Extremal Problems For Polynomials And Power Series

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Abstract

Linear functionals in several special Banach spaces are studied, notably in the space of functions regular in $|z| < 1$ and belonging to the class H_p, and the space of polynomials of fixed degree with maximum modulus in the unit circle as norm. The main problems investigated are the attainment of a maximum by the linear functional on the unit sphere of the space, the value of this maximum (i.e., the norm of the functional) and the set of points on the unit sphere where the maximum is attained (the extremal functions). A familiarly observed duality phenomenon connecting this maximum problem with a problem of closest approximation in the conjugate space is formulated abstractly and shown to be a consequence of the Hahn-Banach theorem. This duality is then exploited in order to obtain results on the above-mentioned problems.

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Table of Notation and Terminology

Notation	Place defined, or definition
associated step function	[2.41]
cases (i),(ii),(iii),(iv)	[2.31]
c.f.	constant factor
duality	[0.5]
H_p, $\overset{0}{H}_p$, $\bar{H}_p$, $H_\infty C$	[0.431]
index	[2.41],[2.5]
interpolating set	[2.42]
$J(t_0,\ldots t_n)$	[2.3]
jumps	[2.41],[2.52]
L_p	[0.421]
limit monomial functional (l.m.f.)	[2.5]
maximum indicator	[2.43]
monomial case, monomial functional (m.f.)	[2.31]
nodes	[2.41],[2.52]
$\Vert\ \Vert_p$	$\Vert u(\theta)\Vert_p = (1/2\pi)\int_0^{2\pi}\vert u(\theta)\vert^p d\theta)^{\frac{1}{p}}$
p'	$1/p + 1/p' = 1$
transpose	[2.22]
types (i),(ii),(iii),(iv)	[2.31]
up to c.f.	[2.3]
V( )	total variation

Introduction

[0.1] In this paper, we study extremal problems of the following type: Let X be a normed linear space (which in our case will consist of either analytic functions, or polynomials of fixed degree) and let T be a linear functional on X. We then wish to determine whether $|Tx|$ attains a maximum as x ranges over X_o, the set of elements of unit norm in X, and if so, the value $\| T \|$ of this maximum, and the set of $x \varepsilon X_o$ for which $|Tx| = \| T \|$.

[0.2] Part I deals with spaces of analytic functions, namely the spaces H_p. Regarding the literature on this subject, we refer the reader to MacIntyre and Rogosinski[1]. Those authors consider the linear functional

$$(1) \quad T: \ f(z) \to (1/2\pi i)\int_C f(z)K(z)dz \quad , \quad f \ \varepsilon \ H_p$$

where the kernel K(z) is a rational function, and C denotes $|z| = 1$. Their method is based upon a duality between the problem of maximizing $|Tf|$ on the unit sphere of H_p, and a certai. minimum problem in $H_{p'}$. By exploiting the relationship between these problems, they obtain extensive information about both. The duality consists in showing that

$$(2) \qquad \operatorname*{Max}_{f \varepsilon H_p^0} |Tf| = \operatorname*{Min}_{g \varepsilon H_{p'}} \| K+g \|_{p'}$$

(H_p^0 is the set of $f \varepsilon H_p$ with $\| f \|_p = 1$); their proof of this is due to Kakeya[2] and exploits the special nature of the kernel K. In part I of this paper we show that the duality is a consequence of a general theorem on normed linear spaces (proved in [0.5]). This enables us to extend the relation (2) to very general kernels K. We also prove related theorems about extremal problems in H_p with linear restrictions on f, and about best approximation of a function of L_p with a function of $\overline{H}_p$.

[0.3] In part II, our general duality principle is applied to spaces of polynomials, notably the space π_n of polynomials of degree n: $f(z) = a_0 + \ldots a_n z^n$, with $\| f \| = \operatorname{Max} |f(e^{i\theta})|$, and the linear functional

$$T: a_0 + \ldots a_n z^n \rightarrow t_0 a_0 + \ldots t_n a_n \qquad \text{on } \pi_n.$$

Both general and special functionals T are studied; in particular several new proofs of S. Bernstein's inequality for the derivative of a polynomial are obtained.

[0.4] We collect in this section for reference some theorems concerning general normed linear spaces, the spaces L_p and H_p, and the reflection principle for analytic functions. More specialized lemmata will be introduced as needed.

[0.41] Hahn-Banach theorem [ ]

Let X be a normed linear space, X_1 a linear subspace. Let T be a linear functional on X_1 of norm M. Then there exists a linear functional $\bar{T}$ on X such that

(1) $\bar{T}x = Tx \quad \text{for} \quad x \in X_1$

(2) $\| \bar{T} \| = M$

[0.42] Properties of the space L_p. [ ]

[0.421] Let L_p denote the set of functions $f(\theta)$ defined for $0 \leqslant \theta \leqslant 2\pi$ and measurable, and such that

(1) $\| f \|_p = [(1/2\pi)\int_0^{2\pi} |f(\theta)|^p d\theta]^{1/p} < \infty.$

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For $p = \infty$, $\|f\|_p$ is defined as ess sup $|f|$.

[0.421] Let $C = C[0,2\pi]$ denote the set of functions $f(\theta)$ which are continuous $(0 \leqslant \theta \leqslant 2\pi)$, and set $\| f \| = \| f \|_\infty = \text{Max } |f|$.

Let $1/p + 1/p' = 1$, for $1 \leqslant p \leqslant \infty$.

[0.422] Let T denote a bounded linear functional on L_p $(1 \leqslant p < \infty)$. Then T has the form

$$Tf = (1/2\pi)\int_0^{2\pi} f(\theta)K(\theta)d\theta$$

for all $f \in L_p$, where $K(\theta) \in L_{p'}$, and $\| T \| = \| K \|_{p'}$.

[0.423] Let T denote a linear functional on C. Then there exists a $G(\theta)$ of bounded variation $(0 \leqslant \theta \leqslant 2\pi)$ such that, for all $f \in C$, $Tf = \int_0^{2\pi} f(\theta)dG(\theta)$; and $\| T \| = V(G)$ = total variation of G.

[0.424] We include here a theorem on weak compactness of L_p: Let $1 < p \leqslant \infty$, and let there be given a sequence $[f_n]$ of functions of L_p whose norms are bounded. Then a subsequence $[f_{n'}]$ exists, and an $f \in L_p$, such that $\int f_{n'} g d\theta \to \int f g d\theta$ for every $g \in L_{p'}$. For $p = 1$ the theorem is false, but an analogous theorem on functions of bounded variation can often be used.

[0.43] The Classes H_p, $\overline{H}_p$

[0.431] Let $1 \leqslant p \leqslant \infty$. Then H_p is defined as the set of functions $f(z)$ regular for $|z| < 1$, and such that $\| f(re^{i\theta}) \|_p$ remains bounded as $r \to 1-0$.

[0.432] If $f(z) \in H_p$, then for almost all θ

$$\lim_{r \to 1-0} f(re^{i\theta})$$

exists, thus defining a "boundary function" $f(e^{i\theta})$ a.e. Further $f(e^{i\theta}) \in L_p$, and

$$(1) \qquad \int_0^{2\pi} f(e^{i\theta}) e^{in\theta} d\theta = 0 \qquad n = 1, 2, \ldots.$$

Conversely, any $f(e^{i\theta}) \in L_p$ satisfying (1) is the boundary function of an $f(z) \in H_p$. Let $\overline{H}_p$ denote the set of boundary functions of H_p. If $1 \leqslant p \leqslant \infty$, H_p and $\overline{H}_p$ are isometric, under the natural correspondence, and also $\| f(re^{i\theta}) - f(e^{i\theta}) \|_p \to 0$ as $r \to 1-0$ for $p < \infty$.

[0.433] Let $H_\infty C$ denote the set of functions regular for $|z| < 1$ and continuous for $|z| \leqslant 1$, and let $\overline{H_\infty C}$ denote the class of boundary functions. Then for $f(z) \in H_\infty C$, $|f(re^{i\theta}) - f(e^{i\theta})| \to 0$ uniformly in θ as $r \to$

[0.433]

1-0. The sets $H_\infty C$ and $\overline{H_\infty C}$ are isometric under the natural correspondence. $\overline{H_\infty C}$ is identical with the set of continuous functions satisfying (0.432:1).

[0.434] If a function of $\overline{H}_p$ vanishes in a set of positive measure, it vanishes identically.

[0.435] More generally, if $f(z) \varepsilon H_p$ and $K(z)$ is regular on an arc of $|z| = 1$ and $f(e^{i\theta}) = K(e^{i\theta})$ in a subset of this arc which has positive measure, then also $K \varepsilon H_p$ and $f \equiv K$.

[0.436] A function $G(e^{i\theta})$ of bounded variation $(0 \leqslant \theta < 2\pi)$ such that

$$(1) \qquad \int_0^{2\pi} e^{in\theta}\, dG(\theta) = 0 \qquad n = 1,2,\ldots$$

is absolutely continuous, and $G'(e^{i\theta}) = h(e^{i\theta})$ is a function of H_1, with $\| h \|_1 = V(G)$.

[0.437] A sequence of functions $[f_n]$ of H_p whose norms are bounded contains a subsequence which converges uniformly on every closed subset of $|z| < 1$ to a function $f \varepsilon H_p$.

[0.44] Two reflection principles.

Let $F(z)$ be regular and single valued in an open arc-wise connected subset D of $|z| < 1$ having an arc C_1: $\theta_1 \leqslant \theta \leqslant \theta_2$ of $|z| = 1$ as part of its boundary. Suppose that about every interior point of C_1 a circle can be drawn whose intersection with $|z| < 1$ lies in D. Suppose further that a function $F(e^{i\theta}) \in L(\theta_1, \theta_2)$ exists so that for every closed sub-arc $\theta_3 \leqslant \theta \leqslant \theta_4$ of C_1 we have

$$\lim_{r \to 1-0} \int_{\theta_3}^{\theta_4} |F(re^{i\theta}) - F(e^{i\theta})| d\theta \to 0.$$

Finally, define the set D' to be the set of all z' for $z \in D$ ($\bar{z}z' = 1$; $0' = \infty$) and the open set $Q = D + C_1 + D'$.

[0.441] If now $F(e^{i\theta})$ is real for almost all $\theta_1 < \theta < \theta_2$, then $F(z)$ can be continued analytically to a function regular and single-valued in Q, and satisfying the functional equation $F(z') = \overline{F(z)}$.

[0.442] If $|F(e^{i\theta})| = 1$ for almost all $\theta_1 < \theta < \theta_2$, and $F(z)$ has no zeros in D, then $F(z)$ can be continued to a function regular and single-valued in all of Q, and satisfying the functional equation $F(z')\overline{F(z)} =$ $=$ const.

[0.443] Further, in either of the previous statements, if $F(z)$ has an isolated branch point or essential singularity at some point z_o of D, the continued function has a singularity of the same character at z'_o. If $F(z)$ has a zero or pole at z_o, the continued function has a zero or pole of the same order at z'_o in the case of [0.441], and vice versa (a pole or zero of the same order at z'_o) in the case of [0.442].

[0.5] Duality Principle; the following simple theorem is the key to the main results of this paper: Let X be a normed linear space and X_1 a linear subspace. Let T be a bounded linear functional on X, and $[T]_1$ the set of bounded linear functionals on X which vanish on all of X_1. Then, the norm of T restricted to the subspace X_1 equals the distance, in the metric of the conjugate space, from T to $[T]_1$. That is, denoting by $\overset{o}{X}_1$ the unit sphere of X_1,

$$(1) \qquad \sup_{x \varepsilon \overset{o}{X}_1} |Tx| = \inf_{U \varepsilon [T]_1} \| T-U \|$$

The proof is very simple. Denote by a and b, the left and right sides of (1). For any $x \varepsilon X_1$, and any $U \in [T]_1$, $|Tx| = |(T-U)x| \leqslant \| T-U \| \; \| x \|$. Hence $a \leqslant \| T-U \|$; taking now inf over $U \in [T]_1$ gives $a \leqslant b$. On the other hand, since $\| T \|_{X_1} = a$, there exists by the Hahn-Banach theorem a linear functional $\bar{T}$ on X such that $\| \bar{T} \| = a$ and $\bar{T}x = Tx$ for $x \in X_1$. Thus, $U = T-\bar{T}$ vanishes on all of X_1, hence $\in [T]_1$ and $a = \| \bar{T} \| = \| T-U \| \geqslant b$; Thus, $a = b$.

In our applications we shall generally not appeal directly to the duality principle, but rather repeat the line of reasoning, modified as necessary, to fit each particular case. Its significance is that it shows the equivalence of solving an extremal problem of the type mentioned in [0.1] for some space, and solving a problem of closest approximation in the conjugate space.

Part I. Extremal problems in H_p.

[1.1] Theorem 1: Let $1 \leq p' \leq \infty$. Let $K(e^{i\theta})$ belong to $L_{p'}[0,2\pi]$. If $p' = 1$ we demand further that K be continuous on $|z| = 1$. Then the expression

$$(1) \quad I(g) = \| K(e^{i\theta}) - g(e^{i\theta}) \|_{p'}$$

attains a minimum m over all $g(z) \in H_{p'}$ for a unique function $g_1(z)$. The expression

$$(2) \qquad Tf = (1/2\pi i) \int_C f(z)K(z)dz \qquad C: |z| = 1$$

attains a maximum (in absolute value) M over all $f \in H_p$ of norm 1, for some $f_1(z)$. Further, $m = M$, and unless $K \in \overline{H}_{p'}$ (in which case $M = m = 0$), or $p = 1$, $f_1(z)$ is unique except for a constant factor c, $|c| = 1$. Further, f_1 and g_1 satisfy the relation:

$$(3) \quad |(1/2\pi i) \int_C f_1(z)[K(z) - g_1(z)]dz| = \|f_1\|_p \, \|K-g_1\|_{p'}$$

Proof: We begin by showing that $|Tf|$ attains a maximum on H_p^o. Let M denote sup $|Tf|$ for $f \in H_p^o$. Assume first $p > 1$. Choose a sequence $[f_n]$ with $|Tf_n| \to M$. Since the functions $f_n(e^{i\theta}) \in L_p$ and have norm 1 there is by [0.424] a function $f(e^{i\theta}) \in L_p$ of norm $\leqslant 1$ and a subsequence $[f_{n'}]$ such that

(4) $$\int_0^{2\pi} f_{n'}(e^{i\theta})u(e^{i\theta})d\theta \to \int_0^{2\pi} f(e^{i\theta})u(e^{i\theta})d\theta$$

for any $u \in L_{p'}[0,2\pi]$. Since by [0.432] we know, for any positive integer q, and each n'

(5) $$\int_0^{2\pi} f_{n'}(e^{i\theta})e^{iq\theta}\, d\theta = 0$$

we deduce from (4), taking $e^{iq\theta}$ for u that (5) holds also when $f_{n'}$ is replaced by f. Thus by [0.432] $f \in \overline{H}_p$. Writing now K for u in (4) we obtain $Tf_{n'} \to Tf$ whence $|Tf| = M$. This f is thus the desired extremal function and we denote it henceforth by f_1. For the case p = 1, we again choose a sequence f_n as above, but introduce the functions

(6) $$F_n(\theta) = \int_0^{\theta} f_n(e^{i\phi})d\phi;$$

we have now

(7) $$V[F_n] = \| f_n \|_1 = 1.$$

Since the F_n have uniformly bounded variation, there exists a subsequence $[F_{n'}]$ and an F of b.v. such that

$$(8) \qquad \int_0^{2\pi} u(e^{i\theta})dF_{n'}(\theta) \to \int_0^{2\pi} u(e^{i\theta})dF(\theta)$$

for every u continuous on $|z| = 1$. Since each $F_{n'}$ satisfies

$$(9) \qquad \int_0^{2\pi} e^{iq\theta}\, dF_{n'}(\theta) = 0 \qquad\qquad q = 1,2,\ldots$$

we obtain from (8), taking $e^{iq\theta}$ for u, that (9) holds when $F_{n'}$ is replaced by F, whence by [0.436] F is absolutely continuous, and $F'(\theta) = f(e^{i\theta})$ where $f \in H_1$, and $\| f \|_1 = V(F) \leq 1$. Employing (8) again, taking K for u, we obtain $Tf_{n'} \to Tf$, so that $|Tf| = M$ and f is the desired extremal, **which we** denote henceforth by f_1.

Next, we prove that the minimum is attained. Assume first $p < \infty$. Then, T defines a bounded linear functional on $\overline{H}_p$ of norm M. By the Hahn-Banach theorem T can be extended to all of L_p in a norm-preserving manner. Denote this extension by $\overline{T}$. Then there exists by [0.422] a function $u \in L_{p'}$ such that

$$(10) \qquad \overline{T}f = (1/2\pi i) \int_C f(z)u(z)dz$$

for all $f(e^{i\theta}) \in L_p$, and

(11) $$\|T\| = \|\bar{T}\| = M = \|u\|_{p'}.$$

Since $\bar{T}$ agrees with T on $\bar{H}_p$, in particular for $f(e^{i\theta}) = 1, e^{i\theta}, e^{2i\theta}, \ldots$ we have

(12) $$\int_C z^q u(z)dz = \int_C z^q K(z)dz$$

whence the function $g(e^{i\theta}) = K(e^{i\theta}) - u(e^{i\theta})$ is in L_p and satisfies

(13) $$\int_C z^q g(z)dz = 0 \qquad q = 0,1,\ldots \quad .$$

By [0.432] this implies that $g \in \bar{H}_{p'}$, and so we may extend $g(e^{i\theta})$ to $|z| < 1$ and obtain a function $g(z) \in H_{p'}$. Since $K-g = u$ on C, we have by the last equality in (11), (14) $\|K-g\|_{p'} = M$. However, for any $g \in H_{p'}$ we have

(15) $$\|K-g\|_{p'} \geq |(1/2\pi i)\int_C f_1(z)[K(z)-g(z)]dz|$$

$$= |(1/2\pi i)\int f_1(z)K(z)dz| = |Tf_1| = M.$$

Thus I(g) does indeed attain a minimum value = M for the g produced above. Denote this extremal g by g_1. Note that since the chain of inequalities (15) becomes equalities for $g = g_1$ we have proved (3), and only the uniqueness assertions remain unproved, in the case $p < \infty$.

In the case $p = \infty$ we must proceed differently due to the absence of a representation theorem for the linear functionals on L_∞. T is a bounded linear functional of norm M on H_∞. On the subspace $H_\infty C$, T is also a b.l.f. of norm say $M_1 \leqslant M$. Let m denote inf $\| K-g \|_1$ over $g \in \bar{H}_1$. For any $f \in H^0_\infty$ and any $g \in H_1$ we have

$$(16) \quad |Tf| = |(1/2\pi i)\int_C f(z)K(z)dz| = |(1/2\pi i)\int f(z)\cdot [K(z)-g(z)]dz| \leqslant \| K-g \|_1.$$

Hence $M = \| T \| \leqslant \| K-g \|_1$ for all $g \in H_1$ whence

$$(17) \qquad M \leqslant m.$$

Again, since T is a b.l.f. on $H_\infty C$ of norm M_1 it can be extended to a functional $\bar{T}$ on C, the space of all functions continuous on $|z| = 1$ such that $\|\bar{T}\| = M_1$. This $\bar{T}$ has then a representation

$$(18) \quad \bar{T}f = (1/2\pi) \int_0^{2\pi} f(e^{i\theta})dG(\theta)$$

where $G(\theta)$ is of bounded variation for $0 \leqslant \theta < 2\pi$, and $(1/2\pi)V(G) = M_1$. Since $\bar{T}$ agrees with T on $H_\infty C$, in particular for $f = e^{iq\theta}$, $q = 0,1,2,\ldots$ we get

$$(19) \qquad \int_0^{2\pi} e^{iq\theta} dG(\theta) = \int_0^{2\pi} e^{iq\theta}\, e^{i\theta}\, K(e^{i\theta})d\theta$$

for $q = 0,1,\ldots$. Let now $H(\theta) = \int_0^\theta e^{i\phi}K(e^{i\phi})d\phi$. Then, setting $Q(\theta) = G(\theta) - H(\theta)$ we may write (19) as

$$(20) \qquad \int_0^{2\pi} e^{iq\theta}\, dQ(\theta) = 0 \qquad q = 0,1,\ldots$$

Hence $Q(\theta)$ is absolutely continuous, and $Q'(\theta) = q(\theta)$ is in H_1 and has mean value zero so we have $q(\theta) = e^{i\theta}h(e^{i\theta})$ with $h \in H_1$. Now, $G = Q+H$ is absolutely continuous, and differentiating gives $G'(\theta) = e^{i\theta}[h(e^{i\theta}) + K(e^{i\theta})]$. But since $M_1 = V(G)/2\pi$, and the latter expression equals $\| h+K \|_1$ we have produced an $h \in H_1$ with $\| h+K \|_1 = M_1$. Setting $g = -h$, this proves that $m \leqslant M_1$. Hence we have $m \leqslant M_1 \leqslant M \leqslant m$ whence $M = m$. Furthermore the g just produced which we now denote by g_1 serves as the extremal function in H_1 required by the theorem. Finally, $M = |Tf_1| = |(1/2\pi i)\int_C f_1(z)K(z)dz|$

$= |(1/2\pi i)\int_C f_1(z)[K(z)-g_1(z)]dz| \leqslant \|f_1\|_\infty \| K-g_1 \|_1 =$

$= m = M$ shows that (3) is satisfied. Only the uniqueness remains to be proved to complete the proof of theorem 1. For this we look at (3). Suppose first $1 < p < \infty$. In order that equality may hold in (3) we must have, setting $K_1 = K - g_1$

$$(21)\quad \left|\int_0^{2\pi} f_1(e^{i\theta})K_1(e^{i\theta})e^{i\theta}\, d\theta\right| = \int_0^{2\pi} |f_1(e^{i\theta})K_1(e^{i\theta})|\, d\theta$$

and

$$(22)\quad A|f_1(e^{i\theta})|^p = |K_1(e^{i\theta})|^{p'} \qquad \text{a.e.}$$

These equations must hold for every pair of extremals f_1, g_1. Since we assume K is not a.e. a function of $\overline{H}_{p'}$, then $M > 0$, and K_1 differs from zero in a set of positive measure. Integrating (22), we see that

$$A = M^{p'/p'} > 0.$$

Hence (22) implies, since by [0.434] f_1 cannot vanish in a set of positive measure, that K_1 cannot vanish in a set of positive measure. (We do not actually require this fact for the uniqueness proof.) Consider now a <u>fixed</u> extremal g_1. From (22) the modulus of any pos-

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$$(21) \quad \left|\int_0^{2\pi} f_1(e^{i\theta})K_1(e^{i\theta})e^{i\theta}\, d\theta\right| = \int_0^{2\pi} |f_1(e^{i\theta})K_1(e^{i\theta})|\, d\theta$$

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These equations must hold for every pair of extremals f_1, g_1. Since we assume K is not a.e. a function of $\overline{H}_{p'}$, then $M > 0$, and K_1 differs from zero in a set of positive measure. Integrating (22), we see that

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$$(23) \quad \operatorname{sgn} f_1(e^{i\theta})K_1(e^{i\theta})e^{i\theta} = c \qquad (c = \text{const}, \ |c| = 1)$$

and this condition determines the sgn of any possible f_1. Hence f_1 is uniquely determined up to the factor c. Reasoning now in reverse, we choose a fixed f_1 and see that K_1 is uniquely determined up to a factor c. Hence g_1 is strictly unique; for if two extremals g_1 and g_2 existed, $K-g_1$ would be a constant multiple of $K-g_2$: $K-g_1 = c(K-g_2)$ with $c \neq 1$, whence $K =$ $= (g_1-cg_2)/(1-c)$ would be in $H_{p'}$, a contradiction.

The uniqueness in the cases $p = 1$ and $p = \infty$ is more delicate. For $p = \infty$, we have still (21) and also

$$(24) \qquad \int_0^{2\pi} |f_1(e^{i\theta})|\,|K_1(e^{i\theta})|\,d\theta = \int_0^{2\pi} |K_1(e^{i\theta})|\,d\theta.$$

If E is the set where $K_1 \neq 0$ then $|f_1(e^{i\theta})| = 1$ a.e. in E. Also we have from (21) that (23) holds for $\theta \in E$. Thus, fixing first a g_1, $f_1(e^{i\theta})$ is determined up to a constant factor in a set of positive measure, whence $f_1(z)$ is determined up to a c.f., since a function of H_∞ is uniquely determined by its values in a set of positive measure on $|z| = 1$ ([0.434]). Having a fixed f_1, it is then easy to see that sgn K_1 is determined in E, but this is not sufficient to prove the uniqueness of g_1. We do this in [1.6].

For $p = 1$, we have still (21), and

$$(25)\quad \int_0^{2\pi} |f_1(e^{i\theta})|\ |K_1(e^{i\theta})|\,d\theta = \int_0^{2\pi} |f_1(e^{i\theta})|\,d\theta \cdot \operatorname{ess\,sup} K_1 .$$

Since f_1 cannot vanish in a set of positive measure, (23) holds a.e., and (25) gives $K_1 = \operatorname{ess\,sup} K_1 = M$ a.e., and this together with (23) gives as before the uniqueness of K_1, hence of g_1. On the other hand we can deduce that sgn f_1 is uniquely determined, but this does not imply uniqueness of f_1. Theorem 1 is completely proved.

[1.11] We collect a few observations about theorem 1.

[1.111] If $p = 1$ genuinely different extremals f_1 can exist. This phenomenon is completely discussed for rational kernels K in Macintyre-Rogosinski[1], p. 280. A simple example is $K(z) = z^{-2}$ whence $Tf = f'(0)$ and it is easily seen that $f_1(z) = z$ and $f_1(z) = (1/2)(1+z)^2$

are extremals. Here $g_1(z) \equiv 0$, $\| T \| = 1$.

[1.112] If we restrict our definition of an extremal f_1 so that $Tf_1 = \| T \|$ (instead of $|Tf_1| = \| T \|$) the set of extremals in H_1 is a convex set, so that if we have two we have infinitely many.

[1.113] We emphasize the following remark which occurred in the proof: For $1 < p < \infty$, if $K \in L_p\, [0,2\pi]$ then the (unique) best $\overline{H}_p$ approximation to K cannot equal K in a set of positive measure. Note we have also shown that for $p = \infty$ the extremal f_1 has modulus 1 in a set of positive measure, and for $p = 1$ the extremal g_1 satisfies $|K(e^{i\theta}) - g_1(e^{i\theta})| =$ const. a.e.

[1.2] Analytic continuation of extremal functions. Because of the conditions (1.1:22) and (1.1:23) which extremal functions satisfy we can under certain regularity assumptions about K deduce (by using the reflection principles of [0.44]) results about the analytic behavior of f_1 and g_1.

[1.21] Theorem 2. Let K(z) be regular for $t < |z| < 1/t$ where $0 < t < 1$. Then

(i) For $p = \infty$, f_1 is a rational function of the form

(1) $$B(z) = cz^q \prod (z-b)(1-\bar{b}z)^{-1}$$

where $q \geq 0$ is an integer, and the b satisfy $0 < |b| < 1$. Every g_1 is regular in $|z| < 1/t$.

(ii) For $p = 1$ every f_1 is regular for $|z| < 1/t$. Further, g_1 is regular in $|z| < 1/t$ except possibly for poles whose only limit points are on $|z| = 1/t$.

(iii) For $1 < p < \infty$ $f_1(z)$ has the form $B(z)[h(z)]^{1/p}$ where $B(z)$ is a rational function of the form (1) and $h(z)$ is regular for $|z| < 1/t$ and does not vanish for $|z| < 1$. g_1 is regular for $|z| < 1/t$ except possibly for branch points in $|z| > 1$ whose only limit points are on $|z| = 1/t$.

Proof: We set

(2) $$J(z) = zf_1(z)[K(z)-g_1(z)];$$

since we may assume W.L.O.G. that the c in (1.1:23) equals 1, $J(e^{i\theta})$ is real a.e. Let us denote by Q the annulus $t < |z| < 1/t$, and the intersections of Q with the interior and exterior of C: $|z| = 1$ by Q_1 and Q_2

respectively (thus $Q = Q_1+C+Q_2$). Also write K_1 for $K-g_1$. Note that by [0.441] J can be continued so as to be regular in Q.

(i) $p = \infty$. Choose t_1 so that $t < t_1 < 1$. Since every zero of $f_1(z)$ in $t_1 < |z| < 1$ is a zero of $J(z)$, and J is regular for $t_1 < |z| < 1$, f_1 has only finitely many zeros there. Again, f_1 has only a finite number in $|z| < t_1$ so the number of zeros of f_1 in $|z| < 1$ is finite. Since also $|f_1(e^{i\theta})| = 1$ a.e. it follows that f_1 has the form (1). Solving (2) for g_1 gives $g = K-(J/zf_1)$. K is regular in Q_2; so is J, and zf_1 has no zeros in Q_2 so that g is regular in Q_2, hence in $|z| < 1/t$.

(ii) $p = 1$; K_1 has modulus 1 a.e. on $|z| = 1$. Since every zero of K_1 in Q_1 is a zero of J these zeros can cluster only at $|z| = t$. Hence K_1 can be continued into all of Q so as to be regular except for poles in Q_2 which can cluster only at $|z| = 1/t$. Thus $g = K-K_1$ has the same property. Again, $f_1 = J/zK_1$; since K_1 does not vanish in Q_2 (because such zeros would correspond to poles of K_1 in Q_1), f_1 is regular in Q, hence for $|z| < 1/t$.

(iii) $1<p<\infty$. As in (i) f_1 has finitely many zeros in $|z| < 1$. Let $f_1(z) = B(z)f_2(z)$ where $B(z)$ is of the type (1) and f_2 is in H_p and $\neq 0$ in $|z| < 1$. From (1.1:22) we deduce

(3) $$M|f_1(e^{i\theta})|^p = |J(e^{i\theta})| \quad \text{a.e.}$$

Since $|f_1| = |f_2|$ on C, (3) is true when we write f_2 for f_1. Let f_3 denote a branch of f_2^p . Then

(4) $$M|f_3(e^{i\theta})| = |J(e^{i\theta})|;$$

Thus f_3/J has constant modulus on C and no zeros in Q_1 (only poles, arising from zeros of J) so we deduce that J and f_3 have the same zeros on $|z| = 1$, and f_3/J can be continued across C into all of Q so as to be regular there except for poles in Q_1. Since f_3/J is regular in Q_2, so is f_3. Recalling that

$$f_1 = Bf_3^{1/p}$$

we have the result for f_1, with $f_3 = h$. Finally the formula $g = K - J/zf_1$ gives the result for g_1.

[1.22] In case the kernel K is regular only on an arc of C, we can use the preceding principles to give continuations across that arc. We do not bother to state the theorems modified to this situation.

[1.23] It is clear that if we assume that K is regular not in an annulus but in any arc-wise connected domain which contains with each point its inverse point $1/\bar{z}$, we can obtain analogous results on continuation of the extremal functions.

[1.3] Minimal solutions to problems of interpolation by functions of H_p.

[1.31] Let $K_0, \ldots K_s$ be functions defined and in $L_{p'}$ on $|z| = 1$, and such that no non-null linear combination of the K is almost everywhere a function of $\bar{H}_{p'}$ (in particular the K are linearly independent). We allow $1 \leqslant p \leqslant \infty$, but if $p = 1$, $p' = \infty$ we demand further that the K_i be continuous.

[1.32] Consider the set X of functions

(1) $$H(e^{i\theta}) = h(e^{i\theta}) + \sum_{j=0}^{s} c_j K_j(e^{i\theta})$$

where h runs through all of $\bar{H}_{p'}$ and the c_j take all possible (complex) values. X is a linear space which we may norm by defining $\| H \| = \| H \|_{p'}$. Let now $t_0, \ldots t_s$ be arbitrary fixed complex numbers, and consider the mapping

(2) $\quad T: H \to t_0 c_0 + \ldots t_s c_s$.

We shall now show that <u>T is a bounded linear functional on H</u>. To prove this we need the following algebraic lemma:

Given a matrix with q columns and ∞ rows

$$\begin{matrix} a_{11} & \cdots & a_{1q} \\ a_{21} & \cdots & a_{2q} \\ \vdots & & \vdots \end{matrix} ,$$

if every qxq submatrix (formed by choice of q arbitrary rows) is singular then the columns are linearly dependent. We leave the simple proof to the reader. Now, let the function K_j have the Fourier expansion

(3) $$K_j(e^{i\theta}) \sim \sum_{-\infty}^{\infty} b_{jq} e^{-iq\theta} \qquad j = 0,1,\ldots s .$$

In view of the assumption made about the K_j in [1.31] the lemma implies the following: From the matrix

$$(4) \qquad \begin{matrix} b_{01} & \cdots & b_{s1} \\ b_{02} & \cdots & b_{s2} \\ \vdots & & \vdots \end{matrix}$$

we can always extract s+1 rows which form a non-singular matrix. Now, return to equation (1); multiplying by $(1/2\pi)e^{iq\theta}$ and integrating with respect to θ from 0 to 2π we get, setting

$$(5) \qquad (1/2\pi)\int_0^{2\pi} H(e^{i\theta})e^{iq\theta}d\theta = d_q,$$ the relation

$$(6) \qquad d_q = \sum_{j=0}^{s} c_j b_{jq} ;$$

now by choosing s+1 values of q such that the corresponding rows in the matrix (4) are a non-singular matrix, we obtain a set of s+1 linear equations of the form (6) which can be solved for the c_j, in particular for c_0, in terms of the d. We thus obtain

$$(7) \qquad c_0 = \Sigma A_q d_q ,$$

the sum being over the particular values of q chosen; the A depend only on the b, hence only on the given functions K. Suppose now $\| H \| < 1$; then from (5) we get $|d_q| \leqslant 1$. Hence from (7) if A denotes max $|A_q|$

we have $|c_0| \leqslant (s+1)A$; that is, the mapping $H \to c_0$ is a bounded linear functional on X. Similarly the map $H \to c_j$ $(j = 1,\dots s)$ is a **b.l.f.**, and so finally is the map T in (2).

[1.321] The mapping $H(e^{i\theta}) \to h(e^{i\theta})$ is a bounded linear transformation of X into $\overline{H}_{p'}$. For, let $\|H\| \leqslant 1$; then, by [1.32] the c_j in (1) satisfy $|c_j| \leqslant B$ where B depends only on the K. Hence $\|h\| \leqslant 1 + \sum_{j=0}^{s} B\|K_j\|$ proving the assertion.

[1.33] <u>Theorem 3</u> Let $K_0,\dots K_s$ be as in [1.31] and $t_0,\dots t_s$ given complex numbers. There exists an $f \in H_p$ satisfying

$$(1) \quad T_j f = (1/2\pi i)\int_C f(z)K_j(z)dz = t_j \qquad j=0,\dots s \ (C: |z|=1)$$

and for which $\|f\|_p$ is a minimum. This f is strictly unique, if $p > 1$, and in any case satisfies a relation

$$(2) \qquad |(1/2\pi i)\int_C f(z)H(z)dz| = \|f\|_p \, \|H\|_{p'}, \text{ where } H$$

is a certain function of the form (1.32:1).

Proof: Let first $p' < \infty$. Consider now the linear functional defined by (1.32:2). It is by

[1.32] a b.l.f. on the linear subspace X of $L_{p'}[0,2\pi]$. Let $\| T \| = M$. Then there is an extension $\bar{T}$ of T from X to all of $L_{p'}$ with $\| \bar{T} \| = M$. $\bar{T}$ can be represented in the form

$$(3) \qquad \bar{T}g = (1/2\pi i)\int_C f(z)g(z)dz \qquad C: |z| = 1$$

for all $g \in L_{p'}$. Here f is a certain function of L_p on C and $\| f \|_p = M$. Since $\bar{T}$ agrees with T on X we must have

$$(4) \qquad \bar{T}\, e^{iq\theta} = 0 \qquad q = 0,1,\ldots \qquad \text{and}$$

$$(5) \qquad \bar{T}\, K_j(e^{i\theta}) = t_j \qquad j = 0,\ldots s \quad .$$

Hence, taking $e^{iq\theta}$ for g in (3) we have

$$(6) \qquad \int_C f(z)z^q dz = 0 \qquad q = 0,1,\ldots$$

whence we deduce $f \in \bar{H}_p$. Further, the conditions (5) tell us that f satisfies (1). We have thus produced an f (which we call henceforth f_1) satisfying the hypotheses of the theorem and with $\| f_1 \|_p = M$. Again, for any $f \in H_p$ which satisfies those conditions we have, for any $H \in X$ of norm 1,

$$\| f \|_p \geqslant |(1/2\pi i)\int_C f(z)H(z)dz| = |\Sigma t_j c_j| = |TH|$$

and taking a sequence of H for which $|TH| \to M$ we get $\| f \|_p \geqslant M$. Hence f_1 is indeed minimal.

In the case $p' = \infty$ we modify the argument by considering instead of X the subspace X_1 consisting of all H of the form (1.32:1) where $h \in \overline{H_\infty C}$. Since also the K_j are assumed continuous in this case, we now proceed to extend the functional T to all continuous functions on $|z| = 1$ etc. as in the proof of theorem 1. We omit these details.

Next, we shall show that the functional T attains a maximum on the unit sphere of X. Suppose first $p' > 1$. Choose a sequence $[H_n]$ of functions of X satisfying

(7) $$\| H_n \| = 1$$

(8) $$|TH_n| \to M \quad .$$

Now, we can because of (7) find an $H(e^{i\theta}) \in L_{p'}$ and a subsequence of the H_n (which for convenience of notation we call again $[H_n]$) such that

(9) $$\int_0^{2\pi} H_n(e^{i\theta})u(e^{i\theta})d\theta \to \int_0^{2\pi} H(e^{i\theta})u(e^{i\theta})d\theta$$

for every $u \in L_p$; further $\| H \| \leqslant 1$. Let now

$$(10) \qquad H_n(e^{i\theta}) = h_n(e^{i\theta}) + \sum_0^s c_{nj}K_j(e^{i\theta}) \quad .$$

Since as we have seen the mapping $H(e^{i\theta}) \to c_j$ is a b.l.f. on X the condition (7) implies that the c_{nj} occurring in (10) are a bounded set; hence numbers $c_1', \ldots c_s'$ exist and a subsequence of the H_n such that $c_{nj} \to c_j$ for $j = 1, \ldots s$ as $n \to \infty$ through this subsequence. Again for convenience of notation suppose this is already the case for the sequence H_n. Now, recalling the notation $K_j(e^{i\theta}) \sim \Sigma\, b_{jq}e^{-iq\theta}$ we get, multiplying (10) by $e^{iq\theta}$ where $q \geqslant 1$ and integrating

$$(11) \qquad (1/2\pi) \int_0^{2\pi} H_n(e^{i\theta})e^{iq\theta}d\theta = \sum_0^s c_{nj}b_{jq}$$

whence, letting $n \to \infty$,

$$(12) \qquad (1/2\pi)\int_0^{2\pi} H(e^{i\theta})e^{iq\theta}d\theta = \sum_0^s c_j'b_{jq}$$

In other words the function $V = H - \Sigma c_j'K_j$ is in $L_{p'}$ and satisfies

$$\int_0^{2\pi} V(e^{i\theta})e^{iq\theta}d\theta = 0 \qquad q = 1, 2, \ldots$$

so that $V \in \overline{H}_{p'}$; but this says that $H \in X$; since $\| H \| \leqslant 1$, and finally $|TH| = |(1/2\pi i)\int_C f_1(z)H(z)dz|$

$= \lim |(1/2\pi i)\int_C f_1 H_n dz| = \lim |TH_n| = M$ we see that T does indeed attain a maximum at H (which implies that actually $\|H\| = 1$, not merely ≤ 1). A look at the last chain of inequalities shows that (2) is satisfied. The case $p' = 1$ introduces complications because of the failure of the weak compactness, but as in the proof of theorem 1 we get around the difficulty by considering instead of the sequence H_n the sequence of indefinite integrals. We omit details.

Finally the uniqueness is deduced in routine fashion by discussing the case of equality in (2). This completes the proof.

[1.34] If we make regularity assumptions about the K_j we can easily obtain precisely the same results about analytic continuation of the extremal f_1 as we did in theorem 2. For example, for $p = \infty$ suppose the K_j are regular on $|z| = 1$. Then f_1 is a rational function of type (1.21:1). In the case where the K_j are of the form $(z-a)^{-m}$ ($m \geq 1$; $|a| < 1$). This was proved by Kakeya[2].

[1.4] <u>Generalization of theorem 1 with homogeneous linear restrictions on f.</u> Theorem 1 can be deduced easily from theorem 3. In fact, the following more general theorem can be so deduced:

<u>Theorem 4.</u> Let K_j, $j = 0,\ldots s$, be kernels as in [1.31]. Among the set of $f \varepsilon H_p$ for which

(1) $\quad \| f \|_p = 1$

(2) $\quad T_j f = (1/2\pi i)\int_C f(z)K_j(z)dz = 0 \qquad j = 1,\ldots n$

there is an $f = f_1$ which makes

(3) $\quad |T_0 f| = |(1/2\pi i)\int_C f(z)K_0(z)dz|$

a maximum M. Also the expression

(4) $\quad I(h;\vec{c}) = \| h(e^{i\theta}) + \sum_1^s c_j K_j(e^{i\theta}) + K_0(e^{i\theta}) \|_{p'}$

attains a minimum m as h runs through all of $\bar{H}_{p'}$ and the c_j take all complex values. Further, $m = M > 0$. Finally, let $h_1, c_1', \ldots, c_s'$ be a set of "values" for which I takes its ~~minimum~~, and set

(5) $$P_1 = h_1 + \sum_1^s c_j' K_j'.$$

Then if $p < \infty$ $h_1, c_1' \ldots c_s'$ are uniquely determined;

if $p > 1$ f_1 is uniquely determined (up to a c.f.); and in any case f_1 and P_1 satisfy

(6) $$|(1/2\pi i)\int_C f_1(z)P_1(z)dz| = \|f_1\|_p \|P_1\|_{p'}$$

Proof: We apply theorem 3 taking $t_0 = 1$, $t_1 = \ldots t_s = 0$. Let f be the extremal function given by theorem 3. Then, it satisfies (2) and

(7) $$T_0 f = 1 \;,$$

and its norm $\| f \| = B$, say, is smallest for all such functions. Setting $f_1 = f/B$ we obtain a function satisfying (1) and (2) and

(8) $|T_0 f| = 1/B$. Again, any $f_2 \in H_p$ satisfying (1) and (2) and $|T_0 f_2| = 1/A > 1/B$ would be such that $f = bAf_2$, for some $|b| = 1$ satisfies (7) and (2) and has norm $A < B$ contradicting the minimal property of f. Thus f_1 is an extremal for our problem, with $M = 1/B$. Also, as was shown in the proof of theorem 3, the number $1/B$ above is the maximum of the linear functional $T: h+\Sigma c_j K_j \rightarrow c_0$ on the unit sphere of the space X, this maximum being attained; in other words it is the maximum of the expression

(9) $$|c_0| \cdot \| h + \sum_0^s c_j K_j \|^{-1} ,$$

or, dividing through by $|c_0|$, it is the reciprocal of the minimum of $\|(h/c_0) + \Sigma\, (c_j/c_0)K_j + K_0 \|_p$, as h (hence h/c_0) ranges over all of H_p, and the numbers $c_0, \ldots c_s$ (hence $c_1/c_0, \ldots c_s/c_0$) take all complex values, and the minimum is attained. But this is precisely the assertion M = m of our theorem, and at the same time we have shown the attainment of the minimum in (4), with $h_1 = h/c_0$, $c_j' = c_j/c_0$ where h, $c_0, \ldots c_s$ are values which make (9) a maximum. Finally the identity (6) follows from theorem 3, and the uniqueness of f_1 and P_1 in the cases asserted follows as usual by considering the conditions for equality in (6). Once P_1 is determined so are $h_1, c_1', \ldots c_s'$.

[1.41] As before the identity (6) will enable us to obtain analytic continuations of f_1 (and h_1, which is now less interesting) in case the K_j are regular. As a token result in this direction, we state: if $p = \infty$, and the K_j are regular on $|z| = 1$ then f_1 is a rational function of type (1.21:1).

[1.5] Extremal problem with non-homogeneous linear restrictions. In the last two theorems the maximum-minimum duality has already become less interesting, notably the "minimal" part of the theorems, and this is still more so if we now consider restrictions of the form $T_j f = t_j$ with $t_j \neq 0$. Nevertheless the existence of the duality, especially of an identity of type (1.33:2) is still important, mainly because it leads to analytic continuation of the extremal f_1. It is this last fact which gives the following theorem (which does not even contain a duality assertion) some interest.

Theorem 5. Let $K_0, \ldots K_s$ be kernels as in [1.31]. Let $t_1, \ldots t_s$ be complex numbers not all zero. Then the set of $f \in H_p$ satisfying

(1) $$\| f \|_p \leqslant 1$$

(2) $$T_j f = (1/2\pi i) \int_C f(z) K_j(z) dz = t_j \qquad j = 1, \ldots s$$

either is empty, or contains an $f = f_1$ for which

(3) $$|T_0 f| = |(1/2\pi i) \int_C f(z) K_0(z) dz|$$

is a maximum. For the extremal f_1 we have

(4) $$\| f_1 \|_p = 1;$$

Further f_1 is the solution to an extremal problem of the type in theorem 3 (with the same kernels K) and so satisfies an identity of the form (1.33:2).

Note: Due to the non-homogeneity (4) is not obvious in this case.

Proof: Suppose there exist f satisfying (1) and (2). Then by the usual argument an f_1 exists maximizing (3). Suppose $\| f_1 \|_p < 1$. Now, there exists a function $g \varepsilon H_p$ satisfying

(5) $$T_j g = 0 \qquad j = 1, \ldots s \quad \text{and}$$

(6) $$T_0 g \neq 0 \;;$$

this follows from theorem 4. Now, by choosing a suitable constant b we can arrange that $f_2 = f_1 + bg$ (which also satisfies (2)) satisfies further

(7) $$\| f_2 \| < 1$$

(8) $$|T_0 f_2| > |T_0 f_1|$$

thus contradicting the maximal property of f_1. Hence we must have $\| f_1 \| = 1$. Next, let $T_0 f_1 = c$. By theorem 3 we can find an $f \varepsilon H_p$

(which we denote by f_3) such that f_3 satisfies (2) and

(9) $$T_0 f_3 = c,$$

and has the least norm of all such functions. Let $\|f_3\| = m$. Now, we cannot have $m < 1$, for the construction just effected would then serve to produce an $f_4 \varepsilon H_p$ of norm < 1 satisfying (2) and

$$|T_0 f_4| > |T_0 f_3| = |c| = |T_0 f_1|$$

which would be a contradiction. Thus $m \geqslant 1$. But since f_1 has norm 1 and satisfies (2) and $T_0 f_1 = c$, we conclude by the uniqueness of the minimal f_3 that $f_1 \equiv f_3$. Hence f_1 is indeed a solution of an extremal problem of the asserted type, and our theorem is proved. Note that we cannot assert uniqueness of f_1 because f_3 is defined in terms of c which depends on f_1.

[1.51] Here again, if the K_j are regular we can obtain analytic continuation of f_1. For example if all the K_j are regular on $|z| = 1$, and $p = \infty$, f_1 is a rational function of type (1.21:1).

[1.6] In theorem 1 we did not discuss uniqueness of g_1 in the case $p = \infty$. We now answer this question in the affirmative, and give a formula enabling us to compute g_1 from f_1. This formula is valid for all p, although our main interest here is $p = \infty$.

We know that f_1 is unique up to a c.f. Suppose f_1 is so normalized that $Tf_1 = i\,\|T\|$ so f_1 is completely fixed (we make this assumption throughout this section). Then, for any extremal g_1 we must have

$$(1/2\pi i)\int f_1(z)[K(z)-g_1(z)]iz d\theta = i\,\|\,T\,\|$$

whence $e^{i\theta}f_1(e^{i\theta})[K(e^{i\theta})-g_1(e^{i\theta})]$ is pure imaginary a.e. Hence, writing $h(z) = zf_1(z)g_1(z)$ (so that $h \in H_1$, $h(0) = 0$) we have $\operatorname{Re} h(e^{i\theta}) =$ $= \operatorname{Re} e^{i\theta}f_1(e^{i\theta})K(e^{i\theta})$ a.e. Now, by the well-known (Schwarz) formula expressing an analytic function in $|z| < 1$ in terms of the boundary values of its real part we obtain, for $|z| < 1$

$$(1)\quad h(z) = (1/2\pi)\int_0^{2\pi}[\operatorname{Re} e^{i\phi}f_1(e^{i\phi})K(e^{i\phi})](e^{i\phi}+z)(e^{i\phi}-z)^{-1}d\phi$$

This formula gives h (and so g_1) in terms of f_1 and K, and so proves the uniqueness of g_1.

[1.7] Examples and applications. The most important applications of theorem 1 are to extremal problems with rational kernels, and a complete and systematic account of the "best-possible" inequalities which result from various kernels is found in Macintyre-Rogosinski[1]. In a later paper we hope to present a similar discussion for other kernels. In the present section we confine ourselves to a few simple applications to illustrate theorems 1,2, and 4.

[1.71] Let $p = \infty$ and take $K(z) = 1$ for $0 \leqslant \theta \leqslant \pi$ and 0 for $\pi < \theta < 2\pi$. Then $Tf = (1/2\pi i)\int_{C_1} f(z)dz$ where C_1: $|z| = 1$, $\operatorname{Im} z > 0$. By Cauchy's theorem $Tf = (i/2\pi)\int_{-1}^{1} f(x)dx$ whence, if $\| f \| \leqslant 1$, $|Tf| \leqslant 1/\pi$. Equality is attained for $f = f_1 \equiv 1$ and this f_1 is already normalized in the sense of [1.6]. Hence by (1.6:1) we obtain for $|z| < 1$

$$zg_1(z) = (1/2\pi)\int_0^{\pi} \cos \phi(e^{i\phi}+z)(e^{i\phi}-z)^{-1}d\phi$$

Simplifying a little we get

$$g_1(z) = (1/\pi)\int_0^{\pi} (\cos \phi)(e^{i\phi}-z)^{-1}d\phi. \tag{1}$$

This integral can easily be evaluated; it is regular in the entire infinite plane except for logarithmic singularities at ± 1. Recalling the minimal property of g_1 we get the curious inequality

$$\int_0^{\pi} |1-g(e^{i\theta})|d\theta + \int_{\pi}^{2\pi} |g(e^{i\theta})|d\theta \geqslant 2 \tag{2}$$

for every $g \in H_1$; equality holds only for the function

g_1 defined by (1). Another curious feature of this example is that it shows that the (unique) best H_1 approximation g_1 to the step function K is unbounded in $|z| < 1$.

[1.72] We saw that the extremal f_1 for the kernel K in [1.71] was constant. It is easy to derive a necessary and sufficient condition for this. Indeed, let K have the Fourier expansion $K(e^{i\theta}) \sim \sum_{-\infty}^{\infty} b_q e^{iq\theta}$; if then f_1 is constant, g_1 must be such that (writing $g_1(z) = \sum_0^{\infty} a_q z^q$) the function $e^{i\theta}[K(e^{i\theta}) - g_1(e^{i\theta})]$ is, aside from a constant factor, non-negative a.e. That is, the series

$$(1) \qquad -\sum_1^{\infty} a_q e^{i(q+1)\theta} + (b_0 - a_0)e^{i\theta} + \sum_1^{\infty} b_{-q} e^{-i(q-1)\theta}$$

must be, aside from a factor c: $|c| = 1$, the Fourier series of a non-negative function. In particular we must have

$$(2) \qquad cb_{-1} > 0$$

$$(3) \qquad c(b_0 - a_0) = \overline{cb_{-2}}$$

$$(4) \qquad -ca_q = \overline{cb_{-(q+2)}}$$

Now solve (2),(3),(4) for c, a_0, a_1, ... and put these values back into (1). We thus conclude: A necessary and sufficient condition that the extremal f_1 in H_∞ be constant is that the series

$$(5) \qquad \cdots + \overline{cb}_{-2}\, e^{+i\theta} + cb_{-1} + cb_{-2}e^{-i\theta} + cb_{-3}e^{-2i\theta} + \cdots$$

be the Fourier series of a non-negative function, where c is the number of modulus unity determined by (2). In that case $\| T \| = |b_{-1}|$ and the extremal function g_1 is given by

$$(6) \qquad g_1(z) = (b_0 - \bar{c}^2 b_{-2}) - \bar{c}^2(\bar{b}_{-3}z + \bar{b}_{-4}z^2 + \cdots)$$

In particular, this analysis applies to the kernel K of [1.71]. Here $b_0 = 1/2$, $b_{-2} = b_{-4} = \ldots 0$ and $b_{-q} = 1/\pi q$ for q odd. Note that $\| T \| = |b_{-1}| = 1/\pi$ agrees with our previous result. We have also $c = -i$, and

$$(7) \qquad g_1(z) = 1/2 - (i/\pi)(z^3/3 + z^5/5 + \ldots)$$

This is the same $g_1(z)$ as that given by the formula (1.71:1).

[1.73] As an application of the principles of theorem 2, take $p = \infty$ and let K be an arbitrary complex-valued step-function whose value on the arc C_q is a_q ($q = 1, \ldots m$; the C_q are disjoint open arcs whose union is $C: |z| = 1$ minus the points $z_1 \ldots z_m$ where the arcs join.) We assume further that the a_q for adjacent arcs are unequal. Then, if f_1 and g_1 denote the extremal functions we have, setting

(1) $$J_q(z) = zf_1(z)[a_q - g_1(z)]$$

the condition that sgn $J_q(e^{i\theta})$ is constant a.e. on C_q. By suitably normalizing f_1 we may assume this constant is 1, so that we have

(2) $J_q(e^{i\theta})$ is real a.e. on C_q.

Hence we may extend J_q so as to be regular and single-valued in the whole infinite plane, minus the arc D_q (D_q denotes the closed complimentary arc to C_q). Since J_q cannot have zeros which cluster at a point of the arc C_q the same is true of f_1. Since this is true for each arc C_q the zeros of f_1 can only cluster at the points $z_1 \ldots z_m$. Since $|f_1(e^{i\theta})| = 1$ a.e. we conclude that $f_1(z)$ is a Blaschke product of the form

(3) $$f_1(z) = cz^k \prod (z-b)(1-\bar{b}z)^{-1}$$ where the b (of which there may be infinitely many) satisfy $|b| < 1$ and have as their only limit points $z_1, \ldots z_m$. Having

thus continued f_1 it follows that for each q, g_1 can be continued across the arc C_q, and because $g_1 = a_q-(J_q/zf_1)$ and f_1 has no zeros in $|z| > 1$ we deduce that g_1 is regular in the entire infinite plane except possibly for singularities at $z_1 \dots z_m$. Now we show that these are indeed singularities, in fact branch points of g_1. We know by [0.441] and [0.442] that J_q satisfies a functional equation

(4) $\quad J_q(z') = \overline{J_q(z)}$ and f_1 satisfies

(5) $\quad f_1(z')f_1(z) = A = \text{const.}, \ |A| = 1$

Consider now a fixed point z_0 in $|z| > 1$ and let w_q denote the value of $g_1(z_0)$ obtained by continuing g_1 from $|z| < 1$ across C_q to z_0 by a direct path (e.g. a straight line from a point of C_q to z_0. Because of the relations (4), (5) we calculate easily

(6) $\quad t_q = A(\bar{t} - \overline{a_q})[z_0f_1(z_0)]^{-2} + a_q$

where t denotes the number $g_1(z_0')$. Hence for adjacent arcs C_q and C_{q+1} we obtain

(7) $\quad t_q - t_{q+1} = u - B\bar{u}$

where we have set $u = a_q-a_{q+1}$ and $B = A[z_0f_1(z_0)]^{-2}$.

Now by hypothesis $a_q \neq a_{q+1}$ so $u \neq 0$. Further $|B| < 1$ since $|f_1(z_0)| > 1$ and $|z_0| > 1$. Thus, from (7), $|t_q - t_{q+1}| > (1-|B|)|u| > 0$. Since $t_q \neq t_{q+1}$ it follows that the point joining arcs C_q and C_{q+1} is a branch point. Since g_1 is the best H_1 approximation to K we conclude: The (unique) best H_1 approximation to a step function on $|z| = 1$ is regular in the whole infinite plane except for branch points on $|z| = 1$ at the jumps of the step function.

[1.741] The reasoning of the preceding section applies equally well to a kernel K which is piecewise analytic on $|z| = 1$, appropriate modifications being made depending on the domains of regularity of the analytic components of K.

[1.75] As a final example we show how theorem 4 can be applied to give a theorem on uniform approximation; We show: Let $a_1, a_2 \ldots$ be a sequence of distinct points in $|z| < 1$. A necessary and sufficient condition that every continuous function on $|z| = 1$ can be uniformly approximated by rational functions having only simple poles all of which occur at the a_j is that the sum $\Sigma(1-|a_j|)$ diverge.

Note: if we do not require the a_j to be distinct the result still holds provided we allow the rational function to have an m-fold pole at an a_j which is repeated m times. For convenience of notation however we prove only the former assertion.

Proof: Let $K_0(e^{i\theta})$ be the given continuous function. Since a function regular in $|z| < 1$ and continuous in $|z| \leq 1$ can be uniformly approximated by polynomials it is sufficient to show we can obtain the approximation by an expression of the form

(1) $$H(e^{i\theta}) = h(e^{i\theta}) + \sum_1^s c_j K_j(e^{i\theta}) \quad \text{where}$$

(2) $$K_j(z) = 1/(z-a_j),$$

and $h \in H_\infty C$. Now, consider the finite set $a_1 \dots a_s$. Among all $f \in H_1$ satisfying

(3) $$(1/2\pi i)\int_C f(z)K_j(z)dz \ [=f(a_j)] = 0 \qquad j = 1,\dots s$$

and

(4) $$\| f \|_1 = 1 \ ;$$

let $f_1 = f_{1s}$ be that one which makes

(5) $$|T_0 f| = |(1/2\pi i)\int_C f(z)K_0(z)dz|$$

a maximum M_s.

By theorem 4 we know M_s is precisely the minimum of

(6) $$\| H(e^{i\theta}) - K_0(e^{i\theta}) \|_\infty$$

as H runs through all functions of type (1). Hence to prove the sufficiency part of the theorem it suffices to show that $\lim M_s = 0$. Now, suppose this were not so; then since the M_s are non-increasing we have $\lim M_s = M > 0$. Now we know from the proof of theorem 1 that if a sequence of functions $[f_n]$ of H_1 has its norms bounded there is an $f \varepsilon H_1$ and a subsequence $f_{n'}$ such that

(7) $$\int_0^{2\pi} f_{n'}(e^{i\theta})u(e^{i\theta})d\theta \to \int_0^{2\pi} f(e^{i\theta})u(e^{i\theta})d\theta$$

for every continuous function u. Applying this to the sequence $[f_{1s}]$ we obtain a subsequence (denoted again, for convenience, by $[f_{1s}]$) and an $f_1 \varepsilon H_1$ such that (7) holds with f_{1s} for $f_{n'}$ and f_1 for f. Taking $u = K_0$ we deduce that

(8) $$|T_0 f_1| = M.$$

Next taking $u = K_j$ where $j > 1$ is arbitrary, we deduce that $f_1(a_j) = 0$. But since f_1 vanishes at all the a_j and belongs to H_1, and (because of (8)) is not identically zero, it follows from a well-known property of H_p functions that $\Sigma(1-|a_j|) < \infty$, contradicting the hypothesis. For the necessity, assume that every continuous K_0 can be approximated in the required manner. Consider an arbitrary fixed K_0. Then given $\varepsilon > 0$ there exists an integer s and an H of type (1) with $\|K_0 - H\|_\infty < \varepsilon$. Hence a fortiori we have for M_s which is the minimum of (6) over all such H, $M_s < \varepsilon$. Now, consider any $f \varepsilon H_1$ vanishing at all the a_i. We have $|T_0 f| < \|f\| M_s < \varepsilon \|f\|$. Since ε is arbitrary we have $T_0 f = 0$. That is, for every continuous function K_0 we have $\int_C f(z) K_0(z) dz = 0$. This however implies $f \equiv 0$. Thus every $f \varepsilon H_1$ vanishing at the a_i vanishes identically. If $\Sigma(1-|a_j|) < \infty$ we could however construct an $f \varepsilon H_1$, namely $f(z) = \prod (z-a_j)(1-\bar{a}_j z)^{-1}$ vanishing at all the a_j and $\not\equiv 0$. Thus we conclude $\Sigma(1-|a_j|) = \infty$ and the proof is complete.

Part II. Extremal problems for polynomials.

[2.1] The general background for what follows is this: Let π_n denote the linear space of polynomials

(1) $$f(z) = a_0 + \dots a_n z_n$$

of fixed degree n with complex coefficients, where z is a complex variable. The most general linear functional T on π_n is clearly

(2) $$T: f \to t_0 a_0 + \dots t_n a_n$$

where the t_i are complex numbers characterizing T. We sometimes also write the functional (2) as $(t_0, \dots t_n)$. If now π_n is normed by $\| f \| = \| f \|_\infty = \max |f(e^{i\theta})|$, our problem is to determine the value of $\| T \|$ and the associated extremal polynomials f_1 satisfying $|Tf_1| = \| T \|$. We could equally well study this problem with π_n normed by $\| f \| = \| f \|_p$ with $1 \leqslant p < \infty$ but $p = \infty$ is by far the most interesting case, and most nearly related to classical results so we consider it exclusively except for one fundamental existence-uniqueness-duality theorem in [2.61].

[2.2] To make the exposition flow smoothly we assemble here a number of preliminary results.

[2.211] Lemma: Let $t_0 > 0$, $t_1, \ldots t_n$ be any complex numbers. A NASC that there exist a non-decreasing function $G(\theta)$ $0 \leq \theta < 2\pi$ satisfying

(1) $$\int_0^{2\pi} e^{ik\theta} dG(\theta) = t_k \qquad k = 0, \ldots n$$

is that for every system of complex numbers $[c_j]$ $j = 0, \ldots n$ with c_0 real, $c_{-j} = \bar{c}_j$ for which

(2) $$P(\theta) = \sum_{-n}^{n} c_j e^{ij\theta}$$

is non-negative everywhere we have

(3) $$\sum_{-n}^{n} t_k c_k \geq 0$$

(t_{-k} is defined as $\bar{t}_k$). This lemma is well known in the theory of the trigonometric moment problem and follows from theorem 1.1 of Shohat and Tamarkin[4], which depends on an extension theorem for "non-negative functionals". A very simple proof can also be based directly on the classical Hahn-Banach theorem as follows: Since the necessity of condition (3) is obvious we need consider only the sufficiency. Now, let X be the space of $P(\theta)$ of the form (2) normed by $\| P \| = \max |P|$. The map

(4) $$T: P \to \sum_{-n}^{n} t_k c_k$$

is clearly a b.l.f. on the space X. Let now

(5) $$-1 \leqslant P(\theta) \leqslant 1;$$

since $1 + P(\theta) \geqslant 0$ we deduce

(6) $$t_0 + \sum_{-n}^{n} t_k c_k \geqslant 0.$$

Since $1-P(\theta) \geqslant 0$ we deduce

(7) $$t_0 - \sum_{-n}^{n} t_k c_k \geqslant 0.$$

(6) and (7) imply that $\|T\| \leqslant t_0$. Hence there exists an extension $\overline{T}$ of T from X to all of C (continuous functions on $[0,2\pi]$) with $\|\overline{T}\| \leqslant t_0$. (Actually equality holds as $P \equiv 1$ shows) By [0.123] there exists a $G(\theta)$ of bounded variation for $0 \leqslant \theta < 2\pi$ with $V(G) = t_0$ and

(8) $$\overline{T}u = \int_0^{2\pi} u(\theta)\,dG(\theta) \text{ for every } u \in C.$$

Since $\overline{T}$ agrees with T on X, in particular for $u = 1, e^{i\theta}, \ldots e^{in\theta}$ we deduce from (8) that G satisfies (1). Finally since $\int dG(\theta) = V(G)\ (= t_0)$, G must be non-decreasing, completing the proof.

[2.212] **Similarly** one proves: let $t_0 > 0$, $t_1 \ldots t_n$ be arbitrary real numbers. A NASC that there exist a non-decreasing $G(\theta)$ $(-1 \leqslant x \leqslant 1)$ satisfying

(1) $$\int x^k dG(x) = t_k \qquad k = 0, \ldots n$$

is that for any real numbers $c_0, \ldots c_n$ which make $g(x) = c_0 + \ldots c_n x^n$ non-negative in $-1 \leqslant x \leqslant 1$ the expression

(2) $$\sum_0^n t_k c_k \text{ is } \geqslant 0.$$

[2.213] Lemma: (Fejer-Riesz) Let $P(\theta) = \sum_{-n}^{n} c_k e^{ik\theta} > 0$ for all θ (c_k complex, $c_{-k} = \bar{c}_k$). Then complex numbers $a_0, \ldots a_n$ exist with $T(\theta) = |\sum_0^n a_k e^{ik\theta}|^2$.

[2.214] It is easy to deduce from [2.213] that the NASC of [2.211] is equivalent to the condition that the Hermitian (Toeplitz) form of matrix $\| t_{j-k} \|$ $j,k = 0, \ldots n$ be non-negative definite. See [4] p. 6(c).

[2.215] One can similarly show that the NASC of [2.212] is equivalent to the following: Define $t_0' = t_0$, $t_1' = t_1$, $t_2' = 2t_2 - t_0, \ldots$ (generally, $t_k' = \Sigma\, d_{kj} t_j$ where the d are defined by $\cos k\theta = \Sigma\, d_{kj} \cos^j \theta$) and $t_{-k}' = \bar{t}_k'$. The condition is then that $\| t_{j-k}' \|$ be non-negative definite.

[2.22] Lemma: Let $[c_k]$ $(-\infty < k < \infty)$ be a sequence of complex numbers which are Fourier-Stieltjes coefficients, that is, for some G of bounded variation

(1) $$c_k = \int_0^{2\pi} e^{ik\theta}\, dG(\theta) \qquad -\infty < k < \infty .$$

Then, if we define the sequence $[d_k]$ by $d_k = c_{k+q}$, where q is any integer, there exists a function $G_q(\theta)$ of b.v. with

(2) $$d_k = \int_0^{2\pi} e^{ik\theta}\, dG_q(\theta).$$

Further this (unique) G_q has the same total variation as G.

Proof: A little thought will show that the lemma follows at once from the following theorem, which is itself an immediate consequence of Banach[6] p. 74 theorem :

Given a sequence $[c_k]$, $-\infty < k < \infty$, a NASC that there exist a $G(\theta)$ of b.v. and total variation $\leqslant M$ satisfying (1) is that for every trigonometric polynomial (of arbitrary order) $P(\theta) = \sum_{-n}^{n} a_k e^{ik\theta}$ satisfying $|P| \leqslant 1$ we have

$$\left|\sum_{-n}^{n} c_k a_k\right| \leqslant M.$$

Definition: the function G_q defined above is called the transpose of order q of the function G.

[2.221] Let $G(\theta)$ denote a function of bounded variation, $0 \leq \theta < 2\pi$. Then a NASC that

$$\int_0^{2\pi} e^{iq\theta} dG(\theta) = V(G)$$

is that G be the transpose of order q of a non-decreasing function.

[2.231] Let $f(z) = a_0 + \dots a_n z^n$. Then $|f(z)|$ attains a maximum on $|z| = 1$ at no more than n distinct points, unless $f(z) = cz^k$ $0 \leq k \leq n$.

Proof: Let $M = \max_{|z|=1} |f|$. Then $M - |f(e^{i\theta})|^2$ is a non-negative trigonometric polynomial of order n and unless it is $\equiv 0$ (whence $f(z) = cz^k$) it can have (algebraically) at most 2n zeros. Being non-negative all zeros are of order ≥ 2, so there cannot be more than n distinct zeros.

[2.232] Let $f(x) = a_0 + \dots a_n x^n$, a_i real, $f \neq$ constant. Then $|f(x)|$ attains a maximum in $-1 \leq x \leq 1$ at no more than n+1 distinct points. If it attains it at n+1 distinct points it must be a constant multiple of the Chebyshev polynomial of order n.

For, since f' has at most n-1 distinct zeros there are at most n-1 maxima in $-1 < x < 1$; in addition a maximum may be attained at ± 1, making $n+1$ points in all.

For the second assertion we reproduce the simple proof of W. Markoff[7] p. 225 (which probably dates back earlier): If $|f|$ attains a maximum (say 1) at $n+1$ points $-1 \leqslant x_1 < x_2 < \dots x_{n-1} \leqslant 1$ we see that $1-[f(x)]^2$ has degree $2n$ and has simple zeros at ± 1 and double zeros at $x_1, \dots x_{n-1}$. Again, $(1-x^2)[f'(x)]^2$ has degree $2n$ and the same zeros whence $(1-x^2)[f'(x)]^2$ is a constant multiple of $1-[f(x)]^2$; examining the coefficient of x^n shows that the constant is n^2 and we get, writing $y = f(x)$

$$(1) \qquad (1-x^2)(y')^2 = n^2(1-y^2);$$

solving (1) for y gives the result.

[2.3] In this section we prove the basic duality theorem for polynomials in $|z| \leqslant 1$. We shall find the situation much more delicate than was the case for extremal problems in H_∞, notably as regards uniqueness. (On the other hand the existence of the extremal polynomial is trivial.) Throughout Part II we use the notation $J = J(t_0, \dots t_n)$ to denote the Hermitian form whose matrix is

$$(1) \qquad \left\| \begin{matrix} t_0 & t_1 & \cdots & t_n \\ t_{-1} & t_0 & \cdots & t_{n-1} \\ \vdots & & & \vdots \\ t_{-n} & \cdots & & t_0 \end{matrix} \right\|$$

where $t_0, \dots t_n$ are given complex numbers with t_0 real, $t_{-j} = \bar{t}_j$ and write $|J|$ to denote the determinant of (1). Finally due to annoying factors of modulus 1 which constantly intervene we use the abbreviation "up to c.f." to indicate that a statement is true once some obvious multiplication by a factor of modulus 1 is made, e.g. the function 1-x is increasing (up to c.f.) in $0 \leqslant x \leqslant 1$, or the polynomial $i-2ix$ has real coefficients (up to c.f.).

[2.31] <u>Theorem 6</u> Let [G] denote the set of (complex valued) functions $G(\theta)$ of bounded variation for $0 \leqslant \theta < 2\pi$ such that

$$(1) \qquad \int_0^{2\pi} e^{ik\theta} dG(\theta) = t_k \qquad k = 0, \dots n$$

where the t are preassigned complex numbers. Then there exists a $G = G_1 \varepsilon$ [G] whose total variation is minimum m, and m is the norm of the functional $T: (t_0, \dots t_n)$ on π_n. We distinguish further the following cases:

(i) $J(t_0, \dots t_n)$ is non-negative definite (up to c.f.)

(ii) $J(t_n, \dots t_0)$ " " " " "

(iii) For some g: $1 \leqslant q \leqslant n-1$ the numbers $t_0 \dots t_n$ have (up to c.f.) the following property: t_q is real, and the sequence $t_0, \dots t_n$ has Hermitian symmetry about t_q, as far as it extends (e.g. i,1,-i, 6 is such a sequence; here 1 plays the role of t_q) and further the form

$$(2) \qquad \begin{array}{ll} J(t_q, \dots t_n) & \text{if } q \leqslant n/2 \\ J(t_q, \dots t_0) & \text{if } q \geqslant n/2 \end{array}$$

is non-negative definite (these forms are identical if n is even, $q = n/2$).

(iv) None of the above forms is non-negative definite.

In case (i) and only then there exists an extremal G_1 which is (up to c.f.) non-decreasing in $[0, 2\pi)$ (in fact, all G_1 have this property; this remark holds also for (ii) and (iii);) the constant 1 is an extremal polynomial, and $\| T \| = |\tau_0|$.

In case (ii) and only then there exists an extremal G_1 which is the transpose of order n of a non-decreasing function (up to c.f.); z^n is an extremal polynomial, and $\|T\| = |t_n|$.

In case (iii) the same statement holds as in (ii) if we replace n by q. Here z^q is an extremal polynomial and $\|T\| = |t_q|$. Further in cases (i) -- (iii) the extremal polynomial is unique if the form is definite.

In case (iv) G_1 is unique; it is a step-function having at most n jumps. No extremal polynomial of the form $z^k (0 \leq k \leq n)$ exists, and $\|T\| > \max |t_k|$.

Remark 1. The distinction between cases (i), (ii) and (iii) is made mainly for convenience of enunciation; the extremal problem has the same character in all these cases, which we henceforth group under the appellation "monomial case" as opposed to "non-monomial" in referring to an extremal problem, or functional for which case (iv) occurrs.

Remark 2. Due to the restrictions of Hermitian symmetry, case (iii) is rare, in the sense that the set of $(t_0, \ldots t_n)$ in complex n+1-space for which it occurs has measure zero. It is thus "rare" to have

z^q with $1 \leqslant q \leqslant n-1$ as an extremal polynomial in π_n.

Proof of theorem 6: Let M denote $\|T\|$. There exists an extension $\bar{T}$ of T from π_n to all of C (continuous functions on $|z| = 1$) with $\|\bar{T}\| = M$. $\bar{T}$ has a representation

$$(3) \qquad \bar{T}u = \int_0^{2\pi} u(e^{i\theta})dG(\theta)$$

for all $u \in C$ where G is a certain function of b.v. in $[0,2\pi)$ with $V(G) = M$. Since $\bar{T}$ agrees with T on π_n, in particular for $u = e^{ik\theta}$, $k = 0,\ldots n$ we see that G satisfies (1). On the other hand any G satisfying (1), and any extremal polynomial f_1 satisfy $|\int f_1(e^{i\theta})dG(\theta)| = |Tf_1| = M$ whence $V(G) \geqslant M$. Thus the G produced above, which we now call G_1 does indeed have least total variation of all G satisfying (1). Further any extremal pair f_1 and G_1 satisfy

$$(4) \qquad \left|\int_0^{2\pi} f_1(e^{i\theta})dG_1(\theta)\right| = \text{Max } |f_1|\cdot V(G_1)$$

and it is the discussion of the conditions for equality in (4) which lead to the remaining assertions of the theorem.

First let us deal with case (i). Consider the following assertions.

(5) $J(t_0, \ldots t_n)$ is non-negative definite.

(6) 1 is an extremal polynomial

(7) there exists a non-decreasing G_1

(8) all G_1 are non-decreasing.

We wish to prove these assertions equivalent. To do this we show (5) → (7) → (6) → (8) → (5). If (5) holds we know by [2.214] that there exists a non-decreasing G_2 satisfying (1). But any G satisfying (1) has $V(G) \geq t_0 = V(G_2)$ so that G_2 is an extremal. Hence (5) → (7). But (7) implies that $\|T\| = t_0$; since for $f \equiv 1$ we have $Tf = t_0$, (6) follows. From (6), setting $f_1 = 1$ in (4) we get $\left|\int dG_1\right| = V(G_1)$ for any extremal G_1, that is any G_1 is non-decreasing (up to c.f.). Finally, (8) → (7) trivially, and (7) → (5) again by [2.214]. To prove the uniqueness in the case where J is definite, we need the following slight sharpening of [2.214] which is easily proved: If $J(t_0, \ldots t_n)$ is positive definite then there exists a non-decreasing function satisfying (1) and having at least n+1 points of increase. Granting this, since every extremal f_1 satisfies $\int f_1 dG = V(G)$, $|f_1|$ must attain a maximum at n+1 points on $|z| = 1$, hence by [2.231] $f_1 = cz^k$,

$|c| = 1$. Since $|Tf_1| = t_0$ we must have $k = 0$.

The treatment of case (ii) and (iii) is entirely similar and we omit it. For these cases we must observe [2.22] and [2.221].

If case (iv) occurs we know that every f_1 can attain a **maximum** at no more than n points. Choose a fixed f_1 and let $z_1, \ldots z_r$ $(1 \leqslant r \leqslant n)$ be the points on $|z| = 1$ where $|f_1(z)| = 1$. Set $z_k = e^{i\theta k}$ $(0 \leqslant \theta_k \leqslant 2\pi)$. Then, for any G_1 we have

(9) $$\left|\int f_1 dG_1\right| = V(G_1).$$

This can only happen if G_1 has as its only points of non-constancy $\theta_1, \ldots \theta_r$, i.e. G_1 is a step-function. Let u_k denote the value of the (complex) jump of G_1 at θ_k. Then the conditions (1) become:

(10) $$\sum_{j=1}^{r} u_j z_j^k = t_k \qquad k = 0, \ldots n .$$

Considered as a system of linear equations for the u_j: (10) can have at most one solution, for the first r equations of the system (10) has a non-vanishing (Van der Monde) determinant. This proves that G_1 is unique, and theorem 6 is proved.

[2.4] In this section we deal exclusively with non-monomial functionals.

[2.41] We prove first the following extension of case (iv) of theorem 6: If T is a non-monomial functional, an integer r: $1 \leqslant r \leqslant n$ and numbers $z_1, \ldots z_r$; $b_1, \ldots b_r$ of modulus one exist such that every extremal polynomial f_1 attains a maximum modulus at $z_1, \ldots z_r$ and its values at these points are proportional to the numbers $b_1, \ldots b_r$.

Proof: Since we know G_1 is unique, the numbers $z_1, \ldots z_r$; $u_1, \ldots u_r$ which give respectively the location and values of the jumps of G_1 are determined by the functional T in unique manner. Consider now any extremal f_1. By (2.31:9) we get

$$\left| \sum_1^r u_j f_1(z_j) \right| = \sum_1^r |u_j|$$

and this implies immediately the result, with $b_j = \operatorname{sgn} \bar{u}_j$.

Definition: For a non-monomial functional T the integer r is called the <u>index</u> of T; the points $z_1, \ldots z_r$ are the <u>nodes</u>; the numbers $u_1, \ldots u_r$ the <u>jumps</u>; G_1 the <u>associated step function</u>.

Remark: We shall show in [2.5] how these concepts can be extended to the monomial case.

[2.42] Knowledge of the jumps of T gives the value of $\| T \| = \Sigma |u_k|$. We shall show here how to compute the jumps from the nodes, so that finding the nodes then becomes the central problem in analyzing a functional T. Let $z_1 \dots z_r$ be the nodes; choose any complex numbers $z_{r+1}, \dots z_{n+1}$ distinct from each other and from $z_1, \dots z_r$. Let

(1) $$P(z) = (z-z_1)\dots(z-z_r)$$

and

(2) $$Q(z) = P(z)(z-z_{r+1})\dots(z-z_{n+1}).$$

For any $f \in \pi_n$ we have by Lagrange interpolation

(3) $$f(z) = \sum_{k=1}^{n+1} \frac{Q(z)}{(z-z_k)Q'(z_k)} f(z_k).$$

Hence

(4) $$Tf = \sum_1^{n+1} [TQ_k/Q'(z_k)]f(z_k)$$

where

(5) $$Q_k(z) = Q(z)/(z-z_k).$$

On the other hand, we know that for all $f \in \pi_n$

(6) $$Tf = \sum_{1}^{r} u_k f(z_k) \quad (= \int f dG_1).$$

Choose now an f taking the value 1 at z_j $(1 \leqslant j \leqslant n+1)$ and 0 at the remaining z_i; equating the right hand sides of (4) and (6) gives

(7) $$(TQ_j)/Q'(z_j) = u_j \qquad j = 1,\ldots r$$

(8) $$TQ_j = 0 \qquad j = r+1,\ldots n+1.$$

Now, since $z_{r+1},\ldots z_{n+1}$ are arbitrary we may let them all tend to zero and get from (7)

(9) $$u_j = T[z^{n-r+1}P(z)/(z-z_j)] \qquad j = 1,\ldots r.$$

Again, from (8) we deduce that for every polynomial R(z) of precise degree n-r we have $T[P(z)R(z)] = 0$ and this implies

(10) $$T[z^j P(z)] = 0 \qquad j = 1,\ldots n-r .$$

Given any linear functional T (monomial not excluded) and any set of points $z_1,\ldots z_r$ on $|z| = 1$ we say that the z_i are an <u>interpolating set</u> for T if there exists an identity

(11) $$Tf = \sum_{1}^{r} u_j f(z_j)$$

valid for all $f \in \pi_n$ (where the u_j depend on T and the z_j but not on f). Given any T, it is an immediate consequence of the Lagrange interpolation theorem that any set containing $\geq n+1$ points is an interpolating set. Again, since the above discussion did not use the nodal properties of the z_j (only the interpolatory property) we see that (10) is a necessary condition for $z_1, \ldots z_r$ to be an interpolating set for T. The converse is also easily proved (we leave this to the reader) and so we can state: For any functional T, a necessary and sufficient condition that a set of $r \leq n$ numbers $z_1, \ldots z_r$ be an interpolating set for T is that they satisfy the condition (10). In that case we have for all $f \in \pi_n$ the identity (11) where the u_j are defined by (9). In particular the nodes satisfy (10) and the jumps are given in terms of the nodes by (9).

Thus, in order to find the nodes of a non-monomial functional T we must in principle examine all interpolating sets of T having r elements for all values of r: $1 \leq r \leq n$. We know from theorem 6 that there will exist a unique r (the index) and a unique interpolating set (the nodes) for which the sum of the absolute values of the u_j defined by (9) is minimum, and the value of this minimum is $\|T\|$. In practice however we cannot carry this through except in the simplest cases. The author has been unable to devise any method for finding the nodes.

[2.43] The author has not been able to determine NASC in order that the extremal polynomial for a given non-monomial functional T be unique. We have however the following partial result which is useful for applications.

<u>Theorem 7.</u> If a (non-monomial) functional T: $(t_0, \ldots t_n)$ where the t_j are real, has index n the extremal polynomial is unique (up to c.f.).

Proof: We introduce first a concept involving polynomials. Given any $f \in \pi_n$ let $M = \max |f(e^{i\theta})|$. Then $M - |f(e^{i\theta})|^2$ is a non-negative trigonometric polynomial of order n, and so, by [2.213] can be represented in the form $|g(e^{i\theta})|^2$ for some $g \in \pi_n$. If we further demand that g(z) be different from zero for $|z| < 1$, g is unique (up to c.f.) as is known from the proof of [2.213]. This polynomial g(z) we call the <u>maximum indicator</u> of f(z). It satisfies the identity

$$(1) \qquad |f(z)|^2 + |g(z)|^2 = M^2 \quad \text{for } |z| = 1$$

and vanishes on $|z| = 1$ at precisely the points where $|f|$ attains its maximum; g 0 if and only if f is a monomial.

Let now f_1 be an extremal and set $P(z)=(z-z_1)\ldots(z-z_n)$, where z_j are the nodes of T. Since the maximum indicator of f_1 must vanish at $z_1,\ldots z_n$ it is a constant multiple of P: (2) $g(z) = aP(z)$. Now let $b_1,\ldots b_n$ be the numbers defined in [2.41] and let S(z) be the (unique) polynomial of degree n-1 with $S(z_j) = b_j$, $j = 1,\ldots n$. Then aside from a c.f. (which we assume is 1) $f_1(z_j) = b_j$, $j = 1,\ldots n$ whence f_1-S vanishes at $z_1,\ldots z_n$ and so is a constant multiple of P:

$$(3) \qquad f_1(z) = S(z) + cP(z).$$

Putting f_1 for f in (1) and replacing g and f_1 by the right sides of (2) and (3) we get (setting $|a|^2 = A > 0$)

$$(4) \qquad |S(z) + cP(z)|^2 + A|P(z)|^2 = 1 \quad (z = e^{i\theta}).$$

Let now $S(z) = s_0 + \dots s_{n-1} z^{n-1}$ and $P(z) = p_0 + \dots p_{n-1} z^{n-1} + z^n$. Then using the identity $|w|^2 = w\bar{w}$ to expand (4) and equating coefficients of $e^{ik\theta}$ on both sides $(k = 0, \dots n)$ we get

$$(5)_n \qquad c(\bar{s}_0 + \overline{cp}_0) + A\bar{p}_0 = 0$$

$$(5)_{n-1} \qquad c(\bar{s}_1 + \overline{cp}_1) + (s_{n-1} + cp_{n-1})(\bar{s}_0 + \overline{cp}_0) + A(\bar{p}_1 + p_{n-1}\bar{p}_0) = 0$$

$$\vdots$$

$$(5)_1 \qquad c(\bar{s}_{n-1} + \overline{cp}_{n-1}) + (s_{n-1} + cp_{n-1})(\bar{s}_{n-2} + \overline{cp}_{n-2}) + \dots (s_1 + cp_1)(\bar{s}_0 + \overline{cp}_0) + A(\overline{p_{n-1}} + p_{n-1}\overline{p_{n-2}} + \dots p_1\bar{p}_0) = 0$$

$$(5)_0 \qquad |c|^2 + |s_{n-1} + cp_{n-1}|^2 + \dots |s_0 + cp_0|^2 + A(1 + |p_{n-1}|^2 + \dots |p_0|^2) = 1.$$

Now, from $(5)_n$ we get (note that $|p_0| = 1$)

$$(6) \qquad -c\bar{s}_0 p_0 = |c|^2 + A.$$

Since the right side of (6) is positive it follows that $c \neq 0$ (also $s_0 \neq 0$), and

$$(7) \qquad \operatorname{sgn} c = -p_0 \operatorname{sgn} \bar{s}_0$$

Now, it is easy to see that if the value of A given by (6) is inserted in each of the equations $(5)_{n-1},\ldots(5)_0$ all terms in $c\bar{c}$ cancel, and the left hand side of each equation has the form $Xc + Y\bar{c} + Z$ where X,Y,Z are polynomials in the s, $\bar{s}$, p, and $\bar{p}$. Hence writing $c = xe^{i\phi}$ where $x = |c|$ and $e^{i\phi}$ is the right side of (7), each of these becomes a linear equation in x. Suppose now one of these equations does not vanish identically; then it can be solved for x, and we will thus have determined both sgn c and $|c|$ (thus c) in terms of the p and s, which in turn depend only on T; that is we will have shown that the c in (3) is uniquely determined by T, hence f_1 is unique. So suppose that all the linear equations are satisfied identically in x; setting the constant terms equal to zero gives (as one readily verifies)

(8)
$$\begin{aligned}
&s_{n-1}\bar{s}_0 = 0\\
&s_{n-1}\bar{s}_1 + s_{n-2}\bar{s}_0 = 0\\
&\quad\vdots\\
&s_{n-1}\overline{s_{n-2}} + \ldots s_1\bar{s}_0 = 0\\
&|s_{n-1}|^2 + \ldots |s_0|^2 = 1.
\end{aligned}$$

The conditions (8) are equivalent to

(9) $$|S(e^{i\theta})|^2 = 1;$$

that is, S must be a monomial Since $|S(z)| \leqslant 1$ for $|z| = 1$ and $|TS| = \sum_1^n u_j S(z_j)| = |\sum_1^n u_j f_1(z_j)| = |Tf_1|$ $= \|T\|$ it follows that S is an extremal polynomial for T, i.e. T admits a monomial extremal, which is a contradiction, and Theorem 7 is proved.

Remark 1. Note that the preceding proof is actually constructive; that is, given the nodes it gives a method of determining the extremal polynomial. It also gives a method (in principle) for determining whether a given set of numbers $z_1, \ldots z_n$ are nodes; one constructs S,P and determines c,A as above; then if and only if the equations (5) are all satisfied are the z_i nodes, and S+cP is the extremal polynomial.

Remark 2. We saw that (6) implies $s_0 \neq 0$. This can be shown to imply $a_0 \neq a_n$ for the extremal polynomial (we omit the proof).

Remark 3. If $t_0, \ldots t_n$ are real f_1 has real coefficients (up to c.f.). For, normalize f_1 such that $Tf_1 = \|T\|$, and write $f_1(z) = a_0 + \ldots a_n z^n$, Re $a_0 = a_0'$, $f_2(z) = a_0' + \ldots a_n' z^n$; we have $2f_2(z) = f_1(z) + \overline{f_1(\overline{z})}$ whence $|f_2(z)| \leqslant 1$ on $|z| = 1$. Since $Tf_2 = \Sigma t_k a_k' = \text{Re } \Sigma t_k a_k = \text{Re } Tf_1 = \|T\|$, f_2 is an extremal with the same normalization as f_1; hence $f_1 \equiv f_2$ and $a_j = a_j'$ is real.

[2.5] In this section we study more deeply the monomial case. We shall confine our study to case (i) of theorem 6, although the results, with the natural modifications, apply to case (ii) and (iii) as well. Unless stated otherwise the functionals of this section are monomial, of type (i). We define further a limit monomial functional $T: (t_0, \dots t_n)$ (of type (i)) as a monomial functional for which the form $J(t_0, \dots t_n)$ is semi-definite. We define the index of such a functional as the smallest integer r for which $|J(t_0, \dots t_r)|$ vanishes (clearly $1 \leqslant r \leqslant n$). We shall see that this definition is a natural extension of the definition for the non-monomial case. We base the proofs in this section on results which Caratheodory[8] proved in a quite different connection. We isolate as lemmas in [2.51] the material we shall use from [8].

[2.51] Lemma 1. Let $J(t_0, \dots t_n)$ be semi-definite and let r be the smallest integer for which $|J(t_0, \dots t_r)| = 0$. Then positive numbers $u_1, \dots u_r$ and distinct numbers $z_1, \dots z_r$ of modulus 1 exist such that

(1) $$\sum_{k=1}^{r} u_k z_k^q = t_q \qquad q = 0, \dots n \;;$$

the u_k and z_k are uniquely determined.

Lemma 2. Under the same assumptions, the matrix $J(t_0, \ldots t_r)$ has the property that if the first column is replaced by $z^r, \ldots z^2, 1$ the resulting matrix has a determinant which is a polynomial $P(z)$ of precise degree r in z; all zeros of P are distinct and lie on $|z| = 1$; and they are precisely the numbers $z_1, \ldots z_r$ of lemma 1.

Lemma 3. A necessary and sufficient condition that there exist a real non-decreasing step-function $G(\theta)$ having precisely r jumps and satisfying

$$(1) \quad \int_0^{2\pi} e^{iq\theta} dG(\theta) = t_q \qquad q = 0, \ldots n$$

is that the form $J(t_0, \ldots t_n)$ satisfy the conditions of lemma 1. (lemma 3 is a simple reformulation of lemma 1; note also the relation to [2.214]).

[2.52] Consider now a l.m.f. T of index r. By lemma 1 it follows at once that there exists an identity

$$(1) \qquad Tf = \sum_1^r u_r f(z_r)$$

for all $f \in \pi_n$ with $u_1, \ldots u_r > 0$; the z_j and u_j are uniquely determined, and $\Sigma u_j = t_0 = \|T\|$. We define the z_j, u_j as the nodes and jumps, respectively of T.

Consider now a strict monomial functional T, i.e. T is a m.f. whose associated form J is definite. Let z_0 be an arbitrary number of modulus 1 and consider the form $J_a = J(t_0-a, t_1-az_0, \ldots t_n-az_0^n)$ where a is positive. It has the determinant

$$(2) \quad |Ja| = \begin{vmatrix} t_0-a & t_1-az_0 & \cdots & t_n-az_0^n \\ \bar{t}_1-a\bar{z}_0 & t_0-a & & \vdots \\ \vdots & \cdots & \cdots & \vdots \\ \bar{t}_n-a\bar{z}_0^n & & & t_0-a \end{vmatrix}$$

Multiplying the first column by z_0^{q-1} and subtracting it from the q^{th} column for q = 2,...n, a disappears except from the first column; hence |Ja| is linear in a. Thus as a decreases |Ja| can vanish at most once; but since Ja cannot remain definite as a is permitted to decrease sufficiently (in particular, for $a \neq t_0$ the form is indefinite) we conclude there is a unique value of a, say a' for which this form becomes semi-definite. We have then by the foregoing, numbers $z_1, \ldots z_r$ of modulus 1 and $u_1, \ldots u_r > 0$ such that

$$(3) \qquad \sum_{k=1}^{r} u_k z_k^q = t_q - a' z_0^q \qquad q = 0, \ldots n$$

or, defining $u_0 = a'$,

$$(4) \qquad \sum_{k=0}^{r} u_k z_k^q = t_q \qquad q = 0,\dots n$$

Now, (4) says that the non-decreasing function $G(\theta)$ with jump u_k at θ_k ($z_k = e^{i\theta_k}$) satisfies

$$(5) \qquad \int_0^{2\pi} e^{iq\theta} dG(\theta) = t_q \qquad q = 0,\dots n.$$

By the sharp version of [2.214] mentioned in [2.31], since J is definite G must have at least n+1 points of increase. Hence we must have $r = n$; once z_0 is given the points $z_1,\dots z_n$ are uniquely determined. Thus we have, rewriting u_0 and z_0 as u_{n+1} and z_{n+1}: If J is positive definite, there exist positive numbers $u_1,\dots u_{n+1}$ and distinct numbers of modulus 1: $z_1,\dots z_{n+1}$ such that

$$(6) \qquad Tf = \sum_{1}^{n+1} u_k f(z_k) \text{ for all } f \in \pi_n.$$

Once one of the z_j is specified the remaining z_j and the u_j are uniquely determined. This makes it natural to define the index of a strict monomial functional as n+1; and we define $z_1,\dots z_{n+1}$ as a set of nodes for T. Here again $\Sigma u_k = \|T\| = t_0$.

If T is a s.m.f. of type (ii) the index is again n+1, and an identity (6) exists with

(7) $$\Sigma|u_k| = \| T \|.$$ Here the u_k are not positive but the numbers $u_k z_k^n$ are. If T is a s.m.f. of type (iii), the index is Max(q,n-q); an identity (6) exists where $u_k z_k^q$ is positive (q=0,...n) and the u satisfy (7).

[2.53] The results of [2.4] and [2.5] can be partially restated in the following form: let $T_{(z_0)}$ for any z_0 on $|z| = 1$ denote the "elementary" functional: $T_{(z_0)}: f(z) \rightarrow f(z_0)$. (Clearly $\| T_{(z_0)} \| = 1$) then given any linear functional T on π_n it can be represented as a linear combination of r elementary functionals ($1 \leqslant r \leqslant n+1$):

(1) $$T = \sum_1^r u_j \, T_{(z_j)}$$

such that $\| T \| = \Sigma|u_j|$. Further this representation is unique if T is non-monomial or limit-monomial and in these cases $r \leqslant n$. The only novelty in this restatement is the obvious transition from identities of type (2.52:6) to "elementary functionals".

[2.6] We collect in this section some analogous results for polynomials normed by $\| f \| = \| f \|_p$, and polynomials in $-1 \leqslant x \leqslant 1$.

[2.61] <u>Theorem 8.</u> Let $1 \leq p < \infty$. Then among all $g(e^{i\theta}) \in L_{p'}$ on $|z| = 1$ and satisfying

$$(1) \quad (1/2\pi)\int_0^{2\pi} g(e^{i\theta})e^{ik\theta}d\theta = t_k \qquad k = 0,\ldots n$$

where the t_k are given complex numbers, there is a unique $g = g_1$ for which $\| g \|_{p'}$ is a minimum m. Further, among all polynomials $f(z) = \sum_0^n a_k z^k$ satisfying $\| f \|_p = 1$ there is an $f = f_1$ for which

$$(2) \qquad |Tf| = |t_0 a_0 + \ldots t_n a_n|$$

is a maximum, and the value of the maximum is m. If $p > 1$ f_1 is unique (up to c.f.). Further, although the set of all $g \in L_1$ satisfying (1) does not in general contain a g for which $\| g \|_1$ is minimum, the lower bound of $\| g \|_1$ over all such g equals $(1/2\pi)\cdot$norm of T: $(t_0,\ldots t_n)$ in the sense of [2.1].

The proof of theorem 8 is so like the proofs of theorems 1 and 6 that we do not write it out, but confine ourselves to a few remarks. The uniqueness of f_1 and g_1 in the cases stated follows because f_1 and g_1 satisfy the identity

$$|(1/2\pi)\int_0^{2\pi} f_1(e^{i\theta})g_1(e^{i\theta})d\theta| = \| f \|_p \, \| g \|_{p'}.$$

The last statement concerning $p = 1$ follows from theorem 6 because of the following: the minimum of $V(G)$ over all G of bounded variation satisfying

(3) $$\int e^{ik\theta} dG(\theta) = t_k \qquad k = 0,\dots n$$

equals the inf of $\|g\|_1$ over all $g \in L_1$ satisfying (1). To see this, denote min $V(G)$ and inf $\|g\|_1$ by a and b. Clearly $a \leqslant b$; again, let G_1 be a G of b.v. satisfying (3) for which $V(G_1) = a$. Let $\sigma_n(e^{i\theta})$ denote the m^{th} Fejer mean of the Fourier-Stieltjes series of dG_1. Since $\int|\sigma_n(e^{i\theta})|d\theta \to V(G_1)$ we can find an m with $\int|\sigma_m|d\theta < a+\varepsilon$. Further choose m so large that $|\int\sigma_m e^{ik\theta}d\theta - t_k| < \varepsilon/2\pi n$ for $k=1,\dots n$. For this fixed value of m, denote $\int\sigma_m e^{ik\theta}d\theta$ by c_k. Then, let

$$g(e^{i\theta}) = 2\pi[\sigma_m(e^{i\theta}) + \sum_1^n (t_k-c_k)e^{-ik\theta}].$$

Clearly g satisfies (1), and $\|g\|_1 < \int|\sigma_m|d\theta + \varepsilon < a+2\varepsilon$. Hence $b < a+2\varepsilon$; since ε is arbitrary, $b \leqslant a$, completing the proof.

[2.62] Polynomials in $-1 \leqslant x \leqslant 1$. Let π_n denote in this paragraph the linear space of $f(x) = a_o + \dots a_n x^n$ with the a_i real, and $\|f\| = \max_{-1\leqslant x\leqslant 1} |f(x)|$.

An analogous theory to that for polynomials in the unit circle can be developed, encompassing classical results of Chebyshev, S. Bernstein, and W. Markoff. We indicate only the essentials of this theory.

Theorem 9. Among all functions $G(x)$ real and of bounded variation in $-1 \leqslant x \leqslant 1$ and satisfying

$$(1) \qquad \int_{-1}^{1} x^k dG(x) = t_k \qquad k = 0, \dots n$$

where the t_k are given real numbers there is a $G = G_1$ whose total variation is a minimum m. This minimum is the maximum of $|t_o a_o + \dots t_n a_n|$ over all $f(x) = a_o + \dots a_n x^n$ with real coefficients satisfying $|f(x)| \leqslant 1$ for $-1 \leqslant x \leqslant 1$, the maximum being attained for some f_1. We distinguish further two cases:

(i) The form $J(t_o', \dots t_n')$ defined in [2.215] is (aside from a possible factor -1) non-negative definite.

(ii) The case (i) does not hold.

In case (i) and only then, $f \equiv 1$ is an extremal, $\|T\| = |t_o|$, and all extremals G_1 are non-decreasing (if $t_o > 0$). If J is definite, ± 1 are the only extremals.

In case (ii), G_1 is unique. It is a step function with at most n+1 jumps.

The proof is just like that of Theorem 6, but simpler due to the fact that we do not have several types of monomial cases. Again the main conclusions of the theorem follow from the identity $|\int f_1 dG_1| = V(G_1)$, and to discuss the cases of equality one uses [2.215] and [2.232]. As before we can define the index and nodes of a non-monomial functional (i.e. one of type (ii)). Because of [2.232] we have here the following striking phenomenon: If a non-monomial functional has index n+1 then the extremal polynomial is (aside from a constant factor) the Chebyshev polynomial of order n. Analogous but far less simple results can be stated for functionals of index n, due to the fact that one can characterize all real polynomials of degree n which attain their maximum modulus n times in $-1 \leq x \leq 1$. This is essentially what is done by

W. Markoff [7] in his investigation of certain special functionals. We shall have more to say about this in [2.8], which deals with applications.

[2.7] Asymptotic properties of the nodes and extremal polynomials. In this section we confine ourselves to the space π_n in $|z| \leqslant 1$ and study functionals of the type

$$(1) \qquad T: f(z) \rightarrow (1/2\pi i)\int_C f(z)K(z)dz, \quad C: |z|=1.$$

We shall study the behaviour of the nodes, extremal polynomials, etc. in π_n for a fixed K (which unless otherwise specified is assumed only to belong to L_1 on C) as we let $n \rightarrow \infty$. We shall not in general find it necessary to distinguish between monomial and non-monomial functionals. Let us suppose in this section that extremals are always normalized so that $Tf_1 = \|T\|$.

The results of Part I will figure prominently in this section.

[2.71] <u>Theorem 10</u>. Let $f_1(z)$ denote the extremal of T in the class H_∞, $f_1^{(n)}$ any extremal in the class π_n. Then $[f_1^{(n)}(z)]$ converges to $f_1(z)$ uniformly in $|z| \leqslant r_0$ for every $r_0 < 1$; consequently for each fixed q the sequence $a_q^{(n)}$ of coefficients of z^q tends to a_q, the coefficient of z^q in $f_1(z)$. Further $M_n \to M$ where M_n denotes the norm of T in π_n and M the norm in H_∞.

Proof: We show first $M_n \to M$. Let $\sigma_n(z)$ denote the n^{th} Fejer mean of the power series for $f_1(z)$. Since $\sigma_n(e^{i\theta}) \to f_1(e^{i\theta})$ a.e. and the convergence is bounded we have $|T\sigma_n| \to Tf_1 = M$; since $|T\sigma_n| \leqslant M_n \leqslant M$ we deduce $M_n \to M$. Suppose now the assertion on uniform convergence is false. Since a uniformly bounded sequence of analytic functions which converges everywhere in $|z| \leqslant r_0$ must converge uniformly, it is enough to prove pointwise convergence in $|z| < 1$ of $f_1^{(n)}(z)$ to $f_1(z)$. Suppose now there is a point z_0 where $f_1^{(n)}(z_0) \not\to f_1(z_0)$. Then the sequence on the left has at least two limit points, and we can choose a subsequence of the $f_1^{(n)}$ which converges at z_0 to a number $a \neq f_1(z_0)$. Denote this subsequence again by $f_1^{(n)}$ for simplicity. Since $\| f_1^{(n)} \| = 1$ we can by [0.424] find a function $f(z) \varepsilon L_\infty$ with $|f(e^{i\theta})| \leqslant 1$ and a subsequence of the $f_1^{(n)}$ (denoted still by $f_1^{(n)}$) such that

$$(1) \qquad \int_C f_1^{(n)}(z)u(z)dz \to \int_C f(z)u(z)dz$$

for every $u(z) \varepsilon L_1$ on C. Choosing $u(z) = 1, z, \ldots$ the left hand side is always zero, and we deduce that $f(e^{i\theta})$ has vanishing negative Fourier coefficients, hence is a boundary function of H_∞ which can be extended to $|z| < 1$. Taking now $u(z) = K(z)$ in (1) we get $Tf = \lim Tf_1^{(n)} = \lim M_n = M$. Thus f is an extremal function in H_∞ and by uniqueness of f_1, $f \equiv f_1$. Using this, and taking now $u(z) = 1/(z-z_0)$ in (1) we get $a = f_1(z_0)$, a contradiction. This proves the theorem.

[2.72] Theorem 11. Let K(z) be regular on $|z| = 1$ and D: $\theta_1 < \theta < \theta_2$ any open arc on $|z| = 1$. For all sufficiently large n, every extremal polynomial of T attains its maximum modulus in D.

Proof: If not, the nodes of T eventually remain outside of D; hence for large n every $f \varepsilon \pi_n$ satisfying $|f(z)| \leqslant 1$ in the closed arc E complementary to D satisfies $|Tf| \leqslant M_n$. Since the M_n are a bounded set, this implies that T is a bounded linear functional on the space X of all polynomials (of arbitrary degree) f(z), normed by

(2) $$\| f \| = \max_{z \varepsilon E} |f(z)|.$$

Hence we can extend T to a b.l.f. $\bar{T}$ on the space Y of all the continuous functions on $|z| = 1$, where Y is normed also by (2). This functional has a representation

(3) $$\bar{T}f = (1/2\pi)\int_E f(e^{i\theta})dG(\theta)$$

for all f ε Y where G is of bounded variation on E. Since $\bar{T}$ agrees with T on X we have, taking $f(e^{i\theta}) = e^{ij\theta}$ $(j \geqslant 0)$

(4) $$\int_E e^{ij\theta}\, dG(\theta) = \int_0^{2\pi} e^{ij\theta}e^{i\theta}\, K(e^{i\theta})d\theta \quad j=0,1,\ldots$$

Proceeding as in the proof of theorem 1 we now set

(5) $$J(\theta) = \int_0^{\theta} e^{i\phi}\, K(e^{i\phi})d\phi$$

Defining $G_1(\theta) = G(\theta)$ in E and zero in D, and writing $L = J-G_1$ we get

(6) $$\int_0^{2\pi} e^{ik\theta}\, dL(\theta) = 0 \qquad k=0,1,2,\ldots$$

Hence L is absolutely continuous by [0.436], and its derivative is a function $e^{i\theta}h(e^{i\theta})$ where $h \varepsilon \bar{H}_1$. Since L-J vanishes in D, so does its derivative, that is $h(z) = K(z)$ for z ε D. By [0.435] this implies $K(e^{i\theta}) \notin \bar{H}_1$ and $h \equiv K$. Since we may assume at the outset $K(e^{i\theta}) \notin \bar{H}_1$ (for otherwise M_n is always zero and the whole extremal problem degenerates) the theorem is proved.

[2.73] <u>Theorem 12</u> If $K(z)$ is a rational function $K(z) = Q(z)/R(z)$ where Q and R are polynomials having no common zeros and $R(z)$ has m zeros in $|z| < 1$, the index of T in π_n is not less than $n - m + 2$.

Note: This is actually a precise, not an asymptotic, theorem.

Proof: We show more generally that there is no interpolating set for T with fewer than $n - m + 2$ points. Let $z_1, \ldots z_r$ be an interpolating set and suppose $r \leq n - m + 1$. Let $P(z) = \prod (z - z_j)$. From (2.42:10) we know that for any polynomial $S(z)$ of degree $n-r$ we have $T[S(z)P(z)] = 0$, that is

$$(1) \qquad (1/2\pi i)\int_C [S(z)P(z)Q(z) \,/\, R(z)]\,dz = 0 .$$

Now, (1) holds for any $S(z)$ of degree $m-1$. In particular let a_1 be a zero of R in $|z| < 1$ and choose $S(z) = R_1(z)/(z-a_1)$ where $R_1(z)$ is $\prod (z-a)$ over the zeros a of $R(z)$ in $|z| < 1$; S has degree $m-1$ and by residues (1) gives $P(a_1)Q(a_1) = 0$. Since P has all zeros on $|z| = 1$ and Q has no zeros in common with R this is a contradiction.

[2.74] <u>Theorem 13</u>. If $K(z)$ is regular on $|z| = 1$ there is a positive number $d < 1$, depending only on K such that for large n, $M_n > M - d^n$ where M_n is the norm of T in π_n, and M the norm in H_∞.

Proof: Let f_1 denote the extremal in H_∞. By theorem 2, f_1 is regular in $|z| \leqslant R$ for some $R > 1$. Let $s_n(z)$ denote the n^{th} partial sum of $f_1(z)$, and $r_n(z)$ the remainder: $f_1 = s_n + r_n$; let $\varepsilon_n = \text{Max}\, |r_n(e^{i\theta})|$. Then we get

(1) $$|s_n(z)| \leqslant 1 + \varepsilon_n \text{ for } |z| = 1;$$

from the relation $|s_n(z) - f_1(z)| \leqslant \varepsilon_n$ on $|z| = 1$ we have, applying T: $|Ts_n - Tf_1| \leqslant \varepsilon_n \| T \|$, whence since $Tf_1 = \| T \|$

(2) $$|Ts_n| \geqslant (1-\varepsilon_n) \| T \|.$$

From (1) and (2) we have $M_n \geqslant |Ts_n|/\text{Max}\, |s_n| \geqslant [(1-\varepsilon_n)/(1+\varepsilon_n)] \| T \|$. Since however $\varepsilon_n < AR^{-n}$ we have $(1-\varepsilon_n)/(1+\varepsilon_n) = 1 - [2\varepsilon_n/(1+\varepsilon_n)] > 1 - BR^{-n}$ (where A, B are constants depending only on f_1, hence only on T) and this last is $> 1 - R_1^{-n}$ for large n, if we choose $1 < R_1 < R$. This completes the proof with $d = 1/R_1$.

[2.75] The following can also be proved: Let $K(z) \varepsilon L_1$

on $|z| = 1$ and let $G_1^{(n)}$ denote the extremal step-function of T in π_n (or an extremal if more than one exists). Then the sequence $[G_1^{(n)}(\theta)]$ converges everywhere $(0 \leqslant \theta < 2\pi)$. If $K(z)$ is regular on $|z| = 1$ then $G_1(\theta)$ is absolutely continuous; the magnitude of the largest jump of $G_1^{(n)}$ tends to zero; and every fixed interval (θ_1, θ_2) contains a jump of $G_1^{(n)}$ for all large n.

The proof is similar to that of theorem 10, and quite dull, and we omit it. Note that the last assertion contains theorem 11.

[2.8] We now consider applications of the theory of Part II. For the convenience of the casual reader we will state our conclusions using as little as possible of the specialized terminology of the paper (e.g. nodes, index).

[2.81] Bernstein's theorem. Let T denote the functional $f \to f'(a)$ where $|a| \geqslant 1$. Suppose first T is a non-monomial functional; then there exists an identity

(1) $$f'(a) = \sum_1^r u_k f(z_k)$$

for all $f \in \pi_n$, where $r \leq n$. Let us in particular replace $f(z)$ by $P(z) = (z-z_1)\ldots(z-z_r) \in \pi_n$. Then the right hand side is zero; hence $P'(a) = 0$. But by a well-known theorem of Gauss (Polya-Szego[5], III Abschnitt, Aufgabe 31) any zero of $P'(z)$ cannot lie outside the least convex polygon containing $z_1, \ldots z_r$. Since P has no double zeros, a cannot be one of the z_j. Thus since $|a| \geq 1$, a lies outside this polygon, contradiction Gauss' theorem. Hence the assumption that T is non-monomial is false, and the extremal polynomial must be one of the functions $1, z, \ldots z^n$. Since z^n has the largest derivative at a, it is the extremal and we conclude: If $f(z)$ is a polynomial of degree n and $|f(z)| \leq 1$ for $|z| \leq 1$ then $|f'(z)| \leq n|z|^{n-1}$ for $|z| \geq 1$; there is equality only for $f(z) = cz^n$, $|c| = 1$. This inequality is the analog for polynomials in the unit circle of a theorem of Bernstein for real trigonometric polynomials which says: If $t(\theta) = a_0 + \sum_1^n a_k \cos k\theta + b_k \sin k\theta$ (where a_j, b_j are real) satisfies $|t(\theta)| \leq 1$ then $|t'(\theta)| \leq n$, and there is equality only for

(2) $$t(\theta) = \pm \cos(n\theta + \phi).$$

To prove this by our methods we can construct for these polynomials an analogous theory to that for π_n. It is readily seen that the index of the functional T: $t(\theta) \to t'(\theta_0)$ has index n, so any extremal polynomial must attain its maximum modulus n times in $0 \leq \theta < 2\pi$; by an argument similar to that of [2.232] this implies $t(\theta)$ has the form (2).

We could also have proved the result in π_n in the case a = 1 by considering the associated form $J(n, n-1, \ldots 2,1,0)$. To prove it is positive definite one has but to show that $|J(n, n-1, \ldots n-k)| > 0$ for $k = 0,1,\ldots n$. But the value of this determinant is easily shown to be $2^{k-1}(2n-k) > 0$. In this simple case we can find explicitly the canonical representation of T by the analysis of [2.52]. We have $|J(n-a, n-1-a, \ldots -a)| = 2^{n-1}(n-2a)$ which vanishes for $a = n/2$. Hence, if 1 is chosen as a node of T the remaining nodes are gotten from the equation

$$\begin{vmatrix} z^n & (n/2)-1 & \cdot & -n/2 \\ z^{n-1} & n/2 & \cdot & \\ \cdot & (n/2)-1 & \cdot & \cdot \\ \vdots & \vdots & \ddots & \\ 1 & 1-(n/2) & & n/2 \end{vmatrix} = 0 .$$

adding the last row to the first we get at once $z^n+1 = 0$. Thus the set $1, \varepsilon, \varepsilon^2, \ldots \varepsilon^n$ is a set of nodes, where ε is a primitive n^{th} root of -1. We recall that all this is for the functional $T': (n, n-1, \ldots 0)$ i.e., $T'f = nf(1) - f'(1)$. Taking $P(z) = (z-1)(z^n+1)$ and applying the analysis of [2.42] we have

$$(3) \qquad T'f = \sum_{k-1}^{n+1} u_k f(z_k)$$

where $z_k = \varepsilon^k$, $k = 1, \ldots n$ and $z_{n+1} = 1$, and u_k are determined by $u_k = T'[P(z)/(z-z_k)] \cdot 1/P'(z_k)$ from which we compute $u_k = (2/n)|1-\varepsilon^k|^{-2}$ $k = 1, \ldots n$ and $u_{n+1} = n/2$. Because of the identity $T[f(z)] = T'[z^n f(1/z)]$ we have from (3) $Tf = \sum_{k=1}^{n+1} u_k z_k^n f(\bar{z}_k)$. Putting the values of z_k, u_k into this we can finally state: If $f(z)$ is a polynomial of degree n bounded in modulus by 1 at the point $z = 1$ and at each n^{th} root of -1, then $|f'(1)| \leq n$; there is equality only for $f(z) = cz^n$, $|c| = 1$. More generally, every polynomial $f(z)$ of degree n satisfies the identity

$$(4) \qquad f'(1) = (n/2)f(1) - (2/n)\sum_{1}^{n}|1-\varepsilon^k|^{-2} f(\varepsilon^k)$$

where ε is a primitive root of -1. (That this is a generalization follows from the identity $\sum_1^n |1-\varepsilon^k|^{-2} = n^2/4$ which we need not prove, as it follows from the general theory, which tells us $\Sigma|u_k| = \|T\|$ always.)

[2.82] Again let $Tf = f'(a)$ but now suppose a is real and positive. We have seen that

(1) $\quad J(na^{n-1},\ldots 2a,1,0)$

is positive definite for $a \geqslant 1$. Hence it remains so as a decreases from 1 to 0, until a reaches a value a' where J becomes semi-definite. For this a' the determinant of (1) vanishes. Thus we have: If $|f(z)| \leqslant 1$ for $|z| \leqslant 1$ then $|f'(z)| \leqslant n|z|^{n-1}$ for $|z| \geqslant a'$ where $a' = a'(n)$ is the largest positive root of $|J(na^{n-1},\ldots 2a,1,0)| = 0$. Again the extremal is z^n and is unique if $|z| > a'$.

If now $0 \leqslant a < a'$, z^n is no longer an extremal. The constant 1 can never be an extremal; other monomials can occur as extremals only when certain Hermitian symmetry conditions are met, which can only happen for a finite number of values of a. Except for these values of a, $T = Ta$ has no monomial extremal.

Since an identity $f'(z) = \sum_1^r u_k f(z_k)$ cannot occur with $r \leqslant n-1$, since then an $f \in \pi_{n-1}$ exists vanishing at the z_k and satisfying $f'(a) = 1$, the index is precisely n, and the only such monomial extremals can be z and z^{n-1}. For z to be an extremal we must have [since $T = (0,1,2,a,\ldots na^{n-1})$] $2a = 0$, i.e. $a = 0$; the form $\mathfrak{J}(1,0,\ldots 0)$ (1 followed by n-1 zeros) is indeed positive definite, and the conclusion here is trivial: $|f'(0)| \leqslant 1$, equality only for $f = cz$, $|c| = 1$. For z^{n-1} to be an extremal we need $na^{n-1} = (n-2)a^{n-3}$, or $a = [(n-2)/n]^{1/2}$. Since finally by theorem 7 an extremal of index n is unique, and by remark 3 of [2.43] has real coefficients we can state: If $0 < a < a'$ where a' is as defined above, there is among all polynomials f of degree n satisfying $|f(z)| \leqslant 1$ for $|z| \leqslant 1$ a unique (up to c.f.) $f = f_1(z)$ for which $|f'(a)|$ is maximum (except possibly for the point $a = [(n-2)/n]^{1/2}$). f_1 has real coefficients and attains a maximum on $|z| = 1$ at precisely n distinct points. If a is held fixed and $n \to \infty$ these points become everywhere dense (by theorem 11).

[2.83] Let $T = (1,1,\ldots 1,0,0,\ldots 0)$ i.e. q+1 ones followed by n-q zeros. Then Tf is the sum of the first q+1 coefficients of f: $Tf = a_0 + \dots a_q$. The analogous problem in H_∞ was completely solved by Landau[10]. From theorem 12, with $K(z) = z^{-1} + \dots z^{-(q+1)}$ the index is $\geqslant n - q + 1$ so every extremal polynomial must attain its maximum at at least this number of points. For $n \geqslant q + 1$ there is no monomial extremal: the form $J(1,1,\dots 1,0)$ cannot be definite because $J(1,1,0)$ is not, and the latter is gotten from the former by striking out all rows but those of order 1,2,n and all columns but those of order 1,2,n. For $q = 1$ the index is n and we have uniqueness. For other values of q the author is unable to obtain the index nor prove uniqueness. For $q = 1$ and 2 the author can show that the nodes become uniformly distributed in the sense of Weyl for large n.

Analogous to what was done in [2.82] we can also show that if $f \in \pi_n$ and $|f(z)| \leqslant 1$ for $|z| \leqslant 1$ then $|a_0 + a_1 z + \dots a_q z^q| \leqslant 1$ for $|z| \leqslant r_0$ where r_0 is the smallest positive root of $|J(1, r, \dots r^q, 0, 0, \dots 0)| = 0$.

[2.84] Consider now polynomials in $-1 \leqslant x \leqslant 1$. Choose a real number x_0, $|x_0| > 1$. The functional $Tf = f(x_0)$ has index n+1 since an identity

$f(x_0) = \sum_1^r u_k f(x_k)$ cannot exist with $r \leqslant n$. Since the constant 1 is not an extremal (as consideration of the polynomial x shows) we deduce from [2.62] that the extremal is equal to $ct_n(x)$ where $t_n(x)$ is n^{th} Chebyshev polynomial and c is a normalizing factor. This fact is well-known. The main contribution which our general theory offers to the study of extremal problems in [-1,1] is a method for determining at how many points in [-1,1] an extremal polynomial must attain its maximum; other writers have used more cumbersome variational arguments to do this. For example, if $Tf = f'(x_0)$ (any real x_0) we know at the outset that every extremal polynomial must attain a maximum at not less than n distinct points in [-1,1]; and from this we can then deduce, by W. Markoff's method, the extremal polynomials. Again, the classical problem of Chebyshev is equivalent to studying the functional
$Tf = a_n$, or (0,...0,1). Now if the index were
$r \leqslant n$, we would have for the polynomial
$P(x) = (x-x_1)\ldots(x-x_r)$ [where x_j are the nodes]
by (2.42:10) (which applies perfectly well to the interval [-1,1]), $TP(x) = 0$. Hence the leading co-

efficient of P must vanish, which is false. Thus the index is n+1 and the extremal is the Chebyshev polynomial.

[2.85] Let us consider the linear transformation

$$\mathcal{T}: a_0 + \ldots a_n z^n \to t_0 a_0 + \ldots t_n a_n z^n .$$

We shall show that, if π_n is normed by $\| f \| = \| f \|_1$ then the norm of $\mathcal{T}$ is $\leqslant$ the norm of T: $(t_0, \ldots t_n)$, and if T is of monomial type then equality holds. Indeed, we have in any case a canonical representation $T[f(z)] = \sum_1^r u_k f(z_k)$ with $\Sigma |u_k| = \|T\|$ for all $f \in \pi_n$; taking $f(z) = g(wz)$ we get

$$t_0 a_0 + t_1 a_1 w + \ldots t_n a_n w^n = \sum_1^r u_k g(z_k w).$$

Noting that the right side is $\leqslant \sum_1^r |u_k| \, |g(z_k w)|$, putting $w = e^{i\phi}$ and integrating gives (since $\| g(z_k w) \|_1 = \| g(w) \|_1$)

$$(1/2\pi) \int_0^{2\pi} |t_0 a_0 \ldots t_n a_n w^n| \, d\phi \leqslant (\Sigma |u_k|) \| g \|_1$$

or, in other words $\| \mathcal{T} \| \leqslant \| T \|$. If T is monomial and z^q is an extremal $\| T \| = |t_q|$; also, $\mathcal{T} z^q = t_q z^q$ shows $\| \mathcal{T} \| \geqslant |t_q|$ so that in this case $\| \mathcal{T} \| = \| T \|$. As an application we have the known inequality $\int_0^{2\pi} |f'(e^{i\theta})| \, d\theta \leqslant n \int_0^{2\pi} |f(e^{i\theta})| \, d\theta$ valid for all $f \in \pi_n$.

References

Banach, S. [6] Opérations Linéaires, Warsaw (1932)

Bernstein, S. [9] "Sur l'ordre de la meilleure approximation..." Mem. de l'Acad. Royale de Belgique (1912).

Carathéodory, C. [8] "Über den Variabilitätsbereich der Fourierschen Konstanten..." Rend. Palermo, 32 (1911).

Kakeya, S. [2] "Maximum modulus of some expressions of limited analytic functions" Trans. Am. Math. Soc. 22 (1921).

Landau, E. [10] "Darstellung und Begründung einiger neuerer Ergebnisse der Funktionen-Theorie" Berlin, (1916).

Macintyre, A. and Rogosinski, W. [1] "Extremum problems in the theory of analytic functions" Acta Math. 82 (1950).

Markoff, W. [7] "Polynome die Möglichst wenig von Null abweichten" Math. Ann. 77 (1916).

Pólya, G. and Szégö, G. [5] "Aufgaben und Lehrsätze aus der Analysis" Berlin, (1925).

Shohat, J. and Tamarkin, J. [4] "The Problem of Moments" A.M.S. Math. Survey 1 (1943)

Zygmund, A. [3] Trigonometrical Series, Warsaw (1935)

Note: Most of the preliminary results compiled in (0.4) can be found is the cited books of Banach and Zygmund.

Harold Seymour Shapiro was born in 1928 in Brooklyn, New York. He attended City College where, after unsuccessful attempts to become interested in engineering and physics, he discovered the existence of pure mathematics and received his B.S. in that field in June 1949. He came to M.I.T. as a research assistant and in 1951 received a master's degree. This past year he was an A.E.C. predoctoral fellow. Starting in September 1952 he will be employed as a mathematician by the Bell Telephone Laboratories.

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