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Reduced-order control of systems for which balanced truncation is Hankel-norm optimal

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Abstract

Various aspects of model and controller reduction are studied for members of a certain class of linear multivariable systems. For this class of systems, it is shown that balanced truncation is Hankel-norm optimal. A number of properties of the balanced approximants are derived, explicitly in terms of the original plant poles. For example, the $\mathcal{H}_\infty$-norm of the error system is precisely the inverse of the distance from the most dominant discarded pole to the origin. The results are exploited to analyze the ability of a particular low-order $\mathcal{H}_\infty$-controller, designed for a reduced system, to control the original full-order system. When the plant is not stable, $\mathcal{H}_\infty$-balanced truncation may be used, for which there exists an *a priori* small-gain type test for the ability of the low-order controller to stabilize the full-order plant. It is shown that if any unstable poles are truncated, then the small-gain condition is *always* violated. That is, $\mathcal{H}_\infty$-balanced truncation would never authorize the removal of an unstable plant pole.

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1 Introduction

A central problem in control systems design is to design a simple controller for a complex plant. This is an important problem because controllers designed using model-based optimization (such as Linear Quadratic Gaussian control, see e.g., [1], or $\mathcal{H}_\infty$-control, see e.g., [2]) typically produce controllers with the same number of state variables as the plant model (which includes any frequency dependent weightings). This may be impracticably large. Thus one often seeks a lower-order controller which does not give up too much of the good properties that have presumably been designed into the full-order controller.

As [3] has observed, the design of low-order controllers falls into two main categories: *direct* methods, where the structure of the controller is fixed *a priori* and the controller parameters are optimized (see e.g, [4]), and *indirect* methods, where reduction is either done to the full-order plant before the controller is designed, or to the full-order controller after it has been designed. In this paper our focus is on indirect methods. Of particular interest to us are controller-reduction methods that provide *a priori* guarantees of the performance of the low-order controller. That is, based on tests on the data obtained from the full-order design only, one seeks to:

- Guarantee that the low-order controller stabilizes the full-order plant;
- Bound the subsequent degradation in closed-loop performance.

To date, there are few methods that provide these guarantees. Typically, one might first approximate the plant (using for example, balanced truncation [5] or Hankel-norm approximation [6]) and then design a controller based on the reduced-plant. Unfortunately, unless the controller is specifically designed to take account of the reduction error, *a priori* guarantees of closed-loop stability and performance are not available. Also, if the plant has unstable poles neither balanced truncation nor Hankel-norm approximation can be readily applied.

The above difficulties can be avoided, however, in certain cases where the controller design and controller reduction steps are compatible: as in, for example, the $\mathcal{H}_\infty$-balanced truncation method of [7] and the balanced truncation of coprime factors method of [8]. Both methods can deal with both stable or unstable plants because the plant approximation amounts to balanced truncation of plant coprime factors. Both methods have an $\mathcal{H}_\infty$-control law that takes account of coprime factor uncertainty. Finally, both methods use a small-gain argument to derive sufficient and *a priori* conditions to guarantee when the reduced-order controller actually stabilizes the full-order plant. Simply put, the main principle of these two methods is that *the controller is robustly stable with respect to plant approximation.*

A noteworthy feature of representing plant reduction as coprime factor uncertainty is that approximation of the coprime factors may change the number of unstable poles in the plant [9]. One of the themes of this paper is that (at least for the class of plants we are considering and for $\mathcal{H}_\infty$-balanced truncation) closed-loop stability guarantees based on the small-gain theorem are lost if an unstable pole is removed in forming the reduced-order plant. Consequently, approval is never given for the removal of an unstable pole. This is in line the generally-held belief that, if successful control is to be done, the reduced-order plant should contain the 'bad' parts of the original system.

To convey our message with precision we focus on $\mathcal{H}_\infty$-balanced truncation for a class of plants that have particularly explicit approximation properties, which are introduced in Section 2. Basically, we study n-state minimal systems with a realization (A, B, C) where A is real-symmetric and $BB^T = C^TC = I_n$. The key parameters are the eigenvalues θ_i of A (i.e., the poles of the system $C(sI - A)^{-1}B$), rather than the Hankel singular values. This class of systems considered is, we believe, rich enough to be give meaningful results, but is restricted enough for the analysis to be transparent and explicit.

In Section 3, we consider the stable members of the class, and in Section 4 we analyze k-state balanced truncations of these systems. Parts of these sections are taken from [10], and are included for completeness. In this case, balanced approximation is shown to be Hankel-norm optimal. That is, the Hankel-norm of the associated error system equals the minimum possible over all stable approximants with no more than k poles (which equals the $(k+1)$-st Hankel singular value of the original system [6]). This result, we believe, is of independent interest: although balanced truncation is based on sound reasoning, in general balanced truncation is not otherwise known to be optimal in any way. Furthermore, rather than the usual $\mathcal{H}_\infty$-norm *bound* on the error system (twice the sum of the of the discarded Hankel singular values [6, 11]), we derive an *exact* expression for the $\mathcal{H}_\infty$-norm of the error system, precisely $-\theta_{k+1}^{-1}$. Other system properties such as the Hankel singular values, the $\mathcal{H}_\infty$-norm, and the $\mathcal{H}_2$-norm, are also derived in terms of the θ_i only. Thus we have a class of systems for which open-loop balanced truncation performs extremely well; a short numerical example is given to to illustrate this.

In Section 5 we turn our attention to not necessarily stable members of the class. Then $\mathcal{H}_\infty$-balanced truncation [7] is analyzed as a way of approximating these plants. Controllers designed for such approximate plants are then tested for their ability to control the full-order plants, using a small-gain argument. Again, the analysis is especially transparent because of the properties of the class of systems under consideration. One of our main results shows that if an unstable pole is removed in creating a low-order plant, then the small-gain sufficient condition that would guarantee that the associated controller stabilizes the full-order plant, is *never* satisfied. This warns us *not* to remove unstable poles using $\mathcal{H}_\infty$-balanced truncation. The key to proving this result is to deduce a lower bound on the approximation error in the case that unstable poles are removed: the lower bound implies violation of the small-gain sufficient condition. A brief numerical example is given to illustrate this fact.

2 The classes $\mathcal{S}_n^{p,m}$ and $\mathcal{S}_{n;stable}^{p,m}$

Let $\mathcal{C}_n^{p,m}$ denote the class of n-state, minimal, p by m systems with n states:

$$\mathcal{C}_n^{p,m} := \{(A,B,C) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{p\times n} \mid (A,B,C) \text{ is minimal}\}.$$

By the notation $G = (A,B,C)$ we mean as usual the transfer function matrix $G(s) = C(sI-A)^{-1}B$, where s is the Laplace transform variable. The subclass of $\mathcal{C}_n^{p,m}$ consisting of asymptotically stable systems is denoted by $\mathcal{C}_{n;stable}^{p,m}$,

$$\mathcal{C}_{n;stable}^{p,m} := \{(A,B,C) \in \mathcal{C}_n^{p,m} \mid \mathrm{Re}(\lambda_i\{A\}) < 0, \quad \text{for } i = 1,\dots,n\}.$$

In this paper we consider the following subclass of $\mathcal{C}_n^{p,m}$,

$$\mathcal{S}_n^{p,m} := \{(A,B,C) \in \mathcal{C}_n^{p,m} \mid A = A^T,\ BB^T = C^TC = I_n\},$$

and the corresponding subclass of $\mathcal{C}_{n;stable}^{p,m}$,

$$\mathcal{S}_{n;stable}^{p,m} := \{(A,B,C) \in \mathcal{C}_{n;stable}^{p,m} \mid A = A^T,\ BB^T = C^TC = I_n\}.$$

That is, systems in $\mathcal{S}_n^{p,m}$ (respectively $\mathcal{S}_{n;stable}^{p,m}$) are systems from $\mathcal{C}_n^{p,m}$ (respectively $\mathcal{C}_{n;stable}^{p,m}$) possessing a realization (A,B,C) such that A is real-symmetric and $BB^T = C^TC = I_n$.

Standard properties of the rank of a matrix (see for example [12, p 13]) give that $\mathrm{rank}(B) = \mathrm{rank}(BB^T) \le \min\{n,m\}$. Now $BB^T = I_n$ implies $\mathrm{rank}(BB^T) = n$ so $n \le \min\{n,m\}$. Hence $BB^T = I_n$ is possible only if $n \le m$. Similarly $C^TC = I_n$ only if $n \le p$. It follows that $\mathcal{S}_n^{p,m}$ is nonempty only if $m \ge n$ and $p \ge n$ which will be assumed henceforth.

It should be noted that $BB^T = C^TC = I$ implies that (A,B,C) is a minimal realization. It is a straightforward exercise to demonstrate this fact (by applying the well-known Popov-Belevitch-Hautus eigenvalue tests, for example.)

An important consequence of having a symmetric A-matrix is the following standard result (as in [12, Theorem 4.1.5], for example).

Lemma 2.1 (Spectral Theorem for Real-Symmetric Matrices)
The following are equivalent:

(i) $A \in \mathbb{R}^{n\times n}$ *satisfies* $A = A^T$.

(ii) There exist real numbers $\theta_1 \ge \cdots \ge \theta_n$ *and a matrix* $W \in \mathbb{R}^{n\times n}$ *such that*

$$A = W\Theta W^T \quad \text{where } WW^T = I_n \text{ and } \Theta = \mathrm{diag}(\theta_1,\dots,\theta_n).$$

When A is written as $W\Theta W^T$ as above, we say this is a *spectral decomposition* of A. It is easy to see that θ_i is the ith eigenvalue of A, and the ith column of W is the corresponding eigenvector. Since the θ_i are the eigenvalues of A and (A,B,C) is minimal it follows that the θ_i are precisely the poles of $C(sI-A)^{-1}B$. It should be noted that because $W \in \mathbb{R}^{n\times n}$ and $WW^T = I_n$ we have that $W^{-1} = W^T$ and $W^TW = I_n$, facts which will often be used in the sequel.

If $A \in \mathbb{R}^{n\times n}$ and $A = A^T$ then we order the eigenvalues $\lambda_1\{A\} \ge \cdots \ge \lambda_n\{A\}$. If f is any function mapping $\mathbb{R} \to \mathbb{R}$ then the definition of f may be extended to real-symmetric matrices in the usual way. Specifically, define

$$f(A) := W\text{diag}(f(\theta_1), \ldots, f(\theta_n))W^T,$$

where $A = W\Theta W^T$ is the spectral decomposition of A (so that $\lambda_i\{A\} = \theta_i$). By inspection the set of eigenvalues of $f(A)$ is precisely the set of $f(\theta_i)$. However, the ordering of the eigenvalues may or may not be preserved. That is, $\theta_i \geq \theta_j$ may, or may not, imply that $f(\theta_i) \geq f(\theta_j)$. It will be useful in the sequel to identify some functions f which do preserve the ordering of the eigenvalues.

Lemma 2.2 *Let $f : \mathsf{R} \to \mathsf{R}$ be monotonically increasing on an interval $\mathsf{I} \in \mathsf{R}$. Let $A \in \mathsf{R}^{n\times n}$ satisfy $A = A^T$ and have eigenvalues $\theta_1 \geq \cdots \geq \theta_n$. Suppose further that $\theta_i \in \mathsf{I}$ for $i = 1, \ldots, n$. Then the eigenvalues of $f(A)$ are precisely*

$$\lambda_i\{f(A)\} = f(\theta_i), \quad \textit{for } i = 1, \ldots, n.$$

Moreover,

$$\lambda_i\{f(A)\} \geq \lambda_j\{f(A)\} \quad \textit{if } \theta_i \geq \theta_j.$$

That is, the function f preserves the ordering of eigenvalues.

Proof By assumption $\theta_1 \geq \cdots \geq \theta_n$ where all $\theta_i \in \mathsf{I}$. The assumption of monotonicity of f on I implies that $f(\theta_1) \geq \cdots \geq f(\theta_n)$. But these are exactly the eigenvalues of $f(A)$ in the same decreasing order as the eigenvalues of A. □

In the sequel the next lemma will be used a number of times to confirm that the ordering of eigenvalues has not been disrupted.

Lemma 2.3 *The following functions $f : \mathsf{R} \to \mathsf{R}$ are monotonically increasing and positive on the indicated intervals $\mathsf{I} \in \mathsf{R}$:*

(i) $f(\theta) = \theta^2$ *on* $\mathsf{I} = (0, \infty)$.

(ii) $f(\theta) = -(\alpha\theta)^{-1}$ *on* $\mathsf{I} = (-\infty, 0)$ *for* $\alpha > 0$.

(iii) $f(\theta) = \beta^{-2}\theta + \beta^{-2}\sqrt{\theta^2 + \beta^2}$ *on* $\mathsf{I} = (-\infty, \infty)$ *for* $\beta > 0$.

(iv) $f(\theta) = -\zeta^{-2}\theta - \zeta^{-2}\sqrt{\theta^2 - \zeta^2}$ *on* $\mathsf{I} = (-\infty, -\zeta]$ *for* $\zeta > 0$.

Consequently, if the definitions of the above functions are extended to cover real-symmetric matrices, then $f(A)$ preserves the ordering of the eigenvalues of any real-symmetric matrix A whose eigenvalues lie in the appropriate interval I.

Proof Verification that the above functions are indeed monotonically increasing and positive on the stated intervals is a straightforward exercise and is therefore omitted. Given monotonicity, apply Lemma 2.2 to complete the proof. □

3 Properties of systems in $\mathcal{S}^{p,m}_{n;stable}$

Systems in $\mathcal{S}^{p,m}_{n;stable}$ have a number of properties that permit explicit analysis. To begin with, the controllability and observability Gramians may be calculated explicitly.

Proposition 3.1 *Let $(A,B,C) \in \mathcal{S}^{p,m}_{n;stable}$, then*

(i) The controllability Gramian of (A,B,C) is $P = -(2A)^{-1}$.

(ii) The observability Gramian of (A,B,C) is $Q = -(2A)^{-1}$.

Proof The Lyapunov equations for P and Q are, respectively,

$$0 = PA^T + AP + BB^T \quad \text{and} \quad 0 = QA + A^TQ + C^TC.$$

Since $(A,B,C) \in \mathcal{S}^{p,m}_{n;stable}$ these become the same equation

$$0 = PA + AP + I \quad \text{and} \quad 0 = QA + AQ + I.$$

Since A is asymptotically stable and (A,B,C) is minimal, there exists a unique solution for P that is positive definite [6, Theorem 3.3] and a unique solution for Q that is positive definite. Since P and Q solve the same equation, we have that $Q = P$.

Now introduce a spectral decomposition $A = W\Theta W^T$ and substitute into the P equation:

$$0 = PW\Theta W^T + W\Theta W^TP + I.$$

Multiply on the left by the (nonsingular) matrix W^T, on the right by W, and use the fact that $WW^T = I$ to give

$$0 = (W^TPW)\Theta + \Theta(W^TPW) + I.$$

Trying $W^TPW = \text{diag}(p_1, \ldots, p_n)$ gives the n independent equations

$$0 = 2p_i\theta_i + 1, \quad \text{for } i = 1, \ldots, n.$$

Noting that $\theta_i < 0$ because $(A,B,C) \in \mathcal{S}^{p,m}_{n;stable}$ we have immediately that $p_i = -(2\theta_i)^{-1} > 0$. Hence $W^TPW = \text{diag}(-(2\theta_1)^{-1}, \ldots, -(2\theta_n)^{-1})$, which gives the unique solution

$$P = W(W^TPW)W^T = W\text{diag}(-(2\theta_1)^{-1}, \ldots, -(2\theta_n)^{-1})W^T = -(1/2)W\Theta^{-1}W^T.$$

But $A^{-1} = W\Theta^{-1}W^T$. □

Next, a definition, as in Section 2 of [6].

Definition 3.2 (Hankel singular values) *Let $G = (A,B,C) \in \mathcal{C}^{p,m}_n$ have controllability Gramian P and observability Gramian Q. Then the* Hankel singular values *of G are defined by*

$$\sigma_i := \lambda_i^{1/2}\{QP\}, \quad \textit{for } i = 1, \ldots, n.$$

As is well-known, the Hankel singular values form a set of *input-output invariants* of the system G. That is, they are unaffected by changes of the coordinate basis of the state-space of G. Calculation of the Hankel singular values of a system in $\mathcal{S}^{p,m}_{n;stable}$ is a simple matter.

Corollary 3.3 *Let $G \in \mathcal{S}^{p,m}_{n;stable}$ have poles $\theta_1 \geq \cdots \geq \theta_n$. Then the Hankel singular values σ_i of G are*

$$\sigma_i = -(2\theta_i)^{-1}, \quad \text{for } i = 1, \ldots, n.$$

Proof From the definition of σ_i and Proposition 3.1, for $G = (A, B, C) \in \mathcal{S}^{p,m}_{n;stable}$

$$\begin{aligned} \sigma_i &= \lambda_i^{1/2}\{PQ\} = \lambda_i^{1/2}\{P^2\} \\ &= \lambda_i\{P\} = \lambda_i\{-(2A)^{-1}\} = -(2\theta_i)^{-1}. \end{aligned}$$

In the above, we used Lemma 2.3(ii). □

There are a number of norms of a system G of interest in control theory. We are particularly interested in the $\mathcal{H}_\infty$-norm, the $\mathcal{H}_2$-norm and the Hankel-norm, which are defined as follows,

$$\begin{aligned} \|G\|_\infty &= \sup_\omega \sigma_1\{G(j\omega)\}, \\ \|G\|_2 &= \left(\frac{1}{2\pi}\int_{-\infty}^{\infty} \text{trace}[G^*(j\omega)G(j\omega)]d\omega\right)^{1/2}, \\ \|G\|_H &= \sigma_1. \end{aligned}$$

In the above, the notation $\sigma_i\{G(j\omega)\}$ stands for the ith *singular value* of $G(j\omega)$, which should not be confused with σ_i, the ith Hankel singular value of G. The singular values of $G(j\omega)$ are defined by

$$\sigma_i\{G(j\omega)\} := \lambda_i^{1/2}\{G^*(j\omega)G(j\omega)\}, \quad \text{for } i = 1, \ldots, n.$$

As elsewhere in this paper the eigenvalues $\lambda_i\{M\}$ of an m by m Hermitian matrix M are ordered $\lambda_1\{M\} \geq \cdots \geq \lambda_m\{M\}$. Before evaluating $\|G\|_\infty$ we need to know $\sigma_i\{G(j\omega)\}$.

Lemma 3.4 *Let $G \in \mathcal{S}^{p,m}_{n;stable}$ have poles $\theta_1 \geq \cdots \geq \theta_n$. Then the non-zero singular values of G are*

$$\sigma_i\{G(j\omega)\} = \frac{1}{\sqrt{\theta_i^2 + \omega^2}}, \quad \text{for } i = 1, \ldots, n.$$

Proof Let (A, B, C) be a realization of G and let $A = W\Theta W^T$ be a spectral decomposition of A. By exploiting the facts that $W^TW = WW^T = I$, $BB^T = C^TC = I$, and that for non-zero eigenvalues $\lambda_i\{MN\} = \lambda_i\{NM\}$ where M and N are any matrices for which MN and NM exist and are square, we obtain the following sequence of equalities:

$$\begin{aligned}
\sigma_i^2\{G(j\omega)\} &= \lambda_i\{G^*(j\omega)G(j\omega)\} \\
&= \lambda_i\{B^T(-j\omega I - A^T)^{-1}C^TC(j\omega I - A)^{-1}B\} \\
&= \lambda_i\{B^T(-j\omega I - A)^{-1}(j\omega I - A)^{-1}B\} \\
&= \lambda_i\{(-j\omega I - A)^{-1}(j\omega I - A)^{-1}BB^T\} \\
&= \lambda_i\{(-j\omega I - A)^{-1}(j\omega I - A)^{-1}\} \\
&= \lambda_i\{(W(-j\omega I - \Theta)W^T)^{-1}(W(j\omega I - \Theta)W^T)^{-1}\} \\
&= \lambda_i\{W(j\omega I - \Theta)^{-1}W^TW(-j\omega I - \Theta)^{-1}W^T\} \\
&= \lambda_i\{(\omega^2 I + \Theta^2)^{-1}\} \\
&= (\omega^2 + \theta_i^2)^{-1},
\end{aligned}$$

where the last line follows because $\Theta = \mathrm{diag}(\theta_1, \ldots, \theta_n)$ with $0 > \theta_1 \geq \cdots \geq \theta_n$. □

Now we are able to give explicit expressions for various norms of $G \in \mathcal{S}^{p,m}_{n;stable}$, purely in terms the system poles θ_i.

Proposition 3.5 *Let $G \in \mathcal{S}^{p,m}_{n;stable}$ have poles $\theta_1 \geq \cdots \geq \theta_n$. Then:*

(i) $\|G\|_\infty = -\theta_1^{-1}$.

(ii) $\|G\|_2^2 = -\sum_{i=1}^n (2\theta_i)^{-1}$.

(iii) $\|G\|_H = -(2\theta_1)^{-1}$.

Proof *Part (i)* Using Lemma 3.4 and the definition of $\|G\|_\infty$, we obtain

$$\|G\|_\infty = \sup_\omega \sigma_1\{G(j\omega)\} = \sup_\omega (\theta_1^2 + \omega^2)^{-1/2} = -\theta_1^{-1},$$

as claimed.

Part (ii) It is well-known that $\|G\|_2^2 = \mathrm{trace}[PC^TC]$ where P is the controllability Gramian of $G = (A, B, C)$. Using Proposition 3.1(i) and $C^TC = I$, one obtains

$$\|G\|_2^2 = \mathrm{trace}[-(2A)^{-1}] = -\sum_{i=1}^n (2\theta_i)^{-1}.$$

Part (iii) Immediate from the definition of Hankel-norm and Corollary 3.3. □

Remark 3.6 It is known that in general $\|G\|_\infty \geq \|G\|_H$. (This follows from e.g., [6, Lemma 6.2].) For systems in $\mathcal{S}^{p,m}_{n;stable}$ we have the stronger statement that

$$\|G\|_\infty = 2\|G\|_H \quad \text{if } G \in \mathcal{S}^{p,m}_{n;stable}.$$

This is obvious on comparing Proposition 3.5(i) and (iii). Also

$$\|G\|_2^2 = \sum_{i=1}^{n} \sigma_i =: \|G\|_N \quad \text{if } G \in \mathcal{S}^{p,m}_{n;stable},$$

where $\|G\|_N$ is the *nuclear* norm of G. This is immediate on substituting for the Hankel singular values from Corollary 3.3 into Proposition 3.5(ii).

Remark 3.7 Note that the $\mathcal{H}_\infty$-norm and Hankel-norm of a system $G \in \mathcal{S}^{p,m}_{n;stable}$ depend only on θ_1, the most dominant pole (i.e., the pole which is furthest to the right in the complex plane).

4 Balanced truncation of systems in $\mathcal{S}^{p,m}_{n;stable}$

4.1 Balanced realization of systems in $\mathcal{S}^{p,m}_{n;stable}$

A realization $(A, B, C) \in \mathcal{C}^{p,m}_{n;stable}$ is said to be *balanced* [5] if its controllability Gramian and observability Gramian are equal to the diagonal matrix of Hankel singular values $\Sigma := \text{diag}(\sigma_1, \ldots, \sigma_n)$. There always exists a nonsingular *balancing state-transformation* $T \in \mathbb{R}^{n \times n}$ that takes a given minimal realization (A, B, C) onto a balanced realization $(\tilde{A}, \tilde{B}, \tilde{C}) = (T^{-1}AT, T^{-1}B, CT)$. For systems in $\mathcal{S}^{p,m}_{n;stable}$ a balanced realization and the associated balancing state-transformation have a particularly explicit form.

Proposition 4.1 *Let $(A, B, C) \in \mathcal{S}^{p,m}_{n;stable}$ and let $A = W\Theta W^T$ be a spectral decomposition of A. Then:*

(i) A balancing transformation for (A, B, C) is W.

(ii) A balanced realization of (A, B, C) is $(\Theta, W^T B, CW)$.

(iii) The balanced Gramian of $(\Theta, W^T B, CW)$ is $\Sigma = -(2\Theta)^{-1}$.

Proof If T is an arbitrary nonsingular state-transformation, then we know that $(A, B, C) \mapsto (T^{-1}AT, T^{-1}B, CT)$. Furthermore, by substituting this into the Lyapunov equations for the Gramians P and Q it can be shown that $P \mapsto T^{-1}PT^{-T}$ and $Q \mapsto T^T QT$. If now we set $T = W$, and recall that $W^{-1} = W^T$, we have that $A \mapsto W^T AW = W^T W\Theta W^T W = \Theta$, $B \mapsto W^T B$, $C \mapsto CW$, and $P \mapsto W^T PW$, $Q \mapsto W^T QW$. But using Proposition 3.1 to write $P = -(2A)^{-1} = -(1/2)W\Theta^{-1}W^T$ it follows that

$$P \mapsto W^T PW = -W^T(2A)^{-1}W = -(2\Theta)^{-1} = \Sigma,$$

where the equality follows on recalling Corollary 3.3. Proposition 3.1 also states that $Q = P$ so also $Q \mapsto W^T QW = W^T PW = \Sigma$. Hence $(\Theta, W^T B, CW)$ is a balanced realization of (A, B, C) with balanced Gramian Σ, and W is the appropriate balancing state-transformation. □

4.2 Review of balanced truncation of systems in $\mathcal{C}^{p,m}_{n;stable}$

Before we study balanced truncation of systems in $\mathcal{S}^{p,m}_{n;stable}$, for comparison and reference we summarize the key properties of balanced truncation of systems in $\mathcal{C}^{p,m}_{n;stable}$.

Definition 4.2 (Balanced truncation [5]) *Let $G \in \mathcal{C}^{p,m}_{n;stable}$ be minimal with Hankel singular values $\sigma_1 \geq \cdots \geq \sigma_n$. Let $(\tilde{A}, \tilde{B}, \tilde{C})$ be a balanced realization of G with balanced Gramian $\Sigma = \text{diag}(\sigma_1, \ldots, \sigma_n)$. Let $k < n$ and partition the balanced Gramian as*

$$\Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix}$$

where $\Sigma_1 := \operatorname{diag}(\sigma_1, \ldots, \sigma_k)$ *and* $\Sigma_2 := \operatorname{diag}(\sigma_{k+1}, \ldots, \sigma_n)$. *Partition the balanced realization conformally:*

$$\tilde{A} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{bmatrix}, \quad \textit{and} \quad \tilde{C} = \begin{bmatrix} \tilde{C}_1 & \tilde{C}_2 \end{bmatrix}.$$

Then the major balanced subsystem

$$\mathbb{BT}(G, k) := (\tilde{A}_{11}, \tilde{B}_1, \tilde{C}_1)$$

*is said to be a k-*state balanced truncation *of G. We also define*

$$\mathbb{BT}'(G, n-k) := (\tilde{A}_{22}, \tilde{B}_2, \tilde{C}_2)$$

which is the corresponding $(n-k)$-state minor balanced subsystem.

The balanced subsystems inherit certain properties from the full-order system, as the next proposition states.

Proposition 4.3 ([13]) *Let $G \in \mathcal{C}^{p,m}_{n;stable}$ be minimal with Hankel singular values $\sigma_1 \geq \cdots \geq \sigma_k > \sigma_{k+1} \geq \cdots \geq \sigma_n$. Then:*

(i) $\mathbb{BT}(G, k)$ *is asymptotically stable and minimal, and has Hankel singular values* $\sigma_1, \ldots, \sigma_k$.

(ii) $\mathbb{BT}'(G, n-k)$ *is asymptotically stable and minimal, and has Hankel singular values* $\sigma_{k+1}, \ldots, \sigma_n$.

Remark 4.4 In general the poles of the reduced model are not a subset of the poles of the full model; whilst $\tilde{A}_{11}$ is a submatrix of $\tilde{A}$ it is not true in general that the eigenvalues of $\tilde{A}_{11}$ are a subset of the eigenvalues of $\tilde{A}$. Also, in general the minor balanced subsystem $\mathbb{BT}'(G, n-k)$ is not equal to the error system $G - \mathbb{BT}(G, k)$, as is easily seen by comparing their realizations.

An important property of the balanced truncation method is the existence of a bound for the $\mathcal{H}_\infty$-norm and Hankel-norm of the error system $G - \mathbb{BT}(G, k)$.

Proposition 4.5 ([6, Theorem 9.6]) *Let $G \in \mathcal{C}^{p,m}_{n;stable}$ be minimal with Hankel singular values $\sigma_1 \geq \cdots \geq \sigma_k > \sigma_{k+1} \geq \cdots \geq \sigma_n$. Then:*

(i) $\|G - \mathbb{BT}(G, k)\|_\infty \leq 2 \sum_{i=k+1}^{n} \sigma_i$.

(ii) $\|G - \mathbb{BT}(G, k)\|_H \leq 2 \sum_{i=k+1}^{n} \sigma_i$.

(Repeated Hankel singular values need to be included in the above summations only once, on their first occurrence.)

4.3 Balanced truncation of systems in $\mathcal{S}^{p,m}_{n;stable}$

As we will now see, balanced truncation of systems in $\mathcal{S}^{p,m}_{n;stable}$ has some attractive properties. In particular, exact expressions for various norms of the error system will be derived, rather than error bounds. Firstly we write realizations for the major and minor balanced subsystems of $G \in \mathcal{S}^{p,m}_{n;stable}$ and show that they are in $\mathcal{S}^{p,m}_{k;stable}$ and $\mathcal{S}^{p,m}_{n-k;stable}$ respectively.

Proposition 4.6 *Let $G = (A,B,C) \in \mathcal{S}^{p,m}_{n;stable}$ have poles $\theta_1 \geq \cdots \geq \theta_k > \theta_{k+1} \geq \cdots \geq \theta_n$. Let $A = W\Theta W^T$ be a spectral decomposition of A and partition*

$$\Theta = \begin{bmatrix} \Theta_1 & 0 \\ 0 & \Theta_2 \end{bmatrix}$$

where $\Theta_1 := \mathrm{diag}(\theta_1, \ldots, \theta_k)$ and $\Theta_2 = \mathrm{diag}(\theta_{k+1}, \ldots, \theta_n)$. Partition W conformally as $W = [W_1 \; W_2]$ (i.e., W_1 is the first k columns of W). Then:

(a) For the major balanced subsystem $\mathbb{BT}(G,k)$:

- *(i) $\mathbb{BT}(G,k) = (\Theta_1, W_1^T B, CW_1)$.*
- *(ii) $\mathbb{BT}(G,k) \in \mathcal{S}^{p,m}_{k;stable}$.*
- *(iii) $(\Theta_1, W_1^T B, CW_1)$ is a balanced realization, and has balanced Gramian $\Sigma_1 = -(2\Theta_1)^{-1}$.*

(b) For the minor balanced subsystem $\mathbb{BT}'(G, n-k)$:

- *(i) $\mathbb{BT}'(G,n-k) = (\Theta_2, W_2^T B, CW_2)$.*
- *(ii) $\mathbb{BT}'(G,n-k) \in \mathcal{S}^{p,m}_{n-k;stable}$.*
- *(iii) $(\Theta_2, W_2^T B, CW_2)$ is a balanced realization, and has balanced Gramian $\Sigma_2 = -(2\Theta_2)^{-1}$.*

Proof By Proposition 4.1 we know that $(\Theta, W^T B, CW)$ is a balanced realization of $G = (A,B,C) \in \mathcal{S}^{p,m}_{n;stable}$. From Definition 4.2 and the definition of W_1 it is obvious that $\mathbb{BT}(G,k) = (\Theta_1, W_1^T B, CW_1)$. Now $\Theta_1 = \Theta_1^T < 0$ because Θ_1 is a diagonal matrix of strictly negative numbers. The fact that $W^T W = I_n$ and the definition $W = [W_1 \; W_2]$ gives that $W_1^T W_1 = I_k$, $W_2^T W_2 = I_{n-k}$ and $W_1^T W_2 = 0$. So $(W_1^T B)(W_1^T B)^T = I_k$ since $BB^T = I_n$, and similarly $(CW_1)^T (CW_1) = I_k$. Hence $\mathbb{BT}(G,k) \in \mathcal{S}^{p,m}_{k;stable}$. Applying Proposition 4.1 to this system shows that the realization $(\Theta_1, W_1^T B, CW_1)$ is in fact balanced with balanced Gramian $\Sigma_1 = -(2\Theta_1)^{-1}$. This proves part (a). The proof of part (b) is analogous. □

Remark 4.7 Observe that Θ_1 is the A-matrix of the balanced realization of the reduced system $\mathbb{BT}(G,k)$ given in Proposition 4.6. It is immediate that the poles of the reduced system $\mathbb{BT}(G,k)$ are $\theta_1, \ldots, \theta_k$, *precisely* the k most dominant poles of the full system. This should be compared to the general case mentioned in Remark 4.4.

In general the error system $G - \mathbb{BT}(G,k)$ and the minor balanced subsystem $\mathbb{BT}'(G,n-k)$ are different, as pointed out in Remark 4.4. However, an important property of balanced truncation in $\mathcal{S}^{p,m}_{n;stable}$ is that these systems are always the same.

Proposition 4.8 *Let $G \in \mathcal{S}^{p,m}_{n;stable}$ have poles $\theta_1 \geq \cdots \geq \theta_k > \theta_{k+1} \geq \cdots \geq \theta_n$. Then:*

$$G - \mathbb{BT}(G,k) = \mathbb{BT}'(G,n-k).$$

Proof As usual write $G = (A,B,C)$ where A has a spectral decomposition $A = W\Theta W^T$. Using Propositions 4.1 and 4.6, a realization for $G - \mathbb{BT}(G,k)$ is

$$\begin{aligned} G - \mathbb{BT}(G,k) &= \left(\begin{bmatrix} \Theta_1 & 0 & 0 \\ 0 & \Theta_2 & 0 \\ 0 & 0 & \Theta_1 \end{bmatrix}, \begin{bmatrix} W_1^T B \\ W_2^T B \\ -W_1^T B \end{bmatrix}, \begin{bmatrix} CW_1 & CW_2 & CW_1 \end{bmatrix} \right) \\ &= \left(\begin{bmatrix} \Theta_1 & 0 & 0 \\ 0 & \Theta_2 & 0 \\ 0 & 0 & \Theta_1 \end{bmatrix}, \begin{bmatrix} W_1^T B \\ W_2^T B \\ 0 \end{bmatrix}, \begin{bmatrix} 0 & CW_2 & CW_1 \end{bmatrix} \right) \end{aligned}$$

after applying a state transformation of

$$\begin{bmatrix} I_k & 0 & 0 \\ 0 & I_{n-k} & 0 \\ -I_k & 0 & I_k \end{bmatrix}.$$

Removing the uncontrollable and unobservable states, which are all asymptotically stable, leaves

$$G - \mathbb{BT}(G,k) = (\Theta_2, W_2^T B, CW_2) = \mathbb{BT}'(G,n-k)$$

where the last equality is from Proposition 4.6b(ii). □

It is now possible to find the singular values of the balanced approximant, and of the resulting error system.

Corollary 4.9 *Let $G \in \mathcal{S}^{p,m}_{n;stable}$ have poles $\theta_1 \geq \cdots \geq \theta_k > \theta_{k+1} \geq \cdots \geq \theta_n$. Then:*

(i) The non-zero singular values of $\mathbb{BT}(G,k)$ are, for $i = 1,\ldots,k$,

$$\sigma_i\{\mathbb{BT}(G,k)(j\omega)\} = \sigma_i\{G(j\omega)\} = (\theta_i^2 + \omega^2)^{-1/2}.$$

(ii) The non-zero singular values of $\mathbb{BT}'(G,n-k)$ are, for $i = 1,\ldots,n-k$,

$$\sigma_i\{\mathbb{BT}'(G,n-k)(j\omega)\} = \sigma_{k+i}\{G(j\omega)\} = (\theta_{k+i}^2 + \omega^2)^{-1/2}.$$

(iii) The non-zero singular values of $G - \mathbb{BT}(G,k)$ are, for $i = 1,\ldots,n-k$,

$$\sigma_i\{G(j\omega) - \mathbb{BT}(G,k)(j\omega)\} = \sigma_{k+i}\{G(j\omega)\} = (\theta_{k+i}^2 + \omega^2)^{-1/2}.$$

Proof Immediate from Lemma 3.4 applied to Proposition 4.6, and then to Proposition 4.8. □

Remark 4.10 The above corollary relates the frequency domain properties of the balanced approximants to those of the full-order system, when the full-order system is in $\mathcal{S}^{p,m}_{n;stable}$. The k-state balanced approximant $\mathbb{BT}(G,k)$ has k non-zero singular values that *exactly* match the k largest singular values of the full-order system *at all frequencies*. The associated error system $G - \mathbb{BT}(G,k)$ has $(n-k)$ non-zero singular values that *exactly* match the $(n-k)$ smallest singular values of the full-order system *at all frequencies*. This tells us, precisely, the effect in the frequency domain of using balanced truncation to approximate a system in $\mathcal{S}^{p,m}_{n;stable}$.

We can now write an explicit expression for the $\mathcal{H}_\infty$-norm, $\mathcal{H}_2$-norm and Hankel-norm of the error system in terms of the poles of the original system only.

Proposition 4.11 *Let $G \in \mathcal{S}^{p,m}_{n;stable}$ have poles $\theta_1 \geq \cdots \geq \theta_k > \theta_{k+1} \geq \cdots \geq \theta_n$. Then:*

(i) $\|G - \mathbb{BT}(G,k)\|_\infty = -(\theta_{k+1})^{-1}$.

(ii) $\|G - \mathbb{BT}(G,k)\|_2^2 = -\sum_{i=k+1}^{n}(2\theta_i)^{-1}$.

(iii) $\|G - \mathbb{BT}(G,k)\|_H = -(2\theta_{k+1})^{-1}$.

Proof From Proposition 4.8 we have $G - \mathbb{BT}(G,k) = \mathbb{BT}'(G, n-k)$. Applying Proposition 4.6 we know that $\mathbb{BT}'(G,n-k) = (\Theta_2, W_2^T B, CW_2)$ and that this is in $\mathcal{S}^{p,m}_{n-k;stable}$. Now apply Proposition 3.5 to $\mathbb{BT}'(G,n-k)$ to obtain the results. □

The above result may be rephrased in terms of the Hankel singular values by substitution from Corollary 3.3. As with Proposition 4.11 the corollary is important because it gives an exact expression for various norms of the error system, not just an upper bound.

Corollary 4.12 *Let $G \in \mathcal{S}^{p,m}_{n;stable}$ have Hankel singular values $\sigma_1 \geq \cdots \geq \sigma_k > \sigma_{k+1} \geq \cdots \geq \sigma_n$. Then:*

(i) $\|G - \mathbb{BT}(G,k)\|_\infty = 2\sigma_{k+1}$.

(ii) $\|G - \mathbb{BT}(G,k)\|_2^2 = \sum_{i=k+1}^{n} \sigma_i$.

(iii) $\|G - \mathbb{BT}(G,k)\|_H = \sigma_{k+1}$.

Remark 4.13 The exact value of the $\mathcal{H}_\infty$-norm and Hankel-norm of the error system $G - \mathbb{BT}(G,k)$ given in Corollary 4.12 should be compared to the usual 'twice the sum of the tail' error bounds given in Proposition 4.5. For the $\mathcal{H}_\infty$-norm, the error bound of Proposition 4.5(i) exceeds the exact value of Corollary 4.12(i) by an amount

$$\delta := \begin{cases} (2\sum_{i=k+1}^{n}\sigma_i) - 2\sigma_{k+1} = 2\sum_{i=k+2}^{n}\sigma_i & \text{if } n-k \geq 2, \\ 0 & \text{if } n-k = 1. \end{cases}$$

Using Corollary 3.3 we can write this in terms of the discarded plant poles:

$$\delta = \begin{cases} -\sum_{i=k+2}^{n}\theta_i^{-1} & \text{if } n-k \geq 2, \\ 0 & \text{if } n-k = 1. \end{cases}$$

This slackness is the error bounds may be large. For example, suppose we are given $G \in \mathcal{S}^{p,m}_{10;stable}$, with poles $\theta_i = -i/10$, $i = 1, \ldots, 10$. Consider a single-state balanced approximation $\mathbb{BT}(G, 1)$. Using Proposition 4.11 the exact value of the $\mathcal{H}_\infty$-norm of the error system is

$$\|G - \mathbb{BT}(G,1)\|_\infty = -\theta_2^{-1} = 5,$$

and the Hankel norm of the error system is

$$\|G - \mathbb{BT}(G,1)\|_H = -(2\theta_2)^{-1} = 5/2.$$

But the usual bound of Proposition 4.5 gives the rather loose estimate

$$\|G - \mathbb{BT}(G,1)\| \leq -\sum_{i=2}^{10} \theta_i^{-1} = \sum_{i=2}^{10} 10 i^{-1} \approx 19.3,$$

for both $\mathcal{H}_\infty$-norm and Hankel-norm.

Remark 4.14 The $\mathcal{H}_\infty$-norm of the error system is precisely the inverse of the distance from the most dominant *discarded* pole to the origin.

4.4 Relations to optimal Hankel-norm approximation

Given $G \in \mathcal{C}^{p,m}_{n;stable}$, the *k-state optimal Hankel norm approximation problem* is to solve

$$\arg \inf_{\hat{G}} \{\|G - \hat{G}\|_H \mid \hat{G} \in \mathcal{C}^{p,m}_{k;stable}\}.$$

In [6] it was shown that

$$\sigma_{k+1} = \inf_{\hat{G}} \{\|G - \hat{G}\|_H \mid \hat{G} \in \mathcal{C}^{p,m}_{k;stable}\}.$$

Consequently, given $G \in \mathcal{C}^{p,m}_{n;stable}$, any $\hat{G}$ that satisfies

$$\sigma_{k+1} = \|G - \hat{G}\|_H \quad \text{where } \hat{G} \in \mathcal{C}^{p,m}_{k;stable}, \tag{1}$$

is a k-state optimal Hankel-norm approximation of G. In [6], state-space formulae were derived for all optimal Hankel-norm approximations of $G \in \mathcal{C}^{p,m}_{n;stable}$.

In general a k-state balanced truncation of G is not known to be an optimal Hankel-norm approximation (nor indeed optimal in any sense). However, if $G \in \mathcal{S}^{p,m}_{n;stable}$ then a k-state balanced truncation is in fact Hankel-norm optimal, as stated in the following result.

Proposition 4.15 *Let $G \in \mathcal{S}^{p,m}_{n;stable}$ have poles $\theta_1 \geq \cdots \geq \theta_k > \theta_{k+1} \geq \cdots \geq \theta_n$. Then the k-state balanced truncation $\mathbb{BT}(G, k)$ of G is also a k-state optimal Hankel-norm approximation of G.*

Proof The assumption that $\theta_k > \theta_{k+1}$ is, by Corollary 3.3, equivalent to $\sigma_k > \sigma_{k+1}$. Corollary 4.12 then gives

$\|G - \mathbb{BT}(G,k)\|_H = \sigma_{k+1}$.

Recalling that $\mathbb{BT}(G,k) \in \mathcal{S}^{p,m}_{k;stable}$ from Proposition 4.6 and that $\mathcal{S}^{p,m}_{k;stable} \subset \mathcal{C}^{p,m}_{k;stable}$, we see that $\hat{G} := \mathbb{BT}(G,k)$ satisfies (1). Hence $\mathbb{BT}(G,k)$ is a k-state optimal Hankel-norm approximation of $G \in \mathcal{S}^{p,m}_{n;stable}$. □

Remark 4.16 In [6, Theorem 7.2] it is stated that a square k-state stable system $\hat{G}$ is a k-state optimal Hankel-norm approximation of a square n-state stable system G if and only if there exists an antistable system $F(s) = \bar{D} + \bar{C}(sI - \bar{A})^{-1}\bar{B}$ such that $E(s) := G(s) - \hat{G}(s) - F(s)$ satisfies $EE^* = \sigma^2_{k+1}I$. (By antistable we mean that $-\bar{A}$ is asymptotically stable). In our case $\hat{G} = \mathbb{BT}(G,k)$ and in Appendix A.1 we show that such an F indeed exists.

4.5 Numerical example of balanced truncation in $\mathcal{S}^{p,m}_{n;stable}$

To illustrate the results of this section we briefly outline a simple numerical example. Consider the system $G = (A,B,C) \in \mathcal{S}^{4,4}_{4;stable}$ where

$$A = \begin{bmatrix} -6 & 1 & -3 & -3 \\ 1 & -8 & -3 & -3 \\ -3 & -3 & -11 & 1 \\ -3 & -3 & 1 & -13 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 0.7071 & -0.7071 \\ 0 & 0 & 0.7071 & 0.7071 \\ 0.7071 & 0.7071 & 0 & 0 \\ -0.7071 & 0.7071 & 0 & 0 \end{bmatrix},$$

and

$$C = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

The A-matrix has a spectral decomposition $W\Theta W^T$ where

$$\Theta = \begin{bmatrix} -1.8595 & 0 & 0 & 0 \\ 0 & -8.0656 & 0 & 0 \\ 0 & 0 & -12.7356 & 0 \\ 0 & 0 & 0 & -15.3393 \end{bmatrix},$$

and

$$W = \begin{bmatrix} -0.6783 & 0.6523 & -0.1275 & 0.3133 \\ -0.4883 & -0.7439 & -0.1958 & 0.4121 \\ 0.4214 & 0.1210 & -0.8404 & 0.3185 \\ 0.3520 & 0.0802 & 0.4890 & 0.7941 \end{bmatrix}.$$

The poles of G are θ_i, the diagonal elements of Θ. A balanced realization is then $(\tilde{A}, \tilde{B}, \tilde{C})$ where

$$\tilde{A} = W^T A W = \Theta,$$

$$\tilde{B} = W^T B = \begin{bmatrix} 0.0491 & 0.5469 & -0.8249 & 0.1343 \\ 0.0288 & 0.1423 & -0.0648 & -0.9873 \\ -0.9400 & -0.2485 & -0.2286 & -0.0483 \\ -0.3363 & 0.7867 & 0.5129 & 0.0699 \end{bmatrix},$$

and

$$\tilde{C} = CW = \begin{bmatrix} 0.3520 & 0.0802 & 0.4890 & 0.7941 \\ 0.4214 & 0.1210 & -0.8404 & 0.3185 \\ -0.4883 & -0.7439 & -0.1958 & 0.4121 \\ -0.6783 & 0.6523 & -0.1275 & 0.3133 \end{bmatrix}.$$

The balanced approximants and their properties can now be read off. For example, the 1-state, 2-state and 3-state balanced approximants will have errors

$$\begin{aligned} \|G - \mathbb{BT}(G,1)\|_\infty &= -(\theta_2)^{-1} = 0.1240, \\ \|G - \mathbb{BT}(G,2)\|_\infty &= -(\theta_3)^{-1} = 0.0785, \\ \|G - \mathbb{BT}(G,3)\|_\infty &= -(\theta_4)^{-1} = 0.0652. \end{aligned}$$

For $\mathbb{BT}(G,3)$, the (only non-zero) singular value of the associated error system is

$$\begin{aligned} \sigma_1\{G(j\omega) - \mathbb{BT}(G,3)(j\omega)\} &= \sigma_4\{G(j\omega)\} \\ &= \frac{1}{\sqrt{\theta_4^2 + \omega^2}} = \frac{1}{\sqrt{235.3 + \omega^2}}. \end{aligned}$$

Similarly for the 1-state and 2-state balanced approximants. These frequency responses of the three balanced approximants and that of the original system are plotted in Figure 1. It is clear in the figure that the k-th order balanced approximation has exactly k non-zero singular values, which are the same as the first k singular values of the full-order system.

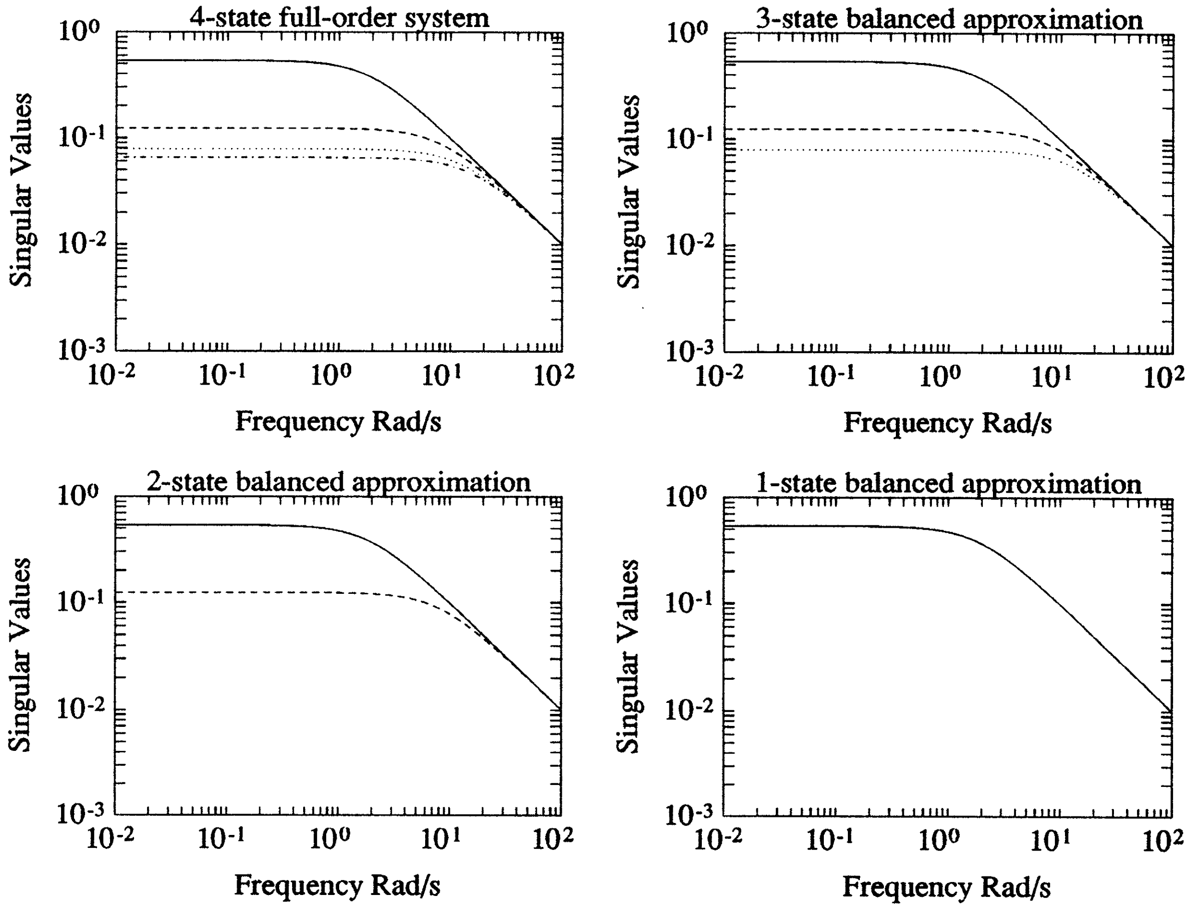

Figure 1: Singular values of the full-order system and its balanced approximants

5 $\mathcal{H}_\infty$-balanced truncation of systems in $\mathcal{S}_n^{p,m}$

Suppose G is a (high-order) plant, and G_r is a (low-order) plant approximation. Suppose a controller K_r is designed for G_r. A central concern is what happens when K_r is used to control G rather than G_r. If $G \in \mathcal{S}_{n;stable}^{p,m}$ then G_r can be obtained by balanced truncation, where some precise statements can be made. Of particular help is the existence of an explicit expression for the $\mathcal{H}_\infty$-norm of the error system (as given in Proposition 4.11). When $G \in \mathcal{S}_n^{p,m}$ but is not asymptotically stable, it is appropriate to consider using $\mathcal{H}_\infty$-balanced truncation [7] to reduce the plant. Firstly a definition is given of the particular control problem of interest.

5.1 The Normalized $\mathcal{H}_\infty$ Controller

We focus on a particular $\mathcal{H}_\infty$ control problem studied in [7, 14]. Here we give a brief description, and refer the reader to [7, 14] for full details. Let $G = (A, B, C)$ be a given n-state minimal system. Define the closed-loop system of interest to be

$$\mathcal{H}(G,K) := \begin{bmatrix} SG & SGK \\ KSG & KS \end{bmatrix} \quad \text{where } S := (I - GK)^{-1}.$$

Given $\gamma > 0$ let $\Xi(G,\gamma)$ be the following set of $\mathcal{H}_\infty$ controllers:

$$\Xi(G,\gamma) := \{K \mid K \text{ stabilizes } G \text{ and } \|\mathcal{H}(G,K)\|_\infty < \gamma\}.$$

The smallest γ such that $\Xi(G,\gamma)$ is nonempty is γ_o, the optimal $\mathcal{H}_\infty$-norm. The next lemma tells us when $\gamma > \gamma_o$ and may be obtained by applying the results of [2] to the problem in hand.

Lemma 5.1 *Let $G = (A, B, C) \in \mathcal{C}_n^{p,m}$. Then $\gamma > \gamma_o$ if and only if $\Xi(G,\gamma)$ is nonempty if and only if the following three conditions all hold:*

(i) There exists $X_\infty = X_\infty^T > 0$ satisfying the $\mathcal{H}_\infty$ Control Algebraic Riccati Equation

$$0 = X_\infty A + A^T X_\infty - (1-\gamma^{-2}) X_\infty B B^T X_\infty + C^T C \qquad \text{(HCARE)}$$

such that $A - (1-\gamma^{-2})BB^T X_\infty$ is asymptotically stable.

(ii) There exists $Y_\infty = Y_\infty^T > 0$ satisfying the $\mathcal{H}_\infty$ Filter Algebraic Riccati Equation

$$0 = Y_\infty A^T + A Y_\infty - (1-\gamma^{-2}) Y_\infty C^T C Y_\infty + B B^T \qquad \text{(HFARE)}$$

such that $A - (1-\gamma^{-2})Y_\infty C^T C$ is asymptotically stable.

(iii) With X_∞ as in (i) and Y_∞ as in (ii) we have $\lambda_1\{X_\infty Y_\infty\} < \gamma^2$.

Note that from [15, Lemma 3.4.1] there exists at most one X_∞ satisfying the HCARE such that $A - (1-\gamma^{-2})BB^T X_\infty$ is asymptotically stable. This is called the *stabilizing solution.* Similarly for Y_∞ and the HFARE.

In the sequel we will often be concerned with plants G having at least one unstable pole. For these systems γ_o cannot be smaller than unity as the next lemma shows.

Lemma 5.2 *If* $G \in \mathcal{C}_n^{p,m}$ *but* $G \notin \mathcal{C}_{n;stable}^{p,m}$ *then* $\gamma_o \geq 1$.

Proof Corollary 5.5 of [7] states that $\gamma_o < 1$ if and only if G is asymptotically stable and has Hankel-norm strictly less than unity. If $G \in \mathcal{C}_n^{p,m}$ but $G \notin \mathcal{C}_{n;stable}^{p,m}$ then G possesses at least one unstable pole so is not asymptotically stable. Hence $\gamma_o \geq 1$ for such systems. □

Now assume $\gamma > \gamma_o$. For every $K \in \Xi(G, \gamma)$ we can then define the *entropy* of the associated closed-loop $H := \mathcal{H}(G, K)$ by

$$I(H, \gamma) := -\frac{1}{2\pi} \int_{-\infty}^{\infty} \ln |\det(I - \gamma^{-2} H^*(j\omega) H(j\omega))| d\omega.$$

For a comprehensive discussion of entropy within the context of $\mathcal{H}_\infty$-control, see [14]. Here it suffices to mention that the entropy may be thought of as a nominal performance measure akin to the usual Linear Quadratic Gaussian cost, which makes sense in an $\mathcal{H}_\infty$-control setting. The control problem we are concerned with in this section can now be stated.

Definition 5.3 (Normalized $\mathcal{H}_\infty$ Controller) *Let G be a given n-state minimal system, and let $\gamma > \gamma_o$. Then the* Normalized $\mathcal{H}_\infty$ Controller *for G is defined by*

$$\mathbb{K}_\gamma(G) := \arg \inf_K \{ I(\mathcal{H}(G, K), \gamma) \mid K \in \Xi(G, \gamma) \}.$$

In other words, $\mathbb{K}_\gamma(G)$ is the stabilizing controller for G which minimizes the entropy of $\mathcal{H}(G, K)$ subject to the $\mathcal{H}_\infty$-norm bound γ. An n-state realization of this controller is given next, as stated in [7, Theorem 4.8].

Proposition 5.4 *Let $G = (A, B, C) \in \mathcal{C}_n^{p,m}$ and let $\gamma > \gamma_o$. Then the Normalized $\mathcal{H}_\infty$ Controller exists, is unique, and has a realization $(\hat{A}, \hat{B}, \hat{C})$ where*

$$\begin{aligned}
\hat{A} &= A - (1 - \gamma^{-2}) Y_\infty C^T C - B B^T X_\infty (I - \gamma^{-2} Y_\infty X_\infty)^{-1} \\
\hat{B} &= Y_\infty C^T \\
\hat{C} &= -B^T X_\infty (I - \gamma^{-2} Y_\infty X_\infty)^{-1}
\end{aligned}$$

where X_∞ and Y_∞ are as defined in Lemma 5.1.

Next, a definition taken from [7].

Definition 5.5 ($\mathcal{H}_\infty$-characteristic values) *Let $G \in (A, B, C) \in \mathcal{C}_n^{p,m}$ and let $\gamma > \gamma_o$. Then the* $\mathcal{H}_\infty$-characteristic values *of G are defined by*

$$\nu_i := \lambda_i^{1/2} \{ X_\infty Y_\infty \}, \quad \textit{for } i = 1, \ldots, n,$$

and satisfy $\nu_i < \gamma$, where X_∞ and Y_∞ are as defined in Lemma 5.1.

Like the Hankel singular values, the $\mathcal{H}_\infty$-characteristic values are a set of input-output invariants for G. As we will soon see, in the (closed-loop) design of reduced-order controllers, the $\mathcal{H}_\infty$-characteristic values play an analogous role to that played by the Hankel singular values in the (open-loop) design of reduced-order plants. That is, if a ν_i (respectively, σ_i) is small enough, the associated state may be truncated without excessive closed-loop (respectively, open-loop) error.

5.2 $\mathcal{H}_\infty$-balanced realization of systems in $\mathcal{S}_n^{p,m}$

A realization $(A,B,C) \in \mathcal{C}_n^{p,m}$ is said to be *$\mathcal{H}_\infty$-balanced* [7] if X_∞ and Y_∞ are equal to the diagonal matrix of $\mathcal{H}_\infty$-characteristic values $N := \mathrm{diag}(\nu_1,\ldots,\nu_n)$. (Here, as elsewhere in this section, X_∞ and Y_∞ are as defined in Lemma 5.1.) Such a realization exists whenever $\gamma > \gamma_o$, in which case there always exists a nonsingular *$\mathcal{H}_\infty$-balancing state-transformation* $T \in \mathbb{R}^{n\times n}$ that takes a given realization $(A,B,C) \in \mathcal{C}_n^{p,m}$ onto an $\mathcal{H}_\infty$-balanced realization $(\tilde{A},\tilde{B},\tilde{C}) = (T^{-1}AT, T^{-1}B, CT)$. Typically, both the $\mathcal{H}_\infty$-balanced realization and the $\mathcal{H}_\infty$-balancing transformation are functions of γ. For systems in $\mathcal{S}_n^{p,m}$ they are independent of γ. Moreover, an $\mathcal{H}_\infty$-balanced realization and the associated $\mathcal{H}_\infty$-balancing state-transformation may be written explicitly. Before doing this it is necessary to derive expressions for X_∞, Y_∞ and ν_i.

Proposition 5.6 *Let $G = (A,B,C) \in \mathcal{S}_n^{p,m}$ have poles $\theta_1 \geq \cdots \geq \theta_n$.*

(i) If $G \in \mathcal{S}_{n;stable}^{p,m}$ then there exists a unique positive definite stabilizing solution X_∞ to the HCARE (respectively, Y_∞ to the HFARE) if and only if $\gamma > (1+\theta_1^2)^{-1/2}$, in which case that solution is

$$X_\infty = Y_\infty = \begin{cases} \beta^{-2}A + \beta^{-2}(\beta^2 I + A)^{1/2} & \text{if } \gamma \neq 1 \\ -(2A)^{-1} & \text{if } \gamma = 1 \end{cases}$$

where $\beta^2 := 1 - \gamma^{-2}$.

(ii) If $G \notin \mathcal{S}_{n;stable}^{p,m}$ then there exists a unique positive definite stabilizing solution X_∞ to the HCARE (respectively, Y_∞ to the HFARE) if and only if $\gamma > 1$, in which case that solution is

$$X_\infty = Y_\infty = \beta^{-2}A + \beta^{-2}(\beta^2 I + A)^{1/2}.$$

Proof See Appendix A.2. □

Having established conditions on γ for suitable X_∞ and Y_∞ to exist, it remains to check if $\lambda_1\{X_\infty Y_\infty\} < \gamma^2$. If so, then Lemma 5.1 says that $\gamma > \gamma_o$. A closed-form expression for γ_o is easily obtained.

Proposition 5.7 *Let $G = (A,B,C) \in \mathcal{S}_n^{p,m}$ have poles $\theta_1 \geq \cdots \geq \theta_n$.*

(i) If $G \in \mathcal{S}_{n;stable}^{p,m}$ then

$$\gamma_o = \max\{\theta_1 + \sqrt{2+\theta_1^2}, (1+\theta_1^2)^{-1/2}\}.$$

(ii) If $G \notin \mathcal{S}_{n;stable}^{p,m}$ then

$$\gamma_o = \theta_1 + \sqrt{2+\theta_1^2}.$$

Proof See Appendix A.3. □

Now we can write closed-form expressions for the $\mathcal{H}_\infty$-characteristic values.

Corollary 5.8 *Let $G \in \mathcal{S}_n^{p,m}$ have poles $\theta_1 \geq \cdots \geq \theta_n$, and let $\gamma > \gamma_o$.*

(i) If $G \in \mathcal{S}_{n;stable}^{p,m}$ then for $i = 1, \ldots, n$,

$$\nu_i = \begin{cases} \beta^{-2}\theta_i + \beta^{-2}\sqrt{\beta^2 + \theta_i^2} & \text{if } \gamma \neq 1 \\ -(2\theta_i)^{-1} & \text{if } \gamma = 1 \end{cases}$$

where $\beta^2 := 1 - \gamma^{-2}$.

(ii) If $G \notin \mathcal{S}_{n;stable}^{p,m}$ then for $i = 1, \ldots, n$,

$$\nu_i = \beta^{-2}\theta_i + \beta^{-2}\sqrt{\beta^2 + \theta_i^2}.$$

Proof Straightforward from the definition of the ν_i, Proposition 5.6 and Lemma 2.3(iii) and (iv). □

If the plant is not asymptotically stable, then this has consequences for γ_o and the ν_i which will be important later on. The key point is that *a priori* lower bounds on γ_o and ν_i become evident.

Proposition 5.9 *Let $G \in \mathcal{S}_n^{p,m}$ but $G \notin \mathcal{S}_{n;stable}^{p,m}$ have poles $\theta_1 \geq \cdots \geq \theta_n$. Then:*

(i) $\gamma_o \geq \sqrt{2}$.

(ii) If $\gamma > \gamma_o$ then $\theta_k \geq 0$ if and only if $\nu_k \geq \beta^{-1}$.

Proof Obvious on examination of Corollary 5.8 and Proposition 5.7 when $\theta_1 \geq 0$. □

Now we can write down an $\mathcal{H}_\infty$-balanced realization and the corresponding solutions to the HCARE and HFARE.

Proposition 5.10 *Let $(A, B, C) \in \mathcal{S}_n^{p,m}$ and let $A = W\Theta W^T$ be a spectral decomposition of A. Then for all $\gamma > \gamma_o$,*

(i) An $\mathcal{H}_\infty$-balancing transformation for (A, B, C) is W.

(ii) An $\mathcal{H}_\infty$-balanced realization of (A, B, C) is $(\Theta, W^T B, CW)$.

(iii) If $\gamma \neq 1$ then the HCARE and HFARE for $(\Theta, W^T B, CW)$ have a unique positive definite stabilizing solution

$$X_\infty = Y_\infty = \beta^{-2}\Theta + \beta^{-2}(\beta^2 I + \Theta^2)^{1/2}.$$

(iv) If $\gamma = 1$ then the HCARE and HFARE for $(\Theta, W^T B, CW)$ have a unique positive definite stabilizing solution

$$X_\infty = Y_\infty = -(2\Theta)^{-1}.$$

(v) If also $G \in \mathcal{S}^{p,m}_{n;stable}$ *then* (Θ, W^TB, CW) *is a balanced realization with balanced Gramian* $\Sigma = -(2\Theta)^{-1}$.

Proof Under a state-transformation of W we know that $A \mapsto W^TAW = \Theta$, $B \mapsto W^TB$, $C \mapsto CW$, $X_\infty \mapsto W^TX_\infty W$ and $Y_\infty \mapsto W^TY_\infty W$. If $\gamma \neq 1$ then Proposition 5.6(b) gives that

$$X_\infty \mapsto W^TX_\infty W = W^T(\beta^{-2}A + \beta^{-2}W^T(\beta^2 I + A^2)^{1/2})W$$
$$= \beta^{-2}\Theta + \beta^{-2}(\beta^2 I + \Theta^2)^{1/2}.$$

The same expression is obtained for $Y_\infty \mapsto W^TY_\infty W$. If $\gamma = 1$ then Proposition 5.6(c) gives

$$X_\infty \mapsto W^TX_\infty W = -W^T(2A)^{-1}W = -(2\Theta)^{-1},$$

and the same same expression is obtained for $Y_\infty \mapsto W^TY_\infty W$. In both of the above cases, (Θ, W^TB, CW) has X_∞ and Y_∞ equal to the diagonal matrix of $\mathcal{H}_\infty$-characteristic values, and therefore (Θ, W^TB, CW) is an $\mathcal{H}_\infty$-balanced realization. This proves (i)–(iv). Now if $G \in \mathcal{S}^{p,m}_{n;stable}$ Proposition 4.1 immediately shows that (Θ, W^TB, CW) is balanced in the ordinary sense, with balanced Gramian $\Sigma = -(2\Theta)^{-1}$. □

Remark 5.11 In general, for systems in $\mathcal{C}^{p,m}_n$, the $\mathcal{H}_\infty$-balanced realization of a system and the $\mathcal{H}_\infty$-balancing transformation are functions of γ. Proposition 5.10 shows that, for systems in $\mathcal{S}^{p,m}_n$, the $\mathcal{H}_\infty$-balanced realization and $\mathcal{H}_\infty$-balancing transformation are independent of γ. Moreover, for systems in $\mathcal{S}^{p,m}_{n;stable}$, the $\mathcal{H}_\infty$-balanced realization is a balanced realization in the ordinary sense. Again, this is not true in general.

5.3 Review of $\mathcal{H}_\infty$-balanced truncation of systems in $\mathcal{C}^{p,m}_n$

Before we study $\mathcal{H}_\infty$-balanced truncation of systems in $\mathcal{S}^{p,m}_{n;stable}$, for comparison and reference we summarize the key properties of $\mathcal{H}_\infty$-balanced truncation of systems in $\mathcal{C}^{p,m}_{n;stable}$.

Definition 5.12 ($\mathcal{H}_\infty$-Balanced truncation of plant [7]) *Let* $G \in \mathcal{C}^{p,m}_{n;stable}$ *be minimal with* $\mathcal{H}_\infty$*-characteristic values* $\nu_1 \geq \cdots \geq \nu_k > \nu_{k+1} \geq \cdots \geq \nu_n$, *for a given* $\gamma > \gamma_o$. *Let* $(\tilde{A}, \tilde{B}, \tilde{C})$ *be an* $\mathcal{H}_\infty$*- balanced realization of* G *with* $X_\infty = Y_\infty = \mathrm{diag}(\nu_1, \ldots, \nu_n) =: N$. *Partition* N *as*

$$N = \begin{bmatrix} N_1 & 0 \\ 0 & N_2 \end{bmatrix}$$

where $N_1 := \mathrm{diag}(\nu_1, \ldots, \nu_k)$ *and* $N_2 := \mathrm{diag}(\nu_{k+1}, \ldots, \nu_n)$. *Partition the* $\mathcal{H}_\infty$*-balanced realization of* G *conformally:*

$$\tilde{A} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \end{bmatrix}, \quad \text{and } \tilde{C} = \begin{bmatrix} \tilde{C}_1 & \tilde{C}_2 \end{bmatrix}.$$

Then the major $\mathcal{H}_\infty$*-balanced subsystem*

$$\mathbb{BT}_\gamma(G,k) := (\tilde{A}_{11}, \tilde{B}_1, \tilde{C}_1)$$

is said to be a k-state $\mathcal{H}_\infty$-balanced truncation of the plant G. *We also define*

$$\mathbb{BT}'_\gamma(G,n-k) := (\tilde{A}_{22}, \tilde{B}_2, \tilde{C}_2)$$

which is the corresponding $(n-k)$*-state minor* $\mathcal{H}_\infty$*-balanced subsystem.*

An analogous definition may be given for a reduced-order controller by $\mathcal{H}_\infty$-balanced truncation. This is irrelevant for our purposes, since it was shown in [7] that controller so obtained is precisely the Normalized $\mathcal{H}_\infty$ Controller for $\mathbb{BT}_\gamma(G,k)$; in the sequel we analyze the latter.

The following is the key result for $\mathcal{H}_\infty$-balanced truncation, and gives an *a priori* sufficient condition that can guarantee closed-loop properties of the reduced-order controller with the full-order plant.

Proposition 5.13 ([7]) *Let* $G \in \mathcal{S}_n^{p,m}$ *have* $\mathcal{H}_\infty$*-characteristic values* $\nu_1 \geq \cdots \geq \nu_k > \nu_{k+1} \geq \cdots \geq \nu_n$. *Let* $\gamma > \max\{1, \gamma_o\}$, *let* $\beta^2 := 1 - \gamma^{-2}$, *and define* $G_r := \mathbb{BT}_\gamma(G,k)$, *with corresponding Normalized* $\mathcal{H}_\infty$ *Controller* $K_r := \mathbb{K}_\gamma(G_r)$. *Define the truncation error*

$$\epsilon := 2 \sum_{i=k+1}^{n} \frac{\nu_i}{\sqrt{1+\beta^2\nu_i^2}}.$$

Then:

(i) K_r *stabilizes* G *if* $\epsilon < (\beta+\gamma)^{-1}$.

(ii) If $\epsilon < (\beta+\gamma)^{-1}$ *then*

$$\|\mathcal{H}(G,K_r)\|_\infty < \gamma + \delta_1$$

where

$$\delta_1 := \frac{\epsilon(\beta+\gamma)(1+\beta+\gamma)}{1-\epsilon(\beta+\gamma)}.$$

It is worth explaining how Proposition 5.13 was derived. Firstly a normalized coprime factorization of βG is performed using Y_∞. That is, write $G = \bar{M}^{-1}\bar{N}$ where $\bar{N}$ and $\bar{M}$ are stable and left-coprime and $\beta^2\bar{N}\bar{N}^* + \bar{M}\bar{M}^* = I$. Now balance and truncate (in the ordinary way) the system $[\beta\bar{N} \;\; \bar{M}]$ to obtain a k-state approximation $[\beta\bar{N}_r \;\; \bar{M}_r]$. Let $\Delta_{\bar{N}} := \bar{N} - \bar{N}_r$ and $\Delta_{\bar{M}} := \bar{M} - \bar{M}_r$. Then Proposition 4.5(i) gives the *a priori* bound

$$\| \begin{bmatrix} \beta\Delta_{\bar{N}} & \Delta_{\bar{M}} \end{bmatrix} \|_\infty \leq 2 \sum_{i=k+1}^{n} \bar{\sigma}_i, \tag{2}$$

where $\bar{\sigma}_i$ is the ith Hankel singular value of $[\beta\bar{N} \;\; \bar{M}]$. It turns out that each $\bar{\sigma}_i$ may be written in terms of ν_i and γ, namely $\bar{\sigma}_i^2 = \beta^2\nu_i^2(1+\beta^2\nu_i^2)^{-1}$. Furthermore, it can be shown that $\bar{G}_r := \bar{M}_r^{-1}\bar{N}_r$ is precisely $\mathbb{BT}_\gamma(G,k)$. So to analyze $\mathbb{K}_\gamma(\mathbb{BT}_\gamma(G,k))$ connected to G we equivalently analyze $\mathbb{K}_\gamma(\bar{G}_r)$ connected to $\bar{G}_r$ with the (stable) perturbation $[\Delta_{\bar{N}} \;\; \Delta_{\bar{M}}]$. An application of the small gain theorem and exploitation of (2) leads to the *a priori* sufficient condition for closed-loop stability given in Proposition 5.13.

The above result may be applied equally to stable or unstable plants, because the coprime factors are always stable. (For further discussion and a numerical example see [7].) A particular feature of using coprime factorization in the above way is that the number of unstable poles of the reduced-order model may differ from that of the full-order model (see, for example [9]). In the sequel we analyze this issue. We will see that (at least if $G \in \mathcal{S}_n^{p,m}$) the small-gain condition in Proposition 5.13(i) is *never* satisfied if an unstable pole is removed. So although the method could be used to remove unstable poles, it is never approved.

5.4 $\mathcal{H}_\infty$-balanced truncation for unstable plants in $\mathcal{S}_n^{p,m}$

Our primary result, now given, shows the implication of using $\mathcal{H}_\infty$-balanced truncation to remove one or more *unstable* poles from the full-order plant.

Proposition 5.14 *Let $G \in \mathcal{S}_n^{p,m}$ have poles $\theta_1 \geq \cdots \geq \theta_k > \theta_{k+1} \geq \cdots \geq \theta_n$, where $\theta_1 \geq 0$. Other definitions as in Proposition 5.13. Then*

$$\theta_{k+1} \geq 0 \implies \epsilon > (\beta+\gamma)^{-1}.$$

In other words, suppose that in using $\mathcal{H}_\infty$-balanced truncation to form G_r from $G \in \mathcal{S}_n^{p,m}$, an unstable pole of G is removed. Then the sufficient condition of Proposition 5.13(i) is not satisfied, so we cannot guarantee that K_r, the Normalized $\mathcal{H}_\infty$-Controller for G_r, will stabilize G.

Proof Using $\gamma > \nu_i \geq \beta^{-1}$ from Definition 5.5 and Propositions 5.8,

$$\begin{aligned}\epsilon &= 2\sum_{i=k+1}^{n}\frac{\nu_i}{\sqrt{1+\beta^2\nu_i^2}}\\ &\geq \frac{2\nu_{k+1}}{\sqrt{1+\beta^2\nu_{k+1}^2}}\\ &> \frac{2\beta^{-1}}{\sqrt{1+\beta^2\gamma^2}}\\ &= 2(\beta\gamma)^{-1}.\end{aligned}$$

Therefore .

$$\epsilon - (\beta+\gamma)^{-1} > \frac{2(\beta+\gamma)-\beta\gamma}{\beta\gamma(\beta+\gamma)} = \frac{\gamma(2-\beta)+2\beta}{\beta\gamma(\beta+\gamma)}.$$

That this is non-negative follows by inspection because $0 < \beta \leq 1$ and $\gamma > 0$. $\square$

Since $\epsilon < (\beta+\gamma)^{-1}$ is a sufficient but not necessary condition for K_r to stabilize G, knowing that $\epsilon > (\beta+\gamma)^{-1}$ means that K_r may, or may not, stabilize G. In fact, the following result shows that K_r does *not* stabilize G when an unstable pole is removed.

Proposition 5.15 *Let $G \in \mathcal{S}_n^{p,m}$ have poles $\theta_1 \geq \cdots \geq \theta_k > \theta_{k+1} \geq \cdots \geq \theta_n$, where $\theta_1 \geq 0$. Other definitions as in Proposition 5.13. Then*

$$\theta_{k+1} \geq 0 \quad \Longrightarrow \quad K_r \text{ does not stabilize } G.$$

In other words, suppose that in using $\mathcal{H}_\infty$-balanced truncation to form G_r from $G \in \mathcal{S}_n^{p,m}$, an unstable pole of G is removed. Then K_r, the Normalized $\mathcal{H}_\infty$-Controller for G_r, does not stabilize G.

Proof See Appendix A.4. $\square$

5.5 Low-order control of balanced truncations in $\mathcal{S}_{n;stable}^{p,m}$

If $G \in \mathcal{S}_{n;stable}^{p,m}$ then $\mathcal{H}_\infty$-balanced truncation is precisely balanced truncation, a result not true in general.

Lemma 5.16 *Let $G \in \mathcal{S}_{n;stable}^{p,m}$ have poles $\theta_1 \geq \cdots \geq \theta_k > \theta_{k+1} \geq \cdots \geq \theta_n$. Then for all $\gamma > \gamma_o$*

$$\mathbb{BT}_\gamma(G,k) = \mathbb{BT}(G,k).$$

In other words, $\mathcal{H}_\infty$-balanced truncation gives the same k-state system as balanced truncation.

Proof From Proposition 5.10(v) that $(\Theta, W^T B, CW)$ is both $\mathcal{H}_\infty$-balanced and balanced in the usual sense when $G \in \mathcal{S}_{n;stable}^{p,m}$. Now Corollary 3.3 gives $\sigma_i = -(2\theta_i)^{-1}$ and Corollary 5.8 gives $\nu_i = \beta^{-2}\theta_i + \beta^{-2}(\beta^2 + \theta_i^2)^{1/2}$ (if $\gamma \neq 1$) and $\nu_i = -(2\theta_i)^{-1}$ (if $\gamma = 1$). In either case, $\sigma_k > \sigma_{k+1}$ if and only if $\nu_k > \nu_{k+1}$ if and only if $\theta_k > \theta_{k+1}$. So the same reduced-order system $(\Theta_1, W_1^T B, CW_1)$ is obtained from both balanced truncation (Definition 4.2) and $\mathcal{H}_\infty$-balanced truncation (Definition 5.12). $\square$

The above result means that we can use the explicit results we derived earlier for balanced truncation to analyze $\mathcal{H}_\infty$-balanced truncation. We will derive a sufficient condition for when the Normalized $\mathcal{H}_\infty$ Controller for the reduced-order plant can be guaranteed to stabilize the full-order plant. Further, a bound on the degradation of the resulting $\mathcal{H}_\infty$-norm of the closed-loop is given. We believe these results are interesting and insightful because they can be put in terms of θ_{k+1} and γ, which are *a priori* data.

Proposition 5.17 *Let $G \in \mathcal{S}_{n;stable}^{p,m}$ have poles $\theta_1 \geq \cdots \geq \theta_k > \theta_{k+1} \geq \cdots \geq \theta_n$. Define $G_r := \mathbb{BT}(G,k)$, let $\gamma > \gamma_o$, and let $K_r := \mathbb{K}_\gamma(G_r)$ be the Normalized $\mathcal{H}_\infty$ Controller for G_r. Define $\hat{\gamma} := \|\mathcal{H}(G_r, K_r)\|_\infty$. Then:*

(i) K_r stabilizes G if $\hat{\gamma} < -\theta_{k+1}$.

(ii) If $\hat{\gamma} < -\theta_{k+1}$ then

$$\|\mathcal{H}(G, K_r)\|_\infty \leq \hat{\gamma} + \hat{\delta}$$

where

$$\hat{\delta} := \frac{(1+\hat{\gamma})^2}{-\theta_{k+1} - \hat{\gamma}}.$$

Note that the above result requires knowledge of the *a posteriori* quantity $\hat{\gamma} = \|\mathcal{H}(G_r, K_r)\|_\infty$. Recall, however, that by construction $\hat{\gamma} < \gamma$. Thus $\hat{\gamma} < -\theta_{k+1}$ if $\gamma < -\theta_{k+1}$. We therefore immediately have the following corollary, which is in terms of *a priori* quantities only. Because the result is based on knowledge of the exact value of $\|G - G_r\|_\infty = -\theta_{k+1}^{-1}$, it is inherently less conservative than would be obtained by applying the same analysis using the usual error bound $\|G - G_r\|_\infty \leq 2\sum_{i=k+1}^{n} \sigma_i$.

Corollary 5.18 *Let $G \in \mathcal{S}^{p,m}_{n;stable}$ have poles $\theta_1 \geq \cdots \geq \theta_k > \theta_{k+1} \geq \cdots \geq \theta_n$. Define $G_r := \mathbb{BT}(G, k)$, let $\gamma > \gamma_o$ and let $K_r := \mathbb{K}_\gamma(G_r)$ be the Normalized $\mathcal{H}_\infty$ Controller for G_r. Then:*

(i) K_r stabilizes G if $\gamma < -\theta_{k+1}$.

(ii) If $\gamma < -\theta_{k+1}$ then

$$\|\mathcal{H}(G, K_r)\|_\infty < \gamma + \delta_2$$

where

$$\delta_2 := \frac{(1+\gamma)^2}{-\theta_{k+1} - \gamma}.$$

The following lemma will be needed in the proof of Proposition 5.17. Proof of the lemma is relegated to Appendix A.5.

Lemma 5.19 *Definitions as in Proposition 5.17. Let $S_{rr} := (I - G_r K_r)^{-1}$ and let $G'_r := G - G_r$. Assume $\hat{\gamma} < -\theta_{k+1}$. Then $(I - G'_r K_r S_{rr})^{-1}$ exists and*

$$\mathcal{H}(G, K_r) = \mathcal{H}(G_r, K_r) + \begin{bmatrix} I \\ K_r \end{bmatrix} S_{rr}(I - G'_r K_r S_{rr})^{-1} G'_r \begin{bmatrix} K_r S_{rr} G_r + I & K_r S_{rr} \end{bmatrix}.$$

Proof of Proposition 5.17 *Part (i)* Write $G = G_r + (G - G_r)$. Note carefully that, since $G \in \mathcal{S}^{p,m}_{n;stable}$ and $G_r = \mathbb{BT}(G, k)$, we have that $G - G_r = \mathbb{BT}'(G, n-k)$ from Proposition 4.8. Hence $\|G - G_r\|_\infty = -\theta_{k+1}^{-1}$ from Proposition 4.11. Viewing $G - G_r$ as an additive perturbation to a nominal system G_r connected to its controller K_r, we may apply the small gain theorem to deduce that K_r stabilizes G if $\|(G - G_r)K_r S_{rr}\|_\infty < 1$ where $S_{rr} := (I - G_r K_r)^{-1}$. Since $K_r S_{rr}$ is a sub-block of $\mathcal{H}(G_r, K_r)$, and $\|\mathcal{H}(G_r, K_r)\|_\infty = \hat{\gamma}$ by definition, we have $\|K_r S_{rr}\|_\infty \leq \hat{\gamma}$. But then

$$\|(G-G_r)K_rS_{rr}\|_\infty \le \|G-G_r\|_\infty \|K_rS_{rr}\|_\infty = -\theta_{k+1}^{-1}\hat{\gamma}.$$

Thus $\|(G-G_r)K_rS_{rr}\|_\infty < 1$ if $-\theta_{k+1}^{-1}\hat{\gamma} < 1$ or equivalently if $\hat{\gamma} < -\theta_{k+1}$ (since $\theta_{k+1} < 0$), in which case K_r stabilizes G.

Part (ii) Since $\hat{\gamma} < -\theta_{k+1}$ we know that $\|G_r'K_rS_{rr}\|_\infty < 1$ from the proof of Proposition 5.17(i). Now take the $\mathcal{H}_\infty$-norm of $\mathcal{H}(G,K_r)$ and substitute for $\mathcal{H}(G,K_r)$ from Lemma 5.19. Apply the triangle and sub-multiplicative properties of the $\mathcal{H}_\infty$-norm, and use the fact [12, p301] that $\|(I-M)^{-1}\|_\infty \le (1-\|M\|_\infty)^{-1}$ if $\|M\|_\infty < 1$ (here M is $G_r'K_rS_{rr}$). Also use that $\hat{\gamma} = \|\mathcal{H}(G_r,K_r)\|_\infty$ by definition. One obtains

$$\begin{aligned}\|\mathcal{H}(G,K_r)\|_\infty &\le \hat{\gamma} + \left\| \begin{bmatrix} I \\ K_r \end{bmatrix} S_{rr} \right\|_\infty \\ &\qquad \times \frac{\|G_r'\|_\infty}{(1-\|G_r'\|_\infty\|K_rS_{rr}\|_\infty)} \left\| \begin{bmatrix} K_rS_{rr}G_r + I & K_rS_{rr} \end{bmatrix} \right\|_\infty . \end{aligned} \tag{3}$$

Now recall that

$$\mathcal{H}(G_r,K_r) = \begin{bmatrix} S_{rr}G_r & S_{rr}G_rK_r \\ K_rS_{rr}G_r & K_rS_{rr} \end{bmatrix}.$$

Taking the $\mathcal{H}_\infty$-norm of this, and using the fact that the $\mathcal{H}_\infty$-norm of a sub-block of a matrix cannot exceed the $\mathcal{H}_\infty$-norm of the whole matrix, we have

$$\left\| \begin{bmatrix} K_rS_{rr}G_r & K_rS_{rr} \end{bmatrix} \right\|_\infty \le \hat{\gamma} \quad \text{and} \quad \|K_rS_{rr}\|_\infty \le \hat{\gamma}.$$

We may also write

$$\mathcal{H}(G_r,K_r) = \begin{bmatrix} I \\ K_r \end{bmatrix} S_{rr} \begin{bmatrix} G_r & I \end{bmatrix} + \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix},$$

from which we deduce

$$\left\| \begin{bmatrix} I \\ K_r \end{bmatrix} S_{rr} \begin{bmatrix} G_r & I \end{bmatrix} \right\|_\infty \le \|\mathcal{H}(G_r,K_r)\|_\infty + 1 = \hat{\gamma} + 1.$$

Consequently,

$$\left\| \begin{bmatrix} I \\ K_r \end{bmatrix} S_{rr} \right\|_\infty \le \hat{\gamma} + 1.$$

Substituting the above bounds into (3) gives

$$\|\mathcal{H}(G,K_r)\|_\infty \le \hat{\gamma} + (1+\hat{\gamma})^2(-\theta_{k+1}^{-1})(1+\hat{\gamma}\theta_{k+1}^{-1})^{-1},$$

and the claim follows. □

5.6 Numerical example of $\mathcal{H}_\infty$-balanced truncation in $\mathcal{S}_n^{p,m}$

Firstly consider applying $\mathcal{H}_\infty$-balanced truncation to the plant used in Section 4.5. The poles are

$$\theta_1 = -1.8595, \quad \theta_2 = -8.0656, \quad \theta_3 = -12.735, \quad \text{and} \quad \theta_4 = -15.3393,$$

so from Proposition 5.7 the optimal $\mathcal{H}_\infty$-norm is

$$\gamma_o = \theta_1 + \sqrt{2 + \theta_1^2} = 0.4767.$$

Now consider some representative values of $\gamma > \max\{1, \gamma_o\}$, so we can use both Proposition 5.13 and Corollary 5.18 to predict closed-loop properties of reduced-order controllers, and compare the results. We will consider $\gamma = 1.1$, 1.5, 2, 10, and 100. From Lemma 5.16 we know $\mathbb{BT}_\gamma(G,k) = \mathbb{BT}(G,k)$ and for simplicity we will focus on $k = 2$. Then from Section 4.5 we have $\mathbb{BT}(G,2) =: (A_r, B_r, C_r)$ where

$$A_r = \begin{bmatrix} -1.8595 & 0 \\ 0 & -8.0656 \end{bmatrix},$$
$$B_r = \begin{bmatrix} 0.0491 & 0.5469 & -0.8249 & 0.1343 \\ 0.0288 & 0.1423 & -0.0648 & -0.9873 \end{bmatrix},$$

and

$$C_r = \begin{bmatrix} 0.3520 & 0.0802 \\ 0.4214 & 0.1210 \\ -0.4883 & -0.7439 \\ -0.6783 & 0.6523 \end{bmatrix}.$$

and we consider two-state controllers $\mathbb{K}_\gamma(\mathbb{BT}(G,2))$. The results are summarized in Table 1. Column (a) checks the condition of Corollary 5.18(i) and predicts $\mathcal{H}(G, \mathbb{K}_\gamma(G,2))$ is stable for $\gamma = 1.1$, 1.5, 2.0 and 10.0 but says nothing for $\gamma = 100.0$; Column (b) checks the condition of Proposition 5.13(i) and predicts $\mathcal{H}(G, \mathbb{K}_\gamma(G,2))$ is stable for $\gamma = 1.1$, 1.5 and 2.0 but says nothing for $\gamma = 10.0$ and 100.0. (The closed-loop $\mathcal{H}(G, \mathbb{K}_\gamma(G,2))$ turned out to be stable for *all* the values of γ.)

In Table 2 we give the predictions and actual value of $\|\mathcal{H}(G, \mathbb{K}_\gamma(G,2))\|_\infty$. Column (c) gives the bound of Proposition 5.13(ii); Column (d) gives the bound of Corollary 5.18(ii). For comparison, Column (e) gives the actual $\mathcal{H}_\infty$-norm of the closed-loop of reduced-order plant with full-order controller, and Column (f) gives the actual $\mathcal{H}_\infty$-norm of the closed-loop of full-order plant with full-order controller.

Now consider a plant $\check{G}$ obtained by negating Θ in the balanced realization given in Section 4.5. That is, consider $\check{G} = (-\Theta, \tilde{B}, \tilde{C})$. This system is in $\mathcal{S}_4^{4,4}$ and all four of its poles are unstable. Thus even a three-state $\mathcal{H}_\infty$-balanced truncation involves removing an unstable pole. For this system

$$\check{\gamma}_o = \check{\theta}_1 + \sqrt{2 + \check{\theta}_1^2} = 30.7437,$$

with an obvious notation. We will consider $\gamma = 33$, 40, 50, and 100. The appropriate results are shown in Table 3 which considers two-state reduced-order controllers. Column (g) shows that, as predicted in Proposition 5.14, the sufficient condition of Proposition 5.13 not satisfied because at least one unstable pole is removed. Finally, Column (h) shows that the actual closed-loop system of reduced-order controller and full-order plant is indeed unstable, as Proposition 5.15 dictates.

γ	(a) Is $\gamma < -\theta_3$?	ν_1	ν_2	ν_3	ν_4	ϵ	$(\beta+\gamma)^{-1}$	(b) Is $\epsilon < (\beta+\gamma)^{-1}$?
1.1	Yes	0.2656	0.0620	0.0392	0.0326	0.1436	0.6594	Yes
1.5	Yes	0.2589	0.0619	0.0392	0.0326	0.1435	0.4454	Yes
2.0	Yes	0.2557	0.0618	0.0392	0.0326	0.1435	0.3489	Yes
10.0	Yes	0.2520	0.0618	0.0392	0.0326	0.1435	0.0910	No
100.0	No	0.2518	0.0618	0.0392	0.0326	0.1435	0.0099	No

Table 1: *A priori* numerical results for $\mathbb{K}_\gamma(\mathbb{BT}_\gamma(G,2))$

γ	(c) $\gamma+\delta_1$	(d) $\gamma+\delta_2$	(e) $\|\mathcal{H}(G,\mathbb{K}_\gamma(\mathbb{BT}_\gamma(G,2)))\|_\infty$	(f) $\|\mathcal{H}(G,\mathbb{K}_\gamma(G))\|_\infty$
1.1	1.4790	1.3784	0.5287	0.5287
1.5	2.0563	1.9754	0.5303	0.5295
2.0	2.8383	2.6986	0.5305	0.5300
10.0	54.2316	——	0.5305	0.5305
100.0	——	——	0.5305	0.5305

Table 2: Predicted and actual values of $\|\mathcal{H}(G,\mathbb{K}_\gamma(\mathbb{BT}_\gamma(G,2))\|_\infty$

γ	ν_1	ν_2	ν_3	ν_4	ϵ	$(\beta+\gamma)^{-1}$	(g) Is $\epsilon < (\beta+\gamma)^{-1}$?	(h) Is $\mathcal{H}(\check{G},\mathbb{K}_\gamma(\mathbb{BT}_\gamma(\check{G},2)))$ stable?
33.0	30.739	25.533	16.208	3.9744	3.9548	0.0294	No	No
40.0	30.730	25.526	16.203	3.9733	3.9472	0.0244	No	No
50.0	30.723	25.521	16.199	3.9724	3.9433	0.0196	No	No
100.0	30.714	25.513	16.195	3.9713	3.9315	0.0099	No	No

Table 3: *A priori* numerical results for $\mathbb{K}_\gamma(\mathbb{BT}_\gamma(\check{G},2))$

6 Conclusion

By restricting our attention to a certain class of systems we have been able to derive some quantitative features of controller and model reduction techniques based on balancing techniques. In particular, for the class of systems we studied in this paper, the following points became evident:

- Balanced truncation is Hankel-norm optimal. Moreover, the $\mathcal{H}_\infty$-norm of the error system is the inverse of the distance from the most dominant discarded pole to the origin.

- The ith Hankel singular value of a stable plant is associated with the ith most dominant pole. Hence it turns out that the poles of the k-state plant obtained by balanced truncation are the k most dominant poles of the original system.

- Whether or not the plant is stable, the ith $\mathcal{H}_\infty$-characteristic value is associated with the ith most dominant pole. Hence it turns out that the poles of k-state plant obtained by $\mathcal{H}_\infty$-balanced truncation are the k most dominant poles of the original system.

- When the plant is stable, $\mathcal{H}_\infty$-balanced truncation gives a reduced-order plant which is identical to that obtained by balanced truncation.

- If a pole of the original plant is unstable, then there exists an *a priori* lower bound on the associated $\mathcal{H}_\infty$-characteristic value.

- If an unstable pole is removed in forming the reduced-order plant, then the appropriate small gain test does *not* guarantee that the controller designed for this reduced-order plant will stabilize the original plant.

- If an unstable pole is removed in forming the reduced-order plant, then in fact the the controller designed for this reduced-order plant will *not* stabilize the original plant.

The above conclusions are true for the class of systems studied in the present paper with controller reduction by $\mathcal{H}_\infty$-balanced truncation. It is stressed that these conclusions are not necessarily true in general. However, we believe the qualitative insights obtained from this investigation are of value in more general situations. Making the claim of the previous sentence precise is a topic for further research, as is the general subject of controller reduction for unstable systems.

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A Appendix

A.1 All-pass dilation of $G - \mathbb{BT}(G,k)$

Here we follow up on Remark 4.16 and demonstrate that given $G = (A, B, C) \in \mathcal{S}^{p,m}_{n;stable}$ with poles $\theta_1 \geq \cdots \geq \theta_k > \theta_{k+1} \geq \cdots \geq \theta_n$ there exists an antistable system $F(s) = \bar{D} + \bar{C}(sI - \bar{A})^{-1}\bar{B}$ such that $E := G - \mathbb{BT}(G,k) - F$ satisfies $EE^* = \sigma_{k+1}^2 I$. By [6, Theorem 7.2] this is a necessary and sufficient condition for $\mathbb{BT}(G,k)$ to be a k-state optimal Hankel-norm approximation of G.

Without loss of generality we assume $m = p$. If $m \neq p$ then B and C may be augmented with, respectively, columns and rows of zeros without affecting $BB^T = C^TC = I$ or the Hankel singular values.

Proposition 4.8 gives that $G - \mathbb{BT}(G,k) = \mathbb{BT}'(G, n-k)$, so $E = \mathbb{BT}'(G, n-k) - F$. Proposition 4.6 gives that a balanced realization for $\mathbb{BT}'(G, n-k)$ is (Θ_2, W_2^TB, CW_2) with balanced Gramian Σ_2. Starting with this system we can directly apply [6, Theorem 6.3] to construct $F = \bar{D} + \bar{C}(sI - \bar{A})^{-1}\bar{B}$ such that $EE^* = \sigma_{k+1}^2 I$. All that remains to be shown is that the $\bar{A}$ so formed is antistable. It is convenient to define

$$
\begin{aligned}
\bar{\Theta} &:= \operatorname{diag}(\theta_{k+r+1}, \ldots, \theta_n), \\
\bar{\Sigma} &:= \operatorname{diag}(\sigma_{k+r+1}, \ldots, \sigma_n), \\
\bar{\Gamma} &:= \bar{\Sigma}^2 - \sigma_{k+1}^2 I = \operatorname{diag}((\sigma_{k+r+1}^2 - \sigma_{k+1}^2), \ldots, (\sigma_n^2 - \sigma_{k+1}^2)),
\end{aligned}
$$

where r is the multiplicity of σ_{k+1}. Since $\bar{\Theta} < 0$, part 3(b) of [6, Theorem 6.3] applies, and says that $\bar{A}$ has the same inertia as $-\bar{\Sigma}\bar{\Gamma}$. But $\bar{\Gamma} < 0$ and $\bar{\Sigma} > 0$ and both are diagonal matrices, so $-\bar{\Sigma}\bar{\Gamma} > 0$. Hence all the eigenvalues of $\bar{A}$ are in the open right-half plane. That is, $\bar{A}$ is antistable as claimed. This completes the construction of an antistable F such that $EE^* = \sigma_{k+1}^2 I$. By [6, Theorem 7.2] this verifies that $\mathbb{BT}(G,k)$ is indeed a k-state optimal Hankel-norm approximation of $G \in \mathcal{S}^{p,m}_{n;stable}$.

A.2 Proof of Proposition 5.6

The HCARE is

$$0 = A^TX_\infty + X_\infty A - (1 - \gamma^{-2})X_\infty BB^TX_\infty + C^TC.$$

Introducing the spectral decomposition $A = W\Theta W^T$ and $BB^T = C^TC = I$ gives

$$0 = W\Theta W^TX_\infty + X_\infty W\Theta W^T - (1 - \gamma^{-2})X_\infty^2 + I.$$

(An identical equation is obtained for the HFARE with Y_∞ in place of X_∞. Thus it suffices to consider the HCARE only.) Multiply on the left by the nonsingular matrix W^T and on the right by W, and use the fact that $WW^T = W^TW = I$ to get

$$0 = \Theta\tilde{X}_\infty + \tilde{X}_\infty\Theta - (1 - \gamma^{-2})\tilde{X}_\infty^2 + I,$$

where $\tilde{X}_\infty := W^TX_\infty W$. Observe that $X_\infty > 0 \iff \tilde{X}_\infty > 0$ and that X_∞ is the stabilizing solution of the HCARE if and only if $\Theta - (1 - \gamma^{-2})\tilde{X}_\infty$ is asymptotically stable. This latter fact follows from

$$A - (1-\gamma^{-2})BB^TX_\infty = A - (1-\gamma^{-2})X_\infty = W(\Theta - (1-\gamma^{-2})\tilde{X}_\infty)W^T.$$

Trying $\tilde{X}_\infty = \text{diag}(x_1, \ldots, x_n)$ gives the n independent equations

$$0 = 2\theta_i x_i - (1-\gamma^{-2})x_i^2 + 1, \tag{4}$$

for $i = 1, \ldots, n$. Obviously, X_∞ is positive definite if and only if $x_i > 0$ for all i, and X_∞ is stabilizing if and only if each x_i is the stabilizing solution of its equation (i.e., $\theta_i - (1-\gamma^{-2})x_i < 0$). Once the positive stabilizing x_i have been found, it is a simple matter to evaluate $X_\infty = W\text{diag}(x_1, \ldots, x_n)W^T$. The following lemma summarizes the relevant properties of equations like (4).

Lemma A.1 *Given $\theta \in \mathbb{R}$ and $\gamma > 0$, consider the quadratic equation*

$$0 = 2\theta x - (1-\gamma^{-2})x^2 + 1. \tag{5}$$

A real solution to (5) is stabilizing if and only if $\theta - (1-\gamma^{-2})x < 0$.

(i) Suppose $\theta < 0$. There exists a real positive stabilizing solution to (5) if and only if $\gamma > (1+\theta^2)^{-1/2}$. In that case, the real positive stabilizing solution is unique and is given by

$$x = \begin{cases} \beta^{-2}\theta + \beta^{-2}(\theta^2+\beta^2)^{1/2} & \text{if } \gamma \neq 1 \\ -(2\theta)^{-1} & \text{if } \gamma = 1 \end{cases}$$

where $\beta^2 := 1 - \gamma^{-2}$.

(ii) Suppose $\theta \geq 0$. There exists a real positive stabilizing solution to (5) if and only if $\gamma > 1$. In that case, the real positive stabilizing solution is unique and is given by

$$x = \beta^{-2}\theta + \beta^{-2}(\theta^2+\beta^2)^{1/2}.$$

Proof *Part (i)* Consider first the case when $\gamma = 1$. Then (5) becomes $0 = 2\theta x + 1$. This has a unique real positive solution $x = -(2\theta)^{-1}$, and this solution is obviously stabilizing.

Now suppose $\gamma \neq 1$ so $1 - \gamma^{-2} \neq 0$. Then (5) has real solutions if and only if $\theta^2 + 1 - \gamma^{-2} \geq 0$. This is true if and only if $\gamma^2 \geq (1+\theta^2)^{-1}$. In that case, the real solutions are

$$x = \frac{\theta \pm \sqrt{\theta^2 + 1 - \gamma^{-2}}}{1-\gamma^{-2}}. \tag{6}$$

One of these solutions is stabilizing if and only if $\gamma > (1+\theta^2)^{-1/2}$. In that case the stabilizing solution is

$$x = \frac{\theta + \sqrt{\theta^2 + 1 - \gamma^{-2}}}{1-\gamma^{-2}}. \tag{7}$$

We must now check when this x is positive. If $\gamma < 1$ then $1 - \gamma^{-2} < 0$ and we define $\zeta := (\gamma^{-2} - 1)^{1/2}$. It follows that $x > 0$ if and only if $f(\theta) := -\zeta^{-2}\theta - \zeta^{-2}(\theta^2 - \zeta^2)^{1/2}$ satisfies $f(\theta) > 0$. From Lemma 2.3(iv), this is true if and only if $\theta \leq -\zeta$, which is is assured since by assumption $\gamma > (1 + \theta^2)^{-1/2}$. So there exists a unique real positive stabilizing solution for $1 > \gamma > (1 + \theta^2)^{-1/2}$, and this solution is given by (7).

Finally assume $\gamma > 1$, so certainly $\gamma > (1 + \theta^2)^{-1/2}$. Then $1 - \gamma^{-2} > 0$ and Lemma 2.3(iii) says that $x > 0$ for any real θ. So there exists a unique real positive stabilizing solution for all $\gamma > 1$, and this solution is given by (7).

Part (ii) Consider first the case when $\gamma = 1$. Then (5) becomes $0 = 2\theta x + 1$. If $\theta = 0$ there is no solution. Otherwise, there is a unique real solution $x = -(2\theta)^{-1}$, but this solution is not positive.

Now suppose $\gamma \neq 1$ so $1 - \gamma^{-2} \neq 0$. Then (5) has real solutions if and only if $\theta^2 + 1 - \gamma^{-2} \geq 0$. This is true if and only if $\gamma^2 \geq (1 + \theta^2)^{-1}$. In that case, the real solutions are as in (6). One of these solutions is stabilizing if and only if $\gamma > (1+\theta^2)^{-1/2}$. In that case the stabilizing solution is given in (7). We must now check when this x is positive. If $\gamma < 1$ then $1 - \gamma^{-2} < 0$ and as above we define $\zeta := (\gamma^{-2} - 1)^{1/2}$. It follows that $x > 0$ if and only if $f(\theta) > 0$. This cannot happen because $\theta \geq 0$. So if $\gamma \leq 1$ there does not exist a real positive stabilizing solution.

Finally assume $\gamma > 1$, so certainly $\gamma > (1 + \theta^2)^{-1/2}$. Then $1 - \gamma^{-2} > 0$ and Lemma 2.3(iii) says that $x > 0$ for any real θ. Hence a real positive stabilizing solution exists for all $\gamma > 1$, and is unique and given by (7). □

Now let us return to the proof of Proposition 5.6.

Part (i) Assume $G \in \mathcal{S}^{p,m}_{n;stable}$, so that $\theta_i < 0$ for $i = 1, \ldots, n$. Apply Lemma A.1(i) to (4) for each $i = 1, \ldots, n$. It follows that there exists a positive definite stabilizing solution to the HCARE if and only if $\gamma > \max_i\{(1 + \theta_i)^{-1/2}\} = (1 + \theta_1)^{-1/2}$. In that case the claimed expression for X_∞ follows immediately on substituting for the x_i given by Lemma A.1(i) into $X_\infty = W\mathrm{diag}(x_1, \ldots, x_n)W^T$.

Part (ii) Assume $G \notin \mathcal{S}^{p,m}_{n;stable}$ so that there is some integer $1 \leq r \leq n$ such that $\theta_1 \geq \cdots \geq \theta_r \geq 0$. Apply Lemma A.1(ii) to (4) for $i = 1, \ldots, r$. We find that there exists a real positive stabilizing solution x_i if and only if $\gamma > 1$, in which case that x_i is given by $x_i = \beta^{-2}\theta_i + \beta^{-2}(\theta_i^2 + \beta^2)^{1/2}$. If $r = n$ then this establishes that there exists a positive definite stabilizing solution X_∞ to the HCARE if and only if $\gamma > 1$. Further, that X_∞ is obviously as given in the claim. If $r < n$ then apply Lemma A.1(i) to (4) for $i = r + 1, \ldots, n$. We find that there exists a real positive stabilizing solution x_i if and only if $\gamma > (1 + \theta_i^2)^{-1/2}$, in which case that x_i is given by $x_i = \beta^{-2}\theta_i + \beta^{-2}(\theta_i^2 + \beta^2)^{1/2}$. Then there exists a positive definite stabilizing solution X_∞ to the HCARE if and only if $\gamma > \max\{1, \max_{i=r+1,\ldots,n}\{(1 + \theta_i^2)^{-1/2}\}\} = 1$, as claimed. Finally, this solution X_∞ clearly is as stated in the proposition. □

A.3 Proof of Proposition 5.7

Just apply Lemma 5.1 to Proposition 5.6. Let γ_{X_∞} be the infimal γ such that Lemma 5.1(i) holds, and let γ_{Y_∞} be the infimal γ such that Lemma 5.1(ii) holds. Then γ_o

is the infimal γ that exceeds $\max\{\gamma_{X_\infty}, \gamma_{Y_\infty}\}$ and is such that $\lambda_1\{X_\infty Y_\infty\} < \gamma^2$. From Proposition 5.6 we immediately have $\gamma_{X_\infty} = \gamma_{Y_\infty}$. Moreover, it follows that

$$\gamma_{X_\infty} = \begin{cases} (1+\theta_1^2)^{-1/2} & \text{if } \theta_1 < 0 \\ 1 & \text{if } \theta_1 \geq 0. \end{cases}$$

Using Corollary 5.8, we can solve $\lambda_1\{X_\infty Y_\infty\} = \gamma^2$ for γ:

$$\begin{aligned} & \gamma = \beta^{-2}\theta_1 + \beta^{-2}\sqrt{\beta^2+\theta_1^2} \\ \Longrightarrow \quad & 0 = (\gamma - \beta^{-2}\theta_1^2)^2 - \beta^{-4}(\beta^2+\theta_1^2) \\ \Longrightarrow \quad & 0 = \gamma^2 - 2\gamma\theta_1 - 2 \end{aligned}$$

The positive solution to this equation is $\tilde{\gamma} := \theta_1 + (2+\theta_1^2)^{1/2}$. Observe that $1 \geq (1+\theta_1)^{-1/2}$ and if $\theta_1 \geq 0$ then $\tilde{\gamma} \geq \sqrt{2}$. We conclude that

$$\gamma_o = \max\{\tilde{\gamma}, \gamma_{X_\infty}\} = \begin{cases} \max\{\theta_1 + \sqrt{2+\theta_1^2}, (1+\theta_1)^{-1/2}\} & \text{if } \theta_1 < 0 \\ \theta_1 + \sqrt{2+\theta_1^2} & \text{if } \theta_1 \geq 0. \end{cases}$$

which completes the proof. □

A.4 Proof of Proposition 5.15

We know from Proposition (5.10) (Θ, W^TB, CW) is an $\mathcal{H}_\infty$-balanced realization of G. Then $G_r := \mathbb{BT}_\gamma(G,k) = (\Theta_1, W_1^TB, CW_1)$, and for this system the HCARE and HFARE both have solution $N_1 = \text{diag}(\nu_1, \ldots, \nu_k)$. The Normalized $\mathcal{H}_\infty$ Controller for this plant is obtained by applying Proposition 5.4 and is given by $K_r = (\hat{A}_r, \hat{B}_r, \hat{C}_r)$ where

$$\begin{aligned} \hat{A}_r &= \Theta_1 - \beta^2 N_1 - N_1(I - \gamma^{-2}N_1^2)^{-1} \\ \hat{B}_r &= N_1 W_1^T C^T \\ \hat{C}_r &= -B^T W_1 N_1 (I - \gamma^{-2}N_1^2)^{-1}. \end{aligned}$$

A simple state-space calculation shows that the A-matrix of $\mathcal{H}(G, K_r)$ is

$$\begin{aligned} \tilde{A} &:= \begin{bmatrix} \Theta & W^TB\hat{C}_r \\ \hat{B}_r CW & \hat{A}_r \end{bmatrix} \\ &= \begin{bmatrix} \Theta_1 & 0 & -N_1(I-\gamma^{-2}N_1^2)^{-1} \\ 0 & \Theta_2 & 0 \\ N_1 & 0 & \hat{A}_r \end{bmatrix} \end{aligned}$$

where we have used the fact that $W_1^TW = [I \quad 0]$. Because $\tilde{A}$ has a certain block structure made up of matrices that are either diagonal or zero, it is in fact now possible to calculate the eigenvalues of $\tilde{A}$ explicitly. One can then deduce necessary and sufficient conditions for $\tilde{A}$ to be asymptotically stable. However, this would take us too far afield, and here it suffices to note that the set of eigenvalues of $\tilde{A}$ is the union of the set of eigenvalues of

$$\Theta_2 \quad \text{and} \quad \begin{bmatrix} \Theta_1 & -N_1(I-\gamma^{-2}N_1^2)^{-1} \\ N_1 & \hat{A}_r \end{bmatrix}.$$

But the eigenvalues of Θ_2 are precisely the discarded poles $\theta_{k+1}, \ldots, \theta_n$. An immediate consequence is that if any of these discarded poles are unstable, then so is $\tilde{A}$, and therefore K_r does not stabilize G. □

A.5 Proof of Lemma 5.19

From the proof of Proposition 5.17(i) we know that the assumption $\hat{\gamma} < -\theta_{k+1}$ ensures $\|G_r' K_r S_{rr}\|_\infty < 1$. This guarantees the existence of $(I - G_r' K_r S_{rr})^{-1}$. By definition

$$S_r^{-1} = I - GK_r = I - G_r K_r - G_r' K_r = S_{rr}^{-1} - G_r' K_r.$$

Hence

$$S_r = S_{rr}(I - G_r' K_r S_{rr})^{-1}.$$

It follows that

$$S_r - S_{rr} = S_{rr}(I - G_r' K_r S_{rr})^{-1} G_r' K_r S_{rr}.$$

It is convenient to write

$$\mathcal{H}(G_r, K_r) = \begin{bmatrix} I \\ K_r \end{bmatrix} S_{rr} \begin{bmatrix} G_r & I \end{bmatrix} + \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix},$$

and similarly for $\mathcal{H}(G, K_r)$. Now we can simplify $\mathcal{H}(G, K_r) - \mathcal{H}(G_r, K_r)$ as follows:

$$\begin{aligned}
\mathcal{H}(G, K_r) &- \mathcal{H}(G_r, K_r) \\
&= \begin{bmatrix} I \\ K_r \end{bmatrix} S_r \begin{bmatrix} G & I \end{bmatrix} - \begin{bmatrix} I \\ K_r \end{bmatrix} S_{rr} \begin{bmatrix} G_r & I \end{bmatrix} \\
&= \begin{bmatrix} I \\ K_r \end{bmatrix} S_r \begin{bmatrix} G_r + G_r' & I \end{bmatrix} - \begin{bmatrix} I \\ K_r \end{bmatrix} S_{rr} \begin{bmatrix} G_r' & 0 \end{bmatrix} \\
&= \begin{bmatrix} I \\ K_r \end{bmatrix} (S_r - S_{rr}) \begin{bmatrix} G_r & I \end{bmatrix} + \begin{bmatrix} I \\ K_r \end{bmatrix} S_r \begin{bmatrix} G_r' & 0 \end{bmatrix} \\
&= \begin{bmatrix} I \\ K_r \end{bmatrix} S_{rr}(I - G_r' K_r S_{rr})^{-1} G_r' \begin{bmatrix} K_r S_{rr} G_r + I & K_r S_{rr} \end{bmatrix},
\end{aligned}$$

as claimed. □