The arrival time for mean curvature flow on a convex domain

by

Nicholas Strehlke

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Signature redacted

Author . . . .
Department of Mathematics
April 9, 2019

Signature redacted

Certified by . . . .
Tobias H. Colding
Cecil and Ida Green Distinguished Professor of Mathematics
Thesis Supervisor

Signature redacted

Accepted by . . . .
Davesh Maulik
Chairman, Department Committee on Graduate Theses

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Abstract

We give asymptotics for the level set equation for mean curvature flow on a convex domain near the point where it attains a maximum. It was shown by Natasa Sesum that solutions are not necessarily C^3, and we recover this result and construct non-smooth solutions which are C^3. We also construct solutions having prescribed behavior near the maximum. We do this by analyzing the asymptotics for rescaled mean curvature flow converging to a stationary sphere.

Thesis Supervisor: Tobias H. Colding
Title: Cecil and Ida Green Distinguished Professor of Mathematics

To Sade and Harry

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Chapter 1

Introduction

Let $\Omega \subset \mathbb{R}^{n+1}$ be a bounded convex domain. Our main object of study is a function $t : \Omega \to \mathbb{R}$ satisfying $t = 0$ on $\partial\Omega$ and

$$|\nabla t|\operatorname{div}\left(\frac{\nabla t}{|\nabla t|}\right) = -1 \tag{1.1}$$

in Ω. This degenerate elliptic boundary value problem is called the level set equation for mean curvature flow, or the arrival time equation. The problem (1.1) admits a unique solution $t \in C^2(\Omega)$, and this solution has a single critical point in Ω at which it attains a maximum. Our main results concern the behavior of t near this critical point: we will linearize the equation (1.1) near the critical point and show that, although the equation (1.1) is degenerate, a solution to the linearized equation provides a good approximation to t. Using this, we will be able to characterize the asymptotics of t near the critical point, that is, the first few nonzero terms in its Taylor expansion. The main result is the following theorem.

THEOREM 1.1. *Suppose $\Omega \subset \mathbb{R}^{n+1}$ is a bounded convex domain and t solves the arrival time problem on Ω. Let x_0 be the unique critical point of t in Ω, where t attains its maximum. Then either Ω is a round ball and $t = t(x_0) - |x - x_0|^2/(2n)$ or there is an integer $k \geq 2$*

and a nonzero homogeneous harmonic polynomial P of degree k such that

$$t(x) = t(x_0) - \frac{|x - x_0|^2}{2n} + |x - x_0|^{k(k-1)/n} P(x - x_0) + E(x), \tag{1.2}$$

where the error term E satisfies $E(x) = O(|x - x_0|^{k(k-1)/n+\varepsilon} P(x - x_0))$ *as* $x \to x_0$ *for some* $\varepsilon > 0$. *Moreover, if P is a homogeneous harmonic polynomial of degree at least two then there exists a bounded convex domain in* $\mathbb{R}^{n+1}$ *on which the solution to the arrival time equation satisfies (1.2) for P.*

The proof of this theorem involves a detailed study of the convergence and asymptotics of rescaled mean curvature flows that are small perturbations of the stationary sphere. The main result about asymptotics of the rescaled mean curvature flow near a sphere is the following theorem.

THEOREM 1.2 (Asymptotics for rescaled MCF near the stationary sphere)**.** *Let* $\mathbf{S}^n \subset \mathbb{R}^{n+1}$ *be the sphere of radius* $(2n)^{1/2}$ *centered at the origin, and suppose* $\{\Sigma_s\}$ *is a rescaled mean curvature flow converging to* $\mathbf{S}^n$. *Then there is a function* $u\colon \mathbf{S}^n \times [s_0, \infty) \to \mathbb{R}$ *with the property that* Σ_s *is the normal graph of u over* $\mathbf{S}^n$ *for sufficiently large s and there is an eigenfunction P for* $-\Delta - 1$ *on the sphere with eigenvalue* $\lambda > 0$ *such that*

$$\|e^{\lambda s} u(\cdot, s) - P\|_{C^k(\mathbf{S}^n)} \leq C_k e^{-\varepsilon s} \tag{1.3}$$

for some $C_k > 0$ *depending on k and a constant* $\varepsilon > 0$ *independent of k. The limit function P may be prescribed, moreover: for each eigenfunction P for* $-\Delta - 1$ *with eigenvalue* $\lambda > 0$, *there exists a normal graph over* $\mathbf{S}^n$ *evolving by rescaled MCF and satisfying (1.3) as* $s \to \infty$.

Remark. The same proof implies a similar asymptotic result for a rescaled MCFs converging exponentially to general compact self-shrinker.

In this section, we introduce the mean curvature flow for hypersurfaces in $\mathbb{R}^{n+1}$ and explain how equation (1.1) arises from the mean curvature flow. We then describe the linearization of (1.1) near a critical point, which motivates the main result. In section 1.2, we explain the strategy behind the proof of Theorem 1.2. In section 1.5, we prove that Theorem 1.1 follows from Theorem 1.2.

In the subsequent sections we prove the main results on the asymptotics of t near the critical point.

1.1 Mean curvature flow

A 1-parameter family of smooth immersions $X : M^n \times [0,T) \to \mathbb{R}^{n+1}$ of an n-dimensional manifold M is a *mean curvature flow* if the position vector X satisfies

$$\nu(X,t) \cdot \frac{\mathrm{d}}{\mathrm{d}t} X(t) = -H(X,t), \tag{1.4}$$

where $\nu(X,t)$ is a unit normal for $M_t = X(M,t)$ at the point X and $H(X,t) = \operatorname{div}_{M_t} \nu(X,t)$ is the scalar mean curvature of M_t at the point X. Notice that the sign of this equation is independent of the choice of unit normal.

EXAMPLE 1.3 (Shrinking sphere). One of the simplest and most important examples of mean curvature flow is a shrinking sphere. If $M_0 = S_R$ is a sphere of radius R centered at a point x_0, then the equation (1.4) becomes an ODE for the radius of a sphere $S_{r(t)}$ centered at x_0, namely, the radius $r(t)$ must satisfy $r' = -n/r$. The unique nonnegative solution to this ODE is $r(t) = (R^2 - 2nt)^{1/2}$, with $R = r(0)$ the initial radius. Thus the sphere of radius R shrinks homothetically to a point at time $T = R^2/(2n)$.

Example 1.3 illustrates an important property of mean curvature flow starting from a compact surface: the evolution becomes singular after a finite amount of time. For a general closed initial surface, this is implied by the following avoidance principle.

PROPOSITION 1.4 (Avoidance principle for mean curvature flow). *Let $\{M_t\}$ and $\{N_t\}$ be two mean curvature flows in $\mathbb{R}^{n+1}$. If M_0 and N_0 are disjoint, then M_t and N_t are disjoint for all t.*

COROLLARY 1.5 (Finite-time singularity formation). *If $\{M_t\}$ is a mean curvature flow starting from a smooth closed surface M_0, then there is a finite time T such that the curvature of M_t becomes infinite as $t \to T$.*

Proof. Since M_0 is compact, we can find a large sphere S_R containing M_0. By the avoidance principle, M_t must not be defined for t larger than the extinction time $R^2/(2n)$ of the sphere. On the other hand, as long as the curvature of M_t is bounded there exists a mean curvature flow starting from M_t and existing for some definite time depending only on the bound for the curvature by the basic existence theorem, Theorem 1.6. So there must be a time between $t = 0$ and $t = R^2/(2n)$ at which the curvature becomes unbounded. ∎

The basic existence theorem for mean curvature flow is the following.

THEOREM 1.6 (Short-time existence and uniqueness for mean curvature flow). *Let M be an n-dimensional smooth embedded closed hypersurface in $\mathbb{R}^{n+1}$. Then there exists $\varepsilon > 0$ depending only on an upper bound for the principal curvatures of M and a unique smooth mean curvature flow $\{M_t\}_{t\in[0,\varepsilon)}$ starting from $M_0 = M$ and defined for $t < \varepsilon$.*

The idea behind the proof of Theorem 1.6 is that a solution to the mean curvature flow starting from a smooth hypersurface with bounded curvature must be a normal graph over the initial surface for a short amount of time. To be precise, let $X : M \to \mathbb{R}^{n+1}$ be an embedding of M into $\mathbb{R}^{n+1}$, with unit normal ν. Then we search for a function $u : M \to \mathbb{R}$ such that $X(p,t) = X(p) + u(p,t)\nu(p)$ satisfies the mean curvature flow equation. Such a function must exist moreover for any mean curvature flow starting from M. The defining equation (1.4) then becomes a quasilinear parabolic equation for the scalar function u, and short time existence and uniqueness then follows from short-time existence and uniqueness for the scalar PDE on M.

1.1.1 Singularity formation and self-similar solutions

Corollary 1.5 shows that surfaces evolving under the mean curvature flow tend to form singularities. The study of these singularities makes up the bulk of the literature on mean curvature flow. Because our main results concern mean curvature flows which are small perturbations of shrinking spheres, we will not dwell on more general singularity models. We will however need some basic ideas in order to understand even solutions to MCF which are nearly spherical.

The basic method of analyzing singularities is, roughly speaking, to magnify the picture near the formation of the singularity in such a way that the curvature of the evolving surface remains bounded. In many cases, there is a specific rate of continuous magnification, that is, a continuous *rescaling* of the flow, which causes the evolving surface to converge toward one of a family of singularity models that are equilibrium points for the rescaled flow. These singularity models are called *self-shrinking surfaces* or *self-shrinkers*, because they shrink homothetically under the mean curvature flow. One simple example is the sphere, discussed in Example 1.3.

The self-shrinking solutions to mean curvature flow are characterized by an elliptic equation relating the position vector and the unit normal, which we derive now. Suppose $X : \Sigma^n \to \mathbb{R}^{n+1}$ is a smooth embedding of a hypersurface in $\mathbb{R}^{n+1}$ that evolves homothetically under the mean curvature flow. This means that there is a function $\lambda : [0, T) \to \mathbb{R}$ with the property that the family of embeddings $X(p,t) = \lambda(t)X(p)$ constitutes a mean curvature flow. Let ν be a unit normal defined in a neighborhood of some point X. Then the defining equation for mean curvature flow gives

$$\lambda' X \cdot \nu = -\operatorname{div}_{\lambda\Sigma} \nu = -\frac{1}{\lambda} \operatorname{div}_{\Sigma} \nu.$$

By translating the embedding if necessary, we can assume $X \cdot \nu(X)$ is nonzero, and it then follows at once that $(X \cdot \nu)^{-1} \operatorname{div}_{\Sigma} \nu$ is constant on Σ, that is, the mean curvature is everywhere proportional to the normal component $X \cdot \nu$ of the position vector. If we rescale Σ by a constant c, then this constant value scales by c^{-2}, and we may therefore assume that it has magnitude $1/2$. The sign is determined by whether λ^2 is decreasing or increasing, in other words, by whether the homothetically evolving MCF is shrinking or expanding.

Thus for a homothetically shrinking MCF, $\lambda\lambda' = -1/2$, and for a homothetically expanding MCF, $\lambda\lambda' = 1/2$. Solving the ODE gives $\lambda(t) = (\lambda(0)^2 - t)^{1/2}$ in the first case and $\lambda(t) = (\lambda(0)^2 + t)^{1/2}$ in the second case. The initial surface satisfies the equation

$$\frac{X \cdot \nu}{2} - \operatorname{div}_\Sigma \nu = 0$$

in case it is shrinking and

$$\frac{X \cdot \nu}{2} + \operatorname{div}_\Sigma \nu = 0$$

in case it is expanding.

We have proved the following.

PROPOSITION 1.7 (Characterization of self-shrinkers). *The hypersurface $\Sigma^n \subset \mathbb{R}^{n+1}$ shrinks homothetically under the mean curvature flow if and only if it can be rescaled to satisfy the equation*

$$\frac{x \cdot \nu}{2} - \operatorname{div}_\Sigma \nu = 0. \tag{1.5}$$

If Σ satisfies this equation, then the surfaces $M_t = (-t)^{1/2}\Sigma$ for $t < 0$ form a mean curvature flow becoming singular at the origin at time $t = 0$.

A similar statement holds for the self-expanding solutions. Notice in particular that homothetically shrinking solutions are *ancient*, that is, they are defined for all backward time, while homothetically expanding solutions are defined for all forward time. Because of this feature, expanding solutions do not arise as singularity models for mean curvature flow. The use of self-shrinkers as singularity models will be explained in the next section.

There are many self-shrinking surfaces, but the embedded mean convex self-shrinking surfaces are very simple: Huisken proved in [Hui93] that they are precisely the generalized cylinders $\mathbf{S}^{n-k} \times \mathbb{R}^k \subset \mathbb{R}^{n+1}$.

1.1.2 Rescaled mean curvature flow

In the preceding section we saw that self-shrinking solutions to the mean curvature flow shrink down to a point like $(T-t)^{1/2}$, where T is the time at which the singularity occurs. Put another way, the curvature of a homothetically shrinking surface grows like $(T-t)^{-1/2}$ as $t \to T$. In fact, many of the singularities that arise in mean curvature flow exhibit curvature blow-up at this rate. In order to study these singularities, we magnify the surface near them at this rate, so that the curvature remains bounded. Under fairly general assumptions the magnified surfaces actually converge to a self-shrinker. In this section we describe this rescaling procedure.

Let $\{M_t\}$ be a mean curvature flow that becomes singular at the origin as $t \to T$, and suppose that the curvature of the rescaled surface $(T-t)^{1/2}M_t$ remains bounded as $t \to T$. Let $s = -\log(T-t)$, and define the rescaled surface Σ_s by $\Sigma_s = e^{-s/2}M_{T-e^s} = (T-t)^{1/2}M_t$. If ν is the unit normal for Σ_s, then its position vector X satisfies the equation

$$\nu \cdot \frac{\mathrm{d}}{\mathrm{d}s}X = -\operatorname{div}_{\Sigma_s} \nu + \frac{1}{2}\nu \cdot X.$$

Notice in particular that, as expected, Σ_s is constant if and only if the position vector satisfies the self-shrinker equation (1.5). Self-shrinkers are therefore equilibrium points for this rescaled flow.

DEFINITION 1.8 (Rescaled mean curvature flow)**.** Let Σ^n be an n-dimensional manifold and suppose $X : \Sigma \times [0,\infty) \to \mathbb{R}^{n+1}$ satisfies the equation

$$\nu \cdot \frac{\mathrm{d}}{\mathrm{d}s}X = -\operatorname{div}_{\Sigma_s} \nu + \frac{1}{2}\nu \cdot X,$$

where ν is a unit normal for the embedding. Let $\Sigma_s = X(\Sigma, s)$. The 1-parameter family $\{\Sigma_s\}_{s\in[0,\infty)}$ of hypersurfaces is a *rescaled mean curvature flow*. If $\{M_t\}$ is a mean curvature flow becoming singular at time T and for which $p \in M_T$, then the surfaces Σ_s defined by $\Sigma_s = e^{s/2}(M_{T-e^s} - p)$ constitute a corresponding rescaled mean curvature flow.

In the convex case, the main convergence theorem was obtained by Huisken in [Hui84]. It states the following.

THEOREM 1.9 (Huisken, [Hui84]). *Suppose $\{M_t\}_{t\in[0,T)}$ is a maximally defined mean curvature flow in $\mathbb{R}^{n+1}$ for which M_0 is a closed convex hypersurface. Then M_t is convex for $t \in [0,T)$, and M_t shrinks down to a point at time T. Moreover, the corresponding rescaled mean curvature flow $\{\Sigma_s\}$ converges to the sphere $\mathbf{S}^n$ of radius $(2n)^{1/2}$ centered at the origin. The convergence is exponential in C^k for any k.*

1.1.3 The rescaled flow near an equilibrium point: normal graphs over self-shrinkers

In this subsection, we explain the ideas behind Theorem 1.2. We first explain the idea of a normal graph and its importance for rescaled MCF near a self-shrinker. Then we explain how Theorem 1.2 follows from the application of a stable manifold theorem for solutions to rescaled MCF converging to a shrinker at a certain rate.

Let $\Sigma^n \subset \mathbb{R}^{n+1}$ be a smooth closed self-shrinker with unit normal ν, and suppose the rescaled mean curvature flow $\{\Sigma_s\}_{s\geq 0}$ converges to Σ as $s \to \infty$ in C^2. Then for large enough s, we can express Σ_s as a normal graph over Σ:

$$\Sigma_s = \{x + u(x,s)\nu(x) :\ x \in \Sigma\}$$

for some scalar function $u : \Sigma \times [s_0,\infty) \to \mathbb{R}$. The function u can be characterized as a solution to a quasilinear parabolic equation of the form

$$\partial_s u = Lu + N(u, \nabla u, \nabla^2 u) \tag{1.6}$$

where L is the linear elliptic operator defined by

$$Lw = \Delta w - \frac{1}{2}x \cdot \nabla w + |\nabla \nu|^2 w + \frac{1}{2}w$$

and the nonlinear error term N has the form

$$N(u, \nabla u, \nabla^2 u) = f(u, \nabla u) + \mathrm{trace}(B(u, \nabla u)\nabla^2 u), \tag{1.7}$$

where f and B are smooth and $f(0,0)$, $df(0,0)$, and $B(0,0)$ are zero. For a detailed proof of this fact, see Section 4.1 of [CMI15].

1.2 Rate of convergence and asymptotics for rescaled MCF near the sphere using a stable manifold theorem

We now explain how Theorem 1.2 is proved. The strategy involves two steps: (1) characterize the possible exponential rates of convergence of the rescaled MCF $\{\Sigma_s\}$ to the self-shrinker Σ and the solutions converging at any particular exponential rate, and (2) rule out the possibility of solutions converging faster than any exponential. In the first step, we use a general stable manifold theorem, Theorem 1.11 below. The second step is Theorem 1.13.

On the stationary sphere $\mathbf{S}^n$ of radius $(2n)^{1/2}$ centered at the origin, the linear operator L simplifies. Because x is proportional to the unit normal on $\mathbf{S}^n$, the term $x \cdot \nabla w$ appearing in Lw disappears. Moreover, $\nabla \nu = (2n)^{-1/2}\Pi_n$ in this case, where Π_n is projection onto the n-dimensional tangent space, and so $|\nabla \nu|^2 = n/(2n) = 1/2$. Putting everything together gives $L = \Delta + 1$ in case $\Sigma = \mathbf{S}^n$. This means the eigenvalues are $\lambda_k = k(k+n-1)/(2n) - 1$ and the eigenfunctions are restrictions of homogeneous harmonic polynomials to the sphere. In particular, the operator L has trivial kernel in this case. We summarize in a proposition.

PROPOSITION 1.10. *If the normal graph of a function $u : \mathbf{S}^n \times [s_0, s_1] \to \mathbb{R}$ evolves by rescaled mean curvature flow, then u satisfies a quasilinear parabolic equation*

$$\partial_s u = (\Delta + 1)u + N(u, \nabla u, \nabla^2 u),$$

where the nonlinear error N has the form (1.7).

Let d_k be the dimension of the space of eigenfunctions corresponding to eigenvalues $\lambda_0, \dots, \lambda_{k-1}$ which are strictly smaller than λ_k. We prove the following result for solutions converging to the sphere.

THEOREM 1.11. *For any integer $r > n/2 + 1$ and any integer $k \geq 2$, there exists an open neighborhood $B = B(k, r)$ of the origin in $H^r(\mathbf{S}^n)$ with the property that the set of initial data $u_0 \in B$ for which the solution u to the rescaled MCF equation (1.6) exists for all time $s \geq 0$ and converges to zero with exponential rate λ_k is a codimension d_k submanifold of B which is invariant for equation (1.6). For such initial data, there exist, for $j \geq k$ with $\lambda_j < 2\lambda_k$, eigenfunctions $P_j \in E_j$ for which the corresponding solution u satisfies*

$$\Big\| u(y, s) - \sum_{\substack{j \geq k \\ \lambda_j < 2\lambda_k}} e^{-\lambda_j s} P_j(y) \Big\|_{H^r(\mathbf{S}^n)} \leq C e^{-2\sigma s}$$

for some constant $C > 0$ and all $\sigma < \lambda_k$.

Remark. The proof involves two steps: constructing the invariant manifold and proving the asymptotics. The construction of the invariant manifold is modeled on the argument of [Nai88] and [EW87], both of which hew closely to the analogous construction for ODE.

We also prove that the leading eigenfunction P_k to which $e^{\lambda_k s} u(x, s)$ converges in $H^r(\mathbf{S}^n)$ may be prescribed.

THEOREM 1.12. *Suppose $k \geq 2$ and let $P \in E_k$ be an eigenfunction for the operator $\Delta + 1$ on the sphere $\mathbf{S}^n$ corresponding to the eigenvalue λ_k. There exists $s_0 \geq 0$ and $u : \mathbf{S}^n \times [s_0, \infty) \to \mathbb{R}$ which solves the rescaled MCF equation (1.6) and satisfies*

$$\| e^{\lambda_k s} u(y, s) - P(y) \|_{H^r(\mathbf{S}^n)} \leq C e^{-\sigma s}$$

for some constants $C > 0$ and $\sigma > 0$ and for all $s \geq s_0$.

Remarks. If $k \geq 3$ or $n = 1$ or 2, then we may take $\sigma = \lambda_{k+1}$ in the statement of the theorem, and if $n \geq 3$ and $k = 2$ we may take any $\sigma < 2\lambda_2 = 2/n$.

The precise asymptotics of the limit, and the prescription of them, are inspired by [AV97]. In fact, the present investigation came from the author's wish to determine similar asymptotics in the simpler compact setting.

The proof of Theorem 1.11 actually shows that any solution to rescaled MCF converging to the sphere lies in one of the invariant manifolds from the statement of the theorem. Therefore, the only remaining step in the proof of Theorem 1.2 is to rule out the possibility of solutions converging to the sphere faster than any exponential. This is the role of Theorem 1.13. As we will see, this translates to a unique continuation result for the arrival time. It rules out solutions converging like e^{-s^2}, for example, and the proof occupies section 4.

THEOREM 1.13 (Ruling out faster-than-exponential convergence). *If $\{\Sigma_s\}$ is a rescaled mean curvature flow converging to the stationary sphere $\mathbf{S}^n$ at a rate that is faster than any exponential, then $\Sigma_s = \mathbf{S}^n$ for all s.*

1.2.1 Asymptotics for rescaled MCF near a general shrinker

The proofs in this thesis are written for the sphere, but they actually apply with little modification to a general compact self-shrinker. Here we state the results in a more general context.

THEOREM 1.14 (Stable manifold theorem for solutions converging at a particular rate). *Suppose $\Sigma^n \subset \mathbb{R}^{n+1}$ is a smooth closed self-shrinker, and suppose E is an eigenspace for $-L$ with eigenvalue $\lambda > 0$. Let d be the dimension of all eigenspaces corresponding to eigenvalues smaller than λ. Then for $r > n/2 + 1$ there is a ball B centered at the origin in $H^r(\Sigma)$ with the property that the set of initial data $u_0 \in B$ for which the solution u to the rescaled MCF equation (1.6) exists for all time $s \geq 0$ and converges to zero with exponential rate λ is a codimension d submanifold of B which is invariant for equation (1.6). For such* *initial data, the corresponding solution u to (1.6) satisfies*

$$\|e^{\lambda s}u(y,s) - \phi(y)\|_{H^r(\Sigma)} \le Ce^{-\varepsilon s}$$

for some $\varepsilon > 0$ *and some eigenfunction* $\phi \in E$. *The limit* ϕ *can be prescribed.*

It is also true that a rescaled MCF cannot converge faster than exponentially to a compact self-shrinker.

THEOREM 1.15. *Suppose* $\Sigma^n \subset \mathbb{R}^{n+1}$ *is a smooth closed self-shrinker and* $\{\Sigma_s\}$ *is a rescaled MCF converging to* Σ *at a rate that is faster than any exponential. Then* $\Sigma_s = \Sigma$ *for all* s.

In case Σ is a closed self-shrinker for which the linear operator L has trivial kernal, these two theorems imply that any solution to rescaled MCF converging to Σ must in fact converge exponentially to the normal graph of some separation of variables solution $e^{-\lambda s}\phi$ to the linear equation $\partial_s w = Lw$ on Σ. However, when L has a kernel there may be solutions that do not converge exponentially to Σ but rather converge at a much slower rate. In this situation, the analysis becomes more complicated (see, for example, [Sch14]).

1.3 The arrival time

Suppose $\Omega \subset \mathbb{R}^{n+1}$ is a smooth bounded domain with mean convex boundary, and let $\{M_t\}$ be a mean curvature flow starting from $M_0 = \partial\Omega$. Then for $x \in \Omega$ near enough to the boundary, there is a unique time $t = t(x)$ at which $x \in M_t$, that is, at which the moving front arrives at the point x. The function t that assigns this time to a point $x \in \Omega$ is called the *arrival time* for mean curvature flow.

The assumption of mean convexity is crucial for this definition, because a mean curvature flow that is not mean convex may touch the same point twice.

As we've stated it, t may not be defined on all of Ω. To see the trouble, consider a mean convex domain for which the mean curvature flow becomes singular before becoming extinct. A simple example is a dumbbell-shaped region in $\mathbb{R}^3$ with a very thin neck. If the neck is thin enough, it will pinch off under the flow before the dumbbells have time to shrink. The issue now is to define the flow in a natural way past the time when the neck pinches off. The simple existence theorem for smooth initial data will not suffice in this situation because at the moment the neck pinches off the evolving surface is not smooth—it has a singularity at the neck pinch. There are in fact several ways to define the flow past the neck pinch, and one of them is to use the arrival time. It was precisely this kind of problem that led to the study of the arrival time function in the first place.

The crucial observation is that, where it is defined, the arrival time t is characterized by the fact that it solves a partial differential equation, namely, the level set equation (1.1) from the introduction. This observation enables us to define t using the equation rather than using a mean curvature flow, thereby giving us another way to construct and study the mean curvature flow: once the function t is constructed, the level sets $M_\tau = \{x : t(x) = \tau\}$ make up a mean curvature flow.

As an example of the utility of this approach, one can show that a solution to equation (1.1) exists on an arbitrary mean convex domain in a certain weak sense. Therefore the mean curvature flow can be defined on the entire dumbbell shaped region described above, for instance, and the arrival time gives us a way to extend the flow beyond the neckpinch singularity. We will not pursue the weak solutions to (1.1) here because our results concern C^2 solutions which satisfy (1.1) exactly away from their critical points.

LEMMA 1.16 (Derivation of the level set equation (1.1)). *Let t be the arrival time for a smooth mean convex mean curvature flow. Away from singularities of the mean curvature flow, the function t satisfies the equation*

$$|\nabla t| \operatorname{div}\left(\frac{\nabla t}{|\nabla t|}\right) = -1. \tag{1.8}$$

Conversely, if $\Omega \subset \mathbb{R}^{n+1}$ is a smooth bounded mean convex region and $t : \Omega \to \mathbb{R}$ satisfies $t = 0$ on $\partial\Omega$ and (1.8) away from critical points in the interior of Ω, then its level sets form *a mean curvature flow* $\{M_t\}$ *starting from* $M_0 = \partial\Omega$.

Proof. Let $X : M \times [0, T) \to \mathbb{R}^{n+1}$ be the mean curvature flow. Then t is characterized by the equation $t(X(p, \tau)) = \tau$. Differentiating both sides in τ and using the equation satisfied by X gives

$$1 = \nabla t \cdot \frac{\mathrm{d}}{\mathrm{d}\tau} X = -\nabla t \cdot \nu H,$$

where ν is the unit normal to $M_t = X(M, t)$ and $H = \operatorname{div}_{M_t} \nu$ is the scalar mean curvature. Now notice that M_t is a level set for t, so away from critical points $\nu = \pm\nabla t/|\nabla t|$. Finally, $\operatorname{div}_{M_t}(\nabla u/|\nabla u|) = \operatorname{div}_{\mathbb{R}^{n+1}}(\nabla u/|\nabla u|)$, in other words, $\nu \cdot \nabla_\nu(\nabla u/|\nabla u|) = 0$, again because $\nabla u/|\nabla u|$ is a unit normal for M_t. Combining everything gives the equation for t.

Next, suppose Ω is a bounded domain in $\mathbb{R}^{n+1}$ with smooth mean convex boundary $\partial\Omega$ and suppose t solves (1.8) in Ω. For small enough ε, define an embedding $X : \partial\Omega \times [0, \varepsilon] \to \mathbb{R}^{n+1}$ by letting $X(\cdot, 0)$ be the identity on $\partial\Omega$ and requiring that $X(p, t)$ satisfy the following ODE:

$$\frac{\mathrm{d}}{\mathrm{d}t} X(p, t) = \frac{\nabla t(X(p, t))}{|\nabla t(X(p, t))|^2}.$$

In other words, X is the flow of $\partial\Omega$ under vector field $\nabla u/|\nabla u|^2$. Then $X(\partial\Omega, \tau)$ is the level set $\{x \in \Omega : t(x) = \tau\}$ because $\partial_\tau t(X(p, \tau)) = 1$ by computation, and on the other hand X forms a mean curvature flow because $\nabla t/|\nabla t|$ is a unit normal for the level set of t and the equation (1.8) says that $|\nabla t|^{-1}$ is the negative mean curvature (the equation says it is the negative of the divergence of the normal). Thus the level sets of t trace out a mean curvature flow. ∎

1.3.1 Background on regularity for the arrival time

If Ω is a bounded convex domain, it was proved by Huisken in [Hui84] that the mean curvature flow $\{M_\tau\}$ starting from $\partial\Omega$ contracts smoothly to a single point $x_0 \in \Omega$ at some finite time T. Moreover, the translated and rescaled flow $(T-\tau)^{-1/2}(M_\tau - x_0)$ converges at time T to the round sphere $\mathbf{S}^n$ of radius $(2n)^{1/2}$ centered at the origin. The function t solving (1.1) for Ω therefore has a single critical point x_0 inside Ω, where $t(x_0) = T$ is the maximum for t. In this case, t is actually C^2 on Ω and the second derivative $\nabla^2 t(x_0)$ of t at this critical point is a multiple of the identity: $\partial_i \partial_j t = -\delta_{ij}/n$.[1]

In the case of a general mean-convex domain, the arrival time t is known to be twice differentiable but not necessarily C^2, see [CMI16], [CMI17], and [CMI18]. In fact, it was shown in [Whi00] (Theorem 1.2) and [Whi15] that any tangent flow of a smooth mean convex mean curvature flow is a generalized cylinder. From this one can figure out what the Hessian of the arrival time function must be if it exists. The remaining issue was to show that the Hessian exists, which is equivalent to the problem of uniqueness of tangent flows. This was solved in [CMI15]. The study of the arrival time is referred to as the *level set method* in the mean curvature flow literature, because it gives a means of rigorously extending mean curvature flow beyond singularities. This point of view was first taken in a computational context by Osher and Sethian, [OS88], and the theory was then developed in [CGG91], [ES91], [ES92a], [ES92b], and [ES95]. We will restrict attention to the case in which the domain of the arrival time function is convex.

In [KS06], Robert Kohn and Sylvia Serfaty proved that the solution to equation (1.1) on a convex planar domain Ω is always C^3, and they asked whether this is true in higher dimensions. Natasa Sesum demonstrated in [Ses08] that the answer is negative: if $n \geq 2$, there exists a convex domain $\Omega \subset \mathbb{R}^{n+1}$ for which the solution t to (1.1) is not three times differentiable. To prove this, she analyzed the rate of convergence of a rescaled MCF $(T-\tau)^{-1/2}M_\tau$, proving the existence of solutions for which this rescaled flow converges to the sphere like $(T-\tau)^{1/n}$.

In a bounded convex domain, Huisken's theorem implies that the arrival time function has precisely one critical point, where it attains its maximum. Huisken showed in addition that the arrival time function is C^2 and agrees to second order at the critical point with the

[1] See Theorem 6.1 of [Hui93].

arrival time for a sphere becoming extinct under MCF at the same time as the boundary of the domain.

We will now prove some basic facts about the arrival time when it is C^2.

LEMMA 1.17. *If the arrival time $t : \Omega \to \mathbb{R}$ is C^2 at a critical point $x_0 \in \Omega$, then $D^2 t(x_0) = -\Pi_{k+1}/k$, where Π_{k+1} is projection onto a $k+1$-dimensional plane through the origin.*

Proof. Suppose the critical point x_0 is at the origin, suppose $t(0) = 0$, and denote by A the second derivative $D^2 t(0)$. If t is C^2 near $x = 0$, then we can write

$$t(x) = \frac{1}{2} x \cdot Ax + e(x),$$

where the error term e is C^2 near $x = 0$ and satisfies $e(0) = De(0) = D^2 e(0) = 0$. I will show that

$$|A\xi|^2 \operatorname{trace} A - A\xi \cdot A(A\xi) + |A\xi|^2 = 0 \tag{1.9}$$

for all $\xi \in S^n$. Suppose this is proved. Since A is a symmetric matrix, we can choose coordinates in which it is diagonalized. Writing $\lambda_1, \ldots, \lambda_{n+1}$ for the eigenvalues, we get

$$\sum_{i=1}^{n+1} \lambda_i^2 \xi_i^2 \left(1 - \lambda_i + \sum_{j=1}^{n+1} \lambda_j \right) = 0$$

for all $\xi \in S^n$. This is true if and only if for each i we have either $\lambda_i = 0$ or $1 - \lambda_i + \Sigma \lambda_j = 0$. Thus if $\lambda_i \neq 0$ we have $\Sigma_{j \neq i} \lambda_j = -1$, meaning that the nonzero λ_i are all equal and satisfy $\lambda_i = -1/k$, where $k+1$ is the number of nonzero λ_i.

It remains to prove (1.9). Multiply the equation (1.1) through by $|Dt|^2$ to see that t satisfies

$$|Dt|^2 \Delta t - Dt \cdot D^2 t Dt + |Dt|^2 = 0.$$

This is also true at critical points if t is C^2. If we now insert the Taylor approximation

$Dt = x \cdot A + De$ into this equation and remember that D^2t is continuous and $|De| = o(|x|)$ as $x \to 0$, we see that

$$|Ax|^2 \operatorname{trace} A - Ax \cdot A(Ax) + |Ax|^2 = o(|x|^2)$$

as $x \to 0$. Therefore, dividing through by $|x|^2$ and making $|x| \to 0$ gives (1.9) for an arbitrary direction. ■

LEMMA 1.18. *Suppose Ω is a connected mean convex domain and $t : \Omega \to \mathbb{R}$ is a C^2 solution to the arrival time equation (1.1). If $D^2t(x_0) = 0$ for some $x_0 \in \Omega$, then D^2t vanishes identically on Ω.*

Proof. Let Z be the zero set of D^2t. Since D^2t is continuous, Z is relatively closed in Ω. We will show that it is also open by arguing that points not in the interior of Z are not contained in Z. So let $y \in \Omega$ be a point that is not in the interior of Z. Then there is a sequence of points $x_n \in \Omega$ with $x_n \to y$ as $n \to \infty$ and $D^2t(x_n) \neq 0$. This implies that we can find points $y_n \in \Omega$ with $y_n \to y$ as $n \to \infty$ and $Dt(y_n) \neq 0$. But then

$$-1 = \Delta t(y_n) - \frac{Dt(y_n)}{|Dt(y_n)|} \cdot D^2t(y_n)\frac{Dt(y_n)}{|Dt(y_n)|}$$

meaning that the right side cannot possibly converge to zero as $n \to \infty$. It follows that $D^2t(y) \neq 0$, that is, that $y \notin Z$. Therefore Z consists only of interior points and is open. Since Ω is connected, $Z = \Omega$ and D^2t vanishes identically. ■

COROLLARY 1.19. *A C^2 solution to the arrival time equation cannot attain a local minimum unless it is constant.*

Proof. By the first lemma, the second derivative at a critical point of a C^2 solution is non-positive. By the second lemma, the second derivative cannot vanish unless the solution is constant. Then either the solution is constant or at any critical point there are directions in which the second derivative is negative definite. In particular, no critical point can be a minimum for a nonconstant solution. ■

Corollary 1.20. *A nonnegative global C^2 solution $t : \mathbb{R}^{n+1} \to [0, \infty)$ to the arrival time equation is constant.*

Proof. Suppose t is not constant and assume $\inf t = 0$. By the preceding corollary, the infimum cannot be attained on $\mathbb{R}^{n+1}$. Thus if x_n are points for which $t(x_n) \to 0$ as $n \to \infty$, then $|x_n| \to \infty$ with n. Let $M_\tau = \{x : t(x) = \tau\}$ be the level set of t so that $\{M_\tau\}$ forms a mean curvature flow. Then if $R_\tau = \min_{x \in M_\tau} |x|/2$, the ball of radius R_τ centered at the origin is disjoint from M_τ and $R_\tau \to \infty$ as $\tau \to 0$. By the comparison principle, $t(0) \geq R_\tau^2/(2n)$ for all τ. Thus t is infinite at the origin, a contradiction. ∎

1.4 Linearization of the arrival time equation

In this section, we linearize the arrival time equation (1.1) on a convex domain and explain why the main results should be expected.

We have already seen that the solution t to the arrival time equation (1.1) on a bounded convex domain $\Omega \subset \mathbb{R}^{n+1}$ is C^2 on Ω with a unique critical point, which we assume to be the origin. Near the critical point, moreover, we have seen that t agrees to second order with the arrival time for a shrinking sphere: if $T = t(0) = \max_\Omega t$, then

$$t(x) = T - \frac{|x|^2}{2n} + E(x),$$

where E is C^2 and $E(x) = o(|x|^2)$ as $x \to 0$. The error term E satisfies the following equation on Ω:

$$\left(-\frac{x}{n} + DE\right) \cdot D^2E \left(-\frac{x}{n} + DE\right) - \Delta E \left|-\frac{x}{n} + DE\right|^2 = 0.$$

Suppose now that $E(x) = Q(x) + E_1(x)$ for some function Q that is homogeneous with degree $d > 2$ and an error term E_1 that satisfies $E_1(x) = o(|x|^d)$, $DE_1(x) = o(|x|^{d-1})$, and $D^2E_1(x) = o(|x|^{d-2})$ as $x \to 0$. Then by equating terms with the same order of vanishing at the origin, we find that Q must satisfy the linear equation

$$|x|^2\Delta Q - x \cdot D^2 Q x = 0.$$

By expressing the Laplacian in polar coordinates, we can easily characterize the homogeneous solutions to this equation.

LEMMA 1.21. *The homogeneous solutions $Q : \mathbb{R}^{n+1} \to \mathbb{R}$ to the equation*

$$|x|^2\Delta Q - x \cdot D^2 Q x = 0 \tag{1.10}$$

are of the form $Q(x) = |x|^{k(k-1)/n} P(x)$ for a homogeneous harmonic polynomial P of degree k.

Proof. Suppose Q is homogeneous of degree d. Then on the unit sphere S^n we get

$$0 = |x|^2\Delta Q - x \cdot D^2 Q x = \Delta_{S^n} Q + n\partial_r Q = \Delta_{S^n} Q + ndQ.$$

This means that Q is an eigenfunction for the Laplacian on S^n with eigenvalue nd. Therefore $d = k(k+n-1)/n$ for some integer $k \geq 0$, and $Q = r^d P(x/r) = r^{d-k} P(x) = r^{k(k-1)/n} P(x)$ for some homogeneous harmonic polynomial P of degree k. ■

Remark. The equation (1.10) is the linearization of the nonlinear arrival time equation at the special solution $t_B(x) = -|x|^2/(2n)$, which is the arrival time for a round sphere becoming extinct at the origin at time zero under mean curvature flow. In other words, if we suppose that $t_\varepsilon = t_B + \varepsilon E$ solves the arrival time equation (1.1) and collect the terms that are linear in ε, then the result is equation (1.10) for the error E. This means that E satisfies this equation up to an error that is order ε^2 provided there are uniform C^2 bounds on the family of solutions t_ε. Thus if t is a solution to the arrival time equation which deviates very little from t_B, then we expect the error $t - t_B$ to approximately satisfy equation (1.10). On the other hand, every solution t to the arrival time equation with convex level sets is very close to t_B near its maximum, and this is why we expect the behavior of the error $t - t_B$ to approximate a solution to (1.10) near the maximum. This is exactly what we have shown above, and we now record the precise result in a proposition. This idea means that another route to proving our main result, Theorem 1.1, might be to derive uniform C^2 bounds for the arrival time equation on a convex domain and thereby regulate the behavior of solutions near the critical point.

PROPOSITION 1.22. *Suppose t is a solution to the arrival time equation (1.1) on the bounded convex domain Ω. If*

$$t(x) = T - \frac{|x|^2}{2n} + Q(x) + E_1(x),$$

where Q is homogeneous of degree $d > 2$ and $E_1(x)$, $|x|DE_1(x)$, and $|x|^2D^2E_1(x)$ are $o(|x|^d)$ as $x \to 0$, then there exists an integer $k \geq 2$ and a homogeneous harmonic polynomial P of degree k such that

$$Q(x) = |x|^{k(k-1)/n} P(x).$$

If the assumptions of Proposition 1.22 were met in all cases, then Theorem 1.1 would be already verified. In order to prove that t actually admits an asymptotic expansion of this form, we relate the arrival time to a normal graph over the sphere that is evolving under the rescaled mean curvature flow. The form of Q actually falls naturally out of the asymptotics for this normal graph.

1.5 Relating convergence of rescaled MCF to asymptotics for the arrival time

Suppose t is the arrival time for a bounded convex mean curvature flow $\{M_t\}$ becoming singular at the origin in $\mathbb{R}^{n+1}$, and let $t(0) = 0$. Thus t is C^2 near the origin and agrees to second order there with the function $-|x|^2/(2n)$, which is the arrival time for a round sphere becoming singular at the origin at time zero. Let $\{\Sigma_s\}$ be the rescaled mean curvature flow corresponding to $\{M_t\}$: $\Sigma_s = e^{s/2} M_{T-e^s} = (-t)^{-1/2} M_t$, where $s = -\log(-t)$. (See definition 1.8.)

Denote by $\mathbf{S}^n$ the sphere of radius $(2n)^{1/2}$ centered at the origin in $\mathbb{R}^{n+1}$. Its outer unit normal at a point $y \in \mathbf{S}^n$ is $y/(2n)^{1/2}$. For large enough s, we may express the surface Σ_s as a normal graph over $\mathbf{S}^n$:

$$\Sigma_s = \left\{ y + u(y, s)\frac{y}{(2n)^{1/2}} : \ y \in \mathbf{S}^n \right\}.$$

If we now unwind the rescaling procedure and set $\tau = -e^s$, we see that the point x defined by

$$x = (-\tau)^{1/2} y \left(1 + \frac{u}{(2n)^{1/2}} \right)$$

lies in the surface $M_\tau = \{x : \ t(x) = \tau\}$. Therefore, we obtain an implicit equation relating the arrival time t to the graph function u, which we summarize in a lemma.

LEMMA 1.23. *If t is the arrival time for a convex MCF $\{M_t\}$ becoming extinct at the origin at time $t = 0$, and the corresponding rescaled MCF $\{\Sigma_s\}$ is the normal graph of a function $u : \mathbf{S}^n \times [0, \infty) \to \mathbb{R}$ over the stationary sphere $\mathbf{S}^n$, then*

$$t(x) = \frac{-|x|^2}{((2n)^{1/2} + u((2n)^{1/2}x/|x|, \log(-t(x))))^2}. \tag{1.11}$$

This implicit equation allows us to relate the asymptotics of t near $x = 0$ to the asymptotics of u as $s \to \infty$. We will use the asymptotics from Theorem 1.14 on the asymptotics for rescaled MCFs in order to obtain asymptotics for t through the implicit equation (1.11).

LEMMA 1.24. *Suppose the normal graph of $u : \mathbf{S}^n \times [0, \infty) \to \mathbb{R}$ over $\mathbf{S}^n$ evolves by rescaled*

MCF and suppose u satisfies

$$\sup_{y \in \mathbb{S}^n} |e^{\lambda s} u(y,s) - P(y)| \le C e^{-\varepsilon s}$$

as $s \to \infty$, for some $C, \lambda > 0$ and a very small $\varepsilon > 0$ and some function $P : \mathbb{S}^n \to \mathbb{R}$. Then the corresponding arrival time satisfies

$$t(x) = -\frac{|x|^2}{2n} + \frac{|x|^{2\lambda+2}}{(2n)^{3/2+\lambda}} P\left((2n)^{1/2}\frac{x}{|x|}\right) + O(|x|^{2\lambda+2+2\varepsilon})$$

as $x \to 0$.

Proof. Using the preceding lemma, we get

$$t(x) = -\frac{|x|^2}{2n} + \frac{|x|^2}{(2n)^{3/2}} u + O(|x|^2 u^2)$$

as $x \to 0$. From the asymptotics for u and the relationship $s = -\log(-t)$ we can write

$$u = (-t)^{\lambda} P(1+e),$$

where $e = O((-t)^{\varepsilon})$ as $t \to 0$. Combining everything gives

$$t = -\frac{|x|^2}{2n} + \frac{|x|^2}{(2n)^{3/2}}(-t)^{\lambda} P + O(|x|^2(-t)^{2\lambda} + |x|^2(-t)^{\lambda+\varepsilon}) \tag{1.12}$$

Since $t \to 0$ with x, we see from this equation that $t = -|x|^2/2n + O(|x|^{2+2\lambda})$ as $x \to 0$. Consequently, $(-t)^{\lambda} = |x|^{2\lambda}/(2n)^{\lambda} + O(|x|^{2+2\lambda})$. Inserting this asymptotic into the right side of equation (1.12) and simplifying then gives

$$t = -\frac{|x|^2}{2n} + \frac{|x|^{2+2\lambda}}{(2n)^{3/2+\lambda}} P + O(|x|^{2+2\lambda+\varepsilon})$$

if ε is small enough, which completes the proof of the lemma. ∎

Chapter 2

Construction of the invariant manifolds

In this section, we adapt the argument of [Nai88], which is a general stable manifold theorem for geometric evolution equations, to our situation in order to construct invariant manifolds of solutions which converge with prescribed exponential rate. We now briefly summarize the main result of [Nai88] and explain how our results differ: Let M be a closed Riemannian manifold of dimension n and let L be an elliptic differential operator on M which is symmetric in the $L^2(M)$ inner product and which has discrete spectrum accumulating only at $+\infty$ (in particular the operator is assumed to be bounded below). Suppose $N = N(u)$ is a nonlinear function defined on $H^{r-1}(M)$ for an integer $r > n/2 + 1$ which satisfies $N(0) = 0$ and a bound of the form we prove in Lemma 2.5. In this situation, Naito proves the following:

THEOREM 2.1 (Naito, [Nai88]). *There exists a ball B centered at the origin in $H^{r+1}(M)$ in which the nonlinear evolution equation*

$$\partial_s u = Lu + N(u)$$

has an invariant stable manifold of finite codimension.

The codimension is equal to the codimension of the space on which L is negative definite (the index of L plus the dimension of the kernel). Naito's argument is modeled on Epstein

& Weinstein's earlier proof of a stable manifold theorem for mean curvature flow in the plane, [EW87], and both of these arguments follow closely the proof of the stable manifold theorem for ODE.[1]

Theorem 2.1 already almost implies part of the conclusion of Theorem 1.11, though it does not include the precise rate of convergence and does not describe the asymptotics of the limit. Using the notation of Theorem 1.11 from the preceding subsection and assuming $k \geq 2$, one would like, in our situation, to replace a solution $u(x, s)$ of (1.6) with $e^{\lambda_k s} u(x, s)$ and to replace the linear term $\Delta + 1$ on the right side of (1.6) with $L = \Delta + 1 + \lambda_k$ and then to apply Naito's theorem. The main issue then is that the nonlinear term will depend on the time parameter s, but this is easy to overcome in this context because the time-dependent nonlinear term satisfies a bound that is uniform in s.

Notice that, assuming this argument is carried out successfully, the stable manifold one obtains in this case from Theorem 2.1 is the set of solutions for which $e^{\lambda_k s} u(s) \to 0$, and it will have the codimension of all eigenspaces corresponding to eigenvalues λ_j with $j \leq k$ (the index plus nullity of $\Delta + 1 + \lambda_k$). If we want precisely the solutions for which $s \mapsto e^{\lambda_k s} u(x, s)$ is bounded, that is, precisely the solutions for which u converges to 0 exponentially at rate λ_k as $s \to \infty$, we must instead apply Theorem 2.1 to $e^{(\lambda_k - \varepsilon)s} u(x, s)$ and $L = \Delta + 1 + \lambda_k - \varepsilon$ for sufficiently small ε. The ultimate conclusion of this analysis is that there exists a codimension d_k invariant submanifold for the equation (1.6) with the property that any solution in this invariant submanifold converges to zero at exponential rate $\lambda_k - \varepsilon$ for all $\varepsilon > 0$. In particular, this argument does not prove that $e^{\lambda_k s} u(s)$ is bounded in $H^{r+1}(M)$, though this can be proved (and we prove it below in Section) using the fact that u converges to zero in every Sobolev space $H^r(\mathbf{S}^n)$.[2] Thus the fact that u converges to zero in Sobolev

[1] For a treatment of the stable manifold theorem in the finite-dimensional ODE context, see, e.g., [Hal09], §III.6.

[2] The assertion is not true for a general nonlinear term, as is already apparent in the finite-dimensional ODE case, for essentially the same reason that a center manifold need not be stable. Consider for example the ODE

$$\dot{x} = -\varepsilon x - \frac{x}{\log |x|}$$

on $\mathbb{R}$. For small initial data, the solution converges to zero like $te^{-\varepsilon t}$ as $t \to \infty$. If the nonlinear term is $O(|x|^{1+\alpha})$ for some $\alpha > 0$ as $x \to 0$ this cannot happen.

spaces of all orders does imply that the rate of convergence is better than shown in [Nai88] or [EW87].[3] The same argument improves the rate of convergence in Naito's general theorem under the additional assumption that the solution converges to zero in Sobolev spaces of every order.

Rather than apply the conclusion of Theorem 2.1 in this way, we prefer to adapt the argument to our situation. This is done in this section (Section 2). Section 2.1 collects some bounds required for the construction in Section 2.3, and both sections follow closely arguments of [Nai88] and [EW87]. We also include, for the convenience of the reader, a proof that a quasilinear nonlinear term N of second order does satisfy the bound required by Naito's hypotheses in [Nai88] and Theorem 2.1. This occupies Section 2.2

In Section 3, we establish the rest of Theorem 1.11, namely, the precise rate of convergence and the asymptotics. This part does not overlap with [Nai88] or [EW87]. We also show that the asymptotics can be prescribed as in Theorem 1.12. Analysis of the asymptotics requires a closer look at the construction of the stable invariant manifold in the first place, and this is part of the reason we prefer to argue directly in the proof of Theorem 1.11 rather than attempt to apply the conclusion of Theorem 2.1 to our situation.

2.1 Linear estimates

Throughout, we write $\langle v, w\rangle$ for the $L^2(\mathbf{S}^n)$ inner product:

$$\langle v, w\rangle = \int_{\mathbf{S}^n} vw.$$

Let L be the linear operator $\Delta + 1$ on the sphere $\mathbf{S}^n$, and let F_k be the subspace of $H^r(\mathbf{S}^n)$

[3] Cf. Proposition 5.2 of [Nai88], where the author establishes convergence to zero with exponential rate σ for any σ smaller than the first positive eigenvalue of the linear operator, and Remark 3, page 136 of [EW87], where the same claim is made.

defined by

$$F_k = \bigoplus_{j=k}^{\infty} E_j$$

with E_j as before the eigenspace for L corresponding to the jth eigenvalue $\lambda_j = j(j+n-1)/(2n)-1$. From now on, we fix an integer $k \geq 2$ so that L is negative definite and bounded above on F_k, satisfying $\langle Lv, v\rangle \leq -\lambda_k \|v\|^2_{L^2(\mathbf{S}^n)}$ for $v \in F_k$.

For $v \in F_k$, we may *define* the $H^\ell(\mathbf{S}^n)$ norm for integer $\ell \geq 0$ by

$$\|v\|_{H^\ell} := \langle (-L)^\ell v, v\rangle.$$

This norm is equivalent to the usual H^ℓ norm.

Lemma 2.2. *If $s \mapsto v(s)$ is a continuously differentiable path in $F_k \cap H^{r+1}(\mathbf{S}^n)$, then for any $\varepsilon > 0$ and any integer $r \geq 1$,*

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}s}\|v(s)\|^2_{H^r(\mathbf{S}^n)} + (1-\varepsilon)\|v(s)\|^2_{H^{r+1}(\mathbf{S}^n)} \leq \frac{1}{4\varepsilon}\|(\partial_s - L)v(s)\|^2_{H^{r-1}(\mathbf{S}^n)}. \tag{2.1}$$

Proof. Write $f = (\partial_s - L)v$ for brevity. Use Cauchy-Schwarz to get, for any $\varepsilon > 0$,

$$\langle (-L)^r v, f\rangle \leq \varepsilon \langle (-L)^{r+1} v, v\rangle + \frac{1}{4\varepsilon}\langle (-L)^{r-1} f, f\rangle.$$

Rearranging and substituting $f = (\partial_s - L)v$ on the left gives

$$\langle (-L)^r v, (\partial_s - (1-\varepsilon)L)v\rangle \leq \frac{1}{4\varepsilon}\langle (-L)^{r-1} f, f\rangle = \frac{1}{4\varepsilon}\|f\|^2_{H^{r-1}},$$

and because $\partial_s \|v\|^2_{H^r}/2 = \langle (-L)^r v, \partial_s v\rangle$ and $\langle (-L)^{r+1} v, v\rangle = \|v\|^2_{H^{r+1}}$, this is equivalent to the conclusion of the lemma. ∎

Corollary 2.3. *If $v(s) \in F_k$ for all $s \geq 0$, then for any σ with $0 < \sigma < \lambda_k$ and any* *integer* $r \geq 1$,

$$e^{2\sigma s}\|v(s)\|^2_{H^r(\mathrm{S}^n)} \leq \|v(0)\|^2_{H^r(\mathrm{S}^n)} + \frac{\lambda_k}{2(\lambda_k - \sigma)} \int_0^s e^{2\sigma\tau}\|(\partial_s - L)v(\tau)\|^2_{H^{r-1}(\mathrm{S}^n)}\, \mathrm{d}\tau.$$

Proof. Notice that the left side of (2.1) can be bounded below for $v \in F_k$ using $\|v(s)\|^2_{H^{r+1}} \geq \lambda_k\|v(s)\|^2_{H^r}$. The result is

$$\frac{\mathrm{d}}{\mathrm{d}s}\|v(s)\|^2_{H^r(\mathrm{S}^n)} + 2(1-\varepsilon)\lambda_k\|v(s)\|^2_{H^{r+1}(\mathrm{S}^n)} \leq \frac{1}{2\varepsilon}\|(\partial_s - L)v(s)\|^2_{H^{r-1}(\mathrm{S}^n)}.$$

This is equivalent to the statement of the corollary with $\sigma = (1-\varepsilon)\lambda_k$ because the left side can be written

$$\frac{\mathrm{d}}{\mathrm{d}s}\|v(s)\|^2_{H^r(\mathrm{S}^n)} + 2(1-\varepsilon)\lambda_k\|v(s)\|^2_{H^{r+1}(\mathrm{S}^n)} = e^{-2(1-\varepsilon)\lambda_k s}\frac{\mathrm{d}}{\mathrm{d}s}\left(e^{2(1-\varepsilon)\lambda_k s}\|v(s)\|^2_{H^r}\right)$$

and we can multiply through by $e^{2(1-\varepsilon)\lambda_k s}$ and integrate. ∎

COROLLARY 2.4. *In the situation of the lemma, if* $r \geq 1$ *is an integer and* $\|v(s_j)\|_{H^r(\mathrm{S}^n)} \to 0$ *for some sequence* s_j *increasing to infinity, then*

$$\int_0^\infty \|v(s)\|^2_{H^{r+1}(\mathrm{S}^n)}\, \mathrm{d}s \leq \|v(0)\|_{H^r(\mathrm{S}^n)} + \int_0^\infty \|(\partial_s - L)v(s)\|^2_{H^{r-1}(\mathrm{S}^n)}\, \mathrm{d}s.$$

2.2 Nonlinear estimate

The nonlinear term $N : \mathbb{R} \times \Gamma(T\mathrm{S}^n) \times \Gamma(T^*\mathrm{S}^n \otimes T\mathrm{S}^n) \to \mathbb{R}$ (here $\Gamma(T\mathrm{S}^n)$ is the space of sections of the tangent bundle, for instance) appearing in the rescaled mean curvature flow equation (1.6) over the sphere has the form

$$N(u, \nabla u, \nabla^2 u) = f(u, \nabla u) + \mathrm{trace}(B(u, \nabla u)\nabla^2 u) \tag{2.2}$$

where $f : \mathbb{R} \times \Gamma(TS^n) \to \mathbb{R}$ is smooth with $f(0,0) = 0$ and $Df(0,0) = 0$, and where $B : \mathbb{R} \times \Gamma(TS^n) \to \Gamma(T^*S^n \otimes TS^n)$ is smooth and satisfies $B(0,0) = 0$.[4]

In this section, we prove the following Sobolev estimate for a nonlinear term N of this form. We abbreviate $N(u, \nabla u, \nabla^2 u)$ by $N(u)$.

LEMMA 2.5. *Let r be an integer with $r > n/2 + 1$, let N be smooth function of the form (2.2), and let $R > 0$ be fixed. There exists a constant C depending on N and R and r with the property that all $v, w \in C^\infty(S^n)$ with $\|v\|_{H^r(S^n)}, \|w\|_{H^r(S^n)} \leq R$ satisfy*

$$\|N(v) - N(w)\|_{H^{r-1}(S^n)} \leq C \left(\|v\|_{H^{r+1}(S^n)} \|v - w\|_{H^r(S^n)} + \|w\|_{H^r(S^n)} \|v - w\|_{H^{r+1}(S^n)} \right).$$

For the proof of Lemma 2.5, we need a Sobolev product lemma which is standard. In this simple case (s an integer) it can be proved using Hölder's inequality and the Sobolev imbedding theorems.

LEMMA 2.6. *Suppose $M = M^n$ is a closed Riemannian manifold of dimension n, and s_1, s_2, and s satisfy $s_i \geq s$ and $s_1 + s_2 \geq s + d/2$. Then there is a constant C depending on s and the Sobolev constant for M such that*

$$\|vw\|_{H^s(M)} \leq C \|v\|_{H^{s_1}(M)} \|w\|_{H^{s_2}(M)}$$

for all $v, w \in C^\infty(M)$.

We now indicate the proof of Lemma 2.5, demonstrating the bound on the f term of N. The other term is similar so we omit the details. For clarity, let us now work in a coordinate chart (it makes no difference in the analysis). Thus let $u_j = \partial_j u$ be the components of the gradient ∇u. Under the preceding assumptions, we can express f as

$$f(u, \nabla u) = g_0(u, \nabla u) u^2 + \sum_{j=1}^{n} g_j(u, \nabla u) u_j^2$$

[4] See [CMI15], Appendix A, for a proof of this fact.

for some smooth functions g_j. In particular,

$$f(u, \nabla u) - f(v, \nabla v) = g_0(u, \nabla u)(u - v)(u + v) + (g_0(u, \nabla u) - g_0(v, \nabla v))v^2 \\ + \sum_{j=1}^{n} g_j(u, \nabla u)(u_j - v_j)(u_j + v_j) + (g_j(u, \nabla u) - g_j(v, \nabla v))v_j^2.$$

Now suppose that u and v are in $H^r(\mathbf{S}^n)$, where $r > n/2+1$. There is a continuous imbedding $H^r(\mathbf{S}^n) \hookrightarrow C^1(\mathbf{S}^n)$, and so the C^1 norms of u and v are controlled by the H^r norms. In this situation, if we assume that $\|u\|_{H^r}, \|v\|_{H^r} \leq R$, we can deduce that the functions $g_j(u, \nabla u)$ satisfy

$$\|g_j(u, \nabla u)\|_{H^\ell} \leq C(1 + \|u\|_{H^{\ell+1}})$$

for any integer $\ell \geq 0$, where C is a constant that depends on the function g_j and on R. (The proof is by induction, and we use the fact that the domain $\mathbf{S}^n$ has finite volume.) In particular, $g_j(u, \nabla u)$ and $g_j(v, \nabla v)$ are in H^{r-1}, and since $r - 1 > n/2$ we may apply the Sobolev product theorem (with $r - 1 = s = s_1 = s_2$) to terms like $g_j(u, \nabla u)(u_j - v_j)(u_j + v_j)$. To deal with the terms $(g_j(u, \nabla u) - g_j(v, \nabla v))v_j^2$, we write

$$g_j(u, \nabla u) - g_j(v, \nabla v) = \int_0^1 \partial_1 g(u + t(v - u), \nabla u + t\nabla(v - u))\, \mathrm{d}t\, (v - u) \\ + \sum_{i=2}^{n+1} \int_0^1 \partial_i g(u + t(v - u), \nabla u + t\nabla(v - u))\, \mathrm{d}t\, (v_i - u_i).$$

The functions $\int_0^1 \partial_i g(u + t(v - u), \nabla u + t\nabla(v - u))\, \mathrm{d}t$ are in H^{r-1} for the same reason that $g_j(u, \nabla u)$ is, and so we may apply the Sobolev product theorem to these terms as well.

Combining everything, we get a bound

$$\|f(u,\nabla u)-f(v,\nabla v)\|_{H^{r-1}} \le \|(u-v)(u+v)\|_{H^{r-1}} + C\sum_{j=1}^{n}\|(u_j-v_j)(u_j+v_j)\|_{H^{r-1}}$$
$$+ C\|(u-v)(v^2+|\nabla v|^2)\|_{H^{r-1}} + C\sum_{j=1}^{n}\|(u_j-v_j)(v^2+|\nabla v|^2)\|_{H^{r-1}}$$

where the constant C depends on f and R and r. We can now apply the Sobolev product theorem to the right side to obtain

$$\|f(u,\nabla u)-f(v,\nabla v)\|_{H^{r-1}} \le C\|u+v\|_{H^r}\|u-v\|_{H^r} + C\|v\|_{H^r}^2\|u-v\|_{H^r}.$$

Since $\|v\|_{H^r} \le R$ by assumption this is bounded by $C\|u-v\|_{H^r}(\|u\|_{H^r}+\|v\|_{H^r})$.

2.3 Constructing the invariant manifolds: contraction argument

Let $\Pi_k : H^r(\mathbb{S}^n) \to F_k$ be orthogonal projection onto F_k. This orthogonal projection operator is the same for all r because of the way we have defined H^r.

Now fix an integer $r \ge 1$. Define $X_{r,\sigma}$ to be the Banach space of paths $v = v(s) : \mathbb{R} \to H^{r+1}(\mathbb{S}^n)$ for which the norm $\|\cdot\|_{r,\sigma}$ defined by

$$\|v\|_{r,\sigma} = \left(\int_0^\infty \|v(s)\|_{H^{r+1}(\mathbb{S}^n)}^2\,\mathrm{d}s\right)^{1/2} + \sup_{s\ge 0} e^{\sigma s}\|v(s)\|_{H^r(\mathbb{S}^n)}$$

is finite.

We define an operator T for $(v(s), u_0) \in X_{r,\sigma}\times F_k$ by requiring that the path $T(s) =$

$T(v; u_0)(s)$ solve the equation

$$
\begin{aligned}
(\partial_s - L)T(s) &= N(v(s)) \\
T(0) &= u_0 - \int_0^\infty e^{-L\tau}(1 - \Pi_k)N(v(\tau))\,\mathrm{d}\tau.
\end{aligned}
\tag{2.3}
$$

The integral in the second equation makes sense pointwise because $1 - \Pi_k$ projects on a finite-dimensional invariant subspace for L. We will see moreover that for N satisfying our requirements it is convergent and defines an element of H^r for $v \in X_{r,\sigma}$ with $\lambda_{k-1} < \sigma < \lambda_k$.

Notice that if v is a fixed point for $T(\cdot; u_0)$, then v solves the nonlinear evolution equation (1.6). If this fixed point lies in the space $X_{r,\sigma}$, then by definition it converges to zero exponentially. We will show that for small enough $u_0 \in F_k$ and for $\lambda_{k-1} < \sigma < \lambda_k$, the mapping $T(\cdot; u_0)$ has precisely one fixed point v in a small ball centered at the origin in $X_{r,\sigma}$. This fixed point depends smoothly in H^r on the parameter u_0, and the initial datum of the corresponding evolution is $v(0)$. The orthogonal projection of $v(0)$ onto F_k is just u_0, and it follows easily that the space of initial data in a small ball of H^r centered at 0 which converges to zero exponentially with rate between λ_{k-1} and λ_k is a graph over F_k. The size of the ball in H^r on which this is true depends on the exponential rate $\sigma \in (\lambda_{k-1}, \lambda_k)$, but since the solution converges to zero and therefore enters every ball centered at zero it is in fact true that the exponential rate of convergence to zero is automatically better than σ for any $\sigma < \lambda_k$.

The main result of this section is the following theorem.

THEOREM 2.7. *If $r > n/2 + 1$ and $\lambda_{k-1} < \sigma < \lambda_k$ and if $u_0 \in F_k$ with $\|u_0\|_{H^r}$ sufficiently small, then $T(\cdot; u_0)$ maps a small ball centered at the origin in $X_{r,\sigma}$ into itself and satisfies*

$$
\|T(v, u_0) - T(w, u_0)\|_{r,\sigma} \le C\left(\|v\|_{r,\sigma} + \|w\|_{r,\sigma}\right)\|v - w\|_{r,\sigma} \tag{2.4}
$$

for some constant $C = C(r, \sigma, k)$ depending on r, σ, and k.

COROLLARY 2.8. *The mapping T is a contraction mapping of a small ball centered at the origin in $X_{r,\sigma}$ into itself. Consequently, it has a unique fixed point in this ball.*

Proof of Theorem 2.3. We first prove the bound (2.4) on a small ball, and then we show that if this ball is small enough it is mapped into itself by T. If v and w are in $X_{r,\sigma}$ and $u_0 \in F_k$, then the difference $D(s) = T(v; u_0)(s) - T(w; u_0)(s)$ is continuously differentiable and satisfies the equation

$$(\partial_s - L)D(s) = N(v(s)) - N(w(s)) \tag{2.5}$$
$$D(0) = -\int_0^\infty e^{-L\tau}(1 - \Pi_k)\,(N(v(\tau)) - N(w(\tau)))\,\mathrm{d}\tau.$$

To bound D, we break it up into components using the orthogonal projection $\Pi_k : H^r \to F_k$. The bound on the component $(1-\Pi_k)D(s)$ is simple, so we take care of that first. The more interesting bound is on $\Pi_k D(s)$, and for this we make use of Corollaries 2.4 and 2.3, which apply because $\Pi_k D(s) \in F_k$ for all $s \geq 0$ (this is why we break D into components in the first place).

We now show how $(1 - \Pi_k)D(s)$ is controlled in $X_{r,\sigma}$. First, $1 - \Pi_k$ projects onto a finite-dimensional subspace of H^r, and $(1 - \Pi_k)D(s)$ can be expressed as an integral

$$\begin{aligned}(1 - \Pi_k)D(s) &= e^{Ls}(1 - \Pi_k)D(0) + \int_0^s e^{L(s-\tau)}\left(1 - \Pi_k\right)(N(v(\tau)) - N(w(\tau)))\,\mathrm{d}\tau \\ &= -\int_s^\infty e^{L(s-\tau)}\left(1 - \Pi_k\right)(N(v(\tau)) - N(w(\tau)))\,\mathrm{d}\tau,\end{aligned}$$

where the second line is obtained from the first by substituting the expression for $D(0)$ and simplifying. For $\tau > s$, the operator $e^{L(s-\tau)}$ has norm $e^{\lambda_{k-1}(\tau-s)}$ on range$(1 - \Pi_k)$. Because the range is finite-dimensional, and all norms on it are equivalent, we may write

$$\begin{aligned}\|(1 - \Pi_k)D(s)\|_{H^r(S^n)} &\leq C\int_s^\infty e^{\lambda_{k-1}(\tau-s)}\|(1 - \Pi_k)\,(N(v(\tau)) - N(w(\tau)))\,\|_{H^{r-1}(S^n)}\,\mathrm{d}\tau \\ &\leq C\int_s^\infty e^{\lambda_{k-1}(\tau-s)}\|N(v(\tau)) - N(w(\tau))\|_{H^{r-1}(S^n)}\,\mathrm{d}\tau\end{aligned}$$

where C is a constant that depends on k and r. Now we just use the nonlinear estimate Lemma 2.5 to bound the right side and obtain

$$\|(1-\Pi_k)D(s)\|_{H^r} \le C\int_s^\infty e^{\lambda_{k-1}(\tau-s)}\left(\|v(\tau)\|_{H^{r+1}}\|v(\tau)-w(\tau)\|_{H^r}+\|w(\tau)\|_{H^r}\|v(\tau)-w(\tau)\|_{H^{r+1}}\right)\,\mathrm{d}\tau.$$

Finally, assuming $\lambda_{k-1}<\sigma<\lambda_k$, we bound the right side by the $\|\cdot\|_{r,\sigma}$ norm straightforwardly as follows (using the first summand for an example):

$$\begin{aligned}
&\int_s^\infty e^{\lambda_{k-1}(\tau-s)}\|v(\tau)\|_{H^{r+1}}\|v(\tau)-w(\tau)\|_{H^r}\,\mathrm{d}\tau\\
&\qquad\le \sup_{\tau\ge s} e^{\sigma(\tau-s)}\|v(\tau)-w(\tau)\|_{H^r}\int_s^\infty e^{-(\sigma-\lambda_{k-1})(\tau-s)}\|v(\tau)\|_{H^{r+1}}\,\mathrm{d}\tau\\
&\qquad\le e^{-\sigma s}\|v-w\|_{r,\sigma}\left(\int_0^\infty e^{-2(\sigma-\lambda_{k-1})\tau}\,\mathrm{d}\tau\right)^{1/2}\left(\int_0^\infty \|v(\tau)\|^2_{H^{r+1}}\,\mathrm{d}\tau\right)^{1/2}\\
&\qquad\le e^{-\sigma s}\|v-w\|_{r,\sigma}\|v\|_{r,\sigma}\frac{1}{(\sigma-\lambda_{k-1})^{1/2}}.
\end{aligned}$$

The passage from the first to the second line is just Cauchy–Schwarz. All told, we obtain

$$e^{\sigma s}\|(1-\Pi_k)D(s)\|_{H^r} \le C(\|v\|_{r,\sigma}+\|w\|_{r,\sigma})\|v-w\|_{r,\sigma}, \tag{2.6}$$

where C depends on k and σ.

Since the H^r and H^{r+1} norms are equivalent on the range of $1-\Pi_k$, we see from the bound (2.6) that

$$\|(1-\Pi_k)D(s)\|_{H^{r+1}} \le e^{-\sigma s}C(\|v\|_{r,\sigma}+\|w\|_{r,\sigma})\|v-w\|_{r,\sigma},$$

and since $e^{-\sigma s}$ is square-integrable over $[0,\infty)$ for $\sigma>0$ we obtain

$$\int_0^\infty \|(1-\Pi_k)D(s)\|^2_{H^{r+1}}\,\mathrm{d}s \le C(\|v\|_{r,\sigma}+\|w\|_{r,\sigma})\|v-w\|_{r,\sigma}. \tag{2.7}$$

Combining (2.6) and (2.7) gives the desired bound

$$\|(1-\Pi_k)D\|_{r,\sigma} \le C(\|v\|_{r,\sigma} + \|w\|_{r,\sigma})\|v-w\|_{r,\sigma},$$

with C depending on k and σ and r.

Let us now bound $\|\Pi_k D(s)\|_{r,\sigma}$. Notice that $\Pi_k D(0) = 0$, so that Corollary 2.3 implies

$$\begin{aligned} e^{2\sigma s}\left\|\Pi_k D(s)\right\|^2_{H^r(\mathbf{S}^n)} &\le \frac{\lambda_k}{2(\lambda_k-\sigma)}\int_0^s e^{2\sigma\tau}\|\Pi_k\left[N(v(\tau))-N(w(\tau))\right]\|^2_{H^{r-1}(\mathbf{S}^n)}\,\mathrm{d}\tau \\ &\le \frac{\lambda_k}{2(\lambda_k-\sigma)}\int_0^s e^{2\sigma\tau}\|N(v(\tau))-N(w(\tau))\|^2_{H^{r-1}(\mathbf{S}^n)}\,\mathrm{d}\tau. \end{aligned}$$

To pass from the first line to the second we just use the fact that Π_k does not increase the H^{r-1} norm. Inserting the bilinear estimate for N into this we bound the integral as

$$\begin{aligned} &\int_0^s e^{2\sigma\tau}\|N(v(\tau))-N(w(\tau))\|^2_{H^{r-1}}\,\mathrm{d}\tau \\ &\qquad \le C\int_0^s e^{2\sigma\tau}\left(\|v(\tau)\|^2_{H^{r+1}}\|v(\tau)-w(\tau)\|^2_{H^r} + \|w(\tau)\|^2_{H^r}\|v(\tau)-w(\tau)\|^2_{H^{r+1}}\right)\,\mathrm{d}\tau, \end{aligned}$$

from which, using the definition of $\|\cdot\|_{r,\sigma}$, we straightforwardly obtain

$$\int_0^s e^{2\sigma\tau}\|N(v(\tau))-N(w(\tau))\|^2_{H^{r-1}}\,\mathrm{d}\tau \le C\left(\|v\|^2_{r,\sigma}+\|w\|^2_{r,\sigma}\right)\|v-w\|^2_{r,\sigma}.$$

Combining this with the H^r estimate for $D(0)$ we get

$$e^{2\sigma s}\|D(s)\|^2_{H^r(\mathbf{S}^n)} \le C\left(1+\frac{\lambda_k}{\lambda_k-\sigma}\right)\left(\|v\|^2_{r,\sigma}+\|w\|^2_{r,\sigma}\right)\|v-w\|^2_{r,\sigma}.$$

By Corollary 2.4 and an analogous use of the nonlinear estimate, we similarly obtain

$$\int_0^\infty \|\Pi_k D(s)\|^2_{H^{r+1}}\,\mathrm{d}s \le \int_0^\infty \|N(v(s))-N(w(s))\|^2_{H^{r-1}}\,\mathrm{d}s \le C\left(\|v\|^2_{r,\sigma}+\|w\|^2_{r,\sigma}\right)\|v-w\|^2_{r,\sigma}.$$

This completes the bound on $\|\Pi_k D(s)\|_{r,\sigma}$.

Combining all of these estimates gives us the final bound:

$$\|D\|_{r,\sigma} \leq \|(1-\Pi_k)D\|_{r,\sigma} + \|\Pi_k D\|_{r,\sigma} \leq C\left(\frac{\lambda_k}{\lambda_k - \sigma}\right)^{1/2} \left(\|v\|_{r,\sigma} + \|w\|_{r,\sigma}\right) \|v - w\|_{r,\sigma}.$$

This proves (2.4).

Now let us show that T maps a small ball centered at the origin in $X_{r,\sigma}$ into itself. Let $U(s) = e^{Ls}u_0$ be the solution to the linear homogeneous equation $(\partial_s - L)U = 0$ with initial data $U(0) = u_0$. First, taking $w = 0$ in (2.4) shows, since $T(0; u_0) = U$ by the definition (2.3) of T, that

$$\|T(v; u_0) - U\|_{r,\sigma} \leq C\|v\|^2_{r,\sigma}.$$

Therefore if $0 < \delta < 1/C$ and $\|U\|_{r,\sigma} < \delta - C\delta^2$, then $\|T(v; u_0)\|_{r,\sigma} < \delta$ whenever $\|v\|_{r,\sigma} < \delta$. That is, $T(\cdot; u_0)$ maps the ball of radius δ centered at zero in $X_{r,\sigma}$ into itself. We need only to show now that $\|U\|_{r,\sigma}$ can be controlled by $\|u_0\|_{H^r(S^n)}$. But this follows immediately from the estimates of Corollaries 2.3 and 2.4 since $U(s) \in F_k$ for all $s \geq 0$. ■

Chapter 3

Asymptotics of the limit

In the preceding section, we constructed, for each $k \geq 2$, a codimension d_k invariant submanifold for equation (1.6) consisting of solutions which converge to zero with exponential rate σ for every $\sigma < \lambda_k$. In this section, we show that any such solution must actually converge to zero with exponential rate λ_k, and we show also that any such solution is approximated well by a solution to the linear equation.

The first result is the following.

PROPOSITION 3.1. *Suppose $k \geq 2$ is an integer and $u : \mathbf{S}^n \times [s_0, \infty) \to \mathbb{R}$ is a solution to (1.6) which satisfies*

$$\sup_{s \geq s_0} e^{\sigma s} \| u(s) \|_{H^{r+2}(\mathbf{S}^n)} < \infty$$

for all $\sigma < \lambda_k$. Then

$$\sup_{s \geq s_0} e^{\lambda_k s} \| u(s) \|_{H^r(\mathbf{S}^n)} < \infty$$

and in fact there exists $P \in E_k$ such that

$$u(y, s) = e^{-\lambda_k s} P(y) + O\left(e^{-2\lambda_k s} + e^{-\lambda_{k+1} s} \right)$$

in $H^r(\mathbf{S}^n)$ as $s \to \infty$.

We now prove a lemma, showing that the first hypothesis of Proposition 3.1 is met automatically for all solutions of (1.6) satisfying

$$\sup_{s \geq s_0} e^{\sigma s} \|u(s)\|_{H^r(\mathbf{S}^n)} < \infty$$

for all $\sigma < \lambda_k$. Notice that Proposition 3.1 requires this condition to hold for the $H^{r+2}(\mathbf{S}^n)$ norm, and not just the $H^r(\mathbf{S}^n)$ norm. We show that it always holds in the $H^{r+2}(\mathbf{S}^n)$ norm if it holds in the $H^r(\mathbf{S}^n)$ norm.

Lemma 3.2. *Suppose $u\colon \mathbf{S}^n \times [s_0, \infty) \to \mathbb{R}$ is a solution to (1.6) converging to zero in $L^2(\mathbf{S}^n)$ as $s \to \infty$. Then either u is identically zero or*

$$\sup_{s \geq s_0} \frac{\|u(s)\|_{H^{r+1}(\mathbf{S}^n)}}{\|u(s)\|_{H^r(\mathbf{S}^n)}} < \infty$$

for every integer $r \geq 0$.

Proof. The crucial feature of rescaled mean curvature flow making this work is that a solution u to (1.6) converging to zero in $L^2(\mathbf{S}^n)$ also converges to zero in $H^r(\mathbf{S}^n)$ for every $r \geq 0$. This follows from Huisken's result, [Hui84] (see Remark (i) after Theorem 1.1), that convergence of a convex mean curvature flow to the sphere is exponential in C^k for any k. The rest of the proof uses generalities about the equation (1.6) satisfied by u.

Since u converges to zero in $H^r(\mathbf{S}^n)$ for every r, it lies in one of the invariant manifolds of Theorem 1.11, as proved in the preceding section. Moreover, u cannot converge to zero faster than any exponential unless it is identically zero, as proved in [Str18] (see Theorem 2.2). Therefore, if u is not identically zero, there is a largest integer $k = k(r) \geq 2$, depending on r, with the property that

$$\sup_{s \geq s_0} e^{\sigma s} \|u(s)\|_{H^r(\mathbf{S}^n)} < \infty$$

for all $\sigma < \lambda_k$. Since $\|\cdot\|_{H^{r+1}(\mathbf{S}^n)} \geq \|\cdot\|_{H^r(\mathbf{S}^n)}$, the integer $k(r)$ does not increase with r. This means that eventually $k(r)$ is constant in r, that is, there exists some r_0 such that $k(r) = k(r_0)$ for $r \geq r_0$.

Then for $r \geq r_0$ we can apply Proposition 3.1 to conclude that

$$\sup_{s \geq s_0} e^{\lambda_k s} \|u(s)\|_{H^r(\mathbf{S}^n)} < \infty,$$

where $k = k(r) = k(r_0)$, and that there exists $P \in E_k$ with the property that $\|e^{\lambda_k s} u(s) - P\|_{H^r(\mathbf{S}^n)} \leq Ce^{-\sigma s}$ for some $\sigma > 0$.

Now P must be nonzero, otherwise $e^{\sigma s}\|u(s)\|_{H^r(\mathbf{S}^n)}$ would be bounded for all $\sigma < \lambda_{k+1}$, and k would not be the largest integer with this property. (It now follows that $k = k(r)$ is the same for all $r \geq 0$ and not just all sufficiently large r.)

This is enough to conclude that $\|u(s)\|_{H^{r+1}(\mathbf{S}^n)}/\|u(s)\|_{H^r(\mathbf{S}^n)}$ is bounded in s for any r, since

$$\frac{\|u(s)\|_{H^{r+1}(\mathbf{S}^n)}}{\|u(s)\|_{H^r(\mathbf{S}^n)}} \leq \frac{e^{-\lambda_k s}\|P\|_{H^{r+1}(\mathbf{S}^n)} + C_1 e^{-(\lambda_k+\sigma)s}}{e^{-\lambda_k s}\|P\|_{H^r(\mathbf{S}^n)} - C_2 e^{-(\lambda_k+\sigma)s}}$$

for some constants $C_1, C_2, \sigma > 0$, and for s sufficiently large the expression on the right is bounded. ■

The proof of Proposition 3.1, to which we now turn, will be the consequence of a series of three lemmas in which we bound the projections of $u(s)$ onto $F_{k+1} = \bigoplus_{j \geq k+1} E_j$, onto E_k, and onto $F_k^\perp = \bigoplus_{j<k} E_j$, with E_j as always the eigenspace for $\Delta + 1$ corresponding to λ_j.

We begin by bounding the projection onto F_{k+1}.

Lemma 3.3. *Under the hypotheses of Proposition 3.1,*

$$\|\Pi_{k+1} u(s)\|_{H^r(\mathbf{S}^n)} = O(e^{-\lambda_{k+1} s} + e^{-2\sigma s})$$

as $s \to \infty$, *for any* $\sigma < \lambda_k$.

Proof. Notice that

$$\begin{aligned}\|\Pi_{k+1}u(s)\|_{H^r}\frac{\mathrm{d}}{\mathrm{d}s}\|\Pi_{k+1}u(s)\|_{H^r} &= \frac{\mathrm{d}}{\mathrm{d}s}\|\Pi_{k+1}u(s)\|_{H^r}^2/2\\ &= \langle\Pi_{k+1}u(s), Lu(s)\rangle_{H^r} + \langle\Pi_{k+1}u(s), N(u(s))\rangle_{H^r}\\ &\le -\lambda_{k+1}\|\Pi_{k+1}u(s)\|_{H^r}^2 + \|\Pi_{k+1}u(s)\|_{H^r}\|N(u(s))\|_{H^r}.\end{aligned}$$

Thus if $\mu \le \lambda_{k+1}$, we get

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(e^{\mu s}\|\Pi_{k+1}u(s)\|_{H^r}\right) \le e^{\mu s}\|N(u(s))\|_{H^r}. \tag{3.1}$$

On the other hand

$$\|N(u(s))\|_{H^r} \le C\|u(s)\|_{H^{r+1}}\|u(s)\|_{H^{r+2}} \le Ce^{-2\sigma s} \tag{3.2}$$

for any $\sigma < \lambda_k$ (the constant may depend on σ). The first inequality is the nonlinear estimate Lemma 2.5 and the second follows from the hypothesis of Proposition 3.1.

Inserting (3.2) into (3.1) and integrating shows that $e^{\mu s}\|\Pi_{k+1}u(s)\|_{H^r}$ is bounded provided $\mu \le \lambda_{k+1}$ and $\mu < 2\lambda_k$. This is the same as the conclusion of the lemma. ■

Next we bound the projection onto $F_k^{\perp}$.

Lemma 3.4. *In the situation of Proposition 3.1,*

$$\|(1-\Pi_k)u(s)\|_{H^r(\mathbf{S}^n)} = O(e^{-2\sigma s})$$

as $s \to \infty$, *for any* $\sigma < \lambda_k$.

Proof. Reasoning as in the proof of Lemma 3.3, we obtain

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s}\|(1-\Pi_k)u(s)\|_{H^r(\mathbf{S}^n)} &\geq -\lambda_{k-1}\|(1-\Pi_k)u(s)\|_{H^r(\mathbf{S}^n)} - C\|N(u(s))\|_{H^r} \\
&\geq -\lambda_{k-1}\|(1-\Pi_k)u(s)\|_{H^r(\mathbf{S}^n)} - Ce^{-2\sigma s}
\end{aligned}
$$

for any $\sigma < \lambda_k$. In other words,

$$
\frac{\mathrm{d}}{\mathrm{d}s}\left(e^{\lambda_{k-1}s}\|(1-\Pi_k)u(s)\|_{H^r}\right) \geq -Ce^{(\lambda_{k-1}-2\sigma)s}.
$$

Integrating this gives, for $s < t$ and $\sigma > \lambda_{k-1}/2$,

$$
\begin{aligned}
e^{\lambda_{k-1}t}\|(1-\Pi_k)u(t)\|_{H^r} &\geq e^{\lambda_{k-1}s}\|(1-\Pi_k)u(s)\|_{H^r} - Ce^{(\lambda_{k-1}-2\sigma)s} \\
&= e^{\lambda_{k-1}s}\left(\|(1-\Pi_k)u(s)\|_{H^r} - Ce^{-2\sigma s}\right).
\end{aligned}
$$

Now make $t \to \infty$. Because $\lambda_{k-1} < \lambda_k$, the hypothesis of Proposition 3.1 tells us that the left side converges to zero. Consequently the right side must be non-positive, or, in other words,

$$
\|(1-\Pi_k)u(s)\|_{H^r} \leq Ce^{-2\sigma s}
$$

for all $s \geq 0$ and any $\sigma < \lambda_k$. As far as we know, C depends on σ of course. ■

Finally, we turn to the projection onto E_k. Write $\pi_k = \Pi_k - \Pi_{k+1}$ for the projection of H^r onto E_k.

Lemma 3.5. *Under the hypotheses of Proposition 3.1 and assuming $s < t$,*

$$
\|e^{\lambda_k t}\pi_k u(t) - e^{\lambda_k s}\pi_k u(s)\|_{H^r(\mathbf{S}^n)} \leq Ce^{-\sigma s}
$$

for any $\sigma < \lambda_k$.

Proof. First,

$$\frac{\mathrm{d}}{\mathrm{d}s} e^{\lambda_k s} \pi_k u(s) = e^{\lambda_k s} \pi_k N(u(s))$$

since $L\pi_k u = -\lambda_k \pi_k u$. Now we integrate this (the equation is in a finite dimensional vector space, namely, the range of π_k) and use the triangle inequality to obtain

$$\begin{aligned} \|e^{\lambda_k t}\pi_k u(t) - e^{\lambda_k s}\pi_k u(s)\|_{H^r} &= \left\| \int_s^t e^{\lambda_k s} \pi_k N(u(\tau))\, \mathrm{d}\tau \right\|_{H^r} \\ &\leq \int_s^t e^{\lambda_k s} \|\pi_k N(u(\tau))\|_{H^r}\, \mathrm{d}\tau \\ &\leq C \int_s^t e^{(\lambda_k - 2\sigma)\tau}\, \mathrm{d}\tau \end{aligned}$$

for $s < t$ and any $\sigma < \lambda_k$. This implies the lemma. ■

An immediate corollary of Lemma 3.5 is that $e^{\lambda_k s}\pi_k u(s)$ converges in H^r norm exponentially fast. The limit of course must be an element of E_k, that is, an eigenfunction P of $\Delta + 1$ with eigenvalue λ_k.

Altogether, Lemmas 3.3, 3.4, and 3.5 imply that

$$u(y, s) = e^{-\lambda_k s} P(y) + O\left(e^{-\lambda_{k+1} s} + e^{-2\sigma s}\right)$$

in $H^r(\mathbf{S}^n)$ as $s \to \infty$, for any $\sigma < \lambda_k$. In particular, $\|u(s)\|_{H^r} \leq Ce^{-\lambda_k s}$ for some $C > 0$, and this fact can be used to improve the asymptotics and take $\sigma = \lambda_k$ in Lemmas 3.4 and 3.5 (but not Lemma 3.3) as follows. The appearance of σ came through H^r bounds on the nonlinear term $N(u(s))$ used in the proofs of Lemmas 3.3, 3.4, and 3.5: We could only say, based on the hypotheses of Proposition 3.1, that $\|N(u(s))\|_{H^r} \leq Ce^{-2\sigma s}$ for any $\sigma < \lambda_k$ and for some $C > 0$ depending on σ. But now that we know $\|u(s)\|_{H^r}$ (hence $\|u(s)\|_{H^{r+1}}$ and $\|u(s)\|_{H^{r+2}}$ by standard parabolic estimates using the fact that u is a solution to (1.6)) is actually $O(e^{-\lambda_k s})$, we can say that $N(u(s)) = O(e^{-2\lambda_k s})$ and then obtain improvements on the error in Lemmas 3.4 and 3.5.

A close look at the proof of Lemma 3.3 reveals that the same method does not work there and we are stuck with the appearance of σ in the conclusion. Of course it does not matter so long as $2\lambda_k > \lambda_{k+1}$, which is the case for most k.

We summarize these observations in a corollary, which states a more precise version of Proposition 3.1.

COROLLARY 3.6. *Under the hypotheses of Proposition 3.1:*

(a) The projection $\Pi_{k+1}u$ onto the sum F_{k+1} of eigenspaces corresponding to eigenvalues λ_j with $j > k$ satisfies

$$\|\Pi_{k+1}u(s)\|_{H^r(S^n)} \leq C\left(e^{-\lambda_{k+1}s} + e^{-2\sigma s}\right)$$

for any $\sigma < \lambda_k$ and some $C > 0$ (depending on σ) and all $s \geq s_0$.

(b) The projection $(1-\Pi_k)u$ onto the sum $F_k^\perp$ of eigenspaces corresponding to eigenvalues λ_j with $j < k$ satisfies

$$\|(1-\Pi_k)u(s)\|_{H^r(S^n)} \leq Ce^{-2\lambda_k s}$$

for some $C > 0$ and all $s \geq s_0$.

(c) The projection $\pi_k u$ onto the eigenspace E_k corresponding to the eigenvalue λ_j satisfies

$$\|e^{\lambda_k s}\pi_k u(s) - P\|_{H^r(S^n)} \leq Ce^{-\lambda_k s}$$

for some $P \in E_k$ and some $C > 0$ and all $s \geq s_0$.

We now obtain more precise asymptotics.

Lemma 3.7. *Suppose u satisfies the hypotheses of Proposition 3.1, and suppose $j \geq k$ and $\lambda_j < 2\lambda_k$. Then there exists P_j in the eigenspace E_j corresponding to λ_j such that*

$$\|e^{\lambda_j s}\pi_j u(s) - P_j\|_{H^r(\mathbf{S}^n)} \leq C e^{(\lambda_j - 2\lambda_k)s}$$

for all $s \geq s_0$.

Proof. We argue as in Lemma 3.5, using

$$\frac{\mathrm{d}}{\mathrm{d}s} e^{\lambda_j s}\pi_j u(s) = e^{\lambda_j s}\pi_j N(u(s))$$

to obtain

$$\|e^{\lambda_j t}\pi_j u(t) - e^{\lambda_j s}\pi_j u(s)\|_{H^r} \leq C \int_s^t e^{\lambda_j \tau}\|N(u(\tau))\|_{H^r}\,\mathrm{d}\tau \leq C \int_s^t e^{(\lambda_j - 2\lambda_k)\tau}\,\mathrm{d}\tau.$$

The right side is $O(e^{(\lambda_j - 2\lambda_k)s})$ independent of t provided that $\lambda_j < 2\lambda_k$, and this gives the conclusion of the lemma. ∎

From Lemma 3.7 we obtain higher order asymptotics in certain cases (when k is large).

Corollary 3.8. *Let u satisfy the hypotheses of Proposition 3.1. Then for $j \geq k$ such that $\lambda_j < 2\lambda_k$, there exists $P_j \in E_j$ such that*

$$u(y, s) = \sum_{\substack{j \geq k \\ \lambda_j < 2\lambda_k}} e^{-\lambda_j s} P_j(y) + O(e^{-2\sigma s})$$

for any $\sigma < \lambda_k$.

3.1 Prescribing the first-order asymptotics

In this section we prove Theorem 1.12. Given $a \in F_k$ sufficiently small in H^r for $r > n/2 + 1$, we constructed in Section 2.3 a unique solution $u(s; a)$ to (1.6) satisfying $\Pi_k u(0; a) = a$

and converging to zero with exponential rate λ_k. In the preceding subsection we showed that

$$P(a) = \lim_{s\to\infty} e^{\lambda_k s} u(s;a)$$

exists and is an element of the eigenspace E_k corresponding to λ_k. Here we will study the map $a \mapsto P(a)$. We will show that the image of this map contains a small ball centered at the origin in E_k. This is enough to conclude that every $P \in E_k$ is attained as the limit $e^{\lambda_k s}u(s)$ of some solution u to (1.6), because we can always replace u with $\tilde{u}(s) = u(s - s_0)$ for $s \geq s_0$ thereby scaling the limit by a factor $e^{\lambda_k s_0}$.

Actually, we do not even need to look at arbitrary $a \in F_k$ to obtain surjectivity: we may even restrict attention to $a \in E_k$. The precise result is the following:

PROPOSITION 3.9. *There exists $\delta > 0$ such that if $b \in E_k$ satisfies $\|b\|_r < \delta$, there exists $a \in E_k$ with $P(a) = b$.*

Proof. Let us first recall the norm $\|\cdot\|_{r,\sigma}$ from Section 2.3:

$$\|v\|_{r,\sigma} = \left(\int_0^\infty \|v(s)\|_{r+1}^2 \, \mathrm{d}s\right)^{1/2} + \sup_{s\geq 0} e^{\sigma s}\|v(s)\|_r.$$

It follows from Theorem 2.7 that if $\sigma < \lambda_k$ and $a \in F_k$ is sufficiently small in H^r, then

$$\|u(s;a) - e^{Ls}a\|_{r,\sigma} \leq C\|u(\cdot;a)\|_{r,\sigma}^2$$

for some constant $C > 0$ depending only on $\|a\|_{H^r}$. By making a smaller if necessary, we may moreover assume that $\|u(\cdot;a)\|_{r,\sigma} < 1/(2C)$ so that we get the bound

$$\|u\|_{r,\sigma} \leq 2\|e^{Ls}a\|_{r,\sigma} \leq 4\|a\|_{H^r}. \tag{3.3}$$

The last inequality is just an H^r estimate for the homogeneous linear equation. Thus we can bound $\|u\|_{r,\sigma}$ by a constant times $\|a\|_{H^r}$ for any $\sigma < \lambda_k$, provided that $\|a\|_{H^r}$ is small enough.

We now make use of the representation

$$e^{\lambda_k s}u(s;a) = e^{(\lambda_k+L)s}a + \int_0^s e^{(\lambda_k+L)(s-t)}e^{\lambda_k t}\Pi_k N(u(t;a))\,\mathrm{d}t - \int_s^\infty e^{(\lambda_k+L)(s-t)}e^{\lambda_k t}(1-\Pi_k)N(u(t;a))\,\mathrm{d}t,$$

which is valid for u because $e^{\lambda_k s}u(s;a)$ is bounded. By taking the H^r norm of both sides and applying the triangle inequality we deduce

$$\left\| e^{\lambda_k s}u(s) - e^{(\lambda_k+L)s}a \right\|_{H^r} \le \int_0^\infty e^{\lambda_k t}\|N(u(t))\|_{H^r}\,\mathrm{d}t \le C\int_0^\infty e^{\lambda_k t}\|u(t)\|_{H^{r+1}}\|u(t)\|_{H^{r+2}}\,\mathrm{d}t.$$

where in the last inequality we've used the nonlinear bound $\|N(u)\|_{H^r} \lesssim \|u\|_{H^{r+1}}\|u\|_{H^{r+2}}$ from Lemma 2.5. Now, the right side can be bounded by $\|u\|^2_{r+2,3\lambda_k/4}$, for example, as follows:

$$\int_0^\infty e^{\lambda_k t}\|u(t)\|_{H^{r+1}}\|u(t)\|_{H^{r+2}}\,\mathrm{d}t \le \int_0^\infty \left(e^{3\lambda_k t/4}\|u(t)\|_{H^{r+2}}\right)^2 e^{-\lambda_k/2}\,\mathrm{d}t \le \frac{2}{\lambda_k}\left(\sup_{t\ge 0} e^{3\lambda_k/4}\|u(t)\|_{H^{r+2}}\right)^2.$$

If we now assume that a is sufficiently small in H^{r+2} rather than in H^r and employ the bound (3.3) (with $r+2$ instead of r) we obtain

$$\left\| e^{\lambda_k s}u(s) - e^{(\lambda_k+L)s}a \right\|_{H^r} \le 4\|a\|^2_{H^{r+2}}.$$

On the left side we let $s \to \infty$. If $\pi_k a$ is the projection of a onto the eigenspace E_k, the result is

$$\frac{1}{\lambda_k^2}\|P(a) - \pi_k a\|_{H^{r+2}} = \|P(a) - \pi_k a\|_{H^r} \le 4\|a\|^2_{H^{r+2}}.$$

The first inequality is just the definition of H^r norm on the eigenspace E_k.

To finish the argument, we restrict attention to $a \in E_k$. For such a we have $\pi_k a = a$ and the foregoing estimate reduces to $\|P(a) - a\| \le C\|a\|^2$ for all sufficiently small a in E_k

(the norm is unimportant because E_k is finite-dimensional). This is enough to prove that the image of P contains a small ball in E_k centered at the origin.

Indeed, we run a standard contraction argument as in one direction of the proof of the inverse function theorem: if $b \in E_k$, we want to solve the fixed point equation $a = b - (P(a) - a)$. If $\delta < 1/(2C)$ and $0 < \|b\| \leq \delta - C\delta^2$, then the map F defined by $F(a) = b - (P(a) - a)$ sends the closed ball $\|a\| \leq \delta$ to itself. Indeed if $\|a\| \leq \delta$ then

$$\|F(a)\| = \|b - (P(a) - a)\| \leq \|b\| + \|P(a) - a\| \leq \delta - C\delta^2 + C\delta^2 = \delta.$$

On the other hand, F depends continuously on a (this follows from the proof of Theorem 2.7) and so, being a continuous mapping of a closed ball into itself, it must have a fixed point a, that is, a solution to $P(a) = b$. ∎

Chapter 4

A unique continuation property

In this section we prove a unique continuation property for the level set equation. Let Ω be a mean-convex domain in $\mathbb{R}^{n+1}$. When $\Omega = B_r(x_0)$ is the ball of radius r centered at the point x_0 in $\mathbb{R}^{n+1}$, the solution t_B to the level set equation (1.1) is

$$t_B(x) = \frac{r^2}{2n} - \frac{|x - x_0|^2}{2n} \tag{4.1}$$

and the corresponding mean curvature flow is a family of homothetically shrinking spheres. The main result is that a solution to the level set equation (1.1) on a mean-convex domain Ω that attains its maximum at x_0 and agrees to infinite order at the point x_0 with the solution t_B to the level set equation for a ball centered at x_0 must actually coincide with t_B everywhere.[1]

THEOREM 4.1. *Suppose $\Omega \subset \mathbb{R}^{n+1}$ is a mean-convex domain and $t : \Omega \to \mathbb{R}$ solves (1.1) and attains its maximum T at the point $x_0 \in \Omega$. If*

$$t(x) = T - \frac{|x - x_0|^2}{2n} + O\left(|x - x_0|^N\right) \tag{4.2}$$

as $x \to x_0$, for every integer $N > 2$, then $t(x) = T - |x - x_0|^2/(2n)$ and $\Omega = B_{(2nT)^{1/2}}(x_0)$ is the ball of radius $(2nT)^{1/2}$ centered at x_0.

[1] A different and more complicated *parabolic* unique continuation property for self-shrinkers was recently proved by Jacob Bernstein in [Ber17].

Remarks.

1. Let Ω be a mean-convex domain and suppose $t : \Omega \to \mathbb{R}$ satisfies (1.1). We already mentioned that t is twice differentiable on its domain (this was proved in [CMI16]).[2] The Hessian of t at a critical point x_0 is then

$$\nabla^2 t(x_0) = -\frac{1}{k}\Pi_{k+1},$$

where k is an integer between 1 and n and Π_{k+1} is a projection onto a $k+1$ dimensional hyperplane through x_0. See [CMI18]. The hypothesis of Theorem 4.1 implies that $k = n$. In this case t is actually C^2 in a neighborhood of the maximum, and the corresponding mean curvature flow becomes extinct at x_0 in such a way that the rescaled mean curvature flow converges to a round sphere. In particular, the isolated point $\{x_0\}$ is a connected component of the level set $\{x : t(x) = T\}$, and nearby level sets are convex. Because a mean curvature flow cannot coincide with a sphere at any time unless it is a shrinking sphere, it is therefore sufficient to prove Theorem 4.1 in case Ω is a convex domain.

2. The hypothesis (4.2) for a fixed $N > 2$ likely implies (for a solution to (1.1)) that t is C^{N-1} near x_0. This is known in case $N = 3$ (Theorem 6.1 of [Hui93]) and in case $N = 4$ (Corollary 5.1 of [Ses08]). This fact is not required for our result, however, and we do not investigate it here. Of course, Theorem 4.1 implies that t is analytic if it satisfies (4.2) for *all* N.

4.1 Proof of Theorem 4.1

To prove Theorem 4.1, we relate the asymptotic behavior of t near its maximum to the behavior of the corresponding mean curvature flow near its singularity. We show that the hypothesis (4.2) for all $N > 2$ implies that the rescaled mean curvature flow converges to a

[2] It was shown in [Whi00] (Theorem 1.2) and [Whi11] that any tangent flow of a smooth mean convex mean curvature flow is a generalized cylinder. From this one can figure out what the Hessian of the arrival time function must be if it exists. The remaining issue was to show that the Hessian exists, which is equivalent to the problem of uniqueness of tangent flows. This was solved in [CMI15].

stationary round sphere at a rate that is faster than any exponential, and then that this cannot happen unless the rescaled flow is identically equal to the stationary sphere. As mentioned in the remarks following the theorem, it is sufficient to prove Theorem 4.1 in case Ω is a convex domain, and we restrict attention to this case from here on.

The following lemma converts Theorem 4.1 to a problem about functions u satisfying this parabolic equation: if t is a solution to the level set equation satisfying the assumption (4.2) for all N, then the corresponding solution to rescaled MCF converges to $\mathbf{S}^n$ faster than any exponential. The problem is then reduced to showing that this cannot happen unless the rescaled MCF is identically $\mathbf{S}^n$.

Note we may assume by translating everything that $x_0 = 0$. Thus we state the lemma for the case when the mean curvature flow becomes extinct at the origin.

LEMMA **4.2.** *Suppose $\Omega \subset \mathbb{R}^{n+1}$ is a convex domain and $t: \Omega \to \mathbb{R}$ solves (1.1) and attains its maximum T at the origin. Suppose*

$$t(x) = T - \frac{|x|^2}{2n} + O\left(|x|^N\right) \tag{4.3}$$

as $x \to 0$, for every integer $N > 2$. Then for any integer $k \geq 0$, the rescaled MCF $\{\Sigma_s\}$ that t defines converges to $\mathbf{S}^n$ in the Sobolev space H^k faster than any exponential.

Proof. We use the following fact: if $\{\Sigma_s\} = \{x + u(x,s)\mathbf{n}(x) : x \in \mathbf{S}^n\}$ is a rescaled MCF converging as $s \to \infty$ to the sphere in L^2, that is, for which u converges to zero in $L^2(\mathbf{S}^n)$, then u is bounded in $H^k(\mathbf{S}^n)$ for every $k \geq 1$ (actually u converges exponentially in H^k). On account of the interpolation inequalities

$$\|u\|_{H^k} \leq \|u\|_{L^2}^{1/2} \|u\|_{H^{2k}}^{1/2},$$

this reduces the proof of the lemma to showing that the function u converges to zero faster than any exponential in $L^2(\mathbf{S}^n)$.

Now we show that (4.3) holding for all N implies that $u(s)$ converges to zero faster than any exponential in $L^\infty(\mathbf{S}^n)$, hence in $L^2(\mathbf{S}^n)$.

First observe: if (4.3) holds for some N, then

$$|x|^2 = (2n)(T - t(x)) + O\left((T - t(x))^{N/2}\right)$$

as $T - t \to 0$, for the same N. Next, use this to write

$$\begin{aligned}(T - t(x))^{-1/2}x &= \left(|x|^2/(2n) + O\left((T - t(x))^{N/2}\right)\right)^{-1/2} x \\ &= (2n)^{1/2}\frac{x}{|x|} + O\left(\frac{(T - t(x))^{N/2}}{|x|^2}\right)\frac{x}{|x|} \\ &= (2n)^{1/2}\frac{x}{|x|} + O\left((T - t(x))^{N/2-1}\right)\frac{x}{|x|}.\end{aligned}$$

This says exactly that if $s = -\log(T - \tau)$ and

$$\{y + u(y,s)\mathbf{n}(y) : \ y \in \mathbf{S}^n\} = \Sigma_s = \left\{(T - \tau)^{-1/2}x : \ t(x) = \tau\right\}$$

is the rescaled MCF defined by t (it will be a graph over $\mathbf{S}^n$ for τ sufficiently close to T), then

$$|u(y,s)| = O\left((T - \tau)^{N/2-1}\right) = O\left(e^{-(N/2-1)s}\right)$$

as $s \to \infty$. Therefore if (4.3) holds for all N, then u converges to zero faster than any exponential in $L^\infty(\mathbf{S}^n)$. Since $\mathbf{S}^n$ has finite volume, u converges to zero faster than any exponential in $L^2(\mathbf{S}^n)$ as well. ■

Having established Lemma 4.2, Theorem 4.1 is a consequence of the following theorem.

Theorem 4.3. *Let $r > n/2 + 1$ be an integer. Suppose $\Sigma_s = \{y + u(y,s)\mathbf{n}(y) : \ y \in \mathbf{S}^n\}$, $s \geq 0$, is a rescaled MCF and suppose that it converges to $\mathbf{S}^n$ in $H^r(\mathbf{S}^n)$ faster than any* *exponential in the sense that*

$$\lim_{s\to\infty} e^{\sigma s}\|u(\cdot, s)\|_{H^r(\mathbf{S}^n)} = 0$$

for all $\sigma > 0$. Then u is identically zero and $\Sigma_s = \mathbf{S}^n$ for all s.

The equation satisfied by u in order for the normal graph $\Sigma_s = \{y + u(y, s)\mathbf{n}(y) : \ y \in \mathbf{S}^n\}$ to evolve by rescaled MCF can be written in the form

$$\partial_s u = \Delta u + u + N(u, \nabla u, \nabla^2 u) \tag{4.4}$$

where Δ is the Laplacian on $\mathbf{S}^n$ and N is a nonlinear term of the form

$$N(u, \nabla u, \nabla^2 u) = f(u, \nabla u) + \mathrm{trace}(B(u, \nabla u)\nabla^2 u), \tag{4.5}$$

where f and B are smooth, $B(0, 0) = f(0, 0) = \mathrm{d}f(0, 0) = 0$. In other words, N vanishes up to quadratic error at zero.

The important feature of this equation is that the linear operator $\partial_s - \Delta - 1$ gives a good approximation to the nonlinear operator in (4.4): in a Sobolev space $H^r(\mathbf{S}^n)$ of high enough order r, a function u the normal graph of which evolves by rescaled MCF satisfies

$$\|(\partial_s - \Delta - 1)u\|_{H^r(\mathbf{S}^n)} \leq C\|u\|_{H^{r+1}(\mathbf{S}^n)}\|u\|_{H^{r+2}(\mathbf{S}^n)}.$$

This bound is what implies that if u converges to zero as $s \to +\infty$, it must do so at an exponential rate (unless it is identically zero). It follows immediately from Lemma 2.5.

Using Lemma 2.5, we now prove Theorem 4.3.

Proof of Theorem 4.3. Throughout the proof, we abbreviate the $H^r(\mathbf{S}^n)$ norm $\|\cdot\|_{H^r(\mathbf{S}^n)}$ by $\|\cdot\|_r$, we abbreviate $u(\cdot, t)$ by $u(t)$, and we abbreviate the nonlinear error $N(u, \nabla u, \nabla^2 u)$ by $N(u)$. Let $\lambda_1 < \lambda_2 < \cdots$ be the eigenvalues of the operator $-\Delta - 1$, and denote by Π_k orthogonal projection onto the direct sum of eigenspaces corresponding to λ_j with $j \geq k$.

We will prove that a solution u satisfying (4.4) obeys, for each positive integer k and each $s_0 \geq 0$, the inequality

$$e^{\lambda_k(s-s_0)}\|u(s)\|_r \leq \|\Pi_k u(s_0)\|_r + \int_{s_0}^{\infty} e^{\lambda_k(t-s_0)}\|N(u(t))\|_r \,\mathrm{d}t. \tag{4.6}$$

Lemma 2.5 implies that there is a constant C depending on N with the property that

$$\|N(u)\|_r \leq C\|u\|_{r+1}\|u\|_{r+2} \leq C\|u\|_{r+2}^2 \leq C\|u\|_r\|u\|_{r+4}.$$

The last inequality follows from Cauchy–Schwarz and integration by parts for example.

It then follows that

$$\begin{aligned} e^{\lambda_k(s-s_0)}\|u(s)\|_r &\leq \|\Pi_k u(s_0)\|_r + C\int_{s_0}^{\infty} e^{\lambda_k(t-s_0)}\|u(t)\|_r\|u(t)\|_{r+4}\,\mathrm{d}t \\ &\leq \|\Pi_k u(s_0)\|_r + C\left(\sup_{t\geq s_0} e^{\lambda_k(t-s_0)}\|u(t)\|_r\right)\int_{s_0}^{\infty}\|u(t)\|_{r+4}\,\mathrm{d}t. \end{aligned}$$

Taking the supremum over $s \geq s_0$ on the left side then gives

$$\left(\sup_{t\geq s_0} e^{\lambda_k(t-s_0)}\|u(t)\|_r\right)\left(1 - C\int_{s_0}^{\infty}\|u(t)\|_{r+4}\,\mathrm{d}t\right) \leq \|\Pi_k u(s_0)\|_r$$

for all k. Because convergence to the sphere is necessarily exponential in C^k for every k, the $H^{r+4}(\mathbf{S}^n)$ norm $\|u(t)\|_{r+4}$ is integrable (as are all other H^s-norms), and therefore we can choose s_0 so large that

$$C\int_{s_0}^{\infty}\|u(t)\|_{r+4}\,\mathrm{d}t \leq 1/2. \tag{4.7}$$

Moreover, we choose s_0 to be the least nonnegative number for which (4.7) holds. For this s_0 we obtain

$$\sup_{t\geq s_0} e^{\lambda_k(t-s_0)}\|u(t)\|_r \leq 2\|\Pi_k u(s_0)\|_r$$

for all positive integers k, and since the right side vanishes in the limit $k \to \infty$ and the left side is non-decreasing in k it follows that $\|u(t)\|_r = 0$ for $t \geq s_0$.

Now we show that $s_0 = 0$. Since $\|u(t)\|_r = 0$ for $t \geq s_0$, we also have $\|u(t)\|_{r+4} = 0$ for $t \geq s_0$. Consequently,

$$C \int_{s_0}^{\infty} \|u(t)\|_{r+4} \, \mathrm{d}t = 0.$$

If $s_0 > 0$, then it cannot possibly be the smallest positive number for which (4.7) holds, and we arrive at a contradiction. Thus u is in fact identically zero for all $s \geq 0$.

Therefore it is enough to prove (4.6). Write $L = \Delta + 1$ for brevity. We will briefly explain how the assumption that u converges to zero faster than any exponential leads to the following representation formula:

$$\begin{aligned} e^{\lambda_k(s-s_0)}u(s) = e^{(\lambda_k+L)(s-s_0)}\Pi_k u(s_0) + \int_{s_0}^{s} e^{(\lambda_k+L)(s-t)}e^{\lambda_k(t-s_0)}\Pi_k N(u(t)) \, \mathrm{d}t \\ - \int_{s}^{\infty} e^{(\lambda_k+L)(s-t)}e^{\lambda_k(t-s_0)}(1-\Pi_k)N(u(t)) \, \mathrm{d}t, \end{aligned}$$

Notice that $\lambda_k + L$ is non-positive definite on the range of Π_k and positive definite on the range of $1-\Pi_k$. Thus (4.6) follows by simply taking the H^r norm of both sides and applying the triangle inequality repeatedly to the right side.

The representation can be derived from the variation of constants formula

$$u(s_1) = e^{L(s_1-s)}u(s) + \int_{s}^{s_1} e^{L(s_1-t)}N(u(t)) \, \mathrm{d}t,$$

where $s_1 \geq s$, in the following way.[3] Apply the projection $1 - \Pi_k$ to both sides of the variation of constants formula. Apply $e^{-L(s_1-s)}$, which is a bounded operator on the finite-

[3]This is a standard trick in the construction of invariant manifolds for ODE.

dimensional image of $1 - \Pi_k$, to both sides and rearrange to obtain

$$(1 - \Pi_k)u(s) = e^{-L(s_1 - s)}(1 - \Pi_k)u(s_1) - \int_s^{s_1} e^{L(s-t)}(1 - \Pi_k)N(u(t))\,\mathrm{d}t.$$

Since $e^{-L(s_1-s)}$ is bounded by $e^{\lambda_{k-1}(s_1-s)}$ on the image of $1 - \Pi_k$, we can send $s_1 \to \infty$ and by our assumption that u vanishes more rapidly than any exponential, the term $e^{-L(s_1-s)}(1 - \Pi_k)u(s_1)$ converges to zero. The integral, meanwhile, is absolutely convergent. Thus we obtain

$$(1 - \Pi_k)u(s) = -\int_s^{\infty} e^{L(s-t)}(1 - \Pi_k)N(u(t))\,\mathrm{d}t,$$

for any k. Substituting this equation into the variation of constants formula (with s_1 replaced by s and s replaced by s_0) gives the representation described above. ∎

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