursion Tests

The shutdown coefficient in an excursion test is inferred from the time behavior of the power distribution. When spatial effects are important the power distribution cannot be measured by one detector only. An improved measurement can be obtained by combining the readings of M detectors in the manner specified by Eq. (6.4). This point is illustrated with the power excursion example of Section 5.3. Figure 6.3 illustrates the actual thermal neutron density (assumed proportional to the power distribution in this example) at \( t = 10 \) msec compared with an experimental estimate of the thermal neutron density constructed from the readings of three neutron detectors located at \( x = a/4, a/2 \) and \( 3a/4 \). The thermal neutron density constructed in this manner is in good agreement with the actual thermal neutron density.
Actual Thermal Neutron Density

Thermal Neutron Density Inferred from Three Neutron Detectors at x = 15 cm, 30 cm and 45 cm.

Figure 6.3 Experimental Measurement of the Space and Time Dependent Thermal Neutron Density in a Power Excursion Experiment.
CHAPTER 7
SUMMARY AND RECOMMENDATIONS FOR FURTHER WORK

7.1 Introduction

In Section 1.1 it is stated that the objective of this thesis is to develop a useful and physically meaningful approximation for space dependent reactor dynamics. It is considered that an approximation is useful and physically meaningful when the approximation (i) is constructed in terms of experimentally verifiable parameters, (ii) can be used to answer questions concerning kinetic behavior of the reactor, and (iii) can be used to interpret kinetic experiments when the observations are functions of space and time. The purposes of this chapter are to present a brief summary of how well this objective was met and, to make recommendations for further work.

7.2 Summary

7.2.1 Methods Used

The objective of the thesis was pursued through the use of a particular space dependent kinetics approximation called the Natural Mode Approximation (NMA). The NMA is based on an expansion of the space and time dependent behavior of the neutron density and other dependent variables of the reactor system into a series of products of unknown, time dependent coefficients and known, spatially dependent expansion vectors. These expansion vectors are called the natural modes of the reactor; they are the eigenvectors of a linear operator derived from the complete set of equations describing the system at
an initial, reference condition. The steady state parameters appearing in the NMA are the eigenvalues of the eigenvalue problem which is solved to generate the natural modes. The formalism of the NMA, which is well known from the work of others\cite{18-20}, is reviewed in Chapter 2. Chapter 3 considers the problem of calculating the natural modes and their eigenvalues.

7.2.2 Experimental Verification of Parameters of the NMA

It is shown in Chapter 4 that the NMA is physically meaningful because certain parameters (the prompt thermal neutron eigenvalues) of the NMA can be verified by means of non-hazardous oscillation tests. The experimental technique is based on simultaneously analyzing the readings of $M$, judiciously located neutron detectors in terms of $M$ natural modes for $M$ expansion coefficients. The prompt thermal neutron eigenvalues are then inferred from the frequency dependent behavior of the expansion coefficients. The technique is applied to numerical experiments and actual experiments with satisfactory results. The number, $M$, of detectors is arbitrary; it ranged from two to five in this work. A procedure is recommended for specifying suitable locations for the detectors. The procedure is based on the results of many numerical experiments.

7.2.3 Use of the NMA to Answer Questions Concerning Kinetic Behavior

It is shown in Chapter 5 that the NMA is a useful approximation for answering questions concerning the kinetic behavior of a large class of problems.
First, a power excursion in a nonuniform, one-dimensional reactor with negative feedback is calculated by Natural Mode Approximations in which three and four ($M=3$ and 4) spatial harmonics are retained. The results are compared with the results given by WIGLE,$^{[52]}$, a code which numerically solves the finite difference form of the space dependent kinetics equations. The NMA results for $M = 4$ are good and the calculation requires less computer time than that used by WIGLE if a number of problems are to be solved for the same reference reactor. The NMA results are improved by using a novel idea called "correction modes". Correction modes effectively include the contributions of higher order spatial harmonics without appreciably increasing the computation time.

Second, it is pointed out that if the eigenvalue spectrum of the natural modes is known, questions about subcriticality, asymptotic stability and spatial stability (oscillations of higher order spatial harmonics) can be answered without having to solve the NMA. An idea developed in this thesis is that the prompt thermal neutron eigenvalues, which can be measured by means of oscillation tests, are of practical significance with regard to answering questions about the space dependent aspects of kinetic behavior. It is shown that the relative magnitudes of the prompt thermal neutron eigenvalues are (i) sensitive functions of the geometrical arrangement of the reactor, and (ii) sensitive indicators of the susceptibility of the neutron density to undergo shape changes following a localized perturbation.

Third, a systematic series of calculations for one-dimensional reactors without feedback is performed with the NMA. The results
indicate that the number of spatial harmonics required for accurate representation of a transient is a function of the magnitude of the perturbation, and the degree of localization of the perturbation. In order to get good results with a perturbation of width, \( \Delta x \), it is necessary to retain \( M + 1 \) spatial harmonics where \( M \) is the index of that spatial harmonic whose wavelength, \( \lambda_M \) (twice the average distance between zero crossings), is less than \( 2\Delta x \). The results of this series of calculations also indicate that the NMA is particularly suitable either for very slow transients (below prompt critical) or for very fast transients (above prompt critical).

7.2.4 Use of the NMA for the Interpretation of Kinetic Experiments

Chapter 6 illustrates how a natural mode expansion can be used to interpret the following kinetic experiments when the experimental observations are functions of space and time: (i) oscillation tests designed to infer subcriticality of the fundamental spatial harmonic, (ii) oscillation tests designed to infer asymptotic stability of the fundamental spatial harmonic, and (iii) reactor excursion tests designed to measure shutdown coefficients. It is shown that the results of these tests may be grossly in error if only a single neutron detector is used. The error is reduced through the use of \( M \) detectors at different locations and analysis of these detector readings in terms of \( M \) appropriate modes of a natural mode expansion.

7.2.5 Application of the NMA to Multidimensional Systems

The application of the NMA to reactors which cannot be approximated by a one-dimensional geometry requires the calculation of multi-
dimensional natural modes. This calculation is treated in Chapter 3 by means of an approximate, synthesis technique\[^{27}\]. Good results are obtained for the multidimensional modes in many cases; but poor results are obtained in other cases. Poor results are indicated by the failure of a synthesized, multidimensional mode to satisfy the orthogonality relationships that a natural mode must satisfy. The synthesis technique effectively reduces an M-dimensional problem to a one-dimensional problem by expanding the M-dimensional mode in a series of products of unknown, one-dimensional coefficients and known trial functions which approximate the behavior of the mode in the remaining (M-1)-dimensions. The success of this technique depends to a large extent on how well the trial functions represent the behavior of the mode; and the ability to choose good trial functions comes from experience with the behavior of the multidimensional modes.

From the preceding summary it can be concluded that the NMA is a useful and physically meaningful kinetics approximation for reactors which may be approximated by a one-dimensional geometry. This conclusion can also be reached by noting that the Approximation Criteria set forth in Chapter 1 are satisfied. Similarly, from an examination of the Approximation Criteria it can be concluded that the NMA is of limited usefulness in multidimensional systems because Criterion 2 (parameters can be calculated, i.e., modes can be calculated) is satisfied to a limited extent. The author is of the opinion that this limitation will be removed by further work.
7.3 Recommendations for Further Work

7.3.1 Multidimensional Natural Modes

The most serious shortcoming of the NMA at the present time is the difficulty in calculating the multidimensional natural modes. It is suggested that further work be done on this problem. One of two possible treatments may permit direct calculation of two-dimensional modes. Three-dimensional modes could then be constructed by synthesis.

First, it is known that the Wielandt Iteration Technique (WIT)[38] can be used to calculate the natural modes in one-dimensional systems. The WIT involves the direct inversion of a matrix, and it has been pointed out[61] that this operation is often impractical for the large matrices associated with two-dimensional descriptions. Nevertheless, the author is not aware of any serious attempt to calculate the two-dimensional modes with the WIT.

Second, the originators of the Stabilized March Technique[28] have pointed out that it is possible to calculate the natural modes in two dimensions with the present version of the SMT; however, the SMT is not well suited for this problem because of the long running times involved and the limitations on computer storage capacity. The originators of the SMT have suggested[57] that it may be possible to extend the present SMT which marches the solution in only one direction to a version which simultaneously marches the solution in two directions. Hence two-dimensional modes could be calculated.
7.3.2 Calculation of Transient Behavior

Chapter 5 introduces the idea of using correction modes to increase the accuracy of a low order NMA \((M \leq 4)\) without a corresponding increase in computational effort. The idea is applied to a specific example and excellent results are obtained. This idea should be examined more closely in order to determine its usefulness as a general technique.

In Chapter 5 a limited series of calculations indicate that the amount of shape change in the neutron density following a perturbation in the same localized, fractional volume of different reactors can be correlated quite well in terms of a tilt parameter, \(\epsilon\). The tilt parameter depends upon the magnitude of the perturbation, and the relative magnitude of prompt, thermal neutron eigenvalues. It is recommended that a more extensive study should be carried out which correlates the amount of shape change in the neutron density as a function of (i) the magnitude of the perturbation, (ii) the relative magnitudes of prompt thermal neutron eigenvalues, (iii) the location of the perturbation, and (iv) the degree of localization of the perturbation.

7.3.3 Experimental Verification of the Parameters of the NMA

Future work may show that the technique proposed here for measuring the prompt thermal neutron eigenvalues is unnecessarily clumsy and involved. It may be possible to measure the eigenvalues in a more direct manner without having to have prior information about the spatial shapes of the prompt thermal neutron modes. The main
point to be recognized is that there is much practical information
about the space dependent dynamic behavior of a reactor which can be
determined by a non-hazardous, diagnostic technique such as an
oscillation test. The problem is specifying (i) what is a meaningful
parameter to measure, and (ii) how to measure it. This thesis has
specified that the prompt thermal neutron eigenvalues are meaningful
parameters to measure and it has specified how to measure them.

As a final recommendation the author suggests that oscillation
tests be performed on future, large reactor systems for the purpose
of measuring the prompt thermal neutron eigenvalues of higher order
spatial harmonics. Such tests may prove to be a valuable means of
continuously monitoring the spatial dynamic characteristics of a
reactor which change during core life because of fuel depletion,
fission product buildup, and changing control rod configurations.
### APPENDIX A

#### GLOSSARY OF PRINCIPAL SYMBOLS

<table>
<thead>
<tr>
<th>English Symbols</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_{mk}$</td>
<td>$m^\text{th}$ time dependent expansion coefficient.</td>
</tr>
<tr>
<td>$A_1$</td>
<td>expansion coefficient defined by Eq. (4.22).</td>
</tr>
<tr>
<td>$a$</td>
<td>extrapolated width of reactor.</td>
</tr>
<tr>
<td>$B^2$</td>
<td>complex buckling defined by equation before Eq. (2.39) or Eq. (F.5).</td>
</tr>
<tr>
<td>$B_T^2$</td>
<td>transverse buckling.</td>
</tr>
<tr>
<td>$B_z^2$</td>
<td>transverse buckling in $z$-direction.</td>
</tr>
<tr>
<td>$C$</td>
<td>vector containing precursor concentrations as components.</td>
</tr>
<tr>
<td>$c_i^{(1)}$</td>
<td>concentration of delayed neutron precursors of type $i$.</td>
</tr>
<tr>
<td>$C_{mk}$</td>
<td>$m^\text{th}$ expansion vector of expansion of $C$.</td>
</tr>
<tr>
<td>$c_{mk}$</td>
<td>coefficient defined by Eq. (4.5).</td>
</tr>
<tr>
<td>$D_g$</td>
<td>diffusion coefficient for group $g$.</td>
</tr>
<tr>
<td>$[D]$</td>
<td>a diagonal matrix of diffusion coefficients.</td>
</tr>
<tr>
<td>$E$</td>
<td>energy.</td>
</tr>
<tr>
<td>$E'_i$</td>
<td>summation of terms defined by Eq. (4.8).</td>
</tr>
<tr>
<td>$E_{Mi}$</td>
<td>summation of terms defined by Eq. (4.8).</td>
</tr>
<tr>
<td>$E_{Mi}'$</td>
<td>$E_{Mi} + E'_i$.</td>
</tr>
<tr>
<td>$F$</td>
<td>a $G$-vector with elements, $\nu\xi_g$.</td>
</tr>
<tr>
<td>$f$</td>
<td>vector of nonlinear terms.</td>
</tr>
</tbody>
</table>
\[ G \] number of neutron energy groups.

\[[G]\] a J by J matrix operator specified by conservation relationships.

\[ G(x,x_0, jw) \] a Green's function defined by equation after Eq. (2.39).

\[[g]\] externally controlled perturbation to \([G]\).

\[[H],[H]^{*}, [H]^{-1}\] an operator describing neutron conservation relationships, its adjoint and its inverse.

\[[H_0]\] matrix operator defined by the operations of Eq. (2.7).

\([H_j]\) vector defined by the operations of Eq. (2.7).

\[[h]\] externally controlled perturbation to \([H]\).

\[[I]\] unit matrix.

\(I\) number of delayed precursor groups.

\(J\) number of feedback variables.

\(j\) imaginary number, \(\sqrt{-1}\).

\(K\) total number of dependent variables.

\[ K(x,x_0, jw) \] Green's function defined by Eq. (F.20).

\(L\) number of trial functions used in synthesis.

\[[L_0]\] matrix operator defined by comparison of Eq. (2.11) with Eq. (2.8).

\[[\ell]\] externally controlled perturbation to \([L_0]\).

\(M\) number of spatial harmonics retained in an expansion, or the number of detectors used.

\(m\) spatial harmonic index.
neutron density.

N \(\text{vector containing group densities as components.}\)

N\(g\) \(\text{neutron density in group } g.\)

N\(_{mk}\) \(\text{m}^k\text{th vector in expansion of } N.\)

[P] \(\text{perturbation matrix whose elements, } P, \text{ are defined by Eq. (2.22) or Eq. (2.32).}\)

P(t) \(\text{dependent variable used in Section 2.6.}\)

Pp \(\text{perturbation parameter defined by Eq. (5.3).}\)

p \(\text{index of azimuthal mode.}\)

[Q] \(\text{a } J \text{ by } K \text{ matrix operator specified by conservation relationships.}\)

[q] \(\text{externally controlled perturbation to } [Q].\)

R\(_i\)(t) \(\text{reading of } i\text{ th detector.}\)

R\(_i\)\(_o\) \(\text{mean value of } R\(_i\)(t).\)

r\(_i\)(t) \(\text{oscillating component of reading of } i\text{ th detector.}\)

r \(\text{position.}\)

S \(\text{vector of external sources.}\)

S' \(\text{source vector defined by comparison of Eq. (2.11) with Eq. (2.8).}\)

S\(_{mk}\) \(\text{m}^k\text{th component of source vector.}\)

s \(\text{Laplace transform variable.}\)

T\(_o\) \(\text{column vector of feedback variables, } T\(_o\)(j).\)

T\(_o\)(j) \(\text{j}^{\text{th}} \text{ feedback variable at a reference condition.}\)

T \(\text{asymptotic period.}\)

t \(\text{time.}\)

[U] \(\text{a matrix defined by Eq. (4.12).}\)

[u] \(\text{a matrix defined by equation after Eq. (5.15).}\)
A diagonal matrix of group velocities.

\( v_g \) neutron velocity in group \( g \).

\( w \) weighting vector.

\([X]\) matrix of trial functions used in synthesis.

\( x_i \) position.

\( x_0 \) position of oscillator.

\( x_i \) position of \( i \)th neutron detector.

\( Z \) expansion vectors for synthesis.

**Greek Symbols**

\( a_1 \)

\( v_1(\Sigma_{al} + \Sigma_r - v\Sigma_f(1 - \beta) + D_BT^2) \).

\( a_2 \)

\( v_2(\Sigma_{a2} + D_2B_T^2) \).

\( \beta \)

\( \text{col}[\beta_1, \beta_2, \ldots, \beta_1] \).

\( \beta \)

\( \sum_{i=1}^{I} \beta_i \).

\( \beta_i \) fraction of fission neutrons appearing as delayed type \( i \).

\( \gamma^2 \) complex buckling defined by Eq. (F.19).

\( \gamma_i \) feedback coefficient in \( i \)th region.

\( \[r] \) \( \text{col}[\lambda_1X_1, \lambda_2X_2, \ldots, \lambda_1X_1] \).

\( \delta(x) \) Dirac delta function.

\( \delta_{ij} \) Kronecker delta.

\( \epsilon \) tilt parameter defined by \( \epsilon = \frac{PP(w_{1p}/w_{2p})}. \)

\( \zeta \) heat capacity.
column vector of incremental feedback variables.

\( \mathbf{q} \)

\( \mathbf{q}_{mk} \)

\( \mathbf{q}(j) \)

departure of \( j \)-th feedback variable from its value, \( T_0(j) \), at a reference condition.

\( \lambda \)

generation time.

\( \lambda_{mk} \)

\( \lambda_m^0 \)

\( [\lambda] \)

\( \lambda_i \)

\( \lambda_M \)

\( \mu \)

\( \nu \)

\( \Xi \)

\( \rho \)

\( \Sigma_{gj} \)

\( \Sigma_{rg} \)

\( \Sigma_{ag} \)

\( \Sigma_{fg} \)

\( \sigma_i \)

parameter defined by Eq. (3.19).

lambda eigenvalue introduced in Eq. (C.2).

diagonal matrix of precursor decay constants.

decay constant of precursors of type \( i \).

wavelength of \( M \)-th spatial harmonic (twice the average distance between zero crossings).

matrix defined by Eq. (3.3).

complex buckling defined by Eq. (F.18)

number of neutrons per fission

vector defined by Eq. (6.4).

reactivity.

macroscopic scattering cross section from group \( g \) to group \( j \) (\( \Sigma_{gg} = 0 \)).

total macroscopic scattering cross section for group \( g \)

(\( \Sigma_{rg} = \sum_j \Sigma_{gj} \)).

total absorption cross section for group \( g \).

total fission cross section for group \( g \).

response function of \( i \)-th neutron detector.
Tilt index defined by Eq. (5.4).

Time constants for heat removal processes.

K-vector containing neutron densities, precursor densities, and feedback variables.

$m_k$th expansion vector of an expansion of $\phi$; more specifically, the $m_k$th natural mode of the operator $[L^0_o]$.

$\psi$-vector containing neutron densities and precursor densities.

$m_k$th expansion vector of an expansion of $\psi$; more specifically, the $m_k$th natural mode of the operator $[H^0_o]$.

Eigenvalue of the $m_k$th natural mode.

Oscillation frequency, radians/second.

Maximum oscillation frequency.
Mathematical Symbols

\( \mathbf{A} \) \hspace{1cm} \) vector.
\( \mathbf{A}^T \) \hspace{1cm} transpose of vector.
\( [\mathbf{A}] \) \hspace{1cm} matrix.
\( [\mathbf{A}]^T \) \hspace{1cm} transpose of matrix.
\( [\mathbf{A}]^* \) \hspace{1cm} adjoint of matrix.
\( \langle \mathbf{A}, \mathbf{B} \rangle \) \hspace{1cm} spatial integral of scalar product of row vector, \( \mathbf{A}^T \), and column vector, \( \mathbf{B} \).
\( \langle \mathbf{A}, \mathbf{B} \rangle_x \) \hspace{1cm} spatial integral of scalar product of row vector, \( \mathbf{A}^T \), and column vector, \( \mathbf{B} \), over \( x \) direction only.
\( \text{col}[\mathbf{A}] \) \hspace{1cm} column matrix.
\( \text{diag}[\mathbf{A}] \) \hspace{1cm} diagonal matrix.
\( \tilde{\mathbf{A}} \) \hspace{1cm} Laplace transform of \( \mathbf{A} \).
\( \delta \mathbf{A} \) \hspace{1cm} increment of \( \mathbf{A} \).
\( \mathbf{A} \) \hspace{1cm} phase of \( \mathbf{A} = \tan^{-1}(\text{Im}(\mathbf{A})/\text{Re}(\mathbf{A})) \).
\( |\mathbf{A}| \) \hspace{1cm} magnitude of \( \mathbf{A} \).
\( \tilde{\mathbf{A}} \) \hspace{1cm} matrix formed by an averaging operation.
\( \text{Re}(\mathbf{A}) \) \hspace{1cm} real part of \( \mathbf{A} \).
\( \text{Im}(\mathbf{A}) \) \hspace{1cm} imaginary part of \( \mathbf{A} \).
\( \nabla \) \hspace{1cm} del operator.
\( \nabla^2 \) \hspace{1cm} Laplacian operator.
\( \sum \) \hspace{1cm} summation operator.
Sub- and Superscripts

\(k\) \hspace{1cm} \text{index which indexes the modes belonging to the}\ m^{\text{th}}\ \text{spatial harmonic.}

\(m\) \hspace{1cm} \text{spatial harmonic index.}

\(o\) \hspace{1cm} A_0, \text{value of } A \text{ at a reference condition.}

\(p\) \hspace{1cm} \text{denotes prompt thermal neutron term.}

\(T\) \hspace{1cm} \text{transpose.}

\(\text{th,2}\) \hspace{1cm} \text{thermal neutron group.}

\(1\) \hspace{1cm} \text{epithermal neutron group.}

\(*\) \hspace{1cm} \text{adjoint.}
APPENDIX B

SUPPORTING CALCULATIONS FOR THE EXAMPLE OF SECTION 2.6

B.1 Space and Frequency Dependent Solution for an Oscillating Plane Perturbation in a Uniform Slab Reactor

The one-dimensional equations to be solved in a uniform, critical slab reactor of extrapolated width, a, are

\[ vD^2 N(x,t) + v \left( \Sigma_t - \Sigma_a \right) N(x,t) + \lambda c(x,t) = \frac{\partial N}{\partial t}, \quad (B.1) \]

and

\[ v\beta \Sigma_t N(x,t) - \lambda c(x,t) = \frac{\partial c(x,t)}{\partial t}, \quad (B.2) \]

where

\[ \delta \Sigma_a(x,t) = - \delta \Sigma_a(x-x_0)e^{j\omega t}, \]

\[ N(0,t) = N(a,t) = 0, \]

and

\[ N(x,0) = N_0(x) = \sin \frac{nx}{a}. \]

It is to be understood that only the real part of the solution is to be retained. It is assumed that \( \delta \Sigma_a \) is small enough so that \( N(x,t) - N_0(x) \ll N_0(x) \) and the "perturbation approximation" which replaces \( \delta \Sigma_a N_0(x,t) \) by \( \delta \Sigma_a N_0(x) \) may be used.

The Laplace transformed version of Eqs. (B.1) and (B.2) may be written as
\[ v \left\{ Dv^2 - \sum_a \frac{s}{v} + \nu \sum_{x} (1-\beta) \right\} \overline{N}(x,s) + \lambda \overline{C}(x,s) = -N_o(x) - \nu \sum_a \delta(x-x_o) \frac{1}{s-j \omega} N_o(x) \]  
(B.3)

and

\[ (\lambda + s) \overline{C}(x,s) = C_o(x) + \nu \beta \nu \sum_{x} \overline{N}(x,s) \]  
(B.4)

where \( \overline{C}(x,s) \) and \( \overline{N}(x,s) \) are the Laplace transforms of \( C(x,t) \) and \( N(x,t) \), respectively. Elimination of the variable, \( \overline{C}(x,s) \), from these equations and use of the steady state condition,

\[ \lambda C_o(x) = \nu \beta \nu \sum_{x} N_o(x) \]

gives the relation,

\[ \nabla^2 \overline{N}(x,s) + B^2(s) \overline{N}(x,s) = -\frac{N_o(x)}{vD} \left( 1 + \frac{\beta \nu \sum_{x} \nu}{\lambda + s} \right) - \frac{\delta \sum_{x} N_o(x)}{D} \delta(x-x_o) \frac{1}{s-j \omega} , \]

(B.5)

where

\[ B^2(s) = \frac{\nu \sum_{x}}{D} - \frac{s}{vD} \left( 1 + \frac{\beta \nu \sum_{x}}{\lambda + s} \right) . \]

Assuming that the solution to Eq. (B.5) can be expressed in terms of a Fourier sine expansion,

\[ \overline{N}(x,s) = \sum_{n=1}^{\infty} b_n(x,s) \sin \frac{\pi x}{a} , \]

(B.6)

leads to the following form of Eq. (B.5):

\[ \sum_{n=1}^{\infty} \left\{ -n^2 \left( \frac{\pi}{a} \right)^2 + B^2(s) \right\} b_n(x,s) \sin \frac{\pi x}{a} = -\frac{N_o(x)}{vD} \left( 1 + \frac{\beta \nu \sum_{x}}{\lambda + s} \right) - \frac{\delta \sum_{x} N_o(x)}{D} \delta(x-x_o) \frac{1}{s-j \omega} . \]

(B.7)
Multiplication by \( \sin \frac{m\xi}{a} \) and integration over \( x \) from \( x = 0 \) to \( x = a \) yield the following expression for the expansion coefficients of Eq. (B.6):

\[
b_m(x_0, s) = \frac{1}{vD} \frac{1 + \frac{\beta \nu \Sigma_r}{A + s}}{m^2 \left( \frac{\pi}{a} \right)^2 - B^2(s)} \delta_{ml} + \frac{2\delta_i \gamma N(x_0)}{aD} \frac{\sin \frac{m\xi}{a}}{(s - j\omega) \left\{ \frac{m^2 \left( \frac{\pi}{a} \right)^2 - B^2(s) }{2} \right\} },
\]

(B.8)

where

\[ \delta_{ml} = 1 \text{ for } m = 1, \text{ and} \]
\[ \delta_{ml} = 0 \text{ otherwise.} \]

Substitution of this expression for the expansion coefficients into Eq. (B.6) and use of the critical condition,

\[
\nu \Sigma_f - \Sigma_{a0} = D(\pi/a)^2,
\]

yield

\[
\bar{N}(x, s) = \frac{1}{s} N_0(x) + \frac{2\delta_i \gamma N(x_0)}{aD} \sum_{m=1}^{\infty} \sin \frac{m\xi}{a} \sin \frac{m\xi}{a} \frac{2^a \gamma N(x_0)}{(s - j\omega) \left\{ \frac{m^2 \left( \frac{\pi}{a} \right)^2 - B^2(s) }{2} \right\} }.
\]

Inversion of this expression gives the solution,

\[
N(x, t) = N_0(x) + \frac{2\delta_i \gamma N(x_0)}{\pi^2 D} \sum_{m=1}^{\infty} \sin \frac{m\xi}{a} \sin \frac{m\xi}{a} e^{j\omega t} \left\{ \frac{2\gamma N(x_0)}{m^2 - \left( \frac{\pi}{a} \right)^2 B^2(j\omega) } \right\}
\]

(B.9)

+ (transient terms which die out).

Since only real terms are to be kept, the imaginary term due to the pole at \( s = 0 \) (for \( m = 1 \)) has been neglected.
The infinite summation may be expressed in closed form as

\[
\sum_{m=1}^{\infty} \frac{\sin a x_m \sin a x_0}{m^2 - \left( \frac{a}{\pi} \right)^2 B^2(j \omega)} = \frac{\pi^2}{4a B^2(j \omega)} \sin B(x+x_0) - \sin B|x-x_0| - 2\cot B \sin Bx_0 \sin Bx
\]

\[
\equiv \frac{\pi^2}{2a} G(x,x_0,j \omega) ,
\]

where

\[
G(x,x_0,j \omega) = \frac{\sin Bx \sin(B(a-x_0))}{B \sin Bx} \quad (x \leq x_0) ,
\]

and

\[
G(x,x_0,j \omega) = \frac{\sin Bx_0 \sin B(a-x)}{B \sin Bx} \quad (x \geq x_0) .
\]

Hence a closed form for the solution given by Eq. (B.9) is

\[
N(x,t) = N_0(x) + \sum_{n=1}^{\infty} a N_0(x_n) G(x,x_0,j \omega)e^{j \omega t} \quad (B.10)
\]

The function, \(G(x,x_0,j \omega)\) is the Green’s function for an equation of the form,

\[
\left\{ \nabla^2 + B^2(j \omega) \right\} N(x,j \omega)e^{j \omega t} = - \frac{\delta(x-x_0) \delta(x-x_0)}{D} e^{j \omega t} ,
\]

which may be found by solving the equation,

\[
\left\{ \nabla^2 + B^2(j \omega) \right\} G(x,x_0,j \omega) = - \delta(x-x_0) ,
\]

subject to the conditions that \(G(0,x_0,j \omega) = G(a,x_0,j \omega) = 0\).
B.2 Evaluation of the Parameter, $\rho / \Lambda$

In the illustration of Section 2.6, the parameter, $\rho / \Lambda$, is given by

$$\rho = \frac{\langle \nu(x)(\nabla^2 + \nu \Sigma_f - \Sigma_{ao} + \delta \sum_a \delta(x-x_0)e^{i \omega t})N(x,t) \rangle}{\frac{1}{\nu} \langle w(x)N(x,t) \rangle} , \quad (B.11)$$

where $w(x)$ is an arbitrary weighting function. This expression may be written as

$$\rho = \frac{\langle w(x)(\nabla^2 + \nu \Sigma_f - \Sigma_{ao})[N(x,t) - N_0(x)] \rangle + \langle w(x)\delta \sum_a \delta(x-x_0)e^{i \omega t}N_0(x) \rangle}{\langle w(x)N_0(x) \rangle} , \quad (B.12)$$

where use has been made of the fact that

$$\{\nabla^2 + \nu \Sigma_f - \Sigma_{ao}\}N_0(x) = 0 ,$$

and of the approximations that

$$\langle w(x)[N(x,t) - N_0(x)] \rangle \ll \langle w(x)N_0(x) \rangle ,$$

and

$$\langle w(x)\delta(x-x_0)[N(x,t) - N_0(x)] \rangle \ll \langle w(x)\delta(x-x_0)N_0(x) \rangle .$$

These approximations are consistent with the approximation that

$$\delta \Sigma_a N(x,t) \sim \delta \Sigma_a N_0(x)$$

which was made in deriving the solution in the first part of this appendix.

In the case where the weighting function is chosen as the response function for the $i^{th}$ detector,

$$w(x) = \nu_0 \delta(x-x_i) ,$$
Eq. (B.12) yields the following result for $p/\Lambda$:

$$
\rho = \frac{\langle \delta(x-x_1)D^2 + \nu \Sigma_f - \Sigma_{ao} \rangle}{\frac{1}{v} \langle \delta(x-x_1)N_o(x) \rangle} e^{jwt} (x_1 \neq x_o)$$

$$= \nu \delta \Sigma a e^{jwt} \left( -B^2(jw) + \frac{1}{a^2} - \frac{N(x)}{N(x_1)} G(x_1,x_0,jw) \right)$$

$$= \nu \delta \Sigma a e^{jwt} \left( \frac{jw(jw + \lambda + \beta \nu \Sigma v)}{vD(jw + \lambda)} G(x_1,x_0,jw) \right).$$

(B.13)

Evaluation of this expression gives the phase relationship between $\frac{p(t)}{\Lambda}$ and $-\delta \Sigma a e^{jwt}$ shown in Fig. 2.1(a).

In the case where the weighting function is chosen as the solution to the steady state adjoint equation,

$$w(x) = N_o^*(x) = \sin \frac{\pi x}{a},$$

(B.14)

Eq. (B.12) yields the following result for $\frac{p}{\Lambda}$:

$$\rho = \frac{\frac{4v}{2} \delta \Sigma a}{\frac{1}{a^2} D} N_o(x_o) \left\langle \sin \frac{\pi x}{a} \left( D^2 + \nu \Sigma_f - \Sigma_{ao} \right) \sum_{m=1}^{\infty} \frac{\sin \frac{\pi x}{a}}{m^2 \left( \frac{\pi}{a} \right)^2 - B^2(jw)} \right\rangle e^{jwt}$$

$$+ \frac{2v \delta \Sigma a}{a} \sin \frac{\pi x}{a} e^{jwt}.$$

The integral yields a non-zero result only for $m = 1$ and in this case the integrand is everywhere zero because of the critical condition,

$$(D^2 + \nu \Sigma_f - \Sigma_{ao}) \sin \frac{\pi x}{a} = 0.$$
Therefore, $P/\Lambda$ is given by

$$
\frac{P}{\Lambda} = \frac{2v_0 \sum_n a}{\pi} \sin \frac{2nx_0}{a} \quad \text{e}^{j\omega t} .
$$

(E.15)

This expression for $P/\Lambda$ is identical to that which would be obtained by replacing $N(x,t)$ in Eq. (B.11) by $N^*_0(x)$ which is defined by Eq. (B.14).

B.3 Evaluation of the Relationship between the Response of a Detector and the Adjoint Weighted Neutron Density

In the case where adjoint weighting is used to develop the space independent kinetics equations, the dependent variable, $P(t)$, is the adjoint weighted, incremental neutron density, $A(t)$, and is given by

$$
P(t) = A(t) = \left\langle N^*_0(x) \{N(x,t) - N_0(x) \} \right\rangle .
$$

For the example of Section 2.6, this expression can be evaluated using Eqs. (B.14) and (B.9). It is

$$
P(t) = A(t) = \left\langle \sin \frac{nx_0}{a} \frac{2\delta(x)}{a} N_0(x) \sum_{m=1}^{\infty} \sin \frac{mx_0}{a} \sin \frac{mx_0}{a} e^{j\omega t} \right\rangle
$$

$$
= \frac{\delta}{D} \sum_n N_0(x_0) \sin \frac{nx_0}{a} \frac{e^{j\omega t}}{\left( \frac{m}{a} \right)^2 - b^2(j\omega)}
$$

$$
= v_0 \sum_n N^*_0(x_0) \frac{(j\omega + \lambda) e^{j\omega t}}{(j\omega + \lambda + \beta \nu^2) j\omega} \sin \frac{nx_0}{a} .
$$

(E.16)

The oscillating portion of the detector reading is given by
\[ r_i(t) = \left\langle \nu \delta(x-x_i)\left[ N(x,t) - N_0(x) \right] \right\rangle \]

\[ = \nu \delta \frac{\delta \Sigma_a N_0(x)}{D} G(x_i,x_o,\omega)e^{j\omega t}. \quad (B.17) \]

Hence the phase relationship between the reading of the \(i^{th}\) detector, \(r_i(t)\), and the adjoint weighted, incremental neutron density, \(A(t)\), is given by

\[ \left( \frac{r_i(j\omega)}{A(j\omega)} \right) = \left( \frac{\sigma_a G(x_i,x_o,j\omega) j\omega}{\pi x_o^2} \right) \left( \frac{j\omega + \lambda + \beta \nu \Sigma_i \lambda}{j\omega + \lambda} \right). \quad (B.18) \]
APPENDIX C

MUDMO-II. A COMPUTER CODE FOR THE SOLUTION OF EIGENVALUE PROBLEMS

C.1 Introduction

MUDMO-II is a computer code which numerically solves for higher order eigenvalues and eigenvectors of two different eigenvalue problems associated with multigroup diffusion theory in one-dimensional systems. MUDMO-II is designed to find the eigenvalues, eigenvectors and adjoint eigenvectors of either the $\lambda$-eigenvalue problem[4] or the natural mode eigenvalue problem. This code, when used in conjunction with the code SYNSIG (See Appendix D), may be used to "synthesize" solutions to these eigenvalue problems in $M$-dimensional systems from the results of $(M-1)$-dimensional calculations. The code is limited to finding real, distinct eigenvalues in systems which have uniform properties by region. MUDMO-II is patterned after the code, MUDMO[57], written by D.R. Edwards.

C.2 Finite Difference Form of the Eigenvalue Problems to be Solved

The eigenvalue problems to be solved may be written in terms of one general formulation,

$$\nabla_x \cdot [D(x) \nabla_{x-mk}(x) + [\mu(x, w_{mk}, \lambda^0_m)]^N_{mk}(x)] = 0 \quad , \quad (C.1)$$

if the definition of $[\mu]$ is modified to include the $\lambda$-eigenvalue, $\lambda^0$. That is,
\[
\begin{align*}
&[\mu(x, w_{mk}, \lambda_m^0)] 
= [\mathbf{x}] - [DB_T^2] + \lambda_m^0 (1 - \beta) [\mathbf{x}_p \cdot \mathbf{F}^T] \\
&+ [\mathbf{n}][w_{mk}[I] + [\lambda]^{-1} \lambda_m^0 [\mathbf{B} \cdot \mathbf{F}^T])[\mathbf{V}^\dagger] - w_{mk}[I]
\end{align*}
\] (C.2)

Note that the \(\lambda\)-eigenvalue multiplies the fission source terms. When a \(\lambda\)-eigenvalue problem is to be solved, all eigenvalues, \(w_{mk}\), are set equal to zero. When a natural mode eigenvalue problem is to be solved, the eigenvalue, \(\lambda_m^0\), is set equal to unity.

The diffusion operator in MUDMC-II is

\[
\nabla_x \cdot [D] \nabla_x = \frac{1}{x^a} \frac{d}{dx} \left\{ x^a [D] \frac{d}{dx} \right\} - \frac{d^2}{x^2} [D],
\] (C.3)

where \(a = 0\) in slab reactors \((p = 0)\), and \(a = 1\) in cylindrical reactors \((p = 0, 1, 2, \ldots)\). For cylindrical cores which are uniform azimuthally, the two-dimensional modes may be obtained. The \(p^{th}\) azimuthal mode of the \(m^{th}\) radial mode is

\[
N_{mp,k}(x, \phi) = N_{m,k}(x) \cos p\phi.
\]

MUDMC-II solves the finite difference form of Eq. (C.1) by the Stabilized March Technique\([^{23}\). The finite difference form of this equation at mesh point \(j\) is obtained by multiplying Eq. (C.1) by \(x^a\) and integrating from the midpoint \(x_-\) of the mesh interval to the left to the midpoint \(x_+\) of the mesh interval to the right. This procedure is patterned after the procedure of Ref. 25. See Fig. C.1 for an illustration of the mesh. The result of this integration may be written as (retaining only those subscripts which identify the mesh point)
\[ N_{j+1} = [A_j(\omega, \lambda^0)] N_j + [B_j] N_{j-1}, \quad (C.4) \]

where

\[ [A_j(\omega, \lambda^0)] = [I] + [D_+]^{-1}[D_-] \frac{\Delta x_+}{\Delta x_-} \beta_j + \]

\[ + \alpha_j \frac{(x_j + \frac{\Delta x_+}{2})^{a+1} - x_j^{a+1}}{x_j} \frac{\Delta x_+}{2} [D_+]^{-1} [v_+] + \]

\[ + \alpha_j \frac{x_j^{a+1} - (x_j - \frac{\Delta x_-}{2})^{a+1}}{x_j} \frac{\Delta x_-}{2} [D_+]^{-1} [v_-] + \]

\[ + \alpha_j^{\Delta x_+} x_j^{p^2} \left[ [I] \ln \left( \frac{x_j + \frac{\Delta x_+}{2}}{x_j} \right) + [D_+]^{-1} \lambda \frac{x_j^{\Delta x_+}}{\Delta x_-^2} (x_j - \frac{\Delta x_-}{2}) \right] + \]

\[ [B_j] = -[D_+]^{-1}[D_-] \beta_j \frac{\Delta x_+}{\Delta x_-}, \]
\[
\alpha_j = \frac{x_j}{(x_j + \frac{\Delta x_+}{2})}, \\
\beta_j = \frac{(x_j - \frac{\Delta x_-}{2})}{(x_j + \frac{\Delta x_+}{2})},
\]

and \([I]\) is the unit matrix. Use has been made of the fact that \(p \neq 0\) only for \(a = 1\). It has been assumed that the derivatives at \(x_+\) and \(x_-\) are approximated by

\[
\frac{dN}{dx}\bigg|_{x_+} = \frac{N_{j+1} - N_j}{\Delta x_+},
\]

and

\[
\frac{dN}{dx}\bigg|_{x_-} = \frac{N_j - N_{j-1}}{\Delta x_-},
\]

and it has been assumed that \(N\) is constant over the range \(x_-\) to \(x_+\).

If the reactor properties at \(x_-\) and \(x_+\) are identical, and if \(\Delta x_- = \Delta x_+ = \Delta x\), Eq. (C.4) may be written as

\[
N_{j+1} = \left[\alpha_j[P] + \gamma_j[I]\right]N_j - \beta_j N_{j-1},
\]

where

\[
[P] = 2[I] + (\Delta x)^2[D]^{-1}[\mu],
\]

and

\[
\gamma_j = \alpha_j \frac{\Delta x}{x_j^2} \left(\frac{1}{\beta_j}ight)\ln(1/\beta_j).
\]
For a zero boundary condition at the initial boundary \((j = 0)\), integration of Eq. (C.1) from \(x = 0\) to \(x = x_+\) yields

\[
\mathbf{N}_1 = \left(1 + \frac{(Ax)^2}{Z(a+1)} \mathbf{[D]}^{-1} \mathbf{[L]} \right) \mathbf{N}_0 .
\]

Note that \(p = 0\) whenever a zero derivative boundary condition is imposed at the origin. A boundary condition of \(\mathbf{N}_o = 0\) in either slab or cylindrical systems is applied by requiring \(\mathbf{N}_o = 0\) in Eq. (C.5) when \(j = 1\). A zero boundary condition at the outer boundary is applied by requiring \(\mathbf{N}_{NPTS} = 0\) in Eq. (C.5) when \(j = NPTS - 1\).

\(NPTS\) is the member of mesh points.

### C.3 The Stabilized March Technique

The Stabilized March Technique (SMT) treats either Eq. (C.4) or (C.5) as an initial value problem. Such a procedure is inherently unstable, but the error growth is controlled by the application of what is called a "conditioning transformation." The error bounds on the final solution are set by the "frequency of conditioning." The details of the SMT are given elsewhere\(^{[28]}\). The brief description of the SMT which follows is for the purpose of allowing a user of MUDMC-II to prepare input and to use the code with efficiency.

The \(G\)-component eigenvector at \(j = 1\) is expanded as \(\mathbf{N}_1 = [\psi_1] G\) where \([\psi_1] = [I]\), the set of unit vectors (\(G\) by \(G\) unit matrix). The vector \(G\) is identically equal to the eigenvector at \(j = 1\). The eigenvector at every mesh point may be represented as \(\mathbf{N}_j = [\psi_j] G\) where the \([\psi_j]\)'s are related by means of the recursion relation, Eq. (C.4); that is,
\[
[\psi_{j+1}] = [A_j(\alpha, \lambda)][\psi_j] + [B_j][\psi_{j-1}]. \tag{C.6}
\]

The boundary condition at the initial boundary allows \([\psi_1]\) and \([\psi_0]\) to be related. Repeated use of Eq. (C.6) allows the vector sets, \([\psi_j]\), \([\psi_{j+1}]\), etc., to be calculated. Due to the inherent instability of this procedure, the vector sets, \([\psi_j]\), become more ill-conditioned (linearly dependent) as \(j\) increases. To prevent \([\psi_j]\) from becoming singular, the procedure is interrupted at selected mesh points (called points of conditioning) and \([\psi_j]\) is transformed by a "conditioning transformation" into the well-conditioned set of unit vectors, \([I]\). This Stabilized March Technique is continued until the outer boundary is reached at \(j = NPTS\). For the boundary condition, \(N_{NPTS} = 0\), the determinant, \(|\psi_{NPTS}|\), must be zero. This condition will be satisfied only for proper choices of the eigenvalue. The calculation of \(|\psi_{NPTS}|\) is repeated for each trial value of the eigenvalue. This determinant changes sign as two successive eigenvalues just bracket the actual eigenvalue. A conventional root finding technique may then be used to locate that eigenvalue for which \(|\psi_{NPTS}| = 0\). The calculation of the eigenvector and the adjoint eigenvector is a direct calculation once the eigenvalue is known.

The eigenvector calculation originates from those points at which a conditioning transformation was performed. The calculation of the eigenvector at mesh points between points of conditioning is done by marching the eigenvector backward from the \(K\)th point of conditioning and marching the eigenvector forward from the \((K-1)\)st point of conditioning. At some mesh point between points of conditioning,
the eigenvector will have been calculated by both a forward march and a backward march. At this point a test is made to see how well these calculations of the eigenvector agree. If the difference in any component of the eigenvector is greater than 0.1 percent, a message is printed which indicates the percent error in the agreement and the mesh point at which the test occurs.

C.4 Features of MUDMC-II

The FORTRAN-II version of MUDMC-II listed in this Appendix may be used for eigenvectors having up to six components. This means that it will handle six neutron energy groups in a one-dimensional problem or six pseudo-groups in a synthesis of a multidimensional solution. The code will handle up to 320 mesh points in a maximum of four regions of different composition. Not only will the code search for eigenvalues but it will also search for a dimension, a poison concentration, or a transverse buckling which corresponds to a specified eigenvalue. At the initial boundary a choice may be made between the boundary condition of \( N_0 = 0 \) or \( \frac{dN_o}{dx} = 0 \). At the outer boundary the only boundary condition available is \( N_{NPTS} = 0 \).

The parameter to be determined (eigenvalue, region thickness, poison concentration, or transverse buckling) is defined when the determinant, \( |\psi|_{NPTS} \), is zero. The initiation of the search for the zero of this determinant is restricted to the situation where the sign of the determinant changes from positive to negative. A plot of the typical behavior of the determinant as a function of the prompt
thermal neutron eigenvalue is shown in Fig. C.2. Note the systematic repetition of zeros and discontinuities. A recommended procedure for locating the parameter(s) to be determined is to make a very rough, overall scan of the entire range of possible parameter values. This rough scan will indicate the approximate magnitudes of the exact parameters. A finer search can then be conducted in such a way to guarantee that the sign change of the determinant from positive to negative locates a zero and not a discontinuity.

The prompt epithermal neutron eigenvalues of Eq. (C.1) are difficult to locate because in a plot such as Fig. C.2 they cannot be distinguished from the prompt thermal neutron eigenvalues of spatial harmonics of very high order. (See Fig. 3.3 and note that low order

![Graph showing typical behavior of the determinant as a function of the eigenvalue.]

Figure C.2 Typical Behavior of the Determinant, $|\psi_{MPTS}|$, as a Function of the Eigenvalue, $\omega$. 
prompt epithermal neutron eigenvalues will eventually be of the same order of magnitude as high order prompt thermal neutron eigenvalues.)

There is an approximation built into MUDMO-II which allows a good estimate of the prompt epithermal neutron eigenvalues when only two neutron groups are used. This approximation involves neglecting the diffusion operator term in the balance equation for thermal neutrons. The approximate equations which are solved are

\[
(\nabla \cdot \mathbf{D}_1 - \nabla \cdot \mathbf{D}_2 - \sum_{i=1}^{l} \Sigma_{r_i} + \nu \Sigma_T \Gamma T (1-\beta) - w_{m3}/v_1) v_{l_1 m3}^{(1)}(x) + \nu \Sigma_{f2} \Gamma T (1-\beta) v_{l_2 m3}^{(2)}(x) = 0
\]

and

\[
v_{l_1 m3}^{(1)}(x) - (\Sigma_{a_2} + \mathbf{D}_2 - w_{m3}/v_2) v_{l_2 m3}^{(2)}(x) = 0 , \quad (C.7)
\]

where the index, \( k = 3 \), denotes the prompt epithermal neutron eigenvalue. This approximation gives good estimates of the eigenvalues, \( w_{m3} \), because \( |w_{m3}| \ll \nabla \cdot \mathbf{D}_2 \nabla N_{m3}^{(2)} \). This approximation does not give good estimates of the eigenvectors because there are discontinuities in \( N_{m3}^{(2)} \) at region boundaries. The eigenvalues introduced by the operator, \( \nabla \cdot \mathbf{D}_2 \nabla \), are now eliminated and first estimates of the \( w_{m3} \) may be easily determined.

MUDMO-II is written as a main program and fourteen subroutines. The subroutines have the following functions: DATA reads all input according to the format given below; SET calculates the matrix \([P]\) given in Eq. (C.5); START initiates the march operation according to the boundary condition at \( x = 0 \); DETINV calculates the determinant or the inverse of a matrix; CONDIT performs the conditioning transformation; CROSS extends the solution across region boundaries according
to Eq. (C.4); MARCH performs the march operations between points of conditioning; EVAL evaluates the determinant at the outer boundary; ZERO performs the search for the zero in the determinant using a three-point Lagrangian Interpolation; RESTRT starts a backward march from the outer boundary towards the initial boundary (a reverse march is a necessary operation for the calculation of the eigenvector); FUNCT calculates the eigenvector after the eigenvalue is known; HOSIEQ solves sets of homogeneous algebraic equations which are part of the eigenvector determination; OUTPUT prints or punches all the required output; and YEGADS prints error messages.

C5. Preparation of Input for MUDMC-II

The input is preceded by a run identity card with any characters permissible in columns 2 through 72. This card is followed by fixed point, control data punched with a 20I3 format. The fixed point control data are:

(FIRST CARD OF FIXED POINT DATA)

NREG = number of uniform regions in the direction of march \( \leq 4 \).

IA = 0 for a slab reactor; = 1 for a cylindrical reactor.

NOG = number of neutron energy groups. \( G \leq 6 \) for a one-dimensional problem, or \( G \leq 6/L \) for an L term synthesis expansion in a two-dimensional problem.

NPTS = number of mesh points, \( \leq 320 \).

KCOND = frequency of conditioning, \( f \). The conditioning transformation is applied at every \((f_c)^{th}\) mesh point. The smallest frequency of conditioning which can be used is 2. Most problems are solved satisfactorily with \( 4 \leq f_c \leq 8 \).
KEV = 1 to calculate the $\lambda$-modes and eigenvalues;
    = 2 to calculate the natural modes of Eq. (C.1) without
delayed neutrons;
    = 3 to calculate the natural modes of Eq. (C.1) with
delayed neutrons;
    = 4 to calculate the first approximations of the prompt
epithermal neutron eigenvalues by Eq. (C.7) (Must be
accompanied by $K1 = 1$, $K2 = 0$ and NOG = 2).

KCHO = 1 to search for a region dimension corresponding
to a specified eigenvalue;
    = 2 to search for an eigenvalue;
    = 3 to search for a poison concentration corresponding
to a specified eigenvalue;
    = 4 to search for a transverse buckling corresponding
to a specified eigenvalue.

KREG = region in which dimension search or poison search
takes place.

IFOUND = 1, search for parameter indicated by KCHO;
    2, no search, parameter indicated by KCHO is read in.

KHUNT = 1, no search for zero of determinant; 2, search for
zero of determinant.

IFUNCT = 1, no calculation of the eigenvector; 2, calculate
eigenvector after parameter indicated by KCHO is
found.

KADJ = 1, calculate eigenvector; 2, calculate adjoint eigenvector.

KAPP = 0, calculate only that vector indicated by KADJ;
    1, calculate both the eigenvector and adjoint eigenvector.

KPOW = 1, no punched card output of eigenvector (and/or adjoint);
    2, punched card output of eigenvector (and/or adjoint).

NFCT = print control; print out the eigenvector (and/or
adjoint) at every (NFCT)$^\text{st}$ mesh point.

KBCI = 1, $N_0 = 0$; 2, $\frac{dN_0}{dx} = 0$.

KBCO = 1, $\frac{N}{N_{\text{NPTS}}} = 0$. (Only choice available)

KL = number of epithermal group trial vectors in a two group
synthesis of a two-dimensional solution (KL = number of
epithermal groups for no synthesis).
The next three cards contain floating point data which is read on a 15X 4E15.8 format. The first 15 spaces allow any comment for identification of the input.

The first card contains:

CEFT = initial guess of the eigenvalue. If KEV = 1, this is a \( \lambda \)-eigenvalue. If KEV \( \neq 1 \), this is a natural mode eigenvalue.

DEL = increment used in changing that parameter indicated by KCHO from one march to the next.
The next card contains:

**EPA** = criterion for convergence of that parameter indicated by KCHO.

**EPB** = criterion for singularity of matrix in DETINV.

The next card contains:

**RAD(I)** = thickness of each region, NREG values.

At this point the input data has NOG + 2 comment cards which have the same format as the run identity card.

Now if NEWP = 1, the following series of input is read. If NEWP = 0, the following input is not read and the problem is solved using the input of the previous problem.

The first card of this series contains fixed point data on a 2013 format. All other cards of this series contain floating point data a 15X 4E15.8 format. Each subscripted variable is begun on a new card. The subscripted data is read as (((S(I,J,K), I = 1, N1), J = 1, N2) K = 1, N3).

The first card of this series contains:

**IND(I)** = the mesh point at the outer boundary of each region; NREG values. IND(NREG) = NPTS + 1 - KBCO. The number of mesh points within each region must be an integer multiple of NFCT.

The next card contains:

**BETTOT** = total fraction of delayed neutron, $\beta$.

The next groups of cards contain:

**B(I)** = fraction of fission neutrons appearing as type i, NFREC values.

**PPREC(I)** = decay constant of precursors of type i, NFREC values.

**F(J,K)** = fission yield of prompt neutrons in Group J in Region K, (K1 + K2) X NREG values.
The final six groups of cards of this series contain those synthesis input parameters which are prepared by SYNSIG if a two-dimensional distribution is being synthesized. If the calculation is for a one-dimensional system, this input is prepared by hand if the $G \times G$ ($G = NOG$) matrices appearing in Eq. (C.2) were full. Zero elements in these matrices must be read in as zero cross sections.

- $ASIG(J,K,L) = \text{absorption cross sections, } \Sigma_{ag}, (K1 + K2) \times (K1 + K2) \times NREG \text{ values.}$
- $FSIG(J,K,L) = \text{fission cross sections times the number of neutrons per fission, } \nu \Sigma_{fg}, (K1 + K2) \times (K1 + K2) \times NREG \text{ values.}$
- $RISG(J,K,L) = \text{removal cross sections, } \Sigma_{rg}, (K1 + K2) \times (K1 + K2) \times NREG \text{ values. All off diagonal entries (}J \neq K\text{) must be read in as negative values.}$
- $D(J,K,L) = \text{diffusion coefficients, } D_g, (K1 + K2) \times (K1 + K2) \times NREG \text{ values (zero for }J \neq K\text{).}$
- $DBUCK(J,K,L) = \text{transverse leakage terms, } D_B R_T^2, (K1 + K2) \times (K1 + K2) \times NREG \text{ values (zero for }J \neq K\text{).}$
- $WINT(J,K,L) = \text{reciprocal group velocities, } v^{-1}, (K1 + K2) \times (K1 + K2) \times NREG \text{ values (zero for }J \neq K\text{).}$

The FORTRAN-II listing of MUDNO-II is given in the following pages.
LIST 8
LABEL
SYMBOL TABLE

MUDMO-II. A computer code for finding the higher order eigenvalues, eigenvectors and adjoint eigenvectors of two eigenvalue problems associated with multigroup diffusion theory. Solutions are found directly in one-dimensional systems. Solutions are found by synthesis in multidimensional systems.

COMMON IND,RAD,CEFT,RSIG,FSIG,D,WABUCK,PSI,PHI,F,T,WAD,TEMP,
2TEMPS,RHOSIG,PHISIG,FISIG,Y,TERRD,FRWDBUCK,KSOLVIHUNT,KQUIT,KPRINT,
3NUMBER,KCOND,KSTOR,ISOLV,IT,HPHH,ISTOP,IREG,KINDM,NPTS,NREG
5,KREG,KIND,MARSTP,LOLim,MNDEX,KBCO,KBCI,IFOUNDI,FUNCT,INDEX,INDEX,
6,NUMAX,KPOW,MORE,NFCT,NOG,KAPP,KCHO,KHUNT,IA,K1,EPA,EPB,DEL,ANA,
7NB*X,VB*X1,XT,XTA,KAKKB,KB,KB,KD,KN,KA,ALPHA,BETA,DT,DTO,ISOLV,KROSS
8,KEND,K2,K3,KG,KG,KG,KG,KG,KG,KG,KPRE,NMPE,NSOL,KeV,WINT,GAMMA,ASIG,DBUCK,
9NPREC,PREC,BETTONT,BV
DIMENSION IND(4),RAD(4),GNU(4),ASIG(6,6,4),RSIG(6,6,4),
2FSIG(6,6,4),F(6,4),D(6,6,4),Z(6,6),WA(6),WB(6),GAM(6),
3EXTP(6),WINT(6,6,4),P(6,6,4),T(6,6),S(6,6),TEMP(6,6),PRE(16)
DIMENSION TEMPS(6,6),RHO(4),PSI(6,6,8),PHI(6,320),PHIC(6),B(16),
2Y(20),V(20),BKWD(6,6,160),FRWD(6,6,160),KSOLV(160),DBUCK(6,6,4)
10 CALL DATA
SET INITIAL PARAMETERS

NREG=NREG
IFOUND=IFOUND
IFUNCT=IFUNCT
KPOW=KPOW
KCHO=KCHO
MORE=MORE
KREG=KREG
INDEX=1
NUMBER=1
15 IHUNT=-1
KQUIT=0
KPRINT=1
KCOND=KSTOR
20 ISOLV=0
IT=1
H=1
KH=1
ISTOP=1
IREG=1
KINDM=NPTS
30 IF(NREG=1) 40,40,50
40 KIND=NPTS
GO TO 60
50 KIND=IND(1)
60 CALL SET
70 IF(KQUIT) 80,80,430
80 CALL START
90 IF(KQUIT) 100, 100, 430
100 CALL CONDIT(MARSTP-1, MARSTP)
110 IF(KH) 120, 120, 170

BACKWARD MARCH

120 IF(KQUIT) 130, 130, 150
130 IF(ISTOP) 400, 400, 140
140 LOLIM=3
   MARSTP=MNDEX-ISTOP+2
   GO TO 260

CONDIT FAILED RESET FREQUENCY OF CONDITIONING

150 KCOND=KCOND-1
   KQUIT=0
160 IF(KCOND=1) 165, 165, 20
165 CALL YEGADS(3, KSTOR, 0)
   GO TO 430

FORWARD MARCH

170 IF(KQUIT) 200, 200, 180
180 MARSTP=MARSTP-1
   KQUIT=0
   MNDEX=MNDEX-1
190 IF(MARSTP-2) 195, 195, 100
195 CALL YEGADS(4, MNDEX, 0)
   GO TO 430

NUMBER OF STEPS TO NEXT CONDITIONING

200 IF(IT) 220, 210, 210
210 LOLIM=3
   MARSTP=3+KBCO
   IT=1
   GO TO 260
220 LOLIM=3
   MARSTP=2+KCOND
230 IF(MNDEX+MARSTP+KBCO-NPTS) 260, 250, 240
240 MARSTP=NPTS-MNDEX-KBCO
250 IT=0
260 CALL MARCH(1)
270 IF(IT) 100, 100, 280
280 GO TO (290, 310), IFOUND
290 CALL EVAL(MARSTP-KBCO+1)
   CALL ZERO
   CALL OUTPUT(1)
   IFOUND=IFOUND
300 GO TO (305, 310), IFOUND
305 IF(INDEX-INDEXM) 20, 430, 430
310 GO TO (380, 320), IFUNCT

START BACKWARD MARCH
IT = 0
IF (NREG = 1) GO TO 340
KIND = 1
GO TO 360
KIND = IND (NREG - 1)
CALL RESTRT
IF (KQUIT) GO TO 100
ARE THERE MORE TRIALS
IF (INDEX - INDEXM) GO TO 390
IF (NUMBER - NUMAX) IFOUND = 1
GO TO (396, 397, 398, 399)
KADJ = 2
KAPP = 0
GO TO 15
GO TO (440, 10)
DO 431 I = 1, NREG
DO 431 K = 1, NOG
DBUCK (K, K, I) = D(K, K, I) * DEL + DBUCK (K, K, I)
GO TO 15
CALL FUNCT
CALL OUTPUT(2)
IF (KAPP) GO TO 388
KADJ = 2
KAPP = 0
GO TO 15
GO TO (440, 10)
CALL EXIT
END
* LIST 8
* LABEL
* SYMBOL TABLE
SUBROUTINE DATA
READS ALL INPUT
COMMON SAME AS MAIN PROGRAM
DIMENSION SAME AS MAIN PROGRAM
READ INPUT TAPE 4, 270, (Y(I), I = 1, 12)
PRINT 300, (Y(I), I = 1, 12)
PUNCH 300, (Y(I), I = 1, 12)
READ INPUT TAPE 4, 280, NREG, IA, NOG, NPTS, KCOND, KEV, KCHO, KREG,
2 IFOUND, KHUNT, IFUNCT, KADJ, KAPP, KPOW, NFCT, KBCI, KBCO, K1, K2, K3, NPREC,
3 INDEXM, NUMAX, KG, KGA, KNA, NEWP, JM, MORE
KSTOR = KCOND
READ INPUT TAPE 4, 290, CEFT, DEL
READ INPUT TAPE 4, 290, EPA, EPB
READ INPUT TAPE 4, 290, (RAD(I), I = 1, NREG)
COMMENT CARDS

80 READ INPUT TAPE 4,270, (Y(I), I=1, 12)
PUNCH 300, (Y(I), I=1, 12)
PRINT 300, (Y(I), I=1, 12)
READ INPUT TAPE 4,270, (Y(I), I=1, 12)
PUNCH 310, (Y(I), I=1, 12)
PRINT 310, (Y(I), I=1, 12)
PRINT 330
90 DO 100 J=1, NOG
READ INPUT TAPE 4,270, (Y(I), I=1, 12)
PUNCH 320, (Y(I), I=1, 12)
100 PRINT 320, (Y(I), I=1, 12)
IF(NEWP) 20, 10, 20
10 RETURN
20 A=FLOATF(IA)
NA=K1+K2
NB=K1+K2
READ INPUT TAPE 4,280, (IND(I), I=1, NREG)
READ INPUT TAPE 4,290, BETTOT
READ INPUT TAPE 4,290, (B(I), I=1, NPRED)
READ INPUT TAPE 4,290, (PREC(I), I=1, NPRED)
NC=NA
IF(KEV==4) 400, 410, 400
410 NC=2*K1
400 READ INPUT TAPE 4,290, (F(J,K), J=1, NC), K=1, NREG)
READ INPUT TAPE 4,290, (ASIG(J,K,L), J=1, NC), K=1, NC), L=1, NREG)
READ INPUT TAPE 4,290, (FSIG(J,K,L), J=1, NC), K=1, NC), L=1, NREG)
READ INPUT TAPE 4,290, (RSIG(J,K,L), J=1, NC), K=1, NC), L=1, NREG)
READ INPUT TAPE 4,290, (DBUCK(J,K,L), J=1, NC), K=1, NC), L=1, NREG)
READ INPUT TAPE 4,290, (WINT(J,K,L), J=1, NC), K=1, NC), L=1, NREG)
DO 420 I=1, NREG
420 WA(I)=FSIG(1,1, I)
200 DO 230 I=1, NA
210 DO 220 J=1, NA
220 Z(I,J)=0.
230 Z(I,I)=1.
240 RETURN
270 FORMAT (1X 11A6, A5)
280 FORMAT (20I3)
290 FORMAT (15X 4E15.8)
300 FORMAT (1H1 11A6, A5)
310 FORMAT (1H0 11A6, A5)
320 FORMAT (1H 11A6, A5)
330 FORMAT (1H )
END
* LIST 8
* LABEL
* SYMBOL TABLE
* SUBROUTINE SET
C
C CALCULATES P MATRIX
COMMON SAME AS MAIN PROGRAM
DIMENSION SAME AS MAIN PROGRAM

10 IF(INDEX-1) 30,30,20
20 GO TO (30,240,240,240),KCHO
30 RHO(I)=RAD(I)/FLOATF(IND(I))
40 IF(NREG-1) 70,70,50
50 DO 60 I=2,NREG
60 RHO(I)=RAD(I)/FLOATF(IND(I)-IND(I-1))
70 DO 90 I=1,NREG
75 IF(RHO(I)) 80,80,90
80 CALL YEGADS(1,1,2)
90 CONTINUE
240 GO TO(100,110,120,130),KEV
100 PEV=CEFT
W=0.
GO TO 140
110 PEV=1.-BETTOT
W=CEFT
GO TO 140
120 PEV=1.0
W=CEFT
DO 121 I=1,NPREC
121 PEV=PEV-CEFT*B(I)/(PREC(I)+CEFT)
GO TO 140
130 PEV=1.-BETTOT
W=CEFT
DO 131 L=1,NREG
131 FSIG(I,J,L)=WA(L)+FSIG(I,J,L)*RSIG(I,J,L)/(ASIG(2,J,L)+CEFT/
2*WINT(2,J,L)+DBUCK(2,J,L))
140 DO 340 L=1,NREG
X=RHO(L)*RHO(L)
DO 270 J=1,N
DO 270 K=1,N
2*FSIG(J,K,L)*PEV
GO TO (273,360), KADJ
273 GO TO (271,272), KPRINT
271 PRINT 400,X*PEV,W
PRINT 400, ((P(I,J,L),J=1,N),I=1,N)
GO TO 272
360 DO 361 J=1,N
DO 361 K=1,N
TEMPA(J,K)=D(J,K,L)
361 TEMP(J,K)=P(J,K,L)
DO 362 J=1,N
DO 362 K=1,N
D(J,K,L)=TEMPA(K,J)
362 P(J,K,L)=TEMP(K,J)
GO TO (326,272), KPRINT
326 PRINT 400, CEFT, CEFT
PRINT 400, ((P(I,J,L),J=1,N),I=1,N)
272 DO 290 J=1,N
DO 290 K=1,N
290 S(J,K)=D(J,K,L)
CALL DETINV(1,2)
DO 320 J=1,NA
DO 320 K=1,NA
TEMP(J,K)=0.0
DO 320 M=1,NA
TEMP(J,K)=TEMP(J,K)+X*T(J,M)*P(M,K,L)
DO 310 J=1,NA
DO 310 K=1,NA
P(J,K,L)=TEMP(J,K)
320 DO 340 I=1,NA
340 P(I,I,L)=2.0+P(I,I,L)
GO TO (341,350),KPRINT
341 PRINT 400,CEFT
PRINT 400,(((P(I,J,L),J=1,NA),I=1,NA),L=1,NREG)
350 RETURN
400 FORMAT (1H 14X 4E15.8)
500 FORMAT (139 12X E15.8)
510 FORMAT (15X 4E15.8)
END
LIST
LABEL
SYMBOL TABLE
SUBROUTINE START
BEGIN MARCH
COMMON SAME AS MAIN PROGRAM
DIMENSION SAME AS MAIN PROGRAM
MNDEX=1
20 DO 40 I=1,NA
30 DO 40 J=1,NA
40 PSI(I,J,1)=Z(I,J)
GO TO (90,190),KBCI
PHI(0) = 0
90 X=FLOATF(2*MNDEX)
ALPHA=X/(X+A)
BETA=(X-A)/(X+A)
GAMMA=2.*FLOATF(JM)**2*LOGF(1./BETA)/X
100 DO 130 I=1,NA
110 DO 130 J=1,NB
PSI(I,J,2)=ALPHA*GAMMA*PSI(I,J,1)
120 DO 130 L=1,NA
130 PSI(I,J,2)=PSI(I,J,2)+ALPHA*P(I,L,1)*PSI(L,J,1)
MNDEX=2
X=FLOATF(2*MNDEX)
ALPHA=X/(X+A)
BETA=(X-A)/(X+A)
GAMMA=2.*FLOATF(JM)**2*LOGF(1./BETA)/X
140 DO 170 I=1,NA
150 DO 170 J=1,NB
PSI(I,J,3)=-BETA*PSI(I,J,1) + ALPHA*GAMMA*PSI(I,J,2)
160 DO 170 L=1,NA
170 PSI(I,J,3)=PSI(I,J,3)+ALPHA*P(I,L,1)*PSI(L,J,2)
MARSTP=3
224

180 RETURN
C
D(\text{PHI}(0))/\text{DR} = 0
C
190 DO 220 I=1,\text{NA}
200 DO 210 J=1,\text{NA}
210 S(I,J)=P(I,J,1)
X=1./[2.*((1.+A)]
230 DO 260 I=1,\text{NA}
240 DO 250 J=1,\text{NA}
250 S(I,J)=X*S(I,J)
CALL DETINV(1,3)
270 IF(KQUIT) 280,280,330
280 X=FLOATF(2*\text{MNDEX})
ALPHA=X/(X+A)
BETA=(X-A)/(X+A)
290 DO 320 I=1,\text{NA}
300 DO 320 J=1,\text{NB}
PSI(I,J,2)=0.
310 DO 320 L=1,\text{NA}
320 PSI(I,J,2)=PSI(I,J,2)+(ALPHA*P(IL,1)-BETA*T(IL))*PSI(L,J,1)
MARSTP=2
330 RETURN
END

LIST 8
LABEL
SYMBOL TABLE
FUNCTION BET(M)
C
C
ALPHA, BETA AND GAMMA
C
COMMON SAME AS MAIN PROGRAM
DIMENSION SAME AS MAIN PROGRAM
10 IF(IA) 20,20,30
20 ALPHA=1.
BET=1.
GAMMA=0.
GO TO 80
30 IF(M-1) 40,40,50
40 X=\text{MNDEX}
GO TO 70
50 X=0.
MM1=M-1
DO 60 I=1,MM1
60 X=X+RAD(I)
X=FLOATF(\text{MNDEX}-\text{IND}(MM1))\times \text{RHO}(M)
70 XA=0.5*H*A
ALPHA=X/(X+XA)
BET=(X-XA)/(X+XA)
GAMMA=(ALPHA/X)*(FLOATF(JM))**2*LOGF((X+0.5)/(X-0.5))
80 RETURN
END
LIST 8
* LABEL
SYMBOL TABLE
SUBROUTINE DETINV(M,II)
C C
INVERTS S TO GIVE T OR EVALUATES DETERMINANT(S)
C C
COMMON SAME AS MAIN PROGRAM
DIMENSION SAME AS MAIN PROGRAM
BIG=1.E25
GO TO (2,2,1,2,2,1,1,2,2,1,1,2,2,2,2,1),II
1 NC=NA
GO TO 10
2 NC=NB
10 GO TO (20,70),M

UNIT MATRIX NEEDED FOR INVERSION

20 DO 40 I=1,NC
30 DO 40 J=1,NC
40 TEMP(I,J)=0.
50 DO 60 I=1,NC
60 TEMP(I,I)=1.
70 IF(ABSF(S(1,1))<EPB) 80,80,130
80 CALL YEGADS(2,1,II)
90 DT=0.
100 RETURN
110 DT=0.
120 RETURN

CROUT REDUCTION

130 IF(NC-1) 140,140,130
140 DO 150 I=2,NC
150 S(I,1)=S(I,1)/S(1,1)
160 GO TO (170,180),M
170 TEMP(1,1)=TEMP(1,1)/S(1,1)
180 IF(NC-1) 190,190,185
185 DO 360 K=2,NC
190 IMIN=K-1
200 DO 210 I=1,IMIN
210 S(K+1)=S(K+1)-S(K,1)*S(1,K)
220 GO TO (230,240),M
230 CALL YEGADS(2,1,II)
240 DT=0.
250 GO TO 540
260 IMAX=K+1
270 IF(NC-IMAX) 280,280,270
280 DO 290 J=1,IMIN
290 S(I,J)=S(I,J)-S(I,K)*S(J,K)
300 S(K+1)=S(K+1)-S(K,J)*S(J,K)
310 GO TO (320,360),M
DO 350 I=1,K
330 DO 340 J=1,IMIN
340 TEMP(K,I)=TEMP(K,I)-S(K,J)*TEMP(J,I)
350 TEMP(K,I)=TEMP(K,I)/S(K,K)
360 CONTINUE
370 GO TO (380,470),M

**Inverse Matrix**

380 DO 390 J=1,NC
390 T(NC,J)=TEMP(NC,J)
400 IF(NC-1) 460,460,420
420 DO 450 K=2,NC
  IMIN=NC+1-K
  IMAX=IMIN+1
430 DO 450 J=1,NC
  T(IMIN,J)=TEMP(IMIN,J)
440 DO 450 I=IMAX,NC
  [IMIN,J]=T(IMIN,J)-S(IMIN,I)*T(I,J)
460 RETURN

**Determinant**

470 DT=S(1,1)/10.
480 IF(NC-1) 540,540,500
500 DO 530 I=2,NC
  DT=DT*S(I,I)/10.
510 IF(ABS(DT)-BIG) 530,530,520
520 DT=SIGNF(BIG,DT)
530 CONTINUE
540 RETURN

END

* LIST 8
* LABEL
* SYMBOL TABLE
SUBROUTINE CONDIT(N,M)

**Conditioning Transformation**

COMMON SAME AS MAIN PROGRAM
DIMENSION SAME AS MAIN PROGRAM
20 DO 40 I=1,NA
30 DO 40 J=1,NA
  S(I,J)=PSI(I,J,M)
40 TEMPA(I,J)=PSI(I,J,N)
90 CALL DETINV(1,5)
100 IF(KQUIT) 110,110,210
110 H=120,210
110 IF(H) 120,120,120

**Forward March**

120 ISOLV=ISOLV+1
  KSOLV(ISOLV)=MNDEX+1
130 DO 160 I=1,NB
140 DO 160 J=1,NB
FRWD(I,J,ISOLV) = 0.150 DO 160 K = 1, NB
   160 FRWD(I,J,ISOLV) = FRWD(I,J,ISOLV) + TEMPA(I,K) * T(K,J)
   180 DO 200 I = 1, NA
   190 DO 200 J = 1, NA
       PSI(I,J,1) = FRWD(I,J,ISOLV)
   200 PSI(I,J,2) = Z(I,J)
   210 RETURN

C C C BACKWARD MARCH

   270 DO 300 I = 1, NB
   280 DO 300 J = 1, NB
       BKWD(I,J,ISOLV) = 0.
   290 DO 300 K = 1, NB
   300 BKWD(I,J,ISOLV) = BKWD(I,J,ISOLV) + TEMPA(I,K) * T(K,J)
   320 DO 340 I = 1, NA
   330 DO 340 J = 1, NA
       PSI(I,J,1) = BKWD(I,J,ISOLV)
   340 PSI(I,J,2) = Z(I,J)
   390 ISOLV = ISOLV - 1
   400 IF(ISOLV) 410, 410, 420
   410 ISTOP = 0
       GO TO 430
   420 ISTOP = KSOLV(ISOLV)
   430 RETURN
END

* LIST 8
* LABEL
* SYMBOL TABLE
SUBROUTINE CROSS(KL, KR, K, KFOP)

C C MARCH ACROSS BOUNDARY

COMMON SAME AS MAIN PROGRAM
DIMENSION SAME AS MAIN PROGRAM
KA = K + 1
KB = K - 1
X = 0.
KTEMP = (KL + KR) / 2
   DO 1 I = 1, KTEMP
       1 X = X + RAD(I)
       XA = 0.5 * H * A * RHO(KR)
       ALPHA = X / (X + XA)
       BETA = (X - 0.5 * A * H * RHO(KL)) / (X + XA)
       IF(KH) 10, 10, 20
   10 ARG1 = ALPHA
       ARG2 = BETA / ALPHA
       GO TO 100
   20 ARG1 = 1. / ALPHA
       ARG2 = ALPHA / BETA
   100 DO 120 I = 1, NA
   110 DO 120 J = 1, NA
       S(I,J) = D(I,J,KR)
   120 CALL DETINV(1, 6)

IF(KQUIT) 60,60,300
60 DO 90 I=1,NA
70 DO 90 J=1,NA
   TEMP(I,J)=0.
80 DO 90 L=1,NA
90 TEMP(I,J)=TEMP(I,J)+T(I,L)*D(L,J,KL)
   BM=FLOATF(JM)**2/X
150 X=RHO(KR)/RHO(KL)
   XA=BETA*X
   Q=ALPHA/2.*
   QQ=(BETA=ALPHA+ALPHA*BM*RHO(KL)*LOGF(ARG2))*X
   QQ=1.-ALPHA+ALPHA*BM*RHO(KR)*LOGF(ARG1)
160 DO 205 I=1,NA
170 DO 200 J=1,NA
   S(I,J)=0.
180 DO 190 L=1,NA
190   S(I,J)=S(I,J)+X*TEMP(I,L)*P(L,J,KL)
   T(I,J)=XA*TEMP(I,J)
205 S(I,I)=S(I,I)+QQQ
210 GO TO (220,270),KFOP

EXPANSION VECTORS

220 DO 250 I=1,NA
230 DO 250 J=1,NA
   PSI(I,J,KA)=0.
240 DO 250 L=1,NA
250   PSI(I,J,KA)=PSI(I,J,KA)+S(I,L)*PSI(L,KA)+T(I,L)*PSI(L,KB)
   GO TO 300

EIGENFUNCTION

270 KA=MNDEX+KH
   KB=MNDEX-KH
275 DO 290 I=1,NA
   PHI(I,KA)=0.
280 DO 290 L=1,NA
290   PHI(I,KA)=PHI(I,KA)+S(I,L)*PHI(L,MNDEX)-T(I,L)*PHI(L,KB)
300 RETURN
END

* LIST 8
* LABEL
* SYMBOL TABLE
SUBROUTINE MARCH(IM)

MARCH BETWEEN POINTS OF CONDITIONING

COMMON SAME AS MAIN PROGRAM
DIMENSION SAME AS MAIN PROGRAM
IREG=IREG
10 DO 240 J=LOLIM,MARSTP
   MNDEX=MNDEX+KH
   BETA=BET(IREG)
TEST FOR BOUNDARY CROSSING

20 IF(MNDEX-KIND) 150,30,150
30 IF(KINDM-KIND) 50,40,50
40 IT=J-1
   GO TO 250
50 CALL CROSS(IREG,IREG+KH•J-1,M)
60 IF(KQUIT) 70,70,250
70 IREG=IREG+KH
   KROSS=KROSS+1
80 IF(KH) 90,90,120
90 IF(IREG-1) 100,100,110
100 KIND=1
   GO TO 240
110 KIND=IND(IREG-1)
   GO TO 240
120 KIND=IND(IREG)
130 IF(NREG-IREG) 140,140,240
140 KIND=KIND+KBCO-1
   GO TO 240
150 GO TO (191,160),M

EIGENFUNCTION

160 KA=MNDEX+KH
   KB=MNDEX-KH
170 DO 190 I=1,NA
   PHI(I,K)=BETA*PHI(I,KB)+GAMMA*PHI(I,MNDEX)
180 DO 190 K=1,NA
190 PHI(I,K)=PHI(I,K)+ALPHA*P(I,K,IREG)*PHI(K,MNDEX)
   GO TO 240

EXANSION VECTORS

191 DO 192 I=1,NA
   DO 192 K=1,NA
192 TEMP(I,K)=ALPHA*P(I,K,IREG)
200 DO 230 I=1,NA
210 DO 230 K=1,NB
   PSI(I,K,J)=-BETA*PSI(I,K,J-2)+GAMMA*PSI(I,K,J-1)
220 DO 230 L=1,NA
230 PSI(I,K,J)=PSI(I,K,J)+TEMP(I,L)*PSI(L,K,J-1)
240 CONTINUE
250 RETURN
END

* LIST 8
* LABEL
* SYMBOL TABLE
* SUBROUTINE EVAL(M)

SET UP MATRIX FOR DETERMINANT EVALUATION

COMMON SAME AS MAIN PROGRAM
DIMENSION SAME AS MAIN PROGRAM
PHI(M) = 0

DO 50 I=1,NA
40 DO 50 J=1,NA
50 S(I,J)=PSI(IJ,M)
290 CALL DETINV(2,8)
300 RETURN
END

* LIST 8
* LABEL
* SYMBOL TABLE
SUBROUTINE ZERO

SEARCH FOR DETERMINANT EQUAL ZERO
THREE POINT INTERPOLATION

COMMON SAME AS MAIN PROGRAM
DIMENSION SAME AS MAIN PROGRAM
KB=1

10 GO TO (20,70),KHUNT

NO SEARCH

20 GO TO (30,40,41,42),KCHO
30 V(1)=RAD(KREG)
   RAD(KREG)=RAD(KREG)+DEL
   GO TO 50
40 V(1)=CEFT
   CEFT=CEFT+DEL
   GO TO 50
41 V(1)=ASIG(KG,KG,KREG)
   ASIG(KG,KG,KREG)=ASIG(KG,KG,KREG)+DEL
   GO TO 50
42 V(1)=DBUCK(KG,KG,KREG)
   DO 43 I=1,NREG
   DO 43 K=1,NOG
   43 DBUCK(K,K,I)=DBUCK(K,K,I)+D(K,K,I)*DEL
50 INDEX=INDEX+1
60 RETURN

SEARCH

70 IF(DT) 90,80,100
80 GO TO (85,86,87,88),KCHO
85 V(1)=RAD(KREG)
   GO TO 210
86 V(1)=CEFT
   GO TO 210
87 V(1)=ASIG(KG,KG,KREG)
   GO TO 210
88 V(1)=DBUCK(KG,KG,KREG)
   GO TO 210
90 KB=2
100 IF(IHUNT) 105,105,250
105 GO TO (110,120,121,122),KCHO
231

110 V(KB)=RAD(KREG)
   GO TO 125
120 V(KB)=CEFT
   GO TO 125
121 V(KB)=ASIG(KG,KG,KREG)
   GO TO 125
122 V(KB)=DBUCK(KG,KG,KREG)
125 V(KB+3)=DT
130 IF(IHUNT) 140,150,250
140 DTO=DT
   IHUNT=0
   GO TO 20
150 X=1.
   X=SIGNF(X,DTO)
   XA=1.
   XA=SIGNF(XA,DT)
   TEST FOR PROPER CHANGE IN SIGN

160 IF(X*X A) 170,180,140
170 IF(XA) 180,180,140
180 IHUNT=1
190 VV=0.5*(V(1)+V(2))
200 IF(ABSF((V(1)-V(2))/VV)-EPA) 210,210,220
210 IFOUND=2
   NUMBER=NUMBER+1
220 GO TO (230,240,241,242),KCHO
230 RAD(KREG)=VV
   GO TO 60
240 CEFT=VV
   GO TO 60
241 ASIG(KG,KG,KREG)=VV
   GO TO 60
242 VVV=DBUCK(KG,KG,KREG)
   DO 243 I=1,NREG
   DO 243 K=1,NOG
243 DBUCK(K,K,I)=DBUCK(K,K,I)*VV/VVV
   DBUCK(KG,KG,KREG)=VV
   GO TO 60
250 V(3)=V(KB)
   V(6)=V(KB+3)
   GO TO (260,270,271,272),KCHO
260 V(KB)=RAD(KREG)
   GO TO 280
270 V(KB)=CEFT
   GO TO 280
271 V(KB)=ASIG(KG,KG,KREG)
   GO TO 280
272 V(KB)=DBUCK(KG,KG,KREG)
280 V(KB+3)=DT
   VV=V(1)*V(5)*V(6)/((V(4)-V(5))*(V(4)-V(6)))+V(2)*V(4)*V(6)/((V(5)-V(4))*(V(5)-V(6)))+V(3)*V(4)*V(5)/((V(6)-V(4))*(V(6)-V(5)))
   IF(V(2)-VV) 190,190,290
290 IF(VV-V(1)) 190,190,200
END
* LIST 8
* LABEL
* SYMBOL TABLE
SUBROUTINE RESTRT

BEGIN BACKWARD MARCH

COMMON SAME AS MAIN PROGRAM
DIMENSION SAME AS MAIN PROGRAM
KH=-1
H=-1.
MNDEX=IND(NREG)

DO 40 I=1,NA
30 DO 40 J=1,NA
40 PSI(I,J,1)=Z(I,J)
80 GO TO (90, 90),KBCO

PHI(M) = 0

90 MNDEX=MNDEX-1
BETA=BET(NREG)

DO 100 I=1,NA
110 DO 100 J=1,NB
PSI(I,J,2)=GAMMA*PSI(I,J,1)
120 DO 100 K=1,NA
130 PSI(I,J,2)=PSI(I,J,2)+ALPHA*(I,K,NREG)*PSI(K,J,1)

ISOLVM=ISOLV
MARSTP=MNDEX+2-KSOLV(ISOLV)
LOLIM=3
IREG=NREG

IF(MARSTP-LOLIM) 470,460,460
460 CALL MARCH(1)
470 RETURN

SOLVES FOR EIGENFUNCTION

COMMON SAME AS MAIN PROGRAM
DIMENSION SAME AS MAIN PROGRAM
V2=WINT(2,2,1)
IREG=NREG
KEND=NPTS+1

DO 520 J=1,ISOLVM

MARCH FORWARD TO LAST VALUE OF PHI

K=ISOLVM+1-J
H=1.
KH=1
N=KSOLV(K)
CALL HOSIEQ(1,K)
20 IF(IREG-1) 50,50,30
30 IF(N=IND(IREG-1)) 40,40,50
40 IREG=IREG-1
GO TO 20
50 MNDEX=N-1
50 MARSTP=KEND-N+1
50 LOLIM=3
60 IF(MARSTP-LOLIM) 110,70,70
70 KROSS=0
70 KIND=IND(IREG)
80 IF(NREG-IREG) 90,90,100
90 KIND=KIND+KBCO-1
100 CALL MARCH(2)
101 IF(KQUIT) 109,109,1000
109 IREG=IREG-KROSS
110 IF(J=1) 300,300,120
120 IF(KEND-N) 130,130,140
130 KN=N
GO TO 145
140 KN=KEND-1
145 KEND=KN+1
150 DO 180 I=1,NREG
160 IF(KEND-IND(I)) 180,170,180
170 KA=I
GO TO 190
180 CONTINUE
180 KA=IREG+KROSS
180 IF(KN-IND(KA)) 186,182,186
182 KA=KA+1
186 KB=KN+1
186 KC=KN+2
186 KD=KN
186 KH=-1
186 H=-1.
GO TO 200
190 KB=KN
190 KC=KN-1
190 KD=KN+1
190 KH=1
190 H=1.
MATCH
200 MNDEX=KB
200 BETA=BET(KA)
210 DO 220 L=1,NA
210 Y(L)=-BETA*PHI(L,KC)+GAMMA*PHI(L,KB)
210 DO 220 L=1,NA
220 Y(L)=Y(L)+ALPHA*P(L,KA)*PHI(I,KB)
220 IF(KH) 222,221,221
221 CM=PHI(KG,KD)/Y(KG)
GO TO 229
222 CM=Y(KG)/PHI(KG,KD)
229 LOLIM=N-1
229 X=FLOATF((KH+1)/2)*CM+FLOATF((1-KH)/2)
240 DO 250 I=1,NA
    Y(I)=X*Y(I)
230 DO 250 L=LOLIM,KN
250 PHI(I,L)=CM*PHI(I,L)
    IPR=0
    DO 255 I=1,NA
    Y(I)=(Y(I)-PHI(I,KD))*100./PHI(I,KD)
    WRITE MATCH ERROR
255 CONTINUE
    IF(IPR) 300,300,257
257 IPR=1
255 PRINT
    PRINT 1010, KN,KG
    PRINT 1020, (Y(I),I=1,NA)
300 DO 310 I=1,NA
310 V(I)=PHI(I,N)
MARCH BACKWARD HALFWAY TO NEXT POINT OF CONDITIONING
    H=-1.
    KH=-1
    CALL HOSIEQ(2,K)
320 IF(K-1) 330,330,340
330 KEND=1
    GO TO 350
340 KEND=N-1-(KSOLV(K)-KSOLV(K-1))/2
350 LOLIM=3
    MARSTP=1+N-KEND
360 IF(MARSTP-LOLIM) 430,370,370
370 MNDEX=N
    KROSS=0
380 IF(IREG-1) 390,390,400
390 KIND=1
    GO TO 410
400 KIND=IND(IREG-1)
410 CALL MARCH(2)
420 IF(KQUIT) 430,430,1000
430 IREG=IREG
    LOLIM=KEND
    CM=V(KG)/PHI(KG,N)
    MARSTP=N-1
440 DO 470 I=1,NA
450 DO 460 L=LOLIM,MARSTP
460 PHI(I,L)=CM*PHI(I,L)
470 PHI(I,N)=V(I)
520 CONTINUE
    KH=1
    H=1.
    PHI(0)
660 ALPHA=(2./((2.-A)
    BETA=(2.+A)/(2.-A)
GAMMA = ALPHA * FLOATF(JM)**2 * LOGF(3.) * A

DO 690 I = 1, NA
PHIC(I) = -BETA * PHI(I, 2) + GAMMA * PHI(I, 1)

DO 690 J = 1, NA
PHIC(I) = PHIC(I) + ALPHA * P(IJ, 1) * PHI(J, 1)

NORMALIZE TO PHI(KGA, KNA) = 1.

IF (KGA) 942, 942, 9946
CM = 1. / PHIC(KGA)
GO TO 950

CM = 1. / PHI(KGA, KNA)
DO 990 I = 1, NA
Y(I) = CM * Y(I)
PHIC(I) = CM * PHIC(I)

DO 990 J = 1, NPTS
PHI(I, J) = CM * PHI(I, J)
IF (KEV <= 4) 1000, 991, 1000

PHIC(2) = PHIC(1) * RSIG(1, 1, 1) / (ASIG(2, 2, 1) + CEFT/V2 + DBUCK(2, 2, 1))  
IREG = 1
DO 992 J = 1, NPTS
IF (J - IND(IREG)) 992, 993
IREG = IREG + 1
993

PHI(2, J) = PHI(1, J) * RSIG(1, 1, IREG) / (ASIG(2, 2, IREG) + CEFT/V2 +  
2DBUCK(2, 2, IREG))
1000 RETURN

FORMAT (1HO 14X 42H HERE IS THE PERCENT ERROR FOR MATCH AT X = 13, 215H FOR Y MATCH AT I3, 1H.)  
1020 FORMAT (1HO 14X 4E15.8)

END

COMMON SAME AS MAIN PROGRAM
DIMENSION SAME AS MAIN PROGRAM
GO TO (40, 90), M

FORWARD MARCH

DO 80 I = 1, NB
DO 75 J = 1, NB
S(I, J) = 0.
DO 70 L = 1, NB
S(I, J) = S(I, J) + FRWD(I, L, K) * BKWD(L, J, K)
T(I, J) = S(I, J)
S(I, I) = S(I, I) - 1.
GO TO 140

BACKWARD MARCH

DO 130 I = 1, NB
236

100 DO 125 J=1,NB
   S(I,J)=0.
110 DO 120 L=1,NB
120  S(I,J)=S(I,J)+BKWD(I,L,K)*FRWD(L,J,K)
125  T(I,J)=S(I,J)
140 IF(NB-1) 150,150,160
150 Y(1)=1.
   GO TO 400

   WILKINSON'S METHOD

160 DO 270 I=2,NB
   KA=I-1
170 DO 270 J=1,KA
180 IF(S(I,J)) 190,270,190
190 IF(ABSF(S(J,J)) - ABSF(S(I,J))) 210,200,200
200  X=S(I,J)/S(J,J)
   GO TO 240
210  X=S(J,J)/S(I,J)
220 DO 230 L=1,NB
   XA=S(J,L)
   S(J,L)=S(I,L)
230  S(I,L)=XA
240  KB=J+1
250 DO 260 L=KB,NB
260  S(I,L)=S(I,L)-X*S(J,L)
270 CONTINUE
   XA=S(NB,NB)
   Y(NB)=1.
280 DO 350 I=2,NB
   KB=NB-I+1
   X=0.
   KA=NB-I+2
290 DO 295 L=KA,NB
295  X=X+S(KB,L)*Y(L)
300 IF(ABSF(S(KB,KB)) - 1. * E-10) 310, 310, 340
310  Y(KB)=1.
   XA=0.
320 DO 330 J=KA,NB
330  Y(J)=0.
   GO TO 350
340  Y(KB)=(XA-X)/S(KB,KB)
350 CONTINUE
351 IF(K3) 359,359,890

   ITERATION TO REDUCE ERROR

890 DO 1000 K=1,K3
   DO 960 I=1,NB
   Y(I)=0.
   KA=I-1
   KB=I+1
   IF(ABSF(T(I,I)-1.)*EPB) 960,960,900
900 IF(KA) 930,930,910
DO 920 J=1,KA
Y(I)=Y(I)+T(I,J)*Y(J)
IF(KB-NB) 940,940,955
DO 950 J=KB,NB
Y(I)=Y(I)+T(I,J)*Y(J)
Y(I)=Y(I)/(1.-T(I,I))
CONTINUE

X=ABSF(Y(1))
XA=1.
XA=SIGNF(XA,Y(1))
DO 980 I=2,NB
IF(X-ABSF(Y(I))) 970,980,980
X=ABSF(Y(I))
XA=1.
XA=SIGNF(XA,Y(I))
CONTINUE
X=XA/X
DO 990 I=1,NB
Y(I)=X*Y(I)
CONTINUE

NORMALIZE TO MAXIMUM VALUE OF Y = 1.

DO 390 I=2,NB
IF(X-ABSF(Y(I))) 380,390,390
X=ABSF(Y(I))
XA=1.
XA=SIGNF(XA,Y(I))
CONTINUE
X=XA/X
DO 395 I=1,NB
Y(I)=X*Y(I)

PHI(J) AND PHI(J-1)

GO TO (410,500),M
FORWARD MARCH

KA=K Sov(K)-1
KB=KA+1
DO 450 I=1,NA
PHI(I,KA)=Y(I)
PHI(I,KB)=0.
DO 450 I=1,NA
PHI(I,KB)=PHI(I,KB)+BKWD(I,J,K)*Y(J)
GO TO 590

BACKWARD MARCH

KA=K Sov(K)
KB=KA-1
520 DO 540 I=1,NA
   PHI(I,KA)=Y(I)
   PHI(I,KB)=0.
530 DO 540 J=1,NA
540 PHI(I,KB)=PHI(I,KB)+FRWD(I,J,K)*Y(J)
590 RETURN
END

* LIST 8
** LABEL
* SYMBOL TABLE
SUBROUTINE OUTPUT(M)
WRITE RESULTS

COMMON SAME AS MAIN PROGRAM
DIMENSION SAME AS MAIN PROGRAM
10 GO TO (20,260),KPRINT

GENERAL INFORMATION

20 KPRINT=2
70 IF(IA) 90,90,100
90 PRINT 570
   IF(K1-1) 91,91,92
91 PRINT 611
   GO TO 110
92 PRINT 610,K1,K2
   GO TO 110
100 PRINT 580
   PRINT 581,JM
   IF(K1-1) 101,101,102
101 PRINT 611
   GO TO 110
102 PRINT 620,K1,K2
110 GO TO (120,130),KBCI
120 PRINT 630
   GO TO 150
130 PRINT 640
150 PRINT 650
170 PRINT 670,NREG,NOG,NPTS
   GO TO (171,172,173,174),KEV
171 PRINT 671
   GO TO 180
172 PRINT 672
   GO TO 180
173 PRINT 673
   GO TO 180
174 PRINT 674
180 GO TO (190,260,260),M
190 PRINT 680,KCOND,EPA
   GO TO (193,192),KADJ
192 PRINT 682
193 PRINT 690
PRINT 700
X=0.
200 DO 210 I=1,NREG
   XA=CEFT
   X=X+RAD(I)
210 PRINT 710, I,XA,IND(I),X
   GO TO (230,240,241,242),KCHO
230 PRINT 720, KREG
   PRINT 700
235 PRINT 740, V(KB),DT
   GO TO 250
240 PRINT 730
   PRINT 700
245 XA=V(KB)
   PRINT 740, XA,DT
   GO TO 250
241 PRINT 830,KG,KG,KREG
   PRINT 700
243 PRINT 740, V(KB),DT
   GO TO 250
242 PRINT 840,KG,KG,KREG
   PRINT 700
244 PRINT 740, V(KB),DT
250 RETURN
260 GO TO (270,290),M
C C EIGENVALUE AND DETERMINANT
270 GO TO (235,245,243,244),KCHO
C C EIGENFUNCTION
290 PRINT 750
   PRINT 690
   PRINT 700
   X=0.
300 DO 310 I=1,NREG
   X=X+RAD(I)
   XA=CEFT
310 PRINT 710, I,XA,IND(I),X
   PRINT 680, KCOND, EPA
   GO TO (311,312),KADJ
311 PRINT 681
   GO TO 313
312 PRINT 682
313 PRINT 760, NA
   PRINT 700
   X=0.
   NC=NA
   IF(KEV=4) 320,321,320
320 IF(NC=4) 314,314,317
314 PRINT 770,X,(PHIC(I),I=1,NC)
   GO TO 315
317 PRINT 770,X,(PHIC(I),I=1,4)
   PRINT 771, (PHIC(I),I=5,NC)
315 GORTOT (332, 331), KPOW 'THIS IS THE ADJOINT EIGENFUNCTION')
319 PUNCH 770 7, (PHI(I), I=1, NC)
GO TO 332
321 IF (NC=4) 318, 318, 319, IREG=1
326 DO 405 J=NFC+1, NFC+1
327 IF (J=IND (IREG)) 380, 380, 370
328 IREG=IREG+1
329 X=X+FLOATF(NFCT)*RHO(IREG)
330 IF (NC=4) 395, 395, 394
331 PRINT 770 7, (PHI(I), I=1, NC)
GO TO 393
332 PUNCH 770, X, (PHI(I), I=1, NC)
333 PRINT 771, (PHI(I), I=1, NC)
334 CONTINUE
335 PRINT 750 SAME AS MAIN PROGRAM
336 RETURN
570 FORMAT (1H19X 36HTHE REACTOR BEING STUDIED IS A SLAB.)
580 FORMAT (1H19X 40HTHE REACTOR BEING STUDIED IS A CYLINDER.)
581 FORMAT (1H19X 51HTHE AZIMUTHAL MODE IS INDEXED BY I2)
610 FORMAT (1H19X 50HTHE MULTIDIMENSIONAL VECTOR EIGENFUNCTION IS BEING SYNTHESIZED BY A MARCH IN THE AXIAL DIRECTION USING 2ING/1H 14X 51HTHERAL DISTRIBUTIONS AND I3, 29HTHERMAL DISTRIBUTIONS)
611 FORMAT (1H19X 49HTHE MULTIDIMENSIONAL VECTOR EIGENFUNCTION IS BEING SYNTHESIZED BY A MARCH IN THE RADIAL DIRECTION USING 2ING/1H 14X 51HTHERAL DISTRIBUTIONS AND I3, 29HTHERMAL DISTRIBUTIONS)
620 FORMAT (1H19X 50HTHE MULTIDIMENSIONAL VECTOR EIGENFUNCTION IS BEING SYNTHESIZED BY A MARCH IN THE RADIAL DIRECTION USING 2ING/1H 14X 51HTHERAL DISTRIBUTIONS AND I3, 29HTHERMAL DISTRIBUTIONS)
630 FORMAT (1H19X 53HTHE INNER BOUNDARY CONDITION IS THE FLUX EQUALS ZERO.)
640 FORMAT (1H19X 51HTHE INNER BOUNDARY CONDITION IS THE GRADIENT OF THE FLUX)
650 FORMAT (1H19X 47HTHE OUTER BOUNDARY CONDITION IS THE HOMOGENEOUS FLUX)
660 FORMAT (1H19X 28HTHE NUMBER OF REGIONS EQUALS I3, 22HTHE NUMBER OF GROUPS EQUALS I3, 39HTHE NUMBER OF SPACE POINTS EQUALS I3, 1H.)
670 FORMAT (1H19X 26HTHE MODES ARE LAMBDA MODES)
671 FORMAT (1H19X 22HTHE MODES ARE WP MODES)
672 FORMAT (1H19X 22HTHE MODES ARE WD MODES)
673 FORMAT (1H19X 43HTHE APPROXIMATE CALCULATION OF EPITHERMAL MODES)
680 FORMAT (1H19X 36HTHE FREQUENCY OF CONDITIONING EQUALS I2, 19HTHE APPROXIMATE EIGENVALUES / 1H 14X E10.3, 1H.)
681 FORMAT (1H19X 25HTHE REACTOR BEING STUDIED IS THE EIGENFUNCTION)
SUBROUTINE YEGADS(N1,N2,N3)

WRITE ERROR MESSAGES

COMMON SAME AS MAIN PROGRAM
DIMENSION SAME AS MAIN PROGRAM
KQUIT=1
10 GO TO (20,30,40,50),N1
20 PRINT 70, N2
GO TO 60
30 PRINT 80, N2,N3
GO TO 60
40 PRINT 90, N2
GO TO 60
50 PRINT 100, N2
60 RETURN
70 FORMAT (1H0 19X 25HRHO IS NEGATIVE IN REGION I2, 1H.)
80 FORMAT (1H0 19X 32HMATRIX INVERSION FAILED AT POINT I3, 22H. THE
2INVERTED MATRIX / 1H 14X 18HWAS FOR SUBROUTINE I3, 1H.)
90 FORMAT (1H0 19X 27HCOND WAS REDUCED TO 1 FROM I3, 22HBY REPEATED
2FAILURE OF / 1H 14X 21HCONDIT ON MARCH BACK.)
100 FORMAT (1H0 19X 28HMARSTP WAS REDUCED TO 2 NEAR I3, 22HBY REPEATED
2 FAILURE OF / 1H 14X 24HCONDIT ON MARCH FORWARD.)
END
APPENDIX D
SYNSIG, A COMPUTER CODE FOR THE PREPARATION OF SYNTHESIS INPUT FOR MUDMO-II

D.1 Introduction

SYNSIG is a computer code which prepares punched card input data for the code, MUDMO-II, described in Appendix C. MUDMO-II numerically solves eigenvalue problems in one-dimensional systems. SYNSIG and MUDMO-II, when used in combination, can construct approximations of the higher order natural modes or \( \lambda \)-modes in multidimensional systems described by two group theory. The ideas of the synthesis technique are developed in detail in Refs. 27 and 31. A description of equations and computer programs used for implementing these ideas is given in Ref. 58. This appendix describes the equations and the details of the synthesis technique which are pertinent to SYNSIG.

D.2 The Synthesis Technique

The eigenvalue problem to be solved by the synthesis technique is Eq. (3.2). The development which follows synthesizes a two-dimensional mode from the results of one-dimensional calculations.

The two-dimensional, two neutron energy group version of Eq. (3.2) is

\[
\nabla \cdot \left[ D(x, z) \nabla N_{mn,k}(x, z) + \mu(x, z, m_{mn}, k, \lambda^{(0)}_{mn}) T_{mn,k}(x, z) \right] = 0 , \quad (D.1)
\]

where
\[
\left[\mu(x,z,\omega_{mn,k},\lambda_{mn}^o)\right] = \left[-[I] + \lambda_{mn}^o(1 - \sum_{i=1}^{I} \frac{\beta_i \omega_{mk}}{\lambda_i^1 + \omega_{mk}})X_p^* F^T\right]
\]

\[
- \omega_{mk}[V]^{-1}[V],
\]

(D.2)

\[
N_{mn,k}(x,z) = \text{col}[N_{mn,k}^1(x,z),N_{mn,k}^2(x,z)],
\]

and the other notation is defined in Chapter 2 and Appendix A. The subscript \(m\) is used to index the spatial harmonic in the \(x\) direction; the subscript \(n\) is used to index the spatial harmonic in the \(z\) direction; and the subscript \(k\) is used to index the \(K\) eigenvectors and eigenvalues of the \(m\)-th to \(n\)-th cluster of eigenvectors. The synthesis technique is based on the idea of trying to find an approximate solution to Eq. (D.1) of the form,

\[
N_{mn,k}^{(g)}(x,z) \approx \sum_{\ell=1}^{Kg} X_{m,k}^{(g,\ell)}(x) Z_{n,k}^{(g,\ell)}(z),
\]

(D.3)

where the \(X_{m,k}^{(g,\ell)}(x)\) are known trial functions and the \(Z_{n,k}^{(g,\ell)}(z)\) are unknown coefficients to be determined. The \(X_{m,k}^{(g,\ell)}(x)\) are usually chosen to represent the \(x\)-shape of the \(g\)-th component of \(N_{mn,k}(x,z)\) in various \(z\)-regions of the reactor.

In a similar fashion, the solution to adjoint equation,

\[
\nabla \cdot [D(x,z)] \nabla \frac{N_{mn,k}^*}{N_{mn,k}}(x,z) + \left[\mu(x,z,\omega_{mn,k},\lambda_{mn}^o)\right] \frac{N_{mn,k}}{N_{mn,k}}(x,z) = 0,
\]

(D.4)

is approximated as
The following arrangements:

\[
X_{n_{m_{k}}}^{(g)}(x) \equiv \text{col}[X_{m_{k}}^{(g,1)}, \ldots, X_{m_{k}}^{(g,K_g)}],
\]

\[
[X_{m_{k}}] = \begin{bmatrix}
X_{m_{k}}^{(1)} & 0 \\
0 & X_{m_{k}}^{(2)}
\end{bmatrix},
\]

\[
Z_{n_{k}}^{(g)}(z) \equiv \text{col}[Z_{n_{k}}^{(g,1)}, \ldots, Z_{n_{k}}^{(g,K_g)}],
\]

\[
Z_{n_{k}} = \text{col}[Z_{n_{k}}^{(1)}, Z_{n_{k}}^{(2)}],
\]

and similar arrangements for the adjoint quantities allow Eqs. (D.3) and (D.5) to be written as

\[
N_{m_{m_{k}}}(x,z) \simeq [X_{m_{k}}]^{T}Z_{n_{k}}^{*},
\]

and

\[
N_{m_{m_{k}}}(x,z) \simeq [X_{m_{k}}]^{T}Z_{n_{k}}^{*},
\]

respectively.

A one-dimensional equation for the unknown \(Z_{n_{k}}^{(g,l)}(z)\) may be obtained by substituting Eq. (D.6) into Eq. (D.1), multiplying the resulting expression by an \(x\) dependent weighting function, and integrating over \(x\). The choice of \([X_{m_{k}}]\) as the weighting function in this procedure leads to
This choice for the weighting function can be motivated by a variational principle. When the following definitions of x-averaged quantities are introduced,

\[
\tilde{\mu}(z, w_{mn}, k, \lambda_{mn}^O) = \left[ -[\tilde{\Sigma}_{r, mk}] - [\tilde{\Sigma}_{a, mk}] + \left( 1 - \sum_{i=1}^{I} \frac{\beta_i w_{mn,k}}{\lambda_i + w_{mn,k}} \right) [X_p \cdot \tilde{F}_{mk}] \right]
\]

\[
- w_{mn,k} \left[ V_{mk} \right]^{-1} - \left[ \tilde{D}_{mk} \tilde{F}_{mk} \right] \left[ V_{mk} \right]^{-1}
\]

\[
\tilde{\Sigma}_{r, mk} = \left[ X_{mk}^{*} \left[ \Sigma_{r}(x, z) \right] X_{mk} \right]_{x} T
\]

\[
\tilde{\Sigma}_{a, mk} = \left[ X_{mk}^{*} \left[ \Sigma_{a}(x, z) \right] X_{mk} \right]_{x} T
\]

\[
[X_p \cdot \tilde{F}_{mk}] T = \left[ X_{mk}^{*} \left[ X_p \cdot F_{x}^{T}(x, z) \right] X_{mk} \right]_{x} T
\]

\[
\tilde{V}_{mk}^{-1} = \left[ X_{mk}^{*} \left[ V^{-1} \right] X_{mk} \right]_{x} T
\]

\[
\tilde{D}_{mk} = \left[ X_{mk}^{*} \left[ D(x, z) \right] X_{mk} \right]_{x} T
\]

\[
\tilde{D}_{mk}^{-2} = \left[ X_{mk}^{*} \left[ D(x, z) \right]^{-2} X_{mk} \right]_{x} T
\]

Equation (D.16) is a one-dimensional equation which may be solved for the vectors, \( Z_{mk}(z) \), by using MUDMO-II. SYNSIG calculates the x-averaged quantities defined by Eqs. (D.10) through (D.15).
A one-dimensional equation for the unknown $Z_{mk}^*(z,k)$ of the adjoint solution, Eq. (D.7), may be constructed by a similar procedure which uses $[X_{mk}]$ as the weighting function. The result of this procedure is the one-dimensional equation,

$$\nabla_z [\tilde{D}_{mk}]^T \nabla_z Z_{mk}^*(z) + [\tilde{\mu}(z, w_{nm}, \lambda_{nm}^0)]^T Z_{mk}^*(z) = 0 \; , \quad (D.17)$$

which may be solved by using MUDMO-II.

D.3 Calculation of $x$-Averaged Quantities

The nuclear parameters appearing in the $x$-averaged quantities are constant by region. Let the subscript $c_{i,j}$ index the composition of the $i^{th}$ $x$-region and the $j^{th}$ $z$-region. The $x$-averaged quantities for the $j^{th}$ region may then be written as

$$[\tilde{\Sigma}_{r, mk}]_j = \begin{bmatrix} [\tilde{\Sigma}_{r, mk}]_j & [0] \\ -[\tilde{\Sigma}_{12, mk}]_j & [0] \end{bmatrix} \; , \quad (D.18)$$

$$[\tilde{\Sigma}_{a, mk}]_j = \begin{bmatrix} [\tilde{\Sigma}_{a1, mk}]_j & [0] \\ [0] & [\tilde{\Sigma}_{a2, mk}]_j \end{bmatrix} \; , \quad (D.19)$$

$$[\tilde{\Lambda}_p, F_{mk}]_j = \begin{bmatrix} [\tilde{\mu}\tilde{\Sigma}_{11, mk}]_j & [\tilde{\mu}\tilde{\Sigma}_{12, mk}]_j \\ [0] & [0] \end{bmatrix} \; , \quad (D.20)$$

$$[\tilde{D}_{mk}]_j = \begin{bmatrix} [\tilde{D}_{1, mk}]_j & [0] \\ [0] & [\tilde{D}_{2, mk}]_j \end{bmatrix} \; , \quad (D.21)$$
\[
[\tilde{V}_{mk}]^{-1}_j = \begin{bmatrix}
[\tilde{V}_{1, mk}]_j & [0] \\
[0] & [\tilde{V}_{2, mk}]_j
\end{bmatrix}, \quad (D.22)
\]

\[
[D_B^{mk}]_j = \begin{bmatrix}
[D_{B1}^{mk}]_j & [0] \\
[0] & [D_{B2}^{mk}]_j
\end{bmatrix}, \quad (D.23)
\]

where

\[
[\tilde{\Sigma}_{r, mk}]_j = \text{a } K_1 \text{ by } K_1 \text{ matrix whose } p^{th} - q^{th} \text{ element is}
\]

\[
\sum_{i=1}^{\text{NXREG}} \Sigma_{r,c}^{i} \langle x_{mk}^{*}(1,p)x_{mk}^{(1,q)} \rangle_{\text{Region } i}, \quad (D.24)
\]

\[
[\tilde{\Sigma}_{12, mk}]_j = \text{a } K_2 \text{ by } K_1 \text{ matrix whose } p^{th} - q^{th} \text{ element is}
\]

\[
\sum_{i=1}^{\text{NXREG}} \Sigma_{12,c}^{i} \langle x_{mk}^{*}(2,p)x_{mk}^{(1,q)} \rangle_{\text{Region } i}, \quad (D.25)
\]

\[
[\tilde{\Sigma}_{ag, mk}]_j = \text{a } K_g \text{ by } K_g \text{ matrix whose } p^{th} - q^{th} \text{ element is}
\]

\[
\sum_{i=1}^{\text{NXREG}} \Sigma_{ag,c}^{i} \langle x_{mk}^{*}(g,p)x_{mk}^{(g,q)} \rangle_{\text{Region } i}, \quad (D.26)
\]

\[
[\nu \tilde{Z}_{fg, mk}] = \text{a } K_1 \text{ by } K_g \text{ matrix whose } p^{th} - q^{th} \text{ element is}
\]

\[
\sum_{i=1}^{\text{NXREG}} \nu_{fg}^{i} \langle x_{mk}^{*}(1,p)x_{mk}^{(g,q)} \rangle_{\text{Region } i}, \quad (D.27)
\]
\[ [D_{g,mk}]_{ij} = \text{a Kg by Kg matrix whose } p^{th} - q^{th} \text{ element is} \]
\[ \sum_{i=1}^{\text{NXREG}} D_{g_i,mk} c_{ij} \left< X_{mk}^*(g,p)X_{mk}(g,q) \right>_{\text{Region } i}, \quad (D.28) \]

\[ [V_{g,mk}]^{-1} = \text{a Kg by Kg matrix whose } p^{th} - q^{th} \text{ element is} \]
\[ \sum_{i=1}^{\text{VXREG}} V_{g_i,mk}^{-1} \left< X_{mk}^*(g,p)X_{mk}(g,q) \right>_{\text{Region } i}, \quad (D.29) \]

\[ [D_{g,mk}^B]^2 = \text{a Kg by Kg matrix whose } p^{th} - q^{th} \text{ element is} \]
\[ \sum_{i=1}^{\text{NXREG}} D_{g_i,mk} c_{ij} \left< \nabla x_{mk}^*(g,p)\nabla x_{mk}(g,q) \right> \quad (D.30) \]

D.4 Features of SYNSIG

Good trial functions, \( X_{mk}^*(g,l) \) and good adjoint trial functions, \( X_{mk}(g,l) \) can often be generated by solving a one-dimensional eigenvalue problem with MUDMC-II. MUDMC-II prepares punched output of one-dimensional, two group eigenvectors in a format which may be used directly as input for SYNSIG.

A case where MUDMC-II cannot be used to prepare trial functions is when the \( l^{th} \) z-region does not contain fissionable matter. All neutrons in the \( l^{th} \) z-region come from adjacent z-regions. The \( x \)-dependent problem in this case is not an eigenvalue problem; it is an inhomogeneous boundary value problem with an external source. The Stabilized March Technique can handle such problems as has been shown by Lucey[59].
If the trial functions or the adjoint trial functions are close to being linearly dependent, difficulty may be encountered. The cures for this difficulty are to avoid using more trial functions than are needed and to make the trial functions linearly independent by subtracting them from one another. The subroutine ADJUST of SYNSIG performs this operation according to the following formulas:

\[ x_{mk}(g, l) = x_{mk}(g, l-1) - x_{mk}(g, l) , \]  
\[ (D.31) \]

for \( l = 2, \ldots, KF \);

and

\[ x^*_{mk}(g, l) = x^*_{mk}(g, l-1) - x^*_{mk}(g, l) , \]  
\[ (D.32) \]

for \( l = 2, \ldots, KA \).

Before these operations are performed, the integrals, \( \langle (x_{mk}(g, l))^a \rangle_x \) are normalized to \( \langle (x_{mk}(g, l))^a \rangle_x \) for \( l = 2, \ldots, KF \); and the integrals \( \langle (x^*_{mk}(g, l))^a \rangle_x \) are normalized to \( \langle (x^*_{mk}(g, l))^a \rangle_x \) for \( l = 2, \ldots, KA \).

The integer, \( a \), is specified by the user.

SYNSIG is written as a main program with three subroutines and one function subprogram. The main program combines the parameters and region integrals in the manner prescribed by Eqs. (D.24) through (D.30); DATA reads all input data; ADJUST performs the operations which lead to the linearly independent expansion functions and weighting functions and prints and punches these adjusted functions; OUTPUT prints and punches the \( x \)-averaged parameters in a format which can be used as input to MNDCII. The function subprogram SINP performs all integrations using Simpson's Rule.
Preparation of Input for SYNSIG

The input is preceded by a run identity card with any characters permissible in columns 2 through 72. This card is followed by fixed point, control data punched with a 20I3 format. The fixed point control data are:

- \( \text{NOG} \) = number of neutron energy groups (restricted to 2).
- \( \text{KI} \) = number of trial functions used in expansion of the epithermal neutron density.
- \( \text{K2} \) = number of trial functions used in expansion of the thermal neutron density \((\text{KI} + \text{K2} \leq 10)\).
- \( \text{NCOMP} \) = number of regions of different composition \((\text{NCOMP} \leq 25)\).
- \( \text{NXREG} \) = maximum number of regions in x-direction \((\text{NXREG} \leq 5)\).
- \( \text{NZREG} \) = maximum number of regions in z-direction \((\text{NZREG} \leq 5)\).
- \( \text{NPTS} \) = number of mesh points in the x dependent distribution \((\text{NPTS} \leq 150)\).
- \( \text{IA} \) = 0 for slab configuration in x-direction; 1 for cylindrical configuration in x-direction.
- \( \text{MORE} \) = 0 for more data, 1 for last set of data.
- \( \text{NICARD} \) = number of identification cards which precede the reading-in of the x dependent distributions.
- \( \text{KF} \) = total number of pairs of expansion functions, \( X_{mk}^{(1,\lambda)} \) and \( X_{mk}^{(2,\lambda)} \), to be read in \((\text{KF} \leq 5)\).
- \( \text{KA} \) = total number of pairs of weighting functions, \( X_{mk}^{* (1,\lambda)} \) and \( X_{mk}^{* (2,\lambda)} \), to be read in \((\text{KA} \leq 5)\).
- \( \text{JPOWER} \) = exponent, \( a \), used in the normalizing integrals which follow Eq. (D.32).
- \( \text{KADJST} \) = 0, use expansion functions and weighting functions as they are read in; 1, perform the operations on the expansion functions and weighting functions which make them linearly independent.
KPI = print control for subroutine ADJUST; = 0, skip print of renormalized expansion function; = 1, print renormalized expansion functions.

The next series of cards reads in the expansion functions.
The first card contains fixed point identification information on a 3I3 format:

IF = identification number for \( X^{(1,IF)}_{mk} \).
IT = identification number for \( X^{(2,IT)}_{mk} \).
JM = index of azimuthal mode when the expansion functions are for a cylindrical configuration in \( x \) direction.

The second group of cards (NICARD in number) contains identification information for the epithermal and thermal expansion functions which are indexed by IF and IT. Any characters are permissible in columns 2 through 72.

Each card of the next group of cards (NPTS + 1 in number) contains \( X^{(1,IF)}_{mk}(x_1) \) and \( X^{(2,IT)}_{mk}(x_1) \) on a 12X 2E15.8 format. This format is the same as that of the punched output from MUDMO-II.

This series which begins with the reading of IF, IT and JM is now repeated KF times.

The next series of cards reads in the weighting functions.
The first card contains fixed point identification information on a 2I3 format:

IF = identification number for \( X^{*}(1,IF)_{mk} \).
IT = identification number for \( X^{*}(2,IT)_{mk} \).
The second group of cards (NICARD in number) contains identification information for the epithermal and thermal weighting functions which are indexed by IF and IT.

Each card of the next group of cards (NPTS + 1 in number) contains $X^*_{mk}(1,IF)(x_i)$ and $X^*_{mk}(2,IT)(x_i)$ on a 12X 2E15.8 format.

This series which begins with the reading of IF and IT is now repeated KA times.

The next series of cards reads values of $D$, $\Sigma_a$, $\Sigma_r$ and $\nu \Sigma_f$ by group and by composition. Each new parameter is begun on a new card.

The subscripted data are read as $((S(I,J),I = 1, NOG), J = 1, NCOMP)$ on a 15X 4E15.8 format. This series contains:

- $D\phi(I,J) = \text{diffusion coefficients, } D_{g,c_i} \text{ NOGxNCOMP values.}$
- $F SIG\phi(I,J) = \text{fission cross sections times the number of neutrons per fission, } \nu \Sigma_f_{ig,c_i} \text{ NOGxNCOMP values.}$
- $A SIG\phi(I,J) = \text{absorption cross sections, } \Sigma_a_{i,j} \text{ NOGxNCOMP values.}$
- $R SIG\phi(I,J) = \text{removal cross section, } \Sigma_r_{i,j} \text{ NOGxNCOMP values.}$

The next series of cards contains two groups of fixed point data read on a 2CI3 format.

The first group of cards in this series reads the composition indices. There are NZREG cards each of which contains:

- $KC(J,I) = \text{the composition index of the } J^{th} \text{ z-region and the } I^{th} \text{ x-region, NXREG values per card.}$

The second group of cards in this series contains:

- $IND(I) = \text{the mesh points at the boundaries of the x-regions, NXREG values. These mesh points must be even numbers for Simpson's Rule to work properly.}$
The next series of cards reads floating point data on a 15X 4E15.8 format. The group of cards of this series contains:

RAD(I) = thickness of each x-region, NXREG values.

The next and final card contains:

V(I) = group velocities, v_g, NOG values.

The FORTRAN-II listing of SYNSIG is given in the pages following the discussion of test results.

D.6 Test Results

The results of a two-dimensional synthesis of a fundamental λ-mode were compared with a PDQ[60] two-dimensional solution. The geometrical configuration and nuclear parameters of the test problem is given in Fig. D.1.

The trial functions and the adjoint trial functions used were one-dimensional solutions in the x direction. The fundamental mode solutions in the rod-in zone and the rod-out zone were generated by MUDMO-II. These x-direction shapes were obtained with transverse bucklings \(B_z^2\) of such a size that the eigenvalues of the x direction problems were approximately the same as that of the final, synthesized, two-dimensional shape. The trial function and adjoint trial function in the reflector zone were both chosen as \(\cos \frac{\pi x}{40}\).

The eigenvalue obtained by PDQ was 1.0171. The eigenvalue obtained by the synthesis using MUDMO-II and SYNSIG was 1.0154. The synthesis result for the thermal neutron distribution is compared with the PDQ two-dimensional solution in Fig. C.2.
**Fig. D.1 Information for Two-Dimensional Test Problem**

<table>
<thead>
<tr>
<th></th>
<th>Fuel</th>
<th>Water</th>
<th>Control Rod</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D_1$</td>
<td>1.7</td>
<td>1.7</td>
<td>0.5</td>
</tr>
<tr>
<td>$\Sigma_{al}$</td>
<td>0</td>
<td>0</td>
<td>0.1</td>
</tr>
<tr>
<td>$\Sigma_{rl}$</td>
<td>0.016</td>
<td>0.041</td>
<td>0</td>
</tr>
<tr>
<td>$\nu\Sigma_{f1}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$D_2$</td>
<td>0.42</td>
<td>0.23</td>
<td>0.1</td>
</tr>
<tr>
<td>$\Sigma_{a2}$</td>
<td>0.055</td>
<td>0.012</td>
<td>1.5</td>
</tr>
<tr>
<td>$\nu\Sigma_{f2}$</td>
<td>0.0832</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>
Fig. D.2 Results of a Two-Dimensional Test Problem
LIST 8
LABEL
SYMBOL TABLE
SYNSIG, A PROGRAM FOR CROSS SECTION AVERAGING FOR FLUX SYNTHESIS
THIS SUBROUTINE PREPARES SIGMAS FOR SYNTHESIS OF MULTI-
DIMENSIONAL SOLUTIONS USING MUDMO-II
COMMON RHO,RAD,IND,NRREG,K1,K2,KK,NZREG,ASIG,FSIG,RSIG,WINT,D,
2DBUCK,KC,DO,ASIGO,FSIGO,RSIGO,V,Y,NOG,NCOMP,NPTS,IA,MORE,JM,H,W,
3RIA,RR,X,SUM1,SUM2,SUM3,SAVE,KP1,KP2,KADJST,JPOWER
DIMENSION RHO(5),RAD(5),IND(5),ASIG(10,10,5),FSIG(10,10,5),
RSIG(10,10,5),WINT(10,10,5),D(10,10,5),DBUCK(10,10,5),
SUM1(10,10,5),SUM2(10,10,5),SUM3(10,10,5),KC(5,5),DO(2,25),
ASIGO(2,25),FSIGO(2,25),RSIGO(2,25),V(2),Y(12),H(5,2,150),
W(5,2,150),X(150),SAVE(10)

CALL DATA

MESH SPACING
NXREG IS NRREG

RHO(1)=RAD(1)/FLOATF(IND(1))
IF(NRREG=1) 10,10,20
20 DO 30 I=2,NRREG
30 RHO(I)=RAD(I)/FLOATF(IND(I)-IND(I-1))

CALL KADJST IF FUNCTIONS ARE NOT LINEARLY INDEPENDENT

IF(KADJST) 31,10,31
31 CALL ADJUST
10 KK=K1+K2

INITIALIZE ELEMENTS

DO 40 L=1,NZREG
DO 40 J=1,KK
DO 40 K=1,KK
ASIG(J,K,L)=0.0
FSIG(J,K,L)=0.0
RSIG(J,K,L)=0.0
WINT(J,K,L)=0.0
D(J,K,L)=0.0
40 DBUCK(J,K,L)=0.0

PREPARE K1 BY K1 MATRICES AND K2 BY K2 MATRICES

NGRP=1
ZE=1.0
KK=K1
KA=0
49 DO 50 J=1,KK
DO 50 K=1,KK
DO 50 I=1,NRREG
SUM1(J,K,I)=SIMP(NGRP,NGRP,J,K,I,1)
SUM2(J,K,I) = SIMP(NGRP,NGRP,J,K,I,2)

SUM3(J,K,I) = SIMP(NGRP,NGRP,J,K,I,3)

DO 60 L=1,NZREG
DO 60 J=1,KK
DO 60 K=1,KK
DO 60 I=1,NRREG
M=KC(L,I)
JPKA=J+KA
KPKA=K+KA
D(JPKA,KPKA,L)=D(JPKA,KPKA,L)+DO(NGRP,M)*SUM1(J,K,I)
ASIG(JPKA,KPKA,L)=ASIG(JPKA,KPKA,L)+ASIGO(NGRP,M)*SUM1(J,K,I)*ZE
FSIG(JPKA,KPKA,L)=FSIG(JPKA,KPKA,L)+FSIGO(NGRP,M)*SUM1(J,K,I)*ZE
RSIG(JPKA,KPKA,L)=RSIG(JPKA,KPKA,L)+RSIGO(NGRP,M)*SUM1(J,K,I)
WINT(JPKA,KPKA,L)=WINT(JPKA,KPKA,L)+SUM1(J,K,I)/V(NGRP)

DBUCK(JPKA,KPKA,L)=DBUCK(JPKA,KPKA,L)+DO(NGRP,M)*(SUM2(J,K,I)

GO TO (70,80),NGRP

70 NGRP=2
ZE=0.0
KK=K2
KA=K1
GO TO 49

PREPARE K1 BY K2 MATRICES

DO 90 J=1,K1
DO 90 K=1,K2
DO 90 I=1,NRREG
SUM1(J,K,I)=SIMP(1,2,J,K,I,1)

DO 100 L=1,NZREG
DO 100 J=1,K1
DO 100 K=1,K2
DO 100 I=1,NRREG
M=KC(L,I)
KPK1=K+K1
FSIG(J,KPK1,L)=FSIG(J,KPK1,L)+FSIGO(2,M)*SUM1(J,K,I)

PREPARE K2 BY K1 MATRICES

DO 110 J=1,K2
DO 110 K=1,K1
DO 110 I=1,NRREG
SUM1(J,K,I)=SIMP(2,1,J,K,I,1)

DO 120 L=1,NZREG
DO 120 J=1,K2
DO 120 K=1,K1
DO 120 I=1,NRREG
M=KC(L,I)
JPK1=J+K1
RSIG(JPK1,K,L)=RSIG(JPK1,K,L)+RSIGO(1,M)*SUM1(J,K,I)

KK=K1+K2

CALL OUTPUT

IF(MORE) 130,140,130

CALL EXIT

END
LIST 8

LABEL

SYMBOL TABLE

SUBROUTINE DATA

READS INPUT

COMMON RHO,RAD,IND,NRREG,K1,K2,KK,NZREG,ASIG,FSIG,RSIG,WINT,D,
2DBUCK,KC,DO,ASIGO,FSIGO,RSIGO,V,Y,NOG,NCOMP,NPTS,IA,MORE,JM,H,W,
3R1A,RR,X,SUM1,SUM2,SUM3,SAVE,KP1,KP2,KADJST,JPOWER,KF

DIMENSION RHO(5),RAD(5),IND(5),ASIG(10,10,5),FSIG(10,10,5),
2RSIG(10,10,5),WINT(10,10,5),D(10,10,5),DBUCK(10,10,5),
3SUM1(10,10,5),SUM2(10,10,5),SUM3(10,10,5),KC(5,5),DO(2,25),
4ASIGO(2,25),FSIGO(2,25),RSIGO(2,25),V(2),Y(12),H(5,2,150),
5W(5,2,150),X(150),SAVE(10)

I.D. CARD
READ INPUT TAPE 4,270, (Y(I),I=1,12)
PUNCH 300, (Y(I),I=1,12)
PRINT 300, (Y(I),I=1,12)
READ INPUT TAPE 4,280,NOG,K1,K2,NCOMP,NRREG,NZREG,NPTS,IA,MORE,
2NICARD,KF,KA,JPOWER,KADJST,KP1,KP2

KK=K1+K2
NO=NPTS+1

READ IN EXPANSION FUNCTIONS

DO 10 J=1,KF
READ INPUT TAPE 4,280, IF,IT,JM
PRINT 281,IF,IT
PUNCH 281,IF,IT
DO 5 M=1,NICARD
READ INPUT TAPE 4,270, (Y(I),I=1,12)
PUNCH 310, (Y(I),I=1,12)
PRINT 310, (Y(I),I=1,12)

10 READ INPUT TAPE 4,285, H(IF,1,I),H(IT,2,I)

READ IN WEIGHTING FUNCTIONS

DO 20 J=1,KA
READ INPUT TAPE 4,280,IF,IT
PRINT 281,IF,IT
PUNCH 281,IF,IT
DO 15 M=1,NICARD
READ INPUT TAPE 4,270, (Y(I),I=1,12)
PUNCH 310, (Y(I),I=1,12)
PRINT 310, (Y(I),I=1,12)

20 READ INPUT TAPE 4,285, W(IF,1,I),W(IT,2,I)

READ IN NUCLEAR DATA

READ INPUT TAPE 4,290, ((DO(I,J),I=1,NOG),J=1,NCOMP)
READ INPUT TAPE 4,290, ((FSIGO(I,J),I=1,NOG),J=1,NCOMP)
READ INPUT TAPE 4,290, ((ASIGO(I,J),I=1,NOG),J=1,NCOMP)
READ INPUT TAPE 4,290, ((RSIGO(I,J),I=1,NOG),J=1,NCOMP)

READ COMPOSITION INDICES

DO 30 L=1,NZREG

30 READ INPUT TAPE 4,280, (KC(L,I),I=1,NRREG)
READ INPUT TAPE 4,280, (IND(I),I=1,NRREG)
READ INPUT TAPE 4,290, (RAD(1), I=1,NRREG)
READ INPUT TAPE 4,290, V(1), V(2)
RETURN
270 FORMAT (1X 11A6, A5)
280 FORMAT (20X)
285 FORMAT (12X 2E15.8)
290 FORMAT (15X 4E15.8)
300 FORMAT (1H1 11A6, A5)
310 FORMAT (1H0 11A6, A5)
281 FORMAT (1H0 1X 22H FAST FUNCTION INDEX ISI3, 29H.) THERMAL FUNCTION
2 INDEX ISI3, 1H(1)
END
*
LIST 8
*
LABEL
*
SYMBOL TABLE
SUBROUTINE ADJUST
C
CONSTRUCTS LINEARLY INDEPENDENT FUNCTIONS
C
COMMON RHO, RAD, IND, NRREG, K1, K2, KK, NZREG, ASIG, FSIG, RSIG, WINT, D,
2 DBUCK, KC, DO, ASIGO, FSIGO, RSIGO, V, Y, NOS, NCOMP, NPTS, IA, MORE, JMN, H, W,
3 RIA, RR, X, SUM1, SUM2, SUM3, SAVE, KP1, KP2, KADJST, JPOWER, KF
DIMENSION RHO(5), RAD(5), IND(5), ASIG(10, 10, 5), FSIG(10, 10, 5),
2 RSIG(10, 10, 5), WINT(10, 10, 5), D(10, 10, 5), DBUCK(10, 10, 5),
3 SUM1(10, 10, 5), SUM2(10, 10, 5), SUM3(10, 10, 5), KC(5, 5), DO(2, 25),
4 ASIGO(2, 25), FSIGO(2, 25), RSIGO(2, 25), V(2), Y(12), H(5, 2, 150),
5 W(5, 2, 150), X(150), SAVE(10)
NO=NPTS+1
DO 100 J=1, NOG
C CALCULATION OF STANDARD NORMALIZATION
YNORMS=0.
WNORMS=0.
DO 20 K=1, NRREG
WNORM=WNORM+SIMP(J, J, 1, 1, K, 4)
20 YNORM=YNORMS+SIMP(J, J, 1, 1, K, 5)
C CALCULATION OF NORMALIZATION OF EACH FUNCTION
DO 100 M=2, KF
YNORM=0.
WNORM=0.
DO 50 K=1, NRREG
WNORM=WNORM+SIMP(J, J, M, M, K, 4)
50 YNORM=YNORMS+SIMP(J, J, M, M, K, 5)
C RENORMALIZATION OF FUNCTION
GO TO (60, 70), JPOWER
60 FACY=YNORMS/YNORM
FACW=WNORMS/WNORM
GO TO 80
70 FACY=SQRTF(YNORMS/YNORM)
FACW=SQRTF(WNORMS/WNORM)
80 DO 90 N=1, NO
H(M, J, N)=H(M, J, N)*FACY
90 W(M, J, N)=W(M, J, N)*FACW
IF(KP1) 81, 100, 81
81 PRINT 699, M, J
PRINT 700, (H(M, J, N), N=1, NO)
CONSTRUCTION OF LINEARLY INDEPENDENT FUNCTIONS

DO 200 K=2*KF
MM=K
MMM1=MM-1
DO 200 J=1,NO
H(M,J,N)=H(MM1,J,N)-H(M,J,N)

PRINT NEW EXPANSION FUNCTIONS
DO 820 M=1,KF
PRINT 600
PRINT 601,M
PRINT 602
PRINT 603
IREG=1
Z=0.
DO 820 N=1,NO
IF(N-IND(IREG)) 800,800,810
810 IREG=IREG+1
800 PRINT 604,Z,H(M,1,N),H(M,2,N),W(M,1,N),W(M,2,N)
PUNCH 604,Z,H(M,1,N),H(M,2,N),W(M,1,N),W(M,2,N)
820 Z=Z+RHO(IREG)
RETURN

FORMAT (1HO 5X 215)
FORMAT (1H 14X 4E15.8)
FORMAT (1H 9X 18) EXPANSION INDEX I5 I5
FORMAT (1H 5X 1HX 13X 19) EXPANSION FUNCTIONS 11X 19 WEIGHTING FUNCTIONS
FORMAT (1H 19X 4HFAST 11X 7HTHERMAL 8X 4HFAST 11X 7HTHERMAL)
FORMAT (1H F10.4, 4X 4E15.8)
END

** LIST 8
** LABEL
** SYMBOL TABLE
FUNCTION SIMP(NW,NE,IW,IE,IREG,NCH)

** PERFORMS INTEGRATIONS

COMMON RHO,RAD,IND,NRREG,K1,K2,KN,NREG,ASIG,FSIG,RSIG,WINT,D,
2DBUCK,KC,DO,ASIGO,FSIGO,RSIGO,V,Y,NOG,NCOMP,NPTS,IA,MORE,JM,H,W,
3RIA,RR,X,SUM1,SUM2,SUM3,SAVE,KP1,KP2,KADJST,JPOWER
DIMENSION RHO(5),RAD(5),IND(5),ASIG(10,10,5),FSIG(10,10,5),
2RSIG(10,10,5),WINT(10,10,5),D(10,10,5),DBUCK(10,10,5),
3SUM1(10,10,5),SUM2(10,10,5),SUM3(10,10,5),KC(5,5),DO(2,25),
4ASIGO(2,25),FSIGO(2,25),RSIGO(2,25),V(2),Y(12),H(5,2,150),
5W(5,2,150),X(150),SAVE(10)
IF(IREG-1) 20,10,20
10 NN=0
NNN=IND(1)+1
GO TO 30
20 NN=IND(IREG-1)
NNN=IND(IREG)-NN+1
30 IF (IA) 50, 40, 50
40 RIA = 1.
   GO TO 60
50 RIA = 0.
   IF (IREG - 1) 70, 60, 70
70 IM1 = IREG - 1
   DO 80 I = 1, IM1
80 RIA = RIA + RAD(I)
   GO TO (90, 160, 110, 200, 250), NCH
C INTEGRATION OF W*H
90 DO 100 I = 1, NNN
   NNPI = NN + I
   X(I) = W(IW, NW, NNPI) * H(IE, NE, NNPI) * RIA
100 RIA = RIA + FLOATF(I) * RHO(IREG)
   GO TO 180
C INTEGRATION OF AZIMUTHAL PORTION OF (DW/DX)*(DH/DX)
110 IF (IREG - 1) 130, 120, 130
120 RR = 0.
   GO TO 140
130 RR = 1./RIA
140 DO 150 I = 1, NNN
   BM = FLOATF(JM)
   NNPI = NN + I
   X(I) = W(IW, NW, NNPI) * H(IE, NE, NNPI) * RR * BM ** 2
   RIA = RIA + FLOATF(I) * RHO(IREG)
150 RR = 1./RIA
   GO TO 180
C INTEGRATION OF (DW/DX)*(DH/DX)
160 AGW = (W(IW, NW, NN + 2) - W(IW, NW, NN + 1)) / RHO(IREG)
   AGH = (H(IE, NE, NN + 2) - H(IE, NE, NN + 1)) / RHO(IREG)
   X(I) = RIA * AGW * AGH
   RIA = RIA + FLOATF(I) * RHO(IREG)
   NNNM1 = NNN - 1
   DO 170 I = 2, NNNM1
      NNPI1 = NN + I + 1
      NNPI1M1 = NN + I - 1
      AGW = (W(IW, NW, NNPI1) - W(IW, NW, NNPI1M1)) / (2. * RHO(IREG))
      AGH = (H(IE, NE, NNPI1) - H(IE, NE, NNPI1M1)) / (2. * RHO(IREG))
      X(I) = RIA * AGW * AGH
170 RIA = RIA + FLOATF(I) * RHO(IREG)
II = IND(IREG)
   AGW = (W(IW, NW, II + 1) - W(IW, NW, II)) / RHO(IREG)
   AGH = (H(IE, NE, II + 1) - H(IE, NE, II)) / RHO(IREG)
   X(NNN) = RIA * AGW * AGH
C SIMPSONS RULE
180 SIMP = X(1)
   DO 190 I = 2, NNN, 2
190 SIMP = SIMP + 4. * X(I) + 2. * X(I + 1)
   SIMP = SIMP - X(NNN)
   SIMP = SIMP * RHO(IREG) / 3.
   RETURN
C INTEGRATION OF W**JPOWER
200 DO 210 I = 1, NNN
   NNPI = NN + I
   X(I) = W(IW, NW, NNPI) ** JPOWER * RIA
262

210 RIA=RIA+FLOATF(IA)*RHO(IREG)
GO TO 180

C INTEGRATION OF H**JPOWER

250 DO 260 I=1,NNN

NNPI=NN+I
X(I)=H(I)E*NE*NNPI)**JPOWER*RHO(IREG)

260 RIA=RIA+FLOATF(IA)*RHO(IREG)
GO TO 180
END

* LIST 8
* LABEL
* SYMBOL TABLE

SUBROUTINE OUTPUT

C PUNCHES OUTPUT FOR MUDMO-II INPUT

COMMON RHO,RAD,IND,NRREG,K1,K2,KK,NZREG,ASIG,FSIG,RSIG,WINT,D,
2DBUCK,KC,DO,ASIGO,RSIGO,V,Y,NOG,NCOMP,NPTS,IA,MORE,JM,H,W,
3RIA,RR,R,XX,SSUM1,SSUM2,SAVE,KP1,KP2,KADJST,JPOWER

DIMENSION RHO(5),RAD(5),ASIG(10,10,10),FSIG(10,10,10),
2RSIG(10,10,10),WINT(10,10,10),D(10,10,10),DBUCK(10,10,10),
3SUM1(10,10,10),SUM2(10,10,10),SUM3(10,10,10),KC(5,5),DO(2,25),
4ASIGO(2,25),FSIG(2,25),RSIGO(2,25),V(2),Y(12),H(5,2,150),
5W(5,2,150),X(150),SAVE(10)

PRINT 100,KK,NZREG
PUNCH 100,KK,NZREG
PRINT 290,(( ASIG(J,K,L),J=1,KK),K=1,KK),L=1,NZREG)
PUNCH 290,(( ASIG(J,K,L),J=1,KK),K=1,KK),L=1,NZREG)
PRINT 101,KK,NZREG
PUNCH 101,KK,NZREG
PRINT 290,(( FSIG(J,K,L),J=1,KK),K=1,KK),L=1,NZREG)
PUNCH 290,(( FSIG(J,K,L),J=1,KK),K=1,KK),L=1,NZREG)
PRINT 102,KK,NZREG
PUNCH 102,KK,NZREG
PRINT 290,(( RSIG(J,K,L),J=1,KK),K=1,KK),L=1,NZREG)
PUNCH 290,(( RSIG(J,K,L),J=1,KK),K=1,KK),L=1,NZREG)
PRINT 103,KK,NZREG
PUNCH 103,KK,NZREG
PRINT 290,(( D(J,K,L),J=1,KK),K=1,KK),L=1,NZREG)
PUNCH 290,(( D(J,K,L),J=1,KK),K=1,KK),L=1,NZREG)
PRINT 104,KK,NZREG
PUNCH 104,KK,NZREG
PRINT 290,(( DBUCK(J,K,L),J=1,KK),K=1,KK),L=1,NZREG)
PUNCH 290,(( DBUCK(J,K,L),J=1,KK),K=1,KK),L=1,NZREG)
PRINT 105,KK,NZREG
PUNCH 105,KK,NZREG
PRINT 290,(( WINT(J,K,L),J=1,KK),K=1,KK),L=1,NZREG)
PUNCH 290,(( WINT(J,K,L),J=1,KK),K=1,KK),L=1,NZREG)
RETURN

290 FORMAT (1H 14X 4E15.8)
100 FORMAT (1H0 1X 4HASIG 5X I3,14HPSEUDO-GROUPS,5X I3,10HZ REGIONS)
101 FORMAT (1H0 1X 4HFSIG 5X I3,14HPSEUDO-GROUPS,5X I3,10HZ REGIONS)
102 FORMAT (1H0 1X 4HRSIG 5X I3,14HPSEUDO-GROUPS,5X I3,10HZ REGIONS)
103 FORMAT (1H0 1X 4HD 5X I3,14HPSEUDO-GROUPS,5X I3,10HZ REGIONS)
104 FORMAT (1H0 1X 4HDBUCK 5X I3,14HPSEUDO-GROUPS,5X I3,10HZ REGIONS)
105 FORMAT (1H0 1X 4HWINT 5X I3,14HPSEUDO-GROUPS,5X I3,10HZ REGIONS)
END
APPENDIX E

SAMPLE INPUT AND OUTPUT FOR MUDMO-II AND SYNSIG

E.1 Introduction

Input and output for sample computer problems are given in the following pages. The sample problems are related in that they lead to the synthesis of a two-dimensional natural mode which is associated with the second spatial harmonic in the x-direction and with the fourth spatial harmonic in the z-direction (denoted as the 2,4-prompt thermal neutron mode).

Section E.2 gives input for a MUDMO-II problem which calculates the prompt thermal neutron mode and adjoint mode for the one-dimensional reactor formed by the region, \(60 \leq z \leq 180\) and \(0 \leq x \leq 240\), in Fig. 3.7. The one-dimensional mode to be calculated is associated with the second spatial harmonic in the x-direction. The results of this problem are used as a trial function and a weighting function in the synthesis of the 2,4-prompt thermal neutron mode.

Section E.3 gives input for a SYNSIG problem which prepares x-averaged parameters. Section E.4 gives output for this calculation. The output consists of (i) a set of parameters which are used as input in Section E.5, and (ii) new, linearly independent versions of the trial functions, \(X\), and the weighting functions, \(X^*\).
Section E.5 gives input for a MUDMO-II problem which calculates the 2,4-prompt thermal neutron mode of a supercritical version of the reactor of Fig. 3.7. Section E.6 gives the output for this problem. The output consists of the eigenvalue and the coefficients \( Z_1, Z_2, Z_3, Z_4, Z_1^*, Z_2^*, Z_3^* \) and \( Z_4^* \). The synthesized epithermal neutron distribution is given by

\[
\phi_{24,p}^{(1)}(x,z) = v_1 N_{24,p}^{(1)} = Z_1 X_{\text{fast-1}} + Z_2 X_{\text{fast-2}}
\]

and the synthesized thermal neutron distribution is given by

\[
\phi_{24,p}^{(2)}(x,z) = v_2 N_{24,p}^{(2)}(x,z) = Z_3 X_{\text{thermal-1}} + Z_4 X_{\text{thermal-2}}
\]

The synthesized adjoints are given by

\[
N_{24,p}^{(1)*}(x,z) = Z_1^* X_{\text{fast-1}} + Z_2^* X_{\text{fast-2}}
\]

and

\[
N_{24,p}^{(2)*}(x,z) = Z_3^* X_{\text{thermal-1}} + Z_4^* X_{\text{thermal-2}}
\]

This example is for a supercritical version of the two-dimensional reactor. It is for this reason that the calculated eigenvalue -5793.9, in Section E.6 is different from the eigenvalue, -5829.2, listed in Table 3.5.
E.2 SAMPLE INPUT FOR MUDMO-II; CALCULATION OF A ONE-DIMENSIONAL MODE

RUN 41D. SEARCH FOR EIGENVALUE. 240 CM NONUNIFORM REACTOR.
3 0 2240 4 3 2 1 1 2 2 1 1 2 2 1 1 1 1 0
1 1 1 1 2 2 2 60 1 0 1
E.V. -500.0 -50.0
EPA EPB 1.0 E-05 1.0 E-05
RADII 60.0 120.0 60.0
THIS PROBLEM SEARCHES FOR THE PROMPT THERMAL NEUTRON EIGENVALUE OF THE SECOND SPATIAL HARMONIC. REACTOR IS THE 240 CM NONUNIFORM SLAB OF YASINSKY-HENRY (REACTOR III) TWO NEUTRON GROUPS; ONE PRECURSOR GROUP

60180240
BETTOT 0.0064
B(I) 0.0064
PREC 0.08
F 1.0 1.0
F ASIG 0.0322365
ASIG 0.0940843
ASIG 0.0322026
FSIG 0.0194962
FSIG 0.0194962
FSIG 0.0194962
RSIG 0.0164444 -0.0164444
RSIG 0.0164444 -0.0164444
D 1.69531
D 1.69531
D DBUCK 1.69531
DBUCK
DBUCK
DBUCK
WINT 0.24630541E-06 0.24630541E-06 0.24630541E-06
WINT 0.24630541E-06 0.24630541E-06 0.24630541E-06
WINT 0.24630541E-06 0.24630541E-06 0.24630541E-06
PREPARATION OF SYNTHESIS SIGMAS FOR 2ND MODE IN X DIRECTION.

1
1
1

1

RUN 41D. SEARCH FOR EIGENVALUE. 240 CM NONUNIFORM REACTOR. 7/7/66

THIS PROBLEM SEARCHES FOR THE PROMPT THERMAL NEUTRON

EIGENVALUE OF THE SECOND SPATIAL HARMONIC. REACTOR

IS THE 240 CM NONUNIFORM SLAB OF YASINSKY-HENRY

TWO NEUTRON GROUPS; ONE PRECURSOR GROUP

-10560280E-06 -63760184E-08 0.
-46595684E 01 -29199043E 00 8.000
-89872691E 01 -56318390E 00 16.000
-12674613E 02 -79424991E 00 24.000
-15458960E 02 -96872905E 00 32.000
-17141858E 02 -10741872E 01 40.000
-17603229E 02 -11030999E 01 48.000
-16810480E 02 -10534212E 01 56.000
-14956195E 02 -93722713E 00 64.000
-12931641E 02 -81036236E 00 72.000
-10882436E 02 -68194298E 00 80.000
-88124162E 01 -55226998E 00 88.000
-67255671E 01 -42145515E 00 96.000
-46258444E 01 -28987696E 00 104.000
-25172928E 01 -15774519E 00 112.000
-40392552E 00 -25310522E-01 120.000
-17115526E 01 -10729384E 00 128.000
-38240645E 01 -23963375E 00 136.000
-59292988E 01 -37155691E 00 144.000
-80231961E 01 -50277043E 00 152.000
-10101778E 02 -63302434E 00 160.000
-12161043E 02 -76206713E 00 168.000
-14197089E 02 -88965510E 00 176.000
-16068387E 02 -10069197E 01 184.000
-16906505E 02 -10594339E 01 192.000
-16518637E 02 -10351340E 01 200.000
-14932881E 02 -93576256E 00 208.000
-12264213E 02 -76853275E 00 216.000
-87061956E 01 -54557047E 00 224.000
-45168225E 01 -28304468E 00 232.000
-13917652E-06 -95640276E-08 240.000

2

THIS IS THE 2ND TRIAL FUNCTION.

IT IS SIN(2*PI*X/240)

TO BE USED IN THE SYNTHESIS OF THE (2,4) PROMPT

THERMAL NEUTRON EIGENVECTOR

EXTRA CARD

0. 00000000E 00 00000000E 00
8.000 20791152E 00 20791152E 00
16.000  40673631E 00  40673631E 00
24.000  58778481E 00  58778481E 00
32.000  74314434E 00  74314434E 00
40.000  86602495E 00  86602495E 00
48.000  95105617E 00  95105617E 00
56.000  99452176E 00  99452176E 00
64.000  99452204E 00  99452204E 00
72.000  95105701E 00  95105701E 00
80.000  86602629E 00  86602629E 00
88.000  74314615E 00  74314615E 00
96.000  58778699E 00  58778699E 00
104.000 40673878E 00  40673878E 00
112.000 20791414E 00  20791414E 00
120.000 26822090E-05  26822090E-05
128.000 -20790889E 00  -20790889E 00
136.000 -40673384E 00  -40673384E 00
144.000 -58778263E 00  -58778263E 00
152.000 -74314255E 00  -74314255E 00
160.000 -86602361E 00  -86602361E 00
168.000 -95105535E 00  -95105535E 00
176.000 -99452148E 00  -99452148E 00
184.000 -99452233E 00  -99452233E 00
192.000 -95105785E 00  -95105785E 00
200.000 -86602765E 00  -86602765E 00
208.000 -74314795E 00  -74314795E 00
216.000 -58778917E 00  -58778917E 00
224.000 -40674123E 00  -40674123E 00
232.000 -20791677E 00  -20791677E 00
240.000 -53681432E-05  -53681432E-05

1 RUN 41D SEARCH FOR EIGENVALUE 240 CM NONUNIFORM REACTOR 7/7/66
1
0

THIS PROBLEM SEARCHES FOR THE PROMPT THERMAL NEUTRON EIGENVALUE OF THE SECOND SPATIAL HARMONIC REACTOR IS THE 240 CM NONUNIFORM SLAB OF YASINSKY-HENRY
THIS IS THE ADJOINT EIGENVECTOR
-34520283E-08  12784365E-07  0.000
-15490277E 00  29199763E 00  8.000
-29876829E 00  56519013E 00  16.000
-42134663E 00  79425471E 00  24.000
-51390416E 00  96872929E 00  32.000
-56984600E 00  10741819E 01  40.000
-58518694E 00  11030998E 01  48.000
-55833254E 00  10534218E 01  56.000
-49719109E 00  93722423E 00  64.000
-42988940E 00  81035848E 00  72.000
-36176694E 00  68194435E 00  80.000
-29295177E 00  55222590E 00  88.000
-22357866E 00  42145402E 00  96.000
-15377742E 00  28987655E 00  104.000
-83682217E-01  15774426E 00  112.000
-13427125E-01  25309571E-01  120.000
-56889148E-01  10723837E 00  128.000
-12710470E 00  23959764E 00  136.000
-19707731E 00  37149878E 00  144.000
-26667428E 00  50269105E 00  152.000
THIS IS THE 2ND TRIAL WEIGHTING FUNCTION.
IT IS SIN(2*PI*X/240)
TO BE USED IN THE SYNTHESIS OF THE (2,2) PROMPT THERMAL NEUTRON EIGENVECTORS.

```
DO 0.000 0.00000000E 00   0.00000000E 00
   8.000 0.20791152E 00   0.20791152E 00
   16.000 0.40673631E 00   0.40673631E 00
   24.000 0.58778481E 00   0.58778481E 00
   32.000 0.74314434E 00   0.74314434E 00
   40.000 0.86602495E 00   0.86602495E 00
   48.000 0.95105617E 00   0.95105617E 00
   56.000 0.99452176E 00   0.99452176E 00
   64.000 0.99452204E 00   0.99452204E 00
   72.000 0.95105701E 00   0.95105701E 00
   80.000 0.86602629E 00   0.86602629E 00
   88.000 0.74314615E 00   0.74314615E 00
   96.000 0.58778699E 00   0.58778699E 00
  104.000 0.40673878E 00   0.40673878E 00
  112.000 0.20791414E 00   0.20791414E 00
  120.000 0.00000000E 00   0.00000000E 00
  128.000 0.00000000E 00   0.00000000E 00
  136.000 0.00000000E 00   0.00000000E 00
  144.000 0.00000000E 00   0.00000000E 00
  152.000 0.00000000E 00   0.00000000E 00
  160.000 0.00000000E 00   0.00000000E 00
  168.000 0.00000000E 00   0.00000000E 00
  176.000 0.00000000E 00   0.00000000E 00
  184.000 0.00000000E 00   0.00000000E 00
  192.000 0.00000000E 00   0.00000000E 00
  200.000 0.00000000E 00   0.00000000E 00
  208.000 0.00000000E 00   0.00000000E 00
  216.000 0.00000000E 00   0.00000000E 00
  224.000 0.00000000E 00   0.00000000E 00
  232.000 0.00000000E 00   0.00000000E 00
  240.000 0.00000000E 00   0.00000000E 00
```

DO 1.69531   0.409718   1.69531   0.409718
DO 1.69531   0.409718
FSIGO 0.0195021  0.498007   0.0195021  0.498007
FSIGO 0.0195021  0.498007
ASIGO 0.0322365  0.265820   0.0340843  0.266275
ASIGO 0.0322026  0.265820
RSIGO 0.0164444  0.266275
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<tr>
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<td></td>
</tr>
<tr>
<td>30 90 120</td>
<td></td>
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<th>60.0</th>
<th>120.0</th>
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<td>E 06</td>
<td>2.2 E 05</td>
</tr>
<tr>
<td>4.06</td>
<td>E 05</td>
<td></td>
</tr>
</tbody>
</table>
PREPARATION OF SYNTHESIS SIGMAS FOR 2ND MODE IN X DIRECTION.

FAST FUNCTION INDEX IS 1. THERMAL FUNCTION INDEX IS 2.
RUN 41D. SEARCH FOR EIGENVALUE. 240 CM NONUNIFORM REACTOR. 7/7/66

1 THIS PROBLEM SEARCHES FOR THE PROMPT THERMAL NEUTRON
0 EIGENVALUE OF THE SECOND SPATIAL HARMONIC. REACTOR

IS THE 240 CM NONUNIFORM SLAB OF YASINSKY-HENRY
TWO NEUTRON GROUPS: ONE PRECURSOR GROUP.

FAST FUNCTION INDEX IS 2. THERMAL FUNCTION INDEX IS 2.
THIS IS THE 2ND TRIAL FUNCTION.
IT IS SIN(2*PI*X/240)
TO BE USED IN THE SYNTHESIS OF THE (2,4) PROMPT
THERMAL NEUTRON EIGENVECTOR
EXTRA CARD

FAST FUNCTION INDEX IS 1. THERMAL FUNCTION INDEX IS 2.
RUN 41D. SEARCH FOR EIGENVALUE. 240 CM NONUNIFORM REACTOR. 7/7/66

1 THIS PROBLEM SEARCHES FOR THE PROMPT THERMAL NEUTRON
0 EIGENVALUE OF THE SECOND SPATIAL HARMONIC. REACTOR

IS THE 240 CM NONUNIFORM SLAB OF YASINSKY-HENRY
THIS IS THE ADJOINT EIGENVECTOR

FAST FUNCTION INDEX IS 2. THERMAL FUNCTION INDEX IS 2.
THIS IS THE 2ND TRIAL WEIGHTING FUNCTION.
IT IS SIN(2*PI*X/240)
TO BE USED IN THE SYNTHESIS OF THE (2,2) PROMPT
THERMAL NEUTRON EIGENVECTOR
EXTRA CARD

EXPANSION TRIAL FUNCTIONS

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<th>FAST 2</th>
<th>THERMAL 1</th>
<th>THERMAL 2</th>
</tr>
</thead>
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<td>-1.0560280E-06</td>
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</tr>
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<td>6.6452022E 00</td>
<td>1.4731003E 00</td>
<td>4.1642469E 01</td>
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<td>1.30595636E 01</td>
<td>2.9199043E 00</td>
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<td>1.8999884E 01</td>
<td>4.3144684E 00</td>
<td>1.1906233E 00</td>
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<td>1.5201483E 00</td>
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<td>5.8484364E 00</td>
<td>1.7939486E 00</td>
</tr>
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<td>3.1925358E 01</td>
<td>7.9424991E 00</td>
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SYNTHESIS OF THE TWO-DIMENSIONAL (2,4) PROMPT THERMAL MODE.

GRP 1 IS 1ST FAST MODE; GRP 2 IS 2ND FAST MODE
GRP 3 IS 1ST THERMAL MODE; GRP 4 IS 2ND THERMAL MODE

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SYNTHESIS OF THE TWO-DIMENSIONAL (2,4) PROMPT THERMAL MODE

THE REACTOR BEING STUDIED IS A SLAB

THE MULTIDIMENSIONAL VECTOR EIGENFUNCTION IS BEING SYNTHESIZED BY A MARCH IN THE AXIAL DIRECTION USING 2 RADIAL FAST DISTRIBUTIONS AND 2 RADIAL THERMAL DISTRIBUTIONS

THE INNER BOUNDARY CONDITION IS THE FLUX EQUALS ZERO.

THE OUTER BOUNDARY CONDITION IS THE HOMOGENEOUS DIRICHLET ONE.


THE MODES ARE WD MODES.

THE FREQUENCY OF CONDITIONING EQUALS 4 AND EPSILON EQUALS 1.000E-06

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<th>LAST PT.</th>
<th>RADIUS-CM</th>
</tr>
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-579374999E 04
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-57939452E 04
-57938476E 04
-57938964E 04

DETERMINANT
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-24374080E-03
-20898746E-03
-44539090E-05
-11385201E-03
-53129741E-04
-24914917E-04
-10059204E-04
-27135585E-05
-34161630E-06
-76501760E-06
HERE IS THE PERCENT ERROR FOR MATCH AT X = 123 FOR Y MATCH AT 3.

HERE IS THE PERCENT ERROR FOR MATCH AT X = 119 FOR Y MATCH AT 3.

THE FREQUENCY OF CONDITIONING EQUALS 4 AND EPSILON EQUALS 1.000E-06

THIS IS THE EIGENFUNCTION

RAD.-CM     GROUP FLUXES, GROUPS 1 THRU 4
| 104.000 | -71715970E 00 | -13703085E 01 | -78116550E 00 | -14926076E 01 |
| 108.000 | -56240435E 00 | -10787714E 01 | -61259890E 00 | -11750516E 01 |
| 112.000 | -38546330E 00 | -78502298E 00 | -41986480E 00 | -85508599E 01 |
| 116.000 | -19327119E 00 | -48973254E 00 | -21052603E 00 | -53344065E 00 |
| 120.000 | -65326718E-02 | -19310310E 00 | -72245808E-02 | -21033712E 00 |
| 124.000 | -20639217E 00 | -10328239E 00 | -22481233E 00 | -11250077E 00 |
| 128.000 | -39825599E 00 | -39943499E 00 | -43379968E 00 | -43508552E 00 |
| 132.000 | -57447835E 00 | -69460814E 00 | -62602220E 00 | -75660333E 00 |
| 136.000 | -72888765E 00 | -98017460E 00 | -79393900E 00 | -10763694E 01 |
| 140.000 | -85465290E 00 | -12795554E 01 | -93092831E 01 | -13937577E 01 |
| 144.000 | -94701792E 00 | -15681021E 01 | -10315367E 01 | -17080526E 01 |
| 148.000 | -10022401E 01 | -18533107E 01 | -10916870E 01 | -20187163E 01 |
| 152.000 | -10179813E 01 | -21346584E 01 | -11088333E 01 | -23251752E 01 |
| 156.000 | -9939814E 00 | -24116767E 01 | -10820564E 01 | -26269238E 01 |
| 160.000 | -92917173E 00 | -26839421E 01 | -10120993E 01 | -29234849E 01 |
| 164.000 | -82749096E 00 | -29510460E 01 | -90134308E 01 | -32144210E 01 |
| 168.000 | -69195743E 00 | -32126684E 01 | -75371275E 00 | -34993975E 01 |
| 172.000 | -52745497E 00 | -34684622E 00 | -57452898E 00 | -37780344E 01 |
| 176.000 | -33996598E 00 | -37181798E 00 | -37030821E 00 | -40500211E 01 |
| 180.000 | -13635349E 00 | -39616809E 01 | -14852305E 00 | -43152541E 01 |
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| 188.000 | -27568356E 00 | -43359794E 00 | -30028755E 00 | -47229582E 00 |
| 192.000 | -46310373E 00 | -44316509E 01 | -50443856E 00 | -48271656E 01 |
| 196.000 | -62676127E 00 | -44560131E 01 | -68269888E 00 | -48537043E 01 |
| 200.000 | -75939393E 00 | -44035377E 00 | -82716874E 00 | -47965406E 01 |
| 204.000 | -85517924E 00 | -42707884E 01 | -93150277E 00 | -46519444E 01 |
| 208.000 | -90994514E 00 | -40562495E 01 | -99117550E 00 | -44182957E 01 |
| 212.000 | -92150045E 01 | -37612625E 01 | -10037455E 01 | -40969506E 01 |
| 216.000 | -88952567E 00 | -33890963E 01 | -96891361E 00 | -36915686E 01 |
| 220.000 | -81570689E 00 | -29454900E 01 | -88850685E 00 | -32083659E 01 |
| 224.000 | -70363796E 00 | -24384486E 01 | -76643607E 00 | -26560748E 01 |
| 228.000 | -55865113E 00 | -18779965E 00 | -60850926E 00 | -20456062E 01 |
| 232.000 | -38757176E 00 | -12758779E 00 | -42216149E 00 | -13897487E 01 |
| 236.000 | -19841027E 00 | -64520245E 00 | -21611811E 00 | -70278525E 00 |
| 240.000 | -18931208E-07 | -50153747E-07 | -10783502E-07 | -70805290E-07 |
HERE IS THE PERCENT ERROR FOR MATCH AT X = 123 FOR Y MATCH AT 3:
- 1.2536565E-04 - 4.4847089E 00 - 1.1504986E-05 - 4.4969592E 00

THE REACTOR BEING STUDIED IS A SLAB.

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THE FREQUENCY OF CONDITIONING EQUALS 4 AND EPSILON EQUALS 1.000E-06.

THIS IS THE ADJOINT EIGENFUNCTION.

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APPENDIX F

DERIVATION OF THE SOLUTION FOR THE SPACE
AND FREQUENCY DEPENDENT NEUTRON DENSITY IN A UNIFORM, SLAB REACTOR

The equation to be solved is Eq. (4.14) subject to the boundary condition: \( \psi(0,t) = \psi(a,t) = 0 \). After a time during which the transient solutions die out, it is assumed that the time behavior of \( \psi(x,t) \) is given as

\[
\psi(x,t) = \psi_0(x) + \psi(x,j\omega)e^{j\omega t}, \quad (F.1)
\]

where

\[
\psi(x,j\omega)e^{j\omega t} = \text{col}[N^{(1)}(x,j\omega), N^{(2)}(x,j\omega), C(x,j\omega)]
\]

Substituting Eq. (F.1) into Eq. (4.14) and eliminating the precursor component yields the equation,

\[
\frac{d^2N(x,j\omega)}{dx^2} + [B^2(j\omega)]N(x,j\omega) = S(x_o), \quad (F.2)
\]

where

\[
N(x,j\omega) = \text{col}[N^{(1)}(x,j\omega), N^{(2)}(x,j\omega)] \quad (F.3)
\]

\[
S(x_o) = \text{col}[0, \delta \Sigma_{a2}N^{(2)}(x)\delta(x-x_o)/D_2] \quad (F.4)
\]

\[
[B^2(j\omega)] = \begin{bmatrix}
B_{11} & B_{12} \\
B_{21} & B_{22}
\end{bmatrix} \quad (F.5)
\]
\[ B_{11}^2 = \frac{v_1}{D_1} \left( v \Sigma_f (1 - \frac{\beta_0}{\lambda^2 + \omega^2}) - \Sigma_r - \Sigma_{a1} - D_1 B_T^2 - D_1 \left( \frac{\eta^2}{a} \right)^2 \right) \tag{F.6} \]

\[ B_{12}^2 = \frac{v_1}{D_1} \left( v \Sigma_f (1 - \frac{\beta_0}{\lambda^2 + \omega^2}) - j\omega \nu \Sigma_f - \frac{\beta \lambda}{\lambda^2 + \omega^2} \right) \tag{F.7} \]

\[ B_{21}^2 = \frac{1}{D_2} \left( \nu \Sigma_f \frac{\eta}{D_2} \right) \tag{F.8} \]

\[ B_{22}^2 = \frac{v_2}{D_2} \left( - D_2 (\frac{\eta}{a})^2 - D_2 B_T^2 - \Sigma_{a2} - j\omega / v_2 \right) \tag{F.9} \]

\[ N_o^{(2)}(x) = \text{the mean value of the thermal neutron density, and} \]

\[ a = \text{the extrapolated width of the slab reactor. Eq. (F.2) is an inhomogeneous equation with homogeneous boundary conditions,} \]

\[ N(0,j\omega) = N(a,j\omega) = 0. \]

It is convenient to work with a homogeneous form of Eq. (F.2) which has the source term included as boundary conditions. Integrate Eq. (F.2) from \( x_0 - \epsilon \) to \( x_0 + \epsilon \) and let \( \epsilon \) become vanishingly small. The result of this is

\[ \frac{dN(x,j\omega)}{dx} \bigg|_{x=x_0^+} - \frac{dN(x,j\omega)}{dx} \bigg|_{x=x_0^-} = \begin{cases} 0 \\ \delta \Sigma_a \frac{N_o^{(2)}(x_0)/D_2}{D_2} \end{cases} \tag{F.10} \]

where \( x_0^+ = \lim(x_0 + \epsilon) \) and \( x_0^- = \lim(x_0 - \epsilon) \).

This is one boundary condition. The other boundary condition which has been introduced in the derivation of Eq. (F.10) is

\[ N(x_0^+,j\omega) = N(x_0^-,j\omega) \tag{F.11} \]
The solution of

\[ \frac{d^2 N(x, jw)}{dx^2} + [B^2(jw)] N(x, jw) = 0 \]  

(F.12)

in the region, \( 0 < x < x^o \), is

\[ N(x, jw) = \begin{cases} N^{(1)}(x, jw) & \text{for } x^+ < x < a \\ N^{(2)}(x, jw) & \text{for } x^+ < x < a \end{cases} \]

(F.13)

and the solution of Eq. (F.12) in the region, \( x^o < x < a \), is

\[ N(x, jw) = \begin{cases} A_s S_{\mu} \sin \mu(a-x) + A_s S_{\gamma} \sin \gamma(a-x) & \text{for } x^+ < x < a \\ A_s S_{\mu} \sin \mu(a-x) + A_s S_{\gamma} \sin \gamma(a-x) & \text{for } x^+ < x < a \end{cases} \]

(F.14)

The unknown quantities, \( \mu \) and \( \gamma \), and the unknown coupling coefficients, \( S_{\mu} \) and \( S_{\gamma} \), may be found by substituting Eq. (F.13) into Eq. (F.12).

In order for Eq. (F.12) to have a nontrivial solution the following requirements must be satisfied:

\[
\begin{vmatrix}
-S_{\mu}^2 + B_{12}^2 S_{\mu}^2 + B_{12}^2 & -S_{\gamma}^2 + B_{12}^2 S_{\gamma}^2 + B_{12}^2 \\
-\mu^2 + B_{22}^2 S_{\mu}^2 + B_{22}^2 & -\gamma^2 + B_{22}^2 S_{\gamma}^2 + B_{22}^2
\end{vmatrix} = 0 ,
\]

(F.15)

\[ S_{\mu} = \frac{\mu^2 - B_{22}^2}{B_{22}} \]

(F.16)

and

\[ S_{\gamma} = \frac{\gamma^2 - B_{22}^2}{B_{22}} \]

(F.17)
The quantities, $\mu^2$ and $\nu^2$, are found from Eq. (F.15) as

$$\mu^2 = \frac{1}{2} \left\{ \frac{B_{11}^2 + B_{22}^2}{(B_{11}^2 + B_{22}^2)^2 - 4(B_{11} B_{22} - B_{12} B_{21})^2} \right\}, \quad (F.18)$$

and

$$\nu^2 = \frac{1}{2} \left\{ \frac{B_{11}^2 + B_{22}^2}{(B_{11}^2 + B_{22}^2)^2 - 4(B_{11} B_{22} - B_{12} B_{21})^2} \right\}, \quad (F.19)$$

The remaining task is to determine the four coefficients, $A_\mu, A_\nu, A'_\mu$ and $A'_\nu$, by using the four conditions given by Eqs. (F.10) and (F.11). The result for the thermal neutron density component is

$$N^{(2)}(x,t) - N^{(2)}(x) = N^{(2)}(x_jw)e^{jwt}$$

$$= K(x,x_jw)e^{jwt}, \quad (F.20)$$

where

$$K(x,x_jw) = \frac{\sqrt{\delta\sigma_a}}{D^2} \frac{N^{(2)}(x_jw)}{N^{(2)}(x)} \left\{ S_{\mu} K_\mu - S_{\mu} K_{\nu} \right\},$$

$$K_\mu = \frac{\sin \mu(a-x_o) \sin \mu x}{\mu \sin \mu a} \quad \text{for} \quad x \leq x_o,$$

$$K_\mu = \frac{\sin \mu(a-x) \sin \mu x_o}{\mu \sin \mu a} \quad \text{for} \quad x \geq x_o,$$

$$K_{\nu} = \frac{\sin \nu(a-x_o) \sin \nu x}{\nu \sin \nu a} \quad \text{for} \quad x \leq x_o,$$

$$K_{\nu} = \frac{\sin \nu(a-x) \sin \nu x_o}{\nu \sin \nu a} \quad \text{for} \quad x \geq x_o.$$
REFERENCES


40. Bliss, H., Personal Communication (1965).


53. DPNV, INDV, SHARE Distributions 413 and 827, Computation Center, Massachusetts Institute of Technology, Cambridge, Mass.


BIOGRAPHICAL NOTE

Personal
Born on April 24, 1937 in Pratt, Kansas
Married to Janice Cheatham, January, 1960
Two children: Laura Lindsey (deceased), Andrew Lan

Academic Background
1943-1955 Public School System of Kiowa, Kansas
1955-1961 Kansas State University (Manhattan, Kansas)
1961-1962 University of Oslo, (Oslo, Norway)
1961-1966 Massachusetts Institute of Technology
(Cambridge, Massachusetts)

Degrees
1960 B.S., Nuclear Engineering; 1961 M.S., Nuclear Engineering

Fellowships Held and Honorary Society Membership
1955-1959 Dow Chemical Co. Scholarship
1960-1961 Atomic Energy Commission Fellowship
1961-1962 Fulbright Scholar to Norway
1962-1964 Atomic Energy Commission Fellowship
1964-1965 General Electric Fellowship
1966 National Science Foundation Traineeship

Member of Phi Eta Sigma, Phi Lambda Upsilon, Phi Kappa Phi,
Sigma Xi, Tau Beta Pi, and Blue Key.

Professional Society Membership
Student Member of the American Nuclear Society

Professional Experience
1957 Dow Chemical Company, Freeport, Texas (Summer)
1959 Argonne National Laboratory, Lemont, Illinois (Summer)
1962 Institut for Atomenergi, Kjeller, Norway
1965 Teaching Assistant, M.I.T., Nuclear Eng. Dept.

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Down and Diffusion Models", Kansas State University Bulletin,
"Investigations in Space-Dependent Reactor Kinetics", Nuclear Sci. and

After completion of this thesis, the author will spend two years in
the U.S. Army Corps of Engineers working with the Army Reactors