New Progress Towards Three Open Conjectures in Geometric Analysis

by

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Abstract

This thesis, like all of Gaul, is divided into three parts.

In Chapter One, I study minimal surfaces in $\mathbb{R}^3$ with quadratic area growth. I give the first partial result towards a conjecture of Meeks and Wolf on asymptotic behavior of such surfaces at infinity. In particular, I prove that under mild conditions, these surfaces must have unique tangent cones at infinity.

In Chapter Two, I give new results towards a conjecture of Schoen on minimal hypersurfaces in $\mathbb{R}^4$. I prove that if a stable minimal hypersurface Σ with weight given by its Jacobi field has a stable minimal weighted subsurface, then Σ must be a hyperplane inside of $\mathbb{R}^4$.

Finally, in Chapter Three, I do an in-depth analysis of the nodal set results of Logonov-Malinnikova. I give explicit bounds for the eigenvalue exponent in terms of dimension, and make a slight improvement on their methodology.

Thesis Supervisor: William P. Minicozzi
Title: Professor

Acknowledgments

There are so many people for me to thank that is it unreasonable to list them all here. However, there are some that I cannot neglect to mention.

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- The other two members of my committee, Toby Colding and Larry Guth, who have taught me so much over my years here both inside and outside the classroom.
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Finally, I must acknowledge my brother, John, and my mother, Phyllis, who have supported me in everything that I have ever done, and everything that I will do.

This thesis is dedicated to the memory of Marcus M. Mundy.

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Chapter 1

New Results Towards the Schoen Conjecture

1.1 Introduction

Let $\gamma \subset \mathbb{R}^3$ be a compact curve without boundary. One could ask the following question:

Plateau's Problem: Does there exist a compact surface Σ so that $\partial\Sigma = \gamma$ and Σ has minimal area out of all surfaces $\widetilde{\Sigma}$ with $\partial\widetilde{\Sigma} = \gamma$?

A solution to Plateau's Problem is called a **minimal surface**. This question was initially posed by Lagrange in 1760, and began the study of Calculus of Variation. Minimal surfaces occur in nature as soap films (see Figure 1-1), and Plateau did soap film experiments to further explore the problem in the mid 1800's. Plateau's problem in three dimensions was fully resolved in the early 1930's independently by Douglas and Rado.

Suppose that $\partial\Sigma = \gamma$. Then a necessary condition for Σ to be area-minimizing is

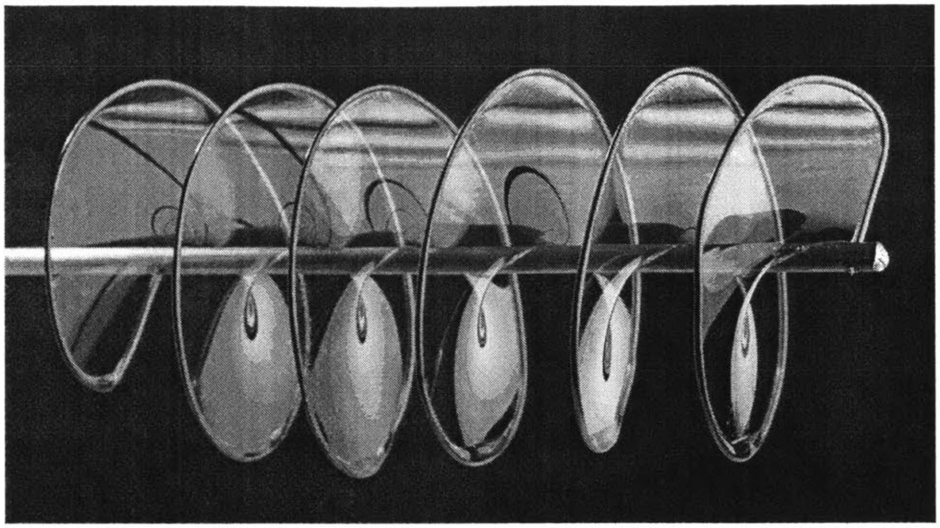

Figure 1-1: A half-helicoid created by a soap film

the following. If Σ_t is a smooth variation of Σ so that $\partial\Sigma_t = \gamma$ and $\Sigma_0 = \Sigma$, then

$$\frac{d}{dt}_{t=0} Area(\Sigma_t) = 0 \tag{1.1}$$

The above equation seems like a global condition on Σ, but it can be interpreted locally. The **second fundamental form** of Σ, denoted A, is defined as follows. Let v, w be vector fields on Σ, let n be a normal field, and let ∇ be the connection on $\mathbb{R}^n$. Then

$$A(v, w) := \langle \nabla_v w, n \rangle$$

Miraculously, A turns out to be a symmetric 2-tensor on the tangent space of Σ. The **mean curvature** H of Σ is defined to be the trace of A.

Equation 1.1 can be show to be equivalent to $H = 0$ on all of Σ – which is a purely local condition. However, this equation implies only that Σ is a critical point for area, not that it is a local minimizer. In order for Σ to be a local minimizer, it is necessary that the second order derivative of all variations be positive, that is,

$$\frac{d^2}{dt^2}\mathcal{H}^{n-1}(\Sigma_t \cap B(p, R)) \geq 0$$

The condition above is called **stability**. After a computation, this winds up being equivalent to the statement that the operator $\Delta_\Sigma + |A|^2$ has all negative eigenvalues, where Δ_Σ is the Laplacian on Σ. Note that we are taking the Laplacian to be negative. This operator is called the Jacobi operator, and solutions to the equation

$$\Delta_\Sigma h + |A|^2 h = 0$$

are called Jacobi functions. Any stable minimal surface is guaranteed to have a positive Jacobi function, and if a complete minimal surface has a positive Jacobi function, it is guaranteed to be stable.

While minimality is a purely local condition, stability is inherantly global – all minimal surfaces are locally area minimizing, but they might still be unstable on a large scale.

A commonly studied question in minimal surface theory is the following:

What do complete stable minimal surfaces look like?

The first results towards this question concerned minimal graphs, which are guaranteed to be stable, since the vertical vector field provides a positive Jacobi function when dotted with the unit normal to the surface.

Theorem 1.1.1 (Bernstein [4]). *Let Σ be a minimal graph in $\mathbb{R}^3$. Then Σ is a plane.*

Since this result was the first to give an effective condition under which a minimal surface was a plane, these types of results are often called **Bernstein theorems**. Bernstein's original result relied heavily on complex analysis. A proof via curvature estimates was presented by Heinz [20].

Bernestein theorems for minimal graphs were totally resolved by a series of results in the 1960's:

Theorem 1.1.2. *In dimension 4 (Fleming [18], De Giorgi [12]), 5 (Almgren [3]), 6, 7 and 8 (Simons [35]), all complete minimal graphs are planes. However, starting in dimension 9, there exist nonplanar minimal graphs (Simons [35], Bomberi, De Giorgi, Giusti [5])*

The general question for the structure of stable minimal hypersurfaces is much less well understood. The only full results are the following in dimensions 3 and 8.

Theorem 1.1.3 (Fischer-Colbrie–Schoen [17] and doCarmo-Peng [13]). *If Σ is a complete stable minimal surface in $\mathbb{R}^3$, then Σ is a plane.*

Theorem 1.1.4 (Simons [35]). *In $\mathbb{R}^8$ the minimal cone over $S^3 \times S^3$ is stable*

In addition, the Simons cone is also area minimizing, as proved by Bomberi, De Giorgi and Giusti [5].

There is virtually nothing known for dimension between 4 and 7. Schoen has conjectured the following:

Conjecture 1.1.5 (Schoen). *If $\Sigma \subset \mathbb{R}^4$ is stable and complete, then Σ is a hyperplane.*

We prove two main results towards proving this conjecture. Both effectively put restrictions on the types of stable minimal hypersurfaces in $\mathbb{R}^4$ which are not planes.

The first theorem restricts integrability of $|A|$.

Theorem 1.1.6. *Let Σ be a stable minimal hypersurface in $\mathbb{R}^4$, and let $\lambda_1^2 \geq \lambda_2^2 \geq \lambda_3^2$ be the three eigenvalues of A^2. Suppose that for some ϵ*

$$\int_\Sigma |A|^{3-\epsilon} < \infty$$

Then Σ is a hyperplane.

The second theorem restricts what minimal subsurfaces of Σ can look like.

Theorem 1.1.7. *Let Σ^3 be a stable minimal hypersurface in $\mathbb{R}^4$, and let h be a positive solution of the Jacobi equation on Σ, that is, let $\Delta_\Sigma h + |A|^2 h = 0$. If there exists a surface $\Gamma^2 \subset \Sigma$ such that Γ is minimal and stable when weighted by h, then Σ is a hyperplane.*

1.2 Proof of Theorem 1.1.6

Schoen-Simon-Yau proved:

Theorem 1.2.1. *[31] For $n \leq 6$, there exists $p(n) > n$ such that if a stable minimal hypersurface $\Sigma \subset \mathbb{R}^n$ satisfies $|\Sigma \cap B_R| \leq CR^p$, then Σ is a hyperplane.*

Therefore, to prove Schoen's conjecture, it is sufficient to show that any stable minimal hypersurface has volume growth slower than some polynomial in R. We will show that we can apply the following result of Carron to get Euclidean volume growth.

Theorem 1.2.2. *[8] Let M^n be a Riemannian manifold, and let $\lambda_1(x)$ be the most negative eigenvalue of $Ric_M(x)$ or 0. Suppose that the following three conditions hold:*

1. *There exists some $\delta > 0$ such that the operator $\Delta_M - (n-2)(1+\delta)\lambda_1$ is negative.*
2. *For some $\epsilon > 0$*
$$\int \lambda_1^{\frac{n}{2}(1-\epsilon)} < \infty$$
3. *M satisfies a Sobolev inequality.*

Then $Vol(\mathcal{B}_R(x)) \leq CR^n$.

As an aside, Carron's result allows for other assumptions besides integrability of Ricci, including the existence of a solution h to

$$\Delta_M h - (n-1)(1+\delta)\lambda_1 h = 0$$

such that $1 \leq h \leq \gamma$ for some constant γ. In the stable minimal surface scenario, this translates to having a Jacobi function which is bounded away from zero and infinity.

Let's begin with a few observations. To start, by the work of Hoffman and Spruck [21] and Michael and Simon [29], minimal surfaces in $\mathbb{R}^n$ always satisfy a Sobolev inequality.

Second, we have that because Σ is minimal,

$$Ric_\Sigma = -A^2$$

so that the most negative eigenvalue of Ric will be the square of the eigenvalue of A with the largest magnitude. Our assumption on the integrability of A immediately grants Carron's condition 2.

Therefore, we will aim to prove that

$$\Delta_\Sigma + (1+\delta)\lambda_1^2$$

is a negative operator – where we have taken $n = 3$. Note, that by stability, we already have that

$$\Delta_\Sigma + |A|^2 = \Delta_\Sigma + \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \leq 0$$

Let $V = (1+\delta)\lambda_1^2$. If we can show that $V \leq |A|^2$ for some $\delta > 0$, we will be done.

Lemma 1.2.3. *If $\delta = 1/2$, then $V \leq |A|^2$.*

Proof. First note that λ_2 and λ_3 must have the same sign, otherwise the inequality $\lambda_1^2 \geq \lambda_2^2 \geq \lambda_3^2$ would be incompatible with minimality.

After applying some algebra to the desired statement, we find that we need only prove $\delta\lambda_1^2 \leq \lambda_2^2 + \lambda_3^2$. Using minimality, we have that $\lambda_1 = -\lambda_2 - \lambda_3$. Substituting, we find that we need

$$\delta(\lambda_2^2 + 2\lambda_2\lambda_3 + \lambda_3^2) \leq \lambda_2^2 + \lambda_3^2$$

and thus

$$\lambda_2\lambda_3 \leq (\delta^{-1} - 1)\frac{\lambda_2^2 + \lambda_3^2}{2}$$

However, when we plug in $\delta = 1/2$, this simply becomes the AMGM inequality, which completes our proof.

$\square$

1.3 Proof of Theorem 1.1.7

This theorem and its proof are inspired by Schoen and Yau's full proof of the Positive Mass Theorem in all dimensions in [34]. Their original proof of the Positive Mass Theorem in [32] and [33] relied on the regularity theory for minimal hypersurfaces which only gave the result in dimensions less than 7. However, in [34], by using clever weights derived from positive solutions of the Jacobi equation, they were able to extend their proof to all dimensions. We use a similar approach here.

Let $\Delta_\Sigma h + |A|^2 h = 0$ and $h > 0$. Use $h = e^\phi$ as a weight on Σ. Then we have the weighted Ricci Ric_ϕ and Perelman scalar R_ϕ curvatures given by:

$$Ric_\phi = Ric_\Sigma - \nabla^2\phi$$
$$R_\phi = R_\Sigma - 2\Delta\phi - |\nabla\phi|^2$$

However, note that $\Delta\phi = -|A|^2 - |\nabla\phi|^2$, and so

$$R_\phi = |A|^2 + |\nabla\phi|^2$$

We therefore expect Σ with the weight h to behave like a 3-manifold with positive scalar curvature.

Let $\Gamma^2 \subset \Sigma^3$ be a weighted stable minimal surface with respect to the weight h, and let $\overline{A}$ be its unweighted second fundamental form as a submanifold of Σ. By Espinar [16] we have the following:

Lemma 1.3.1. *(Main Lemma in [16]) With Γ and Σ as above, and for $\psi \in C_0^\infty(\Gamma)$*

$$\frac{1}{3}\int_\Gamma (V - K_\Gamma)\psi^2 \leq \int_\Gamma |\nabla\psi|^2$$

where

$$V = \frac{1}{2}R_\phi^2 + \frac{1}{2}|\bar{A}|^2 + \frac{1}{8}|\nabla\phi|^2$$

Note that by our formula for R_ϕ we have that

$$V = \frac{1}{2}(|A|^2 + |\bar{A}|^2) + \frac{3}{8}|\nabla\phi|^2$$

We will also use the following result of Espinar:

Theorem 1.3.2. *(4.3 in [16]) Suppose that $\Gamma^2 \subset \Sigma^3$ is a weighted stable minimal surface where $R_\phi + \frac{1}{4}|\nabla\phi|^2 \geq 0$. Then Γ is conformal to either $\mathbb{C}$ or $\mathbb{C}\backslash 0$. In the second case, Γ must be totally geodesic, and $R_\phi + \frac{1}{4}|\nabla\phi|^2 = 0$ along Γ.*

Since $R_\phi \geq 0$ in our setting, the above holds. Note that in both the $\mathbb{C}$ and $\mathbb{C}\backslash 0$ cases, Γ must be parabolic. In particular, this means that there is a sequence of $u_k \in C_0^\infty(\Gamma)$ such that the following holds:

- $u_k = 1$ on $B_k(p)$.
- $\int_\Gamma |\nabla u_k|^2 \to 0$.

We will use these u_k as our test functions in Equation 1.3.1. Taking limits, the right hand side goes to zero, and the left hand side will approach $\frac{1}{3}\int_\Gamma(V - K)$.However, in order to get something useful, we will need that $V \geq K_\Gamma$.

Lemma 1.3.3. *With K_Γ the Gaussian curvature of Γ, we have $K_\Gamma \leq \frac{1}{2}(|A|^2 + |\bar{A}|^2)$.*

Proof. Let $E_1, E_2 \in T\Gamma$ be an orthonormal frame near p, let E_3 be perpendicular to $\Gamma \subset \Sigma$, and let E_4 be perpendicular to Σ. Then applying the Gauss Equation twice, we get

$$\begin{aligned} K_\Gamma &= R^\Sigma_{1212} + \bar{A}_{11}\bar{A}_{22} - \bar{A}_{12}^2 \\ &= A_{11}A_{22} - A_{12}^2 + \bar{A}_{11}\bar{A}_{22} - \bar{A}_{12}^2 \\ &\leq A_{11}A_{22} + \bar{A}_{11}\bar{A}_{22} \end{aligned}$$

However, it's an easy algebra fact to see that $A_{11}A_{22} \leq |A|^2/2$ and the same fact with $\bar{A}$. $\square$

Combining the above lemma with Equation 1.3.1, we get that $V - K_\Gamma = 0$ everywhere. Therefore, all inequalities in the above lemma must be sharp, and $|\nabla\phi| = 0$ along Γ. In particular, the equalities $A_{11}A_{22} = |A|^2/2$ and $\bar{A}_{11}\bar{A}_{22} = |\bar{A}|^2/2$ will only be true if $A_{11} = A_{22}$, $\bar{A}_{11} = \bar{A}_{22}$ and all other terms are zero. Thus, Γ^2 is totally umbilic inside of Σ^3.

However, note that the weighted mean curvature of Γ is $H_\phi = H + \langle N, \nabla\phi\rangle$, and since $\nabla\phi = 0$, Γ is actually minimal in Σ, and so the trace of $\bar{A}$ is zero. Combining this with Γ being totally umbilic, we have that Γ is totally geodesic in Σ. Similarly, since $A_{11} = A_{22}$ and all other terms are zero, and since Σ is minimal, we have that A is also identically zero along Γ. Thus, Γ is a 2-plane, and Σ is totally geodesic along Γ. The next lemma will give our result.

Lemma 1.3.4. *Suppose that $\Sigma^{n-1} \subset \mathbb{R}^n$ is a smooth minimal hypersurface which has $A \equiv 0$ along a $n-2$ plane. Then Σ is flat.*

Proof. First note that since Σ and A are both real analytic, if we can show that A is zero to all orders at a point, it must be zero everywhere.

By the Gauss equation, all sectional curvatures of Σ along Γ must be zero. Combining this with the Codazzi equations, this implies that the tensors $\nabla^k A$ are all totally symmetric along Γ. We will prove $\nabla^k A = 0$ along Γ by induction on k.

First, since $A = 0$ on Γ, we have our base case. Suppose $\nabla^k A = 0$ along Γ. Then because of our choice of the orthonormal frame E_i, we have that

$$\nabla_i \nabla^k A = 0$$

if $i = 1$ or $i = 2$. Since $\nabla^{k+1} A$ is fully symmetric, if any subscripts are equal to 1 or 2, that term must be zero. So, the only one that could be nonzero is the term that has all 3's as subscripts. However, because of minimality, we can replace A_{33} by $-A_{11} - A_{22}$, which will then make the term zero.

Therefore, A and all its derivatives are zero along Γ and so A must be identically zero, so Σ must be flat. $\square$

Chapter 2

Tangent Cones at Infinity for Minimal Surfaces in $\mathbb{R}^3$

2.1 Introduction

Let Σ be an embedded minimal surface in $\mathbb{R}^3$. One of the fundamental properties of minimal surfaces is the following:

Theorem 2.1.1 (Monotonicity). *[9] Let $r > s$. Then*

$$\frac{A(\Sigma \cap B_r)}{r^2} - \frac{A(\Sigma \cap B_s)}{s^2} = \int_{\Sigma \cap B_r \setminus B_s} \frac{|x^N|^2}{|x|^4} \geq 0$$

Note that if we define the **area density** as

$$\Theta(r) := \frac{A(\Sigma \cap B_r)}{\pi r^2}$$

then the monotonicity formula implies that Θ is nondecreasing. If

$$\lim_{r \to \infty} \Theta(r) = \Theta(\infty) = k < \infty,$$

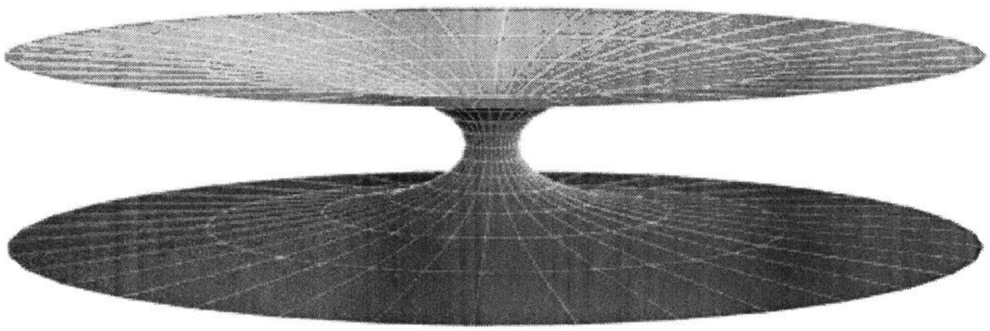

Figure 2-1: Catenoid (from http://www.indiana.edu/ minimal)

we say that Σ has quadratic area growth, or the area growth of k planes.

For surfaces with the growth of 2 planes, there are two canonical examples: the catenoid (Fig 2-1), and Scherk's singly periodic surfaces, which occur in a one parameter family (Fig 2-2 and Fig 2-3), where the parameter is the angle betwee the two leaves. As the angle goes to zero, the Scherk surfaces approach a catenoid on compact sets after an appropriate rescaling. In 2005, Meeks and Wolf proved the following theorem:

Theorem 2.1.2. *[28] Suppose that Σ is an embedded minimal surface in $\mathbb{R}^3$ which has infinite symmetry group and $\Theta(\infty) < 3$. Then Σ is either a catenoid or a Scherk example.*

Meeks has conjectured that the symmetry condition in the above may be removed:

Conjecture 2.1.3. *[27] Let Σ be an embedded minimal surface in $\mathbb{R}^3$ with area growth of 2 planes. Then Σ is either a catenoid or a Scherk example.*

However, an initial difficulty with the above is that it is not yet known that a minimal surface with quadratic growth even needs to be asymptotic to a catenoid or a Scherk example. By the compactness results from Geometric Measure Theory, it is known that if Σ is an embedded minimal surface with quadratic area growth, then for any sequence $r_i \to \infty$, there exists a subsequence ρ_i such that $\Sigma/\rho_i \cap B_1$ converges

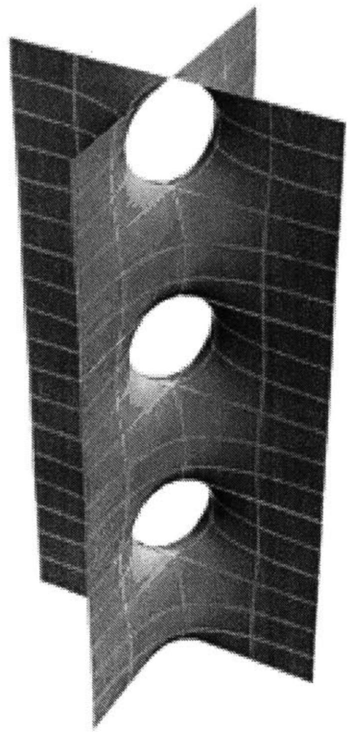

Figure 2-2: Scherk Singly Periodic (from http://www.indiana.edu/ minimal)

to a minimal cone $\mathcal{C}$ in the varifold topology. Such a cone $\mathcal{C}$ is called a **tangent cone at infinity**. A priori, there may be many tangent cones at infinity.

This leads to the following conjecture, also due to Meeks:

Conjecture 2.1.4. *[27] Let Σ be an embedded minimal surface in $\mathbb{R}^3$ with quadratic area growth. Then Σ has a unique tangent cone at infinity.*

In the case of finite genus, this had already been resolved by Collin [11], who proved that any minimal surface with finite genus and quadratic area growth must be asymptotic to a single multiplicity k plane. In particular, when combined with a result of Schoen [30], this resolves Meeks' full conjecture in the case of finite genus - that is, the only minimal surface with the area growth of two planes and finite genus is the catenoid.

In this paper, we prove that Meeks' Conjecture 1.4 holds true under additional

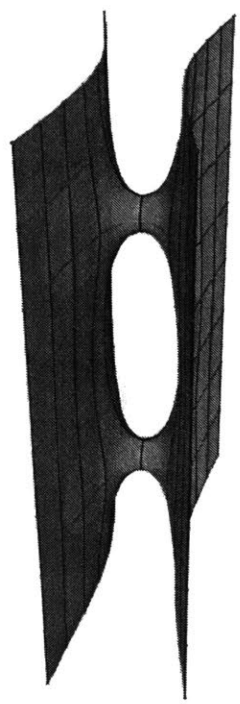

Figure 2-3: Non-orthogonal Scherk (from http://www.indiana.edu/ minimal)

assumptions:

Theorem 2.1.5. *Let Σ be an embedded minimal surface with the area growth of k planes. Suppose that there exists $\alpha < 1$ such that for all R sufficiently large, there exists a line l_R*

$$\Sigma \cap B_R \cap \{d(x, l_R) > R^\alpha\}$$

is a union of at least $2k$ disks Σ_i and such that $\partial\Sigma_i$ is homotopically nontrivial in $\partial(B_R \cap \{d(x, l_R) > R^\alpha\})$. Then Σ has a unique tangent cone at infinity.

This leads to the following:

Theorem 2.1.6. *Let Σ be an embedded minimal surface with quadratic area growth. Let*

$$\mathcal{C}_\alpha = \{x_1^2 + x_2^2 \leq R^{2\alpha}\}.$$

Then if for some R_0, $\Sigma \backslash (B_{R_0} \cup \mathcal{C}_\alpha)$ is a union of $2k$ topological disks Σ_i each with finitely many boundary components, then Σ has a unique tangent cone at infinity.

Note that the corollary substitutes the homotopy requirement from the theorem for the existence of a single line around which we can base our sublinearly growing set. To the author's knowledge, these two theorems are the first progress towards proving Meeks' conjecture.

2.1.1 Summary of Proofs

Both of the above theorems are proved by first showing a lower area bound for the area of Σ inside large balls. This, combined with the upper area bound coming from the monotonicity formula and quadratic area growth, along with a projection argument due to Brian White, leads to uniqueness of tangent cones.

Both theorems prove their lower area bound by working on each leaf of Σ separately. The lower area bound used in Theorem 2.1.5 is rather straightforward to prove using the homotopy requirement. However, bounding the area from below in Theorem 2.1.6 is slightly more detailed, and relies on arguments made in the proof of Lemma 2.2.1, as well as a case by case analysis of the possible shapes of the leaves of Σ.

2.2 Proof of Theorem 2.1.5

The proof of this begins with the following:

Lemma 2.2.1 (Lower Area Bound). *Suppose that Σ satisfies the conditions of Theorem 2.1.5. Then for some $C = C(\Sigma)$*

$$Area(B_R \cap \Sigma) > k\pi R^2 - CR^{\alpha+1}.$$

Proof. We will work on each leaf Σ_i separately, and the lemma will come from adding the area of all the leaves together.

First note that $B_R \cap \{d(x, l_R) > R^\alpha\} = T_R$ is a rotationally symmetric solid torus and (since Σ_i is a disk), $\partial\Sigma_i$ is contractible in T_R. However, since T_R is rotationally symmetric, the smallest spanning disk for any such curve has area at least that of a vertical cross section C. Any such vertical cross section consists of a half-circle of radius R minus a strip of length $2R$ and width CR^α. Thus, we have

$$A(\Sigma_i) \geq A(C) \geq \frac{\pi}{2}R^2 - CR^{\alpha+1}$$

$\square$

Remark 2.2.2. *Note that Lemma 2.2.1 implies that there are in fact exactly $2k$ disks in the statement of Theorem 2.1.5.*

We make a definition:

Definition 2.2.3. *The **error** at scale r of a minimal surfaces with area growth of k planes is defined as*

$$e(r) = \pi k - \frac{Area(\Sigma \cap B_r)}{r^2}$$

Thus, Lemma 2.2.1 is equivalent to the statement:

$$e(r) \leq Cr^{\alpha-1} \tag{2.1}$$

We now apply an argument of Brian White [36] to prove uniqueness of the tangent cone.

Lemma 2.2.4. *Let Σ satisfy the following: $\exists R_0, \alpha < 1$ such that for $R_0 < r < \infty$,*

$$e(r) < Cr^{1-\alpha} \tag{2.2}$$

Then Σ has a unique tangent cone at infinity.

Proof. Define $F(z) = z/|z|$. Then note that $A(F(\Sigma \cap (B_r \backslash B_s)))$ is equal to the area of the projection of $\Sigma \cap (B_r \backslash B_s))$ onto the unit sphere. We will bound this area. We have:

$$\begin{aligned} A(F(\Sigma \cap (B_r \backslash B_s))) &= \int_{\Sigma \cap B_r \backslash B_s} \frac{|x^N|}{|x|^3} d\Sigma \\ &\leq \left[\int_{\Sigma \cap B_r \backslash B_s} \frac{|x^N|^2}{|x|^4} d\Sigma \right]^{1/2} \left[\int_{\Sigma \cap B_r \backslash B_s} \frac{1}{|x|^2} d\Sigma \right]^{1/2} \end{aligned}$$

By the monotonicity formula, 2.1.1 and the fact that the area density of Σ is uniformly bounded by k, we can bound the term inside the first bracket:

$$\begin{aligned} \int_{\Sigma \cap B_r \backslash B_s} \frac{|x^N|^2}{|x|^4} d\Sigma &\leq \frac{A(\Sigma \cap B_r)}{r^2} - \frac{A(\Sigma \cap B_s)}{s^2} \\ &\leq k\pi - \frac{A(\Sigma \cap B_s)}{s^2} = e(s) \end{aligned}$$

For the term in the second bracket, we have

$$\int_{\Sigma \cap B_r \backslash B_s} \frac{1}{|x|^2} d\Sigma \leq \int_{\Sigma \cap B_r \backslash B_s} \frac{1}{s^2} d\Sigma \leq A(B_r \cap \Sigma) s^{-2}.$$

Thus, we get that

$$A(F(\Sigma \cap (B_r \backslash B_s))) \leq e(s)^{1/2}(s^{-2}A(B_r \cap \Sigma))^{1/2}$$

Now, by equation (2.2), along with the fact that $A(B_r \cap \Sigma) < k\pi r^2$, we have that this is bounded by

$$Cs^{(\alpha-1)/2}\left[\left(\frac{r}{s}\right)^2 (r^{-2}A(B_r \cap \Sigma))\right]^{1/2} \leq C\frac{r}{s^{(1-\alpha)/2+1}}$$

Pick s and r such that $s \leq r \leq 2s$. Then

$$A(F(\Sigma \cap (B_r \backslash B_s))) \leq Cs^{(\alpha-1)/2}$$

We then sum the above bound to see

$$\begin{aligned} A(F(\Sigma \cap (B_{2^n r} \backslash B_r))) &= \sum_{k=1}^{n} A(F(\Sigma \cap (B_{2^k r} \backslash B_{2^{k-1} r}))) \\ &\leq C\sum_{k=1}^{n}(2^k r)^{(\alpha-1)/2} \\ &\leq \frac{C}{r^{(1-\alpha)/2}}\frac{1}{1-2^{(1-\alpha)/2}} \end{aligned}$$

As $r \to \infty$, this term goes to zero. Thus, the area of the projection of $\Sigma \backslash B_r$ approaches zero as r gets large, which means that the tangent cone must be unique.

$\square$

2.3 Proof of Theorem 2.1.6

For the reader's convenience, we restate the assumptions: that there exists α, R_0 such that if

$$\mathcal{C}_\alpha = \{x_1^2 + x_2^2 \leq R^{2\alpha}\}$$

and $\Sigma \backslash (B_{R_0} \cup \mathcal{C}_\alpha$ is a union of $2k$ disks Σ_i, each with finitely many boundary components.

Note that the closure of Σ_i in $\mathbb{R}^3$ must be conformally equivalent to $\overline{\mathbb{D}}^2$ with finitely many boundary points removed. Take a neighborhood N of one of these missing boundary points which does not come close to any other missing boundary points. Then $N \subset \Sigma_i$ has exactly one boundary component. There are two options for the shape of ∂N.

1. The function $x_3|_{\partial N}$ is unbounded in both directions.

2. $x_3|_{\partial N}$ is bounded in one direction.

Note that x_3 cannot be bounded in both directions, as then ∂N would be compact, which it is not.

We temporarily assume that Option 1 occurs (see Figure 2-4). Let γ be the portion of ∂N which is not on the boundary of $\mathcal{C}_\alpha \cup B_{R_0}$. Note that we can take R_0 to be large enough so that ∂B_{R_0} is arbitrarily close to the missing point of $\partial \overline{\mathbb{D}}^2$, and thus in particular, $\gamma \subset B_{R_0}$. Redefine N to be $N \cap B_{R_0}^c$, and let $R >> R_0$.

Lemma 2.3.1. *$\partial B_R \cap N$ has a component which starts at the $x_3 \to +\infty$ side of $\partial N \cap \partial \mathcal{C}_\alpha$ and ends at the $x_3 \to -\infty$ side.*

Proof. Suppose not. Then every component of $\partial B_R \cap N$ starts and ends on the same side of the missing point. In particular, there are an even number of points on each side. Consider moving along $\partial \mathcal{C}_\alpha$ towards the missing point. Each point of

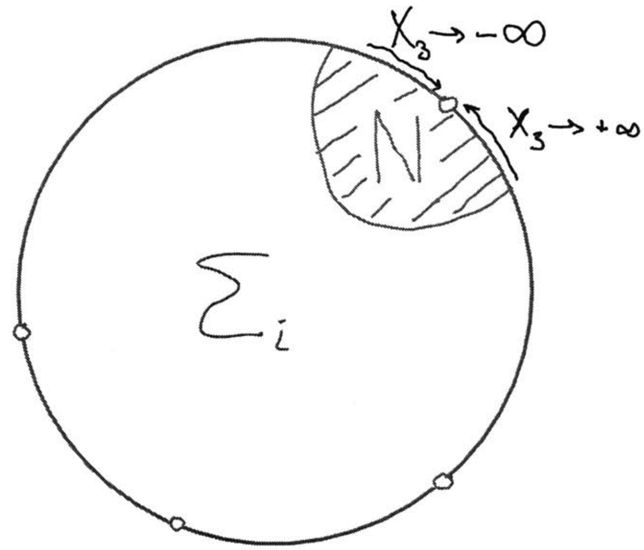

Figure 2-4: N and Σ_i for Option 1 (conformal picture)

$\partial B_R \cap N \cap \partial \mathcal{C}_\alpha$ represents a change from radius smaller than R to radius larger than R. However, since the radius started at $R_0 < R$, there cannot be an even number of these points. $\square$

The above lemma implies that some component of $N \cap B_R \cap C_\alpha^c$ will satisfy the homotopy conditions of Theorem 2.1.5. This implies that it is possible to prove the Lower Area Bound lemma for this component, and in particular, the area must be asymptotic to $\pi R^2/2$.

The following lemma will complete our proof:

Lemma 2.3.2. *Under our assumptions, Option 2 is not possible.*

Proof. Suppose that Option 2 occurs. WLOG, let $x_3|_{\partial N}$ be bounded below by 0, and let $(x_1, x_2, 0) \in \partial N$ be the point at which that minimum is achieved. Let $\rho = (x_1^2 + x_2^2)^{1/2}$. Let C be a catenoid where the radius of the center geodesic is strictly larger than 2ρ. Then by a simple application of the maximum principle, N must intersect C. In particular, this implies that $\inf_{\partial B_R} x_3|_N < C_0 + \log R$.

Now, consider a sequence of R_i such that $\Sigma \cap B_{R_i}$ converges to a tangent cone at infinity. By compactness, $R_i^{-1}N \cap \partial B_{R_i}$ must either converge to a union of geodesics on B_1 or must disappear at infinity. However, due to the discussion of the previous paragraph, N cannot disappear at infinty, and so must converge to a nontrivial union of geodesics Γ_j, possibly with endpoints at the north or south poles. We aim to show that these Γ_j are all great circles.

Let p be a nonsmooth point on $\cup\Gamma_j$. Then there must exist a neighborhood S of p such that $|A|$ restricted to $S \cap R_i^{-1}N$ is unbounded as $i \to \infty$. However, since N is a minimal disk with quadratic area growth bounds, $|A|(x)$ must be bounded by $C/d(x)$, where $d(x)$ is the distance of x from the boundary of N.

Suppose that our nonsmooth p is not equal to the south pole. Then we can choose our neighborhood S of p to stay away from the x_3 axis, so we will have that $|A| < C$ uniformly on $S \cap R_i^{-1}N$. Suppose that p is equal to the south pole. Then by the assumption of Option 2, ∂N is only contained in the region $x_3 \geq 0$. So, we can choose $S = B_{1/2}(p)$, and this implies the same uniform $|A|$ bound.

Therefore, there will be no nonsmooth points of $\cup\Gamma_j$, which implies that Γ_j consists of a single great circle passing through the north pole.

In particular, this implies that there are some $\epsilon(R_i) \to 0$ such that the area of $R_i^{-1}N \cap B_1$ is greater than $\pi - \epsilon(R_i)$, where $\epsilon \to 0$ as $R_i \to \infty$. Thus, we have at least $2k$ components of $\Sigma \backslash \mathcal{C}_\alpha$, each of which has area growth at least $\pi R^2/2$ by the discussion of Option 1. However, since the global area growth is $k\pi R^2$, no component can have growth πR^2. $\square$

2.4 Future Directions

2.4.1 Expanding on Current Work

There are several potential extensions of the work above. Theorem 2.1.5 and Corollary 2.1.6 effectively assume that all tangent cones of Σ are unions of planes *with a common axis.* It is likely not significantly more difficult to show that the same result holds in the case when the one-dimensional singular set is more complicated, as long as away from a sublinearly growing neighborhood, Σ is a union of disks. That is, we have the following as another potential step towards the resolution of Meeks' Conjecture:

Conjecture 2.4.1. *Let Σ have the area growth of k planes, and suppose that there exists a uniform $\alpha < 1$ such that for each $R > R_0 >> 1$, the following is true: There exist line segments $L_i(R)$, $1 \leq i \leq m(R) < M$ such that outside of an $\alpha-$sublinearly growing neighborhood of $\cup L_i(R)$, $\Sigma \cap B_R$ is a union of disks. Then Σ has a unique tangent cone at infinity.*

There are likely other simple conditions which can be put on Σ to force Lemma 2.2.1 to hold.

However, it may be possible to prove theorems approaching Conjecture 1.4 without factoring through some kind of lower area bound.

2.4.2 Dealing with Higher Multiplicity

Many of the problems in uniqueness for tangent cones stem from the fact that higher multiplicity tangent cones exist. Very little is known about this situation in general. The multiplicity one case is simplified by the Allard Regularity Theorem, a consequence of which we state here:

Theorem 2.4.2. *[2] There exists ϵ such that if Σ^{n-1} is a minimal surface in $B_1 \subset \mathbb{R}^n$ with $\theta(1) < 1 + \epsilon$, then Σ is smooth and $|A| < c(\epsilon)$ inside $B_{1/2}$.*

However, such an estimate is obviously not true if multiplicity is large. For example, a rescaled catenoid has curvature blowing up at a point, and a properly rescaled sequence of Scherk surfaces will have curvature blowing up along a line while approaching a multiplicity 2 plane. This is not even the worst case scenario. By the sphere doubling work of Kapouleas and McGrath ([22] and [23]), there exists a sequence of minimal surfaces $\Sigma_i^2 \subset B_1 \subset \mathbb{R}^3$ which approach the plane $x_3 = 0$ with multiplicity 2, but if p is any point on the plane $x_3 = 0$, and $\epsilon > 0$, then

$$\sup_{B(p,\epsilon)\cap\Sigma_i} |A| \to \infty \text{ as } i \to \infty$$

Note – the results of Kapouleas and McGrath concern the convergence of a sequence of minimal surfaces in S^3 to an equatorial S^2 with multiplicity 2. In their examples, the curvature must do one of the following two things (up to a subsequence):

1. $|A| \to 0$ away from a set with finite $\mathcal{H}^1$ measure
2. $|A| \to \infty$ everywhere.

Therefore, I believe the following conjecture to be true:

Conjecture 2.4.3. *Let Σ_i be a sequence of minimal surfaces in $B_1 \subset \mathbb{R}^3$ which approach a multiplicity k plane. Let Γ_0 be a Lipschitz set of finite $\mathcal{H}^1$ measure, and suppose that for all $\epsilon > 0$*

$$\limsup_{(B_1\setminus(B_{1/2}\cup N_\epsilon(\Gamma_0))\cap\Sigma_i} |A| = 0$$

Then after taking a subsequence Σ_j, there exists a Lipschitz set Γ of finite $\mathcal{H}^1$ measure such that for all ϵ,

$$\limsup_{(B_1\setminus N_\epsilon(\Gamma))\cap\Sigma_j} |A| = 0$$

That is to say, as long as the Σ_i converge smoothly away from a set of curves outside $B_{1/2}$, they will also converge smoothly away from a set of curves **inside** $B_{1/2}$.

Chapter 3

Explicit Constants for the Logunov-Malinnikova Method

3.1 Introduction

Let (N, g) be a compact Riemannian manifold without boundary, and let ϕ be an eigenfunction of the laplacian on N, that is:

$$\Delta_N \phi = -\lambda \phi$$

where $\lambda > 0$. We define the nodal set

$$Z(\phi) := \{x \in N | \phi(x) = 0\}$$

A simple example of this is the following. Let $N = S^1$. Then $\Delta = \frac{d^2}{dx^2}$, and the eigenfunctions of Δ are just $\sin(nx)$ and $\cos(nx)$ with eigenvalues n^2. Both $\sin(nx)$ and $\cos(nx)$ have exactly n zeros on S^1.

In two dimensions the nodal set can be visualized by **Chladni plates** (see Figure

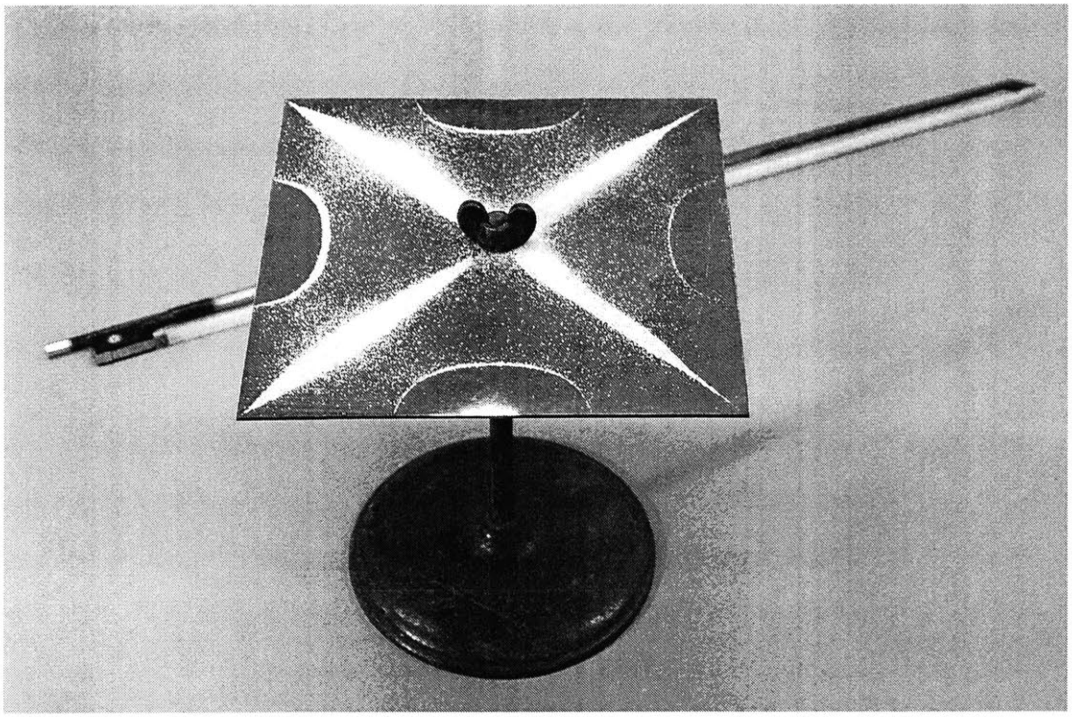

Figure 3-1: A Chladni Plate showing the nodal set of a square – Smithsonian

3-1). Developed by Ernst Chladni in the late 1700's, they are generally made of metal or stiff wood, suspended at a single point, and then made to vibrate, either by an external speaker playing a pure frequency, or by using a bow along an edge. Sand is then sprinkled over the top of the plate. As the plate vibrates, the sand will accumulate along the lines where the vibration is lowest, which are exactly the nodal sets of the plate at that frequency. As the frequency increases, the nodal set becomes larger.

In [37], Yau conjectured that the following statment about the zero set of ϕ should hold:

Conjecture 3.1.1 (Yau). *With ϕ and λ as above,*

$$C^{-1}\lambda^{1/2} \leq \mathcal{H}^{n-1}(Z(\phi)) \leq C\lambda^{1/2}$$

where $C = C(N, g)$.

This conjecture has a rich history. For real analytic manifolds with real analytic metrics, the full conjecture was proven by Donnelly and Fefferman in 1988

Theorem 3.1.2 (Donnelly-Fefferman [14]). *If $(N, g) \in C^\omega$, then*

$$C^{-1}\lambda^{1/2} \leq Z(\phi) \leq C\lambda^{1/2}$$

An initial upper bound for any manifold and metric was achieved by Robert Hardt and Leon Simon in 1989.

Theorem 3.1.3 (Hardt-Simon [19]). *With ϕ and λ as above,*

$$Z(\phi) \leq c\lambda^{c\sqrt{\lambda}}$$

For surfaces, the following was already known, with the full lower bound above proved by Brüning [6], and the upper bound achieved by Donnelly and Fefferman [15].

Theorem 3.1.4.

$$C^{-1}\lambda^{1/2} \leq Z(\phi) \leq C\lambda^{3/4}$$

Colding and Minicozzi proved another lower bound with dimension-dependent exponent:

Theorem 3.1.5 (CM [10]).

$$Z(\phi) \geq C\lambda^{\frac{3-n}{4}}$$

The lower bound was achieved in full generality by Logunov and Malinnikova in [25]. In [24] they also proved an upper bound of the form

$$\mathcal{H}^{n-1}(\{x|\phi(x) = 0\}) \leq C\lambda^{\alpha}$$

where α is some function of dimension only. The result of Logunov and Malinnikova in Equation 3.1 is the first estimate that works for C^∞ manifolds that is polynomial in λ.

In this chapter, we refine their technique, fill in missing proofs of lemmas from Euclidean geometry, and in so doing, achieve an almost explicit formula for α in terms of dimension.

3.2 Preliminaries

We utilize the standard trick of turning Laplace eigenfunctions into harmonic functions.

If $\Delta\phi = -\lambda\phi$, then we let u be a function defined on $M = N \times \mathbb{R}$, and define

$$u(x,t) = \phi(x)e^{\sqrt{\lambda}t}$$

where $x \in N$ and $t \in \mathbb{R}$. Then u will be harmonic on the product manifold M, and here on out we will mostly work with zero sets of u in M and relate it back to N at the very end.

Let u be a harmonic function on M, fix a base point O, and define

$$H(r) = \int_{\partial B(O,r)} u^2$$

for r smaller than the injectivity radius of M.

Definition 3.2.1. *The **frequency function** of a harmonic function is defined as*

$$\beta(r) = \frac{rH'(r)}{2H(r)}$$

For harmonic functions in $\mathbb{R}^n$, the frequency is monotone, and constant for harmonic polynomials. In general, the frequency is almost monotone, see [26].

Lemma 3.2.2. *For any $\epsilon > 0$, there exists $R_0(M, g, \epsilon, O)$ such that*

$$\beta(r_1) < \beta(r_2)(1+\epsilon)$$

for $r_1 < r_2 < R_0$

This begins a common theme throughout the work of Logunov and Malinnikova

- before doing anything, we must zoom in on our point of interest until our metric looks nearly Euclidean. The amount that we initially zoom in will be irrelevant to the final exponent α in Equation 3.1, so we will do it as much as we please.

Similar to 3.2.2, we will decrease R_0 enough so that in normal coordinates in $B(O, R_0)$, we can treat Δ_M as a uniformly elliptic operator in $\mathbb{R}^n$, and so that if d is Euclidean distance in these coordinates, and d_g is distance in the metric,

$$1 - \epsilon < \frac{d_g(x,y)}{d(x,y)} < 1 + \epsilon$$

This will allow us to pretend like we are living inside some large compact domain in $\mathbb{R}^n$. Again, this step will have no effect on our α.

Rather than working with the frequency function β, we choose instead to use the following L^∞ version:

Definition 3.2.3. *For a ball $B \subset \mathbb{R}^n$, the* ***doubling index*** *of B, notated $N(B)$, is given so that*

$$2^{N(B)} = \frac{\sup_{2B} |u|}{\sup_B |u|}$$

If $B = B(p, r)$ a ball centered at p with radius r, then $N(p, r) := N(B)$.

The proof of the following lemma can be found in [24]. We will use it unchanged.

Lemma 3.2.4 (Quantitative Doubling). *For any $\epsilon \in (0, 1)$ there exist C and R_0 such that the following holds: If $t > 2$ and $B(p, t\rho) \subset B(O, R_0)$, then*

$$t^{N(p,\rho)(1-\epsilon)-C} \leq \frac{\sup_{B(p,t\rho)} |u|}{\sup_{B(p,\rho)} |u|} \leq t^{N(p,t\rho)(1+\epsilon)+C} \tag{3.1}$$

In addition, if $N(p, \rho) > N_0(M, g, \epsilon, O)$, then both bounds are improved, i.e.

$$t^{N(p,\rho)(1-\epsilon)} \leq \frac{\sup_{B(p,t\rho)} |u|}{\sup_{B(p,\rho)} |u|} \leq t^{N(p,\rho)(1+\epsilon)} \tag{3.2}$$

Note that if we just take the exponents, we get that as long as $N(p, \rho) > N_0$,

$$N(p, \rho)(1 - \epsilon) < N(p, t\rho)(1 + \epsilon)$$

The equation above implies that the doubling index, N, satisfies the same monotonicity properties as the frequency, β.

Here we see a second important assumption. Just as at points we will need to assume that all radii are smaller than some initial radius R_0, we also need to assume that the doubling index of our function u is greater than some initial N_0. Again, increasing this N_0 will have no effect on our final value for α, so we will try to push as much uncertainty into N_0 and R_0 throughout the course of this proof. Therefore, R_0 will continue to shrink, and N_0 will continue to increase.

We will also need the following technical lemma which will allow us to bound the doubling index at one point by the doubling index at another point at a smaller scale.

Lemma 3.2.5 (Lemma 7.4 of [24]). *As long as R_0 is small enough and N_0 is large enough, if $p_1, p_2 \in B(O, R_0)$, and $d(p_1, p_2) < \rho < R_0$,*

$$N(p_2, 1000\rho) > \frac{99}{100} N(p_1, \rho)$$

Proof. First, note that there exist C, δ depending on nothing so that the following containments hold:

$$B(p_2, \frac{C\rho}{2}(1 - \delta)) \subset B(p_1, \frac{C\rho}{2}(1 - \frac{\delta}{10}))$$
$$B(p_1, C\rho(1 - \frac{\delta}{10})) \subset B(p_2, C\rho)$$

In particular, choosing $\delta = 1/100$ and $C = 1000$ will be sufficient for our purposes.

Now, we apply Equations 3.2, 3.2, as well as the above containments to get:

$$
\begin{aligned}
\left(\frac{2}{1-\delta}\right)^{N(p_2,\rho C)(1+\epsilon)} &\geq \frac{\sup_{B(p_2,C\rho)}|u|}{\sup_{B(p_2,C\rho(1-\delta)/2}|u|} \\
&\geq \frac{\sup_{B(p_1,C\rho(1-\delta/10)}|u|}{\sup_{B(p_1,C\rho(1-\delta/10)/2)}|u|} \\
&\geq 2^{N(p_1,C\rho(1-\delta/10)/2)(1-\epsilon)} \\
&\geq 2^{N(p_1,\rho)(1-\epsilon)^2}
\end{aligned}
$$

Thus, we have:

$$N(p_2, C\rho)(1+\delta)(1+\epsilon) \geq N(p_1,\rho)(1-\epsilon)^2$$

Substituing $\delta = 1/100$ and choosing ϵ small enough, we get our desired result. $\square$

3.3 The Simplex Lemmas

The following lemma will allow us to increase the scale at which we are analyzing frequency by combining information coming from many different points.

Lemma 3.3.1 (Euclidean Geometry Lemma). *Let S be a simplex in $\mathbb{R}^n$ with diameter 1 and scale invariant width greater than a. Then there exist $c_1 > 0$, $K > 2/a$ depending only on a and dimension n such that*

$$B(p_0, K(1 + c_1)) \subset \cup_{i=1}^{n+1} B(p_i, K)$$

Furthermore, we can let

$$c_1 = \frac{a^2}{5n^4} \text{ and } K = \frac{2\sqrt{2}n^2}{a}$$

Proof. We begin by finding the worst possible simplex for a fixed diameter and scale invariant width. Let $S_0(\epsilon)$ be the simplex with the following vertices:

$$p_i = (0, \cdots, 0, 1, 0, \cdots, 0) \text{ and } p_{n+1} = \frac{1+\epsilon}{n}(1, \cdots, 1)$$

We will show that S_0 is the worst simplex for this lemma, and then compute explicit constants for S_0.

First note that the width is always achieved by a pair of planes such that every point is contained in exactly one plane. So, potentially one plane could contain k points, and the other plane will contain $n + 1 - k$ points. We start with the scenario where $k = 1$, and show later that this will be the worst case scenario.

Let F_i be the face of S opposite p_i. Let $Z_i(K)$ be the point at distance K from all p_j except p_i on the non-S side of F_i, and let H_i be the plane containing Z_i parallel to F_i. Without loss of generality, let the diameter be achieved by $d(p_1, p_2)$, and let the width be achieved by $d(p_{n+1}, F_{n+1})$. Then, reorient the simplex so that $p_1, \cdots, p_n$ lie in $\mathbb{R}^{n-1}$, and so that p_{n+1} has positive nth coordinate.

Now, as long as K is large enough, $\min_i d(p_0, Z_i(K)) = d(p_0, Z_{n+1}(K))$. Let $\rho = d(p_0, Z_{n+1}(K))$, so that $B(p_0, \rho) \subset \cup B(p_i, K)$. Let L be the line perpendicular to F_{n+1} passing through $Z_{n+1}(K)$. Note that L is the set of points which are equidistant from $p_1 \cdots p_n$. If p_{n+1} is placed (preserving the nth coordinate) so that p_0 does not lie on L, then ρ will be strictly greater than if p_{n+1} is placed to make p_0 lie on L. Therefore, the worst case simplex with $p_1, \cdots, p_n$ fixed will have p_{n+1} placed to make p_0 lie on L.

Now, in order to find the worst possible simplex, we need to find the arrangement of points $p_1, \cdots, p_n$ in $\mathbb{R}^{n-1}$ so that the x_n coordinate of Z_{n+1} is as small as possible when compared with the diameter of the simplex. This is obviously going to happen when all p_i are equidistant from each other. This completes the proof that S_0 is the worst possible simplex for this lemma.

Now, we compute with S_0. S_0 has diameter $\sqrt{2}$, and scale invariant width equal to $w = \epsilon\sqrt{n}/\sqrt{2}$. The barycenter of S_0 is located at

$$p_0 = \left(\frac{1}{n} + \frac{1+\epsilon}{(n+1)n}\right)(1, \cdots, 1)$$

and the line L described above is just the line where all coordinates are equal to each other.

Suppose that $Z_{n+1}(K) = (-b, \cdots, -b)$. Then by scaling,

$$\begin{aligned} K\sqrt{2} &= d(Z_{n+1}(K), (1, 0, \cdots, 0)) \\ &= \left[(1+b)^2 + (n-1)b^2\right]^{1/2} = (1 + 2b + nb^2)^{1/2} \end{aligned}$$

and

$$\begin{aligned} \rho\sqrt{2} &= d(Z_{n+1}(K), p_0) \\ &= \sqrt{n}\left(\frac{1}{n}(1 + \frac{\epsilon}{n+1}) + b\right) \end{aligned}$$

We need $\rho > K$, and then $c_1 < \rho/K - 1$. Substituing our expressions for ρ and K and simplifying, we get

$$b > \frac{n+1}{2\epsilon} \text{ and so}$$
$$K\sqrt{2} > \left(1 + \frac{n+1}{\epsilon} + \frac{n(n+1)}{4\epsilon^2}\right)^{1/2}$$

We will choose $K\sqrt{2} = \frac{8n^{3/2}}{\epsilon}$. Therefore, solving for b, we get

$$b = \frac{1}{n}\left((1 - n(1 - \frac{64n^3}{\epsilon^2}))^{1/2} - 1\right)$$

and so substituting this into the equation for ρ, we get

$$1 + c_1 < \frac{\rho}{K} = \frac{1}{8n^2}\left(\frac{\epsilon^2}{(1+n)} + \left((1-n)\epsilon^2 + 64n^4\right)^{\frac{1}{2}}\right)$$

Taking the expansion around $\epsilon = 0$, we find that we need

$$1 + c_1 < 1 + \frac{15n^1 + 1}{128n^4(n+1)}\epsilon^2 + h.o.t.$$

so that we can take

$$c_1 = \frac{\epsilon^2}{10n^3}$$

Now, we use the fact that $\epsilon = \sqrt{2}w/\sqrt{n}$ to get our stated formulae for K and c_1.

Let's return to the cases where k is not equal to 1, so that if Π_1 and Π_2 are the planes that achieve the width, then Π_1 contains $k > 1$ points and Π_2 contains $n + 1 - k \geq k$ points. Then by arguing as above, the worst case scenario will again happen when the barycenter of the full simplex lines up with the barycenter of the sub-simplices contained in Π_i. Thus, we get to reduce dimension, and since our constants will get worse as dimension increases, the worst case scenario is when $k = 1$. $\square$

Lemma 3.3.2 (The Simplex Lemma). *Let S be a simplex in $\mathbb{R}^n$ with diameter 1 and scale invariant width greater than a. Let B_i be balls centered at the vertices p_i of S*

with radii less than or equal to K. Then there exist $c_s(a,n)$ and $C_s(a,n)$ such that if $N(B_i) > N$ for all i, then $N(p_0, C_s) > N \cdot (1 + c_s)$.

Furthermore, we can take

$$C_s = \frac{20n^4}{a^2} \text{ and } c_s = \frac{a^2}{300n^4|\log(an^2)|}$$

Proof. Throughout the following, $c_1(a,n)$ and $K(a,n)$ are as in the Euclidean Geometry Lemma. Note that by almost monotonicity of the doubling index, we can set the radii of all balls to be equal to K, possibly by increasing N_0.

Let $M = \sup_{\cup B_i} |u|$. Then, by the ball containment implied by the above Euclidean Geometry Lemma, $|u| < M$ in $B(x_0, K(1+c_1))$. Let $t > 2$ and $\epsilon > 0$ be parameters to be specified later. Suppose that $|u| = M$ somewhere in B_i, and assume that 3.2 holds on B_i. Then,

$$\sup_{B(p_i, Kt)} |u| \geq M t^{N(1-\epsilon)}$$

We use the following fact: if $\delta = 1/(Kt)$, then

$$B(p_i, Kt) \subset B(p_0, Kt(1+\delta))$$

Let $\widetilde{N}$ be the doubling index for $B(p_0, Kt(1+\delta))$. Eventually, we will take $C = Kt(1+\delta)$. Suppose that Equation 3.1 holds for the pair of balls $B(p_0, Kt(1+\delta)$ and $B(p_0, K(1+c_1))$, that is, $t(1+\delta) > 2(1+c_1)$ and both are contained in $B(O, R_0)$. Then, by Equations 3.1 and 3.2,

$$\left[\frac{t(1+\delta)}{1+c_1}\right]^{\widetilde{N}(1+\epsilon)+C} \geq t^{N(1-\epsilon)} \tag{3.3}$$

Now, choose t so that $\delta = c_1/2$, that is, choose $t = \frac{2}{Kc_1}$, so that

$$\frac{t(1+\delta)}{1+c_1} \leq t^{1-c_2}$$

Solving for c_2, and substituting our chosen values for t as well as the formulae for c_1 and K, we see that we can take

$$c_2 = \frac{a^2}{50n^4|\log(an^2)|}.$$

Then choose $\epsilon(c_2) = c_2/6 > 0$ so that

$$\frac{1-\epsilon}{(1+\epsilon)(1-c_2)} > 1 + 2\epsilon. \tag{3.4}$$

Simplifying Equation 3.3 using these inequalities, we get that

$$\widetilde{N} \geq N\frac{1-\epsilon}{(1+\epsilon)(1-c_2)} - C_1$$

where $C_1 = C_1(a, M, g)$.

Therefore, since $N > N_0$ and by Equation 3.4,

$$\widetilde{N} \geq N(1+\epsilon) + (\epsilon N_0 - C_1)$$

Now, take $N_0 > C_1/\epsilon$ so that

$$\widetilde{N} \geq N(1+\epsilon).$$

Retracing our steps, and aligning our constants to those in the Euclidean Geometry Lemma, we find that we can take

$$\epsilon = \frac{c_2}{6} = \frac{c_1}{24|\log(a)|} = \frac{a^2}{300n^4|\log(an^2)|}$$

This is our c in the Simplex Lemma statement. Second, we have

$$Kt(1+\delta) = \frac{2}{c_1}(1 + \frac{c_1}{2}) < \frac{20n^4}{a^2}$$

which we take to be our C. $\square$

There is one more lemma we will need in order to apply the Simplex Lemma to general sets with positive width.

Lemma 3.3.3 (Simplex Embedding). *Let X be a convex body with diameter d, and let F be a subset with positive width. Define $\widetilde{w}(F) = \frac{1}{d}width(F)$ as the relative width of F in X. Then there exists a constant $a(\widetilde{w}(F), n) = \widetilde{w}(F)2^{-n}$ and a simplex $S \subset F$ such that $w(S) > a$ and $diam(S) > a \cdot d$.*

Proof. We will construct a sequence of simplices S_i where S_i is an $(i-1)$-simplex, and $S_i \subset S_{i+1}$. First, take 2 points p_1 and p_2 that achieve the diameter of F. We will let S_2 be the segment formed by these two points. To create S_{i+1} from S_i, we will add on a point taken from F which is as far as possible from the $i-1$ plane that contains S_i.

Let's look at some properties of our new simplex. First note that due to the way we chose our points, if we take the hyperplane Π defined by any n points, then the distance of the final point to this plane must be at least $width(F)/2$. Lets call the minimum distance from a single point to the plane generated by the other points the **1-width** $width_1(S)$, and in general if we have two planes, onen of which contains k points, the minimimum distance between them will be the k-width $width_k(S)$. Therefore, if we can prove that

$$width(S) > c(n)width_1(s)$$

we will be done with this lemma.

Let Π_1 and Π_2 be the two planes that achieve our width, and let Π_1 contain k points $p_1, \cdots, p_k$. First note the following. Since the final width of S is positive, then Π_1 and Π_2 must be fully skew, that is to say, if we move Π_1 to intersect Π_2, they will intersect at exactly one point, and have exactly one direction which is mutually normal. Therefore, if we translate everything so that Π_2 contains zero, and quotient $\mathbb{R}^n$ by Π_2, we will not lose any of the dimension of Π_1, and the width of S will be achieved by the distance of Π_1 to the origin.

Our strategy for proving Equation 3.3 will essentially do this process in reverse, starting with one point, and all other points quotiented down to 0, and successively splitting points of S away from the points quotiented to zero. At each step, we will have a bound on how far the width can decrease in terms of the previous width.

Let p_1 be the point that acheives $width_1(S)$. Without loss of generality, suppose that $p_1 \in \Pi_1$. Quotient out by the plane that contains all the other points besides p_1, so that we have just a line with a point p_1, and the 1-width of S is the distance from p_1 to 0. Now, break off another point $p_2 \in \Pi_1$. At this stage, we have three points in $\mathbb{R}^2$, p_1, p_2, and p_{others} the last of which contains the projection of all the other vertices in S. w_1 is equal to the distance between p_1 and the line containing p_2 and p_{others}. Let's make sure that we've chosen p_2 so as to minimize the possible distance from p_{others} to the line containing p_1 and p_2.

Let's note a few things about the 2D geometry of these three points. These will generalize to the higher dimensional case, but we will only do all details in this 2D case.

1. The distance from p_2 to p_{others} must be at least the distance from p_1 to p_{others}, otherwise p_2 would have been the point that achieves the width.
2. The worst case scenario occurs when p_2 and p_{others} are on the same side of the perpendicular dropped from p_1.

See Figure 3-2 for a picture.

Now, if we call w_2 the distance from p_{others} to the line connecting p_1 and p_2, we see that even in the worst case scenario we still have that $w_2 > w_1/2$. Continuing in this fashion, we get that $w_i > w_{i-1}/2$, and so

$$width(S) = w_k > \frac{w_1}{2^{n-1}}$$

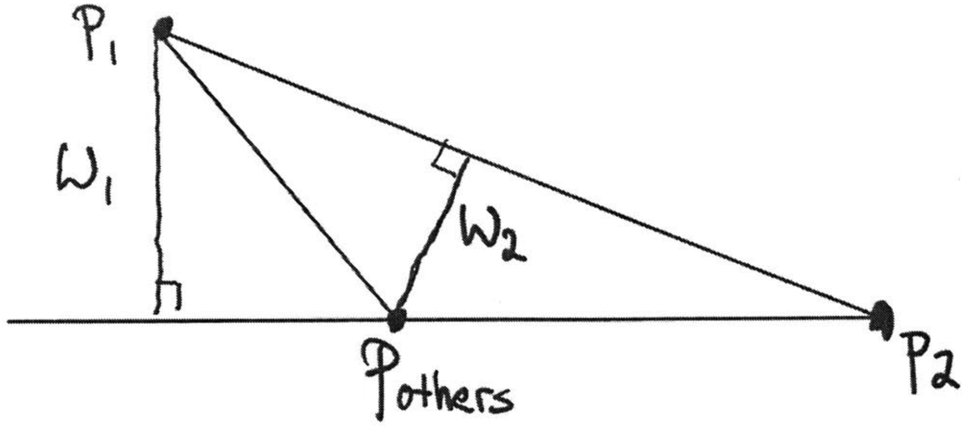

Figure 3-2: 2D case of Lemma 3.3.3

Therefore, returning to the intial formulation of the Lemma, we can say that

$$a(\widetilde{w}(F), n) = \frac{\widetilde{w}(F)}{2^n}$$

$\square$

3.4 Propogation of Smallness

In order to understand dimension dependence for propogation of smallness for elliptic PDE, we must revisit the work of Alessandrini, Rondi, Rosset and Vessella in Theorem 1.7 of [1] as used by Logunov and Malinnikova. Looking through their work, one discovers that a precise formula for η relies heavily on every constant in the Schauder estimates – which goes beyond the scope of this thesis. We choose instead to black-box the formula for η and leave that little bag of self-loathing for another enterprising Minicozzi student.

Theorem 3.4.1. *Let Q be a cube with side length r. There exists η depending only on dimension such that if $|u| \leq \epsilon$ and $|\nabla u| \leq \epsilon/r$ on some face F of Q, then on $Q/2$, $|u| \leq \epsilon^\eta$.*

3.5 Hyperplane Lemma

The doubling index of a cube Q is defined as follows:

$$N(Q) = \sup_{x \in Q, r < diam(Q)} N(x,r) \tag{3.5}$$

This allows frequency to be monotone non-decreasing by containment.

The following lemma should be interpreted as the following statement: If a cube Q has doubling index $2N$, then at least one small subcube has doubling index smaller than N, though Logunov's statement of the lemma uses the contrapositive of this.

Lemma 3.5.1. *Let Q be a cube $[-1,1]^n \subset \mathbb{R}^n$. Divide Q into $(2A+1)^n$ subcubes q_i with side length $2/(2A+1)$. Consider the subcubes $q_{i,0}$ which intersect the plane $x_n = 0$. Suppose that for each $q_{i,0}$, there exists $p_i \in q_{i,0}$ and $r_i < diam(q_{i,0}) = 2\sqrt{n}/(2A+1)$ such that $N(p_i, r_i) > N$. Then, there exists $A_0(n)$ and N_0 such that if $A > A_0$ and $N > N_0$, then $N(Q) > 2N$.*

In particular, we can choose

$$A_0 = 32^{64} n^{32} e^{1/\eta}$$

Proof. Let $B = B(0,1)$, and let $M = \sup_B |u|$. Note that we have the following sequence of containments for every $p_i \in B(1, 1/16)$ as long as $A_0 > 100\sqrt{n}$:

$$2q_{i,0} \subset B(p_i, \frac{4\sqrt{n}}{2A+1}) \subset B(p_i, \frac{1}{32}) \subset B(1, 1/8)$$

Now, we apply these containments and reformulate Equation 3.2 as

$$\sup |u|_{B(x,\rho)} \leq t^{-N(x,\rho)(1-\epsilon)} \sup_{B(x,t\rho)} |u|$$

and choose

$$\rho = \frac{4\sqrt{N}}{2A+1}, t = \frac{2A+1}{128\sqrt{n}}, \text{ and } \epsilon = \frac{1}{2}$$

to get the following:

$$\sup_{2q_{i,0}} |u| \leq \sup_{B(p_i, \frac{4\sqrt{n}}{2A+1})} |u| \leq \sup_{B(p_i, \frac{1}{32})} |u| \left(\frac{128\sqrt{n}}{2A+1}\right)^{N/2} \leq Me^{-cN\log(A)} \tag{3.6}$$

Let's try and get a bound on c. Taking the log of the righthand most inequality and solving for c, we find that

$$c \leq \frac{\log(2A+1) - \log(128\sqrt{n})}{2\log(A)}$$

If we make $A \geq 128^2 n$, then the term involving dimension can be absorbed into the $\log(2A+1)$ term, so it is sufficient to take

$$c \leq \frac{\log(2A+1)}{4\log(A)}$$

However, since $\log(2A+1)/\log(A) > 1$, we can let $c = 1/4$.

Now, we use the standard elliptic gradient estimate to say

$$\sup_{q_{i,0}} |\nabla u| \leq C(n)A \sup_{2q_{i,0}} |u| \leq C(n)AMe^{-N\log(A)/4} \leq Me^{-c_1(n)N\log(A)}$$

Again, let's put bounds on $c_1(n)$. We need

$$\log(C) + \log(A) - \frac{\log(A)}{4}N \leq -c_1(n)N\log(A)$$

Note that if we take N to be sufficiently large, both the $\log(C)$ and the $\log(A)$ terms can be absorbed into the term involving N, and so we can take $c_1(n)$ to be any number less than $1/4$. We choose to let $c_1 = 1/8$.

Note that we have now proven that both $|u|$ and $|\nabla u|$ are smaller than $e^{-N\log(A)/8}$.

Let q be a cube with side length $\frac{1}{16\sqrt{n}}$ with a face F centered at 0 on the hyperplane $x_n = 0$. Note that because of our choice of side length, we have the following containments:

$$B(0, \frac{1}{32\sqrt{n}}) \subset q \subset B(0, 1/8)$$

Let $v = u/M$, so that $\sup_q |v| \leq 1$. Let $\epsilon = e^{-N\log(A)/16}$. Then we have both

$$|v| \leq \epsilon \text{ and } |\nabla v| \leq 2A\epsilon$$

so v satisfies the conditions to apply propogation of smallness. Thus,

$$\sup_{\frac{1}{2}q} |v| \leq \epsilon^{\eta} \text{ and so } \sup_{\frac{1}{2}q} |u| \leq Me^{-\eta N\log(A)/16}$$

where η is as in Section 3.4. Thus, if p is the center of q,

$$\sup_{B(p,\frac{1}{64\sqrt{n}})} |u| \leq Me^{-\eta N\log(A)/16}$$

However, $B(0, 1/8) \subset B(p, 1/2)$, and so

$$\sup_{B(p,1/2)} \geq M$$

Combining these two facts, we have that

$$\frac{\sup_{B(p,1/2)} |u|}{\sup_{B(p,\frac{1}{64\sqrt{n}})}} \geq e^{\eta N\log(A)/16}$$

Let $\widetilde{N} = N(p, 1/2)$ so that, in particular, $N(Q) > \widetilde{N}$. Then by Equation 3.1 with $t = 32\sqrt{n}$, we have

$$\frac{\sup_{B(p,1/2)} |u|}{\sup_{B(p,\frac{1}{64\sqrt{n}})}} \leq (32\sqrt{n})^{\widetilde{N}\frac{3}{2}+C}$$

If we choose $N_0 > 2C$, then this right hand side becomes less than $(32\sqrt{N})^{2\widetilde{N}}$. Combining these last two inequalities, we have

$$(32\sqrt{n})^{2\widetilde{N}} \geq e^{\eta N\log(A)/16}$$

If we want $\widetilde{N} \geq 2N$, it is sufficient to pick $A > 32^{64} n^{32} e^{1/\eta}$ $\square$

The following lemma will wind up being used in its contrapositive, but is easier to state this direction.

Lemma 3.5.2. *Let $Q = [-1,1]^n$ such that $N(Q) \leq 2N$ and $N > N_0$. For any $\epsilon > 0$, there exists $A_1(n, \epsilon)$ such that the following is true. Divide Q into A_1^n subcubes each with side length $1/A_1$, and let $q_{i,0}$ be those subcubes that intersect the hyperplane $x_n = 0$. Then the proportion of subcubes where $N(q_{i,0}) > N$ is less than ϵ - and so the number of subcubes where that holds is less than ϵA_1^{n-1}.*

We can choose

$$A_1(n, \epsilon) = \left(\gamma n^{\gamma} e^{1/\eta}\right)^{|\log \epsilon| \left(\gamma n^{\gamma} e^{1/\eta}\right)^{n-1}}$$

where γ is independent of n.

Please Note: this constant is terrible. Therefore, we will refrain from writing out every single constant throughout the rest of the chapter in terms of explicit n dependence. Instead, we will make sure to use only constants whose n dependence we have already computed.

Proof. We are going to apply Lemma 3.5.1 k times, each time on smaller subcubes. We will start by dividing Q into $(2A_0 + 1)^n$ subcubes, and then continue dividing each of those subcubes into $(2A_0 + 1)^n$, applying Lemma 3.5.1 each time to restrict the number of subcubes at each level which can have large doubling index.

In particular, suppose that after the kth subdivision (so now there are a total of $(2A_0 + 1)^{kn}$ subcubes) we have M_k total subcubes $q_{k,0}$ with $N(q_{k,0}) > N$. Applying Lemma 3.5.1 to each one of these, we see that each $q_{k,0}$ has at most $(2A_0 + 1)^{n-1} - 1$ subsubcubes that line up with $x_n = 0$ with doubling larger than N.

$$M_{k+1} \leq M_k((2A_0 + 1)^{n-1} - 1)$$

so that

$$M_k \leq ((2A_0+1)^{n-1}-1)^k = (2A_0+1)^{(n-1)k}(1-\frac{1}{(2A_0+1)^{n-1}})$$

We need to find a k such that $M_k < \epsilon(2A_0+1)^{k(n-1)}$, and then $A_1 = (2A_0+1)^k$. So, choosing k so that this is true, and substituting the formula for A_0, we get our value for A_1. □

3.5.1 Comments on Dimension Dependence

Note that there are essentially 3 places where n dependence appears in the formula for A_1.

1. The exponent η in propogation of smallness has dependence on n through the constants in the elliptic Schauder estimates and coming from the geometry of an n-cube.

2. The n^γ in the formula for A_0 comes from the fact that in order to fit a cube inside a ball, the radius of the ball has to be $\sqrt{n}$ times the diameter of the cube.

3. Even if we could remove the n dependence from η and A_0, there would still be an n dependence of the form $A_0^{A_0^{n-1}}$ due to the fact that we have to apply Lemma 3.5.1 A_0^{n-1} times.

3.6 Bounding number of cubes with large doubling index

This is Theorem 5.1 from Logunov - as we have done up to this point, we carefully trace the dimension dependence.

Theorem 3.6.1. *There exist constants $c(n)$ and $A(n)$, and $N_0(M, g)$ such that if $Q = [-1, 1]$, and we partition Q into A^n subcubes q, then the number of subcubes such that $N(q) > \max(N(Q)/(1 + c), N_0)$ is smaller than $\frac{1}{2}A^{n-1}$.*

In addition,

$$A = A_1(2^{-n-1}, n)^{8\log(A_1)} \text{ and } c = \frac{1}{2}c_s(2^{-n}A_1^{-1}, n)$$

where c_s is from the Simplex Lemma, and A_1 is from Lemma 3.5.2.

Proof. Just as we proved Lemma 3.5.2 via multiple applications of 3.5.1, so we will prove this theorem by applying 3.5.2 many times at many scales.

Fix some ϵ to be chosen later, and choose $A_1(n, \epsilon)$ as in Lemma 3.5.2. Subdivide Q into A_1^n subcubes j times, so that we now have A_1^{jn} subcubes q each with diameter on the order of $A_1^j\sqrt{n}$. Choose some q and subdivide it again into q_i. We will say that q_i is **bad** if $N(q_i) > N(Q)/(1 + c)$. We want to bound the number of bad cubes q_i contained in q.

Once we prove the following lemma, it will be simple to complete the proof of Theorem 3.6.1

Lemma 3.6.2. *If $\epsilon = 2^{-n-1}$, then the number of bad $q_i \subset q$ is less than $\frac{1}{2}A_1^{n-1}$*

Proof. Let F be the set of points $p \in q$ such that for some $r < diam(q_i)$, $N(p, r) > N(Q)/(1 + c)$. Note that every bad q_i contains a point form F. As in Lemma 3.3.3, we let $\widetilde{w}(F)$ be the relative width of F in q. Let S be the simplex guaranteed by Lemma 3.3.3 (Simplex Embedding).

Suppose that $\widetilde{w}(F) > 1/A_1$. Then

$$w(S) > 1/(2^n A_1) \text{ and } diam(S) > diam(q)/(2^n A_1)$$

Since the vertices p_k of S are contained in F, we have r_k such that

$$r_k < diam(q) < diam(S)2^n A_1 \text{ and } N(p_k, r_k) > N(Q)/(1+c)$$

Note that $2^n A_1$, while rather large, is still less than K from Lemma 3.3.1 when we take $a = 1/(2^n A_1)$. Therefore, we can apply Lemma 3.3.2, the Simplex Lemma, in order to get

$$N(p_0, C_S(a,n)diam(S)) > \frac{(1 + c_S(a,n))N(Q)}{1+c}$$

If $C_S(a,n)diam(S) < diam(Q)$, and $c < c_S(a,n)$, then this will be a contradiction. However, since

$$diam(S) < diam(q) < \frac{diam(Q)}{A_1^j}$$

However, by our formulas, A_1^3 is orders of magnitude larger than $C_s(a,n)$, and so as long as we have subdivided at least 3 times, that is, $j > 2$, we must have that

$$\widetilde{w}(F) < 1/A_1$$

This means that there exists a hyperplane P such that all of F is contained within a $\frac{diam(q)}{A_1}$ neighborhood of P. Take a new cube $\widetilde{q}$ such that its center is contained in $P \cap q$ and so that the diameter of $\widetilde{q}$ is $20n/A_1^j = 10\sqrt{n}diam(q)$. In particular, this implies that $q \subset \widetilde{q}$. We will be applying Lemma 3.5.2 to $\widetilde{q}$. Divide $\widetilde{q}$ into A_1^n subcubes. Let $\widetilde{q}_{i,0}$ be the subcubes that intersect P. Then because of the scales we have chosen, every bad q_i is fully covered by at most 2^{n-1} $\widetilde{q}_{i,0}$. Thus, we have the following inequality:

$$\#\{\text{bad } \widetilde{q}_{i,0}\} \leq 2^{n-1}\#\{\text{bad } q_i\}$$

Now suppose that $\#\{\text{bad } q_i\} > \frac{1}{2}A_1^{n-1}$. Then

$$\#\{\text{bad } \widetilde{q}_{i,0}\} \geq 2^{-n}A_1^{n-1}$$

Choose $\epsilon = 2^{-n-1}$. Then by Lemma 3.5.2, and since the number of bad $\widetilde{q}_{i,0}$ is larger than ϵA_1^{n-1}, we must have that $N(\widetilde{q}) > 2N(Q)/(1+c)$. Since c is stupid small, there will be a $\widetilde{p} \in \widetilde{q}$ such that

$$N(\widetilde{p}, diam(\widetilde{q}) > \frac{3}{2}N(Q)$$

However, $\widetilde{p}$ is not necessarily contained in Q, so this is not yet a contradiction.

We will apply Lemma 3.2.5 to get a lower bound on doubling index for some point inside of Q. Pick $p \in Q$ so that $d(p, \widetilde{p}) < diam(\widetilde{q})$. Then applying Lemma 3.2.5 to these two points with $\rho = diam(\widetilde{q}) \approx \sqrt{n}/A_1$ gives that

$$N(p, 1000 diam(\widetilde{q})) > \frac{4}{3}N(Q)$$

Since $diam(\widetilde{q})$ is much smaller than $diam(Q)/1000$, this is the contradiction we need.

This completes the proof of Lemma 3.6.2 $\square$

Now it is straightforward to prove Theorem 3.6.1. As in the proof of Lemma 3.5.2, let K_j be the number of bad subcubes after we do j subdivisions. By Lemma 3.6.2, and as long as $j > 2$, we have

$$K_{j+1} \leq \frac{1}{2}A_1^{n-1}K_j$$

and so, letting $A = A_1^j$

$$K_j \leq K_2 A_1^{(j-2)(n-1)} 2^{2-j} \qquad\qquad = K_2 A^{n-1} A_1^{-2(n-1)} 2^{2-j}$$

So, we need only show that

$$K_2 A_1^{-2(n-1)} 2^{2-j} \leq 1/2$$

for a sufficiently large choice of j. However, note that K_2 is at most A_1^{2n} (in the case when every subcube after 2 divisions is bad). Therefore, after some algebra, we find that it is sufficient to take $j = 8\log(A_1)$. $\square$

3.7 Upper Bounds on the Nodal Set

Theorem 3.7.1. *Let u be harmonic on M, and let $O \in M$. Then there exist $R_0(M, g, O)$ and $C(M, g, O)$ such that if Q is a cube contained in $B(O, R_0)$, then there exists $\alpha_1 = \alpha_1(n)$*

$$\mathcal{H}^{n-1}(\{u = 0\} \cap Q) \leq C diam(Q)^{n-1} N_u^{\alpha_1}(Q)$$

where

$$\alpha_1(n) = \frac{\log(4A(n))}{\log(1 + c(n))}$$

Proof. We follow Logunov [24]. Define

$$F(N) := \sup \frac{\mathcal{H}^{n-1}(\{u = 0\} \cap Q)}{diam(Q)^{n-1}}$$

where the supremum is taken over all harmonic functions u with $N_u(Q) < N$. Note that F is nondecreasing in N. Then the estimate we need to prove is that

$$F(N) \leq CN^{\alpha_1}$$

We say that N is **bad** if

$$F(N) > 4AF(N/(1 + c))$$

Logunov shows that the set of bad N is bounded above by some $N_0(M, g)$ by using Theorem 3.6.1. We use this fact to get an explicit formula for α_1.

Let $N > N_0$. Then there exists $k \geq 0$ such that

$$N_0(1 + c)^{k+1} > N \geq N_0(1 + c)^k$$

and so

$$F(N) \le F(N_0(1+c)^{k+1}) \le (4A)^{k+1}F(N_0)$$

Now, by 3.7,

$$k \le \frac{\log(N) - \log(N_0)}{\log(1+c)}$$

and so

$$\begin{aligned} F(N) &\le 4A^{\frac{\log(N)-\log(N_0)}{\log(1+c)}+1}F(N_0) \\ &= C(M,g,O)N^{\frac{\log(4A)}{\log(1+c)}} \end{aligned}$$

□

Now, we use the trick described in the first section to extend this to eigenfunctions.

Theorem 3.7.2. *Let $\Delta_N\phi + \lambda\phi = 0$. Then there exists a $C(N,g)$ such that*

$$\mathcal{H}^{n-1}(\{\phi = 0\}) \le C\lambda^{\alpha}$$

where $\alpha = \alpha_1/2$.

Proof. We let u be defined on $M = N \times \mathbb{R}$ as

$$u(x,t) = \phi(x)e^{\sqrt{\lambda}t}$$

so that u is harmonic on M.

By Donnelly-Fefferman [14], we have that

$$\sup_{B(p,2r)} |\phi| \le 2^{C\sqrt{\lambda}} \sup_{B(p,r)} |\phi|$$

as long as r is small enough. Note that this also implies that

$$N_u(p,r) \leq C\lambda^{1/2}$$

as long as $p \in N \times \{0\}$ and r is small enough. Thus, we can apply Theorem 3.7.1 to say that

$$\mathcal{H}^{n-1}(\{u=0\} \cap B(p,r)) \leq C\lambda^{\alpha_1/2}$$

However, since the nodal set of u is just equal to $\{\phi = 0\} \times \mathbb{R}$, we get that

$$\mathcal{H}^{n-1}(\{\phi=0\} \cap B(p,r)) \leq C\lambda^{\alpha_1/2}$$

By covering N with balls of radius r, we get our desired result. □

To finish, let's trace through our proofs and compute the formula for α in terms of n. For A, we have that

$$\begin{aligned}\log(A) &= 8\log(A_1(2^{-n-1}, n)^2 \\ &= \left[(n+1)[\gamma n^{\gamma} e^{1/\eta}]^n\right]^2 \\ &\approx C n^{\gamma n} e^{2n/\eta}\end{aligned}$$

At this stage, it is unclear which of $n^{\gamma n}$ and $e^{n/\eta}$ is larger.

Computing $\log(1+c)$ is slightly easier.

$$\begin{aligned}\log(1+c) &\approx c_s(2^{-n-1}) \\ &\approx C4^{-n-1}n^{-4}\end{aligned}$$

Combining these two, we see that the dominant term comes from the $\log(A)$ portion, so, by increasing γ if needed, we have that

$$\alpha = C n^{\gamma n} e^{2n/\eta}$$

where C and γ are independent of n.

3.8 Thoughts on Improvement

Logunov and Malinnikova's methods worked perfectly to prove the lower bound in Yau's conjecture. I do not believe that they can be extended to a full proof of the conjectured upper bound. However, I do believe that they can be refined to prove the following:

Conjecture 3.8.1. *Let ϕ solve $\Delta_N \phi + \lambda \phi = 0$. Then for all $\epsilon > 0$ there exists a $C_\epsilon = C(M, g, \epsilon)$ such that*

$$\mathcal{H}^{n-1}(\{\phi = 0\}) \leq C_\epsilon \lambda^{1/2+\epsilon}$$

First, note that in order to prove the full upper bound, we would need to show that $\alpha_1 = 1$ from Theorem 3.7.1. This would require three steps.

1. Remove the 4.
2. Make $A = 1$
3. Make $c = 0$

To resolve the first problem, we can simply make N_0 larger, and bring the coefficient of A as close to 1 as we like.

Resolving the second and third problems are much more difficult. Let's start with discussing a process by which we can begin to make A smaller. Many of the factors of n come from having to do a sequence of containments of the form

$$q \subset B \subset Q$$

where q and Q are cubes, and B is a ball. This process is guaranteed to bump up the scale by at least $\sqrt{n}$. But, if we could find a tile T of $\mathbb{R}^n$ so that the containment

$$t \subset B \subset T$$

only increases the diameter by a factor of μ independent of dimension, then we could remove at least a portion of the dimension dependence from A. Fortunately, by work of Butler, such a tile exists:

Theorem 3.8.2. *[7] There exists tiles T_n on $\mathbb{R}^n$ so that if*

$$t_n \subset B \subset T_n$$

then the ratio $diam(T_n)/diam(t_n)$ can be made to approach 2 as $n \to \infty$.

In particular, this theorem can probably be used to remove all dimension dependence from A_0 except for the dependence coming from the elliptic theory.

Note that Lemma 3.5.1 could potentially be improved. Recall, that lemma essentially said that if Q has doubling equal to $2N$, then along the center hyperplane, there is at least one smaller cube with doubling less than N. I conjecture the following stronger version:

Conjecture 3.8.3. *Let $N(Q) = 2N$. Let $N_i = 2N - 2^{-i}N$. Then for a subset $\nu \subset \{0, \cdots, \Xi(n)\}$ with size $\Omega(n)$, for each $i \in \nu$, there exists at least one cube q so that $N(q) \in [N_{i-1}, N_i)$.*

A theorem of this form might allow an improvement in the proof of Lemma 3.5.2 – in particular, it could remove the necessity of applying Lemma 3.5.1 many many times.

In order to remove the n dependence from the simplex lemmas, we would need to possibly use a shape Σ other than a simplex, and a different measure h for distance from a plane, so that the following version of Lemma 3.3.3 is true:

If F is a set with positive h distance from a plane, then there exists a $\Sigma \subset F$ with the same h distance from the same plane.

We would also need a reformulation of 3.3.2.

Let Σ have h distance a from a plane, then there exists a point p, and constants $c(a)$ and $C(a)$ so that the result of Lemma 3.3.2 holds.

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