A restriction estimate in $\mathbb{R}^3$

by

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Abstract

In this thesis, I proved a restriction estimate for paraboloid in $\mathbb{R}^3$ based on the polynomial partitioning method introduced by Larry Guth and the "two ends argument" introduced by Wolff and Tao.

Thesis Supervisor: Larry Guth
Title: Professor

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Contents

1 Introduction **9**

1.1 Notations 14

2 Preliminary **15**

2.1 Wave packet decomposition 15

2.2 Broad L^p–norm 16

2.3 Some basic reductions. 17

2.4 Polynomial partitioning in [2] 17

3 A white lie version of the proof of Theorem 2 **27**

3.1 Tangential case in the cells 28

3.2 Analysis of the broom 35

4 Polynomial Structure lemma **55**

5 Two ends argument and some easy cases **61**

6 Estimate about the L^2–norm **67**

6.1 Planes 68

6.2 Brooms 72

7 Proof of Theorem 4 **85**

Chapter 1

Introduction

My research is focused on the restriction theory, which is about functions whose Fourier transform supported on some curved submanifold in $\mathbb{R}^n$. One example of those functions is the following.

Definition 1.1. *Given a complex valued function f supported in the unit disk in $\mathbb{R}^{n-1}$, we define*

$$Ef(x_1,\dots,x_n):=\int_{|\xi|\le 1} e^{i(\xi_1 x_1+\cdots+\xi_{n-1}x_{n-1}+|\xi|^2 x_n)} f(\xi)d\xi.$$

The function Ef is a typical object of study in the restriction theory, whose Fourier transform in $\mathbb{R}^n$ is supported on the truncated paraboloid $\{(\xi,|\xi|^2),|\xi|\le 1\}$. Elias Stein [6] made the following restriction conjecture about Ef in the 1960s.

Conjecture 1.2 (Stein [6]). *For any $p>\frac{2n}{n-1}$ and $q\ge\frac{(n+1)q'}{n-1}$, we have*

$$\|Ef\|_{L^p(\mathbb{R}^n)}\le C_{p,q}\|f\|_{L^q(\mathbb{R}^{n-1})}. \tag{1.1}$$

Stein's restriction conjecture is an open problem about the L^p–norm of Ef. We refer to a survey of Tao [10] for a good presentation of Stein's conjecture. The conjecture in $\mathbb{R}^2$ was proven by Fefferman and Cordoba. The conjecture in $\mathbb{R}^3$ has been studied by various mathematicians but has not been proven for all p and q. We are particularly interested in the case where $q=\infty, p>\frac{2n}{n-1}$ since it implies other cases by the Stein-Nikishin factorization theorem. From now on, we focus on the three dimensional case with $p > 3, q = \infty$.

Various mathematicians have studied this problem and different techniques were introduced and lead to development of harmonic analysis. In 1975, Tomas and Stein [12] [7] proved for $p \geq 4$ using TT^* method, which leads to Strichartz estimates in dispersive PDEs. In 1991, Bourgain [1] made progress on the Kakeya problem and showed how to transfer it into the progress on restriction. Around 2000, Wolff [13] and Tao [9] introduced a new method called the two ends argument and made a big improvement on restriction estimate for $p > \frac{10}{3}$. In 2014, Guth [2] introduced polynomial partitioning techniques to better exploit the L^∞–norm on the righthandside and proved for $p > 3.25$.

We strengthen the result of Guth[2] about the restriction conjecture in $\mathbb{R}^3$. In this thesis, we study the conjecture in $\mathbb{R}^3$ and for $q = \infty$, which asks if

$$\|Ef\|_{L^p(\mathbb{R}^3)} \leq C(p,S)\|f\|_{L^\infty(\mathbb{R}^2)} \tag{1.2}$$

for all $p > 3$.

We provide a small improvement and prove for the range $p > 3 + 3/13$ based on the polynomial partitioning techniques and the "two ends argument".

Theorem 1. *The restriction estimate 1.2 holds for all $p > 3 + 3/13$.*

To prove his estimate, Guth [2] splits Ef into a narrow part and a broad part. The narrow part can be estimated using induction on scales. The broad part, which is more difficult, is handled by an estimate of $\|Ef\|_{BL^p(B_R)}$, the broad L^p–norm of Ef inside a large ball B_R. In this paper, we prove a stronger estimate about $\|Ef\|_{BL^p(B_R)}$ in Theorem 2. Obtaining Theorem 1 from Theorem 2 requires no modification to Guth's argument, so we omit the proof of Theorem 1 from this paper.

Theorem 2. *For any small $\epsilon > 0$, there exists a large constant C_ϵ depending only on ϵ such that for any large ball B_R of radius R and any $p > 3 + 3/13$,*

$$\|Ef\|_{BL^p(B_R)} \leq C_\epsilon R^\epsilon \|f\|_{L^2(\mathbb{R}^2)}^{2/p} \|f\|_{L^\infty(\mathbb{R}^2)}^{1-2/p}.$$

The broad L^p–norm of Ef is introduced in Guth's papers [2] and [3] to capture the difficult part of $\|Ef\|_{L^p}$. We give its full description in chapter 2. One could think of $\|Ef\|_{BL^p(B_R)}$ as $\|Ef\|_{L^p(B_R)}$ for most of this paper and especially in the introduction.

We begin by highlighting some of Guth's arguments to illustrate how our idea grows out of Guth's ideas. The argument of Guth relies on induction on both scale (the radius of the spatial ball B_R) and the L^2 mass $\|f\|_{L^2}$. His argument consists of a one-time application as opposed to a multiple iteration of the following trichotomy. As a result of the polynomial partitioning, the term Ef can be split into a cellular, a transverse and a tangential term. The cellular and the transversal contribution is estimated similarly, using the induction hypothesis on mass and radius. The tangent term is estimated directly, without appealing to induction.

The unconditional estimate for the tangent term remains favorable if the induction hypothesis is changed to accommodate this paper's Theorem 2, which is reflected in reducing the exponent weight on L^2–mass. However, in this new setup, the estimate for the cellular part is no longer an immediate consequence of induction on radius. Most of the novelty of this paper goes into finding a new way to deal with the cellular contribution. Our argument contains a multi-step iteration. And the scale of the wave packets changes throughout the iteration process. Essentially, at each step we split the cellular component into two parts, using Wolff's "two ends argument": a local part and a global part. The local part will be estimated using induction on the radius. The global part of Ef is mainly concentrated in thin neighborhoods of algebraic varieties and needs a delicate analysis that forces us to introduce a new geometric object we call "broom". A large part of the argument in this paper goes into quantifying how brooms interact with thin neighborhoods of algebraic varieties.

To describe a broom, we recall the wave packet decomposition of Ef introduced by Bourgain. The wave packet decomposition says that inside a large ball of radius R, we can decompose Ef into a sum over wave packets $Ef_{\theta,v}$. Each wave packet $Ef_{\theta,v}$ is essentially supported in a tube $T_{\theta,v}$ of length R, radius $R^{1/2}$. The axis of $T_{\theta,v}$ points in a direction depending only on θ and the location of $T_{\theta,v}$ is described by v. The absolute value of a wave packet $|Ef_{\theta,v}|$ is approximately a constant function on

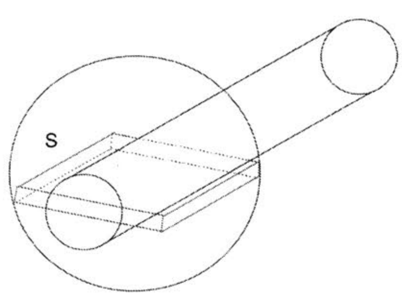
$T_{\theta,v}$.

Roughly speaking, a broom is a collection of large wave packets $Ef_{\theta,v}$ satisfying certain properties with respect to a thin neighborhood of a plane. More precisely, for some $r < R$ let S denote the $r^{1/2}$ neighborhood of a plane Σ inside a ball of radius r. The dimensions of S are $r \times r \times r^{1/2}$.

In particular, S is an example of the thin neighborhood of an aforementioned algebraic variety where Ef is concentrated on. For simplicity, we assume in the introduction that every such thin neighborhood is in the form of S. Now we are ready to give a better description of a broom with respect to S as illustrated in Figure 1-1. A broom consists of wave packets which

- intersect at a common point on S;
- lie inside the $R^{1/2}$–neighborhood of a plane perpendicular to S;
- form an angle with S of at most $r^{-1/2}$.

The key ingredient of our proof is an improved estimate about the L^2–mass of Ef on a typical S compared to the L^2–mass of Ef on the ball B_r containing S. We explain using two examples how brooms help to obtain such an estimate. The first example is when a typical broom contains only one wave packet $Ef_{\theta,v}$. Since $|Ef_{\theta,v}|$ is approximately a constant function on $T_{\theta,v}$, its L^2–mass is evenly distributed on $T_{\theta,v}$. So the L^2–mass contained in S is only a small proportion because the volume of S is much smaller than the volume of $T_{\theta,v} \cap B_r$.



The second example is when a typical broom consists of a large number of wave packets. Since the wave packets in a broom intersect in S on one end and spread

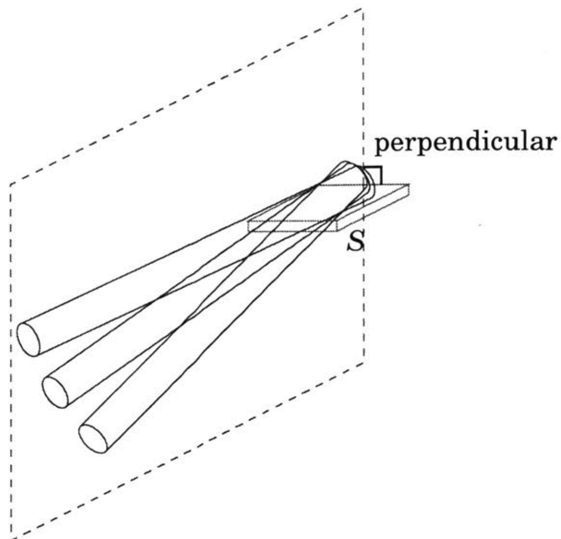
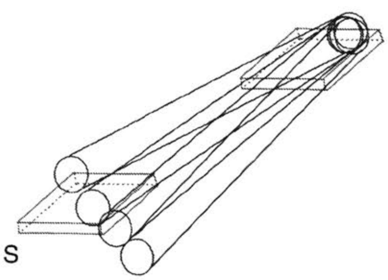


Figure 1-1: a broom

out on the other end, a typical S' on the other end intersects the broom in only a few wave packets, which implies that the L^2–mass of Ef on S' is small because many wave packets miss S'.



The rest of the thesis is organized in the following way. In Chapter 2, we recall Guth's proof for $p > 3.25$ in [2] using one time polynomial partitioning, which is the starting point of our proof. We apply the polynomial partitioning iteratively and prove a polynomial structure lemma, Lemma 4.3, which says that Ef is mainly concentrated on thin neighborhoods of a collection of algebraic surfaces. For simplicity, we assume those algebraic surfaces are planes and provide a "white lie" proof of Theorem 2 in Chapter 3. The "white lie" proof contains the main idea and is close to my initial thoughts about this problem.

The rest of the paper is devoted to the proof of Theorem 2. In Chapter 4, we prove the aforementioned polynomial structure lemma. In Chapter 5, we apply the "two ends" argument to reduce to the study of the global part of Ef. In Chapter 6, we estimate the global part using the broom structure. More precisely, in Section 6.1 we prove a geometric lemma, Lemma 6.4, saying that the algebraic surfaces in Lemma 4.3 can be viewed as planes. In Section 6.2, we define the broom structure from those planes and estimate the global part of Ef using brooms. Finally, in Chapter 7, we conclude the proof using lemmas in previous chapters.

1.1 Notations

If X is a finite set, we use $|X|$ to denote its cardinality; if X is a measurable set, we use $|X|$ to denote its Lebesgue measure. We use $d(\theta)$ to denote the diameter of a small disk or the diameter of a small cap on a paraboloid. We use B_r to denote a ball of radius r. We use $A \lesssim B$ or $A = O(B)$ to denote the estimate $A \le CB$ where C is an absolute constant. We use $A \ll B$ to denote the estimate $A \le B^\epsilon$ for some arbitrarily small $\epsilon > 0$. For a dyadic number λ, when we say a quantity $A \sim \lambda$, we mean $\lambda < A \le 2\lambda$.

Chapter 2

Preliminary

We first recall the wave packet decomposition and the definition of BL^p–norm. Then we proceed to the proof sketch of Guth's result.

2.1 Wave packet decomposition

Definition 2.1. *We decompose the unit disk into finitely overlapping small disks θ of radius $R^{-1/2}$. We can do wave packet decomposition of Ef in B_R, a ball of radius R in $\mathbb{R}^3$, according to the covering $\{\theta\}$:*

$$Ef = \sum_{\theta,v} Ef_{\theta,v}.$$

Each $Ef_{\theta,v}$ is a wave packet *essentially supported on a tube $T_{\theta,v}$ of length R, radius $R^{1/2}$, with its direction in $G(\theta) = \{(-2\xi, 1) \mid \xi \in \theta\}$.*

Since the directions in $G(\theta)$ are within a range of $R^{-1/2}$, for a fixed θ and v, any direction in $G(\theta)$ defines essentially the same tube.

The wave packets $Ef_{\theta,v}$ are the building blocks for Ef satisfying two properties. The first property is the so-called "locally constant" property: the absolute value $|Ef_{\theta,v}|$ is essentially a constant function on $T_{\theta,v}$. The second property is the orthogonality, which is due to the oscillation of wave packets: $\sum_{\theta,v} \|f_{\theta,v}\|_{L^2}^2 \approx \|f\|_{L^2}^2$. We build our heuristics and proof on these two properties.

2.2 Broad L^p–norm

We recall the broad L^p–norm $\|Ef\|_{BL^p(B_R)}$ defined in Guth's papers [2] and [3].

We decompose the unit disk into finitely overlapping small disks α of radius K^{-1}, where $1 \ll K \ll R^\epsilon$. We decompose $f = \sum_\alpha f_\alpha$ by partition of unity with f_α supported in α. The wave packets in Ef_α are those $Ef_{\theta,v}$ with $\theta \subset \alpha$.

Now we are ready to recall the definition of the broad L^p–norm. For each small ball B_K, with $1 \ll A \ll K^\epsilon$, we define

$$\mu_{Ef}(B_K) := \min_{V_1,\dots,V_A:\text{ lines of } \mathbb{R}^3} \Big(\max_{\alpha:\text{Angle}(G(\alpha),V_a)\ge K^{-1} \text{ for all } 1\le a\le A} \int_{B_K} |Ef_\alpha|^p \Big), \tag{2.1}$$

where $G(\alpha) = \{(-2\xi,1) \mid \xi \in \alpha\}$ is defined in the same way as $G(\theta)$. If a set U is (finitely overlapping) tiled by B_K, then we define $\mu_{Ef}(U)$ by summing over the tiling: $\mu_{Ef}(U) = \sum_{B_K \subset U} \mu_{Ef}(B_K)$.

We define the broad L^p–norm on B_R as:

$$\|Ef\|^p_{BL^p_A(B_R)} = \mu_{Ef}(B_R).$$

Usually we neglect the parameter A unless it plays a role in the proof.

The broad L^p–norm is closely related to the bilinear norm, which was studied by Tao [9]. By the uncertainty principle, the absolute value $|Ef_\alpha|$ is essentially a constant on any B_K. Hence, locally

$$\mu_{Ef}(B_K) \le \max_{\alpha_1,\alpha_2 \text{ nonadjacent}} \int_{B_K} |Ef_{\alpha_1} Ef_{\alpha_2}|^{p/2} \le \sum_{\alpha_1,\alpha_2 \text{ nonadj}} \int_{B_K} |Ef_{\alpha_1} Ef_{\alpha_2}|^{p/2}$$

and the BL^p–norm can be bounded by the sum over bilinear norms:

$$\|Ef\|^p_{BL^p(B_R)} \le \sum_{\alpha_1,\alpha_2 \text{ nonadjacent}} \|(Ef_{\alpha_1} Ef_{\alpha_2})^{1/2}\|^p_{L^p(B_R)}. \tag{2.2}$$

One can also view the BL^p–norm as approximately an L^p–norm with broadness: if f is supported inside a small cap of radius K^{-1}, then $\|Ef\|_{BL^p(B_R)} = 0$. Similar to

L^p–norm, the usual Hölder's inequality and triangle inequality hold for BL^p–norm with a modification of A, we refer to Guth's paper[2] for a proof of these inequalities.

2.3 Some basic reductions.

We sort the wave packets $Ef_{\theta,v}$ according to the size of $\|f_{\theta,v}\|_{L^2}$. Since there are at most R^2 wave packets, the contribution from those wave packets with $\|f_{\theta,v}\|_{L^2} \le R^{-10}$ satisfies the inequality in Theorem 2. So it suffices to consider the wave packets with $\|f_{\theta,v}\|_{L^2} \in [R^{-10}, 1]$. For each dyadic λ between R^{-10} and 1, let Ef_λ denote the sum over wave packets with $\|f_{\theta,v}\|_{L^2} \sim \lambda$.

Since there are $\lesssim \log R$ choices of λ, there exists a λ_0 such that

$$\|Ef_{\lambda_0}\|_{BL^p(B_R)} \gtrsim (\log R)^{-1} \|Ef\|_{BL^p(B_R)}.$$

From now on, we assume $Ef = Ef_{\lambda_0}$. In particular, each wave packet in Ef is either zero or satisfies $\|f_{\theta,v}\|_{L^2} \sim \lambda_0$. We use $\mathbb{T}_0$ to denote the collection of tubes $T_{\theta,v}$ with nonzero $\|f_{\theta,v}\|_{L^2}$.

2.4 Polynomial partitioning in [2]

In this section, we provide a proof sketch of Guth's theorem [2].

Theorem 3. *If f is supported on the unit disk in $\mathbb{R}^2$, then for any small $\epsilon > 0$, there exists a large constant C_ϵ depending only on ϵ such that for any large radius R, and for any $p > 13/4$,*

$$\|Ef\|_{BL^p(B_R)} \le C_\epsilon R^\epsilon \|f\|_{L^2}^{12/13} \max_{d(\theta)=R^{-1/2}} \|f_\theta\|_{L^2_{avg}(\theta)}^{1/13},$$

where $d(\theta)$ means the radius of θ, and $f_\theta = f\phi_\theta$ is f multiplied by a bump function ϕ_θ which is 1 in θ and supported in 2θ, and $\|f_\theta\|_{L^2_{avg}(\theta)}^2 = \mathrm{Vol}(\theta)^{-1}\|f_\theta\|_{L^2(\theta)}^2$.

The idea is to use the zero set of a polynomial to partition $\mu_{Ef}(B_R) = \|Ef\|_{BL^p(B_R)}^p$.

After partitioning, we obtain a cellular part, a tangential part and a transversal part of Ef. We shall estimate the tangential part directly and use induction on scale R for the cellular part and the transversal part.

We apply Theorem 0.6 in [2] to find the partitioning polynomial. Theorem 0.6 says that for any degree $d \geq 1$, we can find a non-zero polynomial P of degree at most d so that the complement of its zero set $Z(P)$ in B_R is a union of $O(d^3)$ disjoint cells U_i':

$$B_R \setminus Z(P) = \bigsqcup U_i',$$

and the BL^p–norm is roughly the same in each cell

$$\|Ef\|^p_{BL^p(U_i')} \approx d^{-3}\|Ef\|^p_{BL^p(B_R)}.$$

Now we introduce a slightly different treatment of the cells than Guth's argument, which do not change the conclusion of Guth's theorem but will be beneficial when we introduce our new idea in the next section.

The cell U_i' might have various shape, in order to apply induction on scale we would like to put it inside a smaller ball of radius $\frac{R}{d}$. To do so, it suffices to multiply P by another polynomial G of degree $3d$, and consider the cells cut-off by the zero set of $P \cdot G$. More precisely, let G_k, $k = 1,2,3$, be the product of linear equations whose zero set is a union of planes parallel to x_k–axis, of spacing $\frac{R}{d}$ and intersecting B_R. The degree of G_k is at most d. Let G be the product of G_1, G_2 and G_3, so $\deg G = 3d$. Let the polynomial $Q = P \cdot G$ be our new partitioning polynomial, then we have a new decomposition of B_R,

$$B_R \setminus Z(Q) = \bigsqcup O_i'.$$

The zero set $Z(Q)$ decomposes B_R into at most $O(d^3)$ cells O_i' by the Milnor-Thom Theorem [5], [11].

The general idea is to do induction on scale in each cell, and sum up the contributions of Ef in cells. To efficiently sum up the contributions, we need to understand how each wave packet $Ef_{\theta,v}$ interacts with the cells. By the definition of wave packets, a wave packet $Ef_{\theta,v}$ has negligible contribution to a cell O_i' if its essential support $T_{\theta,v}$ does not intersect O_i'. To analyze how $T_{\theta,v}$ intersects a cell O_i', we need to shrink O_i' further. We define the wall W as the $R^{1/2}$–neighborhood of $Z(Q)$ in B_R and define the new cells as $O_i = O_i' \setminus W$.

To summarize, we decompose $B_R = W \bigsqcup O_i$ and we have the following inequalities:

$$\|Ef\|^p_{BL^p(B_R)} = \|Ef\|^p_{BL^p(W)} + \sum_i \|Ef\|^p_{BL^p(O_i)} \tag{2.3}$$

and

$$\|Ef\|^p_{BL^p(O_i)} \lesssim d^{-3} \|Ef\|^p_{BL^p(B_R)}. \tag{2.4}$$

Cellular case. We are in the cellular case if $\|Ef\|^p_{BL^p(B_R)} \lesssim \sum_i \|Ef\|^p_{BL^p(O_i)}$.

Since wave packets $Ef_{\theta,v}$ with $T_{\theta,v} \cap O_i = \emptyset$ have negligible contribution to $\|Ef\|_{BL^p(O_i)}$, it is helpful to define

$$Ef_i = \sum_{T_{\theta,v} \cap O_i \neq \emptyset} Ef_{\theta,v}.$$

And we have $\|Ef\|_{BL^p(O_i)} = \|Ef_i\|_{BL^p(O_i)} + \text{RapDec}(R)\|f\|_{L^2}$. Here the notation RapDec(R) means that the quantity is bounded by $O_N(R^{-N})$ for any large integer $N > 0$.

Since each wave packet intersects at most $d+1$ cells O_i, we have

$$\begin{aligned} \sum_i \|f_i\|^2_{L^2} &= \sum_i \sum_{T_{\theta,v} \cap O_i \neq \emptyset} \|f_{\theta,v}\|^2_{L^2} \\ &\lesssim d \sum_{\theta,v} \|f_{\theta,v}\|^2_{L^2} \lesssim d \|f\|^2_{L^2} \end{aligned}$$

By inequality 2.4 and the definition of the cellular case, there are at least $O(d^3)$ cells O_i such that

$$\|Ef\|^p_{BL^p(B_R)} \lesssim d^3 \|Ef_i\|^p_{BL^p(O_i)}. \tag{2.5}$$

Since there are $O(d^3)$ cells, for 99% of the cells, we have

$$\|f_i\|_{L^2} \lesssim d^{-1} \|f\|_{L^2}. \tag{2.6}$$

Now we are ready to apply the induction of Theorem 3 at scale $\frac{R}{d}$ for the function

Ef_i,

$$\|Ef_i\|_{BL^p(O_i)} \lesssim (\frac{R}{d})^{\epsilon p} \|f_i\|_{L^2}^{12/13} \max_{d(\tau)=(\frac{R}{d})^{-1/2}} \|f_{i,\tau}\|_{L^2_{avg}(\tau)}^{1/13},$$

where τ denotes a cap of radius $(\frac{R}{d})^{-1/2}$.

To estimate $\max_{d(\tau)=(\frac{R}{d})^{-1/2}} \|f_{i,\tau}\|_{L^2_{avg}(\tau)}$, we shall apply the following two lemmas.

Lemma 2.2. *Let τ be a cap of radius $r^{-1/2}$ with $r < R$, then*

$$\|f_\tau\|_{L^2_{avg}(\tau)} \le \max_{\theta\subset\tau, d(\theta)=R^{-1/2}} \|f_\theta\|_{L^2_{avg}(\theta)}.$$

Proof.

$$\begin{aligned} \|f_\tau\|_{L^2_{avg}(\tau)}^2 &= \operatorname*{Avg}_{\theta\subset\tau, d(\theta)=R^{-1/2}} \|f_\theta\|_{L^2_{avg}(\theta)}^2 \\ &\le \max_{\theta\subset\tau, d(\theta)=R^{-1/2}} \|f_\theta\|_{L^2_{avg}(\theta)}^2. \end{aligned}$$

□

Lemma 2.3. *If $\mathbb{T}$ is a collection of tubes $T_{\theta,v}$ and $f^{\mathbb{T}} = \sum_{T_{\theta,v}\in\mathbb{T}} f_{\theta,v}$, then $\|f^{\mathbb{T}}\|_{L^2} \le \|f\|_{L^2}$.*

Proof. By the L^2–orthogonality of wave packets,

$$\begin{aligned} \|f^{\mathbb{T}}\|_{L^2}^2 &\le \sum_{T_{\theta,v}\in\mathbb{T}} \|f_{\theta,v}\|_{L^2}^2 \\ &\le \sum_{T_{\theta,v}} \|f_{\theta,v}\|_{L^2}^2 = \|f\|_{L^2}^2. \end{aligned}$$

□

We apply Lemma 2.2 on $f_{i,\tau}$ and then Lemma 2.3 on $f_{i,\theta}$,

$$\max_{d(\tau)=(\frac{R}{d})^{-1/2}} \|f_{i,\tau}\|_{L^2_{avg}(\tau)} \le \max_{d(\theta)=R^{-1/2}} \|f_{i,\theta}\|_{L^2_{avg}(\theta)} \le \max_{d(\theta)=R^{-1/2}} \|f_\theta\|_{L^2_{avg}(\theta)}.$$

We use a cell O_i satisfying both inequality 2.6 and inequality 2.5 to close the induction,

$$\begin{aligned}\|Ef\|^p_{BL^p(B_R)} &\lesssim d^3\|Ef_i\|^p_{BL^p(O_i)}\\ &\lesssim d^3(\frac{R}{d})^{\epsilon p}\|f_i\|_{L^2}^{\frac{12p}{13}}\max_{d(\tau)=(\frac{R}{d})^{-1/2}}\|f_{i,\tau}\|^{\frac{p}{13}}_{L^2_{avg}(\tau)}\\ &\lesssim d^{3-\frac{12p}{13}}(\frac{R}{d})^{\epsilon p}\|f\|_{L^2}^{\frac{12p}{13}}\max_{d(\theta)=R^{-1/2}}\|f_\theta\|^{\frac{p}{13}}_{L^2_{avg}(\theta)}\\ &\lesssim R^{\epsilon p}\|f\|_{L^2}^{\frac{12p}{13}}\max_{d(\theta)=R^{-1/2}}\|f_\theta\|^{\frac{p}{13}}_{L^2_{avg}(\theta)}.\end{aligned}$$

The last inequality holds when $p > \frac{13}{4}$.

Algebraic case. If we are not in the cellular case, then $\|Ef\|_{BL^p(B_R)} \lesssim \|Ef\|_{BL^p(W)}$. We call it the algebraic case because the BL^p–norm of Ef is concentrated on the neighborhood of an algebraic surface. As argued previously, only wave packets $Ef_{\theta,v}$ whose essential support $T_{\theta,v}$ intersects W contribute to $\|Ef\|_{BL^p(W)}$. Depending on how they intersect, we identify a tangential part, which consists of the wave packets tangential to W, and a transversal part, which consists of the wave packets intersecting W transversely.

We give the precise definition of the tangential tubes and the transversal tubes as follows. We cover W with finitely overlapping balls B_k of radius $\rho := R^{1-\delta}$. For each B_k, we define $\mathbb{T}_{k,\text{tang}}$, the collection of the tangential tubes in B_k, and $\mathbb{T}_{k,\text{trans}}$, the collection of the transversal tubes in B_k. The definitions are the same as in Guth's paper[2].

Definition 2.4. *$\mathbb{T}_{k,\text{tang}}$ is the set of all tubes T obeying the following two conditions:*

- $T \cap W \cap B_k \neq \emptyset$.
- *If z is any non-singular point of $Z(P)$ lying in $2B_k \cap 10T$, then*

$$\text{Angle}(v(T), T_z Z) \leq R^{-1/2+2\delta}.$$

We note $\mathbb{T}_{\text{tang}} = \cup_k \mathbb{T}_{k,\text{tang}}$.

Definition 2.5. *$\mathbb{T}_{k,\text{trans}}$ is the set of all T obeying the following two conditions:*

- $T \cap W \cap B_k \neq \emptyset$.
- *There exists a non-singular point z of $Z(P)$ lying in $2B_k \cap 10T$, so that*

$$\text{Angle}(v(T), T_z Z) > R^{-1/2+2\delta}.$$

We note $\mathbb{T}_{\text{trans}} = \cup_k \mathbb{T}_{k,\text{trans}}$.

For the fixed B_k, let

$$f_{k,\text{tang}} = \sum_{T_{\theta,v} \in \mathbb{T}_{k,\text{tang}}} f_{\theta,v}, \quad f_{k,\text{trans}} = \sum_{T_{\theta,v} \in \mathbb{T}_{k,\text{trans}}} f_{\theta,v}.$$

We estimate the algebraic part by

$$\|Ef\|_{BL^p(W)} \le \sum_{B_k} \|Ef_{k,\text{tang}} + Ef_{k,\text{trans}}\|^p_{BL^p(W \cap B_k)} \tag{2.7}$$

$$\lesssim \sum_{B_k} \|Ef_{k,\text{tang}}\|^p_{BL^p(W \cap B_k)} + \sum_{B_k} \|Ef_{k,\text{trans}}\|^p_{BL^p(W \cap B_k)}. \tag{2.8}$$

The last inequality is a result of the following triangle inequality for BL^p–norm, proved in Section 4 in [3].

Lemma 2.6. *If $A = A_1 + A_2$, then $\|Ef + Eg\|_{BL^p_A} \lesssim \|Ef\|_{BL^p_{A_1}} + \|Eg\|_{BL^p_{A_2}}$.*

Here is a remark regarding the parameter A in the BL^p–norm. We only apply the triangle inequality at the algebraic cases. There are at most $N = \frac{\log \delta}{\log(1-\delta)}$ steps in the algebraic cases because otherwise the radius would be smaller than $R^{(1-\delta)^N} = R^\delta$ and we could apply the trivial bound. Hence, we need to reduce A at most N times, which is a constant independent of R. We set $A \ge N^2$ and omit the discussion about it from our arguments.

Back to estimating the algebraic case, we split it into the tangential case and the transversal case. For the tangential case we are going to estimate directly, and for the transversal case we apply induction on scale as in the cellular case.

Tangential case.

We are in the tangential case if $\sum_k \|Ef_{k,\text{tang}}\|^p_{BL^p(W\cap B_k)} \gtrsim \|Ef\|^p_{BL^p(W)} \gtrsim \|Ef\|^p_{BL^p(B_R)}$. By inequality 2.2, the BL^p–norm is bounded by some bilinear norms. To bound those bilinear norms, we shall interpolate Tao's bilinear restriction estimate [8]

$$\|Ef_k\|_{BL^{\frac{10}{3}}(W\cap B_k)} \lesssim \|f_{k,\text{tang}}\|_{L^2}$$

with the L^2 energy estimate

$$\|Ef_k\|_{BL^2(W\cap B_k)} \lesssim \rho^{1/2}\|f_{k,\text{tang}}\|_{L^2}.$$

Hence,

$$\|Ef_k\|^p_{BL^p(W\cap B_k)} \lesssim \rho^{\frac{5}{2}-\frac{3p}{4}}\|f_{k,\text{tang}}\|^p_{L^2}. \tag{2.9}$$

$Ef_{k,\text{tang}}$ lives in the thin neighborhood of a 2–dim surface, which heuristically implies that the support of $f_{k,\text{tang}}$ is small. We shall apply the following crucial geometric lemma (Lemma 4.9 in [2]) to efficiently bound $\|f_{k,\text{tang}}\|_{L^2}$ by its L^∞–norm.

Lemma 2.7. *The support* $\operatorname{supp} f_{k,\text{tang}}$ *lies in a union of at most* $R^{1/2+O(\delta)}$ *caps* θ *of radius* $R^{-1/2}$.

By Lemma 2.7 and Lemma 2.3,

$$\|f_{k,\text{tang}}\|^2_{L^2} \lesssim R^{-1/2+O(\delta)}\|f_{k,\text{tang},\theta}\|^2_{L^2_{avg}(\theta)} \lesssim R^{-1/2+O(\delta)} \max_{d(\theta)=R^{-1/2}} \|f_\theta\|^2_{L^2_{avg}(\theta)}. \tag{2.10}$$

By Lemma 2.3, we also have $\|f_{k,\text{tang}}\|_{L^2} \le \|f\|_{L^2}$. We apply the above bounds about $\|f_{k,\text{tang}}\|_{L^2}$ in inequality 2.9, when $p > 13/4$,

$$\begin{aligned}\|Ef_{k,\text{tang}}\|^p_{BL^p(B_R)} &\lesssim \rho^{\frac{5}{2}-\frac{3p}{4}-\frac{p}{52}}\|f\|^{\frac{12p}{13}}_{L^2} \max_{d(\theta)=R^{-1/2}} \|f_\theta\|^{\frac{p}{13}}_{L^2_{avg}(\theta)}\\ &\lesssim R^{O(\delta)}\|f\|^{\frac{12p}{13}}_{L^2} \max_{d(\theta)=R^{-1/2}} \|f_\theta\|^{\frac{p}{13}}_{L^2_{avg}(\theta)}.\end{aligned}$$

Since we need at most $O(R^{3\delta})$ balls B_k to cover B_R and $\delta \ll \epsilon$, we have

$$\|Ef\|^p_{BL^p(B_R)} \lesssim \sum_{B_k} \|Ef_{k,\text{tang}}\|^p_{BL^p(W\cap B_k)} \lesssim R^\epsilon \|f\|^{12p/13}_{L^2} \max_{d(\theta)=R^{-1/2}} \|f_\theta\|^{p/13}_{L^2_{avg}(\theta)}.$$

Transversal case.

We are in the transversal case when $\|Ef\|^p_{BL^p(B_R)} \lesssim \sum_{B_k} \|Ef_{k,\text{trans}}\|^p_{BL^p(B_k\cap W)}$. By inequality 2.3 and inequality 2.8, if $\|Ef\|_{BL^p(B_R)}$ is not dominated by the cellular part or the tangential part, then it must be dominated by the transversal part. To deal with the transversal case, we shall apply induction of Theorem 3 on $Ef_{k,\text{trans}}$ at scale ρ.

The treatment of the transversal case is similar to the cellular case. After induction in each B_k, we need to sum up $\|f_{k,\text{trans}}\|_{L^2}$, which requires the following Lemma 2.8 in the place of Lemma 2.6.

Lemma 2.8. *[Lemma 3.5 [2],] Each tube T belongs to at most* Poly(d) *different sets* $\mathbb{T}_{k,\text{trans}}$. *Here* Poly($d$) *means a quantity bounded by a polynomial power of d and the degree of the polynomial is a constant independent of d.*

By Lemma 2.8, we have

$$\sum_{B_k} \|f_{k,\text{trans}}\|^2_{L^2} \lesssim \text{Poly}(d)\|f\|^2_{L^2}. \tag{2.11}$$

We apply induction on scale and then sum up over the balls B_k,

$$\begin{aligned}\|Ef\|^p_{BL^p(B_R)} &\lesssim \sum_k \|Ef_{k,\text{trans}}\|^p_{BL^p(B_k)} \\ &\lesssim \rho^{\epsilon p} \sum_{B_k} \|f_{k,\text{trans}}\|^{\frac{12p}{13}}_{L^2} \max_{d(\tau)=\rho^{-1/2}} \|f_{k,\text{trans},\tau}\|^{\frac{p}{13}}_{L^2_{avg}(\tau)}.\end{aligned}$$

To estimate $\|f_{k,\text{trans},\tau}\|_{L^2_{avg}(\tau)}$, we apply Lemma 2.2 on $f_{k,\text{trans},\tau}$ and then Lemma 2.3 on $\|f_{k,\text{trans},\theta}\|_{L^2}$. So

$$\|f_{k,\text{trans},\tau}\|_{L^2_{avg}(\tau)} \lesssim \max_{\theta\subset\tau, d(\theta)=R^{-1/2}} \|f_{k,\text{trans},\theta}\|_{L^2_{avg}(\theta)} \lesssim \max_{d(\theta)=R^{-1/2}} \|f_\theta\|_{L^2_{avg}(\theta)}.$$

Then we sum up $\|f_{k,\text{trans}}\|_{L^2}$ using the fact that $l^q \le l^2$ when $q \ge 2$,

$$
\begin{aligned}
\|Ef\|_{BL^p(B_R)}^p &\lesssim \sum_k \|Ef_{k,\text{trans}}\|_{BL^p(B_k)}^p \\
&\lesssim C_\epsilon \rho^{\epsilon p} (\sum_{B_k} \|f_{k,\text{trans}}\|_{L^2}^{\frac{12p}{13}}) \max_{d(\theta)=R^{-1/2}} \|f_\theta\|_{L^2_{avg}(\theta)}^{\frac{p}{13}} \\
&\lesssim C_\epsilon \rho^{\epsilon p} (\sum_{B_k} \|f_{k,\text{trans}}\|_{L^2}^2)^{\frac{6p}{13}} \max_{d(\theta)=R^{-1/2}} \|f_\theta\|_{L^2_{avg}(\theta)}^{\frac{p}{13}} \\
&\lesssim C_\epsilon R^{\epsilon p} \|f\|_{L^2}^{\frac{12p}{13}} \max_{d(\theta)=R^{-1/2}} \|f_\theta\|_{L^2_{avg}(\theta)}^{\frac{p}{13}}.
\end{aligned}
$$

The last inequality is because $\sum_{B_k} \|f_{k,\text{trans}}\|_{L^2}^2 \lesssim \text{Poly}(d)\|f\|_{L^2}^2$ by inequality 2.11. And since $d = \log R$, we choose R large enough such that $R^{\epsilon\delta} \gg \text{Poly}(d)$.

Chapter 3

A white lie version of the proof of Theorem 2

In this section, we provide a white lie version of the proof of Theorem 2. Indeed, we prove a slightly stronger version of Theorem 2 which works better for our induction.

Theorem 4. *If f is supported on the unit disk in $\mathbb{R}^2$, then for any small $\epsilon > 0$, there exists a large constant C_ϵ depending only on ϵ such that for any large enough radius R, and for $p > 3 + 3/13$, we have*

$$\|Ef\|_{BL^p(B_R)} \leq C_\epsilon R^\epsilon \|f\|_{L^2}^{2/p} \max_{d(\theta)=R^{-1/2}} \|f_\theta\|_{L^2_{avg}(\theta)}^{1-2/p}. \tag{3.1}$$

A direct corollary of Theorem 4 is Theorem 2 because

$$\max_{d(\theta)=R^{-1/2}} \|f_\theta\|_{L^2_{avg}(\theta)} \leq \|f\|_{L^\infty}.$$

In Theorem 4, we reduce the exponent weight on the L^2–norm compared to Theorem 3 and we prove for a slightly larger range of p.

3.1 Tangential case in the cells

Our idea comes from analyzing which part of Guth's proof is not tight. To do so, we write Guth's iteration in multi-steps. More precisely, we treat the polynomial partitioning iteration as an algorithm which stops in the tangential cases. We apply the polynomial partitioning on $\|Ef\|_{BL^p(B_R)}^p$ as in Subsection 2.4. If we are in the tangential case, then we stop; otherwise we are in the cellular case or the transversal case, we repeat the previous process in the cells, which are in $B_{\frac{R}{d}}$ or in balls B_k.

We assume two white lies in this subsection.

- **White lie 1.** We assume that the zero set of the partitioning polynomial is a plane.
- **White lie 2.** We assume that if we are in the transverse case, then the tangential part is zero; if we are in the tangential case, then the transverse part is zero. In reality, this might not happen and we are going to treat it carefully in the following sections.

The white lie 1 assumption will be removed using Lemma 6.4 in the actual proof. And the white lie 2 assumption will be removed using Lemma 5.3.

Next we are going to set up some notations to describe the polynomial partitioning multi-step iteration.

Initial step. We run the first polynomial partitioning as in Subsection 2.4, and we are either in the tangential case, the cellular case or the transversal case.

If we are in the tangential case, we stop and write $r = R$, $O = B_R$ and $f_O = f$.

If we are in the cellular case, we keep those cells satisfying inequality 2.6 and inequality 2.5. And we denote them by O_1, the cells at the first iteration step. Each cell O_1 lies inside a ball of radius $r_1 = R/d$. The number of cells O_1 is at least $O(d^3)$.

Otherwise, we are in the transversal case. We select the B_k with a popular BL^p–norm in the following way. For a dyadic number λ, we say that $B_k \in \Lambda_\lambda$ if $\|Ef_{k,\text{trans}}\|_{BL^p(B_k\cap W)} \sim \lambda$. It suffices to consider $R^{-10}\|f\|_{L^2} \le \lambda \le R\|f\|_{L^2}$. The lower bound is accounted for by the fact that the contribution from those B_k with $\|Ef_{k,\text{trans}}\|_{BL^p(B_k\cap W)} \le$

$R^{-10}\|f\|_{L^2}$ is negligible for our proof. The upper bound is justified by the Stein-Tomas inequality, $\|Ef_{k,\text{trans}}\|_{BL^p(B_k)} \le R\|f_{k,\text{trans}}\|_{L^2} \le R\|f\|_{L^2}$. Hence, there exists a popular dyadic λ, such that

$$\|Ef\|^p_{BL^p(B_R)} \lesssim \log R \sum_{B_k \in \Lambda_\lambda} \|Ef_{k,\text{trans}}\|^p_{BL^p(B_k)}. \tag{3.2}$$

For those $B_k \in \Lambda_\lambda$, we define $O_1 = B_k \cap W$ and we call them cells as well. We define $r_1 = R^{1-\delta}$, each cell O_1 lies inside a ball of radius r_1. We define

$$Ef_{O_1} = \sum_{T_{\theta,v} \cap O_1 \neq \emptyset} Ef_{\theta,v}. \tag{3.3}$$

Restricted on O_1, we have $Ef_{O_1} = Ef + \text{RapDec}(R)\|f\|_{L^2}$. Under our white lie 2 assumption , when we are in the transversal case, the tangential part is zero. So $f_{\text{trans}} = f_{O_1}$.

We recall the information about f_{O_1} from polynomial partitioning. In the cellular case, by inequality 2.6,

$$\sum_{O_1} \|f_{O_1}\|^2_{L^2} \lesssim d\|f\|^2_{L^2}; \tag{3.4}$$

and by inequality 2.5

$$\|Ef\|^p_{BL^p(B_R)} \lesssim \sum_{O_1} \|Ef_{O_1}\|^p_{BL^p(O_1)}. \tag{3.5}$$

In the transversal case, by inequality 2.11,

$$\sum_{O_1} \|f_{O_1}\|^2_{L^2} \lesssim \text{Poly}(d)\|f\|^2_{L^2}; \tag{3.6}$$

and by inequality 3.2,

$$\|Ef\|^p_{BL^p(B_R)} \lesssim \log R \sum_{O_1} \|Ef_{O_1}\|^p_{BL^p(O_1)}. \tag{3.7}$$

Iteration step.

Assume that we have m steps in the cellular cases and n steps in the transversal cases, $r_{m+n} \ge R^\epsilon$, and we obtain $\gtrsim (\log R)^{-n} d^{3m}$ cells O_{m+n}. Each cell has approximately the same BL^p–norm and is contained in a ball of radius $r_{m+n} \geq R^\delta$. Restricted on each O_{m+n}, we have $Ef_{O_{m+n}} = Ef + \text{RapDec}(R)\|f\|_{L^2}$. Furthermore, we can bound the BL^p–norm of Ef by

$$\|Ef\|_{BL^p(B_R)}^p \lesssim (\log R)^{2n} \sum_{O_{m+n}} \|Ef_{O_{m+n}}\|_{BL^p(O_{m+n})}^p, \tag{3.8}$$

and the L^2–norm of $f_{O_{m+n}}$ by

$$\sum_{O_{m+n}} \|f_{O_{m+n}}\|_{L^2}^2 \lesssim d^m (\text{Poly}(d))^n \|f\|_{L^2}^2. \tag{3.9}$$

We do wave packet decomposition in each $B_{r_{m+n}}$

$$Ef = \sum_{d(\tau)=r_{m+n}^{-1/2}, w} Ef_{\tau,w}.$$

The scale of the wave packets changes throughout the iteration process. We shall also pay attention to how the large wave packets $Ef_{\theta,v}$ and the small wave packets $Ef_{\tau,w}$ interchange. We apply polynomial partitioning inside each cell O_{m+n} at scale r_{m+n}.

If for more than 1/3 fraction of the cells O_{m+n} we are in the cellular case, then we keep only those O_{m+n}. We choose children cells O_{m+n+1} inside parent cells O_{m+n} as in the initial step, and we write $r_{m+n+1} = r_{m+n}/d$. There are at last $O(d^3)$ children cells in each parent cell. Hence there are $\gtrsim (\log R)^{-n} d^{3(m+1)}$ cells in total. We define

$$Ef_{O_{m+n+1}} = \sum_{T_{\tau,w} \cap O_{m+n+1} \neq \emptyset} Ef_{\tau,w}.$$

Restricted on each O_{m+n+1}, we have $Ef = Ef_{O_{m+n+1}} + \text{RapDec}(R)\|f\|_{L^2}$ and the corresponding inequality 3.8 and inequality 3.9 for $m+1$ in the place of m.

If for more than 1/3 fraction of the cells O_{m+n} we are in the transversal case, then we select the children cells as follows.

- As in the initial step, we obtain a collection of children cells O_{m+n+1} inside each parent cell O_{m+n}, and the children cells have BL^p–norm about $\lambda(O_{m+n})$.

- We sort the parent cells O_{m+n} according to $\lambda(O_{m+n})$, then by pigeonholing we can find a popular dyadic number λ such that more than $(\log R)^{-1}$ of the parent cells $O_{m+n})$ have $\lambda(O_{m+n}) \sim \lambda$.

- We keep only the parent cells O_{m+n} with $\lambda(O_{m+n}) \sim \lambda$ and their children cells O_{m+n+1}. Hence, the number of remaining children cells in total is $\gtrsim (\log R)^{n+1} d^{3m}$.

We define $r_{m+n+1} = r_{m+n}^{1-\delta}$, and each O_{m+n+1} lies inside a ball of radius r_{m+n+1}. By the white lie 2 assumption,

$$Ef_{O_{m+n+1}} = \sum_{T_{\tau,w} \cap O_{m+n+1} \neq \emptyset} Ef_{\tau,w}$$

and restricted on O_{m+n+1} we have $Ef = Ef_{O_{m+n+1}} + \mathrm{RapDec}(R)\|f\|_{L^2}$. We have as well the corresponding inequality 3.8 and inequality 3.9 with $n+1$ in the place of n.

Stopping condition: small radius. We stop if $r_{m+n+1} \leq R^{\delta}$. And we write $O = O_{m+n+1}$, $r = r_{m+n+1}$. We show in Lemma 3.3 that if we stop when $r \leq R^{\delta}$, then the restriction estimate holds for $p > 3$.

Stopping condition: tangential case. Otherwise, for more than 1/3 fraction of the cells O_{m+n}, we are in the tangential case:

$$\|Ef_{O_{m+n}}\|_{BL^p(O_{m+n})} \lesssim \|Ef_{O_{m+n},\mathrm{tang}}\|_{BL^p(O_{m+n})} \tag{3.10}$$

and $Ef_{O_{m+n},\mathrm{tang}}$ is concentrated in the $r_{m+n}^{1/2}$–neighborhood of a low degree algebraic surface. We stop and write $O = O_{m+n}$, $r = r_{m+n}$. Restricted on O we have

$$Ef_O = \sum_{T_{\tau,w} \cap O \neq \emptyset} Ef_{\tau,w} = \sum_{T_{\theta,v} \cap O \neq \emptyset} Ef_{\theta,v} + \mathrm{RapDec}(R)\|f\|_{L^2}.$$

We write $D = d^m$. The number of cells is greater than

$$\#O \gtrsim (\log R)^{-n} D^3. \tag{3.11}$$

Since $r \geq R^{\delta}$, we have $n \leq \delta^{-2}$ because $R^{(1-\delta)^{\delta^{-2}}} \leq R^{\delta}$. So $(\log R)^n \lesssim R^{\delta}$ when R is large enough. We would like to state the BL^p–bound and the L^2–bound about Ef_O.

By inequality 3.8 and inequality 3.10,

$$\|Ef\|^p_{BL^p(B_R)} \lesssim R^{\delta} \sum_O \|Ef_O\|^p_{BL^p(O)} \lesssim R^{\delta} \sum_O \|Ef_{O,\text{tang}}\|^p_{BL^p(O)}. \tag{3.12}$$

Since each cell has approximately the same BL^p–norm, we have as well

$$\|Ef\|^p_{BL^p(B_R)} \lesssim R^{\delta} \#\{O\} \|Ef_{O,\text{tang}}\|^p_{BL^p(O)} \tag{3.13}$$

for each cell O.

For the L^2–bound, by inequality 3.9,

$$\sum_O \|f_{O,\text{tang}}\|^2_{L^2} \leq \sum_O \|f_O\|^2_{L^2} \lesssim D(\text{Poly}(d))^n \|f\|^2_{L^2} \lesssim DR^{\delta} \|f\|^2_{L^2}. \tag{3.14}$$

The last inequality is because $d \sim \log R$, we have $(\text{Poly}(d))^n \leq R^{\delta}$ when R is large enough.

Lemma 3.1. *If D, r are defined as in the above polynomial partitioning iteration, we have $D \leq R/r$.*

Proof. Assume that we have m steps in the cellular case and n steps in the transversal case. By definition $D = d^m$ and $d \sim \log R$. Let $r_0 = R$. If the ith step is celluar, then $r_i = r_{i-1}/d$. Hence, $r = r_{m+n} \leq R/d^m = R/D$. □

Now we have finished the polynomial partitioning iteration. The BL^p–norm of Ef is concentrated in thin neighborhoods of low degree algebraic surfaces in some balls B_r. When r is too large or too small, we have good bounds about $\|Ef\|_{BL^p(B_R)}$ by inequality 3.12, inequality 3.14 and Lemma 2.7 at scale r.

Lemma 3.2. *When $r \geq R^{\frac{13}{16}}$, for any $p > \frac{42}{13}$,*

$$\|Ef\|_{BL^p(B_R)} \leq C_{\epsilon} R^{\epsilon} \|f\|^{2/p}_{L^2} \max_{d(\theta)=R^{-1/2}} \|f_{\theta}\|^{1-2/p}_{L^2_{avg}(\theta)}.$$

Proof. By inequality 3.12 and inequality 2.9 at scale r,

$$\|Ef\|^p_{BL^p(B_R)} \lesssim R^\delta \sum_O \|Ef_{O,\mathrm{tang}}\|^p_{BL^p(O)} \lesssim R^\delta \sum_O r^{\frac{5}{2}-\frac{3p}{4}} \|f_{O,\mathrm{tang}}\|^p_{L^2}. \tag{3.15}$$

By inequality 3.14, $\sum_O \|f_{O,\mathrm{tang}}\|^2_{L^2} \lesssim DR^\delta \|f\|^2_{L^2}$. We can also bound each $\|f_{O,\mathrm{tang}}\|_{L^2}$ by its approximate L^∞–norm using Lemma 2.7 and inequality 2.10 at scale r,

$$\begin{aligned}\|f_{O,\mathrm{tang}}\|_{L^2} &\lesssim r^{-\frac{1}{4}} \max_{d(\tau)=r^{-1/2}} \|f_{O,\mathrm{tang},\tau}\|_{L^2_{avg}(\tau)} \\ &\lesssim r^{-\frac{1}{4}} \max_{d(\theta)=R^{-1/2}} \|f_\theta\|_{L^2_{avg}(\theta)}\end{aligned}$$

The last inequality is a result of Lemma 2.2 and Lemma 2.3.

Back to estimating $\|Ef\|_{BL^p(B_R)}$,

$$\begin{aligned}\|Ef\|_{BL^p(B_R)} &\lesssim R^\delta r^{\frac{5}{2}-\frac{3p}{4}} \sum_O \|f_{O,\mathrm{tang}}\|^p_{L^2} \\ &\lesssim R^\delta r^{\frac{5}{2}-\frac{3p}{4}-\frac{p-2}{4}} \sum_O \|f_{O,\mathrm{tang}}\|^2_{L^2} \max_{d(\tau)=r^{-1/2}} \|f_{O,\mathrm{tang},\tau}\|^{p-2}_{L^2_{avg}(\tau)} \\ &\lesssim R^{O(\delta)} r^{\frac{5}{2}-\frac{3p}{4}-\frac{p-2}{4}} D \|f\|^2_{L^2} \max_{d(\theta)=R^{-1/2}} \|f_\theta\|^{p-2}_{L^2_{avg}(\theta)} \\ &\lesssim R^\epsilon \|f\|^2_{L^2} \max_{d(\theta)=R^{-1/2}} \|f_\theta\|^{p-2}_{L^2_{avg}(\theta)}\end{aligned}$$

The last inequality is because when $r \geq R^{13/16}$, by Lemma 3.1 we have $D \leq R/r \leq r^{3/13}$. Hence, the constant term $R^{O(\delta)} r^{\frac{5}{2}-\frac{3p}{4}-\frac{p-2}{4}} D$ is bounded by R^ϵ. □

Lemma 3.3. *When $r \leq R^\delta$ with $\delta \ll \epsilon$, then for any $p > 3$,*

$$\|Ef\|_{BL^p(B_R)} \leq C_\epsilon R^\epsilon \|f\|_{L^2}.$$

Proof. We apply inequality 3.8, and we have $\gtrsim R^{-\delta} D^3$ cells O such that

$$\begin{aligned}\|Ef\|^p_{BL^p(B_R)} &\lesssim R^{O(\delta)} \#\{O\} \|Ef_O\|^p_{BL^p(O)} \\ &\lesssim R^{O(\delta)} \#\{O\} \|f_O\|^p_{L^2}.\end{aligned}$$

The last inequality is because $r \le R^{\delta}$.

We apply inequality 3.9 to estimate $\|f_O\|_{L^2}^2$. For a typical cell, we have

$$\|f_O\|_{L^2}^2 \lesssim \frac{DR^{O(\delta)}}{\#\{O\}} \|f\|_{L^2}^2.$$

Since the number of cells is greater than $R^{-\delta}D^3$, we have

$$\begin{aligned}\|Ef\|_{BL^p(B_R)}^p &\lesssim R^{O(\delta)}\#\{O\}\|f_O\|_{L^2}^p \\ &\lesssim R^{O(\delta)}\#\{O\}^{1-p/2} \cdot D^{p/2}\|f\|_{L^2}^p \\ &\lesssim R^{O(\delta)}D^{3-p}\|f\|_{L^2}^p \lesssim R^{\epsilon}\|f\|_{L^2}^p.\end{aligned}$$

□

After Lemma 3.2 and Lemma 3.3, it suffices to consider the case when $R^{\delta} \le r \le R^{13/16}$. This is also the technical case. We have gathered some information from polynomial partitioning: Ef is concentrated on the $r^{1/2}$–neighborhood of algebraic surfaces in the cells in B_r, the number of cells is greater than $D^3R^{-\delta}$. To obtain the improved range of p, we need information in addition to inequality 3.13 and inequality 3.14.

In order to obtain the extra information, we shall apply Wolff's two ends argument to split Ef to a global part $Ef^{\nsim}$ and a local part $Ef^{\sim}$. For the local part, we use the induction on scale as in Wolff's original arguments. For the global part, we introduce the new geometric object called "brooms" and use the brooms to analyze the finer structure of Ef.

We recall Wolff's two ends argument here. We cover B_R with balls B_k of radius $\rho = R^{1-\epsilon_0}$ with $\delta \ll \epsilon_0 \ll \epsilon$. The idea is to give each tube $T_{\theta,v}$ a few balls B_k one can "exclude" from the analysis, which we define as $T_{\theta,v} \sim B_k$ later in the next subsection. For now, it suffices that the relation $\sim$ satisfies the following property:

for each $T_{\theta,v}$, the number of B_k with $B_k \sim T_{\theta,v}$ is bounded by $O_{\delta}(1)$.

And we choose R large enough such that $O_{\delta}(1) \le R^{\delta^2}$.

For each B_k, we define the local part of Ef as $Ef_k^{\sim} = \sum_{T_{\theta,v} \sim B_k} Ef_{\theta,v}$ and the global part of Ef as $Ef_k^{\nsim} = Ef - Ef_k^{\sim}$. Since each $T_{\theta,v}$ is related to at most $O_\delta(1)$ balls B_k, we have

$$\sum_{B_k} \|f_k\|_{L^2}^2 \lesssim_\delta \|f\|_{L^2}^2. \tag{3.16}$$

By triangle inequality,

$$\|Ef\|_{BL^p(B_R)}^p \lesssim \sum_{B_k} \|Ef_k^{\sim}\|_{BL^p(B_k)}^p + \sum_{B_k} \|Ef_k^{\nsim}\|_{BL^p(B_k)}^p.$$

The local part is proved by induction on scale.

Lemma 3.4. *If* $\sum_{B_k} \|Ef_k^{\sim}\|_{BL^p(B_k)}^p \gtrsim R^{-\delta} \|Ef\|_{BL^p(B_R)}^p$, *then*

$$\|Ef\|_{BL^p(B_R)}^p \le C_\epsilon R^\epsilon \|f\|_{L^2}^2 \max_{d(\theta)=R^{-1/2}} \|f_\theta\|_{L^2_{avg}(\theta)}^{p-2}.$$

Proof. We apply induction on scale of Theorem 4 at scale ρ,

$$\begin{aligned}
\|Ef\|_{BL^p(B_R)}^p &\lesssim R^\delta \sum_{B_k} \|Ef_k^{\sim}\|_{BL^p(B_k)}^p \\
&\lesssim R^\delta C_\epsilon \rho^\epsilon \sum_{B_k} \|f_k\|_{L^2}^2 \max_{|\theta'|=\rho^{-1/2}} \|f_{k,\theta'}\|_{L^2_{avg}(\theta')}^{p-2} \\
&\lesssim C_\epsilon R^\epsilon R^{-\epsilon\epsilon_0 + O(\delta)} \|f\|_{L^2}^2 \max_{d(\theta)=R^{-1/2}} \|f_\theta\|_{L^2_{avg}(\theta)}^{p-2}
\end{aligned}$$

The last inequality is because of the L^2–bound 3.16, Lemma 2.2 and Lemma 2.3. Since $\delta \ll \epsilon_0 \ll \epsilon$, the constant term is bounded by $C_\epsilon R^\epsilon$. □

3.2 Analysis of the broom

In this subsection, we give the full definition of the relation $B_k \sim T_{\theta,v}$ and prove for the case when the global part $Ef_k^{\nsim}$ dominates.

Recall that after polynomial partitioning iteration, Ef is dominated by the union of Ef_O. And each Ef_O is dominated by its tangential part $Ef_{O,\text{tang}}$, which lives in the $r^{1/2}$ of a plane inside B_r by the white lie 1 assumption. Restricted on O, Ef_O is the same as a sum over large wave packets:

$$Ef_O = \sum_{d(\theta)=R^{-1/2}, T_{\theta,v}\cap O\neq\emptyset} Ef_{\theta,v} + \text{RapDec}(R)\|f\|_{L^2}.$$

Since $O \subset B_r$, we can do wave packet decomposition of Ef_O inside B_r,

$$Ef_O = \sum_{d(\tau)=r^{-1/2},w} Ef_{O,\tau,w},$$

and $Ef_{O,\text{tang}}$ is obtained by summing over those small wave packets $Ef_{O,\tau,w}$ tangential to the partitioning algebraic surface:

$$Ef_{O,\text{tang}} = \sum_{T_{\tau,w}\in\mathbb{T}_{O,\text{tang}}} Ef_{O,\tau,w},$$

where $\mathbb{T}_{O,\text{tang}}$ denotes the collection of small tubes $T_{\tau,w}$ tangential to the partitioning algebraic surface.

For each ball B_k of radius $R^{1-\delta}$, we define $Ef_k^{\sim} = \sum_{T_{\theta,v}\sim B_k} Ef_{\theta,v}$ and $Ef_k^{\nsim} = \sum_{T_{\theta,v}\nsim B_k} Ef_{\theta,v}$.

For each cell $O \subseteq B_r \subseteq B_k$, we do wave packet decomposition of $Ef_k^{\sim}$ on B_r:

$$Ef_k^{\sim} = \sum_{d(\tau)=r^{-1/2},v} Ef_{k,\tau,w}^{\sim}.$$

We define $Ef_O^{\sim} = \sum_{T_{\tau,w}\cap O\neq} Ef_{k,\tau,w}^{\sim}$ and we have $Ef_O^{\sim} = Ef_k^{\sim}|_O + \text{RapDec}(R)\|f\|_{L^2}$. We define $Ef_O^{\nsim}$ in the same way with $Ef_k^{\nsim}$.

We define $Ef_{O,\text{tang}}^{\nsim}$ be the tangential part of $Ef_O^{\nsim}$ with respect to the polynomial partitioning for Ef_O:

$$Ef_{O,\text{tang}}^{\nsim} = \sum_{T_{\tau,w}\in\mathbb{T}_{O,\text{tang}}} Ef_{k,\tau,w}^{\nsim}.$$

Since the global part $Ef_k^{\nsim}$ dominates, for most of the cells we have $\|Ef_O^{\sim}\|_{BL^p(O)}^p \lesssim R^{-\epsilon}\|Ef_O\|_{BL^p(O)}^p$. Under the white lie 2 assumption, we have $\|Ef_{O,\text{tang}}^{\nsim}\|_{BL^p(O)}^p \gtrsim \|Ef_{O,\text{tang}}\|_{BL^p(O)}^p$.

By inequality 3.12,

$$\|Ef\|^p_{BL^p(B_R)} \lesssim R^\delta \sum_O \|Ef^\sim_{O,\mathrm{tang}}\|^p_{BL^p(O)}. \tag{3.17}$$

The key inequality we are going to prove is the following, for most cells O,

$$\|f^\sim_{O,\mathrm{tang},\tau}\|^2_{L_{avg}(\tau)^2} \lesssim (\frac{R}{r})^{-1/2} \max_{\theta\subset\tau} \|f_\theta\|^2_{L^2_{avg}(\theta)}. \tag{3.18}$$

One can see that this inequality is an improvement over Lemma 2.2 and Lemma 2.3, which was the previous treatment of $\|f_{O,\mathrm{tang},\tau}\|_{L^2_{avg}(\tau)}$ in [2]. Combining this extra information with previous estimates 3.13, 3.17 and 3.14, we conclude the proof of Theorem 4. We leave the calculations in Lemma 3.14.

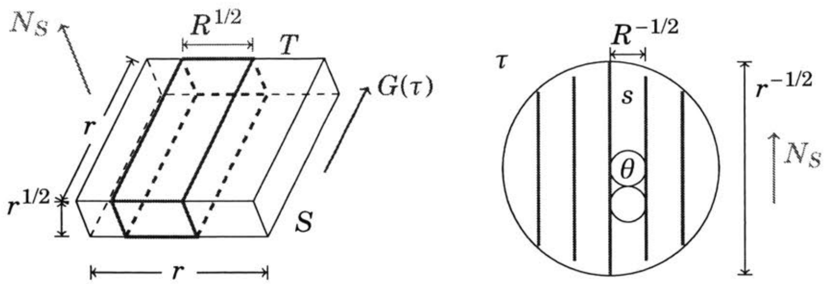
We discuss separately the case when $r \geq R^{1/2}$ and the case when $r \leq R^{1/2}$.

The case when $r \geq R^{1/2}$.

By the white lie 1 assumption, $Ef_{O,\mathrm{tang},\tau}$ is tangential to the $r^{1/2}$–neighborhood of a plane Σ inside a ball B_r, which we denote by the fat plane S. The dimensions of a fat plane S are $r \times r \times r^{1/2}$. Recall that every nonzero $Ef_{\theta,v}$ has about the same L^2–norm by our reduction in Subsection 2.3.

We would like to study how large tubes $T_{\theta,v}$ intersect a fat plane S. A large tube $T_{\theta,v}$ intersect S into a plank T if the directions in $G(\theta)$ are parallel to S up to $r^{-1/2}$–error, which we denote $G(\theta) \parallel S$. In particular, the dimensions of a plank T are $r \times R^{1/2} \times r^{1/2}$. If two large tubes $T_{\theta,v}, T_{\theta',v'}$ intersect at a common point in S and $\theta, \theta' \subseteq \tau$, $d(\tau) = r^{-1/2}$ and $G(\tau) \parallel S$, then $T_{\theta,v} \cap S \approx T_{\theta',v'} \cap S \approx T$.

For a fixed direction τ such that $G(\tau) \parallel S$, we decompose S into disjoint union of parallel planks T with direction in $G(\tau)$ up to $r^{-1/2}$–error.



Now that we have decomposed S into union of planks T in the physical space, we can also decompose the cap τ into union of strips s in the frequency space, such that each s is the intersection of a dual convex of T and τ. More precisely, we decompose the cap τ into union of parallel strips s of length $r^{-1/2}$, width $R^{-1/2}$. Each s is parallel to the normal direction of S up to angle difference $(\frac{R}{r})^{-1/2}$. The smallest box covering s is of dimensions $r^{-1/2} \times R^{-1/2} \times r^{-1}$, which corresponds to the dual convex of T. By the uncertainty principle, we have the L^2–orthogonality on each plank T:

$$\int_T |Ef_{O,\text{tang},\tau}|^2 \lesssim \sum_s \int |Ef_{O,\text{tang},s}|^2 w_T, \tag{3.19}$$

where the weight function w_T is essentially supported on T and rapidly decays elsewhere. One might treat w_T as the characteristic function on T for simplicity.

Now we are ready to give the definition of a broom.

Definition 3.5. *For each s and plank T described in the last three paragraphs, we define a* broom $\mathscr{B}$ *as the collection of large wave packets $Ef_{\theta,v}$ with $\theta \subseteq s$ and the essential support $T_{\theta,v} \cap T \neq \emptyset$. We call T the* root *of the broom $\mathscr{B}$, and we denote it by $T_{\mathscr{B}}$. We say that a broom $\mathscr{B}$ is rooted at fat plane S if $T_{\mathscr{B}} \subset S$. The size of the broom $\mathscr{B}$ is defined by the number of nonzero large wave packets it contains, which we denote by $|\mathscr{B}|$.*

Notice that a root $T_{\mathscr{B}}$ does not uniquely determine the broom $\mathscr{B}$. For example, brooms $\mathscr{B}_1$ and $\mathscr{B}_2$ corresponding to different $s_1, s_2 \subset \tau$ might share the same plank $T = T_{\mathscr{B}_1} = T_{\mathscr{B}_2}$.

Lemma 3.6. *The size of a broom is bounded by $(\frac{R}{r})^{1/2}$.*

Proof. A broom $\mathscr{B}$ is associated to a unique pair $(T_{\mathscr{B}}, s)$. Each cap θ corresponds to at most one large wave packet in a broom $\mathscr{B}$ because every large wave packet in the broom must intersects $T_{\mathscr{B}}$. The lemma follows because there are at most $(\frac{R}{r})^{1/2}$ caps θ in the strip s. □

Remark 3.7. *Given a fat plane S and τ with $G(\tau) \parallel S$, all large wave packets $Ef_{\theta,v}$ with $\theta \subset \tau$ and $T_{\theta,v} \cap S \neq \emptyset$ are organized into brooms and each $T_{\theta,v}$ belongs to a unique broom rooted at S.*

Remark 3.8. *The tubes in a broom $\mathscr{B}$ span on the normal direction of the plane, see Figure 1-1. In particular, the tubes in $\mathscr{B}$ lie in the $R^{1/2}$–neighborhood of a plane $\Sigma^{\perp}$, where $\Sigma^{\perp}$ is parallel to the normal direction N_S of S and $G(\theta)$ with any $\theta \subset s$.*

By inequality 3.19, we have

$$\int |Ef_{O,\mathrm{tang},\tau}|^2 \lesssim \sum_{\mathscr{B}} \int |Ef_{O,\mathrm{tang},\mathscr{B}}|^2 w_{T_{\mathscr{B}}}, \tag{3.20}$$

where $\mathscr{B}$ is the broom determined by plank $T = T_{\mathscr{B}}$ and s in definition 3.5 and $Ef_{O,\mathrm{tang},\mathscr{B}} := Ef_{O,\mathrm{tang},s}|_{T_{\mathscr{B}}}$.

The L^2–mass of a broom is proportional to its size.

Lemma 3.9. *For any $R^{1/2} \le r \le R$, if $\mathscr{B}$ is a broom of size b with root $T_{\mathscr{B}}$ and $Eg = \sum_{T_{\theta,v} \in \mathscr{B}} Eg_{\theta,v}$ then*

$$\|Eg\|^2_{L^2(w_{T_{\mathscr{B}}})} \lesssim (\frac{R}{r})^{-1/2} b \|Eg\|^2_{L^2(B_r)}.$$

Proof. Since $|Eg_{\theta,v}|$ is essentially constant on $T_{\theta,v}$,

$$\int |Eg_{\theta,v}|^2 w_{T_{\mathscr{B}}} \lesssim \frac{|T_{\mathscr{B}}|}{|B_r \cap T_{\theta,v}|} \int_{B_r} |Eg_{\theta,v}|^2 \lesssim (\frac{R}{r})^{-1/2} \int_{B_r} |Eg_{\theta,v}|^2.$$

By Cauchy-Schwartz inequality,

$$\begin{aligned}\int |Eg|^2 w_{T_B} &= \int |\sum_{T_{\theta,v}\in\mathscr{B}} Eg_{\theta,v}|^2 w_{T_B} \\ &\le b \sum_{T_{\theta,v}\in\mathscr{B}} \int |Eg_{\theta,v}|^2 w_{T_B} \\ &\lesssim (\frac{R}{r})^{-1/2} b \sum_{T_{\theta,v}\in\mathscr{B}} \int_{B_r} |Eg_{\theta,v}|^2 \\ &\lesssim (\frac{R}{r})^{-1/2} b \int_{B_r} |Eg|^2\end{aligned}$$

The last inequality is because $r \ge R^{1/2}$. □

Combining with inequality 3.20, Lemma 3.9 says that if $Ef_{O,\text{tang},\tau}$ is dominated by brooms of small size, then it has small L^2–norm. So we sort the brooms $\mathscr{B}$ according to their sizes and decompose

$$Ef_{O,\text{tang},\tau} = \sum_{\text{dyadic } b} \sum_{|\mathscr{B}|\sim b} Ef_{O,\text{tang},\mathscr{B}}.$$

Since $0 \le b \le (\frac{R}{r})^{1/2}$, there exists some dyadic number b such that the brooms of size about b dominates $\|Ef_{O,\text{tang},\tau}\|^2_{L^2(B_r)}$:

$$\int_{B_r} |Ef_{O,\text{tang},\tau}|^2 \lesssim \log R \int_{B_r} |\sum_{|\mathscr{B}|\sim b} Ef_{O,\text{tang},\mathscr{B}}|^2.$$

Lemma 3.10. *Let Eh_τ be a function whose Fourier transform is supported on the R^{-1} neighborhood of a paraboloid. Let $R^{1/2} \le r \le R$, and $S \subset B_r$ be a fat plane of thickness $r^{1/2}$, and τ be a cap of radius $r^{-1/2}$ such that $G(\tau)$ is parallel to S. If the large wave packets $Eh_{\theta,v}$ of Eh_τ are organized into brooms of size about b,*

$$Eh_\tau = \sum_{|\mathscr{B}|\sim b} Eh_{\tau,\mathscr{B}},$$

then

$$\int_S |Eh_\tau|^2 \lesssim (\frac{R}{r})^{-1/2} b \int_{B_r} |Eh_\tau|^2.$$

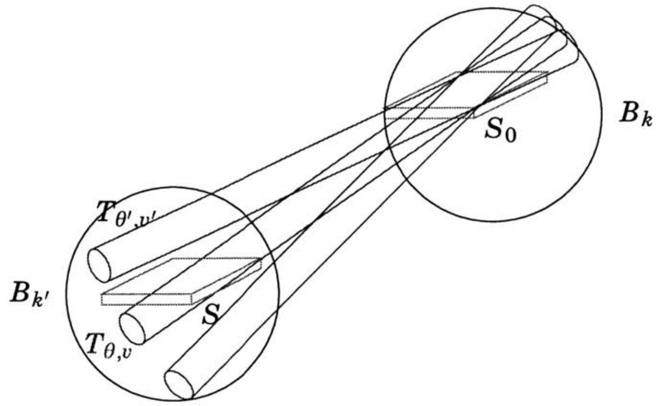


Figure 3-1: A tube $T_{\theta,v}$ related to B_k.

Proof. For a fixed cap τ, we decompose S into tiling of plank T whose direction is parallel to $G(\tau)$. By inequality 3.20 and then by Lemma 3.9,

$$\begin{aligned}\int_S |Eh_\tau|^2 &\lesssim \sum_{\mathscr{B}} \int |Eh_{\tau,\mathscr{B}}|^2 w_{T_{\mathscr{B}}} \\ &\lesssim \sum_{\mathscr{B}} (\frac{R}{r})^{-1/2} b \int_{B_r} |Eh_{\tau,\mathscr{B}}|^2 \\ &\lesssim (\frac{R}{r})^{-1/2} b \int_{B_r} |Eh_\tau|^2.\end{aligned}$$

□

Remark 3.11. *We observe that the size of $\|f_{O,\text{tang},\tau}\|_{L^2}$ is influenced by two independent factors: the number of large wave packets $Ef_{\theta,v}$ intersecting S tangentially and the size of brooms they construct.*

$$\|f_{O,\text{tang},\tau}\|_{L^2} \overset{(A)}{\leq} \|f_{O,\tau}\|_{L^2} \overset{(B)}{\leq} \|f_\tau\|_{L^2}. \tag{3.21}$$

By Lemma 3.6, the coefficient in Lemma 3.10 is $(\frac{R}{r})^{-1/2} b \leq 1$. If the size of the brooms constructing $f_{O,\text{tang},\tau}$ is small, then we have some gain on part (A) of the inequality 3.21 by Lemma 3.10. If the fraction of large wave packets within direction $G(\tau)$ intersecting O is small, then we have some gain on part (B) of the inequality 3.21.

The heuristic for the rest of the proof is the following. Assume that for a typical cell O, $\|Ef_{O,\mathrm{tang},\tau}\|_{L^2}$ is dominated by brooms of size about b. If $b \ll (\frac{R}{r})^{-1/2}$, then we gain in part (A) of inequality 3.21. Otherwise, we cover B_R with balls B_k of radius $\rho = R^{1-\delta}$, and we define $T_{\theta,v} \sim B_k$ if $Ef_{\theta,v}$ belongs to many brooms rooted in B_k. By Remark 3.7, a wave packet belongs to a unique broom rooted at a fat plane S. However, since there are many fat planes S inside a ball B_k of radius $R^{1-\delta}$, a wave packet might belongs to many brooms rooted in B_k. Now we look at a $B_{k'}$ with distance at least $R^{1-\delta}$ from B_k. After the "two ends" reduction, for each fat plane $S \subset B_{k'}$, it suffices to consider only wave packets with $T_{\theta,v} \nsim B_{k'}$.

Each wave packet $T_{\theta,v} \nsim B_{k'}$ carries many other wave packets $T_{\theta',v'}$ (see Figure 3-1) not intersecting S. This is because S misses a large fraction of wave packets $T_{\theta',v'}$ if the size of a typical broom rooted in B_k is large. Since the fraction of wave packets hitting S is small, we gain by part (B) of the inequality.

Now we continue the proof . Recall that after polynomial partitioning, Ef is concentrated on a collection of $r^{1/2}$–neighborhood of algebraic surfaces. In the white lie proof, we assume that those algebraic surfaces are planes Σ. So Ef is concentrated on a collection of fat planes S, the $r^{1/2}$–neighborhood of Σ inside B_r. Each cell O is associated with one Σ. We should think Σ as a piece of plane in O, in particular, it is contained in a B_r. Let N_Σ denote the normal direction of Σ. We divide the unit sphere $\mathbb{S}^2$ into $O(1)$ caps α of radius 1/100. There exists a cap α, such that the cells whose corresponding planes Σ have normal directions N_Σ in α dominate:

$$\|Ef\|^p_{BL^p(B_R)} \lesssim \sum_O \|Ef^\sim_{O,\mathrm{tang}}\|^p_{BL^p(O)} \lesssim \sum_{N_\Sigma \in \alpha} \|Ef^\sim_{O,\mathrm{tang}}\|^p_{BL^p(O)}.$$

From now on, we assume that for every cell O, the corresponding partitioning plane Σ has normal direction in α. Hence, for any pair of partitioning planes Σ_1 and Σ_2, their normal directions form an angle of at most 1/100.

After some dyadic pigeonholing, which we omit from the white lie proof, we might assume that

- every non-empty broom has size about b.
- each tube $T_{\theta,v}$ intersects about γ planes Σ.

The main idea of counting wave packets using brooms is included in this special case. We deal with the general case in Section 6.2.

We define the function $\chi(T_{\theta,v},\Sigma)=1$ if $T_{\theta,v}$ intersects Σ and $G(\theta)$ is parallel to Σ, otherwise $\chi(T_{\theta,v},\Sigma)=0$. We say that a broom is rooted at Σ if so it is with the fat plane S containing $\Sigma\cap O$. It is helpful to rewrite the assumptions above using the function χ. For any tube $T_{\theta,v}$, we have:

1. $\sum_{T_{\theta',v'}\in\mathscr{B}}\chi(T_{\theta',v'},\Sigma)\sim b$ for the broom $\mathscr{B}$ containing $T_{\theta,v}$ rooted at Σ.

2. $\sum_{\Sigma}\chi(T_{\theta,v},\Sigma)\sim\gamma$ where we sum over all the planes intersecting $T_{\theta,v}$.

Now we are ready to give the definition of $T_{\theta,v}\sim B_k$. We cover B_R with finitely overlapping balls B_k of radius $\rho=R^{1-\delta}$.

Definition 3.12. *For each $T_{\theta,v}$, let B_k^* be the ball that maximizes the quantity*

$$\sum_{\Sigma\cap O\subseteq B_k}\chi(T_{\theta,v},\Sigma).$$

If there are multiple maximizers, we choose only one. We define $T_{\theta,v}\sim B_k$ if $B_k\subseteq 10B_k^*$. Otherwise, we define $T_{\theta,v}\nsim B_k$.

By our definition, each tube $T_{\theta,v}$ is related to at most $O(1)$ balls B_k. Now we are ready to prove the key lemma in this white lie proof.

Lemma 3.13. *If $R^{1/2}\le r\le R^{1-\delta}$, then*

$$\|f_{O,\mathrm{tang},\tau}^{\nsim}\|_{L^2_{avg}(\tau)}^2\lesssim(\frac{R}{r})^{-1/2}R^{O(\delta)}\max_{\theta\subseteq\tau}\|f_\theta\|_{L^2_{avg}(\theta)}^2.$$

Proof. We assume that there are about $(\frac{R}{r})^{\beta_0}$ nonzero wave packets $Ef_{\theta,v}$ with $\theta\subseteq\tau$. After the reduction in Subsection 2.3, we might assume that a wave packet is either zero or has about the same L^2–norm.

Let B_k be the ball of radius $\rho=R^{1-\delta}$ containing O and let Σ_1 be the partitioning plane in O. We say that $\Sigma_2\not\subseteq B_k$ if Σ_2 is associated to some cell O_2 outside of $5B_k$.

The idea is to double count the number of wave packets shared by Σ_1 and Σ_2 associated to cells far apart, specifically, the quantity

$$\sum_{\Sigma_2 \not\subseteq B_k} \sum_{\theta \subseteq \tau, v} \chi(T_{\theta,v}, \Sigma_1)\chi(T_{\theta,v}, \Sigma_2). \tag{3.22}$$

For each $T_{\theta,v} \not\sim B_k$,

$$\sum_{\Sigma_2 \not\subseteq B_k} \chi(T_{\theta,v}, \Sigma_2) \gtrsim \sum_{\Sigma} \chi(T_{\theta,v}, \Sigma), \tag{3.23}$$

otherwise the B_k^* that maximizes $\sum_{\Sigma \cap O \subseteq B_k^*} \chi(T_{\theta,v}, \Sigma)$ should belong to $5B_k$, and we should have $T_{\theta,v} \sim B_k$ by definition 3.12.

Inequality 3.23 is the only information we need from the two ends reduction. Since $\sum_{\Sigma} \chi(T_{\theta,v}, \Sigma) \sim \gamma$, we have

$$\sum_{\Sigma_2 \not\subseteq B_k} \chi(T_{\theta,v}, \Sigma_2) \gtrsim \gamma. \tag{3.24}$$

Assume that the number of nonzero wave packets $Ef_{\theta,v}$ satisfying $\theta \subset \tau$, $T_{\theta,v} \not\sim B_k$ and $\chi(T_{\theta,v}, \Sigma_1) = 1$ is $(\frac{R}{r})^{\beta_1}$. Then we have the following lower bound for 3.22 by combining inequality 3.23 with inequality 3.24,

$$\sum_{\Sigma_2 \not\subseteq B_k} \sum_{\theta \subseteq \tau, v} \chi(T_{\theta,v}, \Sigma_1)\chi(T_{\theta,v}, \Sigma_2) \gtrsim \gamma (\frac{R}{r})^{\beta_1}. \tag{3.25}$$

Next we are going to give an upper bound for 3.22. To do so, we introduce the following geometric observation.

When the distance between O_1 and O_2 is at least $R^{1-\delta}$ and the normal directions of the corresponding Σ_1 and Σ_2 form an angle at most 1/100, a broom $\mathscr{B}$ rooted at Σ_2 can intersect with Σ_1 in at most $R^{O(\delta)}$ tubes $T_{\theta,v}$. The reason is the following. Since O_1 and O_2 have distance at least $R^{1-\delta}$, near O_1 the tubes $T_{\theta,v} \in \mathscr{B}$ are almost disjoint (up to $R^{O(\delta)}$ tubes overlapping at one point). Moreover, since the normal vectors of Σ_1 and Σ_2 form an angle of at most 1/100 and the tubes in $\mathscr{B}$ span on the normal direction of Σ_2 (See Remark 3.8), there are at most $R^{O(\delta)}$ tubes from $\mathscr{B}$ intersecting Σ_1.

By our assumption that every non-empty broom has size about b, the number of wave packets $T_{\theta,v}$ shared by Σ_1 and Σ_2 is at most $R^{O(\delta)}b^{-1}$ times the number of wave packets intersecting Σ_2.

For each $\Sigma_2 \not\subseteq B_k$, we have

$$\sum_{\theta\subseteq\tau,v} \chi(T_{\theta,v},\Sigma_1)\chi(T_{\theta,v},\Sigma_2) \lesssim R^{O(\delta)}b^{-1} \sum_{\theta\subseteq\tau,v} \chi(T_{\theta,v},\Sigma_2). \tag{3.26}$$

Since each wave packet $T_{\theta,v}$ interacts roughly γ planes Σ_2: $\sum_{\Sigma_2}\chi(T_{\theta,v},\Sigma_2) \sim \gamma$ and there are about $(\frac{R}{r})^{\beta_0}$ nonzero wave packets $Ef_{\theta,v}$ with $\theta \subseteq \tau$, so

$$\sum_{\theta\subseteq\tau,v}\sum_{\Sigma_2}\chi(T_{\theta,v},\Sigma_2) \lesssim \gamma(\frac{R}{r})^{\beta_0}. \tag{3.27}$$

We sum over $\Sigma_2 \not\subseteq B_k$ with inequality 3.26 and then apply inequality 3.27, we have the following upper bound for 3.22,

$$\sum_{\Sigma_2\not\subseteq B_k}\sum_{\theta\subseteq\tau,v} \chi(T_{\theta,v},\Sigma_1)\chi(T_{\theta,v},\Sigma_2) \lesssim R^{O(\delta)}(\frac{R}{r})^{\beta_0}\gamma b^{-1}. \tag{3.28}$$

We compare the lower bound 3.25 and upper bound 3.28 for 3.22 to obtain

$$(\frac{R}{r})^{\beta_1} b \lesssim R^{O(\delta)}(\frac{R}{r})^{\beta_0}. \tag{3.29}$$

To finish the proof, we need to apply Lemma 3.10. We define

$$Eh_\tau = \sum_{T_{\theta,v}\cap\Sigma_1\neq\emptyset,\theta\subset\tau} Ef^{\sim}_{O,\theta,v}. \tag{3.30}$$

Here $Ef^{\sim}_{O,\theta,v} = Ef_{\theta,v}$ if $T_{\theta,v} \not\sim B_k$, with $O \subset B_k$ and $T_{\theta,v}\cap O \neq \emptyset$. Otherwise $Ef^{\sim}_{O,\theta,v} = 0$. Restricted on S, the $r^{1/2}$–neighborhood of Σ_1, we have

$$Eh_\tau = Ef^{\sim}_{O,\text{tang},\tau} + \text{RapDec}(R)\|f\|_{L^2}.$$

We apply Lemma 3.10 with

$$\int_S |Ef^{\sim}_{O,\mathrm{tang},\tau}|^2 \lesssim \int_S |Eh_\tau|^2 \lesssim (\frac{R}{r})^{-1/2} b \int_{B_r} |Eh_\tau|^2.$$

Since $\|Ef^{\sim}_{O,\mathrm{tang},\tau}\|^2_{L^2(S)} \approx \|Ef^{\sim}_{O,\mathrm{tang},\tau}\|^2_{L^2(B_r)} \approx r\|f^{\sim}_{O,\mathrm{tang},\tau}\|^2_{L^2}$ and $\|Eh_\tau\|^2_{L^2(B_r)} \approx r\|h_\tau\|^2_{L^2}$, we have

$$\|f^{\sim}_{O,\mathrm{tang},\tau}\|^2_{L^2} \lesssim (\frac{R}{r})^{-1/2} b \|h_\tau\|^2_{L^2}. \tag{3.31}$$

There are $(\frac{R}{r})^{\beta_1}$ out of $(\frac{R}{r})^{\beta_0}$ nonzero large wave packets $Ef_{\theta,v}$ with $\theta \subseteq \tau$ intersecting Σ_1 and $T_{\theta,v} \not\sim B_k$, hence

$$\|h_\tau\|^2_{L^2} \lesssim (\frac{R}{r})^{\beta_1-\beta_0} \|f_\tau\|^2_{L^2}.$$

Combining with inequality 3.31 and inequality 3.29, we obtain

$$\|f^{\sim}_{O,\mathrm{tang},\tau}\|^2_{L^2} \lesssim R^{O(\delta)} (\frac{R}{r})^{-1/2} \|f_\tau\|^2_{L^2}.$$

□

Using Lemma 3.13 we prove Theorem 4 for the case $R^{1/2} \le r \le R^{1-\delta}$ in Lemma 3.14. The case $r \ge R^{1-\delta}$ was covered in Lemma 3.2. We discuss the case $r \le R^{1/2}$ in the next subsection.

Lemma 3.14. *When $R^{1/2} \le r \le R^{1-\delta}$ and $p > 3 + \frac{3}{13}$,*

$$\|Ef\|^p_{BL^p(B_R)} \le C_\epsilon R^\epsilon \|f\|^2_{L^2} \max_{d(\theta)=R^{-1/2}} \|f_\theta\|^{p-2}_{L^2_{avg}(\theta)}.$$

Proof. After the two ends reduction and Lemma 3.4, it suffices to consider when $Ef^{\sim}_O$ dominates Ef_O for most of the cells O. Furthermore, using the white lie 2 assumption, we assume that $Ef^{\sim}_O \approx Ef^{\sim}_{O,\mathrm{tang}}$ on the fat plane. In reality this might not be true, and we are going to treat it carefully using Lemma 5.3. So we have

$$\|Ef\|^p_{BL^p(B_R)} \lesssim R^{O(\delta)} \sum_O \|Ef^{\sim}_{O,\mathrm{tang}}\|^p_{BL^p(O)}$$

and we apply inequality 2.9 and Lemma 2.7 at scale r,

$$\begin{aligned}\|Ef\|^p_{BL^p(B_R)} &\lesssim R^{O(\delta)}\sum_O \|Ef^{\sim}_{O,\text{tang}}\|^p_{BL^p(O)}\\ &\lesssim R^{O(\delta)}\sum_O r^{\frac{5}{2}-\frac{3p}{4}}\|f^{\sim}_{O,\text{tang}}\|^p_{L^2}\end{aligned}$$

We are going to estimate $\sum_O \|f^{\sim}_{O,\text{tang}}\|^p_{L^2}$ in two ways. We need to apply the following L^2–estimate similar to inequality 3.14, which we prove later in Lemma 3.15,

$$\sum_O \|f^{\sim}_{O,\text{tang}}\|^2_{L^2} \le \sum_O \|f^{\sim}_O\|^2_{L^2} \lesssim DR^{\delta}\|f\|^2_{L^2}. \tag{3.32}$$

Firstly, we combine with Lemma 3.13 for $\|f^{\sim}_{O,\text{tang},\tau}\|_{L^2_{avg}(\tau)}$ we obtain the following estimate,

$$\sum_O \|f^{\sim}_{O,\text{tang}}\|^p_{L^2} \lesssim r^{-\frac{p-2}{4}}\sum_O \|f^{\sim}_{O,\text{tang}}\|^2_{L^2}\max_{d(\tau)=r^{-1/2}}\|f^{\sim}_{O,\text{tang},\tau}\|^{p-2}_{L^2_{avg}(\tau)} \tag{3.33}$$

$$\lesssim R^{-\frac{p-2}{4}+O(\delta)}D\|f\|^2_{L^2}\max_{d(\theta)=R^{-1/2}}\|f_\theta\|^{p-2}_{L^2_{avg}(\theta)}. \tag{3.34}$$

Secondly, by Lemma 3.11, there are more than $D^3R^{-\delta}$ cells O, each one has approximately the same $\|Ef^{\sim}_{O,\text{tang}}\|_{BL^p(O)}$. Then for a typical cell O,

$$\|f^{\sim}_{O,\text{tang}}\|^2_{L^2} \le \frac{DR^{\delta}}{\#\{O\}}\|f\|^2_{L^2}.$$

Hence,

$$\sum_O \|f^{\sim}_{O,\text{tang}}\|^p_{L^2} \lesssim R^{O(\delta)}\#\{O\}^{1-\frac{p}{2}}D^{p/2}\|f\|^p_{L^2} \tag{3.35}$$

$$\lesssim R^{O(\delta)}D^{3-p}\|f\|^p_{L^2}. \tag{3.36}$$

We compare estimate 3.34 and estimate 3.36, the worst case happens when $D^{3-p} = DR^{-\frac{p-2}{4}}$, which is $D = R^{1/4}$. In this case, by the definition of r, we know that $r \le \frac{R}{D} \le$

D^3, so

$$\begin{aligned}\|Ef\|^p_{BL^p(B_R)} &\lesssim R^{O(\delta)} r^{\frac{5}{2}-\frac{3p}{4}} R^{-\frac{p-2}{4}} D \|f\|^2_{L^2} \max_{d(\theta)=R^{-1/2}} \|f_\theta\|^{p-2}_{L^2_{avg}(\theta)} \\ &\lesssim R^{O(\delta)} D^{3(\frac{5}{2}-\frac{3p}{4})} D^{-(p-2)+1} \|f\|^2_{L^2} \max_{d(\theta)=R^{-1/2}} \|f_\theta\|^{p-2}_{L^2_{avg}(\theta)} \\ &\lesssim R^{O(\delta)} D^{\frac{21}{2}-\frac{13p}{4}} \|f\|^2_{L^2} \max_{d(\theta)=R^{-1/2}} \|f_\theta\|^{p-2}_{L^2_{avg}(\theta)} \\ &\lesssim R^{\epsilon} \|f\|^2_{L^2} \max_{d(\theta)=R^{-1/2}} \|f_\theta\|^{p-2}_{L^2_{avg}(\theta)}.\end{aligned}$$

The constant term is bounded by R^ϵ when $p > \frac{42}{13}$. □

Finally, we prove the L^2–inequality for $f_O^\sim$.

Lemma 3.15. *If $O \subseteq B_r$ are cells with $r \le R^{1-\delta}$, then $\sum_O \|f_O^\sim\|^2_{L^2} \lesssim DR^\delta \|f\|^2_{L^2}$.*

Proof. Assume that $O \subset B_k$, where B_k is a ball of radius $R^{1-\delta}$. Assume that there are m steps in the celluar case and n steps in the transversal case with $n \le \delta^{-2}$.

Then O is contained in the parent cells O_t in such a way:

$$O = O_{n+m} \subset \cdots \subset O_j \subset B_k$$

and $O_t \subset B_{r_t}$. We do wave packet decomposition of $Ef_k^\sim$ in B_{r_t}:

$$Ef_k^\sim = \sum_{d(\tau_t)=r_t^{-1/2}, w_t} Ef^\sim_{k,\tau_t,w_t}.$$

By definition $Ef^\sim_{O_t} = \sum_{T_{\tau_t,w_t} \cap O \neq \emptyset} Ef^\sim_{k,\tau_t,w_t}$.

If the tth step is the cellular case, aach T_{τ_t,w_t} passes through at most d cells O_{t+1} in the same parent cell O_t.

Otherwise, the tth step is the transversal case, each T_{τ_t,w_t} passes through at most Poly(d) cells O_{t+1} in the same parent cell O_t.

Hence,

$$\sum_{O} \|f_O^{\sim}\|_{L^2}^2 \lesssim d^m \operatorname{Poly}(d)^n \|f\|_{L^2}^2$$
$$\lesssim DR^{O(\delta)} \|f\|_{L^2}^2.$$

□

The case when $r \le R^{1/2}$

Now we discuss the case when $R^{\delta} \le r \le R^{1/2}$. Notice that in this case, a cell can be completely contained in a large tube $T_{\theta,v}$. In particular, two cells with distance $R^{1-\delta}$ share at most $R^{O(\delta)}$ large wave packets. Because of this property, we define a bush structure and use bushes to count wave packets. The arguments are similar but simpler.

Definition 3.16. *When $r \le R^{1/2}$, given a cell $O \subset B_r$ and a cap τ of radius $r^{-1/2}$, we define a* bush *$\mathscr{U}$ to be the collection of nonzero large wave packets $Ef_{\theta,v}$ such that $T_{\theta,v} \cap O \neq \emptyset$ and $\theta \subseteq \tau$. We say that the bush $\mathscr{U}$ is rooted at cell O and we define the number of wave packets in $\mathscr{U}$ as the size of bush $\mathscr{U}$.*

The bush is an analogue of the broom structure, and they share several common properties. Similar to Lemma 3.10, the size of a bush is proportional to the L^2–mass near a plane.

Lemma 3.17. *If $r \le R^{1/2}$ and $\mathscr{U}$ is a bush of size u rooted at cell $O \subseteq B_r$ in direction τ, and $Eg_{\mathscr{U}} = \sum_{T_{\theta,v} \in \mathscr{U}} Eg_{\theta,v}$, then for any plane Σ intersecting B_r and its $r^{1/2}$–neighborhood S, we have*

$$\|Eg_{\mathscr{U}}\|_{L^2(S)}^2 \le r^{-1/2} u \int_{B_r} |Eg_{\mathscr{U}}|^2.$$

Proof. Inside B_r, we decompose $Eg_{\mathscr{U}} = \sum_{|\theta'|=r^{-1}} Eg_{\mathscr{U},\theta'}$. Since there are at most u nonzero wave packets $Ef_{\theta,v}$ in bush $\mathscr{U}$, the number of θ' such that $g_{\mathscr{U},\theta'} \neq 0$ is at most u. We apply Cauchy-Schwartz inequality,

$$\begin{aligned}
\int_S |Eg_{\mathscr{U}}|^2 &\le \int_S |\sum_{|\theta'|=r^{-1}} Eg_{\mathscr{U},\theta'}|^2 \\
&\le u \int_S \sum_{|\theta'|=r^{-1}} |Eg_{\mathscr{U},\theta'}|^2 \\
&\lesssim r^{-1/2} u \int_{B_r} \sum_{|\theta'|=r^{-1}} |Eg_{\mathscr{U},\theta'}|^2 \\
&\lesssim r^{-1/2} u \int_{B_r} |Eg_{\mathscr{U}}|^2.
\end{aligned}$$

We applied the property that each $|Eg_{\mathscr{U},\theta'}|$ is essentially constant on B_r. □

We note that Lemma 3.17 is only useful when the size of a bush is smaller than $r^{1/2}$.

After some dyadic pigeonholing, it suffices to consider the following special case.

- for a fixed cap τ of radius $r^{-1/2}$ in the support of $f^{\sim}_{O,\mathrm{tang}}$, every bush in the direction $G(\tau)$ has size about u,
- every nonzero wave packet $Ef_{\theta,v}$ with $\theta \subseteq \tau$ intersects about γ cells, in other words, it belongs to about γ bushes.

We leave the discussion of the general case in Section 6.2. We introduce the function χ to count the bushes. We define $\chi(T_{\theta,v}, O) = 1$ if $T_{\theta,v}$ intersects O, otherwise $\chi(T_{\theta,v}, O) = 0$. We describe the above special case with function χ:

1. if the cap $\tau \subseteq \mathrm{supp} f^{\sim}_{O,\mathrm{tang}}$, then $\sum_{\theta\subseteq\tau,v} \chi(T_{\theta,v}, O) \sim u$,
2. for each nonzero wave packet $Ef_{\theta,v}$, we have $\sum_O \chi(T_{\theta,v}, O) \sim \gamma$.

We define the relation $T_{\theta,v} \sim B_k$ between a tube $T_{\theta,v}$ and a ball B_k of radius $\rho = R^{1-\delta}$ using the bush structure.

Definition 3.18. *For a fixed nonzero wave packets $Ef_{\theta,v}$, let B_k^* be the ball of radius ρ that maximizes the quantity $\sum_{O\subseteq B_k^*} \chi(T_{\theta,v}, O)$. If there are multiple candidates for B_k^*, we arbitrarily choose one of them. We define $T_{\theta,v} \sim B_k$ if $B_k \subseteq 10B_k^*$, otherwise we define $T_{\theta,v} \nsim B_k$.*

A nonzero wave packet is related to at most $O(1)$ balls B_k. We prove an analogue of Lemma 3.13 when $r \le R^{1/2}$ using the function χ defined above and Lemma 3.17.

Lemma 3.19. *When $r \le R^{1/2}$, under the assumption* (1) *and* (2)*x, we have*

$$\|f^{\sim}_{O,\text{tang},\tau}\|^2_{L^2_{avg}(\tau)} \lesssim r^{-1/2} R^{O(\delta)} \max_{d(\theta)=R^{-1/2},\theta\subseteq\tau} \|f_\theta\|^2_{L^2_{avg}(\theta)}.$$

Proof. We apply similar arguments as in Lemma 3.13. We count the number of large wave packets $Ef_{\theta,v}$ within the direction $G(\tau)$ shared by two far apart cells $O_1 = O$ and O_2. More specifically, we would like to double count the quantity

$$\sum_{O_2 \not\subseteq 5B_k} \sum_{\theta\subseteq\tau,v} \chi(T_{\theta,v},O_1)\chi(T_{\theta,v},O_2). \tag{3.37}$$

First we would like to give a lower bound for the quantity 3.37. For each tube $T_{\theta,v} \sim B_k$, we have

$$\sum_{O_2 \not\subseteq 5B_k} \chi(T_{\theta,v},O_2) \gtrsim \sum_{O'} \chi(T_{\theta,v},O'). \tag{3.38}$$

Otherwise, the ball B_k^* that maximizes $\sum_{O\subseteq B_k^*} \chi(T_{\theta,v},O)$ should belong to $5B_k$, which violates the assumption that $T_{\theta,v} \sim B_k$. By our assumption, we have

$$\sum_{O'} \chi(T_{\theta,v},O') \gtrsim \gamma. \tag{3.39}$$

Assume that there are $(\frac{R}{r})^{\beta_1}$ nonzero wave packets $Ef_{\theta,v}$ such that $\theta \subseteq \tau$, $T_{\theta,v}$ intersects O_1 and $T_{\theta,v} \sim B_k$. Combine inequality 3.38 with inequality 3.39, we obtain a lower bound for 3.37,

$$\sum_{O_2 \not\subseteq 5B_k} \sum_{\theta\subseteq\tau,v} \chi(T_{\theta,v},O_1)\chi(T_{\theta,v},O_2) \gtrsim (\frac{R}{r})^{\beta_1}\gamma. \tag{3.40}$$

We point out that $(\frac{R}{r})^{\beta_1}$ might be smaller than the size of bush u, since we have added an extra condition $T_{\theta,v} \sim B_k$.

Next we are going to give an upper bound for the quantity 3.37. Fix a pair of cells O_1 and O_2 with distance $R^{1-\delta}$. Since each cell lies inside a ball of radius $r \le R^{1/2}$, the number of large wave packets shared by two cells is at most $R^{O(\delta)}$. Since the bush in direction $G(\tau)$ rooted at O_2 has size about u, for a pair of far apart cells O_1 and O_2, we have

$$\sum_{\theta\subseteq\tau,v}\chi(T_{\theta,v},O_1)\chi(T_{\theta,v},O_2)\lesssim R^{O(\delta)}u^{-1}\sum_{\theta\subseteq\tau,v}\chi(T_{\theta,v},O_2). \tag{3.41}$$

By our assumption, for each nonzero wave packet $Ef_{\theta,v}$, we have $\sum_{O_2}\chi(T_{\theta,v},O_2)\lesssim\gamma$. Assume that there are $(\frac{R}{r})^{\beta_0}$ nonzero wave packets $Ef_{\theta,v}$ with $\theta\subseteq\tau$, we have

$$\sum_{\theta\subseteq\tau,v}\sum_{O_2\not\subseteq 5B_k}\chi(T_{\theta,v},O_2)\lesssim\gamma(\frac{R}{r})^{\beta_0}. \tag{3.42}$$

We sum inequality 3.41 over all the cells $O_2\not\subseteq 5B_k$ and apply inequality 3.42 to obtain the following upper bound for 3.37,

$$\sum_{O_2\not\subseteq 5B_k}\sum_{\theta\subseteq\tau,v}\chi(T_{\theta,v},O_1)\chi(T_{\theta,v},O_2)\lesssim R^{O(\delta)}u^{-1}(\frac{R}{r})^{\beta_0}\gamma. \tag{3.43}$$

Compare inequality 3.40 to inequality 3.43, we have

$$(\frac{R}{r})^{\beta_1-\beta_0}u\lesssim R^{O(\delta)}. \tag{3.44}$$

We define

$$Eg_{\mathcal{U}}=\sum_{T_{\theta,v}\cap O\neq\emptyset,\theta\subset\tau}Ef^{\sim}_{\theta,v}$$

. Here $Ef^{\sim}_{\theta,v}=Ef_{\theta,v}$ if $T_{\theta,v}\sim B_k$ with $O\subset B_k$. Otherwise, $Ef^{\sim}_{\theta,v}=0$. We apply Lemma 3.17 with $g_{\mathcal{U}}$. Then $Ef^{\sim}_{O,\text{tang},\tau}$ is $Eg_{\mathcal{U}}$ restricted on the $r^{1/2}$–neighborhood of a plane in O. Hence,

$$\|f^{\sim}_{O,\text{tang},\tau}\|^2_{L^2}\lesssim r^{-1/2}u\|g_{\mathcal{U}}\|^2_{L^2}.$$

For the fixed τ, there are $(\frac{R}{r})^{\beta_1}$ nonzero wave packets $Ef_{\theta,v}$ such that $T_{\theta,v}$ intersects O and $T_{\theta,v}\sim B_k$,

$$\|g_{\mathcal{U}}\|^2_{L^2}\lesssim(\frac{R}{r})^{\beta_1-\beta_0}\|f_\tau\|^2_{L^2}.$$

We apply inequality 3.44,

$$\|f^{\sim}_{O,\mathrm{tang},\tau}\|^2_{L^2} \lesssim r^{-1/2} R^{O(\epsilon_0)} \|f_\tau\|^2_{L^2}.$$

□

After Lemma 3.4, it suffices to consider when $Ef^{\sim}_O$ dominates. Again by the white lie, $Ef_{O,\mathrm{trans}}$ is zero, we may assume that $Ef^{\sim}_{O,\mathrm{tang}} = Ef^{\sim}_O$.

Lemma 3.20. *If $r \le R^{1/2}$ and $Ef^{\sim}_{O,\mathrm{tang}}$ dominates for most of the cells O, then for any $p > 3 + 1/5$ and for any small $\epsilon > 0$,*

$$\|Ef\|_{BL^p(B_R)} \le C_\epsilon R^\epsilon \|f\|^{2/p}_{L^2} \max_{d(\theta)=R^{-1/2}} \|f_\theta\|^{1-2/p}_{L^2_{avg}(\theta)}.$$

Proof. The proof is separated into two cases.

The first case is for $D \ge r^{1/2}$. We need only the information from polynomial partitioning. We apply inequality 3.13 and inequality 2.9, for a typical cell O,

$$\begin{aligned}\|Ef\|^p_{BL^p(B_R)} &\lesssim R^{O(\delta)} \#\{O\} \|Ef_{O,\mathrm{tang}}\|^p_{BL^p(O)} \\ &\lesssim R^{O(\delta)} r^{\frac{5}{2}-\frac{3p}{4}} \#\{O\} \|f_{O,\mathrm{tang}}\|^p_{L^2}.\end{aligned}$$

We apply the L^2 inequality 3.14, $\sum_O \|f_{O,\mathrm{tang}}\|^2_{L^2} \lesssim DR^\delta \|f\|^2_{L^2}$. There exists a cell O, such that

$$\|f_{O,\mathrm{tang}}\|^2_{L^2} \lesssim DR^\delta \#\{O\}^{-1} \|f\|^2_{L^2}.$$

Since the number of cells O is at least $D^3 R^{-O(\delta)}$, we have

$$\begin{aligned}\|Ef\|^p_{BL^p(B_R)} &\lesssim R^{O(\delta)} r^{\frac{5}{2}-\frac{3p}{4}} \#\{O\}^{1-p/2} D^{p/2} \|f\|^2_{L^2} \\ &\lesssim D^{3-p} R^{O(\delta)} r^{\frac{5}{2}-\frac{3p}{4}} \|f\|^2_{L^2}.\end{aligned}$$

Since $D \geq r^{1/2}$ and $p > 3$, we have

$$\begin{aligned}\|Ef\|^p_{BL^p(B_R)} &\lesssim R^{O(\delta)} r^{\frac{3-p}{2}+\frac{5}{2}-\frac{3p}{4}} \|f\|^2_{L^2} \\ &\lesssim R^{O(\delta)} r^{4-\frac{5p}{4}} \|f\|^2_{L^2}.\end{aligned}$$

When $p > 16/5$, the constant term is bounded by R^{ϵ}.

The second case is for $D \leq r^{1/2}$. We would like to apply the extra information, Lemma 3.19. After Lemma 3.4, it suffices to consider when the global part dominates, which corresponds to $\|Ef_O\|_{BL^p(O)}$ dominated by $\|Ef^{\sim}_{O,\text{tang}}\|_{BL^p(O)}$ for most of the cells. By inequality 3.12 and inequality 2.9,

$$\begin{aligned}\|Ef\|^p_{BL^p(B_R)} &\lesssim R^{\delta} \sum_O \|Ef^{\sim}_{O,\text{tang}}\|^p_{BL^p(O)} \\ &\lesssim R^{\delta} r^{\frac{5}{2}-\frac{3p}{4}} \sum_O \|f^{\sim}_{O,\text{tang}}\|^p_{L^2}.\end{aligned}$$

By Lemma 2.7 and Lemma 3.19,

$$\begin{aligned}\|f^{\sim}_{O,\text{tang}}\|^2_{L^2} &\lesssim r^{-1/2+O(\delta)} \max_{d(\tau)=r^{-1/2}} \|f^{\sim}_{O,\text{tang},\tau}\|^2_{L^2_{avg}(\tau)} \\ &\lesssim r^{-1} R^{O(\delta)} \max_{d(\theta)=R^{-1/2}} \|f_\theta\|^2_{L^2_{avg}(\theta)}\end{aligned}$$

Hence,

$$\|Ef\|^p_{BL^p(B_R)} \lesssim R^{O(\delta)} r^{\frac{5}{2}-\frac{3p}{4}-\frac{p-2}{2}} \sum_O \|f^{\sim}_{O,\text{tang}}\|^2_{L^2} \max_{d(\theta)=R^{-1/2}} \|f_\theta\|^{p-2}_{L^2_{avg}(\theta)}.$$

We apply Lemma 3.15,

$$\|Ef\|^p_{BL^p(B_R)} \lesssim R^{O(\delta)} D r^{\frac{5}{2}-\frac{3p}{4}-\frac{p-2}{2}} \|f\|^2_{L^2} \max_{d(\theta)=R^{-1/2}} \|f_\theta\|^{p-2}_{L^2_{avg}(\theta)}.$$

Since $D \leq r^{1/2}$,

$$\|Ef\|^p_{BL^p(B_R)} \lesssim R^{O(\delta)} r^{4-\frac{5p}{4}} \|f\|^2_{L^2} \max_{d(\theta)=R^{-1/2}} \|f_\theta\|^{p-2}_{L^2_{avg}(\theta)}.$$

When $p > 16/5$, the constant term is bounded by R^{ϵ}. □

Chapter 4

Polynomial Structure lemma

In the white lie version proof, the key properties of $f_{O,\mathrm{tang}}$ and $Ef_{O,\mathrm{tang}}$ are the following:

(a) $\sum_O \|f_{O,\mathrm{tang}}\|_{L^2}^2 \lesssim DR^\delta \|f\|_{L^2}^2$,

(b) Restricted on O, Ef_O can be written as the sum over some large wave packets $Ef_{\theta,v}$.

And $Ef_{O,\mathrm{tang}}$ is obtained by redoing wave packet decomposition in B_r

$$Ef_O = \sum_{d(\tau)=r^{-1/2},w} Ef_{O,\tau,w}$$

and by restricting Ef_O on the tangential wave packets to some low degree algebraic surface

$$Ef_{O,\mathrm{tang}} = \sum_{T_{\tau,w}\in\mathbb{T}_{O,\mathrm{tang}}} Ef_{O,\tau,w}. \tag{4.1}$$

In general, it is difficult for a function to satisfy both properties without the white lie 2 assumption. In this section, we state a structure lemma that decomposes the function Ef into functions Ef_{S_t} and $Ef_{\Pi_{S_t}}$ satisfying the above properties separately.

Definition 4.1. *Fix a large integer $d \sim \log R$ and some $R^\delta \le r \le R$, a* fat *r–surface S is the $r^{1/2+\delta}$–neighborhood of a degree d algebraic surface, which we denote by S_0, inside a ball B_r of radius r.*

Definition 4.2. *Let $T_{\tau,w}$ be a tube of length r, radius $r^{1/2}$, we say that $T_{\tau,w}$ is tangential to S if it satisfies that $2T_{\tau,w} \cap S_0 \neq \emptyset$ and*

$$\mathrm{Angle}(G(\tau), T_x(S_0)) \leq r^{-1/2+2\delta}$$

for any nonsingular point $x \in 10T_{\tau,w} \cap 2B_r \cap S_0$. Recall that $G(\tau)$ is the direction of the tube $T_{\tau,w}$. We define $\mathbb{T}_{S,\text{tang}}$ as the collection of tubes $T_{\tau,w}$ tangential to S. We define $\mathbb{T}_{S,\text{trans}}$ as the collection of tubes $T_{\tau,w}$ such that $2T_{\tau,w} \cap S_0 \neq \emptyset$ and $T \notin \mathbb{T}_{S,\text{tang}}$.

The following lemma says that outside a collection of fat surfaces, we have a good bound about Ef. Similar statement can also be found in discrete geometry. For example, the paper by Guth [4] says that either we have a good incidence bound, or the lines are contained in a collection of low degree algebraic surfaces.

Lemma 4.3. *If f is supported in the unit disk and Ef is supported in B_R, then there exists a collection of disjoint cells O and n collections $\mathscr{S}_t$ of fat r_t–surfaces S_t, $1 \leq t \leq n \leq \delta^{-2}$ and $r_n < \cdots < r_1$, such that*

$$\|Ef\|^p_{BL^p(B_R)} \lesssim R^{O(\delta)} \sum_O \|Ef\|^p_{BL^p(O)} \tag{4.2}$$

and

1. *Each O is contained in a ball of radius $r_0 \approx R^\delta < r_n$ and $\|Ef\|^p_{BL^p(O)}$ has about the same size for all O, the number of cells is greater than $D^3R^{-\delta}$.*
2. *Each collection $\mathscr{S}_t, 1 \leq t \leq n$, consists of more than $D_t^3 R^{-\delta}$ disjoint the fat r_t–surfaces S_t with $D_t \leq R/r_t$. Every S_t contains about the same number of S_{t+1} for $1 \leq t \leq n-1$ and every S_n contains about the same number of O.*
3. *For each O, there exists a chain $O \subseteq S_n \cdots \subseteq S_1$ with*

$$Ef|_O = Ef_O + \sum_{t=1}^n Ef_{S_t} + \mathrm{RapDec}(R)\|f\|_{L^2},$$

where the function Ef_{S_t} is a sum ave wave packets tangential to S_t. The functions f_O and f_{S_t} are defined in inequalities 4.3, 4.6 and 4.4, 4.7 in the proof.

(4) *For each* S_t*, there exists a chain* $S_t \subseteq S_{t-1} \cdots \subseteq S_1$ *with* $Ef_{\Pi_{S_t}} = Ef_{S_t} + \sum_{l=1}^{t-1} Ef_{S_l,S_t} + \mathrm{RapDec}(R)\|f\|_{L^2}$ *where* $Ef_{S_l,S_t} := \sum_{T_{\tau_t,w_t} \in \mathbb{T}_{S_t,\mathrm{tang}}} Ef_{S_l,\tau_t,w_t}$.

(5) *We have the* L^2*–bounds* $\sum_{S_t} \|f_{S_t}\|_{L^2}^2 \lesssim D_t R^{\delta} \|f\|_{L^2}^2$ *and* $\sum_O \|f_O\|_{L^2}^2 \le DR^{\delta}\|f\|_{L^2}^2$.

Proof. We apply the polynomial partitioning on Ef iteratively until the diameter of a cell is reduced to R^{δ}. We record the tangential parts along the iteration process.

Initial Step. We apply polynomial partitioning on Ef in B_R as in Subsection 2.4. Let Z_1 be the zero set of the degree d partitioning polynomial and W_1 be the $R^{1/2+\delta}$–neighborhood of Z_1. If we are in the cellular case, then each cell O_1 lies inside a ball of radius $R_1 = R/d$ and

$$\sum_{O_1} \|Ef\|_{BL^p(O_1)}^p \gtrsim \|Ef\|_{BL^p(B_R)}^p,$$

$$\|Ef\|_{BL^p(O_1)}^p \lesssim d^{-3}\|Ef\|_{BL^p(B_R)}^p.$$

We define

$$Ef_{O_1} = \sum_{T_{\theta,v} \cap O_1 \neq \emptyset} Ef_{\theta,v}, \tag{4.3}$$

and we have $\sum_{O_1} \|f_{O_1}\|_{L^2}^2 \lesssim d\|f\|_{L^2}^2$ and $\|Ef\|_{BL^p(O_1)}^p \le \|Ef_{O_1}\|_{BL^p(O_1)}^p + \mathrm{RapDec}(R)\|f\|_{L^2}^p$.

Otherwise, we are in algebraic case, $\|Ef\|_{BL^p(W_1)}^p \gtrsim \|Ef\|_{BL^p(B_R)}^p$. We cover W_1 with balls B_k of radius $R_1 = R^{1-\delta}$ and we define $r_1 = R_1$, $S_1 = B_k \cap W_1$ and

$$Ef_{S_1} = Ef_{k,\mathrm{tang}}. \tag{4.4}$$

One can see that S_1 is a fat r_1–surface in B_k. Ef_{S_1} is a sum of wave packets tangential to S_1 and $Ef_{S_1} = Ef_{\Pi_{S_1}} + \mathrm{RapDec}(R)\|f\|_{L^2}$.

We also define $D_1 = 1$, $O_1 = S_1$ and $Ef_{O_1} = Ef_{k,\mathrm{trans}}$. The Ef_{O_1} satisfies: $\sum_{O_1} \|f_{O_1}\|_{L^2}^2 \lesssim \mathrm{Poly}(d)\|f\|_{L^2}^2$. Restricted on $O_1 \subseteq S_1$, we have $Ef = Ef_{O_1} + Ef_{S_1} + \mathrm{RapDec}(R)\|f\|_{L^2}$.

Iteration Step. Assume that we have run the polynomial partitioning j steps, and we have defined $O_j \subseteq B_{R_j}$ and $Ef_{O_j}, Ef_{S_1}, \ldots, Ef_{S_t}$ satisfying:

- there exists a chain $O_j \subseteq S_t \subseteq \cdots \subseteq S_1$. Restricted on each O_j, we have $Ef = Ef_{O_j} + Ef_{S_1} + \cdots + Ef_{S_t} + \mathrm{RapDec}(R)\|f\|_{L^2}$;

- we have the following L^2 estimates: $\sum_{O_j} \|f_{O_j}\|_{L^2}^2 \lesssim d^j \mathrm{Poly}(d)^t \|f\|_{L^2}^2$ and $\sum_{S_l} \|f_{S_l}\|_{L^2}^2 \lesssim R^\delta D_l \|f\|_{L^2}^2$ for $1 \le l \le t$;

- $Ef_{\Pi_{S_l}} = Ef_{S_l} + Ef_{S_1,S_l} + \cdots + Ef_{S_{l-1},S_l} + \mathrm{RapDec}(R)\|f\|_{L^2}$ for all $1 \le l \le t$;

- $\sum_{O_j} \|Ef\|_{BL^p(O_j)}^p \gtrsim \|Ef\|_{BL^p(B_R)}^p$ and

$$\|Ef\|_{BL^p(O_j)}^p \lesssim d^{-3(j-t)} \|Ef\|_{BL^p(B_R)}^p. \tag{4.5}$$

We apply polynomial partitioning on $\|Ef\|_{BL^p(O_j)}^p$ in each O_j. Let Z_{j+1} be the zero set of the partitioning polynomial and W_{j+1} be the $R_j^{1/2+\delta}$–neighborhood of Z_{j+1}. We do wave packet decomposition of Ef_{O_j} inside B_{R_j}: $Ef_{O_j} = \sum_{\tau_j,w_j} Ef_{O_j,\tau_j,w_j}$.

If for more than 1/2 fraction of the cells O_j we are in the cellular case, then we keep only those O_j and we define $Ef_{O_{j+1}} = \sum_{T_{\tau_j,w_j} \cap O_{j+1} \neq \emptyset} Ef_{O_j,\tau_j,w_j}$ and we write $R_{j+1} = R_j/d$. We have the following L^2–estimate,

$$\sum_{O_{j+1}} \|f_{O_{j+1}}\|_{L^2}^2 \le \sum_{O_j} d\|f_{O_j}\|_{L^2}^2 \le d^{j+1} \mathrm{Poly}(d)^t \|f\|_{L^2}^2.$$

We have the following BL^p–estimates,

$$\sum_{O_{j+1}} \|Ef\|_{BL^p(O_{j+1})}^p \gtrsim \|Ef\|_{BL^p(B_R)}^p,$$

and

$$\|Ef\|_{BL^p(O_{j+1})}^p \lesssim d^{-3} \|Ef\|_{BL^p(O_j)}^p \lesssim d^{-3(j+1-t)} \|Ef\|_{BL^p(B_R)}^p.$$

Otherwise, for more than 1/2 fraction of the cells we are in the algebraic case, then we keep only those O_j and we define $R_{j+1} = R_j^{1-\delta}$ and we cover W_{j+1} with balls of radius R_{j+1}, we denote by $r_{t+1} = R_{j+1}$ and $S_{t+1} = O_{j+1} = W_{j+1} \cap B_{R_{j+1}}$. Here S_{t+1} is a fat r_{t+1}–surface inside $B_{r_{t+1}}$. We define

$$Ef_{O_{j+1}} = \sum_{T_{\tau_j,w_j} \in \mathbb{T}_{S_{t+1},\mathrm{trans}}} Ef_{O_j,\tau_j,w_j} \tag{4.6}$$

and

$$Ef_{S_{t+1}} = \sum_{T_{\tau_j,w_j} \in \mathbb{T}_{S_{t+1},\text{tang}}} Ef_{O_j,\tau_j,w_j}. \tag{4.7}$$

Restricted on each $O_{j+1} = S_{t+1}$, we have $Ef_{O_j} = Ef_{O_{j+1}} + Ef_{S_{t+1}}$, so

$$Ef = Ef_{O_{j+1}} + Ef_{S_{t+1}} + \cdots + Ef_{S_1} + \text{RapDec}(R)\|f\|_{L^2}. \tag{4.8}$$

We do wave packets decomposition of each function in equation 4.8 inside B_{R_j} and we take only the wave packets tangential to S_{t+1}, then equation 4.8 leads to

$$Ef_{\Pi_{S_{t+1}}} = Ef_{S_{t+1}} + Ef_{S_t,S_{t+1}} + \cdots + Ef_{S_1,S_{t+1}} + \text{RapDec}(R)\|f\|_{L^2}.$$

We define $r_{t+1} = R_{j+1}$ and $D_{t+1} = d^{j-t}$. From the definition of r_{t+1}, we know that $D_{t+1} \le R/r_{t+1}$. We have the following L^2–estimates for $f_{O_{j+1}}$ and $f_{S_{t+1}}$:

$$\sum_{O_{j+1}} \|f_{O_{j+1}}\|_{L^2}^2 \lesssim \sum_{O_j} d\|f_{O_j}\|_{L^2}^2 \le d^j \text{Poly}(d)^{t+1}\|f\|_{L^2}^2,$$

$$\sum_{S_{t+1}} \|f_{S_{t+1}}\|_{L^2}^2 \lesssim \sum_{O_j} \sum_{S_{t+1} \subseteq O_j} \|f_{S_{t+1}}\|_{L^2}^2 \lesssim R^\delta \sum_{O_j} \|f_{O_j}\|_{L^2}^2 \le R^\delta d^j \text{Poly}(d)^t \|f\|_{L^2}^2.$$

We have as well the BL^p–estimate:

$$\sum_{O_{j+1}} \|Ef\|_{BL^p(O_{j+1})}^p \gtrsim \|Ef\|_{BL^p(B_R)}^p$$

and

$$\|Ef\|_{BL^p(O_{j+1})}^p \lesssim \|Ef\|_{BL^p(O_j)}^p \lesssim d^{-3(j+1-(t+1))}\|Ef\|_{BL^p(B_R)}^p.$$

When $R_{j+1} \le R^\delta$, we stop and define $r_0 = R_{j+1}$, $O = O_{j+1}$ and $D = d^{j+1-t}$. For each algebraic case, we lower the radius R_j of the spatial ball to $R_j^{1-\delta}$. So the number of algebraic steps is bounded by n with $R^{(1-\delta)^n} \le R^\delta$. In particular, $n \le \delta^{-2}$. Recall that we choose $d = \log R$, and we can choose R large enough such that $\text{Poly}(d)^n \le R^{\delta^2}$.

Now we have more than D^3 cells O with

$$\sum_O \|Ef\|^p_{BL^p(O)} \gtrsim \|Ef\|^p_{BL^p(B_R)}, \tag{4.9}$$

and

$$\|Ef\|^p_{BL^p(O)} \lesssim D^{-3}\|Ef\|^p_{BL^p(B_R)}. \tag{4.10}$$

So far we have verified properties (3), (4), (5) in the lemma. In order to have property (1) and (2), we need to sort the cells O and the fat r_t–surfaces S_t. We sort the cells O according to $\|Ef\|^p_{BL^p(O)}$, which we denote by λ. There exists a dyadic λ with about Y cells O such that $\|Ef\|^p_{BL^p(O)} \sim \lambda$ and $Y\lambda \gtrsim (\log R)^{-1}\|Ef\|^p_{BL^p(B_R)}$. We keep only those O. By inequality 4.10, $Y \gtrsim (\log R)^{-1}D^3$.

Now we have fixed our choice of O, we then sort S_n according to the number of cells O it contains. After dyadic pigeonholing, there exists a collection $\mathscr{S}_n$ such that: each $S_n \in \mathscr{S}_n$ contains about the same number of O, and a logarithmic fraction of cells O are contained in $S_n \in \mathscr{S}_n$. By inequality 4.5, the number of S_n in $\mathscr{S}_n$ is at least $(\log R)^{-2}D_n$.

We sort $S_{n-1},\dots,S_1$ in the same way. In the end, we have collections $\mathscr{S}_t$, $1 \le t \le n$ satisfying:

$$\#\mathscr{S}_t \gtrsim (\log R)^{-n-1}D_t^3 \gtrsim R^{-\delta}D_t^3$$

because $n \le \delta^{-2}$.

Now our set of O and the collections $\mathscr{S}_t$ satisfy property (1) and (2). □

Corollary 4.4. *If $f = f_1 + f_2$ are supported in the unit disk, then for S_t and O defined in Lemma 4.3, then we can define Ef_{i,S_t}, $Ef_{i,\Pi_{S_t}}$ and $Ef_{i,O}$, $i = 1,2$ satisfying property (3), (4), (5) in Lemma 4.3 with $Ef_{S_t} = Ef_{1,S_t} + Ef_{2,S_t}$, $Ef_{\Pi_{S_t}} = Ef_{1,\Pi_{S_t}} + Ef_{2,\Pi_{S_t}}$ and $Ef_O = Ef_{1,O} + Ef_{2,O}$.*

Proof. By the proof of Lemma 4.3, we can see that the construction of Ef_O, Ef_{S_t} and $Ef_{\Pi_{S_t}}$ for $1 \le t \le n$ is linear and only depends on O and S_t, $1 \le t \le n$. □

Chapter 5

Two ends argument and some easy cases

We cover B_R with balls B_k of radius $R^{1-\epsilon_0}$, $\delta \ll \epsilon_0 \ll \epsilon$. For each B_k, we define in the next section some relation $\sim$ between a large tube $T_{\theta,v}$ and B_k. The relations $\sim$ has the following property: for a fixed $T_{\theta,v}$, the number of balls $B_k \sim T_{\theta,v}$ is bounded by $O_\delta(1)$. For each ball B_k, we define $Ef_k^{\sim} = \sum_{T_{\theta,v} \sim B_k} Ef_{\theta,v}$ and $Ef_k^{\nsim} = \sum_{T_{\theta,v} \nsim B_k} Ef_{\theta,v}$. We also define the functions $Ef^{\sim}$ and $Ef^{\nsim}$ on B_R by $Ef^{\sim}|_{B_k} = Ef_k^{\sim}$ and $Ef^{\nsim}|_{B_k} = Ef_k^{\nsim}$.

Restricted on each cell O, we have $Ef = Ef^{\sim} + Ef^{\nsim} + \text{RapDec}(R)\|f\|_{L^2}$. If for most cells O, $\|Ef\|_{BL^p(O)} \lesssim \|Ef^{\sim}\|_{BL^p(O)}$, then we apply Lemma 3.4 to conclude the proof of Theorem 4.

Otherwise, for most cells O, $\|Ef\|_{BL^p(O)} \lesssim \|Ef^{\nsim}\|_{BL^p(O)}$. By Corollary 4.4, we have $Ef^{\nsim}|_O = Ef_O^{\nsim} + \sum_{t=1}^n Ef_{S_t}^{\nsim}$ for $O \subset S_n \subset \cdots \subset S_1$.

By triangle inequality, we have

$$\|Ef^{\nsim}\|_{L^p(O)} \le \|Ef_O^{\nsim}\|_{BL^p(O)} + \sum_{t=1}^n \|Ef_{S_t}^{\nsim}\|_{BL^p(S_t)}.$$

If there exists $R^{-\delta}$ fraction of the cells O such that $\|Ef^{\nsim}\|_{BL^p(O)} \lesssim \|Ef_O^{\nsim}\|_{BL^p(O)}$, then we apply the following lemma 5.1.

Lemma 5.1. *If there exists $R^{-\delta}$ fraction of the cells O with*

$$\|Ef\|_{BL^p(O)} \lesssim \|Ef^{\sim}\|_{BL^p(O)} \lesssim R^{\delta}\|Ef^{\sim}_O\|_{BL^p(O)},$$

then Theorem 4 holds for Ef and for all $p > 3$.

Proof. By (1) and (5) in Lemma 4.3, Corollary 4.4 and our assumptions, we have

$$\|Ef\|^p_{BL^p(B_R)} \lesssim R^{\delta}\#\{O\}\|Ef\|^p_{BL^p(O)} \lesssim R^{O(\delta)}\#\{O\}\|Ef^{\sim}_O\|^p_{BL^p(O)} \tag{5.1}$$

$$\sum_O \|f^{\sim}_O\|^2_{L^2} \lesssim DR^{\delta}\|f\|^2_{L^2} \tag{5.2}$$

and $\#\{O\} \gtrsim D^3R^{-\delta}$, each $O \subseteq B_{R^{\delta}}$.

The rest of the proof is the same as Lemma 3.3. □

If for $r_t^{-\delta}$ fraction of the cells O, $\|Ef\|_{BL^p(O)} \lesssim r_t^{\delta}\|Ef^{\sim}_{S_t}\|_{BL^p(O)}$ with $r_t \geq R^{13/16}$, then we apply the following lemma 5.2

Lemma 5.2. *If there exists $r_t^{-\delta}$ fraction of the cells O such that*

$$\|Ef\|_{BL^p(O)} \lesssim \|Ef^{\sim}\|_{BL^p(O)} \lesssim r_t^{\delta}\|Ef^{\sim}_{S_t}\|_{BL^p(O)}$$

for an S_t with $r_t \geq R^{13/16}$, then Theorem 4 holds for Ef for any $p > 3 + 3/13$.

Proof. We apply Lemma 4.3 and Corollary 4.4, we have

$$\|Ef\|^p_{BL^p(B_R)} \lesssim R^{-\delta}\sum_O \|Ef^{\sim}_{S_t}\|^p_{BL^p(O)} \tag{5.3}$$

and

$$\sum_{S_t} \|f^{\sim}_{S_t}\|^2_{L^2} \lesssim D_t R^{\delta}\|f\|^2_{L^2}. \tag{5.4}$$

The rest of the proof is the same as Lemma 3.2 using

$$\|f^{\sim}_{S_t}\|^2_{L^2} \lesssim r_t^{-1/2+O(\delta)} \max_{d(\tau_t)=r_t^{-1/2}} \|f^{\sim}_{\tau_t}\|^2_{L^2_{avg}(\tau_t)}.$$

□

If $\|Ef^{\sim}\|_{BL^p(O)} \gtrsim R^{\delta}\|Ef^{\sim}_{O}\|_{BL^p(O)}$ for more than $1-R^{-\delta}$ fraction of the cells, then we choose the smallest t, such that there exists $R^{-\delta}$ fraction of the cells O such that

$$\|Ef^{\sim}\|_{BL^p(O)} \le r_t^{\delta}\|Ef^{\sim}_{S_t}\|_{BL^p(O)}, \tag{5.5}$$

and

$$\|Ef^{\sim}\|_{BL^p(O)} \ge r_l^{\delta}\|Ef^{\sim}_{S_l}\|_{BL^p(O)}, \text{ for all } l<t. \tag{5.6}$$

Such a t exists by pigeonholing. We first look at S_1, if there exists $R^{-\delta}$ fraction of the O with

$$\|Ef^{\sim}\|_{BL^p(O)} \lesssim r_1^{\delta}\|Ef^{\sim}_{S_1}\|_{BL^p(O)}, \tag{5.7}$$

then we choose $t=1$. Otherwise, it remains at least $1-R^{-\delta}$ fraction of the cells O. And we look at whether inequality 5.5 holds for $t=2$ for more than $r_1^{-\delta}$ fraction of the remaining cells. If so, we choose $t=2$; otherwise, there exists more than $1-R^{-\delta}-r_1^{-\delta}$ cells. We keep going. Either we find a t satisfies both inequality 5.5 and inequality 5.6, or

$$\|Ef^{\sim}\|_{BL^p(O)} \ge r_l^{\delta}\|Ef^{\sim}_{S_l}\|_{BL^p(O)}$$

for more than $1-nR^{-\delta^2} \gtrsim 1$ fraction of the cells. By triangle inequality and our assumption in the beginning of last paragraph, this can not be true.

We might assume that $R^{\delta} < r_t < R^{13/16}$, otherwise it suffices to apply Lemma 5.1 and Lemma 5.2.

We observe that $Ef^{\sim}_{\Pi_{S_t}} = Ef^{\sim}_{S_t} + \sum_{l=1}^{t-1} Ef^{\sim}_{S_l,S_t} + \mathrm{RapDec}(R)\|f\|_{L^2}$ satisfies property (b) at the beginning of Section 4, which enables us to use the broom structure and the bush structure. Using inequalities 5.5 and 5.6, we show that $Ef^{\sim}_{\Pi_{S_t}}$ and $Ef^{\sim}_{S_t}$ have about the same BL^p–norm for $R^{-\delta}$ fraction of the cells O.

Lemma 5.3. *Assume that inequality 5.5 and inequality 5.6 hold for some $1\le t\le n$, we have $\|Ef^{\sim}_{\Pi_{S_t}}\|_{BL^p(O)} \sim \|Ef^{\sim}_{S_t}\|_{BL^p(O)}$ for most of the cells O.*

Proof. By property (4) of Lemma 4.3 and Corollary 4.4, $Ef^{\sim}_{\Pi_{S_t}} = Ef^{\sim}_{S_t} + \sum_{l=1}^{t-1} Ef^{\sim}_{S_l,S_t}$. By inequalities 5.5 and 5.6, for a typical cell O, $\|Ef^{\sim}_{S_l}\|_{BL^p(O)} \le r_l^{-\delta} r_t^{\delta}\|Ef^{\sim}_{S_t}\|_{BL^p(O)}$ for all $l > t$. We show in Lemma 5.4 that

$$\|Ef^{\sim}_{S_l,S_t}\|_{BL^p(O)} \le \|Ef^{\sim}_{S_l}\|_{BL^p(O)}.$$

Since $l < t$, $r_l^{1-\delta} > r_t > R^\delta$ and $t < \delta^{-2}$, all the $Ef^{\sim}_{S_l,S_t}$ are negligible terms compared to Ef_{S_t}, by triangle inequality we finish the proof. □

Lemma 5.4. *Given a fat r_1–surface $S_1 \subset B_{r_1}$ and a fat r_2–surface $S_2 \subset B_{r_2}$ with $B_{r_1} \subseteq B_{r_2}$, $r_2^{1-\delta} \ge r_1$. Assume that Ef is tangential to S_2. We decompose $Ef|_{S_1} = Ef_{\text{tang}} + Ef_{\text{trans}}$ into the corresponding tangential and the transverse component to S_1. The following estimates hold for any ball $B \subseteq S_1$ of radius K, where K is a large constant defined in Definition 2.1.*

- $\|Ef_{\text{tang}}\|_{BL^p_A(B)} \le \|Ef\|_{BL^p_A(B)}$,
- $\|Ef_{\text{trans}}\|_{BL^p_A(B)} \le \|Ef\|_{BL^p_A(B)}$.

Proof. From the definition of fat r–surface, we know that S_j is the $r_j^{1/2+\delta}$–neighborhood of a degree d algebraic surface $S_{j,0}$, $j = 1,2$. For any smooth point $z_j \in (B + B^{1/2}_{r_j}(0)) \cap S_{j,0}$, let Σ_j be the tangent plane of $S_{j,0}$ at z_j, $j = 1,2$. By the definition of Ef_{tang}, for any $T_{\theta_j,v_j} \in \mathbb{T}_{S_j,\text{tang}}$, and $T_{\theta_j,v_j} \cap B \ne \emptyset$, we have

$$\text{Angle}(G(\theta_j),\Sigma_j) \lesssim r_j^{-1/2+2\delta} \le r_1^{-1/2+2\delta}.$$

The directions parallel to Σ_j can be represented as circles $\mathscr{C}_j$ in the unit sphere in $\mathbb{R}^3$. We decompose $\mathscr{C}_2$ into the tangential part, $\mathscr{C}_{2,\text{tang}} = \mathscr{C}_2 \cap N_{r_1^{-1/2}}\mathscr{C}_1$, and the transversal part, $\mathscr{C}_{2,\text{trans}} = \mathscr{C}_2 \setminus \mathscr{C}_{2,\text{tang}}$. The directions in the tangential part contains the directions of wave packets tangential to S and passing through B. When $r_1^{-1/2+2\delta} \le \text{Angle}(\Sigma_1,\Sigma_2) \le K r_1^{-1/2+2\delta}$, we cover $\mathscr{C}_2$ with caps $\{\alpha\}$ of radius K^{-1}. A cap α lies in either $\mathscr{C}_{2,\text{tang}}$ or $\mathscr{C}_{2,\text{trans}}$. (Note that this is not necessarily true in higher dimensions.) By the definition of the BL^p–norm,

$$\mu_{Ef}(B_K) := \min_{V_1,\dots,V_A:\text{ lines of }\mathbb{R}^3} \Big(\max_{\tau:\text{Angle}(G(\tau),V_a)\ge K^{-1}\text{ for all }a} \int_{B_K} |Ef_\tau|^p \Big).$$

The quantity $\mu_{Ef}(B) = \|Ef\|^p_{BL^p_A(B)}$ takes the $(A+1)$th largest value of $\int_B |Ef_\alpha|^p$. A cap α either belongs to the tangential part or to the transversal part, so

$$\|Ef\|^p_{BL^p_A(B)} \geq \max(\|Ef_{\text{tang}}\|^p_{BL^p_A(B)}, \|Ef_{\text{trans}}\|^p_{BL^p_A(B)}). \tag{5.8}$$

□

Recall that in the definition of broad L^p–norm, we have an underlying A:

$$\mu_{Ef}(B_K) := \min_{V_1,\ldots,V_A:\text{ lines of }\mathbb{R}^3} \Big(\max_{\tau:\text{Angle}(G(\tau),V_a)\geq K^{-1}\text{ for all }a} \int_{B_K} |Ef_\tau|^p \Big).$$

The A changes from line to line because of the subadditive property:

$$\|Ef + Eg\|_{BL^p_{A_1+A_2}(B_K)} \leq \|Ef\|_{BL^p_{A_1}(B_K)} + \|Eg\|_{BL^p_{A_2}(B_K)}.$$

Everytime we use the triangle inequality of the broad L^p–norm, we need to reduce A. In inequality 5.8, both sides have the same A in the BL^p–norm: $\|Ef\|^p_{BL^p_A(B)} \geq \|Ef_{\text{tang}}\|^p_{BL^p_A(B)}$.

In order to deal with the change of A, it suffices to make inequalities 5.5 and 5.6 to

$$\|Ef^{\sim}\|_{BL^p_A(O)} \leq r_t^\delta \|Ef^{\sim}_{S_t}\|_{BL^p_{A_t}(O)}$$

and

$$\|Ef^{\sim}\|_{BL^p_A(O)} \geq r_l^\delta \|Ef^{\sim}_{S_l}\|_{BL^p_{A_l}(O)}, \text{ for all } l < t$$

with $A_l \geq 2^l A_{l-1}$ and $A \gg A_n$.

Chapter 6

Estimate about the L^2–norm

In this section, we discuss the case when for $R^{-\delta}$ fraction of the O, $\|Ef\|^p_{BL^p(O)} \lesssim \|Ef^{\sim}\|^p_{BL^p(O)}$ and there exists a t satisfying 5.5 and 5.6,

$$\|Ef\|_{BL^p(O)} \lesssim \|Ef^{\sim}\|_{BL^p(O)} \lesssim \|Ef^{\sim}_{S_t}\|_{BL^p(O)} \sim \|Ef^{\sim}_{\Pi_{S_t}}\|_{BL^p(O)}. \tag{6.1}$$

The main lemmas we prove in this section are the following.

Lemma 6.1. *If for $R^{-\delta}$ fraction of the cells O,*

$$\|Ef\|_{BL^p(O)} \lesssim \|Ef^{\sim}\|_{BL^p(O)} \lesssim \|Ef^{\sim}_{S_t}\|_{BL^p(O)} \sim \|Ef^{\sim}_{\Pi_{S_t}}\|_{BL^p(O)}$$

and $r_t \geq R^{1/2}$, then for $p > 3 + 3/13$,

$$\|Ef\|^p_{BL^p(B_R)} \lesssim R^{\epsilon}\|f\|^2_{L^2} \max_{d(\theta)=R^{-1/2}} \|f_\theta\|^{p-2}_{L^2_{avg}(\theta)}.$$

Lemma 6.2. *If for $R^{-\delta}$ fraction of the cells O,*

$$\|Ef\|_{BL^p(O)} \lesssim \|Ef^{\sim}\|_{BL^p(O)} \lesssim \|Ef^{\sim}_{S_t}\|_{BL^p(O)} \sim \|Ef^{\sim}_{\Pi_{S_t}}\|_{BL^p(O)}$$

and $r_t \leq R^{1/2}$, then for $p > 3 + 1/5$,

$$\|Ef\|^p_{BL^p(B_R)} \lesssim R^{\epsilon}\|f\|^2_{L^2} \max_{d(\theta)=R^{-1/2}} \|f_\theta\|^{p-2}_{L^2_{avg}(\theta)}.$$

Recall that in the white lie proof, the key fact is that the L^2–norm of Ef_{tang} near a plane Σ is small unless the large wave packets $Ef_{\theta,v}$ are organized into large brooms rooted at Σ. After the polynomial partitioning iteration, we obtain collections of fat r_t–surfaces S_t. Lemma 6.4 in the next subsection says that we can treat S_t as at most $r^{O(\delta)}$ thin neighborhoods of planes if we only look at the wave packets with directions in a τ_t.

6.1 Planes

Let Z be a smooth degree d algebraic surface and let S be the $R^{1/2+\delta}$ neighborhood of Z in B_R. By definition, S is a fat R–surface. Let $Ef = Ef_{\text{tang}}$ be a function tangential to S in B_R.

Definition 6.3. *We define $\mathbb{T}_{S,\text{tang},ess}$ to be the set of tubes $T_{\theta,v} \in \mathbb{T}_{S,\text{tang}}$ such that there exists another $T_{\theta',v'} \in \mathbb{T}_{S,\text{tang}}$ with*

- $T_{\theta,v} \cap T_{\theta',v'} \cap S \neq \emptyset$ *and*
- $\text{Angle}(G(\theta), G(\theta')) \geq K^{-1}$.

We define the $T_{\theta',v'}$ as $T^*_{\theta,v}$. For a fixed $T_{\theta,v}$, there might be multiple $T^*_{\theta,v}$.

We define $Ef_{\text{tang},ess} = \sum_{T_{\theta,v} \in \mathbb{T}_{S,\text{tang},ess}} Ef_{\theta,v}$. By the definition of broad L^p–norm, $Ef_{\text{tang},ess}$ essentially represents Ef_{tang} in the sense that

$$\|Ef_{\text{tang}}\|^p_{BL^p(B_\rho \cap W)} \leq \|Ef_{\text{tang},ess}\|^p_{BL^p(B_\rho \cap W)} + \text{RapDec}(R)\|f\|^p_{L^2}.$$

From now on, we write $Ef_{\text{tang}} = Ef_{\text{tang},ess}$ and $\mathbb{T}_{S,\text{tang}} = \mathbb{T}_{S,\text{tang},ess}$. Fix a direction θ, we show that all the wave packets $Ef_{\theta,v}$ from θ have their essential supports $T_{\theta,v}$ tangential to $\lesssim R^{O(\delta)}$ planes.

Lemma 6.4. *Let S be a fat R–surface. For a fixed cap θ of radius $R^{-1/2}$, there exists at most $R^{O(\delta)}$ planes, such that every $T_{\theta,v} \in \mathbb{T}_{S,\text{tang}}$ is $R^{-1/2+\delta}$–tangential to one of them.*

Proof. We choose two unit vectors $\mathbf{v}_1$ and $\mathbf{v}_2$, such that $\mathbf{v}_1 \perp \mathbf{v}_2$ and they are both orthogonal to $G(\theta)$ up to $R^{-1/2}$–angle difference. So we have some room to choose $\mathbf{v}_1$ and $\mathbf{v}_2$ and we shall apply Sard's theorem multiple times. Let $\mathbb{T}_\theta$ denote the collection of $T_{\theta,v}$ in $\mathbb{T}_{S,\text{tang}}$. We define $\mathbb{T}_{\theta,i}$, $i = 1,2$, as the collection of tubes $T_{\theta,v}$ such that there exists a $T_{\theta',v'} \in \mathbb{T}_{S,\text{tang}}$ with

$$|G(\theta') \wedge G(\theta) \wedge \mathbf{v}_i| \geq K^{-1} \tag{6.2}$$

and $T_{\theta,v} \cap T_{\theta',v'} \cap S \neq \emptyset$. Since $\mathbf{v}_1$, $\mathbf{v}_2$ and $G(\theta)$ are pairwise orthogonal and by Definition 6.3

$$\mathbb{T}_\theta \subseteq \mathbb{T}_{\theta,1} \cup \mathbb{T}_{\theta,2}.$$

We show that the tubes in $\mathbb{T}_{\theta,1}$ can be covered by the $R^{1/2+\delta}$–neighborhood of at most $R^{O(\delta)}$ planes. Same arguments apply to $\mathbb{T}_{\theta,2}$.

We consider the projection $\Pi_{\mathbf{v}_1}$ along $\mathbf{v}_1$ to the plane $\Sigma_{\mathbf{v}_1}$, which is perpendicular to $\mathbf{v}_1$ and passes through the origin. We would like to decompose Z into at most Poly(d) pieces Z_j, such that the projection $\Pi_{\mathbf{v}_1}$ on Z_j is injective. And we show in Lemma 6.5 that the tubes tangential to Z_j can be covered by at most $R^{O(\delta)}$ $R^{1/2}$–neighborhood of planes.

We consider the set of planes $\Sigma_{\mathbf{v}_2,t}$ perpendicular to $\mathbf{v}_2$ parameterized by the coordinate t along $\mathbf{v}_2$ direction. By our choice of $\mathbf{v}_1$ and $\mathbf{v}_2$, the planes $\Sigma_{\mathbf{v}_2,t}$ are parallel to $G(\theta)$ and $\mathbf{v}_1$.

We define $Z_{\mathbf{v}_1}$ as the set of points p in Z such that $\mathbf{v}_1 \in T_pZ$. Then for a generic $\mathbf{v}_1$, $Z_{\mathbf{v}_1}$ is an algebraic curve of degree at most Poly(d). We color $Z_{\mathbf{v}_1}$ with red and blue. For any point $p \in Z_{\mathbf{v}_1}$, we color it with red if p is singular or if the tangent direction $T_pZ_{\mathbf{v}_1}$ satisfies $\text{Angle}(T_pZ_{\mathbf{v}_1}, \Sigma_{\mathbf{v}_2,t}) \leq R^{-1/2+10\delta}$, otherwise we color it with blue. There are at most Poly(d) singular points on $Z_{\mathbf{v}_1}$ for a generic $\mathbf{v}_1$. We decompose $Z_{\mathbf{v}_1}$ into connected components, such that points on each component has the same color. There are at most Poly(d) red components. The end points of red components are those p such that $\text{Angle}(T_pZ_{\mathbf{v}_1}, \Sigma_{\mathbf{v}_2,t}) = R^{-1/2+10\delta}$. The number $R^{-1/2+10\delta}$ can be chosen generically between $R^{-1/2+5\delta}$ and $R^{-1/2+15\delta}$. So there exists a choice such that there are at most Poly(d) red components.

For each red component $Z_{\mathbf{v}_1,red}$ of $Z_{\mathbf{v}_1}$, we can cover $Z_{\mathbf{v}_1,red}$ with the $R^{1/2+O(\delta)}$–neighborhood of a plane $\Sigma_{\mathbf{v}_2,t}$ for some t. Since we are inside a B_R, and the red component forms an angle $\lesssim R^{-1/2+10\delta}$ with $\Sigma_{\mathbf{v}_2,t}$, the red component is trapped inside the $R^{1/2+O(\delta)}$–neighborhood of some plane $\Sigma_{\mathbf{v}_2,t}$. We also add two planes $\Sigma_{\mathbf{v}_2,t_A}$ and $\Sigma_{\mathbf{v}_2,t_B}$, $t_A \le t_B$, such that B_R lies between them.

There exists Poly(d) planes $\Sigma_{\mathbf{v}_2,t_1},\dots,\Sigma_{\mathbf{v}_2,t_M}$, $M \le$ Poly(d), such that their $R^{1/2+O(\delta)}$–neighborhoods cover all red components $Z_{\mathbf{v}_1,red}$. Then $t_0 = t_A < t_1 < \cdots < t_M < t_B$.

We remove those $R^{1/2+O(\delta)}$–neighborhoods of planes from Z, let Z_0 denote the remaining part. We also remove all tubes $T_{\theta,v}$ intersecting those $R^{1/2+O(\delta)}$–neighborhoods from $\mathbb{T}_{\theta,1}$. Since $\Sigma_{\mathbf{v}_2,t}$ is parallel to $G(\theta)$, the removed tubes $T_{\theta,v}$ are $R^{-1/2+O(\delta)}$–tangential to one of the planes $\Sigma_{\mathbf{v}_2,t_1},\dots,\Sigma_{\mathbf{v}_2,t_M}$. We can view the curve $Z_{\mathbf{v}_1}$ as the zero set of a function $t = f(x,y)$, where t is the coordinate on the $\mathbf{v}_2$ direction, x is the coordinate on the center direction in $G(\theta)$, and y is the coordinate on the $\mathbf{v}_1$ direction. Then all the local extrema of the function $t = f(x,y)$ are covered by the $R^{1/2+O(\delta)}$–neighborhood of $\Sigma_{\mathbf{v}_2,t_m}$, $1 \le m \le M$.

The blue components $Z_{\mathbf{v}_1,blue}$ cut Z_0 into Poly(d) connected components Z_j. Restricted on each Z_j, the orthogonal projection $\Pi_{\mathbf{v}_1}$ is injective.

Each component Z_j is bounded by two planes $\Sigma_{\mathbf{v}_2,t_a}$ and $\Sigma_{\mathbf{v}_2,t_b}$ and some $Z_{\mathbf{v}_1,blue}$. The curve $Z_{\mathbf{v}_1}$ intersects $\Sigma_{\mathbf{v}_2,t_a}$ with at most Poly(d) points: $p_1,\dots,p_N$. Furthermore, the angle Angle$(T_{p_l}Z_{\mathbf{v}_1},\Sigma_{\mathbf{v}_2,t}) \ge R^{-1/2+10\delta}$ for $1 \le l \le N$. Let Z_t denote the intersection of Z and $\Sigma_{\mathbf{v}_2,t}$. Z_t is a smooth curve of degree d. The points $\{p_l\}$ decompose Z_{t_a} into Poly(d) components $Z_{t,l}$, each component is bounded by some points p_{l_1} and p_{l_2}. The projection $\Pi_{\mathbf{v}_1}$ restricted on each $Z_{t,l}$ is injective.

When we move the plane $\Sigma_{\mathbf{v}_2,t}$ from t_a to t_b, each point p_l has a unique trajectory. In particular, the number of points in $Z_{\mathbf{v}_1} \cap \Sigma_{\mathbf{v}_2,t}$ for t between t_a and t_b stays the same. The boundary of Z_j contains the trajectory of some p_{l_1} and p_{l_2} and two planes $\Sigma_{\mathbf{v}_2,t_a}$ and $\Sigma_{\mathbf{v}_2,t_b}$. In particular, the projection $\Pi_{\mathbf{v}_1}$ restricted on Z_j is injective since $\Pi_{\mathbf{v}_1}$ is injective at each $Z_j \cap \Sigma_{\mathbf{v}_2,t}$ for t between t_a and t_b.

Now we project Z_j to $\Sigma_{\mathbf{v}_1}$, the plane perpendicular to $\mathbf{v}_1$. We consider the set of tubes

$$\mathbb{T}_{1,\theta,j} := \{T_{\theta,v} \in \mathbb{T}_{S,\text{tang}} : \exists T_{\theta',v'} \in \mathbb{T}_{S,\text{tang}}, T_{\theta,v} \cap T_{\theta',v'} \cap Z_j \neq \emptyset \text{ and } |G(\theta) \wedge G(\theta') \wedge \mathbf{v}_1| \gtrsim 1/K\}. \tag{6.3}$$

We claim that for any $T_{\theta,v} \in \mathbb{T}_{1,\theta,j}$, $T_{\theta,v} \cap \partial Z_j = \emptyset$. In otherwords, the projection image $\Pi_{\mathbf{v}_1}(T_{\theta,v})$ is entirely contained in $\Pi_{\mathbf{v}_1}(Z_j)$. If $T_{\theta,v} \cap \partial Z_j \neq \emptyset$, then $T_{\theta,v} \cap Z_{\mathbf{v}_1,blue} \neq \emptyset$ because we have removed all tubes $T_{\theta,v}$ intersecting $\Sigma_{\mathbf{v}_2,t_a}$ and $\Sigma_{\mathbf{v}_2,t_b}$. Let $p \in T_{\theta,v} \cap Z_{\mathbf{v}_1,blue}$, then the tangent plane T_pZ contains the direction $\mathbf{v}_1$ and $T_pZ_{\mathbf{v}_1,blue}$ and $G(\theta)$ up to $R^{-1/2+\delta}$–error. This is impossible because $\Sigma_{\mathbf{v}_2,t}$ is parallel to $G(\theta)$ and $\mathbf{v}_1$ up to $R^{-1/2}$–error, while Angle$(T_pZ_{\mathbf{v}_1,blue}, \Sigma_{\mathbf{v}_2,t}) \geq R^{-1/2+10\delta}$.

We apply Lemma 6.5 to conclude the proof.

□

Lemma 6.5. *If the orthogonal projection $\Pi_{\mathbf{v}_1} : Z_j \to \Sigma_{\mathbf{v}_1}$ is injective, then the tubes in $\mathbb{T}_{1,\theta,j}$ defined in 6.3 are $R^{-1/2+\delta}$–tangent to $\lesssim R^{O(\delta)}$ planes.*

Proof. We might cover Z_j with balls B_ρ of radius $\rho = R^{1-\delta}$. It suffices to show Lemma 6.5 assuming $Z_j \subset B_\rho$.

On $\Sigma_{\mathbf{v}_1}$, we take a line segment I of length ρ centered at the center of $\Pi_{\mathbf{v}_1}(B_\rho)$, with direction orthogonal to $G(\theta)$. For any $T_{\theta,v} \in \mathbb{T}_{1,\theta,j}$, we have $\Pi_{\mathbf{v}_1}(T_{\theta,v}) \cap I \neq \emptyset$.

By Definition 6.3, for any tube $T_{\theta,v} \in \mathbb{T}_{1,\theta,j}$, there exists $T = T_{\theta',v'}$ such that $T_{\theta,v} \cap T_{\theta',v'} \cap Z_j \neq \emptyset$ and $|G(\theta') \wedge G(\theta) \wedge \mathbf{v}_1| \gtrsim 1/K$.

Let I_T be the projection of $\Pi_{\mathbf{v}_1}(T)$ along $G(\theta)$ to the line containing I. The line segment I_T has length at least $R^{1-2\delta}$ because $|G(\theta') \wedge G(\theta) \wedge \mathbf{v}_1| \gtrsim 1/K$.

For any tube $T_{\theta,v^*} \in \mathbb{T}_{1,\theta,j}$, if the projection image $\Pi_{\mathbf{v}_1}(T_{\theta,v^*}) \cap I_T \neq \emptyset$, then the tubes $T_{\theta,v^*} \cap T \neq \emptyset$ because $\Pi_{\mathbf{v}_1}$ is injective on Z_j.

Since I_T has length at least $R^{1-2\delta}$, there exists at most $O(R^\delta)$ T, such that the union of I_T covers $\bigcup_{T_{\theta,v} \in \mathbb{T}_{1,j,\theta}} \Pi_{\mathbf{v}_1}(T_{\theta,v}) \cap I$. And the tubes $T_{\theta,v} \in \mathbb{T}_{1,j,\theta}$ intersecting T are tangential to the same plane spanned by T and $G(\theta)$ direction.

□

Corollary 6.6. *Let $\mathbb{T}_{S,\text{tang}}$ be the collection of tubes tangential to a fat r–surface S as defined in Definition 6.3. Then for each cap τ of radius $r^{-1/2}$, there exist at most $O_d(r^{O(\delta)})$ planes whose $r^{1/2}$–neighborhoods contain all $T_{\tau,w} \in \mathbb{T}_{S,\text{tang}}$ in $G(\tau)$ direction.*

6.2 Brooms

In this subsection, we would like to give definition to $T_{\theta,v} \sim B_k$ according to the broom structure.

To simplify the notation, for any $R^{\delta} \le r_t \le R^{13/16}$, we let $r = r_t, S = S_t, \tau = \tau_t$ and $d(\tau) = r^{-1/2}$. By Lemma 6.4 and Corollary 6.6, there exists a collection $\Omega_{S,\tau}$ of at most $r^{O(\delta)}$ planes Σ such that

$$\|Ef^{\sim}_{\Pi_S,\tau}\|^2_{L^2(B_r)} \le \sum_{\Sigma\in\Omega_{S,\tau}} \|Ef^{\sim}_{\Pi_S,\tau}\|^2_{L^2(N\Sigma)} \tag{6.4}$$

where $N\Sigma$ denotes the $r^{1/2}$–neighborhood of Σ in B_r.

Definition 6.7. *Let τ be a cap of radius $r^{-1/2}$. For any $\Sigma \in \Omega_{S,\tau}$, we define a broom $\mathscr{B}$ rooted at Σ as in Definition 3.5.*

Recall that in the special case in Subsection 3.2, the brooms rooted at Σ have about the same size, and each wave packets belongs to about the same number of brooms. We are going to reduce to the special case via several steps of dyadic pigeonholing.

Let Γ_b denote the set of numbers $\{1,(\frac{R}{r})^{\delta},(\frac{R}{r})^{2\delta},\dots,(\frac{R}{r})^{1/2}\}$ and let Γ_γ denote the set of numbers $\{1,(\frac{R}{r})^{\delta},(\frac{R}{r})^{2\delta},\dots,R\}$. Since $r \le R^{13/16}$, we have $\#\Gamma_b, \#\Gamma_\gamma \lesssim \delta^{-1}$.

We define brooms when $r \ge R^{1/2}$. We decompose the unit sphere in $\mathbb{R}^3$ into caps α of radius 1/100. As in the white lie proof, we would like to consider only the interaction between planes with normal direction in one $G(\alpha)$. Let Ω_τ be the union of all $\Omega_{S,\tau}$.

For each $b_1 \in \Gamma_b$ and $\Sigma \in \Omega_\tau$, we define $\chi_{\alpha,b_1}(T_{\theta,v},\Sigma) = 1$ if $T_{\theta,v}$ belongs to a broom $\mathscr{B}$ rooted at Σ such that

1. the normal direction of Σ lies in $G(\alpha)$,

2. $b_1 \le |\mathscr{B}| \le (\frac{R}{r})^\delta b_1$.

Otherwise, we define $\chi_{\alpha,b_1}(T_{\theta,v},\Sigma)=0$.

For each $\gamma_1 \in \Gamma_\gamma$, we define $\chi_{\alpha,b_1,\gamma_1}(T_{\theta,v},\Sigma)=1$ if $T_{\theta,v}$ belongs to a broom $\mathscr{B}$ such that

1. $\chi_{\alpha,b_1}(T_{\theta,v},\Sigma)=1$,

2. $\gamma_1 \le \sum_{\Sigma'\in\Omega_\tau} \chi_{\alpha,b_1}(T_{\theta,v},\Sigma') \le (\frac{R}{r})^\delta \gamma_1$.

Otherwise, we define $\chi_{\alpha,b_1,\gamma_1}(T_{\theta,v},\Sigma)=0$.

Let $\kappa=(\alpha,t,b_1,\gamma_1,\dots,b_l,\gamma_l)$ and $\kappa'=(\alpha,t,b_1,\gamma_1,\dots,b_l)$ for some $2\le l\lesssim\delta^{-1}$. We define χ_κ inductively as above, and we stop when

$$b_l \ge (\frac{R}{r})^{-\delta} b_{l-1}, \gamma_l \ge (\frac{R}{r})^{-\delta}\gamma_{l-1}. \tag{6.5}$$

From our definition of χ_κ, we have $b_1\ge b_2\ge\cdots b_l$ and $\gamma_1\ge\cdots\gamma_l$. Then there are at most $O_\delta(1)$ vectors κ and κ' satisfying inequality 6.5.

For any $T_{\theta,v}\cap\Sigma\neq\emptyset$ and $G(\theta)\parallel\Sigma$ up to angle difference $r^{-1/2}$, there exists κ and κ' satisfying 6.5 such that $\chi_\kappa(T_{\theta,v},\Sigma)=1$ and $\chi_{\kappa'}(T_{\theta,v},\Sigma)=1$.

Let B_k be a ball of radius $R^{1-\epsilon_0}$, $\delta\ll\epsilon_0\ll\epsilon$. We define $T_{\theta,v}\sim_\kappa B_k$ according to the function χ_κ in the same way as Definition 3.12.

Definition 6.8. *Fix κ and a tube $T_{\theta,v}$ with $\theta\subseteq\tau$, let B_k^* be the ball that maximizes the quantity*

$$\sum_{\Sigma\in\Omega_\tau}\chi_\kappa(T_{\theta,v},\Sigma).$$

If there are multiple maximizer balls B_k, then we choose only one. We say that $T_{\theta,v}\sim_\kappa B_k$ if B_k lies inside $10B_k^*$. We define $T_{\theta,v}\sim_{\kappa'} B_k$ according to the same rule with function $\chi_{\kappa'}$ instead.

There are at most $O_\delta(1)$ choices for κ and κ'.

We define bushes when $r\le R^{1/2}$ as in the Definition 3.16.

Definition 6.9. *For a fat r–surface S with $r \le R^{1/2}$ and a cap τ of radius $r^{-1/2}$, a bush $\mathcal{U}$ rooted at S is defined as the collection of large tubes $T_{\theta,v}$ passing through S with $\theta \subseteq \tau$ and the corresponding wave packet $Ef_{\theta,v}$ nonzero.*

Let Γ_u denote the collection of numbers $\{1, (\frac{R}{r})^{\delta}, (\frac{R}{r})^{2\delta}, \dots, \frac{R}{r}\}$.

For any fat r–surface S with $r \le R^{1/2}$, any $T_{\theta,v}$ and $u_1 \in \Gamma_u$, we define $\chi_{u_1}(T_{\theta,v}, S) = 1$ if $Ef_{\theta,v}$ belongs to a bush $\mathcal{U}$ rooted at S and the size of $\mathcal{U}$ is between u_1 and $u_1(\frac{R}{r})^{\delta}$. Otherwise, we define $\chi_{u_1}(T_{\theta,v}, S) = 0$.

For each large tube $T_{\theta,v}$ and each $\gamma_1 \in \Gamma_\gamma$, we define $\chi_{u_1,\gamma_1}(T_{\theta,v}, S) = 1$ if $\chi_{u_1}(T_{\theta,v}, S) = 1$ and if the number $\sum_{S'} \chi_{u_1}(T_{\theta,v}, S')$ is between γ_1 and $\gamma_1(\frac{R}{r})^{\delta}$.

Let ι denote the vector $(u_1, \gamma_1, u_2, \gamma_2, \dots, u_l, \gamma_l)$ and let ι' denote the vector $(u_1, \gamma_1, u_2, \gamma_2, \dots, u_l)$. We define χ_ι and $\chi_{\iota'}$ inductively as above. We stop if ι and ι' satisfy

$$u_l \ge u_{l-1}(\frac{R}{r})^{-\delta} \text{ and } \gamma_l \ge \gamma_{l-1}(\frac{R}{r})^{-\delta}. \tag{6.6}$$

If $T_{\theta,v} \cap S \neq \emptyset$, then $\chi_\iota(T_{\theta,v}, S) = 1$ and $\chi_{\iota'}(T_{\theta,v}, S) = 1$ for some ι and ι' satisfying 6.6. We define $T_{\theta,v} \sim_\iota B_k$ and $T_{\theta,v} \sim_{\iota'} B_k$ using the function χ_ι and $\chi_{\iota'}$.

Definition 6.10. *For a fixed $T_{\theta,v}$, let B_k* denote the ball that attains the maximum quantity $\sum_{S \subseteq B_k} \chi_\iota(T_{\theta,v}, S)$. If there are multiple choices for B_k*, we choose only one. We define $T_{\theta,v} \sim_\iota B_k$ if B_k belongs to $10B_k^*$. We define $T_{\theta,v} \sim_{\iota'} B_k$ according to the same rule with the function $\chi_{\iota'}$ instead.*

Finally, we define $T_{\theta,v} \sim B_k$.

Definition 6.11. *We say that $T_{\theta,v} \sim B_k$ if there exists some $r = r_t$ with $R^{\delta} < r_t \le R^{13/16}$ and κ, κ', ι or ι', such that one of the following is true: $T_{\theta,v} \sim_\kappa B_k$, $T_{\theta,v} \sim_{\kappa'} B_k$, $T_{\theta,v} \sim_\iota B_k$ and $T_{\theta,v} \sim_{\iota'} B_k$.*

Lemma 6.12. *For each $T_{\theta,v}$, the number of B_k such that $T_{\theta,v} \sim B_k$ is bounded by $O_\delta(1)$.*

Proof. By the definition of $\sim_\kappa$, for a fixed $T_{\theta,v}$, the number of B_k such that $T_{\theta,v} \sim_\kappa$ is bounded by $O(1)$. The number of B_k such that $T_{\theta,v} \sim_\iota B_k$ is also bounded by $O(1)$ for a fixed $T_{\theta,v}$. There are at most $O_\delta(1)$ choices for κ and ι. Same arguments apply to κ' and ι'. $\square$

We might assume that each small tube $T_{\tau,w}$ is tangential to at most one $\Sigma \in \Omega_\tau$. Otherwise, we assign $T_{\tau,w}$ to only one Σ and pretend that $T_{\tau,w}$ does not exist for other plane.

Recall that in the beginning of this section, there exists a $t \in 1,\dots,n$ such that for $R^{-\delta}$ fraction of the cells O, we have inequality 6.1

$$\|Ef\|_{BL^p(O)} \lesssim \|Ef^{\sim}\|_{BL^p(O)} \lesssim \|Ef^{\sim}_{S_t}\|_{BL^p(O)} \sim \|Ef^{\sim}_{\Pi_{S_t}}\|_{BL^p(O)}.$$

From now on, we fix a cap τ of radius $r^{-1/2}$.

For $r = r_t$, $S = S_t$ and $\Sigma \in \Omega_{S,\tau}$, let $\mathbb{T}_\Sigma \subseteq \mathbb{T}_{S,\mathrm{tang}}$ denote the set of small tubes $T_{\tau,w}$ $r^{-1/2}$–tangential to Σ.

When $r \geq R^{1/2}$, for each κ and $\Sigma \in \Omega_{S,\tau}$, we define

$$Ef^{\sim}_{\kappa,\Sigma} = \sum_{\chi_\kappa(T_{\theta,v},\Sigma)=1, T_{\theta,v}\sim B_k} Ef_{\theta,v}$$

and

$$Ef^{\sim}_{\kappa,\Sigma,\mathrm{tang}} = \sum_{T_{\tau,w}\in\mathbb{T}_\Sigma} Ef^{\sim}_{\kappa,\Sigma,\tau,w}.$$

Then we have

$$Ef^{\sim}_{\Pi_S,\tau} = \sum_{\Sigma\in\Omega_{S,\tau}} \sum_{\kappa} Ef^{\sim}_{\kappa,\Sigma,\mathrm{tang}} + \mathrm{RapDec}(R)\|f\|_{L^2}. \tag{6.7}$$

Here $\sum_{\kappa}$ means sum over all the κ satisfying inequality 6.5. Since the number of κ is bounded by $O_\delta(1)$ and there exists a κ such that for $\gtrsim_\delta 1$ fraction of the S such that

$$\|Ef^{\sim}_{\Pi_S,\tau}\|_{L^2(B_r)} \lesssim_\delta \sum_{\Sigma\in\Omega_{S,\tau}} \|Ef^{\sim}_{\kappa,\Sigma,\mathrm{tang}}\|_{L^2(B_r)}. \tag{6.8}$$

When $r \le R^{1/2}$, for each $\Sigma \in \Omega_{S,\tau}$, we define

$$Ef_{\iota,\Sigma}^{\sim} = \sum_{\chi_\iota(T_{\theta,v},S)=1, T_{\theta,v}\cap\Sigma\neq\emptyset, T_{\theta,v}\sim B_k} Ef_{\theta,v}$$

and

$$Ef_{\iota,\Sigma,\mathrm{tang}}^{\sim} = \sum_{T_{\tau,w}\in\mathbb{T}_{S,\mathrm{tang}}, T_{\tau,w}\cap\Sigma\neq\emptyset} Ef_{\iota,\Sigma,\tau,w}^{\sim}.$$

We have

$$Ef_{\Pi_S,\tau}^{\sim} = \sum_{\Sigma\in\Omega_{S,\tau}} \sum_{\iota} Ef_{\iota,\Sigma,\mathrm{tang}}^{\sim} + \mathrm{RapDec}(R)\|f\|_{L^2}. \tag{6.9}$$

Since the number of ι is bounded by $O_\delta(1)$ and the number of $\Sigma \in \Omega_{S,\tau}$ is bounded by $r^{O(\delta)}$, there exists a ι such that for $\gtrsim_\delta 1$ fraction of the S, we have

$$\|Ef_{\Pi_S,\tau}^{\sim}\|_{L^2(B_r)} \lesssim r^{O(\delta)} \|Ef_{\iota,\Sigma,\mathrm{tang}}^{\sim}\|_{L^2(B_r)} \tag{6.10}$$

for some $\Sigma \in \Omega_{S,\tau}$,

We prove the following Lemma 6.13, which corresponds to Lemma 3.13 in the white lie proof.

Lemma 6.13. *If $r \ge R^{/12}$ and τ is a cap of radius $r^{-1/2}$, $\Sigma \in \Omega_\tau$, then for each κ satisfying 6.5,*

$$\|f_{\kappa,\Sigma,\mathrm{tang}}^{\sim}\|_{L^2(\tau)}^2 \lesssim (\frac{R}{r})^{-1/2} R^{O(\epsilon_0)} \|f_\tau\|_{L^2}^2.$$

Proof. The proof is similar to Lemma 3.13. The only change is that we have χ_κ and $\chi_{\kappa'}$ instead of χ.

As in Lemma 3.13, we assume that there are about $(\frac{R}{r})^{\beta_0}$ nonzero wave packets $Ef_{\theta,v}$ with $\theta \subseteq \tau$. Let B_k be the ball of radius $R^{1-\epsilon_0}$ containing B_r and let $\Sigma_1 = \Sigma$.

We say that $\Sigma_2 \not\subseteq B_k$ if Σ_2 is associated to some fat r–surface S_2 outside of $5B_k$. The main idea is to double count the number of wave packets shared by Σ_1 and those far apart Σ_2, specifically, the quantity

$$\sum_{\Sigma_2\not\subseteq B_k} \sum_{\theta\subseteq\tau,v} \chi_\kappa(T_{\theta,v},\Sigma_1)\chi_{\kappa'}(T_{\theta,v},\Sigma_2). \tag{6.11}$$

By the definition of $T_{\theta,v} \not\sim B_k$, for each $T_{\theta,v}$,

$$\sum_{\Sigma_2 \not\subseteq B_k} \chi_{\kappa'}(T_{\theta,v}, \Sigma_2) \gtrsim \sum_{\Sigma'} \chi_{\kappa'}(T_{\theta,v}, \Sigma'). \tag{6.12}$$

If inequality 6.12 is not true, then the B_k^* that maximizes $\sum_{\Sigma \cap S \subseteq B_k^*} \chi_{\kappa'}(T_{\theta,v}, \Sigma)$ belongs to $5B_k$ and $T_{\theta,v} \sim_{\kappa'} B_k$, which violates the assumption $T_{\theta,v} \not\sim B_k$. This is the only part we need to use the information that $T_{\theta,v} \not\sim B_k$. For each $T_{\theta,v} \not\sim B_k$ and $T_{\theta,v}$ satisfying $\chi_\kappa(T_{\theta,v}, \Sigma_1) = 1$, we have

$$\sum_{\Sigma'} \chi_{\kappa'}(T_{\theta,v}, \Sigma') \gtrsim \gamma_l. \tag{6.13}$$

Assume that $\sum_{\theta \subset \tau, v} \chi_\kappa(T_{\theta,v}, \Sigma_1) \approx (\frac{R}{r})^{\beta_1}$. Then we have the following lower bound for 6.11 by combining inequality 6.12 and inequality 6.13,

$$\sum_{\Sigma_2 \not\subseteq B_k} \sum_{\theta \subseteq \tau, v} \chi_\kappa(T_{\theta,v}, \Sigma_1) \chi_{\kappa'}(T_{\theta,v}, \Sigma_2) \gtrsim \gamma_l (\frac{R}{r})^{\beta_1}. \tag{6.14}$$

Next we are going to give an upper bound for 6.11. We shall apply the same geometric observation as in the proof of Lemma 3.13. When $S_1 = S$ and S_2 are $R^{1-\epsilon_0}$ apart and the normal directions of Σ_1 and Σ_2 are both in α, a broom rooted at Σ_2 can intersect with Σ_1 in at most $R^{O(\epsilon_0)}$ large tubes $T_{\theta,v}$.

By Remark 3.7, for a fixed Σ_2, each tube $T_{\theta,v}$ belongs to at most one broom rooted at Σ_2. The function $\chi_{\kappa'}$ counts brooms of size about b_l.

Hence, for each $\Sigma_2 \not\subseteq B_k$, the number of tubes $T_{\theta,v}$ shared by Σ_1 and Σ_2 is at most b_l^{-1} fraction of the $T_{\theta,v}$ through Σ_2

$$\sum_{\theta \subseteq \tau, v} \chi_\kappa(T_{\theta,v}, \Sigma_1) \chi_{\kappa'}(T_{\theta,v}, \Sigma_2) \lesssim R^{O(\epsilon_0)} b_l^{-1} \sum_{\theta \subseteq \tau, v} \chi_{\kappa'}(T_{\theta,v}, \Sigma_2). \tag{6.15}$$

By the definition of $\chi_{\kappa'}$, each wave packet $T_{\theta,v}$ satisfies $\sum_{\Sigma_2} \chi_{\kappa'}(T_{\theta,v}, \Sigma_2) \leq \gamma_{l-1} (\frac{R}{r})^\delta$.

Since there are $(\frac{R}{r})^{\beta_0}$ nonzero wave packets $Ef_{\theta,v}$ with $\theta \subseteq \tau$, we have

$$\sum_{\theta\subseteq\tau,v}\sum_{\Sigma_2}\chi_{\kappa'}(T_{\theta,v},\Sigma_2) \le \gamma_{l-1}(\frac{R}{r})^{\beta_0+\delta}. \tag{6.16}$$

We sum over the $\Sigma_2 \not\subseteq B_k$ with inequality 6.15 and apply inequality 6.16, we have the following upper bound for 6.11,

$$\sum_{\Sigma_2\not\subseteq B_k}\sum_{\theta\subseteq\tau,v}\chi_\kappa(T_{\theta,v},\Sigma_1)\chi_{\kappa'}(T_{\theta,v},\Sigma_2) \lesssim R^{O(\epsilon_0)}(\frac{R}{r})^{\beta_0}\gamma_{l-1}b_l^{-1} \tag{6.17}$$

Since κ satisfies inequality 6.5, we have $\gamma_l \ge \gamma_{l-1}(\frac{R}{r})^{-100\delta}$. Compare inequality 6.14 with inequality 6.17,

$$(\frac{R}{r})^{\beta_1}b_l \le R^{O(\epsilon_0)}(\frac{R}{r})^{\beta_0}. \tag{6.18}$$

We apply Lemma 3.10 with $Eh_\tau = Ef^{\sim}_{\kappa,\Sigma}$ and $b = b_l$,

$$\int |Ef^{\sim}_{\kappa,\Sigma,\text{tang}}|^2 \le \int_{N\Sigma}|Ef^{\sim}_{\kappa,\Sigma}|^2 \lesssim (\frac{R}{r})^{-1/2}b_l\int_{B_r}|Ef^{\sim}_{\kappa,\Sigma}|^2 \tag{6.19}$$

Hence,

$$\|f^{\sim}_{\kappa,\Sigma,\text{tang}}\|^2_{L^2} \lesssim (\frac{R}{r})^{-1/2}b_l\|f^{\sim}_{\kappa,\Sigma}\|^2_{L^2}.$$

Since there are $(\frac{R}{r})^{\beta_1}$ out of $(\frac{R}{r})^{\beta_0}$ nonzero large wave packets $Ef_{\theta,v}$, $\theta \subseteq \tau$, intersecting Σ, we have

$$\|f^{\sim}_{\kappa,\Sigma}\|^2_{L^2} \lesssim (\frac{R}{r})^{\beta_1-\beta_0}\|f_\tau\|^2_{L^2}.$$

By inequality 6.18, we have

$$\|f^{\sim}_{\kappa,\Sigma,\text{tang}}\|^2_{L^2} \le R^{O(\epsilon_0)}(\frac{R}{r})^{-1/2}\|f_\tau\|^2_{L^2}. \tag{6.20}$$

□

We prove Lemma 6.1 using Lemma 6.13.

Proof. By our assumption 6.1 and inequality 2.9 on $\|Ef_S^{\sim}\|_{BL^p(S)}^p$ and $\|Ef_{\Pi_S}^{\sim}\|_{BL^p(S)}^p$,

$$\begin{aligned}
\|Ef\|_{BL^p(B_R)}^p &\lesssim R^{\delta}\sum_{S\in\mathscr{S}}\sum_{O\subseteq S}\|Ef_S^{\sim}\|_{BL^p(O)}^p \\
&\lesssim R^{\delta}\sum_{S\in\mathscr{S}}\sum_{O\subseteq S}\|Ef_S^{\sim}\|_{BL^p(O)}^2\|Ef_{\Pi_S}^{\sim}\|_{BL^p(O)}^{p-2} \\
&\lesssim R^{\delta}\sum_{S\in\mathscr{S}}\|Ef_S^{\sim}\|_{BL^p(S)}^2\|Ef_{\Pi_S}^{\sim}\|_{BL^p(S)}^{p-2} \\
&\lesssim R^{O(\delta)}r^{\frac{5}{2}-\frac{3p}{4}}\sum_{S\in\mathscr{S}}\|f_S^{\sim}\|_{L^2}^2\|f_{\Pi_S}^{\sim}\|_{L^2}^{p-2}
\end{aligned}$$

By Lemma 4.3 and Corollary 4.4,

$$\sum_{S\in\mathscr{S}}\|f_S^{\sim}\|_{L^2}^2\lesssim DR^{\delta}\|f\|_{L^2}^2. \tag{6.21}$$

To estimate $\|f_{\Pi_S}^{\sim}\|_{L^2}$, we apply Lemma 2.7, inequality 6.8 and Lemma 6.13,

$$\begin{aligned}
\|f_{\Pi_S}^{\sim}\|_{L^2}^2 &\lesssim r^{-1/2+O(\delta)}\max_{d(\tau)=r^{-1/2}}\|f_{\Pi_S,\tau}^{\sim}\|_{L^2_{avg}(\tau)}^2 \\
&\lesssim r^{-1/2+O(\delta)}\max_{d(\tau)=r^{-1/2},\Sigma\in\Omega_{S,\tau}}\|f_{\kappa,\Sigma,\mathrm{tang}}^{\sim}\|_{L^2_{avg}(\tau)}^2 \\
&\lesssim R^{-1/2+O(\epsilon_0)}\max_{\tau}\|f_\tau\|_{L^2_{avg}(\tau)}^2.
\end{aligned}$$

So we have

$$\|Ef\|_{BL^p(B_R)}^p\lesssim r^{\frac{5}{2}-\frac{3p}{4}}R^{-\frac{p-2}{4}+O(\epsilon_0)}D\|f\|_{L^2}^2\max_{d(\theta)=R^{-1/2}}\|f_\theta\|_{L^2}^{p-2}. \tag{6.22}$$

Estimate 6.22 is good when D is small. When D is large, estimate 6.21 becomes more favorable. Since each $S\in\mathscr{S}$ contains about the same number of O, by Lemma 4.3, we have

$$\begin{aligned}
\|Ef\|_{BL^p(B_R)}^p &\lesssim R^{\delta}\#\{O\}\|Ef\|_{BL^p(O)}^p \\
&\lesssim R^{O(\delta)}\#\mathscr{S}\|Ef_S^{\sim}\|_{BL^p(S)}^p \\
&\lesssim R^{O(\delta)}r^{\frac{5}{2}-\frac{3p}{4}}\#\mathscr{S}\|f_S^{\sim}\|_{L^2}^p
\end{aligned}$$

and the number of S is greater than $D^3R^{-\delta}$.

By inequality 6.21,

$$\|f_S^{\sim}\|_{L^2}^2 \le DR^{O(\delta)}(\#S)^{-1}\|f\|_{L^2}^2.$$

Hence,

$$\|Ef\|_{BL^p(B_R)}^p \le R^{O(\delta)}r^{\frac{5}{2}-\frac{3p}{4}}D^{p/2}(\#S)^{1-p/2}\|f\|_{L^2}^p \le R^{O(\delta)}r^{\frac{5}{2}-\frac{3p}{4}}D^{3-p}\|f\|_{L^2}^p. \tag{6.23}$$

We compare inequality 6.23 with inequality 6.22, the worst case happens when $D = R^{\frac{1}{4}}$ and $r \le R/D \le D^3$. When $p \ge 3+3/13$, the constant term is bounded by

$$R^{O(\epsilon_0)}r^{\frac{5}{2}-\frac{3p}{4}}D^{3-p} \lesssim D^{3(\frac{5}{2}-\frac{3p}{4})+3-p} \lesssim R^{\epsilon}.$$

□

Now we discuss the case when $r \le R^{1/2}$. Our main ingredient the following Lemma 6.14, which corresponds to Lemma 3.19.

Lemma 6.14. *If S is a fat r–surface with $r \le R^{/12}$ and τ is a cap of radius $r^{-1/2}$, $\Sigma \in \Omega_{S,\tau}$, then for each $\iota = (u_1,\gamma_1,\dots,u_l,\gamma_l)$ satisfying inequality 6.6, we have*

$$\|f_{\iota,\Sigma,\text{tang}}^{\sim}\|_{L^2(\tau)}^2 \lesssim r^{-1/2}R^{O(\epsilon_0)}\|f_\tau\|_{L^2}^2.$$

Proof. Let $\iota' = (u_1,\gamma_1,\dots,u_l)$. We count the number of large wave packets shared by two far apart fat r–surface $S_1 = S$ and S_2:

$$\sum_{S_2 \not\subseteq 5B_k}\sum_{\theta\subset\tau,v}\chi_\iota(T_{\theta,v},S_1)\chi_{\iota'}(T_{\theta,v},S_2). \tag{6.24}$$

For each tube $T_{\theta,v} \not\sim B_k$ we have

$$\sum_{S_2 \not\subseteq 5B_k}\chi_{\iota'}(T_{\theta,v},S_2) \gtrsim \sum_{S'}\chi_{\iota'}(T_{\theta,v},S'). \tag{6.25}$$

Otherwise, the ball B_k^* that maximizes $\sum_{S'\subseteq B_k^*}\chi_{\iota'}(T_{\theta,v},S')$ must belong to $5B_k$, which violates the assumption $T_{\theta,v} \nsim B_k$. For each tube $T_{\theta,v}$ satisfying $\chi_\iota(T_{\theta,v},S_1) = 1$, by the definition of χ_ι, we know

$$\sum_{S'} \chi_{\iota'}(T_{\theta,v},S') \gtrsim \gamma_l. \tag{6.26}$$

Assume that $\sum_{\theta\subset\tau,v} \chi_\iota(T_{\theta,v},S_1) = (\frac{R}{r})^{\beta_1}$. Combine inequality 6.25 and inequality 6.26, we have a lower bound for the quantity 6.24,

$$\sum_{S_2 \not\subseteq 5B_k} \sum_{\theta\subset\tau,v} \chi_\iota(T_{\theta,v},S_1)\chi_{\iota'}(T_{\theta,v},S_2) \gtrsim (\frac{R}{r})^{\beta_1}\gamma_l \tag{6.27}$$

Next we are going to give an upper bound for the quantity 6.24. Fix a pair of fat r–surfaces S_1 and S_2 with distance $R^{1-\epsilon_0}$, and each one lies inside a ball of radius $r \le R^{1/2}$, the number of large wave packets shared by two fat r–surfaces is at most $R^{O(\epsilon_0)}$. Specifically,

$$\sum_{\theta\subseteq\tau,v} \chi_\iota(T_{\theta,v},S_1)\chi_{\iota'}(T_{\theta,v},S_2) \lesssim R^{O(\epsilon_0)}. \tag{6.28}$$

Since $\chi_{\iota'}$ counts bushes of size about u_l, inequality 6.28 can be written as

$$\sum_{\theta\subseteq\tau,v} \chi_\iota(T_{\theta,v},S_1)\chi_{\iota'}(T_{\theta,v},S_2) \lesssim R^{O(\epsilon_0)} u_l^{-1} \sum_{\theta\subseteq\tau} \chi_{\iota'}(T_{\theta,v},S_2) \tag{6.29}$$

Assume that there are $(\frac{R}{r})^{\beta_0}$ nonzero wave packets $Ef_{\theta,v}$ with $\theta\subseteq\tau$. By definition of $\chi_{\iota'}$,

$$\sum_{S'} \chi_{\iota'}(T_{\theta,v},S') \lesssim \gamma_{l-1}(\frac{R}{r})^{100\delta}.$$

We sum over all the nonzero wave packets $Ef_{\theta,v}$ with $\theta\subseteq\tau$,

$$\sum_{\theta\subseteq\tau,v} \sum_{S'} \chi_{\iota'}(T_{\theta,v},S') \lesssim \gamma_{l-1}(\frac{R}{r})^{\beta_0+\delta}. \tag{6.30}$$

We sum inequality 6.29 over all the cells $S_2 \not\subseteq 5B_k$ and apply inequality 6.30 to obtain the following upper bound for 6.24

$$\sum_{S_2 \not\subseteq 5B_k} \sum_{\theta\subseteq\tau,v} \chi_\iota(T_{\theta,v},S_1)\chi_{\iota'}(T_{\theta,v},S_2) \lesssim R^{O(\epsilon_0)} u_l^{-1} (\frac{R}{r})^{\beta_0}\gamma_{l-1} \tag{6.31}$$

Since ι satisfies inequality 6.6, $\gamma_l \geq \gamma_{l-1}(\frac{R}{r})^{-\delta}$. Compare inequality 6.27 with inequality 6.31,

$$(\frac{R}{r})^{\beta_1-\beta_0} u_l \lesssim R^{O(\epsilon_0)} \tag{6.32}$$

Now we are ready to estimate $\|f^{\sim}_{\iota,\Sigma,\mathrm{tang}}\|_{L^2(\tau)}$. We apply Lemma 3.17 with $f^{\sim}_{\iota,\Sigma} = g_{\mathscr{U}}$ and $u \leq u_l$. By the definition of $f^{\sim}_{\iota,\Sigma,\mathrm{tang}}$, which takes the tangential part to S and Σ, we have

$$\|f^{\sim}_{\iota,\Sigma,\mathrm{tang}}\|^2_{L^2(\tau)} \lesssim r^{-1/2} u_l \|f^{\sim}_{\iota,\Sigma}\|^2_{L^2(\tau)}.$$

Since there are $\lesssim (\frac{R}{r})^{\beta_1}$ nonzero wave packets $Ef_{\theta,v}$ such that $\chi_\iota(T_{\theta,v}, S) = 1$,

$$\|f^{\sim}_{\iota,\Sigma}\|^2_{L^2(\tau)} \leq (\frac{R}{r})^{\beta_1-\beta_0} \|f_\tau\|^2_{L^2}.$$

We apply inequality 6.32,

$$\|f^{\sim}_{\iota,\Sigma,\mathrm{tang}}\|^2_{L^2(\tau)} \lesssim r^{-1/2} R^{O(\epsilon_0)} \|f_\tau\|^2_{L^2}.$$

□

We prove Lemma 6.2 with Lemma 6.14.

Lemma 6.2 corresponds to Lemma 3.20 in the white lie proof.

Proof. When $D \geq r^{1/2}$, by Lemma 4.3 and the assumptions, there exist more than $R^{-\delta}|\mathscr{S}| \gtrsim D^3 R^{-O(\delta)}$ fat r–surfaces S, such that

$$\|Ef\|^p_{BL^p(B_R)} \lesssim |\mathscr{S}| R^{O(\delta)} \|Ef^{\sim}_S\|^p_{BL^p(S)}.$$

We apply inequality 2.9 at scale r,

$$\|Ef\|^p_{BL^p(B_R)} \lesssim |\mathscr{S}| R^{O(\delta)} r^{\frac{5}{2}-\frac{3p}{4}} \|f^{\sim}_S\|^p_{L^2}.$$

By Lemma 3.15,

$$\sum_{S\in\mathscr{S}} \|f^{\sim}_S\|^2_{L^2} \lesssim D R^{O(\delta)} \|f\|^2_{L^2}.$$

There exists an S, such that $\|f_S^{\sim}\|_{L^2}^2 \lesssim DR^{O(\delta)}|\mathscr{S}|^{-1}\|f\|_{L^2}^2$. We use this S to estimate $\|Ef\|_{BL^p(B_R)}^p$,

$$\begin{aligned}\|Ef\|_{BL^p(B_R)}^p &\lesssim |\mathscr{S}|R^{O(\delta)}r^{\frac{5}{2}-\frac{3p}{4}}\|f_S^{\sim}\|_{L^2}^p \\ &\lesssim |\mathscr{S}|^{1-\frac{p}{2}}D^{\frac{p}{2}}R^{O(\delta)}r^{\frac{5}{2}-\frac{3p}{4}}\|f\|_{L^2}^p \\ &\lesssim D^{3-p}r^{\frac{5}{2}-\frac{3p}{4}}\|f\|_{L^2}^p \\ &\lesssim r^{\frac{5}{2}-\frac{3p}{4}+\frac{3-p}{2}}\|f\|_{L^2}^p.\end{aligned}$$

We used the inequality $|\mathscr{S}| \gtrsim D^3R^{O(\delta)}$ and the assumption that $D \geq r^{1/2}$. When $p > \frac{16}{5}$, the constant term is bounded by R^{ϵ}.

When $D \leq r^{1/2}$, we shall apply the estimate in Lemma 6.14.

By our assumptions and inequality 2.9 on scale r,

$$\begin{aligned}\|Ef\|_{BL^p(B_R)}^p &\lesssim R^{\delta}\sum_{O}\|Ef\|_{BL^p(O)}^p \\ &\lesssim R^{\delta}\sum_{S\in\mathscr{S}}\sum_{O\subseteq S}\|Ef_S^{\sim}\|_{BL^p(O)}^2\|Ef_{\Pi_S}^{\sim}\|_{BL^p(O)}^{p-2} \\ &\lesssim R^{\delta}\sum_{S\in\mathscr{S}}\|Ef_S^{\sim}\|_{BL^p(S)}^2\|Ef_{\Pi_S}^{\sim}\|_{BL^p(S)}^{p-2} \\ &\lesssim R^{\delta}r^{\frac{5}{2}-\frac{3p}{4}}\sum_{S\in\mathscr{S}}\|f_S^{\sim}\|_{L^2}^2\|f_{\Pi_S}^{\sim}\|_{L^2}^{p-2}.\end{aligned}$$

For the term $\|f_{\Pi_S}^{\sim}\|_{L^2}$, we apply Lemma 2.7,

$$\|f_{\Pi_S}^{\sim}\|_{L^2}^2 \lesssim r^{-\frac{1}{2}}\max_{d(\tau)=r^{-1/2+O(\delta)}}\|f_{\Pi_S,\tau}^{\sim}\|_{L^2_{avg}(\tau)}^2.$$

Then we apply inequality 6.10, for some ι satisfying inequality 6.6 and some $\Sigma \in \Omega_{S,\tau}$ we have

$$\|f_{\Pi_S,\tau}^{\sim}\|_{L^2}^2 \lesssim r^{O(\delta)}\|f_{\iota,\Sigma,\mathrm{tang}}^{\sim}\|_{L^2(\tau)}^2.$$

We apply Lemma 6.14 to obtain

$$\|f_{\Pi_S}^{\sim}\|_{L^2}^2 \lesssim r^{-1+O(\delta)}\|f_\tau\|_{L^2_{avg}(\tau)}^2. \tag{6.33}$$

We apply Lemma 3.15 to bound $\sum_{S\in\mathscr{S}} \|f_S^{\sim}\|_{L^2}^2$,

$$\begin{aligned}
\|Ef\|_{BL^p(B_R)}^p &\lesssim R^{\delta} r^{\frac{5}{2}-\frac{3p}{4}} \sum_{S\in\mathscr{S}} \|f_S^{\sim}\|_{L^2}^2 \|f_{\Pi_S}^{\sim}\|_{L^2}^{p-2} \\
&\lesssim R^{O(\epsilon_0)} r^{\frac{5}{2}-\frac{3p}{4}-\frac{p-2}{2}} \sum_{S\in\mathscr{S}} \|f_S^{\sim}\|_{L^2}^2 \max_{d(\tau)=r^{-1/2}} \|f_\tau\|_{L^2_{avg}(\tau)}^{p-2} \\
&\lesssim R^{O(\epsilon_0)} r^{\frac{5}{2}-\frac{3p}{4}-\frac{p-2}{2}} D \|f\|_{L^2}^2 \max_{d(\theta)=R^{-1/2}} \|f_\theta\|_{L^2_{avg}(\theta)}^{p-2} \\
&\lesssim R^{O(\epsilon_0)} r^{\frac{5}{2}-\frac{3p}{4}-\frac{p-2}{2}+\frac{1}{2}} \|f\|_{L^2}^2 \max_{d(\theta)=R^{-1/2}} \|f_\theta\|_{L^2_{avg}(\theta)}^{p-2}.
\end{aligned}$$

Since $D \le r^{1/2}$, when $p > \frac{16}{5}$, the constant term is bounded by R^{ϵ}. □

Chapter 7

Proof of Theorem 4

The proof of Theorem 4 combines the lemmas in previous chapters. First we apply Lemma 4.3 to obtain $\|Ef\|^p_{BL^p(B_R)} \lesssim \sum_O \|Ef\|^p_{BL^p(O)}$ and

$$Ef|_O = Ef_O + \sum_{t=1}^{n} Ef_{S_t} + \text{RapDec}(R)\|f\|_{L^2}.$$

Then we decompose $Ef = Ef^{\sim} + Ef^{\nsim}$ and apply the two ends argument. If $Ef^{\sim}$ dominates, then Lemma 3.4 gives the answer. Otherwise, $Ef^{\nsim}$ dominates. By Corollary 4.4,

$$Ef^{\nsim}|_O = Ef^{\nsim}_O + \sum_{t=1}^{n} Ef^{\nsim}_{S_t} + \text{RapDec}(R)\|f\|_{L^2}.$$

If $Ef^{\nsim}_O$ dominates, then we apply Lemma 5.1. Otherwise, there exists a t such that $Ef^{\nsim}_{S_t}$ dominates for most of the O and $\|Ef^{\nsim}_{S_t}\|_{BL^p(O)} \sim \|Ef^{\nsim}_{\Pi_{S_t}}\|_{BL^p(O)}$. We apply

- Lemma 5.2 when $r_t \geq R^{13/16}$,
- Lemma 6.1 when the corresponding $R^{1/2} \leq r_t \leq R^{13/16}$,
- and Lemma 6.2 when $r_t \leq R^{1/2}$.

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