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Probability Bounds on the Peak Intensity of Optical Ultrashort Light Pulse CDMA[1]

by

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Abstract

Optical CDMA is being proposed as a suitable multiaccess technique to take advantage of the large bandwidth available in single-mode fiber-optic communications. In a CDMA system the communication link is shared by a number of users, each of which has a unique address code called signature. Recently an all-optical CDMA system based upon encoding and decoding coherent ultrashort light pulses was introduced. An important factor in the performance of the CDMA system is the peak intensity of the encoded signals. We consider using random signatures for the phase encoding of the light pulses in this system and prove that they result in a fairly low peak intensity of the signals. In particular, we obtain an upper bound on the probability that the peak intensity of the encoded signal is greater than some value. This bound is strictly better than previously reported bounds and decreases exponentially with the length of the signatures. We also find a lower bound on the maximum number of "minimally interfering" signatures (or, to be more accurate, λ-separated signatures, which will be defined) of length N_0 that can be designed. Assuming that the communication is reliable if and only if any pair of signatures in the system is λ-separated, a notion of capacity is defined and a lower bound on it is found.

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1. INTRODUCTION

Optical CDMA is being proposed as a suitable multiaccess technique to take advantage of the large bandwidth available in single-mode fiber-optic communications. In a CDMA system the communication link is shared by a number of users, each of which is given a unique address code called *signature.* If a transmitter wants to communicate with some receiver then the receiver has to know the signature used by the transmitter. If user l puts something on the channel which is not destined for user m then under certain orthogonality conditions on the signatures of l and m, the receiver of m is going to perceive this signal as a low intensity noise. In order to have small interference between different communication pairs, the address codes of the users have to be designed so that they are "minimally interfering". The choice of the signatures is one of the main issues in the design of a CDMA system and several possibilities have been proposed in the litterature (e.g. Optical Orthogonal Codes in [CSW89]). In this paper we argue that random signatures can also have good performance.

We call two signatures *λ-separated* (or minimally interfering) if when a transmitter uses one of them in order to CDMA encode a signal and the receiver uses the other one to decode it, then the peak intensity of the (interference) signal at the output of the decoder is smaller than some value $\lambda\sqrt{P_0}$, where P_0 is the peak intensity of the signals used and $\lambda << 1$. We assume that the noise and the number of users in the system is such that the communication is "reliable" if and only if any two user signatures are λ-separated. In general the value of the parameter λ which is adequate decreases with the intensity of the noise and the number of users in the system. However, in our model we will treat λ as a known parameter and we will find a lower bound on the maximum number of λ-separated signatures which can be designed. Given the constraint that the communication is reliable only when any pair of user signatures are λ-separated, it makes sense to define the notion of a maximum transmission rate per user on the channel, which we call λ-capacity. For given λ, we find a lower bound on this capacity. Note here that we do not have to assume any particular probabilistic characteristics for the noise of the channel and its intensity, or any model for the number of users present on the system. These factors are hidden in the parameter λ.

The CDMA system that we analyze is illustrated in Figure 1. The transmitter consists of a bandlimited signal source that generates a train of ultrashort light pulses. Each ultra-short light pulse has duration t_c and the time interval between two pulses is T_b seconds. The pulse is multiplied by 0 or 1 depending on the data bit (ON-OFF Keying). If the data bit is a 0 no energy is transmitted. If the data bit is a 1, then the ultrashort pulse is sent to to the spectral-phase encoder which adds a determinate phase shift to each spectral component of the ultrashort pulse. This is accompliced by passing the pulse through an optical encoder composed of a pair of diffraction gratings with a phase mask inserted between them. The ultra-short pulse is decomposed into its spectral components when passing through the first diffraction grating. The phase mask introduces a phase shift into each spectral component; these phase shifts are in general different for each spectral component. The vector of phase shifts introduced by a particular user is the signature of the user. Following the phase mask, there is a second grating which reassembles the spectral components into a single optical beam, which is transmitted on the channel. When a pseudorandom set of spectral phase shifts is selected, spectral phase coding spreads the ultrashort pulse into a longer, lower intensity pseudonoise burst ([SHW90]).

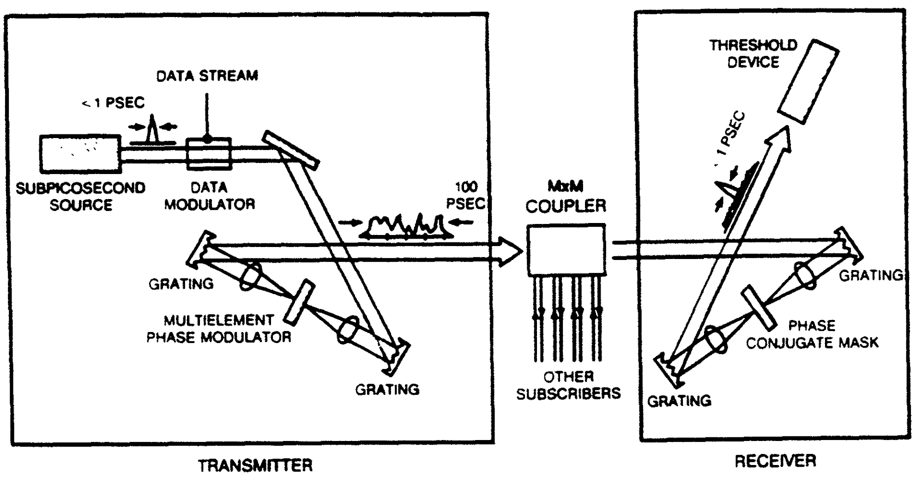

Figure 1: The communication system.

The receiver is assumed to be a correlation decoder. The optical decoder is similar to the optical encoder with the difference that the decoder's phase mask is the conjugate of that of the encoder, that is the phase shifts introduced by the decoder are the opposite of those introduced at the encoder. A decoder with this property is said to be *matched* to the encoder. Then assuming no noise in the channel, the phase shifts introduced at the encoder are removed at the decoder and the initial ultra-short light pulse is recovered. On the other hand, if the phase mask of the decoder is not the conjugate of the one of the encoder (thus the encoder and the decoder are not matched), the phase shifts are rearranged but not removed and a low intensity pseudonoise burst is generated at the output of the decoder. A threshold device detects data corresponding to intense, properly decoded pulses and rejects low intensity, improperly decoded pseudonoise bursts.

It is important to note here that in the system described above the encoding and the decoding takes place in the optical domain. It is well known that, with current technology, the real bottleneck in fiber-optic communications is the slow electronic components. An advantage of the system under investigation is that it is in its largest part optical and, thus, unnecessary photon to electron and electron to photon conversions are avoided.

In optical CDMA we want to design the signatures of the users so that if the transmitter and the receiver use phase masks that correspond to different signatures, the peak of the (interference) signal at the output of the decoder is small. We consider generating these signatures randomly. By using Chernoff's bound we prove that the peak $\mathcal{D}_{lm}$ of the interference signal at the output of decoder m due to the transmission of a one by user l is greater than some value $\lambda\sqrt{P_0}$ with a probability that goes to 0 exponentially as the length N_0 of the signatures goes to infinity. Randomly generated signatures have also been examined in the past in the work of [HaS89], but the bounds found there were weaker than the bounds which we will present. The upper bound on the maximum number of λ-separated signatures of length N_0 which we also derive, and the lower bound on the λ-capacity have not been considered in the past in a CDMA context and constitute a new way of looking at the performance of such a system. These results are reminiscent, in a different context, of the work of [Roo68] and [HSGB90] (these authors related the capacity of a channel to its impulse response).

The organization of the paper is the following. In Section 2 we present the mathematical model that corresponds to the optical CDMA system under investigation. In Section 3 we prove the upper bound on the probability that the peak intensity of the interference signal at receiver m due to transmissions of user l is above some value. In Section 4 we prove the lower bound on the maximum number of λ-separated signals of length N_0 that can be designed,and we find the lower bound on the λ-capacity. Finally, in the appendices we resolve some technical issues that arised in the other sections.

2. MATHEMATICAL MODEL

In this section we give the mathematical model that describes the communication system. We assume that the ultra-short light pulses correspond to an ideal rectangular spectrum of bandwidth W. Thus, in the time domain the light pulses are given by

$$a(t) = \sqrt{P_0}\ \mathrm{sinc}\left(\frac{W}{2}t\right). \tag{2}$$

The peak intensity of an ultrashort light pulse is P_0 and its duration is $t_c \approx \frac{2\pi}{W}$. Pulses of this form have also been used in the experiments reported in [WHS88].

The encoding described in Figure 1 is taking place in the frequency domain. The signal bandwidth is divided into $N_0 = 2N + 1$ equal chips, each of length

$$B = \frac{W}{N_0}.$$

The spectral components in the interval $[nB, (n+1)B)$, $n = -N, \ldots, N$ experience, when passing through the phase masks, a phase shift of $e^{j\phi_n}$, where

$$\overline{\phi} = \{\phi_n\}_{n=-N}^{N}$$

is the signature of the transmitter. In the time domain the encoded signal is the product

$$G(t) \cdot V(t, \overline{\phi}),$$

where

$$V(t, \overline{\phi}) = \frac{1}{N_0} \sum_{n=-N}^{N} \exp\{-j(nBt + \phi_n)\}, \tag{3}$$

and

$$G(t) = \sqrt{P_0}\ \mathrm{sinc}\left(\frac{B}{2}t\right).$$

Since $G(t)$ varies much slower in time than $V(t, \overline{\phi})$, it can be viewed as the envelope of the encoded signal.

When the receiver is matched to the transmitter then the signal at the output of the decoder is

$$G(t) \cdot \frac{1}{N_0} \sum_{n=-N}^{N} \exp\{-jnBt\}. \tag{4}$$

The peak of (4) occurs for $t = 0$ and is equal to $\sqrt{P_0}$. Assume now that the encoder l and the decoder m use phase masks that correspond to two different signatures $\overline{\phi^l}$ and $\overline{\phi^m}$, respectively. Then the output of the decoder at the absense of other users or noise is equal to

$$D_{lm}(t) = G(t)V(t, \overline{\phi^l} - \overline{\phi^m}) = G(t)\frac{1}{N_0}\sum_{n=-N}^{N} \exp\{-j(nBt + \phi_n^l - \phi_n^m)\}.$$

When encoder l and decoder m are unmatched, $D_{lm}(t)$ is an interference signal and we would like it to be small at all possible sample times. We are interested in this signal being small at all possible sample times, and not only at a particular time, because in our system the transmitter and the receiver are not synchronized. The supremum of $D_{lm}(t)$ for $t \in [-\frac{\pi}{B}, \frac{\pi}{B}]$ is the peak of the mutual interference signal between l and m and is defined as

$$\mathcal{D}_{lm} = \|D_{lm}(t)\|_\infty = \sup_t D_{lm}(t),$$

where $\|\cdot\|_\infty$ denotes the supremum norm. Ideally, we would like to assign the signatures so that $\mathcal{D}_{lm}$ is minimized for every pair of users (l, m). Unfortunately, this has been an intractable task so far. In what follows we are going to look at randomly generated signatures (codes). We assume that the ϕ_n^l components, $n = -N, \ldots, N$, are independent and identically distributed random variables, and are equal to 0 or π with probability 1/2. We are especially interested in the probability that the peak $\mathcal{D}_{lm}$ of the interference signal, for $l \neq m$, is greater than $\lambda\sqrt{P_0}$ for some $\lambda < 1$, that is

$$\Pr(\mathcal{D}_{lm} \geq \lambda\sqrt{P_0}) = \Pr\left(\phi : \|G(t) \cdot V(t, \overline{\phi^l} - \overline{\phi^m})\|_\infty \geq \lambda\sqrt{P_0}\right). \quad (5)$$

Since $\|G(t)\|_\infty = \sqrt{P_0}$ we get that

$$\Pr(\mathcal{D}_{lm} \geq \lambda\sqrt{P_0}) \leq \Pr\left(\phi : \|V(t, \overline{\phi^l} - \overline{\phi^m})\|_\infty \geq \lambda\right).$$

Let

$$\overline{\phi} = (\overline{\phi^l} - \overline{\phi^m}) \bmod (2\pi),$$

where mod is to be interpreted componentwise. Since the components ϕ_n^l and the ϕ_n^l, $n = -N, \ldots, N$ are i.i.d. and equal to 0 or π with probability 1/2, the ϕ_n's, are also i.i.d. with the same probability distribution. Then

$$\Pr(\mathcal{D}_{lm} \geq \lambda\sqrt{P_0}) \leq \Pr\left(\phi : \|V(t, \overline{\phi})\|_\infty \geq \lambda\right).$$

Note that when random coding is used, the probability in (5) is the same with the probability that the peak of the signal at the output of the encoder of some user is greater that $\lambda\sqrt{P_0}$. Thus, whenever we talk about $\mathcal{D}_{lm}$ we can view it as either referring to the peak of the interference signal, or to the peak of the encoded signal (both have the same probability distribution).

In the next section we use Chernoff's bound to find an upper bound on (5).

3. A PROBABILISTIC UPPER BOUND ON THE PEAK OF THE ENCODED SIGNALS

In this section we obtain an upper bound on the probability that the peak $\mathcal{D}_{lm}$ is larger than some value $\lambda\sqrt{P_0}$, with $\lambda < 1$ (note that $\mathcal{D}_{lm} \leq \lambda\sqrt{P_0}$ always). We define

$$R_n = \cos(nt + \phi_n)$$

and

$$H_n = \sin(nt + \phi_n).$$

We also define

$$R = \frac{1}{N_0} \sum_{n=-N}^{N} R_n$$

and

$$H = \frac{1}{N_0} \sum_{n=-N}^{N} H_n.$$

Then clearly (3) can be rewritten as

$$V(t, \overline{\phi}) = R + jH.$$

Consider a particular sample time $t_i \in [-\frac{\pi}{B}, \frac{\pi}{B}]$. We will first find a bound on the probability that at time t_i the magnitude $|V(t_i, \overline{\phi})|$ is greater than λ. This bound will be later used to bound the probability that $|V(t, \overline{\phi})| \leq \lambda$ for all t. It is easy to see that

$$\Pr\left(\overline{\phi} : |V(t, \overline{\phi})| \geq \lambda\right) = \Pr\left(R^2 + H^2 \geq \lambda^2\right) \leq$$
$$\leq \Pr\left(|R| \geq \frac{\lambda}{\sqrt{2}}\right) + \Pr\left(|H| \geq \frac{\lambda}{\sqrt{2}}\right). \tag{6}$$

In Appendix 1 we prove, by using the Chernoff bound, that

$$\Pr\left(\overline{\phi} : |V(t_i, \overline{\phi})| \geq \lambda\right) \leq 4 \cdot \exp\left(-\frac{s\lambda}{\sqrt{2}}\right) \cdot \left[\cosh\left(\frac{s}{N_0}\right)\right]^{N_0}. \tag{7}$$

Inequality (7) is true for all positive values of s. The right side of (7) is minimized by chosing

$$s = N_0 \tanh^{-1}\left(\frac{\lambda}{\sqrt{2}}\right) = \frac{N_0}{2} \ln\left(\frac{\sqrt{2} + \lambda}{\sqrt{2} - \lambda}\right), \tag{8}$$

when $\lambda < \sqrt{2}$. (For $\lambda > \sqrt{2}$ the right hand side of (7) is minimized for $s = \infty$. In this case the right hand side of (7) becomes zero which agrees with the fact that the peak of $|V(t, \overline{\phi})|$

cannot be more than $\sqrt{2}$; in fact it is always less than 1). By substituting (8) into (7) we get after some calculations, described in Appendix 2, that

$$\Pr\left(\overline{\phi} : |V(t_i, \overline{\phi})| \geq \lambda\right) \leq 4 \cdot e^{-N_0 E}, \tag{9}$$

where

$$E = E(\lambda) = \frac{\lambda}{2^{3/2}} \ln\left(\frac{\sqrt{2}+\lambda}{\sqrt{2}-\lambda}\right) - \frac{1}{2}\ln\left(\frac{2}{2-\lambda^2}\right). \tag{10}$$

It is interesting to look at the exponent E as a function of λ. Straightforward calculation gives

$$\frac{\delta E}{\delta \lambda} = \frac{1}{2^{3/2}} \ln\left(\frac{\sqrt{2}+\lambda}{\sqrt{2}-\lambda}\right).$$

By evaluating the second derivative of E with respect to λ, it can also be seen that $E(\lambda)$ is an increasing convex function. It rises slowly at the neighborhood of zero (for $\lambda = 0$ we have $\frac{\delta E}{\delta \lambda} = 0$) and its inclination increases rapidly with λ. Figure 2 illustrates E as a function of λ.

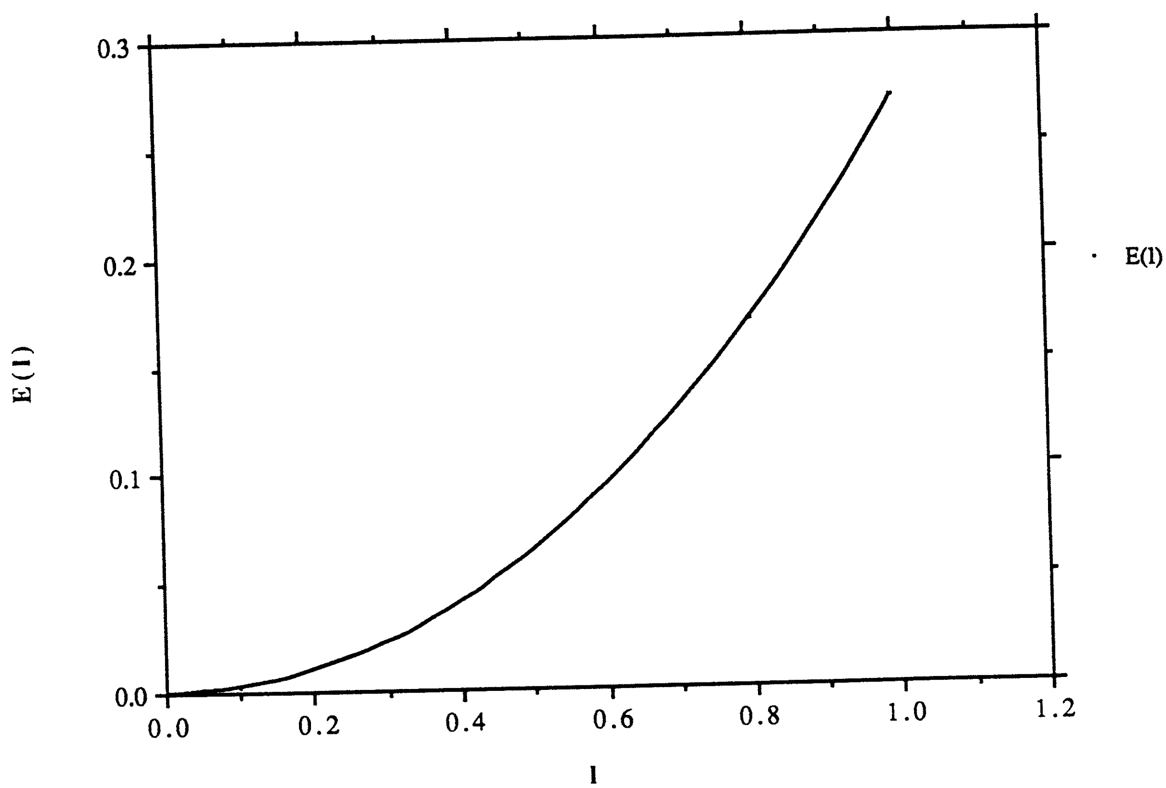

Figure 2: The (negative of) the exponent $E(\lambda)$ as a function of λ.

If we write the logarithms at the right hand side of (9) as sums of powers of λ we obtain

$$E = \sum_{k=1}^{\infty} \frac{1}{k(k-1)} \frac{\lambda^{2k}}{2^{k+1}} = \frac{\lambda^2}{4} + \frac{\lambda^4}{48} + \cdots. \tag{11}$$

By comparing equations (9) and (11) with the upper bound found in [HaS89], we see that the two bounds are close when λ is very small (and N_0 not too large) so that the powers of λ greater than 2 can be ignored. The correponding bound found in [HaS89] (equation (9) there) was decreasing exponentially with the exponent being $-N_0\frac{\lambda^2}{4}$. Thus, the upper bound found here is tighter than the bound in [SaH89] by a factor of

$$\exp\left(N_0\sum_{k=2}^{\infty}\frac{1}{k(k-1)}\frac{\lambda^{2k}}{2^{k+1}}\right)=\exp\left(N_0(\frac{\lambda^4}{48}+\frac{\lambda^6}{240}+\cdots)\right).$$

Our analysis so far has focused on the magnitude $|V(t_i,\overline{\phi})|$ at some particular sample time t_i. We are interested, however, in this magnitude being small at all the possible sample times or, else, in the probability that the supremum norm $||V(t,\overline{\phi})||_\infty$ is less than λ. As we have already mentioned this requirement comes from the fact that the beginning of times $t=0$ is neither common nor known for all users. In order to account for this we consider kN_0 equally spaced points t_j, $j=1,\ldots,kN_0$ in the interval $[-\frac{T}{2},\frac{T}{2}]$, where $T=\frac{2\pi}{B}$. It is proved in [HaJ89] that if $k>4$ then the following implication holds

$$|V(t_j,\overline{\phi})|<\lambda\ \forall\ j=1,\ldots,kN_0\implies||V(t,\overline{\phi})||_\infty\le\frac{k}{k-\pi}\lambda.$$

Therefore,

$$\Pr\left(\overline{\phi}:||V(t,\overline{\phi})||_\infty\le\lambda\right)\ge\Pr\left(\overline{\phi}:|V(t_j,\overline{\phi})|\le\frac{k-\pi}{k}\lambda,\ \forall\ j=1,\ldots,kN_0\right),$$

or

$$\Pr\left(\overline{\phi}:||V(t,\overline{\phi})||_\infty\ge\lambda\right)\le\Pr\left(\overline{\phi}:|V(t_j,\overline{\phi})|\ge\frac{k-\pi}{k}\lambda,\ \text{for some } j=1,\ldots,kN_0\right).\tag{12}$$

By applying the union bound we find that

$$\Pr\left(\overline{\phi}:|V(t_j)|\ge\frac{k-\pi}{k}\lambda,\ \text{for some } j=1,\ldots,kN_0\right)\le\left(kN_0\Pr\left(\overline{\phi}:|V(t_i)|\ge\frac{k-\pi}{k}\lambda\right)\right)^\rho,\tag{13}$$

for $\rho\in\{0,1\}$. The role of ρ in the previous expression is to ensure that the right hand side is always less than or equal to 1 as a probability function should be (when the expression in the outer parenthesis is greater than 1 we select $\rho=0$, otherwise we select $\rho=1$). By combining (12), (13) and equation (9) with λ replaced by $\frac{k-\pi}{k}\lambda$ we then get

$$\Pr\left(\overline{\phi}:||V(t,\overline{\phi})||_\infty\ge\lambda\right)\le\left(4kN_0\cdot\exp\left(-N_0E(\frac{k-\pi}{k}\lambda)\right)\right)^\rho,\ \forall\rho\in\{0,1\},\tag{14}$$

where $E(\lambda)$ is given by (10) or (11). The upper bound in (14) decreases almost exponentially with the length of the signatures N_0. This bound can have a number of possible applications. In the next section we use it to find a lower bound on the maximum number of λ-separated (or "minimally interfering") signals that can be designed for a given length N_0 of the signatures. We also define the λ-capacity of the CDMA system and find a lower bound on it.

3. A LOWER BOUND ON THE MAXIMUM NUMBER OF λ-SEPARATED SIGNALS

We call two signatures $\overline{\phi}^l$ and $\overline{\phi}^m$ λ-separated (or, sometimes, "minimally interfering") if and only if they satisfy the following relation

$$\|V(t, \overline{\phi}^l - \overline{\phi}^m)\|_\infty \geq \lambda.$$

Thus, if the signatures of user l and m are λ-separated then the peak of their mutual interference signal is less than $\lambda\sqrt{P_0}$ (this is why such signatures are also called "minimally interfering"). Naturally, a set of signatures is called λ-separated if every two signatures in it are λ-separated.

Let $M(N_0)$ be the maximum number of λ-separated signatures of length N_0 and consider the space $X = \{0, \pi\}^{N_0}$ of all possible signatures $\overline{\phi}$. On this space we can define (for the discrete topology) a counting measure μ. Then $\mu(X) = 2^{N_0}$, and for each subset A of X we have

$$\mu(A) = 2^{N_0} \Pr(\overline{\phi} : \overline{\phi} \in A).$$

For each $\overline{\phi}_0 \in X$ we define the set

$$B(\overline{\phi}_0) = \{\overline{\phi} \in X : \|V(t, \overline{\phi} - \overline{\phi}_0)\|_\infty \geq \lambda\}.$$

We call such a region a "sphere" centered at $\overline{\phi}_0$, although it is not a sphere in the usual sense, because $\|V(t, \overline{\phi})\|_\infty$ is not a metric with respect to $\overline{\phi}$. Then it can be seen that the measure of $\mu(B(\overline{\phi}_0))$ is

$$\mu(B(\overline{\phi}_0)) = 2^{N_0} \Pr\left(\overline{\phi} : \|V(t, \overline{\phi} - \overline{\phi}_0)\|_\infty \geq \lambda\right),$$

or, by using (14),

$$\mu(B(\overline{\phi}_0)) \leq 2^{N_0} \left(4kN_0 \cdot \exp\left(-N_0 E(\frac{k-\pi}{k}\lambda)\right)\right)^\rho, \ \forall \rho \in \{0, 1\}. \tag{15}$$

Let now $S = \{\overline{\phi}_0, \overline{\phi}_1, \ldots, \overline{\phi}_{|S|}\}$ be a maximal set of λ-separated signatures in the space X, where $|S|$ is the cardinality of S. Since S is a maximal set, all the points in X belong to a set $B(\overline{\phi}_i)$ for some $i \in \{1, 2, \ldots, |S|\}$; therefore

$$X = \cup_{i=1}^{|S|} B(\overline{\phi}_i).$$

Note that the sets $B(\overline{\phi}_i)$ are not necessarily disjoint. However, from the subadditivity of the counting measure μ we get that

$$2^{N_0} = \mu(X) = \mu\left(\cup_{i=1}^{|S|} B(\overline{\phi}_i)\right) \leq \sum_{i=1}^{|S|} \mu\left(B(\overline{\phi}_i)\right).$$

Using (15), the last equation becomes

$$|S| \geq \left(\frac{1}{4kN_0} \cdot \exp\left(N_0 E(\frac{k-\pi}{k}\lambda)\right)\right)^{\rho}, \ \forall \rho \in \{0,1\}.$$

Since S is a maximal set we get that the maximum number $M(N_0)$ of λ-separated signatures of length N_0 satisfies

$$M(N_0) \geq |S| \geq \left(\frac{1}{4kN_0} \cdot \exp\left(N_0 E(\frac{k-\pi}{k}\lambda)\right)\right)^{\rho}, \ \forall \rho \in \{0,1\}.$$

This relation shows that the maximum number of λ-separated signatures ("good" signatures) grows almost exponentially with the length N_0 of the signatures.

Assume now that we have a specific number U of active users in the system (which can cause interference to each other) and that the noise has some particular but unknown characteristics. These two factors (noise and number of users) determine a particular maximum value of the parameter λ which is adequate for reliable communication (we keep U constant because the adequate value of λ changes in general with U). In other words, for our model, if any two user signatures are λ-separated then the noise and the interference of other users is not enough to cause an error (or, alternatively, the probability of an error is sufficiently small). We define the *λ-capacity* C per user as the maximum reliable transmission rate per user that can be achieved under the previous conditions. Then we claim that as N_0 tends to infinity the communication link can sustain a transmission rate of C bits per user, per t_c seconds with reliable communication guaranteed, where C satisfies

$$C \geq \lim_{N_0 \to \infty} \frac{\log_2 M(N_0)}{N_0} = E\left(\frac{k-\pi}{k}\lambda\right).$$

The transmission rate at the right hand side of the previous inequality can be achieved in the following way. We assign the $M(N_0)$ λ-separated signatures to the users so that each of them is given roughly $M(N_0)/U$ signatures which can be used to represent $1 + M(N_0)/U$ different symbols (thus we do not limit ourselves in ON-OFF Keying any more). Then the total number of bits (from all users) that can be reliably transmitted on the link per t_c seconds is

$$U\frac{\log_2(1 + M(N_0)/U)}{N_0}.$$

For fixed U and N_0 large enough the previous expression assymptotically tends to

$$U\frac{\log_2 M(N_0)}{N_0}.$$

When we allow N_0 to go to infinity (this corresponds to the case where we have maximum reliable transmission rate with infinite delay at the decoder) we obtain that the maximum transmission rate C per user satisfies

$$C \geq E\left(\frac{k-\pi}{k}\lambda\right)$$

The total (for all users) capacity of the link is $UC(\lambda)$ bits per t_c seconds for any fixed U and λ. Note also that in general the value of the parameter λ decreases as the number of users U increases.

APPENDIX 1

In this appendix we prove equation (7).

Equation (6) can be rewritten as

$$\Pr\left(\phi : |V(t,\overline{\phi})|\lambda\right) \leq$$

$$\leq \Pr\left(R \geq \frac{\lambda}{\sqrt{2}}\right) + \Pr\left(H \geq \frac{\lambda}{\sqrt{2}}\right) + \Pr\left(R \leq -\frac{\lambda}{\sqrt{2}}\right) + \Pr\left(H \leq -\frac{\lambda}{\sqrt{2}}\right). \quad (15)$$

Let $\Phi_R(s)$ be the characteristic function of R, that is

$$\Phi_R(s) = \overline{e^{sR}},$$

where the bar denotes expectation. By applying Chernoff's bound ([Gal68]) we get that

$$\Pr\left(R \geq \frac{\lambda}{\sqrt{2}}\right) \leq \exp\left(-\frac{s_1\lambda}{\sqrt{2}}\right)\Phi_R(s_1),$$

for $s_1 > 0$ and

$$\Pr\left(R \leq -\frac{\lambda}{\sqrt{2}}\right) \leq \exp\left(\frac{s_2\lambda}{\sqrt{2}}\right)\Phi_R(s_2),$$

for $s_2 < 0$. By substituting $s_1 = -s_2 = s$ we obtain

$$\Pr\left(R \geq \frac{\lambda}{\sqrt{2}}\right) + \Pr\left(R \leq -\frac{\lambda}{\sqrt{2}}\right) \leq \exp\left(-\frac{s\lambda}{\sqrt{2}}\right)\left(\Phi_R(s) + \Phi_R(-s)\right), \quad (16)$$

for all $s > 0$. A similar relation holds for H:

$$\Pr\left(H \geq \frac{\lambda}{\sqrt{2}}\right) + \Pr\left(H \leq -\frac{\lambda}{\sqrt{2}}\right) \leq \exp\left(-\frac{s\lambda}{\sqrt{2}}\right)\left(\Phi_H(s) + \Phi_H(-s)\right), \quad (17)$$

for all $s > 0$. Since the ϕ_n's are i.i.d. and equal to 0 or π with probability 1/2, we have that the characteristic function of R is

$$\Phi_R(s) = \overline{e^{sR}} = \overline{\exp\left(\frac{s}{N_0}\sum_{n=-N}^{N} R_n\right)} =$$

$$= \prod_{n=-N}^{N}\left(\frac{1}{2}\exp\left(\frac{s}{N_0}\cos nt\right) + \frac{1}{2}\exp\left(-\frac{s}{N_0}\cos nt\right)\right) = \prod_{n=-N}^{N}\cosh\left(\frac{s\cos nt}{N_0}\right). \tag{18}$$

Similarly we find that

$$\Phi_H(s) = \overline{e^{sH}} = \prod_{n=-N}^{N}\cosh\left(\frac{s\sin nt}{N_0}\right). \tag{19}$$

With the use of (16)-(19), (15) is transformed to (20):

$$\Pr\left(\phi : \|V(t,\overline{\phi})\| \geq \lambda\right) \leq$$

$$\leq 2\cdot\exp\left(-\frac{s\lambda}{2^{1/2}}\right)\left(\prod_{n=-N}^{N}\cosh\left(\frac{s\cos nt}{N_0}\right) + \prod_{n=-N}^{N}\cosh\left(\frac{s\sin nt}{N_0}\right)\right). \tag{20}$$

Since $\cosh\left(\frac{s\cos nt}{N_0}\right) \leq \cosh\left(\frac{s}{N_0}\right)$, $\cosh\left(\frac{s\sin nt}{N_0}\right) \leq \cosh\left(\frac{s}{N_0}\right)$ and $N_0 = 2N+1$ we get that

$$\Pr\left(\phi : |V(t,\overline{\phi})| \geq \lambda\right) \leq 4\cdot\exp\left(-\frac{s\lambda}{\sqrt{2}}\right)\cdot\left[\cosh\left(\frac{s}{N_0}\right)\right]^{N_0}$$

for every $s > 0$ This proves equation (7). **Q.E.D.**

APPENDIX 2

In this Appendix we give the intermediate steps through which equation (9) when substituted into (8) gives (10) and (11). By inspection of these formulas one can see that it is enough to prove that

$$\cosh\left(\frac{1}{2}\ln\left(\frac{\sqrt{2}+\lambda}{\sqrt{2}-\lambda}\right)\right) = \exp\left(\frac{1}{2}\ln\left(\frac{2}{2-\lambda^2}\right)\right).$$

This is true because

$$\cosh\left(\frac{1}{2}\ln\left(\frac{\sqrt{2}+\lambda}{\sqrt{2}-\lambda}\right)\right) = \frac{\left(\frac{\sqrt{2}+\lambda}{\sqrt{2}-\lambda}\right)^{1/2} + \left(\frac{\sqrt{2}-\lambda}{\sqrt{2}+\lambda}\right)^{1/2}}{2}$$

$$= \left(\frac{2}{2-\lambda^2}\right)^{1/2} = \exp\left(\frac{1}{2}\ln\left(\frac{2}{2-\lambda^2}\right)\right).$$

Q.E.D.

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