Random sorting networks: local statistics via random matrix laws

RANDOM SORTING NETWORKS: LOCAL STATISTICS VIA RANDOM MATRIX LAWS

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Abstract. This paper finds the bulk local limit of the swap process of uniformly random sorting networks. The limit object is defined through a deterministic procedure, a local version of the Edelman-Greene algorithm, applied to a two dimensional determinantal point process with explicit kernel. The latter describes the asymptotic joint law near 0 of the eigenvalues of the corners in the antisymmetric Gaussian Unitary Ensemble. In particular, the limiting law of the first time a given swap appears in a random sorting network is identified with the limiting distribution of the closest to 0 eigenvalue in the antisymmetric GUE. Moreover, the asymptotic gap, in the bulk, between appearances of a given swap is the Gaudin-Mehta law – the limiting universal distribution for gaps between eigenvalues of real symmetric random matrices.

The proofs rely on the determinantal structure and a double contour integral representation for the kernel of random Poissonized Young tableaux of arbitrary shape.

1. Introduction

1.1. **Overview.** The main object of this article is the uniformly random sorting network, as introduced by Angel, Holroyd, Romik, and Virág in [AHRV]. Let $\mathfrak{S}_n$ denote the symmetric group and τ_i denote the transposition between i and $i+1$ for $1 \le i \le n-1$. The τ_i are called *adjacent swaps.* Let $\mathrm{rev} = n, n-1, \ldots, 1$ denote the reverse permutation of $\mathfrak{S}_n$. A sorting network of $\mathfrak{S}_n$ is a sequence of permutations $\sigma_0 = \mathrm{id}, \sigma_1, \ldots, \sigma_N = \mathrm{rev}$ of shortest length with the property that for every k,

$$\sigma_{k+1} = \sigma_k \circ \tau_i \text{ for some } i.$$

In other words, the permutations change by swapping adjacent labels at each step and must go from the identity to the reverse in the shortest number of swaps. The number of adjacent swaps required in any sorting network of $\mathfrak{S}_n$ is $\binom{n}{2}$. See Figure 1 for an example of a sorting network in the wiring diagram representation. We identify a sorting network of $\mathfrak{S}_n$ by its sequence of swaps

$$\left(s_1, \ldots, s_{\binom{n}{2}}\right),$$

where s_i denotes the adjacent swap $(s_i, s_i + 1)$.

A *random sorting network* of $\mathfrak{S}_n$ is a sorting network of $\mathfrak{S}_n$ chosen uniformly at random. Computer simulations were used to conjecture many beautiful asymptotic properties of

Key words and phrases. Sorting network, reduced decomposition, Gaudin-Mehta law, GUE corners, Young tableau, determinantal point process.

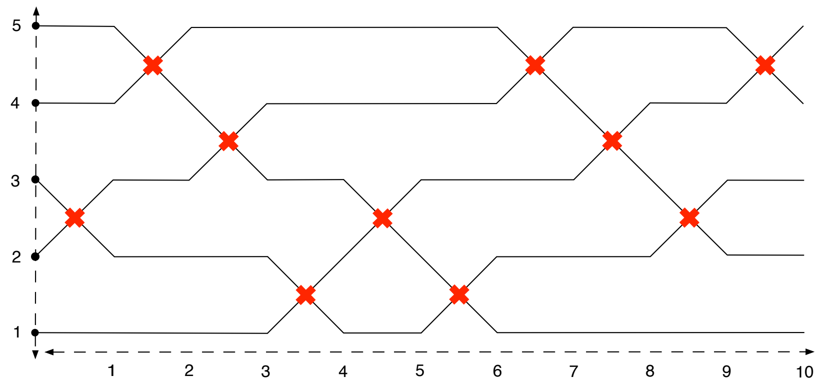

FIGURE 1. Wiring diagram of a sorting network of $\mathfrak{S}_5$ with swap sequence (2,4,3,1,2,1,4,3,2,4). Intersection of two paths at location $(i-1/2, j+1/2)$ indicates a swap at time i between labels at positions j and $j+1$. The intersection locations (red crosses) of a random sorting network of $\mathfrak{S}_n$ has a distributional limit in windows of unit order in the vertical direction and order n in the horizontal direction.

random sorting networks. See [AHRV] for an account of these statements and the first rigorous results, and also [ADHV, AGH, AH, AHR, DVi, Ko, RVV, R] for other asymptotic theorems. The proofs of the conjectures from [AHRV] were recently announced in [D].

In many examples, random combinatorial structures built out of symmetric groups are known to exhibit the same asymptotic behavior as random matrices. The most famous result of this sort due to Baik–Deift–Johansson [BDJ] identifies the fluctuations of longest increasing subsequences of random permutations with the fluctuations of largest eigenvalue of random Hermitian matrices. Its further upgrades, [BOO, J, O] link fluctuations of several first rows of the Young diagram distributed according to the Plancherel measure for symmetric groups to those of several largest eigenvalues. A connection also exists for "bulk" (i.e. not largest) rows and eigenvalues, see [BOO].

On the other hand, up to now no such connections were known for random sorting networks. In the present article we find such a connection. It exists for a sort of local limit of random sorting networks. Indeed, we find the bulk local limit of random sorting networks, by proving that it is given by a simple, local, deterministic algorithm (the local Edelman-Greene algorithm) applied to a specific random point process on $\mathbb{Z} \times \mathbb{R}_{\geq 0}$. In turn, we describe this point process by showing that its correlation functions have determinantal form and provide explicit expressions for the corresponding correlation kernel. The very same point process appeared in the work of Forrester–Nordenstam [FN] (see also Deffosseux [De]) as the hard edge limit of antisymmetric GUE–corners process; it describes the asymptotic distribution of the closest to 0 eigenvalues of the corners of large antisymmetric matrix with i.i.d. (modulo symmetry) Gaussian entries of mean 0.

A corollary of our theorem is that the rescaled, asymptotic distribution of the first time when the swap between $\lfloor \frac{n(1+\alpha)}{2} \rfloor$ and $\lfloor \frac{n(1+\alpha)}{2} \rfloor + 1$ appears, for $\alpha \in (-1, 1)$, is the same as the rescaled, asymptotic distribution of the closest to 0 eigenvalue of an antisymmetric-GUE random matrix. Another corollary is that within the bulk, the asymptotic gap between appearances of the aforementioned swap is described by the Gaudin–Mehta law — the asymptotic universal distribution of the gap between eigenvalues of real symmetric random matrices in the bulk. Complete statements are given in the next section.

In an independent and parallel work, Angel, Dauvergne, Holroyd, and Virág [ADHV] also study the bulk local limit of random sorting networks. Their approach is very different from ours. We deduce explicit formulas for the prelimit local structure of random sorting networks, and then analyze the asymptotic of these formulas in the spirit of *Integrable Probability*, see [BG, BP] and also [Ro]. On the other hand, [ADHV] argue probabilistically, analyzing a Markov chain (whose transition probabilities are expressed through the hook formula for dimensions) for sampling random Young tableaux. The connection to random matrices remains invisible in the results of [ADHV]. It would be interesting to match these two approaches, but it has not been done so far.

1.2. **Bulk limit of random sorting networks.** We now describe our main result. Informally, we study the asymptotics of the point process (s_i, i), $i = 1, \dots, \binom{n}{2}$, in a window of finite height and order n width, so that the number of points in the window remains finite; see Figure 1. Here $(s_1, \dots, s_{\binom{n}{2}})$ are swaps of a random sorting network of $\mathfrak{S}_n$.

In [AHRV] it is proven that the point process (s_i, i) is stationary with respect to the second coordinate. Therefore, it suffices to study windows adjacent to 0 in second coordinate, which we do.

The limiting object S_{local} is a point process on $\mathbb{Z} \times \mathbb{R}_{\geq 0}$ defined by a two–step procedure. First, we introduce an auxiliary point process $\mathcal{X}_{\text{edge}}$ on $\mathbb{Z} \times \mathbb{R}_{\geq 0}$ through its correlation functions.

Definition 1.1. *$\mathcal{X}_{\text{edge}}$ is the (unique) determinantal point process on $\mathbb{Z} \times \mathbb{R}_{\geq 0}$ with correlation kernel*

$$K_{\text{edge}}(x_1, u_1; x_2, u_2) = \begin{cases} \dfrac{2}{\pi} \displaystyle\int_0^1 t^{x_2 - x_1} \cos\left(tu_1 + \frac{\pi}{2} x_1\right) \cos\left(tu_2 + \frac{\pi}{2} x_2\right) dt, & \text{if } x_2 \geq x_1; \\ -\dfrac{2}{\pi} \displaystyle\int_1^\infty t^{x_2 - x_1} \cos\left(tu_1 + \frac{\pi}{2} x_1\right) \cos\left(tu_2 + \frac{\pi}{2} x_2\right) dt, & \text{if } x_2 < x_1. \end{cases}$$

We refer to [B] and Section 2.4 for more detailed discussions of determinantal point processes. We note that the particles of $\mathcal{X}_{\text{edge}}$ on adjacent lines $\{x\} \times \mathbb{R}_{\geq 0}$ and $\{x+1\} \times \mathbb{R}_{\geq 0}$ almost surely interlace, see Figure 2. The point process $\mathcal{X}_{\text{edge}}$ has appeared in the random matrix literature before in [FN], [De]. In more details, let G be an infinite random matrix with rows and columns indexed by $\mathbb{Z}_{>0}$, and whose entries are independent and identically distributed, real-valued, standard Gaussians. Let $A = \frac{G - G^T}{\sqrt{2}}$. The top–left $m \times m$ corner of A almost surely has $2\lfloor m/2 \rfloor$ non-zero eigenvalues of the form

$$\pm \mathbf{i}\lambda_1^m,\ \pm \mathbf{i}\lambda_2^m,\ \ldots,\ \pm \mathbf{i}\lambda_{\lfloor m/2 \rfloor}^m,$$

where $0 < \lambda_1^m < \lambda_2^m < \cdots < \lambda_{\lfloor m/2 \rfloor}^m$. Forrester and Nordenstam prove that $\mathcal{X}_{\text{edge}}$ is the weak limit of the point process $\{(j, \sqrt{2M}\lambda_i^{2M+j})\} \subset \mathbb{Z} \times \mathbb{R}_{\geq 0}$, $i \in \mathbb{Z}_{>0}$, $j \in \mathbb{Z}$, as $M \to \infty$.

Particle configurations of S_{local} are obtained from $\mathcal{X}_{\text{edge}}$ by a deterministic procedure, which is a local version of the well-known Edelman–Greene bijection [EG] between staircase shaped tableaux and sorting networks. In the following we describe this procedure. A rigorous definition of the procedure utilizes properties of $\mathcal{X}_{\text{edge}}$ that are not immediate from Definition 1.1. We provide the rigorous construction in Section 6.3 where the description is given in the language of Young tableau, which is the more standard setup for defining the Edelman-Greene bijection.

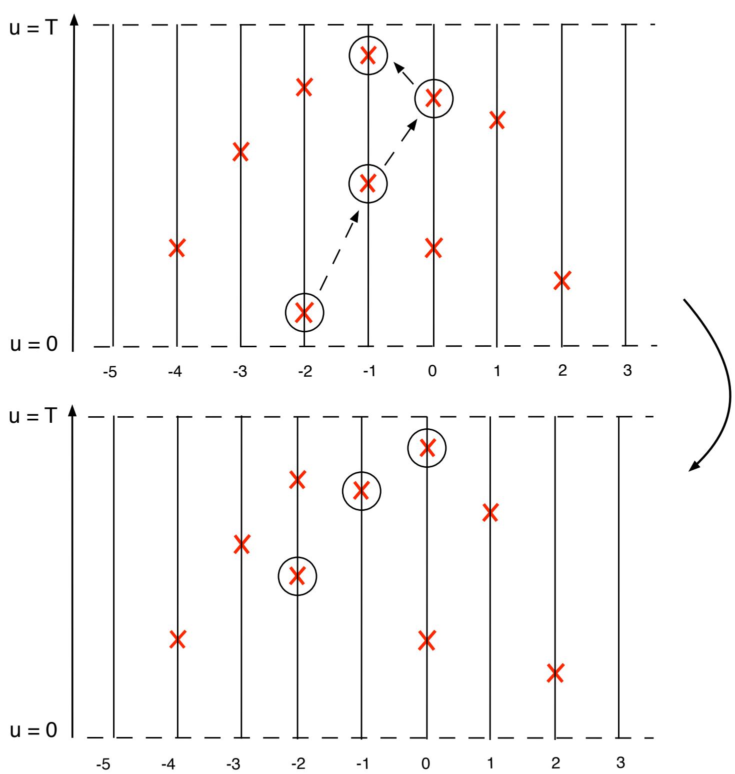

FIGURE 2. Top: A possible configuration (red crosses) of $\mathcal{X}_{edge}$ restricted to $[-5, 3] \times [0, T]$ with no particles on $\{-5, 3\} \times [0, T]$. Particles on consecutive lines interlace. Encircled points represent the sliding path during the first step of the Edelman-Greene algorithm. Bottom: The result after the first step of the Edelman-Greene algorithm.

Local Edelman-Greene algorithm. Fix a configuration X of $\mathcal{X}_{\text{edge}}$ and suppose that we want to define the positions of all particles of S_{local} inside the rectangle $[a, b] \times [0, T]$ with $a < 0 < b$. Then almost surely there are two integers $\hat{a} < 2a$ and $\hat{b} > 2b$ such that X has no particles on the segments $\hat{a} \times [0,T]$ and $\hat{b} \times [0,T]$. The particles of X outside $[\hat{a},\hat{b}] \times [0,T]$ are further ignored.

We now define a particle configuration Y – the restriction of S_{local} onto $[\hat{a},\hat{b}] \times [0,T]$ – through an iterative procedure. Start by declaring $Y = \emptyset$, and setting $\hat{X}$ to be the restriction of X onto $[\hat{a},\hat{b}] \times [0,T]$. Repeat the following until $\hat{X}$ is empty:

(1) Let (x,u) be an element of $\hat{X}$ with smallest second coordinate. The parity of x will be even. Add $(x/2,u)$ to Y, i.e., redefine $Y := Y \cup \{(x/2,u)\}$.

(2) Define the *sliding path* $(x_1,u_1),(x_2,u_2),\dots$ as a unique collection of points in $\hat{X}$ (of maximal length) such that
- $(x_1,u_1) = (x,u)$,
- $u_1 < u_2 < \cdots < u_k$ and $|x_i - x_{i+1}| = 1$ for $i = 1,\dots,k-1$,
- For each $i = 1,\dots,k-1$, the only points of $\hat{X}$ in the rectangle $[x_i - 1, x_i + 1] \times [u_i, u_{i+1}]$ are (x_i,u_i) and (x_{i+1},u_{i+1}).

In other words, (x_{i+1},u_{i+1}) is the point in $[x_i - 1, x_i + 1] \times (u_i, T]$, which is closest to (x_i,u_i). See Figure 2 for an illustration.

(3) Remove the k points $(x_1,u_1),\dots,(x_k,u_k)$ from $\hat{X}$ and replace them by $k-1$ points (x_1,u_2), (x_2,u_3), $\dots$, (x_{k-1},u_k).

(4) Go back to Step (1), unless $\hat{X}$ is empty.

The first coordinates of the particles of Y will be integral; this follows from the interlacing property of the particles of $\hat{X}$, which is preserved throughout the steps of the procedure.

One immediate property of the just defined map $\mathcal{X}_{\text{edge}} \mapsto S_{\text{local}}$ is that the position of the first particle of S_{local} in the ray $\{a\} \times \mathbb{R}_{\geq 0}$ almost surely coincides with the position of the first particle of $\mathcal{X}_{\text{edge}}$ in the ray $\{2a\} \times \mathbb{R}_{\geq 0}$. Therefore, the joint law of the positions of the first particles of S_{local} in the rays $\{a_i\} \times \mathbb{R}_{\geq 0}$, for $i = 1,\dots,k$, can be explicitly evaluated as a Fredholm determinant. See Corollary 1.3 for the case $k = 1$ and [B] for general statements.

We also show that S_{local} is invariant under translations and reflections of the first ($\mathbb{Z}$–valued) coordinate, ergodic with respect to translations of the first coordinate, and stationary in the second ($\mathbb{R}_{\geq 0}$–valued) coordinate; see Proposition 6.4.

We are ready to formulate the main result.

Theorem 1.2 (Local random sorting network)**.** *Fix $\alpha \in (-1,1)$, and let $s_1, s_2, \dots, s_{\binom{n}{2}}$ be swaps of a random sorting network of $\mathfrak{S}_n$. Define the point process $S_{\alpha,n}$ of rescaled swaps near the point $(\frac{n(\alpha+1)}{2}, 0)$ through*

$$S_{\alpha,n} = \left\{ \left(s_i - \left\lfloor \frac{n(\alpha+1)}{2} \right\rfloor, \sqrt{1-\alpha^2} \cdot \frac{2i}{n} \right) \right\}_{i=1}^{\binom{n}{2}}.$$

Then as $n \to \infty$, the point process $S_{\alpha,n}$ converges weakly to S_{local}.

It is proven in [AHRV, Theorem 2] that the global scaling limit of the space-time swap process of random sorting networks is the product of the semicircle law and Lebesgue measure. The $\sqrt{1-\alpha^2}$ scaling of Theorem 1.2 is consistent with the semicircle result.

We emphasize that Theorem 1.2 states both that $S_{\alpha,n}$ converges and the limit is obtained by applying the localized Edelman-Greene algorithm to $\mathcal{X}_{\mathrm{edge}}$. Theorem 1.2 does not cover the case $|\alpha| = 1$, where the asymptotic behavior changes. It is plausible that the methods of the present article can be adapted to this remaining case, but we not address it here; see [R] for another approach to $|\alpha| = 1$ case.

Theorem 1.2 implies that the first swap times in random sorting networks converge to a one–dimensional marginal of S_{local}; the distribution of the latter can be expressed as a Fredholm determinant. Figure 3 shows the approximate sample distribution of the rescaled first swap time and (6.2) shows the tail asymptotics.

Corollary 1.3 (First swap law)**.** . *Let $\mathbf{T}_{FS,\alpha,n}$ be the first time the swap interchanging $\left\lfloor \frac{n(\alpha+1)}{2} \right\rfloor$ with $\left\lfloor \frac{n(\alpha+1)}{2} \right\rfloor + 1$ appears in a random sorting network of $\mathfrak{S}_n$. The following convergence in law holds:*

$$\lim_{n\to\infty} \frac{2\sqrt{1-\alpha^2}}{n} \mathbf{T}_{FS,\alpha,n} = \mathbf{T}_{\mathrm{FS}},$$

where

$$\mathbb{P}\left[\mathbf{T}_{\mathrm{FS}} > t\right] = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \int\limits_{[0,t]^k} \det[K_{\mathrm{edge}}(u_i,u_j)]\, du_1 \cdots du_k, \text{ and} \tag{1.1}$$

$$K_{\mathrm{edge}}(u_1,u_2) = \frac{\sin(u_1-u_2)}{\pi\,(u_1-u_2)} + \frac{\sin(u_1+u_2)}{\pi\,(u_1+u_2)}\,.$$

Connection to the Gaudin-Mehta law. A further consequence deals with the limiting law of the gap between swaps on the same horizontal line in random sorting networks. Fix $\beta \in (0,1)$. Given a random sorting network of $\mathfrak{S}_n$, let $\mathbf{T}_+$ be the distance between $\left\lfloor \beta\binom{n}{2} \right\rfloor$ and the closest to its right swap interchanging $\left\lfloor \frac{n(\alpha+1)}{2} \right\rfloor$ with $\left\lfloor \frac{n(\alpha+1)}{2} \right\rfloor + 1$. Let $\mathbf{T}_-$ be the analogous distance to the closest to its left swap.

Due to stationarity of random sorting networks, the joint law $\mathbf{T}_-$ and $\mathbf{T}_+$ is given by

$$\mathbb{P}\left[\mathbf{T}_- > a, \mathbf{T}_+ > b\right] = \mathbb{P}\left[\mathbf{T}_{FS,\alpha,n} > a+b\right]. \tag{1.2}$$

Indeed, due to stationarity, both sides of (1.2) give the probability of the event that there are no swaps in the interval $[-a,b]$ after the appropriate re-centerings. Equation 1.2 shows that the law of $(\mathbf{T}_+,\mathbf{T}_-)$, and hence, of the gap $\mathbf{T}_- + \mathbf{T}_+$, is determined by the law of the first swap time $\mathbf{T}_{FS,\alpha,n}$. In particular, their limiting law after rescaling by $\sqrt{1-\alpha^2}/n$ is uniquely determined from the distribution function (1.1).

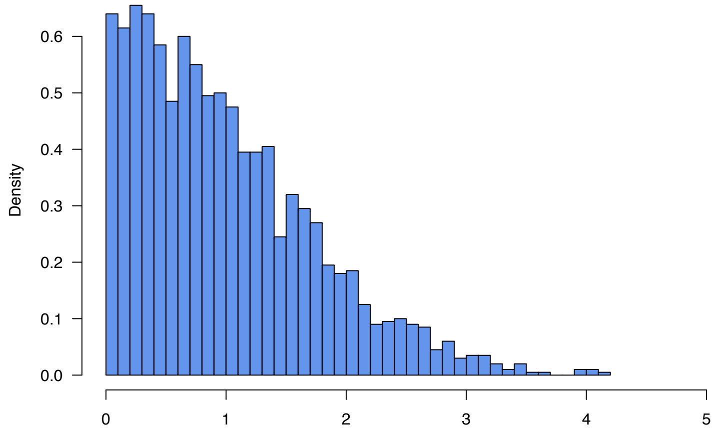

FIGURE 3. Density histogram of the rescaled first swap time for element 500 in a 1000 element random sorting network.

This is connected to the scaling limit of the point process of eigenvalues of GOE random matrices in the bulk. The scaling limit of the eigenvalues of GOE random matrices near 0 is stationary. Let $-\Lambda_-$ and Λ_+ be, respectively, the closest to 0 negative and closest to 0 positive point in the limit process. If the matrices are scaled so that the mean eigenvalue gap near 0 is 1, then (1.1) is the distribution function of $(\pi/2)\Lambda_+$. In other words, (1.1) is the asymptotic probability to see no eigenvalues in an interval of length $(2/\pi)t$ for large GOE random matrices, normalized so that the mean eigenvalue gap around the interval is 1; see e.g. [G], [Dy], [F, (8.139) and (9.81)]. The gap between points, $\Lambda_- + \Lambda_+$, has its law determined from that of Λ_+ according to (1.2). This is the celebrated *Gaudin–Mehta* law, originally put forward by Wigner as a model for the gap between energy levels in heavy nuclei and later found in numerous systems. We arrive at the following corollary.

Corollary 1.4 (Gap law)**.** *For $\alpha \in (-1, 1)$ and $\beta \in (0, 1)$, let $\mathrm{Gap}_{\alpha,\beta,n}$ be the distance in a random sorting network of $\mathfrak{S}_n$ between the two swaps interchanging $\left\lfloor \frac{n(\alpha+1)}{2} \right\rfloor$ with $\left\lfloor \frac{n(\alpha+1)}{2} \right\rfloor + 1$: the one closest from the left to time $\beta\binom{n}{2}$ and the one closest from the right to $\beta\binom{n}{2}$. Then, the distributional limit*

$$\lim_{n\to\infty} \frac{4\sqrt{1-\alpha^2}}{\pi n} \mathrm{Gap}_{\alpha,\beta,n}$$

is the Gaudin–Mehta law, i.e. the asymptotic gap in the bulk between eigenvalues of real symmetric random matrices with mean gap one.

The proof of Theorem 1.2 builds upon two ideas. The first one (which is also used in most of the rigorous results on sorting networks) is to reduce the study of random sorting networks to uniformly random staircase shaped standard Young tableaux via the Edelman–Greene bijection [EG] (see also [HY]). Our observation is that if we *Poissonize* uniformly random standard Young tableaux (of arbitrary shape!), then the result can be described by a determinantal point process with an explicit correlation kernel written as a double contour integral. We further show that the Poissonization does not change the local statistics, and therefore, the limit theorem is reduced to the asymptotic analysis of the aforementioned double contour integral, which we perform.

Our results on the correlations and limiting behavior of random standard Young tableaux might be of independent interest, and so we present them in the next section.

1.3. **Random Standard Young Tableaux.** A partition λ is a sequence of non-negative integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq 0$ such that $|\lambda| := \sum_{i=1}^{\infty} \lambda_i < \infty$. The *length* of λ, denoted $\ell(\lambda)$, is the number of positive λ_i and the *size* of λ is $|\lambda|$.

We identify a partition with a *Young diagram* (YD), which is the set of lattice points

$$\{(i,j) \in \mathbb{Z}^2 : \, i \geq 1, \, 1 \leq j \leq \lambda_i\}.$$

The points of the Young diagram λ are its *cells* and we say the Young diagram has *shape* λ. Given a pair of YDs λ and μ, we write $\lambda \preceq \mu$ if the cells of λ are contained within the cells of μ. If the containment is strict then $\lambda \prec \mu$. If $\lambda \preceq \mu$ then $\mu \setminus \lambda$ denotes the cells of μ that are not in λ. A *standard Young tableau* (SYT) of shape λ is an insertion of the numbers $1, 2, \ldots, |\lambda|$ into the cells of λ such that they strictly increase along the rows (from left to right) and also along the columns (from bottom to top). The numbers within a SYT are its *entries*. The set of SYTs of shape λ is in bijection with the set of increasing sequences of YDs

$$\emptyset = \lambda^{(0)} \prec \lambda^{(1)} \prec \lambda^{(2)} \prec \cdots \prec \lambda^{(|\lambda|)} = \lambda \tag{1.3}$$

such that the entry k is inserted into the singleton cell of $\lambda^{(k)} \setminus \lambda^{(k-1)}$.

A staircase shaped SYT of length $n-1$ (or also $n-1$ rows) is a SYT of shape $(n-1, n-2, \ldots, 2, 1)$, which we denote Δ_n. The Edelman–Greene bijection [EG] gives a one-to-one correspondence between staircase shaped SYTs and sorting networks; see Section 6.1 for the details. This is the reason for our interest in SYTs.

A Poissonized Young tableau (PYT) of shape λ is an insertion of distinct real numbers from the interval $(0,1)$ into the cells of λ such that they strictly increase along the rows and along the columns. Note that if we replace the entries of a PYT by their relative ranks then we get a SYT. The set of PYTs of shape λ is in bijection with the set of increasing sequences of YDs indicating the times of jumps:

$$\emptyset = \lambda^{(0)} \overset{t_1}{\prec} \lambda^{(1)} \overset{t_2}{\prec} \lambda^{(2)} \overset{t_3}{\prec} \cdots \overset{t_{|\lambda|}}{\prec} \lambda^{(|\lambda|)} = \lambda \tag{1.4}$$

such that the entry t_k is inserted in the singleton cell of $\lambda^{(k)} \setminus \lambda^{(k-1)}$. These increasing sequences of Young diagrams with labels were discussed in [BO] in the connection to the *Young bouquet*; see also [N].

We would like to identify a PYT with a collection of non-intersecting paths. For that we first map a Young diagram λ to a countable particle configuration $\{\lambda_i - i + 1/2\}_{i=1,2,\ldots} \subset \mathbb{Z} + 1/2$. This procedure can be viewed as projecting the boundary of the Young diagram in *Russian notation* onto a horizontal line, see Figure 4. The empty Young diagram $\emptyset$ corresponds to $\{-1/2, -3/2, -5/2, \ldots\}$.

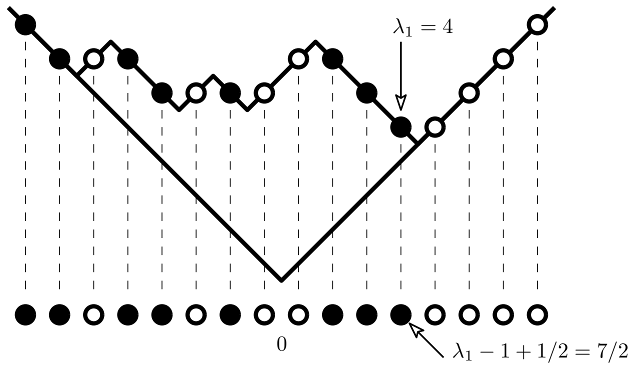

FIGURE 4. Young diagram $(4, 4, 4, 2, 1, 1)$ and corresponding particle configuration $(7/2, 5/2, 3/2, -3/2, -7/2, -9/2, -13/2, -15/2, \dots)$.

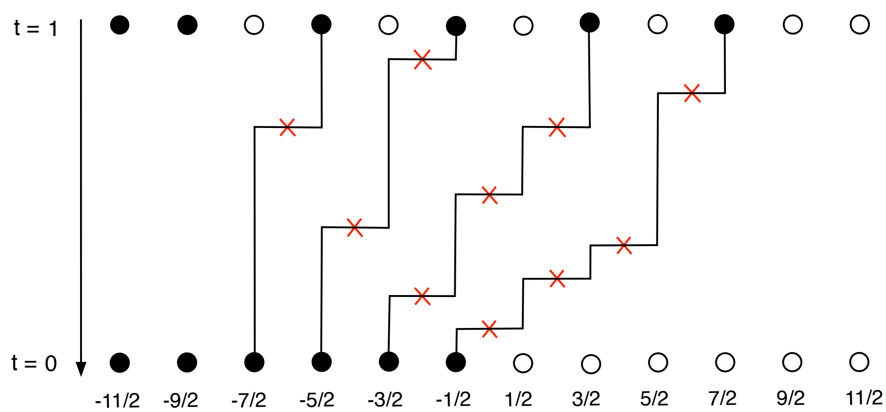

FIGURE 5. Particle system associated to a staircase shaped PYT of size 10. Particles move along non-intersecting paths. The jumps (red crosses) of a random PYT of fixed shape is a determinantal point process. The local window of a PYT of shape Δ_n consists of the trajectories of a group of L successive particles traced down from $t = 1$ to $t = 1 - (u/n)$.

Give a PYT, for each t consider the countable particle configuration corresponding to the Young diagram filled with the entries $\leq t$ in the PYT. The trajectories of particles then form a collection of paths, making jumps to the right at the times indexed by the entries t_k of the tableau (equivalently, labels in (1.4)). Let us draw a cross at a point (x, t), $x \in \mathbb{Z}$, $0 < t < 1$, if a particle jumps from $(x - 1/2)$ to $(x + 1/2)$ at time t; see Figure 5. Although there are infinitely many particles, the only ones that move are the $\ell(\lambda)$ particles that correspond to the rows of λ with positive size.

Theorem 1.5 (Poissonized tableaux)**.** *Given a finite Young diagram λ, consider the point process $\mathcal{X}_\lambda$ of jumps of a uniformly random Poissonized Young tableau of shape λ. $\mathcal{X}_\lambda$ is a determinantal point process on $\mathbb{Z} \times [0, 1]$ with correlation kernel $K_\lambda(x_1, t_1; x_2, t_2)$ as follows.*

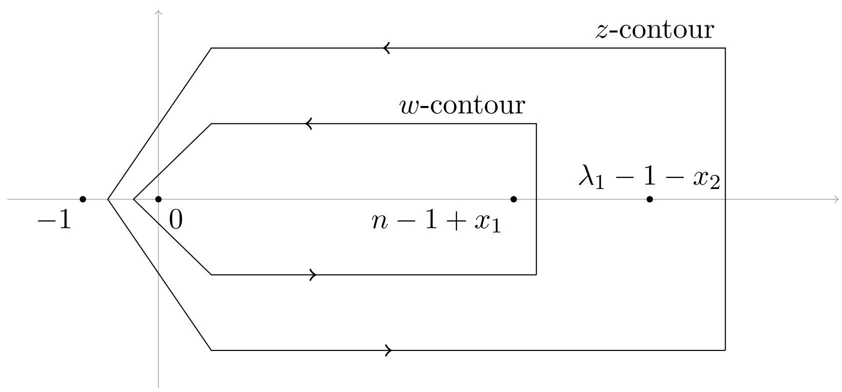

FIGURE 6. The contours in the statement of Theorem 1.5.

For $x_1, x_2 \in \mathbb{Z}$ *and* $t_1, t_2 \in [0,1]$,

$$K_\lambda(x_1, t_1; x_2; t_2) = \mathbf{1}_{\{t_2 > t_1,\, x_1 > x_2\}} \frac{(t_1 - t_2)^{x_1 - x_2 - 1}}{(x_1 - x_2 - 1)!} +$$

$$\frac{1}{(2\pi \mathbf{i})^2} \oint\limits_{C_z[0,\lambda_1 - x_2)} dz \oint\limits_{C_w[0,n+x_1)} dw \, \frac{\Gamma(-w)}{\Gamma(z+1)} \cdot \frac{G_\lambda(z + x_2)}{G_\lambda(x_1 - 1 - w)} \cdot \frac{(1-t_2)^z (1-t_1)^w}{w + z + x_2 - x_1 + 1},$$

$$\text{where } G_\lambda(u) = \Gamma(u+1) \prod_{i=1}^{\infty} \frac{u+i}{u - \lambda_i + i} = \frac{\Gamma(u+1+n)}{\prod_{i=1}^{n}(u - \lambda_i + i)}, \quad n \geq \ell(\lambda).$$

The contours $C_z[0, \lambda_1 - x_2)$ *and* $C_w[0, n + x_1)$ *are as shown in Figure 6. Both are counter-clockwise, encloses only the integers in the respective half open intervals* $[0, \lambda_1 - x_2)$ *and* $[0, n + x_1)$*, and arranged such that* $w + z + x_2 - x_1 + 1$ *remains uniformly bounded away from 0 along the contours.*

Remark 1.6. When t_1 or t_2 equals 1, K_λ is to be understood in the limit as t_1 or t_2 tends to 1. The contours $C_z[0, \lambda_1 - x_2)$ and $C_w[0, n - x_2)$ may also be replaced by unbounded contours $C_z[0, \infty)$ and $C_w[0, \infty)$ with bounded imaginary parts, respectively.

The proof of Theorem 1.5 is through a limit transition in the correlation kernel of [P] (see also [DM]) for the uniformly random Gelfand–Tsetlin patterns; the proof is in Section 3. Such a limit transition can be viewed as a degeneration of the combinatorial structures related to the representation theory of the unitary groups $U(N)$ to those related to the symmetric groups $\mathfrak{S}_n$, see [BO] for a discussion.

Let us emphasize that the very same procedure can be used to identify a uniformly random SYT with a point process of jumps, however, the resulting process will *not* be determinantal – this is why we need to pass from SYTs to PYTs.

1.4. **From Poissonized tableaux to local statistics.** We close the introduction with an outline of the argument that takes us from Poissonized tableaux to local statistics of sorting networks. By the nature of the Edelman-Greene bijection, the swaps of a sorting network of $\mathfrak{S}_n$ near time 0 are determined by the location of the largest entries of an SYT of shape Δ_n. These entries reside within unit order distance of the *edge* of Δ_n, which consist of the cells $(i, n-i)$ for $1 \leq i \leq n-1$. As a result, the first step to deriving local statistics of random sorting networks is to derive the statistics of a uniformly random PYT of shape Δ_n near its edge.

Let $\mathcal{T}_{\Delta_n}$ denote a uniformly random PYT of shape Δ_n. We are interested in the statistics of the entries of $\mathcal{T}_{\Delta_n}$ that lie within the following windows. A window is parameterized by a center $\alpha \in (-1,1)$ (corresponding to the center $\frac{(1+\alpha)n}{2}$ of the swaps of a sorting network), a length L, and an entry height u. The window then consist of entries $\mathcal{T}_{\Delta_n}(i,j)$ that satisfy $|i-(1+\alpha)n/2| \leq L$ and $\mathcal{T}_{\Delta_n}(i,j) \geq 1 - \frac{u}{n}$. In other words, roughly the largest nu entries of $\mathcal{T}_{\Delta_n}$ with row indices in the interval $[(1+\alpha)n/2 - L, (1+\alpha)n/2 + L]$.

We study the statistics of $\mathcal{T}_{\Delta_n}$ in a window in terms of its associated process of jumps $\mathcal{X}_{\Delta_n}$, rescaled accordingly. For each integer $n \geq 1$, let c_n be an integer having the same parity as n and such that $|c_n - \alpha n| = O(1)$ as $n \to \infty$. Consider the rescaled process of jumps

$$\mathcal{X}_{\alpha,n} = \left\{ (x,u) \in \mathbb{Z} \times \mathbb{R}_{\geq 0} : \left(x + c_n, 1 - \frac{u}{n\sqrt{1-\alpha^2}} \right) \in \mathcal{X}_{\Delta_n} \right\}. \tag{1.5}$$

Theorem 4.1 and Proposition 4.3 together imply that $\mathcal{X}_{\alpha,n}$ converges weakly to the point process $\mathcal{X}_{\text{edge}}$ from Definition 1.1. It is the building block for the proof of Theorem 1.2.

In Section 5 we construct the local staircase shaped tableau $\mathcal{T}_{\text{edge}}$ by using the local jump process $\mathcal{X}_{\text{edge}}$. We prove in Theorem 5.2 that it provides the limiting statistics of $\mathcal{T}_{\Delta_n}$ in local windows. Using a de-Poissonization argument we conclude in Theorem 5.3 that uniformly random staircase shaped SYTs also converge within local windows to $\mathcal{T}_{\text{edge}}$. Although Poissonization is not important for the local limit, it is important for the proof.

In Section 6 we prove Theorem 1.2. First, we give a proof of Corollary 1.3 in Section 6.2. In Section 6.3 we define the local version of the Edelman-Greene algorithm that maps $\mathcal{T}_{\text{edge}}$ to S_{local}. In Section 6.4 we complete the proof of Theorem 1.2 and conclude with some statistical properties of S_{local}.

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2. Preliminaries

This section presents basic facts about Young tableaux, Poissonization and determinantal point processes. Some material from the Introduction is repeated for convenience.

2.1. Gelfand-Tsetlin patterns.

A *semi-standard Young tableau* of shape $\lambda = (\lambda_1, \ldots, \lambda_M)$, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_M \geq 0$ are integers, is an insertion of numbers from $\{1, \ldots, M\}$ into the cells of the YD λ such that the entries weakly increase along each row and strictly increase along every column. It is important to emphasize that while for the definitions of Young diagrams and standard Young tableaux the value of M is not important, here the object substantially depends on M. Semi-standard Young tableaux (SSYTs) are in bijection with interlacing particle systems, often known as *Gelfand-Tsetlin patterns* (or *schemes*).

A Gelfand-Tsetlin pattern (GTP) with M rows is a triangular array of non-negative integers $[a(i,j)]$ with row i containing i entries $a(i,1), \ldots, a(i,i)$. The array satisfies the following order and interlacing constraints.

$$\text{Order \& Interlace :} \quad a(i,j) \geq a(i-1,j) \geq a(i,j+1) \ \text{ for every } i \text{ and } j.$$

Let $a^{(i)} = (a(i,1), \ldots, a(i,i))$ denote the i-th row of the GTP. Each row corresponds to a YD due to the order constraints. The interlacing conditions ensure that $a^{(i-1)} \preceq a^{(i)}$, and in fact, $a^{(i)} \setminus a^{(i-1)}$ is a horizontal strip which means that the cells in any row of $a^{(i)} \setminus a^{(i-1)}$ are to the left of the cells in the previous row. Figure 7 provides an example.

$$\begin{array}{ccccccc} 9 & & 5 & & 3 & & 2 \\ & 8 & & 5 & & 2 & \\ & & 6 & & 3 & & \\ & & & 5 & & & \end{array}$$

Figure 7. A Gelfand-Tsetlin pattern with 4 rows.

The set of GTPs with a fixed top row $a^{(M)}$ is in bijection with the set of SSYTs of shape $\lambda = a^{(M)}$. Indeed, given a GTP $[a(i,j)]$ with top row λ, such a tableaux is obtained by inserting the value i into the cells of $a^{(i)} \setminus a^{(i-1)}$ for every $1 \leq i \leq M$ (set $a^{(0)} = \emptyset$). If $a^{(i)} \setminus a^{(i-1)}$ is empty then i is not inserted. In the reverse direction, given a SSYT of shape λ, a GTP with top row λ is obtained by setting $a^{(i)}$ to be the YD consisting of the cells of λ with entries $\leq i$ and removing trailing zero rows to ensure that $a^{(i)}$ has i entries.

A GTP may also be represented as an interlacing particle system on $(\mathbb{Z}+\frac{1}{2})\times\mathbb{Z}$ as follows. Given a GTP $[a(i,j)]$ with M rows, the particle system $[\nu(i,j)]$ associated to it has M rows of particles, with particles on row i being placed on the horizontal line $\{y = i\}$ of the plane, and the position of the j-th particle on row i is

$$\Big(\nu(i,j),\, i\Big) = \Big(a(i,j) - j + \frac{1}{2},\, i\Big) \ \text{ for every } \ 1 \leq j \leq i.$$

The transformation $a(i,j) \to a(i,j) - j + \frac{1}{2}$ makes the order constraints strict and the interlacing constraints semi-strict:

$$\nu(i,j) \geq \nu(i-1,j) > \nu(i,j+1).$$

The *jumps* of an interlacing particle system ν with M rows is a set of points in $\mathbb{Z} \times \{1,\ldots,M-1\}$, defined inductively from the top row to the bottom as follows. Given two consecutive rows $[\nu(i,\cdot)]$ and $[\nu(i-1,\cdot)]$, the jumps on row i consist of particles at the positions

$$(k, i-1) \in \mathbb{Z}^2 \text{ for every integer } k \in [\nu(i-1,j), \nu(i,j)] \text{ and every } 1 \leq j \leq i-1.$$

In other words, the jumps of row i are placed on the horizontal line $\{y = i-1\}$ and fill out integers in the intervals $[\nu(i-1,j), \nu(i,j)]$ for every $1 \leq j \leq i-1$. Note that ν may determined from its top row and set of jumps.

2.2. **Poissonized Young tableaux.** For a YD λ, let $[0,1]^\lambda$ denote the set of functions from the cells of λ into $[0,1]$. Let $\mathrm{PYT}(\lambda)$ denote the set of all functions $T \in [0,1]^\lambda$ that satisfy the following tableau constraints.

$$\begin{aligned} &(1)\ T(i,j) \leq T(i,j+1) \text{ for every } (i,j) \text{ and } (i,j+1) \in \lambda, \\ &(2)\ T(i,j) \leq T(i+1,j) \text{ for every } (i,j) \text{ and } (i+1,j) \in \lambda. \end{aligned} \tag{2.1}$$

The *Poissonized tableau* (PYT) of shape λ is an element of $\mathrm{PYT}(\lambda)$. The Poissonized staircase shaped tableau of size $N = \binom{n}{2}$ is an element of $\mathrm{PYT}(\Delta_n)$.

Let $\mathcal{T}_\lambda$ denote a uniformly random element of $\mathrm{PYT}(\lambda)$. Then $\mathcal{T}_\lambda$ is related to a uniformly random SYT of shape λ in the following way. First, the entries of $\mathcal{T}_\lambda$ are distinct with probability 1. Given that, consider the random SYT $\mathbf{T}_\lambda$ obtained by inserting k into the cell that contains the k-th smallest element of $\mathcal{T}_\lambda$. Then $\mathbf{T}_\lambda$ is a uniformly random SYT of shape λ. In the other direction, $\mathcal{T}_\lambda$ can be generated by first sampling $\mathbf{T}_\lambda$, then independently sampling a uniformly random $Y \in [0,1]^\lambda$, and setting $\mathcal{T}_\lambda(i,j)$ to be the $\mathbf{T}_\lambda(i,j)$-th smallest entry of Y.

Throughout the paper, $\mathcal{T}_\lambda$ denotes a uniformly random element of $\mathrm{PYT}(\lambda)$ and $\mathbf{T}_\lambda$ denotes a uniformly random SYT of shape λ.

2.3. **Jumps of Poissonized tableaux and local limit.** Any $T \in \mathrm{PYT}(\lambda)$ can be represented as an interlacing particles system with a fixed top row in the following manner. Consider $0 \leq t \leq 1$ and let

$$\mathrm{YD}^{(t)} = \{(i,j) \in \Delta_n : T(i,j) \leq t\}.$$

The tableau constraints (2.1) ensure that $\mathrm{YD}^{(\mathrm{t})}$ is a YD for every t. Recall that a YD can be made to have an infinite number of rows by appending rows of size 0 after the last positive row. Encode λ as particle configuration on $\mathbb{Z} + \frac{1}{2}$ by placing a particle at position

$$\nu_j = \lambda_j - j + \frac{1}{2} \text{ for } j \geq 1. \tag{2.2}$$

This is an infinite particle configuration on $\mathbb{Z}+\frac{1}{2}$ such that $\nu_1 > \nu_2 > \cdots$ and $\nu_j - \nu_{j+1} = 1$ for $j > \ell(\lambda)$ (shown in Figure 4). Let $\nu^{(t)}$ be the particle configuration associated to $\mathrm{YD}^{(\mathrm{t})}$ via (2.2) and let $\nu = (\nu^{(t)}; 0 \le t \le 1)$ be the particle system on $(\mathbb{Z}+\frac{1}{2}) \times [0,1]$ with a particle at position (x,t) if and only if $x \in \nu^{(t)}$.

The particle system $\nu^{(t)}$ viewed in reverse time, i.e., from $t=1$ to $t=0$, can be interpreted as an ensemble of non-intersecting and non-increasing paths $p(i,u)$, for $1 \le i \le \ell(\lambda)$. Let $p(i,u)$ be the $(\mathbb{Z}+\frac{1}{2})$-valued path starting from $p(i,0) = \nu_i^{(0)}$ and decreasing an integer unit at the times $1-T(i,\lambda_i), 1-T(i,\lambda_i-1), \ldots, 1-T(i,1)$. If some of the entries are equal then $p(i,u)$ decreases by the number of consecutive equal entries. The paths should be left continuous so that the jumps occur immediately after the jump times. The path $p(i,u)$ decreases by λ_i units with final position $p(i,1) = -i+\frac{1}{2}$. Due to the columns of T being non-decreasing – condition (2) of (2.1) – the paths are *non-intersecting*: $p(i,u) > p(i+1,u)$ for every i and u. Figure 5 shows the paths associated to a staircase shaped PYT.

The *jumps* of $p(i,u)$ consists of points $(x,t) \in \mathbb{Z} \times [0,1]$ such that

(1) $1-t$ is a discontinuity point of $p(i,u)$, i.e., t equals some entry of T on row i.

(2) x in an integer in the interval $\big[p(i,(1-t)+),\, p(i,1-t)\big]$, where $p(i,u+) = \lim_{s\downarrow u} p(i,s)$.

The paths can be reconstructed from their jumps and initial positions. The *jumps* of ν, and also of T, is the (possibly) multiset of $\mathbb{Z} \times [0,1]$ defined by

$$X = \{(x,t) : (x,t) \text{ is a jump of some path } p(i,u)\}. \tag{2.3}$$

X may be a multiset because two adjacent paths may jump at the same time by amounts that causes some of their jumps to coincide. The coinciding jumps has be counted with multiplicity. However, if the entries of T are distinct then X is a simple set. The tableau can be reconstructed from its jumps and the initial position of the paths.

Let $\mathcal{X}_\lambda$ denote the jumps of a uniformly random element $\mathcal{T}_\lambda$ of $\mathrm{PYT}(\lambda)$. $\mathcal{X}_\lambda$ is simple almost surely since $\mathcal{T}_\lambda$ has distinct entries almost surely. Theorem 1.5 asserts that $\mathcal{X}_\lambda$ is a determinantal point process on $\mathbb{Z} \times [0,1]$.

2.4. **Determinantal point processes.** We describe some basic notions about point processes; for a thorough introduction see [B, DVe]. Let S be a locally compact Polish space. A *discrete subset* X of S is a countable multiset of S with no accumulation points. By identifying X with the measure $\sum_{x\in X} \delta_x$, the space of discrete subsets can be given the topology of weak convergence of Borel measures on S. This means that $X_n \to X_\infty$ if for every compact subset $C \subset S$, $\limsup_n \#(C \cap X_n) \le \#(C \cap X_\infty)$, where cardinality is taken with multiplicity.

A discrete set is simple if every point in it has multiplicity one. A point process on S is a Borel-measurable random discrete set of S. All point processes considered in this paper will be simple almost surely.

Throughout the paper we denote $\#_{\mathbb{Z}}$ to be counting measure on $\mathbb{Z}$ and $\mathcal{L}(A)$ to be Lebesgue measure on a measurable subset $A \subset \mathbb{R}$. Also, $\mu_1 \otimes \mu_2$ denotes the product of measures μ_1 and μ_2, and $\mu^{\otimes k}$ denotes the k-fold product of μ.

A *determinantal point process* $\mathcal{X}$ on S is a simple point process for which there is a correlation kernel $K : S \times S \to \mathbb{R}$, and a Radon measure μ on S, called the reference measure, with the following property. For every continuous $f : S^k \to \mathbb{R}$ of compact support,

$$\mathbb{E}\left[\sum_{\substack{(x_1,\ldots,x_k)\in\mathcal{X}^k \\ x_1,\ldots,x_k \text{ distinct}}} f(x_1,\ldots,x_k)\right] = \int_{S^k} \det\left[K(x_i,x_j)\right] f(x_1,\ldots,x_k)\, \mu^{\otimes k}(dx_1,\ldots,dx_k). \tag{2.4}$$

Expectations of the form given by the l.h.s. of (2.4) determine the law of $\mathcal{X}$ under mild conditions on K [Le]. This will be the case in this paper as the correlation kernels we consider will be continuous. If S is discrete then it is customary to take the reference measure to be counting measure. In this case $\mathcal{X}$ is determinantal if for every finite $A \subset S$,

$$\mathbb{P}\left[A \subset \mathcal{X}\right] = \det\left[K(x,y)\right]_{x,y\in A}.$$

Remark 2.1. The correlation kernel of a determinantal point process is not unique. If $\mathcal{X}$ is a determinantal point process with correlation kernel K then K may be replaced by $\frac{g(x)}{g(y)}K(x,y)$, for any non-vanishing function g, without changing determinants on the r.h.s. of (2.4). Thus the new kernel determines the same process. This observation will be used multiple times.

The determinantal point processes that we consider will be on spaces of the form $S = \mathbb{Z}\times\{1,\ldots,M\}$, or $S = \mathbb{Z}\times[0,1]$, or $S = \mathbb{Z}\times\mathbb{R}_{\geq 0}$, with reference measures being, respectively, counting measure, $\#_{\mathbb{Z}} \otimes \mathcal{L}[0,1]$ and $\#_{\mathbb{Z}} \otimes \mathcal{L}(\mathbb{R}_{\geq 0})$. The following lemma records some facts that will be used in deriving weak limits of determinantal point processes. We do not include the proof as it is rather standard; see [DVe, Le].

Lemma 2.2. *I) Let $\mathcal{X}_M$ be a determinantal point process on $\mathbb{Z}\times\{1,\ldots,M\}$ with correlation kernel K_M. For $x_1, x_2 \in \mathbb{Z}$ and $0 \leq t_1, t_2 \leq 1$, let*

$$k_M(x_1,t_1;x_2,t_2) = MK_M(x_1,\lceil Mt_1\rceil;x_2,\lceil Mt_2\rceil).$$

Suppose that $k_M \to k$ uniformly on compact subsets of $\mathbb{Z}\times(0,1)$. Then the point process

$$\mathcal{X}_M^{\text{scaled}} = \{(x,t/M) : (x,t) \in \mathcal{X}_M\}$$

restricted to $\mathbb{Z}\times(0,1)$ converges weakly to a determinantal point process $\mathcal{X}$ whose reference measure is $\#_{\mathbb{Z}} \otimes \mathcal{L}(0,1)$ and whose correlation kernel is k.

II) Let $\mathcal{X}_n$ be a determinantal point process on $\mathbb{Z}\times(0,1)$ with reference measure $\#_{\mathbb{Z}}\otimes\mathcal{L}(0,1)$ and correlation kernel K_n. For $c_n \in \mathbb{Z}$ and $\beta > 0$, define a point process on $\mathbb{Z}\times\mathbb{R}_{>0}$ by

$$\mathcal{X}_n^{\text{scaled}} = \left\{\big(x - c_n,\ \beta n(1-t)\big) : (x,t) \in \mathcal{X}_n\right\}.$$

The correlation kernel of $\mathcal{X}_n^{\text{scaled}}$ with reference measure $\#_{\mathbb{Z}}\otimes\mathcal{L}(\mathbb{R}_{>0})$ is

$$k_n(x_1,u_1;x_2,u_2) = (\beta n)^{-1}K_n\left(x_1+c_n, 1-\frac{u_1}{\beta n};\ x_2+c_n, 1-\frac{u_2}{\beta n}\right).$$

If $k_n \to k$ uniformly on compact subsets of $\mathbb{Z} \times \mathbb{R}_{>0}$ then $\mathcal{X}_n^{\text{scaled}}$ converges weakly to a determinantal point process $\mathcal{X}$ with reference measure $\#_{\mathbb{Z}} \otimes \mathcal{L}(\mathbb{R}_{>0})$ and correlation kernel k.

Extend $\mathcal{X}$ to a point process on $\mathbb{Z} \times \mathbb{R}_{\geq 0}$ without additional points. Then $\mathcal{X}_n^{\text{scaled}}$ converges weakly to $\mathcal{X}$ on $\mathbb{Z} \times \mathbb{R}_{\geq 0}$ if the points of $\mathcal{X}_n^{\text{scaled}}$ do not accumulate at the boundary in the sense that for every $x \in \mathbb{Z}$,

$$\lim_{\epsilon \to 0} \limsup_{n \to \infty} \mathbb{P}\left[\mathcal{X}_n^{\text{scaled}} \cap (\{x\} \times [0, \epsilon]) \neq \emptyset\right] = 0.$$

3. Determinantal representation of Poissonized tableaux

3.1. **Determinantal representation of discrete interlacing particle systems.** In order to prove Theorem 1.5 we use a determinantal description of discrete interlacing particle systems due to Petrov. This is the main tool behind the proof.

Let $\nu = (\nu(M,1) + \frac{1}{2} > \cdots > \nu(M,M) + \frac{1}{2})$ be a fixed particle configuration on $\mathbb{Z} + \frac{1}{2}$. Here we abuse notation from Section 2.1 to have the $\nu(M,j)$s be integers instead of half-integers. Let $\mathbb{P}_\nu$ be the uniform measure on all interlacing particle systems or, equivalently, GTPs as described in Section 2.1, with fixed top row ν. Let $\mathcal{X}_\nu$ be the point process of jumps of an interlacing particle system sampled according to $\mathbb{P}_\nu$, where the jumps are as described in Section 2.1.

Petrov [P, Theorem 5.1] proves that $\mathbb{P}_\nu$ is a determinantal point process on $(\mathbb{Z} + \frac{1}{2}) \times \{1, \ldots, M\}$ with an explicit correlation kernel. According to the notation there, particles live on $\mathbb{Z}$ but we have translated particle systems by $1/2$ so that the jumps are integral. In particular, in the notation of [P, Theorem 5.1], one has $N = M$, $x^{M,j} = \nu(M,j)$ and the variables x_1, x_2 take integer values. In [P, Section 6.1] it is explained that the point process of jumps, $\mathcal{X}_\nu$, is also determinantal on $\mathbb{Z} \times \{1, \ldots, M-1\}$ and its correlation kernel is given in terms of the correlation kernel of $\mathbb{P}_\nu$ in [P, Theorem 6.1] (up to the translation by $1/2$).

In particular, [P, Theorem 6.1] proves that the correlation kernel of the jumps is

$$K_{\mathcal{X}_\nu}(x_1, m_1; x_2, m_2) = (-1)^{x_2 - x_1 + m_2 - m_1} K_{\mathbb{P}_\nu}(x_1 - 1, m_1 + 1; x_2, m_2),$$

where $K_{\mathbb{P}_\nu}$ is the kernel presented in [P, Theorem 5.1]. The discussion there is in terms of lozenge tilings of polygonal domains using three types of lozenges as depicted in Figure 8. It is proved that the positions of any of the three types of lozenges in such a uniformly random tiling is a determinantal point process. The jumps of an interlacing particle system are given by the positions of the lozenges of the rightmost type from Figure 8, where as the particles themselves are given by the positions of lozenges of the leftmost type. Jumps occur when a lozenge of the leftmost type is glued along its bottom diagonal to a lozenge of the rightmost type; see [P, Figure 3] for such a tiling. By Remark 2.1, $(-1)^{x_2 - x_1 + m_2 - m_1} K_{\mathbb{P}_\nu}(x_1 - 1, m_1 + 1; x_2, m_2)$ defines the same determinantal point process as $K_{\mathbb{P}_\nu}(x_1 - 1, m_1 + 1; x_2, m_2)$, and we will use the latter kernel.

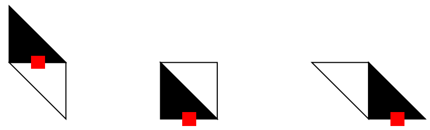

FIGURE 8. The 3 types of lozenges used in tiling polygonal domains that correspond to interlacing particle systems. The position of a lozenge is given by the midpoint of the horizontal side of the black triangular part (red square). The positions have integer coordinates. Particles correspond to positions of the leftmost lozenges but translated by 1/2 in the x-coordinate (in our notation). Jumps correspond to positions of the rightmost lozenges.

Some notation is needed in order to express the kernel for the point process of jumps. For integers a and b, let $C[a,b)$ denote a closed, counter-clockwise contour on $\mathbb{C}$ that encloses only the integers $a, a+1, \dots, b-1$ if $a > b$, and empty otherwise. Throughout the paper, all contours intersect the real line at points which have distance at least $1/10$ from the integers. This ensures that the integrands of all contour integrals will be a uniform distance away from their poles.

For $z \in \mathbb{C}$ and an integer $m \geq 1$, let

$$(z)_m = z(z+1)\cdots(z-m+1), \quad (z)_0 = 1.$$

Theorem 3.1 ([P, Theorem 5.1]). *The process of jumps, $\mathcal{X}_\nu$, of a uniformly random interlacing particle system with fixed top row $\nu = \left(\nu(M,1)+\frac{1}{2} > \nu(M,2)+\frac{1}{2} > \cdots > \nu(M,M)+\frac{1}{2}\right)$ is a determinantal point process on $\mathbb{Z} \times \{1,\dots,M-1\}$ with correlation kernel K as follows. For $x_1, x_2 \in \mathbb{Z}$ and $1 \leq m_1, m_2 \leq M-1$,*

(3.1)

$$K(x_1,m_1;\,x_2,m_2) = -\mathbf{1}_{\{m_2 \leq m_1,\, x_2 < x_1\}} \frac{(x_1-x_2)_{m_1-m_2}}{(m_1-m_2)!} +$$
$$+ \frac{1}{(2\pi\mathbf{i})^2} \oint_{C_z[x_2,\nu(M,1)+1)} dz \oint_{C_w[x_1-M,\nu(M,1)+1)} dw \left[\frac{(z-x_2+1)_{M-m_2-1}}{(w-x_1+1)_{M-m_1}} \cdot \frac{(M-m_1-1)!}{(M-m_2-1)!} \times \right.$$
$$\left. \times \frac{1}{w-z} \cdot \prod_{j=1}^{M} \frac{w-\nu(M,j)}{z-\nu(M,j)} \right].$$

The contour $C_w[-M,\nu(M,1)+1)$ contains $C_z[x_2,\nu(M,1)+1)$ without intersecting it.

3.2. **Proof of Theorem 1.5.** Let λ be a Young diagram with at most n rows of positive length, that is, $\ell(\lambda) \leq n$. For $M \geq n$, consider semi-standard Young tableaux of shape

$$\lambda_M = (\lambda_1, \dots, \lambda_n, \underbrace{0,\dots,0}_{M-n \text{ zeroes}}).$$

The effect of adding $M-n$ zero rows is to allow the entries in the non-zero rows of λ to be between 1 to M. The law of a uniformly random PYT of shape λ is the weak limit of a uniformly random semi-standard Young tableau of shape λ_M, as $M \to \infty$, after the entries are rescaled onto the interval $[0,1]$. Indeed, the law of a uniformly random PYT of shape λ can be approximated by the uniform distribution on points $[T(i,j)] \in [0,1]^\lambda$ that satisfy the tableau constraints (2.1), with each $T(i,j) = k/M$ for some $1 \le k \le M$, and the column constraints being strict.

The top row of particle systems associated to semi-standard Young tableaux of shape λ_M under the bijection described in Section 2.1 is

(3.2)
$$\nu_\lambda^{(M)} = \left((\lambda_1 - 1) + \frac{1}{2}, \dots, (\lambda_n - n) + \frac{1}{2}, -(n+1) + \frac{1}{2}, -(n+2) + \frac{1}{2}, \dots, -M + \frac{1}{2}\right).$$

Due to the approximation scheme above, and the bijection between semi-standard Young tableaux and interlacing particle systems discussed in Section 2.2, the process of jumps, $\mathcal{X}_\lambda$, of a uniformly random PYT of shape λ is the weak limit of the process of jumps, $\mathcal{X}_{\lambda_M}$, of a uniformly random interlacing particle system with top row $\nu^{(M)}$ after these jumps are rescaled onto $\mathbb{Z} \times \{1/(M-1), \dots, 1\}$. Thus, we derive a determinantal description of the rescaled jumps of $\mathcal{X}_{\lambda_M}$ in the large M limit. Since $\mathcal{X}_\lambda$ almost surely contains no jumps on the boundary $\mathbb{Z} \times \{0,1\}$, it suffices to derive the determinantal description with $\mathcal{X}_\lambda$ restricted to $\mathbb{Z} \times (0,1)$.

Let K_M denote the kernel from Theorem 3.1 for the process $\mathscr{X}_{\lambda_M}$. As explained in Remark 2.1, the kernel $(M-1)^{x_2 - x_1} K_M(x_1, m_1; x_2, m_2)$ determines the same determinantal point process. By Lemma 2.2, in order to prove the theorem it suffices to show that

$$(M-1)^{x_2 - x_1 + 1} K_M(x_1, \lceil (M-1)t_1 \rceil; x_2, \lceil (M-1)t_2 \rceil) \longrightarrow K_\lambda,$$

uniformly over compacts subsets of $x_1, x_2 \in \mathbb{Z}$ and $t_1, t_2 \in (0,1)$.

We begin by deforming the contours in the double contour integral from (3.1) that defines K_M. This will simplify the representation of K_M for taking the large M limit. Deform the w-contour, $C_w[x_1 - M, \nu_{\lambda_M}(M,1)+1)$, by pushing it leftward past the z-contour, $C_z[x_2, \nu_{\lambda_M}(M,1)+1)$, so that it encloses the consecutive integers $\min\{x_1, x_2\} - 1, \dots, x_1 - M$. The deformation results in picking up residues at $w = z$ and also at the consecutive integers $w = x_1 - 1, \dots, x_2$, if $x_2 < x_1$.

Let $J_M(w, x_1, m_1; z, x_2, m_2)$ denote the integrand of the double contour integral from (3.1) but without the factor of $1/(w-z)$. Note that $\nu_{\lambda_M}(M,1) + 1 = \lambda_1$. Calculating the residues at $w = z$ and leaving the remaining residues as a contour integral provides the following representation of K_M.

(3.3)
$$K_M(x_1, m_1; x_2, m_2) = -\mathbf{1}_{\{m_2 \le m_1,\, x_2 < x_1\}} \frac{(x_1 - x_2)_{m_1 - m_2}}{(m_1 - m_2)!} + \qquad (Ia)$$
$$+ \frac{1}{2\pi \mathbf{i}} \oint_{C_z[x_2, \lambda_1)} dz \, \frac{(z - x_2 + 1)_{M - m_2 - 1}}{(z - x_1 + 1)_{M - m_1}} \cdot \frac{(M - m_1 - 1)!}{(M - m_2 - 1)!} + \qquad (Ib)$$
$$+ \frac{1}{(2\pi \mathbf{i})^2} \oint_{C_z[x_2, \lambda_1)} dz \oint_{C_w[x_1 - M, \min\{x_1, x_2\}) \cup C_w[x_2, x_1)} dw \, \frac{J_M(w, x_1, m_1; z, x_2, m_2)}{w - z} \qquad (II).$$

The following lemma simplifies $(Ia) + (Ib)$.

Lemma 3.2. *For* $0 \le m_1, m_2 \le M-1$,

$$\frac{1}{2\pi i}\oint_{C_z[x_2,\lambda_1)} dz\, \frac{(z-x_2+1)_{M-m_2-1}}{(z-x_1+1)_{M-m_1}}\cdot\frac{(M-m_1-1)!}{(M-m_2-1)!} - \mathbf{1}_{\{m_2\le m_1,\, x_2<x_1\}}\frac{(x_1-x_2)_{m_1-m_2}}{(m_1-m_2)!}$$
$$= \mathbf{1}_{\{m_2>m_1,\, x_1>x_2\}}\frac{(m_1-m_2+1)_{x_1-x_2-1}}{(x_1-x_2-1)!}.$$

Proof. The integral (Ib) is evaluated in [P, Lemma 6.2] by summing over residues at $z = x_2, \ldots, x_1-1$ and evaluating the resulting sum in closed form via a hypergeometric identity. We have

$$(Ib) = \mathbf{1}_{\{x_1>x_2\}}\frac{(m_1-m_2+1)_{x_1-x_2-1}}{(x_1-x_2-1)!}.$$

Therefore,

$$(Ia)+(Ib) = \mathbf{1}_{\{x_1>x_2\}}\left[\frac{(m_1-m_2+1)_{x_1-x_2-1}}{(x_1-x_2-1)!} - \mathbf{1}_{\{m_2\le m_1\}}\frac{(x_1-x_2)_{m_1-m_2}}{(m_1-m_2)!}\right].$$

If $m_2 \le m_1$ then the above is 0 because both terms in the difference are equal to $\frac{(m_1-m_2+x_1-x_2-1)!}{(m_1-m_2)!(x_1-x_2-1)!}$. Hence, $(Ia)+(Ib)$ in non zero only if $x_1 > x_2$ and $m_2 > m_1$ and equals what is given in the statement of the lemma. □

Now consider the expression (II) from (3.3). Observe that the zeroes of $(w-x_1+1)_{M-m_1}$ are at consecutive integers $x_1-1, x_1-2, \ldots, x_1-(M-m_1)$. On the other hand, the polynomial $\prod_j (w-\nu_{\lambda_M}(M,j))$ also has zeroes at $\nu_{\lambda_M}(M,n+i) = -(n+i)$ for $1 \le i \le M-n$. Therefore, the only poles of J_M in the w variable are at the integers $x_1-1, x_1-2, \ldots, -n$ so long as $x_1+m_1 \ge 0$. For fixed x_1 and $m_1 \ge \sqrt{M}$, say, the condition $x_1+m_1 \ge 0$ is satisfied for all large M. Thus, the contour integral over $C_w[x_1-M, \min\{x_1,x_2\})$ may be shortened to $C_w[-n, \min\{x_1,x_2\})$ for all large M if x_1 remains fixed and $m_1 \ge \sqrt{M}$.

Having done so, (II) becomes the following integral after changing variables $z \mapsto z+x_2$ and $w \mapsto -w+x_1-1$:
(3.4)

$$(II) = \frac{1}{(2\pi\mathbf{i})^2}\oint_{C_z[0,\lambda_1-x_2)} dz \oint_{C_w[0,n+x_1)} dw\, \frac{(z+1)_{M-m_2-1}\,(M-m_1-1)!\,\prod_{j=1}^{M}\frac{-w+x_1-1+\nu_{\lambda_M}(M,j)}{z+x_2-\nu_{\lambda_M}(M,j)}}{(-w)_{M-m_1}\,(M-m_2-1)!\,(w+z+x_2-x_1+1)}.$$

By a slight abuse of notation, let J_M henceforth denote the integrand of (3.4) without the factor $1/(w+z+x_2-x_1+1)$. The following lemma provides the asymptotic form of J_M.

Lemma 3.3. *Fix integers* x_1 *and* x_2. *Suppose* $0 < t_1, t_2 < 1$ *and* w *and* z *remain bounded and have distance at least* $1/10$ *from the integers. Then for* $m_1 = \lceil t_1(M-1)\rceil$ *and* $m_2 =$

$\lceil t_2(M-1) \rceil$,

$$(M-1)^{x_1-x_2-1} J_M(w,x_1,m_1;z,x_2,m_2) = (1-t_1)^w (1-t_2)^z \, \frac{\Gamma(-w)\, G_\lambda(z+x_2)}{\Gamma(z+1)\, G_\lambda(x_1-1-w)} \, (1+O(M^{-1})).$$

The big O term is uniform over z, w so long as the stated assumptions hold and t_1, t_2 remain in compact subsets of $(0,1)$. The function G_λ is as stated in Theorem 1.5.

Proof. We will use the following identity:

$$(y)_m = \frac{\Gamma(y+m)}{\Gamma(y)}, \quad y \notin \{0,-1,-2,\ldots\}. \tag{3.5}$$

We will also use Stirling's approximation of the Gamma function in the following form:

$$\frac{\Gamma(y+m)}{(m-1)!} = m^y\,(1+O(m^{-1})), \quad m \geq 1. \tag{3.6}$$

The big O term is uniform in m so long as y is bounded and bounded away from negative integers. Using these two properties, if $m_1 = \lceil t_1(M-1) \rceil$ and $m_2 = \lceil t_2(M-1) \rceil$ then

$$\frac{(z+1)_{M-m_2-1}}{(-w)_{M-m_1}} \cdot \frac{(M-m_1-1)!}{(M-m_2-1)!} = (1-t_1)^w (1-t_2)^z \frac{\Gamma(-w)}{\Gamma(z+1)} (M-1)^{z+w}\,(1+O(M^{-1})) \tag{3.7}$$

Now consider the term

$$\prod_{j=1}^{M} \left(w - \nu_{\lambda_M}(M,j)\right) = \prod_{j=1}^{n} (w - \lambda_j + j) \cdot (w+n+1)_{M-n}$$
$$= \prod_{j=1}^{n} (w - \lambda_j + j) \cdot \frac{\Gamma(w+M+1)}{\Gamma(w+n+1)}.$$

Applying (3.6) to $\Gamma(w+M+1)$ and $\Gamma(z+M+1)$ gives

$$\prod_j \frac{w - \nu_{\lambda_M}(M,j)}{z - \nu_{\lambda_M}(M,j)} = \frac{\Gamma(z+n+1)}{\Gamma(w+n+1)} \cdot \prod_{j=1}^{n} \frac{w-\lambda_j+j}{z-\lambda_j+j} \cdot M^{w-z}\,(1+O(M^{-1})).$$

Substituting in $z + x_2$ and $-w + x_1 - 1$ then gives

$$\prod_j \frac{-w+x_1-1-\nu_{\lambda_M}(M,j)}{z+x_2-\nu_{\lambda_M}(M,j)} = \frac{G_\lambda(z+x_2)}{G_\lambda(x_1-1-w)}\,(M-1)^{-(w+z)+x_1-x_2-1}(1+O(M^{-1})). \tag{3.8}$$

Combining (3.7) with (3.8) provides the desired conclusion of the lemma. □

We now prove that $(M-1)^{x_2-x_1+1} K_M$ converges to K_λ. Suppose x_i are fixed and $m_i = \lceil t_i(M-1) \rceil$ for $i = 1,2$ and $t_i \in [\delta, 1-\delta]$ for some $\delta > 0$. Recall that K_M is given in (3.3). For all sufficiently large values of M, Lemma 3.2 and then the identity (3.5) followed by the estimate (3.6) show that $(Ia)+(Ib)$ of (3.3) equals

$$(Ia)+(Ib) = \mathbf{1}_{\{m_2>m_1, x_1>x_2\}} \frac{(m_1-m_2+1)_{x_1-x_2-1}}{(x_1-x_2-1)!} = \mathbf{1}_{\{t_2>t_1, x_1<x_2\}} \frac{(t_1-t_2)^{x_1-x_2-1}}{(x_1-x_2-1)!} \cdot (M-1)^{x_1-x_2-1}(1+O(M^{-1})). \tag{3.9}$$

Now we consider (II) in the form given in (3.4) following the change of variables. Lemma 3.3 implies that as $M \to \infty$,

$$(M-1)^{x_2-x_1+1}\, J_M(w,x_1,t_1;z,x_2,t_2) \to (1-t_1)^w(1-t_2)^z \frac{\Gamma(-w)\, G_\lambda(z+x_2)}{\Gamma(z+1)\, G_\lambda(x_1-1-w)}. \tag{3.10}$$

The convergence is uniform over compact subsets of w and z so long as w and z are uniformly bounded away from the integers. For all large values of M the contours of integration of (II) become free of M, namely, $z \in C_z[0, \lambda_1 - x_2)$ and $w \in C_w[0, n+x_1)$. The contours may also be arranged such that they remain bounded away from the integers and $|w+z+x_2-x_1+1| \geq 1/10$ throughout, say. This implies that as $M \to \infty$, $(M-1)^{x_2-x_1+1} \cdot (II)$ converges to

$$\frac{1}{(2\pi \mathbf{i})^2} \oint\limits_{C_z[0,\lambda_1-x_2)} dz \oint\limits_{C_w[0,n+x_1)} dw\, (1-t_1)^w(1-t_2)^z \frac{\Gamma(-w)\, G_\lambda(z+x_2)}{\Gamma(z+1)\, G_\lambda(x_1-1-w)} \times \times \frac{1}{(w+z+x_2-x_1+1)}.$$

We have thus concluded that $(M-1)^{x_2-x_1+1}K_M$ converges to the kernel K_λ given in Theorem 1.5. Moreover, the estimates show that the converge is uniform over compact subsets of $\mathbb{Z} \times (0,1)$. Indeed, so long at $t_1, t_2 \in [\delta, 1-\delta]$ and $|x_1|, |x_2| \leq B$, the error term in the convergence is of order $O_{B,\delta}(M^{-1})$, by Lemma 3.3, because the double contours eventually become free of M and the integrand converges uniformly over the contours. Part (I) of Lemma 2.2 now implies that the rescaled process of jumps, $\mathcal{X}^{\text{scaled}}_{\lambda_M}$, converges weakly on $\mathbb{Z} \times (0,1)$ to a determinantal point process with kernel as given in Theorem 1.5. □

4. Bulk local limit of the jumps of Poissonized staircase shaped tableaux

In this section we prove that the point process $X_{\alpha,n}$ from (1.5) converges weakly to the point process $\mathcal{X}_{\text{edge}}$ from Definition 1.1. This is done in a two-step procedure. First, we prove in Theorem 4.1 that the limit of $X_{\alpha,n}$ is a determinantal point process whose kernel is given in terms of a double contour integral. Second, we identifying this kernel with the one from Definition 1.1 in Proposition 4.3.

Theorem 4.1. *The point process $\mathcal{X}_{\alpha,n}$ from (1.5) converges weakly to a limiting determinantal point process $\mathcal{X}_{\text{edge}}$ on $\mathbb{Z} \times \mathbb{R}_{\geq 0}$. The correlation kernel of $\mathcal{X}_{\text{edge}}$ with respect to* *reference measure* $\#_{\mathbb{Z}} \otimes \mathcal{L}(\mathbb{R}_{\geq 0})$ *is given as follows. For* $u_1, u_2 \in \mathbb{R}_{\geq 0}$ *and* $x_1, x_2 \in \mathbb{Z}$,

$$K_{\text{edge}}(x_1, u_1; x_2, u_2) = \mathbf{1}_{\{u_2 < u_1, x_2 < x_1\}} \frac{(u_2 - u_1)^{x_1 - x_2 - 1}}{(x_1 - x_2 - 1)!} + \frac{1}{(2\pi\mathbf{i})^2} \oint_{C_z[0,\infty)} dz \oint_{C_w[0,\infty)} dw \, \frac{\pi}{2} \cdot \frac{G(w; x_1, u_1)\, G(z; x_2, u_2)}{w + z + x_2 - x_1 + 1}, \quad \textit{where}$$

$$G(z; x, u) = \frac{u^z}{\Gamma(z+1) \sin(\frac{\pi}{2}(z + x_2))}. \tag{4.1}$$

The contours $C_z[0, \infty)$ *and* $C_w[0, \infty)$ *are unbounded, contain the non-negative integers but remain uniformly bounded away from them and are arranged such that* $w + z + x_2 - x_1 + 1$ *remains uniformly bounded away from 0. They may also be arranged such that their imaginary parts remain bounded and* C_z *contains* C_w.

The value of K_{edge} *when* u_1 *or* u_2 *equals 0 is to be understood in the sense of the limit as* u_1 *or* u_2 *tends to 0.*

Proof. The proof proceeds in two steps, each verifying the conditions of part (II) of Lemma 2.2. First, we will show that the correlation kernel of $\mathcal{X}_{\alpha,n}$ converges to K_{edge} uniformly on compact subsets of $x_1, x_2 \in \mathbb{Z}$ and $u_1, u_2 \in \mathbb{R}_{>0}$. Then we will argue that points of $\mathcal{X}_{\alpha,n}$ do not accumulate on the boundary $\mathbb{Z} \times \{0\}$ as $n \to \infty$.

Let $\beta = \sqrt{1 - \alpha^2}$. Part (II) of Lemma 2.2 and Theorem 1.5 imply that the correlation kernel of $\mathcal{X}_{\alpha,n}$ with reference measure $\#_{\mathbb{Z}} \otimes \mathcal{L}(\mathbb{R}_{\geq 0})$ is

$$K_n(x_1, u_1; x_2, u_2) = (\beta n)^{-1} K_{\Delta_n} \left(x_1 + c_n, 1 - \frac{u_1}{\beta n}; x_1 + c_n, 1 - \frac{u_1}{\beta n} \right).$$

The kernel $(\beta n)^{x_1 - x_2} K_n$ determines the same point process by Remark 2.1. Using part (II) of Lemma 2.2 it suffices to show that $(\beta n)^{x_1 - x_2} K_n$ converges uniformly over compact subsets of $\mathbb{Z} \times \mathbb{R}_{>0}$ to K_{edge} as $n \to \infty$ in order to deduce convergence of $\mathcal{X}_{\alpha,n}$ to $\mathcal{X}_{\text{edge}}$ on $\mathbb{Z} \times \mathbb{R}_{>0}$.

Let G_n denote the function $2^{n-1} G_{\Delta_n}$, where G_{Δ_n} is as in Theorem 1.5. Then

$$G_n(z) = \frac{2^{n-1} \Gamma(z + n + 1)}{\prod_{j=1}^{n-1} (z - n + 2j)} = \frac{\Gamma(z + n + 1)}{(\frac{z-n}{2} + 1)_{n-1}} = \frac{\Gamma(z + n + 1)\, \Gamma\left(\frac{z-n+2}{2}\right)}{\Gamma\left(\frac{z+n}{2}\right)}. \tag{4.2}$$

Substitute in $x_i + c_n$ for the variables x_i and $1 - (\beta n)^{-1} u_i$ for the variables t_i in K_{Δ_n}. Then,

$$K_n(x_1, u_1; x_2, u_2) = \mathbf{1}_{\{u_2 < u_1, x_2 < x_1\}} \frac{(u_2 - u_1)^{x_1 - x_2 - 1}}{(x_1 - x_2 - 1)!} (\beta n)^{x_2 - x_1} + \frac{1}{(2\pi\mathbf{i})^2} \oint_{C_z[0, n-1-c_n-x_2)} dz \oint_{C_w[0, n-1+c_n+x_1)} dw \, \frac{\Gamma(-w)\, G_n(z + x_2 + c_n)\, u_1^w\, u_2^z\, (\beta n)^{-w-z-1}}{\Gamma(z+1)\, G_n(x_1 - 1 + c_n - w)(w + z + x_2 - x_1 + 1)}. \tag{4.3}$$

Using the formula for G_n from (4.2) and applying the identity

$$\Gamma(1-y)\,\Gamma(y) = \frac{\pi}{\sin(\pi y)},\ y \notin \{0,-1,-2,\ldots\},\ \text{to}\ \ y = \frac{n-c_n-z}{2}\ \ \text{gives} \tag{4.4}$$

$$G_n(z+c_n) = \frac{\Gamma(z+n+c_n+1)}{\Gamma\left(\frac{n+c_n+z}{2}\right)\Gamma\left(\frac{n-c_n-z}{2}\right)} \cdot \frac{\pi}{\sin\left(\frac{\pi}{2}(n-c_n-z)\right)}. \tag{4.5}$$

The estimate for $\Gamma(z)$ from (3.6) along with the observation that $n \pm c_n = (1\pm\alpha)n + O(1) \to +\infty$ implies the following asymptotic bevaviour as $n \to \infty$. The symbol $\sim$ denotes a multiplicative term $1+O(n^{-1})$ where the big O error is uniform over z in compact subsets of $\mathbb{C} \setminus \mathbb{Z}$.

$$\Gamma\left(\frac{n \pm (c_n+z)}{2}\right) \sim \Gamma\left(\frac{n \pm c_n}{2}\right) \cdot \left(\frac{n\pm c_n}{2}\right)^{\pm \frac{z}{2}}, \tag{4.6}$$

$$\Gamma(n+c_n+z+1) \sim \Gamma(n+c_n) \cdot (n+c_n)^{z+1}. \tag{4.7}$$

Therefore,

$$\begin{aligned}\frac{\Gamma(n+c_n+z+1)}{\Gamma\left(\frac{n+c_n+z}{2}\right)\Gamma\left(\frac{n-c_n-z}{2}\right)} \cdot \frac{\Gamma\left(\frac{n+c_n+w}{2}\right)\Gamma\left(\frac{n-c_n-w}{2}\right)}{\Gamma(n+c_n+w+1)} &\sim \left(\frac{n+c_n}{n-c_n}\right)^{\frac{w-z}{2}} (1+\alpha)^{z-w}\, n^{z-w} \\ &\sim \left(\frac{1-\alpha}{1+\alpha}\right)^{\frac{z-w}{2}} (1+\alpha)^{z-w}\, n^{z-w} \\ &= (\beta n)^{z-w}.\end{aligned}$$

Substituting in $z+x_2$ for z and $-w+x_1-1$ for w in this estimate gives

$$\frac{G_n(z+x_2+c_n)}{G_n(-w+x_1-1+c_n)} \sim \frac{\sin\left(\frac{\pi}{2}(-w+x_1-1-n-c_n)\right)}{\sin\left(\frac{\pi}{2}(z+x_2-n-c_n)\right)} (\beta n)^{w+z+x_2-x_1+1}. \tag{4.8}$$

The error in the above estimate vanishes as $n \to \infty$ so long as w, z and the x_i lie in compact subsets of their respective domains. We also have that

$$\frac{\sin\left(\frac{\pi}{2}(n-c_n-w)\right)}{\sin\left(\frac{\pi}{2}(n-c_n-w)\right)} = \frac{\sin\left(\frac{\pi}{2}w\right)}{\sin\left(\frac{\pi}{2}z\right)} \quad \text{if}\ \ n-c_n\ \ \text{is even}.$$

Since $n-c_n$ is assumed to be even we conclude from (4.8) that so long w and z are bounded and remain uniformly bounded away from the integers then

$$\lim_{n\to\infty} (\beta n)^{x_1-x_2+1} \frac{G_n(z+x_2+c_n)(\beta n)^{-w-z-1}}{G_n(x_1+c_n-1-w)} = \frac{-\sin\left(\frac{\pi}{2}(w-x_1+1)\right)}{\sin\left(\frac{\pi}{2}(z+x_2)\right)}. \tag{4.9}$$

The above displays the pointwise limit of the part of the integrand from (4.3) that depends on n. In order to interchange the pointwise limit with the contour integral we must show that the integrand is bounded uniformly over n by a function that is integrable over the contours $z \in C_z[0, n-1-c_n-x_2)$ and $w \in C_w[0, n-1+c_n+x_1)$, also uniformly over n. Then we may apply the dominated convergence theorem.

Towards this end suppose z is such that (1) $|\Re(z)| \le 2n$, (2) $|\Im(z)|$ is uniformly bounded over n, say by 100, and (3) z remains bounded away from $\mathbb{Z}$ by distance at least $1/10$. In this case Stirling approximation to the Gamma function implies that modulus of the ratio of the l.h.s. of (4.6) to its r.h.s. is bounded above and below by exponential factors of uniform rate in $|\Re(z)|$. The same holds for the ratio of the l.h.s. of (4.7) to its r.h.s. That is, for some constant C,

$$e^{-C\,(|\Re(z)|+1)} \leq \left| \frac{\Gamma\left(\frac{n\pm(c_n+z)}{2}\right)}{\Gamma\left(\frac{n\pm c_n}{2}\right)\cdot\left(\frac{n\pm c_n}{2}\right)^{\pm\frac{z}{2}}} \right| \leq e^{C\,(|\Re(z)|+1)},$$

$$e^{-C\,(|\Re(z)|+1)} \leq \left| \frac{\Gamma\left(n+c_n+z+1\right)}{\Gamma\left(n+c_n\right)\cdot(n+c_n)^{z+1}} \right| \leq e^{C\,(|\Re(z)|+1)}.$$

Throughout the following C denotes a constant that is free of n but its value may change from line to line. We combine the estimates above with the equation for $G_n(z+c_n)$ from (4.5) and observe that there is a C such that $1/C \leq |\sin(n-c_n-z)| \leq C$ due to the assumptions on z. This in turn implies that there is a C such that

$$e^{-C\,(|\Re(z)|+1)} \leq \left| \frac{G_n(z+c_n)}{(\beta n)^z} \right| \leq e^{C\,(|\Re(z)|+1)}.$$

The contours $C_z[0, n-1-c_n-x_2)$ and $C_w[0, n-1+c_n+x_1)$ can certainly be arranged such that for fixed x_1 and x_2 the variables $z+x_2$ and $-w+x_1-1$ satisfy the aforementioned assumptions (1)–(3) uniformly over n. Thus, we get the following uniform estimate over n with $z \in C_z[0, n-1-c_n-x_2)$ and $w \in C_w[0, n-1+c_n+x_1)$:

$$\left| \frac{G_n(z+x_2+c_n)}{G_n(-w+x_1-1+c_n)} \right| \leq (\beta n)^{\Re(z+w)+x_2-x_1+1}\, e^{C\,(|\Re(z)|+|\Re(w)|+1)}.$$

The contours may also be arranged such that $|w+z+x_2-x_1+1| \geq 1/10$, say. Then the modulus of the integrand of the double contour integral from (4.3) satisfies

$$(\beta n)^{x_1-x_2} \left| \frac{\Gamma\left(-w\right) G_n(z+x_2+c_n)\, u_1^w\, u_2^z\, (\beta n)^{-w-z-1}}{\Gamma\left(z+1\right) G_n(x_1-1+c_n-w)\,(w+z+x_2-x_1+1)} \right| \leq \left| \frac{\Gamma\left(-w\right)}{\Gamma\left(z+1\right)} \right| \, |u_1^w|\, |u_2^z|\, e^{C\,(|\Re(z)|+|\Re(w)|+1)}. \tag{4.10}$$

Stirling's approximation implies that if $\Re(z) \geq 1/10$ and $|\Im(z)|$ remains bounded then

$$\frac{|u^z|}{|\Gamma\left(z+1\right)|} = e^{-\Re(z)\log\Re(z)+\Re(z)(\log u+O(1))}. \tag{4.11}$$

Applying (4.4) with $y = -w$ also gives $\Gamma\left(-w\right) = -\pi[\Gamma\left(w+1\right)\sin(\pi w)]^{-1}$. Note that $|\sin(\pi w)|^{-1} \leq C$ so long as w remains uniformly bounded away from the integers. Combining this with (4.11) shows that the r.h.s. of (4.10) is integrable over unbounded double contours $C_z[0,\infty) \ni z$ and $C_w[0,\infty) \ni w$ as long as the contours are arranged such that z, w remain uniformly bounded away from the integers, have uniformly bounded imaginary parts, and $z+w+x_2-x_1+1$ remains uniformly bounded away from 0. Thus, the limit

(4.9), upper bound (4.10) and the dominated convergence theorem implies that as $n \to \infty$

$$(\beta n)^{x_1-x_2} K_n(x_1,u_1;x_2,u_2) \to \mathbf{1}_{\{u_2<u_1, x_2<x_1\}} \frac{(u_2-u_1)^{x_1-x_2-1}}{(x_1-x_2-1)!} +$$

$$\frac{1}{(2\pi\mathbf{i})^2} \oint\limits_{C_z[0,\infty)} dz \oint\limits_{C_w[0,\infty)} dw\, u_1^w u_2^z \cdot \frac{-\Gamma(-w)\sin\left(\frac{\pi}{2}(w-x_1+1)\right)}{\Gamma(z+1)\sin\left(\frac{\pi}{2}(z+x_2)\right)} \cdot \frac{1}{w+z+x_2-x_1+1}.$$

Furthermore, our estimates show that the convergence is uniform over x_i in compact subsets of $\mathbb{Z}$ and u_i in compact subsets of $\mathbb{R}_{>0}$. (In fact, when some $u_i \to 0$ the integral contributes only through residues at the origin. Lemma 4.2 computes the limit as $u_i \to 0$.) Comparing the limit integrand with the one presented in (4.1) we observe that the proof of the kernel convergence will be complete once it is shown that

$$-\Gamma(-w)\sin\left(\frac{\pi}{2}(w-x_1+1)\right) = \frac{(\pi/2)}{\Gamma(w+1)\sin\left(\frac{\pi}{2}(w+x_1)\right)}.$$

From (4.4), $-\Gamma(-w) = \pi[\Gamma(w+1)\sin(\pi w)]^{-1}$. Also, $\sin(\frac{\pi}{2}(w-x+1)) = \cos(\frac{\pi}{2}(w-x))$. Finally, double angle trigonometric formulae imply $\sin(\pi w) = 2\sin(\frac{\pi}{2}(w+x))\cos(\frac{\pi}{2}(w-x))$. Substituting these equations into the l.h.s. of the above verifies the equality with the r.h.s.

To complete the proof of convergence of $\mathcal{X}_{\alpha,n}$ to $\mathcal{X}_{\text{edge}}$ on $\mathbb{Z}\times\mathbb{R}_{\geq 0}$ it is enough to show, using Lemma 2.2, that for every $x \in \mathbb{Z}$,

$$\lim_{\epsilon\to 0} \limsup_{n\to\infty} \mathbb{E}\left[\#\,\mathcal{X}_{\alpha,n} \cap (\{x\}\times[0,\epsilon])\right] = 0. \tag{4.12}$$

In other words, points do not accumulate at the boundary in the limit. From relation (2.4) for determinantal point processes and (4.3) we get

$$\mathbb{E}\left[\#\,\mathcal{X}_{\alpha,n} \cap (\{x\}\times[0,\epsilon])\right] = \int_0^\epsilon K_n(x,t;x,t)\,dt$$

$$= \frac{1}{(2\pi\mathbf{i})^2} \oint\limits_{C_z[0,n-1-c_n-x)} dz \oint\limits_{C_w[0,n-1+c_n+x)} dw\, \frac{\Gamma(-w)\,G_n(z+x+c_n)\left(\int_0^\epsilon t^{w+z}\,dt\right)(\beta n)^{-w-z-1}}{\Gamma(z+1)\,G_n(x-1+c_n-w)(w+z+1)}$$

$$= \frac{1}{(2\pi\mathbf{i})^2} \oint\limits_{C_z[0,n-1-c_n-x)} dz \oint\limits_{C_w[0,n-1+c_n+x)} dw\, \frac{\Gamma(-w)\,G_n(z+x+c_n)\,\epsilon^{w+z+1}\,(\beta n)^{-w-z-1}}{\Gamma(z+1)\,G_n(x-1+c_n-w)(w+z+1)^2}.$$

The quantity above is of the form $\epsilon I_{\epsilon,n}$. Arguing exactly as in the derivation of the limit kernel, $I_{\epsilon,n} \to I_\epsilon$, where I_ϵ is given by the double contour integral in the definition of $K_{\text{edge}}(x,\epsilon;x,\epsilon)$ from (4.1) but with an additional factor of $w+z+1$ in the denominator of the integrand. Indeed, $\epsilon I_\epsilon = \int_0^\epsilon K_{\text{edge}}(x,t;x,t)\,dt$, which is the expected number of points of $\mathcal{X}_{\text{edge}}$ on $\{x\}\times[0,\epsilon]$. The quantity I_ϵ remains uniformly bounded near $\epsilon = 0$ since, as $\epsilon \to 0$, the contribution to the integral that defines I_ϵ comes from the residues at $w, z = 0$ and these residues do not depend on ϵ. (See Lemma 4.2 where $\lim_{\epsilon\to 0} K_{\text{edge}}(x,\epsilon;x,\epsilon)$ is derived analogously.) Consequently, $\epsilon I_\epsilon \to 0$ as $\epsilon \to 0$ and thus the condition from (4.12) holds. □

4.1. **Integral representation of the edge kernel.** We begin with an auxiliary lemma.

Lemma 4.2. *Let $C_w[0,\infty)$ and $C_z[0,\infty)$ be contours as in the statement of Theorem 4.1 and let $G(z;x,u)$ be as in (4.1). For $t>0$ let*

$$I(t) = \frac{1}{(2\pi\mathbf{i})^2} \oint_{C_z[0,\infty)} dz \oint_{C_w[0,\infty)} dw \, \frac{\pi}{2} \cdot \frac{G(w;x_1,tu_1)\, G(z;x_2,tu_2))}{w+z+x_2-x_1+1}.$$

Then,

$$\lim_{t\to 0} I(t) = \begin{cases} \frac{2}{\pi}\,\frac{\cos(\frac{\pi}{2}x_1)\cos(\frac{\pi}{2}x_2)}{x_2-x_1+1} & \text{if } x_2 \neq x_1 - 1 \\ -\mathbf{1}_{\{x_1 \text{ even}\}} & \text{if } x_2 = x_1 - 1. \end{cases}$$

Moreover, I is continuously differentiable on $\mathbb{R}_{>0}$ and can be differentiated by interchanging differentiation with the contour integration.

Proof. The integrand of I is continuously differentiable in t. Observe that the contours of integration contain no singularities of the integrand, and in fact, are arranged to be a positive distance from all zeroes of $\sin(\frac{\pi}{2}(z+x_2))$ and $\sin(\frac{\pi}{2}(w+x_1))$ in the denominator. The estimate for $|u^z|/|\Gamma(z+1)|$ from (4.11) shows that the derivative of the integrand in the variable t is absolutely integrable over the contours as long as t lies in a compact subset of $\mathbb{R}_{\geq 0}$. Consequently, by the dominated convergence theorem, I is continuously differentiable and the derivative may be interchanged with integration.

Let us now consider the limiting value of $I(t)$ as $t\to 0$. Decomposing C_w as $C_w[0,1)\cup C_w[1,\infty)$, and similarly for C_z, gives

$$\oint_{C_z}\oint_{C_w} = \oint_{C_z[0]}\oint_{C_w[0]} + \oint_{C_z[0]}\oint_{C_w[1,\infty]} + \oint_{C_z[1,\infty]}\oint_{C_w[0]} + \oint_{C_z[1,\infty]}\oint_{C_w[1,\infty]}.$$

These four contours may also be arranged such that $\Re(w+z)>0$ unless both $w\in C_w[0]$ and $z\in C_z[0]$. Recall that

$$G(z;x,u) = \frac{u^z}{\Gamma(z+1)\sin(\frac{\pi}{2}(z+x))}.$$

Thus, the integrand of $I(t)$ converges to 0 as $t\to 0$ so long as $w\notin C_w[0]$ and $z\notin C_z[0]$. So each of the double contour integrals above except for the first has a limit value of 0 as $t\to 0$ (the limit operation may be interchanged with integration as argued above). To complete the proof it suffices to calculate

$$\lim_{t\to 0}\frac{1}{(2\pi\mathbf{i})^2}\oint_{C_z[0]} dz \oint_{C_w[0]} dw\,\frac{\pi}{2}\cdot\frac{G(w;x_1,tu_1)\,G(z;x_2,tu_2))}{w+z+x_2-x_1+1}. \tag{4.13}$$

The integral above is evaluated via residues at $w=0$ and $z=0$. If $x_2\neq x_1-1$ then (4.13) equals

$$\mathrm{Res}_{z=0}\left(\mathrm{Res}_{w=0}\left(\frac{(\pi/2)\,G(w;x_1,tu_1)\,G(z;x_2,tu_2)}{w+z+x_2-x_1+1}\right)\right) = \frac{2}{\pi}\frac{\cos(\frac{\pi}{2}x_2)\cos(\frac{\pi}{2}x_1)}{x_2-x_1+1}.$$

This is the limit value of $I(t)$ in the statement of the lemma for $x_2\neq x_1-1$.

Now consider the limit (4.13) in the case $x_2 = x_1 - 1$. As the contour $C_w[0]$ can be arranged to be contained inside $C_z[0]$, the integral in w equals the residue of the integrand at the only possible pole at $w = 0$. This equals

$$\mathrm{Res}_{w=0}\left(\frac{(\pi/2)\,G(w;x_1,tu_1)\,G(z;x_2,tu_2)}{w+z}\right) = \frac{\cos(\frac{\pi}{2}x_1)(tu_2)^z}{z\Gamma\,(z+1)\sin(\frac{\pi}{2}(z+x_2))}.$$

If x_1 is odd then the above equals 0. Otherwise, $\cos((\pi/2)x_1) = (-1)^{x_1/2}$ and the integral of the above over $C_z[0]$ is given by its residue at the pole $z = 0$ (note that $\sin(\frac{\pi}{2}x_2) \neq 0$ since $x_2 = x_1 - 1$ is odd). The residue equals

$$\mathrm{Res}_{z=0}\left(\frac{\cos(\frac{\pi}{2}x_1)(tu_2)^z}{z\Gamma\,(z+1)\sin(\frac{\pi}{2}(z+x_2))}\right) = \frac{\cos(\frac{\pi}{2}x_1)}{\sin(\frac{\pi}{2}x_2)} = -1.$$

Thus, if $x_2 = x_2 - 1$ then (4.13) equals $-\mathbf{1}_{\{x_1 \text{ even}\}}$ and this completes the proof. $\square$

Proposition 4.3. *The kernel K_{edge} has the following form.*

$$K_{\text{edge}}(x_1,u_1;x_2,u_2) = \begin{cases} \dfrac{2}{\pi}\displaystyle\int_1^0 t^{x_2-x_1}\cos\left(tu_1+\frac{\pi}{2}x_1\right)\cos\left(tu_2+\frac{\pi}{2}x_2\right)\,dt, & \text{if } x_2 \geq x_1; \\ -\dfrac{2}{\pi}\displaystyle\int_1^\infty t^{x_2-x_1}\cos\left(tu_1+\frac{\pi}{2}x_1\right)\cos\left(tu_2+\frac{\pi}{2}x_2\right)\,dt, & \text{if } x_2 < x_1. \end{cases}$$

Proof. For $t > 0$ let

$$f(t) = K_{\text{edge}}(x_1,tu_1;x_2,tu_2) = t^{x_1-x_2-1}\mathbf{1}_{\{u_1>u_2,x_1>x_2\}}\frac{(u_2-u_1)^{x_1-x_2-1}}{(x_1-x_2-1)!} + I(t),$$

where $I(t)$ is as defined in Lemma 4.2. By Lemma 4.2, f is continuous differentiable on $\mathbb{R}_{>0}$ and the function $t^{x_2-x_1+1}f(t)$ may be differentiated by interchanging differentiation with integration. Differentiating $t^{x_2-x_1+1}f(t)$ and clearing common powers of t gives

$$\begin{aligned}(x_2-x_1+1)f + tf' &= \frac{1}{(2\pi\mathbf{i})^2}\oint_{C_z[0,\infty)} dz \oint_{C_w[0,\infty)} dw\,\frac{\pi}{2}\cdot G(w;x_1,tu_1)\,G(z;x_2,tu_2)) \\ &= \frac{\pi}{2}\cdot\frac{1}{2\pi\mathbf{i}}\oint_{C_w[0,\infty)} dw\,G(w;x_1,tu_1)\cdot\frac{1}{2\pi\mathbf{i}}\oint_{C_z[0,\infty)} dz\,G(z;x_2,tu_2)).\end{aligned}$$

The contour integrals can be evaluated by summing over residues of G. Inside the contour $C_z[0,\infty]$, the function $G(z,x,u)$ has simple poles at integers z such that $\sin(\pi(z+x)/2) = 0$. In other words, z has to have the same parity as x. Let $\bar{x} = \mathbf{1}_{\{x\,\text{odd}\}}$. The residues of the integral come from integers $z = 2k+\bar{x}$ for $k \geq 0$. Since $\mathrm{Res}_{y=2k}(\sin(\pi y/2)^{-1}) = (2/\pi)(-1)^k$, summing over the residues gives

$$\frac{1}{2\pi\mathbf{i}}\oint_{C_z[0,\infty)} dw\,\frac{(tu)^z}{\Gamma(z+1)\sin(\frac{\pi}{2}(z+x))} = \frac{2(-1)^{\frac{x+\bar{x}}{2}}}{\pi}\sum_{k\geq 0}(-1)^k\frac{(tu)^{2k+\bar{x}}}{(2k+\bar{x})!}$$

$$= \frac{2(-1)^{\frac{x+\bar{x}}{2}}}{\pi}\cos\left(ut-\frac{\pi}{2}\bar{x}\right)$$

$$= \frac{2}{\pi}\cos\left(ut+\frac{\pi}{2}x\right).$$

Consequently,

$$(x_2-x_1+1)f(t)+tf'(t) = \frac{2}{\pi}\cos\left(u_1t+\frac{\pi}{2}x_1\right)\cos\left(u_2t+\frac{\pi}{2}x_2\right). \tag{4.14}$$

Multiplying (4.14) by $t^{x_2-x_1}$ then implies that

$$[t^{x_2-x_1+1}f]' = \frac{2}{\pi}\,t^{x_2-x_1}\cos\left(u_1t+\frac{\pi}{2}x_1\right)\cos\left(u_2t+\frac{\pi}{2}x_2\right). \tag{4.15}$$

For $x_2\geq x_1$, the r.h.s. of (4.15) is integrable over t in $[0,1]$. Moreover, $t^{x_2-x_1+1}f(t)\to 0$ as $t\to 0$ because $\lim_{t\to 0}f(t)=\lim_{t\to 0}I(t)$, and the latter limit is finite whereas $t^{x_2-x_1+1}\to 0$ as $t\to 0$. Therefore, (4.15) implies

$$f(1) = \frac{2}{\pi}\int_0^1 t^{x_2-x_1}\cos\left(u_1t+\frac{\pi}{2}x_1\right)\cos\left(u_2t+\frac{\pi}{2}x_2\right)\,dt.$$

Next, consider the case $x_2<x_1-1$. Now the relation from (4.15) should be integrated from 1 to ∞, which is convergent since $x_2-x_1\leq -2$. The formula follows so long as $\lim_{t\to\infty}t^{x_2-x_1+1}f(t)=0$. Rather than derive this limit we take a slightly indirect approach by considering the limit of $f(t)$ near $t=0$. For $t>0$ define

$$\begin{aligned} g(t) &= -\frac{2}{\pi}\int_1^\infty s^{x_2-x_1}\cos(u_1st+\frac{\pi}{2}x_1)\cos(u_2st+\frac{\pi}{2}x_2)\,ds \\ &= -\frac{2}{\pi}\,t^{x_1-x_2-1}\int_t^\infty s^{x_2-x_1}\cos(u_1s+\frac{\pi}{2}x_1)\cos(u_2s+\frac{\pi}{2}x_2)\,ds. \end{aligned} \tag{4.16}$$

Upon differentiating g it follows readily that g satisfies the same differential equation as f displayed in (4.14). Therefore, $f(t)=g(t)+C$ for some constant C. In order to identify C as zero it suffices to show that $\lim_{t\to 0}f(t)-g(t)=0$. Since $t^{x_1-x_2-1}\to 0$ as $t\to 0$, due to $x_2<x_1-1$, both $f(t)$ and $I(t)$ have the same limit as $t\to 0$. Thus, utilizing Lemma 4.2, showing $C=0$ amounts to proving

$$\lim_{t\to 0}g(t) = \frac{2}{\pi}\,\frac{\cos(\frac{\pi}{2}x_1)\cos(\frac{\pi}{2}x_2)}{x_2-x_1+1}.$$

The limit of $g(t)$ can be found using L'Hôspital's rule, which shows that

$$\lim_{t\to 0} g(t) = \lim_{t\to 0} \frac{-\frac{2}{\pi}\int_t^\infty s^{x_2-x_1}\cos(u_1 s + \frac{\pi}{2}x_1)\cos(u_2 s + \frac{\pi}{2}x_2)\,ds}{t^{x_2-x_1+1}}$$
$$= \lim_{t\to 0} \frac{\frac{2}{\pi}t^{x_2-x_1}\cos(u_1 t + \frac{\pi}{2}x_1)\cos(u_2 t + \frac{\pi}{2}x_2)}{(x_2-x_1+1)t^{x_2-x_1}}$$
$$= \frac{2}{\pi}\,\frac{\cos(\frac{\pi}{2}x_1)\cos(\frac{\pi}{2}x_2)}{x_2-x_1+1}.$$

We conclude that $g(t) = f(t)$ for $t \in \mathbb{R}_{>0}$, and in particular that $f(1) = g(1)$, as required.

Finally, consider the case $x_2 = x_1 - 1$. The r.h.s. of (4.15) is continuous for t in $[0,1]$ because one of $\cos(u_1 t + \frac{\pi}{2}x_1)$ or $\cos(u_2 t + \frac{\pi}{2}x_2)$ has a zero at $t = 0$ depending upon the parity of x_1. From Lemma 4.2, $\lim_{t\to 0} f(t) = \mathbf{1}_{\{u_1>u_2\}} - \mathbf{1}_{\{x_1 \text{ even}\}}$. Therefore, (4.15) implies that

$$f(1) = \mathbf{1}_{\{u_1>u_2\}} - \mathbf{1}_{\{x_1 \text{ even}\}} + \frac{2}{\pi}\int_0^1 t^{-1}\cos(u_1 t + \frac{\pi}{2}x_1)\cos(u_2 t + \frac{\pi}{2}x_2)\,dt. \tag{4.17}$$

We now express (4.17) as an integral over $t \in [1,\infty)$ as given in the proposition. First, note K_{edge} may be modified on the measure zero set consisting of $(x_1,u_1;x_2,u_2)$ such that $u_1 = u_2$ without changing determinants in (2.4), and thus, this does not affect the law of $\mathcal{X}_{\text{edge}}$. We will modify the kernel on this zero set after the following calculations to get the form given in the proposition.

Observe that $\cos(u_2 t + \frac{\pi}{2}x_2) = \sin(u_2 t + \frac{\pi}{2}x_1)$ if $x_2 = x_1 - 1$. Using trigonometric formulae the integrand of (4.17) becomes

$$\frac{2\cos(u_1 t + \frac{\pi}{2}x_1)\sin(u_2 t + \frac{\pi}{2}x_1)}{\pi t} = \begin{cases} \frac{\sin((u_1+u_2)t)+\sin((u_2-u_1)t)}{\pi t}, & x_1 \text{ even} \\ \frac{-\sin((u_1+u_2)t)+\sin((u_2-u_1)t)}{\pi t}, & x_1 \text{ odd.} \end{cases} \tag{4.18}$$

Using the fact that $\int_0^\infty \frac{\sin t}{t} = \frac{\pi}{2}$, we get that for $a \in \mathbb{R}$,

$$\int_0^1 \frac{\sin(at)}{\pi t}\,dt = \operatorname{sgn}(a)\int_0^{|a|}\frac{\sin t}{\pi t}\,dt = \frac{\operatorname{sgn}(a)}{2} - \int_1^\infty \frac{\sin(at)}{\pi t}\,dt. \tag{4.19}$$

Using (4.19) and the representation of the integrand in (4.18) we infer that if x_1 is even then

$$\mathbf{1}_{\{u_1>u_2\}} - \mathbf{1}_{\{x_1 \text{ even}\}} + \frac{2}{\pi}\int_0^1 t^{-1}\cos(u_1 t + \frac{\pi}{2}x_1)\sin(u_2 t + \frac{\pi}{2}x_1)\,dt =$$
$$\mathbf{1}_{\{u_1>u_2\}} - 1 + \frac{\operatorname{sgn}(u_1+u_2) + \operatorname{sgn}(u_2-u_1)}{2} - \int_1^\infty \frac{\sin((u_1+u_2)t) + \sin((u_2-u_1)t)}{\pi t}\,dt =$$
$$-\mathbf{1}_{\{u_1=u_2\}}(\frac{1+\mathbf{1}_{\{u_1=0\}}}{2}) - \frac{2}{\pi}\int_1^\infty t^{-1}\cos(u_1 t + \frac{\pi}{2}x_1)\sin(u_2 t + \frac{\pi}{2}x_1)\,dt.$$

This shows that (4.17) equals the expression given in the statement of the proposition for $x_2 = x_1 - 1$ and x_1 even except for the additive term $-\mathbf{1}_{\{u_1=u_2\}}(\frac{1+\mathbf{1}_{\{u_1=0\}}}{2})$. By modifying K_{edge} on the zero set $\{u_1 = u_2, x_2 = x_1 - 1, x_1 \text{ even}\}$ we may ignore this term.

For x_1 being odd we argue in the same manner as above to infer that (4.17) equals

$$\mathbf{1}_{\{u_1>u_2\}} - \mathbf{1}_{\{x_1 \text{ even}\}} + \frac{2}{\pi}\int_0^1 t^{-1}\cos(u_1 t + \frac{\pi}{2}x_1)\sin(u_2 t + \frac{\pi}{2}x_1)\,dt =$$
$$\mathbf{1}_{\{u_1>u_2\}} - \frac{\operatorname{sgn}(u_1+u_2)+\operatorname{sgn}(u_1-u_2)}{2} + \int_1^\infty \frac{\sin((u_1+u_2)t)+\sin((u_1-u_2)t)}{\pi t}\,dt =$$
$$= -\frac{\mathbf{1}_{\{u_1=u_2>0\}}}{2} - \frac{2}{\pi}\int_1^\infty t^{-1}\cos(u_1 t + \frac{\pi}{2}x_1)\sin(u_2 t + \frac{\pi}{2}x_1)\,dt.$$

Once again, we modify K_{edge} on the zero set $\{u_1 = u_2, x_2 = x_1 - 1, x_1 \text{ odd}\}$ to ignore the additive term $-\frac{1}{2}\mathbf{1}_{\{u_1=u_2>0\}}$ and get the expression of the kernel given in the proposition. □

4.2. **Statistical properties of $\mathcal{X}_{\text{edge}}$.** This section derives certain properties of $\mathcal{X}_{\text{edge}}$, namely, Proposition 4.4, Proposition 4.6, Lemma 4.8 and Lemma 4.9, that will be used to derive the local limit of staircase shaped tableaux and of sorting networks.

Proposition 4.4. *The process $\mathcal{X}_{\text{edge}}$ has the following statistical properties.*

I) Translation and reflection invariance: For any integer h the translated process

$$\mathcal{X}_{\text{edge}} + (2h, 0) = \{(x + 2h, u) : (x, u) \in \mathcal{X}_{\text{edge}}\}$$

and the reflected process

$$(-1, 1) * \mathcal{X}_{\text{edge}} = \{(-x, u) : (x, u) \in \mathcal{X}_{\text{edge}}\}$$

have the same law as $\mathcal{X}_{\text{edge}}$.

II) One dimensional marginals: For any $x \in \mathbb{Z}$ and $u_1, u_2 \in \mathbb{R}_{\geq 0}$,

$$K_{\text{edge}}(x, u_1; x, u_2) = \frac{\sin(u_1 - u_2)}{\pi\,(u_1 - u_2)} + (-1)^x \frac{\sin(u_1 + u_2)}{\pi\,(u_1 + u_2)}. \tag{4.20}$$

Therefore, $\mathcal{X}_{\text{edge}} \cap (\{x\} \times \mathbb{R}_{\geq 0})$ is a determinantal point process with reference measure $\mathcal{L}(\mathbb{R}_{\geq 0})$ and correlation kernel (4.20).

Proof. Part I) From Lemma 2.2 the correlation kernel of $\mathcal{X}_{\text{edge}} + (2h, 0)$ equals $K_{\text{edge}}(x_1 - 2h, u_1; x_2 - 2h, u_2)$. The integral representation of K_{edge} in Proposition 4.3 implies that

$$K_{\text{edge}}(x_1 - 2h, u_1; x_2 - 2h, u_2) = K_{\text{edge}}(x_1, u_1; x_2, u_2)$$

upon observing that $\cos(x + \pi h) = (-1)^h \cos(x)$, which implies that the integrands do not change after the kernel is transformed. Consequently, the translated point process has the same law as the original. Similarly, the correlation kernel for the reflected process is $K_{\text{edge}}(-x_1, u_1; -x_2; u_2) = (-1)^{x_1 - x_2} K_{\text{edge}}(x_2, u_2; x_1, u_1)$. The latter kernel defines the same determinantal point process as $\mathcal{X}_{\text{edge}}$ in law.

Part II) Proposition 4.3 gives that

$$K_{\text{edge}}(x, u_1; x, u_2) = \frac{2}{\pi} \int_0^1 \cos(tu_1 + \frac{\pi}{2}x) \cos(tu_2 + \frac{\pi}{2}x)\, dt$$
$$= \frac{1}{\pi} \left(\frac{\sin(tu_1 - tu_2)}{u_1 - u_2} + \frac{\sin(tu_1 + tu_2 + \pi x)}{u_1 + u_2} \right) \Bigg|_{t=0}^{t=1}$$
$$= \frac{1}{\pi} \left(\frac{\sin(u_1 - u_2)}{u_1 - u_2} + (-1)^x \frac{\sin(u_1 + u_2)}{u_1 + u_2} \right).$$

The fact hat $\mathcal{X}_{\text{edge}} \cap (\{x\} \times \mathbb{R}_{\geq 0})$ is determinantal with kernel as stipulated follows from the relation (2.4) for determinantal point processes. □

Lemma 4.5. *There is a universal constant C such that for $x_1, x_2 \in \mathbb{Z}$ and $u_1, u_2 \in \mathbb{R}_{\geq 0}$,*

$$|K_{\text{edge}}(x_1, u_1; x_2, u_2)| \leq \frac{C}{\max\{|x_1 - x_2|, |u_1 - u_2|\} + 1}. \tag{4.21}$$

Proof. Throughout this argument C denotes a universal constant whose value may change from line to line. We begin with the case $x_2 \neq x_1 - 1$. From the integral representation of K_{edge} we see that if $x_2 \geq x_1$ then

$$|K_{\text{edge}}(x_1, u_1; x_2, u_2)| = \frac{2}{\pi} \left| \int_0^1 t^{x_2 - x_1} \cos(tu_1 + \frac{\pi}{2}x_1) \cos(tu_2 + \frac{\pi}{2}x_2)\, dt \right| \tag{4.22}$$
$$\leq \frac{2}{\pi} \int_0^1 t^{x_2 - x_1}\, dt \leq \frac{C}{|x_2 - x_1| + 1}.$$

Similarly, if $x_2 < x_1$ then $x_2 \leq x_1 - 2$ and

$$|K_{\text{edge}}(x_1, u_1; x_2, u_2)| = \frac{2}{\pi} \left| \int_1^\infty t^{x_2 - x_1} \cos(tu_1 + \frac{\pi}{2}x_1) \cos(tu_2 + \frac{\pi}{2}x_2)\, dt \right| \tag{4.23}$$
$$\leq \frac{C}{|x_2 - x_1| + 1}.$$

Combining these bounds we deduce that if $x_2 \neq x_1 - 1$ then

$$|K_{\text{edge}}(x_1, u_1; x_2, u_2)| \leq \frac{C}{|x_1 - x_2| + 1}. \tag{4.24}$$

Now we consider decay in the u-variables, assuming that $x_2 \neq x_1 - 1$. Define $v(t)$ as

$$v(t) = \begin{cases} \dfrac{\sin(tu_1 - tu_2 + \frac{\pi}{2}(x_1 - x_2))}{\pi(u_1 - u_2)} + \dfrac{\sin(tu_1 + tu_2 + \frac{\pi}{2}(x_1 + x_2))}{\pi(u_1 + u_2)} & \text{if } u_1 \neq u_2 \\[2ex] \dfrac{t \cos(\frac{\pi}{2}(x_1 - x_2))}{\pi} + \dfrac{\sin(tu_1 + tu_2 + \frac{\pi}{2}(x_1 + x_2))}{\pi(u_1 + u_2)} & \text{if } u_1 = u_2. \end{cases}$$

Then $v'(t) = \frac{2}{\pi} \cos(tu_1 + \frac{\pi}{2}x_1) \cos(tu_2 + \frac{\pi}{2}x_2)$. Using that $|\sin(y)/y| \leq 1$, and the formula for $v(t)$, we observe that there is a C such that

$$|v(t)| \leq \frac{C}{|u_1 - u_2| + 1} \quad \text{if } |u_1 - u_2| \geq 1. \tag{4.25}$$

Applying integration by parts to the integral form of K_{edge} gives, for $x_2 \geq x_1$,

$$K_{\text{edge}}(x_1, u_1; x_2, u_2) = v(1) - v(0)\mathbf{1}_{\{x_1 = x_2\}} - \int_0^1 (x_2 - x_1)\, t^{x_2 - x_1 - 1}\, v(t)\, dt.$$

Now the triangle inequality and (4.25) imply that if $|u_1 - u_2| \geq 1$ then

$$\begin{aligned} |K_{\text{edge}}(x_1, u_1; x_2, u_2)| &\leq \frac{C}{|u_1 - u_2| + 1} + \frac{C\,|x_2 - x_1|}{|u_1 - u_2| + 1} \int_0^1 t^{x_2 - x_1 - 1}\, dt \\ &\leq \frac{2C}{|u_1 - u_2| + 1}. \end{aligned}$$

If $|u_1 - u_2| < 1$, then we use the bound (4.24) to reach the same conclusion as above. An entirely analogous bound holds when $x_2 < x_1$ because then $x_2 \leq x_1 - 2$, and $t^{x_2 - x_1}$ is integrable over $t \in [1, \infty)$. Therefore, for $x_2 \neq x_1 - 1$,

$$|K_{\text{edge}}(x_1, u_1; x_2, u_2)| \leq \frac{C}{|u_1 - u_2| + 1}. \tag{4.26}$$

Combining (4.24) with (4.26) implies the required inequality (4.21) for $x_2 \neq x_1 - 1$.

The case $x_2 = x_1 - 1$ requires some care. The representation (4.18) for the integrand of $K_{\text{edge}}(x_1, u_1; x_1 - 1, u_2)$ gives

$$\begin{aligned} K_{\text{edge}}(x_1, u_1; x_1 - 1, u_2) &= -\frac{2}{\pi} \int_1^\infty t^{-1} \cos(tu_1 + \frac{\pi}{2} x_1) \sin(tu_2 + \frac{\pi}{2} x_1)\, dt \\ &= -\int_1^\infty \frac{(-1)^{x_1} \sin((u_1 + u_2)t) + \sin((u_2 - u_1)t)}{\pi t}\, dt. \end{aligned} \tag{4.27}$$

Integration by parts and the triangle inequality imply that for $a \geq 1$,

$$\left| \int_1^\infty dt\, \frac{\sin(at)}{t} \right| = \left| \frac{\cos(a)}{a} - \int_1^\infty dt\, \frac{\cos(at)}{at^2} \right| \leq \frac{C}{a}.$$

For $0 \leq a \leq 1$, we have

$$\left| \int_1^\infty dt\, \frac{\sin(at)}{t} \right| = \left| \frac{\pi}{2} - \int_0^1 dt\, \frac{\sin(at)}{t} \right| \leq C + \int_0^1 dt\, a \leq C.$$

Together, these bounds imply that for $a \in \mathbb{R}$,

$$\left| \int_1^\infty dt\, \frac{\sin(at)}{t} \right| \leq \frac{C}{|a| + 1}. \tag{4.28}$$

Separating (4.27) naturally into two integrals and applying (4.28) implies that

$$|K_{\text{edge}}(x_1, u_1; x_1 - 1, u_2)| \leq \frac{C}{|u_1 - u_2| + 1}.$$

This establishes (4.21) for the case $x_2 = x_1 - 1$ and completes the proof. □

Spatial ergodicity of $\mathcal{X}_{\text{edge}}$. For $h \in \mathbb{Z}$, denote by τ^h the translation that maps $(x, u) \mapsto (x + 2h, u)$ for $(x, u) \in \mathbb{Z} \times \mathbb{R}_{\geq 0}$. So $\mathcal{X}_{\text{edge}}$ is invariant under the action of every τ^h by Proposition 4.4. An event E associated to $\mathcal{X}_{\text{edge}}$ is invariant if for every $h \in \mathbb{Z}$, $E = \tau^h E$, where $\tau^h E = \{\tau^h(\omega) : \omega \in E\}$ and $\tau^h(\omega)$ is the action of τ^h on a sample outcome ω of $\mathcal{X}_{\text{edge}}$. The invariant sigma-algebra of $\mathcal{X}_{\text{edge}}$ is the sigma-algebra $\mathscr{F}_{\text{inv}}$ consisting of all the invariant events.

Proposition 4.6. *$\mathcal{X}_{\text{edge}}$ is ergodic w.r.t. spatial translations in that if $E \in \mathscr{F}_{\text{inv}}$ then $\mathbb{P}[E] \in \{0, 1\}$.*

Proof. For $A \subset \mathbb{Z} \times \mathbb{R}_{\geq 0}$, let $\mathscr{F}(A) = \sigma(\mathcal{X}_{\text{edge}} \cap A)$ be the sigma-algebra generated by the points of $\mathcal{X}_{\text{edge}}$ restricted to A. For $A, B \subset \mathbb{Z} \times \mathbb{R}_{\geq 0}$, let

$$\text{dist}(A, B) = \inf \big\{ \max\{|x - y|, |u - v|\} : (x, u) \in A, (y, v) \in B \big\}.$$

For $k \geq 1$, suppose $f : (\mathbb{Z} \times \mathbb{R}_{\geq 0})^k \to \mathbb{R}$ is continuous and compactly supported. Let

$$N(f) = \sum_{\substack{(x_1,u_1),\ldots,(x_k,u_k) \in \mathcal{X}_{\text{edge}} \\ (x_i,u_i) \text{ all distinct}}} f(x_1, u_1; \cdots ; x_k, u_k).$$

Now suppose $f, g : (\mathbb{Z} \times \mathbb{R}_{\geq 0})^k \to \mathbb{R}$ are continuous and compactly supported such that there are disjoint subsets $A, B \subset \mathbb{Z} \times \mathbb{R}_{\geq 0}$ with $\text{support}(f) \subset A^k$ and $\text{support}(g) \subset B^k$. This implies that if $(x_1, u_1; \cdots ; x_k, u_k) \in \text{support}(f)$ and $(x_{k+1}, u_{k+1}; \cdots ; x_{2k}, u_{2k}) \in \text{support}(g)$, then $(x_i, u_i) \neq (x_{k+j}, u_{k+j})$ for every $1 \leq i, j \leq k$. We first show that in this case

$$|\mathbb{E}[N(f)N(g)] - \mathbb{E}[N(f)]\,\mathbb{E}[N(g)]| \leq \frac{(2k)!C^{2k}}{\text{dist}(A, B)^2 + 1}\, ||f||_1\, ||g||_1, \tag{4.29}$$

where C is the universal constant from Lemma 4.5 and $||f||_1$ is the L^1-norm of f with respect to $(\#_{\mathbb{Z}} \otimes \mathcal{L}(\mathbb{R}_{\geq 0}))^{\otimes k}$.

Indeed, the assumption on the supports of f and g imply from (2.4) that

$$\mathbb{E}[N(f)N(g)] = \int_{(\mathbb{Z}\times\mathbb{R}_{\geq 0})^{2k}} \det[K_{\text{edge}}(x_i, u_i; x_j, u_j)]_{1 \leq i,j \leq 2k}\, f(x_1, u_1; \cdots ; x_k, u_k) \times$$
$$g(x_{k+1}, u_{k+1}; \cdots ; x_{2k}, u_{2k})\, d\big(\#_{\mathbb{Z}} \otimes \mathcal{L}(\mathbb{R}_{\geq 0})\big)^{\otimes 2k}.$$

Let us expand the determinant of the $(2k) \times (2k)$ matrix above as a sum over all permutations. We break up the permutations into two types: permutations that map the subsets $\{1, \ldots, k\}$ and $\{k + 1, \ldots, 2k\}$ into themselves and those that do not. When summed over permutations of the first type the integral above equals $\mathbb{E}[N(f)]\,\mathbb{E}[N(g)]$. For a permutation σ of the second type, observe that there are two indices i and j, with $i \leq k$ and $j > k$, such that $\sigma(i) > k$ and $\sigma(j) < k$. Then for $\ell \in \{i, j\}$, Lemma 4.5 gives

$$|K_{\text{edge}}(x_\ell, u_\ell; x_{\sigma(\ell)}, u_{\sigma(\ell)})| \leq \frac{C}{\max\{|x_\ell - x_{\sigma(\ell)}|, |u_\ell - u_{\sigma(\ell)}|\} + 1} \leq \frac{C}{1 + \text{dist}(A, B)}.$$

For all other indices ℓ we have $|K_{\text{edge}}(x_\ell, u_\ell; x_{\sigma(\ell)}, u_{\sigma(\ell)})| \leq C$. Consequently, the term involving σ contributes at most $C^{2k-2}(1 + \text{dist}(A, B))^{-2}$ in absolute value to the determinant above for every $(x_1, u_1; \cdots ; x_k, u_k) \in \text{support}(f)$ and $(x_{k+1}, u_{k+1}; \cdots ; x_{2k}, u_{2k}) \in$

support(g). Since there are $(2k)! - (k!)^2$ such permutations σ, we conclude that

$$|\mathbb{E}[N(f)N(g)] - \mathbb{E}[N(f)]\,\mathbb{E}[N(g)]| \le \frac{(2k)!C^{2k}}{\mathrm{dist}(A,B)^2+1} \int\limits_{(\mathbb{Z}\times\mathbb{R}_{\ge 0})^{2k}} |fg|\, d\big(\#_{\mathbb{Z}} \otimes \mathcal{L}(\mathbb{R}_{\ge 0})\big)^{\otimes 2k}$$
$$= \frac{(2k)!C^{2k}}{\mathrm{dist}(A,B)^2+1} ||f||_1 ||g||_1.$$

Let $\mathscr{F}^k(A)$ be the sigma-algebra generated by the random variables $N(f)$, where $f : A^k \to \mathbb{R}$ is continuous and compactly supported. The bound (4.29) implies that if A and B are disjoint, X is $\mathscr{F}^k(A)$-measurable and Y is $\mathscr{F}^k(B)$-measurable, then,

$$|\mathbb{E}[XY] - \mathbb{E}[X]\,\mathbb{E}[Y]\,| \le \frac{(2k)!C^{2k}}{\mathrm{dist}(A,B)^2+1} \mathbb{E}[|X|]\,\mathbb{E}[|Y|]\ . \tag{4.30}$$

The bound in (4.30) implies ergodicity of $\mathcal{X}_{\mathrm{edge}}$ as follows. Let $E \in \mathscr{F}_{\mathrm{inv}}$. Given $0 < \epsilon < 1$, we may choose an event $E' \in \mathscr{F}^k([-n,n]\times\mathbb{R}_{\ge 0})$, for some k and n, such that $\mathbb{P}[E\Delta E'] < \epsilon$. Since $\mathcal{X}_{\mathrm{edge}}$ is invariant under τ^h, we have that $\mathbb{P}\left[\tau^h E \Delta \tau^h E'\right] = \mathbb{P}[E\Delta E']$ for every h. Therefore by the triangle inequality,

$$|\mathbb{P}\left[E' \cap \tau^h E'\right] - \mathbb{P}\left[E \cap \tau^h E\right]| \le \mathbb{P}\left[E'\Delta E\right] + \mathbb{P}\left[\tau^h E \Delta \tau^h E'\right] \le 2\epsilon.$$

Due to invariance of E this implies that $|\mathbb{P}\left[E' \cap \tau^h E'\right] - \mathbb{P}[E]\,| \le 2\epsilon$.

Set $h = n + m$ for an integer $m \ge 1$. Then $\tau^h E' \in \mathscr{F}^k([n+2m, 3n+2m]\times\mathbb{R}_{\ge 0})$. We now apply (4.30) with $A = [-n,n]\times\mathbb{R}_{\ge 0}$ and $B = [n+2m, 3n+2m]\times\mathbb{R}_{\ge 0}$, observing that $\mathrm{dist}(A,B) = 2m$. Since $\mathbb{P}\left[\tau^h E'\right] = \mathbb{P}[E']$ by translation invaraince, we infer that

$$\left|\mathbb{P}\left[E' \cap \tau^h E'\right] - \mathbb{P}\left[E'\right]^2\right| \le \frac{(2k)!C^{2k}}{4m^2}.$$

Since $|\mathbb{P}[E'] - \mathbb{P}[E]\,| \le \epsilon$, we conclude that

$$|\mathbb{P}[E] - \mathbb{P}[E]^2\,| \le \frac{(2k)!C^{2k}}{4m^2} + 5\epsilon.$$

Letting $m \to \infty$ followed by $\epsilon \to 0$ shows that $\mathbb{P}[E] = \mathbb{P}[E]^2$, as required. □

Remark 4.7. The proof above may be used to deduce that $\mathcal{X}_{\mathrm{edge}}$ is in fact space-time mixing.

Lemma 4.8. *Almost surely, $\mathcal{X}_{\mathrm{edge}}$ has an unbounded collection of points on every line $\{x\}\times\mathbb{R}_{\ge 0}$. In fact, the following holds. Let $N_x(t) = \#\big(\mathcal{X}_{\mathrm{edge}} \cap (\{x\}\times[0,t])\big)$. For every x, the sequence $N_x(t)/t \to 1/\pi$ in probability as $t \to \infty$.*

Proof. Fix $x \in \mathbb{Z}$. Using part (II) of Proposition 4.4 we see that for any interval $[a,b] \subset \mathbb{R}_{\ge 0}$,

$$\mathbb{E}[N_x([a,b])] = \int_a^b K_{\mathrm{edge}}(x,u,x,u)\,du = \frac{b-a}{\pi} + \frac{(-1)^x}{2\pi}\int_{2a}^{2b} du\, \frac{\sin u}{u}\,. \tag{4.31}$$

Observe from (4.31) that $\mathbb{E}[N_x(t)/t] = 1/\pi + O(1/t)$ as $t \to \infty$.

From (4.20) we see that $K_{\mathrm{edge}}(x,u_1;x,u_2)$ is symmetric in the variables u_1 and u_2. Thus,

$$\rho(u_1,u_2) := K_{\mathrm{edge}}(x,u_1;x,u_2)K_{\mathrm{edge}}(x,u_2;x,u_1) = K_{\mathrm{edge}}(x,u_1;x,u_2)^2 \geq 0.$$

From the relation (2.4) for determinantal point processes we have that

$$\begin{aligned}\mathbb{E}\left[N_x(t)\cdot(N_x(t)-1)\right] &= \int_0^t\int_0^t \det[K_{\mathrm{edge}}(x,u_i;x,u_j)]_{i,j=1,2}\,du_1du_2 \\ &= \left(\int_0^t K_{\mathrm{edge}}(x,u;x,u)\,du\right)^2 - \int_0^t\int_0^t \rho(u_1,u_2)\,du_1du_2 \\ &\leq \mathbb{E}\left[N_x(t)\right]^2.\end{aligned}$$

This inequality implies that $\mathrm{Var}(N_x(t)) \leq \mathbb{E}\left[N_x(t)\right]$. Since $\mathbb{E}\left[N_x(t)\right] = (t/\pi)+O(1)$, Chebyshev's inequality implies that for any $\epsilon > 0$,

$$\mathbb{P}\left[\left|\frac{N_x(t)}{t} - \frac{1}{\pi}\right| > \epsilon\right] = \frac{O(1)}{\epsilon^2\,t}.$$

This provides the claimed convergence in probability.

Convergence in probability implies that there is a sequence of times $t_k \to \infty$ such that $N_x(t_k)/t_k \to 1/\pi$ almost surely as $k \to \infty$. This in turn implies that there is an unbounded collection of points of $\mathcal{X}_{\mathrm{edge}}$ on $\{x\} \times \mathbb{R}_{\geq 0}$ almost surely. An union bound over x provides the claim in the lemma. □

Lemma 4.9. *The following event occurs almost surely. For every $t > 0$ there exists a doubly infinite sequence of integers x_i, $i \in \mathbb{Z}$, such that $\mathcal{X}_{\mathrm{edge}}$ contains no points on each of the segments $\{2x_i\} \times [0,t]$.*

Proof. By monotocity and an union (or rather intersection) bound over rational values of t, it suffices to show that the event occurs almost surely for every fixed $t > 0$. Given a fixed t, let X_i be the indicator of the event that $\mathcal{X}_{\mathrm{edge}}$ has no points on $\{2i\} \times [0,t]$. It suffices to show that almost surely infinitely many of the X_is equal 1 for $i \geq 0$. Then, reflection invariance of $\mathcal{X}_{\mathrm{edge}}$ and another union bound imply that almost surely a doubly infinite collection of the X_is are equal to 1, as required.

Due to translation invariance of $\mathcal{X}_{\mathrm{edge}}$ the sequence X_i, $i = 0,1,2,\ldots$, is stationary in that $(X_0, X_1, \ldots)$ has the same law as $(X_1, X_2, \ldots)$. It is also ergodic by Proposition 4.6. Therefore, by the Ergodic Theorem,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} X_i \;=\; \mathbb{P}\left[X_0 = 1\right], \text{ almost surely.}$$

The probability that $X_0 = 1$ is the probability that $\mathcal{X}_{\mathrm{edge}}$ has no points in $\{0\} \times [0,t]$. This is strictly positive by (6.2) below. As a result, an infinite number of the X_is equal 1 whenever the limit in the above holds. □

5. The local staircase shaped tableau

The local staircase shaped tableau, henceforth, local tableau, is a random function on

$$\Delta_\infty = \left\{(x,y) \in \mathbb{Z}^2 : y \geq 0,\, x \equiv y \pmod 2\right\}. \tag{5.1}$$

Figure 9 provides an illustration. The *rows* and *columns* of Δ_∞ are given by the diagonal lines

$$\text{row } 2x = \{(2x-k,k) : k \geq 0\}, \quad \text{column } 2x = \{(2x+k,k) : k \geq 0\}.$$

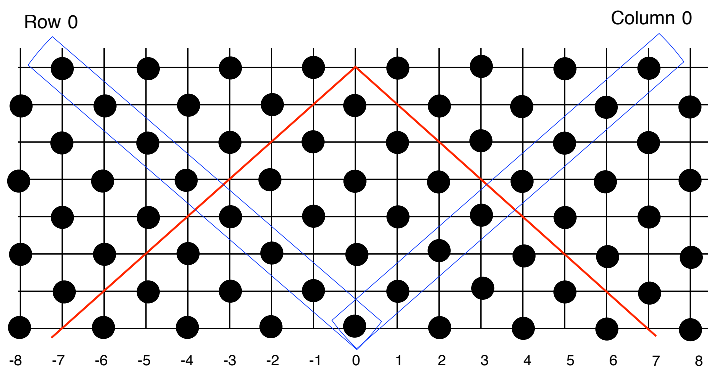

FIGURE 9. Black dots depicts cells of Δ_∞. Rows and columns are lines that start at even integers and go in the directions $(-1,1)$ and $(1,1)$, respectively. The region bounded by the red lines is the embedding of Δ_8 into Δ_∞.

In order to define tableaux on Δ_∞ and their convergence, we first explain the topology on $[0,\infty]$ since tableau entries will take values there (we allow the value ∞). The topology on $[0,\infty]$ is the usual topology on $\mathbb{R}_{\geq 0}$ extended in the natural way by stipulating that a sequence converges to ∞ if its values diverge to ∞, possibly stabilizing to the value ∞. In this case we will say that the sequence *grows to* ∞. For example, $1,2,3,4,\ldots$ grows to ∞, as does $1,\infty,2,\infty,\ldots$, as well as $1,\infty,\infty,\infty,\ldots$.

A *tableau* is a function $T : \Delta_\infty \to [0,\infty]$ such that it satisfies the *tableau constraints*

(5.2) I) $T(x,y) \leq \min\{T(x-1,y+1),\, T(x+1,y+1)\}$ for every $(x,y) \in \Delta_\infty$.

II) Along every row and column of T the entries grow to ∞.

The YD Δ_n embeds into Δ_∞ via $(i,j) \in \Delta_n \mapsto (j-i-\mathbf{1}_{\{n\,\text{odd}\}}, n-i-j) \in \Delta_\infty$. This is a rotation that puts row r of Δ_n on row $2(\lfloor n/2\rfloor - r)$ of Δ_∞; see Figure 9. In this manner any PYT T of shape Δ_n embeds as a tableau $F_T : \Delta_\infty \to [0,\infty]$ by setting

$$F_T(x,y) = \begin{cases} n\left(1 - T\left(\lfloor \frac{n}{2}\rfloor - \frac{x+y}{2}, \lfloor \frac{n}{2}\rfloor + \frac{x-y}{2}\right)\right), & \text{if } \left(\lfloor \frac{n}{2}\rfloor - \frac{x+y}{2}, \lfloor \frac{n}{2}\rfloor + \frac{x-y}{2}\right) \in \Delta_n; \\ \infty, & \text{otherwise.} \end{cases} \tag{5.3}$$

By an abuse of notation we denote F_T by T.

We say that a sequence of tableaux T_n converges if there is a tableau T such that, in the aforementioned topology on $[0,\infty]$, $T_n(x,y) \to T(x,y)$ for every $(x,y) \in \Delta_\infty$. Note we stipulate that a limit of tableaux remain a tableau.

A random tableau $\mathbf{T} : \Delta_\infty \to [0,\infty]$ is a Borel probability measure on tableaux with respect to the topology above. Convergence of a sequence of random tableaux means weak convergence with respect to this topology.

5.1. **Bulk local limit of staircase shaped tableaux.** Section 2.3 describes how PYTs of a given shape are in bijection with ensembles of non-increasing and non-intersecting paths whose initial positions are given in terms of the shape. We describe the bijection explicitly for tableaux defined on Δ_∞ as it will be useful in the proof of the local limit theorem.

Consider an ensemble of paths $\{p(2x,u)\}$, for $x \in \mathbb{Z}$ and $u \in \mathbb{R}_{\geq 0}$, that satisfy the following.

$$\begin{aligned} &\text{I) } p(x,\cdot) : \mathbb{R}_{\geq 0} \to \mathbb{Z} + \frac{1}{2} \text{ is left continuous, non-increasing with } p(x,0) = 2x + \frac{1}{2}. \\ &\text{II) } p(x,\cdot) \text{ are non-intersecting: } p(x,u) > p(x-1,u) \text{ for every } x \in \mathbb{Z}, u \in \mathbb{R}_{\geq 0}. \\ &\text{III) The jumps of the paths as defined by (2.3) is a discrete subset of } \mathbb{Z} \times \mathbb{R}_{\geq 0}. \end{aligned} \tag{5.4}$$

For a tableau $T : \Delta_\infty \to [0,\infty]$, paths satisfying (5.4) are obtained by setting $p(x,u) = 2x + \frac{1}{2}$ for $0 \leq u \leq T(2x,0)$, and for $k \geq 1$,

$$p(x,u) = p(x,0) - k \text{ if } T(2x-k+1,k-1) < u \leq T(2x-k,k).$$

In other words, $p(x,\cdot)$ is left continuous and decreases by integer units at times indexed by row $2x$ of T. The paths are non-intersecting due to the columns of T being non-decreasing. Indeed, $p(x,u) - p(x-1,u) = 2 + N_{x-1}(u) - N_x(u)$, where $N_x(u)$ is the number of entries of T on row $2x$ with value at most u. Due to the columns being non-decreasing, $N_{x-1}(u) \geq N_x(u) - 1$, and thus, $p(x,u) - p(x-1,u) \geq 1$. The jumps of the paths form a discrete set due to the rows and columns of T growing to ∞. When a row entry equals ∞ then the corresponding path jumps only a finite number of times.

Let X denote the jumps for the ensemble of paths associated to a tableau T on Δ_∞. The jumps can be read off from T in the following manner. For every $x \in \mathbb{Z}$, the jumps on the line $\{x\} \times \mathbb{Z}_{\geq 0}$ are the entries of T whose cells have first coordinate x in Δ_∞. More precisely, if u is the k-th smallest point of X on $\{x\} \times \mathbb{R}_{\geq 0}$ then

$$u = T\big(x,\, 2k - 1 - \mathbf{1}_{\{x\,\text{even}\}}\big). \tag{5.5}$$

If there are less than k points on $\{x\} \times \mathbb{R}_{\geq 0}$, there is no such u and the tableau entry above equals ∞.

To see this, observe that u is the time when the path starting at initial position $(x + \frac{1}{2}) + 2k - 1 - \mathbf{1}_{\{x\,\text{even}\}}$ jumps for the $(2k - \mathbf{1}_{\{x\,\text{even}\}})$-th time. Indeed, this is the k-th path starting at or to the right of position $x + \frac{1}{2}$ and it hits position $x - \frac{1}{2}$ after jump number $2k - \mathbf{1}_{\{x\,\text{even}\}}$. The first k jumps on $\{x\} \times \mathbb{R}_{\geq 0}$ are the times when the first k paths starting at or to the right of position $x + \frac{1}{2}$ hits position $x - \frac{1}{2}$. Also, when there is no such u it means that the path starting from $(x + \frac{1}{2}) + 2k - 1 - \mathbf{1}_{\{x\,\text{even}\}}$ has exhausted its jumps and it does not get to position $x - \frac{1}{2}$.

Let $M_{T\to X}$ denote the map from tableaux defined on Δ_∞ to jumps of paths satisfying (5.4). This map is invertible with the inverse given by the relation (5.5). Namely, $T(x, y)$ is the $\big[(y + 1 + \mathbf{1}_{\{x\,even\}})/2\big]$–th smallest jump of X on $\{x\} \times \mathbb{R}_{\geq 0}$ with the convention that $T(x, y) = \infty$ if no such jump exists. Let $M_{X\to T}$ denote the inverse map.

Lemma 5.1. *The set of jumps of paths satisfying (5.4) is closed in the topology on discrete subsets of $\mathbb{Z} \times \mathbb{R}_{\geq 0}$. The map $M_{T\to X}$ is a homeomorphism from the set of tableaux on Δ_∞ to the set of jumps of paths satisfying (5.4), with inverse given by $M_{X\to T}$.*

Proof. The maps $M_{T\to X}$ and $M_{X\to T}$ are inverses by design. We must show that they are continuous. We begin with continuity of $M_{T\to X}$. Let T_n be a sequence of tableaux such that T_n converges to a tableau T_∞. Let $X_n = M_{T\to X}(T_n)$ and $X_\infty = M_{T\to X}(T_\infty)$. Recall from Section 2.4 that convergence of X_n to X_∞ requires that for every $x \in \mathbb{Z}$ and $k \geq 1$, the k-th smallest point of X_n on $\{x\} \times \mathbb{R}_{\geq 0}$ must converge to the corresponding point of X_∞, while accounting for the case that there may be less than k points.

Let $y = 2k - 1 - \mathbf{1}_{\{x\,\text{even}\}}$. Then, by (5.5), $T_n(x, y)$ is the k-th smallest point of X_n on $\{x\} \times \mathbb{R}_{\geq 0}$ and similarly for $T_\infty(x, y)$. There are two cases: $T_\infty(x, y) < \infty$ or $T_\infty(x, y) = \infty$. In the former case, the k-th smallest point of X_n on $\{x\} \times \mathbb{R}_{\geq 0}$ is eventually finite and the same for X_∞. Moreover, we have convergence of these points since $T_n(x, y) \to T_\infty(x, y)$. In the latter case, given any bounded subset of $\{x\} \times \mathbb{R}_{\geq 0}$, the k-th smallest point of X_n eventually escapes the set or is non-existent due to $T_n(x, y) \to \infty$. This is as required since X_∞ has no k-th smallest point on $\{x\} \times \mathbb{R}_{\geq 0}$. This proves that $M_{T\to X}$ is continuous.

Now we show that the set of jumps of paths satisfying (5.4) is a closed set in the space of discrete subsets of $\mathbb{Z} \times \mathbb{R}_{\geq 0}$, as well as that that $M_{X\to T}$ is continuous. Suppose X_n is a sequence of such jumps sets and that it converges to a discrete subset X_∞.

Let $T_n = M_{X\to T}(X_n)$. First, we show that $T_n(x, y)$ converges for every $(x, y) \in \Delta_\infty$. Indeed, $T_n(x, y)$ is the k-th smallest point of X_n on $\{x\} \times \mathbb{R}_{\geq 0}$ for $k = (y + 1 + \mathbf{1}_{\{x\,\text{even}\}})/2$. Therefore, convergence of X_n to X_∞ implies that T_n must converge to some function $T_\infty : \Delta_\infty \to [0, \infty]$. Note that $T_\infty(x, y) = \infty$ if and only if X_∞ has less than k points on $\{x\} \times \mathbb{R}_{\geq 0}$.

The function T_∞ is a tableau because the tableau inequalities from (5.2) continue to hold in the entry-wise limit, and the rows and columns will grow to ∞ due to X_∞ being a discrete set. Thus, consider $\hat{X}_\infty = M_{T\to X}(T_\infty)$. By the first part of the proof, $X_n \to \hat{X}_\infty$. But then, $\hat{X}_\infty = X_\infty$ because limits of discrete subsets of $\mathbb{Z} \times \mathbb{R}_{\geq 0}$ are unique. This shows both the closure property of sets of jumps for paths satisfying (5.4) and the continuity of $M_{X\to T}$. □

We now state the local limit theorem for Poissonized staircase shaped tableaux in the bulk. For $\alpha \in (-1,1)$, let $c_n = 2(\lfloor \frac{n}{2} \rfloor - \lfloor \frac{(1+\alpha)n}{2} \rfloor)$. For a PYT T having shape Δ_n, embed it as a tableau on Δ_∞ according to (5.3) and consider the rescaled tableau $T_{\alpha,n} : \Delta_\infty \to [0,\infty]$ defined by

$$T_{\alpha,n}(x,y) = \sqrt{1-\alpha^2}\, T(x+c_n, y). \tag{5.6}$$

Let $\mathcal{T}_{\Delta_n}$ be a uniformly random PYT of shape Δ_n and denote by $\mathcal{T}_{\alpha,n}$ the random tableau associated to $\mathcal{T}_{\Delta_n}$ by (5.6).

Theorem 5.2. *The sequence of random Poissonized tableaux $\mathcal{T}_{\alpha,n}$ converges weakly to a random tableau $\mathcal{T}_{\text{edge}}$. Moreover, the law of $\mathcal{T}_{\text{edge}}$ is $M_{X\to T}(\mathcal{X}_{\text{edge}})$.*

Proof. Observe that $|c_n - \alpha n| \leq 2$ for every n. With this choice of c_n, the jump process associated to $\mathcal{T}_{\alpha,n}$ has law $\mathcal{X}_{\alpha,n}$ from (1.5) because these jumps are simply the jumps of $\mathcal{T}_{\Delta_n}$ rescaled onto $\mathbb{Z} \times \mathbb{R}_{\geq 0}$ as in (1.5). Theorem 4.1 asserts that $\mathcal{X}_{\alpha,n}$ converges weakly to $\mathcal{X}_{\text{edge}}$. Due to being a weak limit of the jumps of ensembles of paths satisfying (5.4), $\mathcal{X}_{\text{edge}}$ is also almost surely the jumps of such an ensemble of paths by the closure property given in Lemma 5.1. The continuity of $M_{X\to T}$ then implies that $M_{X\to T}(\mathcal{X}_{\alpha,n})$ converges weakly to $M_{X\to T}(\mathcal{X}_{\text{edge}})$. Therefore, $\mathcal{T}_{\alpha,n}$ converges weakly to a random tableau $\mathcal{T}_{\text{edge}}$ having the law of $M_{X\to T}(\mathcal{X}_{\text{edge}})$. □

Bulk local limit theorem for random staircase shaped SYT. Recall $\mathbf{T}_{\Delta_n}$ denotes a uniformly random SYT of shape Δ_n and $N = \binom{n}{2}$. Consider the rescaled tableau

$$\mathbf{T}^{\text{rsc}}_{\Delta_n}(i,j) = \frac{\mathbf{T}_{\Delta_n}(i,j)}{N+1}. \tag{5.7}$$

Theorem 5.3. *The random tableau $\mathbf{T}^{\text{rsc}}_{\Delta_n}$ converges to $\mathcal{T}_{\text{edge}}$ in the bulk local limit, that is, under the embedding and rescaling from* (5.6).

Proof. Consider the following coupling between $\mathbf{T}_{\Delta_n}$ and $\mathcal{T}_{\Delta_n}$. Given $\mathbf{T}_{\Delta_n}$, independently sample $P_{(1)} < P_{(2)} < \cdots < P_{(N)}$ according to the order statistics of N i.i.d. random variables distributed uniformly on $[0,1]$. Insert the entry $P_{(k)}$ into the cell of Δ_n that contains entry k of $\mathbf{T}_{\Delta_n}$. The resulting tableau has the law of $\mathcal{T}_{\Delta_n}$. Using this coupling, and due to the manner the scaling from (5.6) is defined, it suffices to show the following in order to conclude that $\mathbf{T}^{\text{rsc}}_{\Delta_n}$ converges to $\mathcal{T}_{\text{edge}}$ by way of Theorem 5.2.

Fix an $L > 0$ and consider any $(x,y) \in \Delta_\infty$ such that $(x,y) \in [-L,L] \times [0,L]$. Let $P_{(k)}$ be the entry of $\mathcal{T}_{\Delta_n}$ inside (x,y) under the embedding from (5.6). Then as $n \to \infty$, we need to show that

$$n \left| P_{(k)} - \frac{k}{N+1} \right| \longrightarrow 0 \text{ in probability.} \tag{5.8}$$

The number k is random, its distribution depends on $\mathbf{T}$ as well as n and α.

In order to establish (5.8) we will use the following auxiliary fact, which is a byproduct of [AHRV, Theorem 11]. There is a number δ_n of order $o(N)$ as $n \to \infty$, such that with probability tending to 1 as $n \to \infty$, every entry of $\mathbf{T}_{\Delta_n}$ within the cells of $\Delta_\infty \cap [-L, L] \times [0, L]$ under the embedding (5.6) has value at least $N - \delta_n$. As a consequence, $k \geq N - \delta_n$ with probability tending to 1. We write

$$\mathbb{P}\left[n\left|P_{(k)} - \frac{k}{N+1}\right| > \epsilon\right] \leq \frac{n^2}{\epsilon^2}\,\mathbb{E}\left[\left|P_{(k)} - \frac{k}{N+1}\right|^2 \;\Big|\; k \geq N - \delta_n\right] + \mathbb{P}\left[k < N - \delta_n\right]. \tag{5.9}$$

For a fixed deterministic j, $P_{(j)}$ has a Beta distribution with parameters j and $N+1-j$, which has mean $j/(N+1)$ and variance

$$\mathbb{E}\left[\left|P_{(j)} - \frac{j}{N+1}\right|^2\right] = \frac{j(N+1-j)}{(N+1)^2(N+2)}.$$

Since the $P_{(j)}$s are independent of $\mathbf{T}$, employing the bound above for $j \geq N - \delta_n$ and summing over the probabilities of k give

$$\mathbb{E}\left[\left|P_{(k)} - \frac{k}{N+1}\right|^2 \;\Big|\; k \geq N - \delta_n\right] \leq \frac{\delta_n}{N^2}.$$

The latter quantity is of order $o(1)/N$ as $n \to \infty$. Since $N = \binom{n}{2}$, we conclude that both terms on the right hand side of (5.9) tend to 0 as $n \to \infty$. □

Statistical properties of the local staircase shaped tableau. The set Δ_∞ can be made into a directed graph by putting directed edges from each vertex $(x, y) \in \Delta_\infty$ to the vertices $(x-1, y+1)$ and $(x+1, y+1)$. The automorphisms of this graph consists of translations ϕ_h, for $h \in \mathbb{Z}$, given by $\phi_h(x, y) = (x + 2h, y)$, as well as a reflection ϕ_- given by $\phi_-(x, y) = (-x, y)$. Tableaux are preserved by these automorphisms.

A random tableau $\mathbf{T}$ is *translation invariant* if $\mathbf{T} \circ \phi_h$ has the same law as $\mathbf{T}$ for every translation ϕ_h. The random tableau is reflection invariant if $\mathbf{T} \circ \phi_-$ has the same law as $\mathbf{T}$. The translation invariant sigma-algebra of $\mathbf{T}$ is the sigma-algebra of events that remain invariant under every translation:

$$\mathscr{F}_{\text{inv}} = \{\text{Events } E \text{ associated to } \mathbf{T} \text{ s.t. } \phi_h E = E \text{ for every } h \in \mathbb{Z}.\}.$$

(Recall that $\phi_h E = \{\omega \circ \phi_h : \omega \in E\}$.) We say $\mathbf{T}$ is ergodic under translations if $\mathscr{F}_{\text{inv}}$ is the trivial sigma-algebra.

Proposition 5.4. *The local tableau $\mathcal{T}_{\text{edge}}$ has the following statistical properties.*

1. *Almost surely, $\mathcal{T}_{\text{edge}}(x, y)$ is finite for every $(x, y) \in \Delta_\infty$ and the entries of $\mathcal{T}_{\text{edge}}$ are all distinct.*
2. *The law of $\mathcal{T}_{\text{edge}}$ is both translation and reflection invariant.*
3. *$\mathcal{T}_{\text{edge}}$ is ergodic under translations.*
4. *Almost surely, for every $t > 0$ there are infinitely many positive and negative $x \in \mathbb{Z}$ such that $T_{\text{edge}}(2x, 0) > t$.*

Proof. Almost surely, $\mathcal{X}_{\text{edge}}$ has an infinite and unbounded collection of points on every line $\{x\} \times \mathbb{R}_{\geq 0}$ by Lemma 4.8. Also, almost surely, $\mathcal{X}_{\text{edge}}$ does not contain two points of the form (x, u) and (y, u) with $x \neq y$. To see this, observe from the relation (2.4) for determinantal point processes that the expected number of such pairs of points in $\mathcal{X}_{\text{edge}}$ is 0 due to the set of such pairs having measure zero with respect to the measure $\big(\#\mathbb{Z} \otimes \mathcal{L}(\mathbb{R}_{\geq 0})\big)^{\otimes 2}$. When both these properties hold, $\mathcal{T}_{\text{edge}}$ satisfies (1).

The law of $\mathcal{T}_{\text{edge}}$ is invariant under translations because for every translation ϕ_h, the tableau $\mathcal{T}_{\text{edge}} \circ \phi_h$ is constructed from the jump process $\mathcal{X}_{\text{edge}} + (2h, 0)$, which has the same law of $\mathcal{X}_{\text{edge}}$ by Proposition 4.4. Similarly, reflection invariance of $\mathcal{T}_{\text{edge}}$ follows from reflection invariance of $\mathcal{X}_{\text{edge}}$. This establishes (2).

The ergodicty of $\mathcal{T}_{\text{edge}}$ under translations follows from the ergodicity of $\mathcal{X}_{\text{edge}}$ under translations (Proposition 4.6). This is because a translation invariant event for $\mathcal{T}_{\text{edge}}$ is the image of a translation invariant event for $\mathcal{X}_{\text{edge}}$ under the map $M_{X \to T}$. Finally, (4) is the statement of Lemma 4.9. □

6. Random sorting networks

6.1. **Sorting networks, Young tableaux and Edelman-Greene bijection.** Stanley [S] enumerated the number of sorting networks of $\mathfrak{S}_n$, which equals

$$\frac{\binom{n}{2}!}{\prod_{j=1}^{n-1}(2n-1-2j)^j}.$$

Following Stanley, Edelman and Greene [EG] provided an explicit bijection between sorting networks and staircase shaped SYT. An account of further combinatorial developments may be found in [Ga, HY]. We describe the part of the Edelman-Greene bijection that maps staircase shaped tableaux to sorting networks. The inverse map is a modification of the RSK algorithm; we do not describe it here since it is not used in the paper. See [EG] or [AHRV, Section 4] for a full description of the bijection.

Recall that a sorting network of $\mathfrak{S}_n$ is identified by its sequence of adjacent swaps $(s_1, \ldots, s_N)$, where $N = \binom{n}{2}$. For the rest of the paper we will use N to denote $\binom{n}{2}$. For $T \in \mathrm{SYT}(\Delta_\mathrm{n})$, we adopt the convention that $T(i,j) = -\infty$ if $(i,j) \notin \Delta_n$.

The Schützenberger operator. Let $(i_{\max}(T), j_{\max}(T))$ denote the cell containing the maximum entry of a SYT T. The Schützenberger operator $\Phi : \mathrm{SYT}(\Delta_n) \to \mathrm{SYT}(\Delta_n)$ is a bijection defined as follows. Given $T \in \mathrm{SYT}(\Delta_\mathrm{n})$, construct the *sliding path* of cells $c_0, c_1, \ldots, c_{d-1} \in \Delta_n$ iteratively in the following manner. Set $c_0 = (i_{\max}(T), j_{\max}(T))$ and $c_d = (1, 1)$. Then set

$$c_{r+1} = \operatorname{argmax}\big\{T(c_r - (1,0)),\, T(c_r - (0,1))\big\}.$$

Let $\Phi(T) = [\hat{T}(i,j)]$ where $\hat{T}(c_r) = T(c_{r+1}) + 1$ for $0 \leq r \leq d-1$, $\hat{T}(c_d) = 1$, and $\hat{T}(i,j) = T(i,j) + 1$ for all other cells $(i,j) \in \Delta_n \setminus \{c_0, \ldots, c_d\}$. Figure 10 provides an illustration.

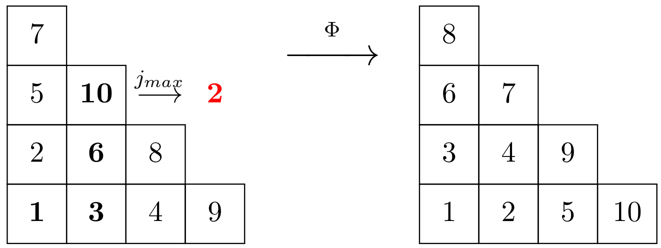

FIGURE 10. First step of the Edelman-Greene algorithm on a staircase shaped SYT of size 10. Sliding path is bolded. The associated sorting network of $\mathfrak{S}_5$ has swap sequence (2,4,3,1,2,1,4,3,2,4).

The *Edelman-Greene map* $\mathrm{EG} : \mathrm{SYT}(\Delta_\mathrm{n}) \mapsto \{\text{sorting networks of } \mathfrak{S}_\mathrm{n}\}$ is defined by

$$\mathrm{EG}(T) = \Big(j_{\max}\big(\Phi^k(T)\big)\Big)_{0\le k\le N-1}\,, \tag{6.1}$$

where Φ^k is the k-th iterate of Φ. Edelman and Greene [EG, Theorem 5.4] proved that EG indeed maps to sorting networks and that is has an inverse.

6.2. **First swap times of random sorting networks: proof of Corollary 1.3.** Let $T_{\mathrm{FS}}(s)$ be the first time the adjacent swap $(s, s+1)$ appears in a sorting network ω of $\mathfrak{S}_n$. According to the Edelman-Greene bijection, this time is recorded in the entry $(n-s, s)$ of $\mathrm{EG}(\omega)$. Thus,

$$T_{\mathrm{FS}}(s) = N + 1 - \mathrm{EG}(\omega)(n-s, s).$$

In terms of the rescaled tableau $\mathbf{T}^{\mathrm{rsc}}_{\Delta_n}$ from (5.7) we have that

$$\mathbf{T}_{\mathrm{FS}}(s) \ \overset{law}{=}\ (N+1)\cdot(1-\mathbf{T}^{\mathrm{rsc}}_{\Delta_n}(n-s,s)).$$

This implies the following for $\mathbf{T}_{\mathrm{FS},\alpha,n}$ – the first time an adjacent swap between $\lfloor\frac{n(1+\alpha)}{2}\rfloor$ and $\lfloor\frac{n(1+\alpha)}{2}\rfloor + 1$ appears in a random sorting network of $\mathfrak{S}_n$:

$$\frac{2\sqrt{1-\alpha^2}}{n}\,\mathbf{T}_{\mathrm{FS},\alpha,n} \ \overset{law}{=}\ \frac{2N+2}{n^2}\,\mathbf{T}^{\mathrm{rsc}}_{\alpha,\Delta_n}(0,0).$$

Here, $\mathbf{T}^{\mathrm{rsc}}_{\alpha,\Delta_n}$ is the tableau $\mathbf{T}^{\mathrm{rsc}}_{\Delta_n}$ rescaled and embedded into Δ_∞ according to (5.6). Theorem 5.3 implies that $\mathbf{T}^{\mathrm{rsc}}_{\alpha,\Delta_n}(0,0)$ converges weakly to $\mathcal{T}_{\mathrm{edge}}(0,0)$. Since $(2N+2)/n^2 \to 1$, we conclude that the rescaled $\mathbf{T}_{\mathrm{FS},\alpha,n}$ from above converges weakly to $\mathcal{T}_{\mathrm{edge}}(0,0)$. Thus, $\mathbf{T}_{\mathrm{FS}}$ has the law of $\mathcal{T}_{\mathrm{edge}}(0,0)$.

Now we explain how to get the distribution function of $\mathbf{T}_{\mathrm{FS}}$ given in (1.1). Observe that the event $\{\mathcal{T}_{\mathrm{edge}}(0,0) > t\}$ is the event $\{\mathcal{X}_{\mathrm{edge}} \cap (\{0\}\times[0,t]) = \emptyset\}$. The probability of the latter (often known as "gap probability") has the representation given by (1.1), which is the Fredholm determinant of K_{edge} over $L^2(\{0\}\times[0,t])$. This is a well-known property of determinantal point processes under the condition that the kernel be of trace class [DVe]. The kernel K_{edge} is of trace class on $L^2(\{0\}\times[0,t])$ simply because $|K_{\mathrm{edge}}(0,u_1;0,u_2)| \le 2/\pi$.

The asymptotic behaviour of the distribution function of $\mathbf{T}_{\mathrm{FS}}$ is well-known:

$$\log \mathbb{P}\left[\mathbf{T}_{\mathrm{FS}} > t\right] = -\frac{1}{4}t^2 - \frac{1}{2}t - \frac{1}{8}\log t + \frac{7}{24}\log 2 + \frac{3}{2}\zeta'(-1) + o(1) \text{ as } t \to \infty. \tag{6.2}$$

The formula (6.2) has a history. In theoretical physics literature, the leading term in (6.2) was first studied in [dCM], while the full expansion was given in [Dy]. The complete mathematical treatment was developed in [DIZ, Kr, E1, DIKZ, E2]; the present form of (6.2) is given in the last reference.

We will only need the simple corollary of (6.2) that $\mathbb{P}\left[\mathbf{T}_{\mathrm{FS}} > t\right] > 0$ for every t.

6.3. **Edelman-Greene algorithm on the local tableau.** The procedure described here is the same as the one given in the Introduction except that it is in the language of tableaux instead of their jumps. In order to define the Edelman-Greene algorithm on the local tableau we first introduce some concepts that allow us to define Edelman-Greene algorithm on tableaux defined on Δ_∞.

A *directed path* from $(x, y) \in \Delta_\infty$ to $(x', y') \in \Delta_\infty$ is a sequence of cells $c_0 = (x, y), c_1, \ldots, c_k = (x', y')$ of Δ_∞ such that $c_{i+1} - c_i \in \{(-1, 1), (1, 1)\}$ for every i. The cells of Δ_∞ can be partially ordered as follows: $(x, y) \leq (x', y')$ if there is a directed path from (x, y) to (x', y'). Recall that Δ_∞ is a directed graph with edges from (x, y) to $(x \pm 1, y + 1)$. It can also be thought of as an undirected graph by forgetting the direction of the edges. A *connected* subset of Δ_∞ is a connected subgraph of Δ_∞ in the undirected sense.

A *Young diagram* (YD) of Δ_∞ is a connected subset λ that is downward closed in the partial order, that is, if $(x, y) \in \lambda$ and $(x', y') \leq (x, y)$ then $(x', y') \in \lambda$. For example, Δ_n is a YD of Δ_∞. The *boundary* of λ, $\partial\lambda$, consists of cells $(x, y) \notin \lambda$ such that there is a directed edge from some cell $(x', y') \in \lambda$ to (x, y). The *peaks* of λ consists of the maximal cells of λ in the partial order.

Let $T : \Delta_\infty \to [0, \infty]$ be a tableau as in (5.2). A *sub-tableau* is the restriction of T to a YD λ; we say λ is the *support* of the sub-tableau. Let $T^{\mathrm{finite}} = \{T(x, y) : T(x, y) \neq \infty\}$. We take the support of T to be the support of T^{finite}. Observe that T^{finite} is a countable disjoint union of sub-tableaux of T, say $T_1, T_2, \ldots$. Indeed, the support of the T_is are the connected components of the subgraph spanned by cells (x, y) such that $T(x, y) \neq \infty$. We will call the T_is the *clusters* of T. The tableau T is *EG-admissible* if all the entries of T^{finite} are distinct and every cluster T_i is supported on a YD of **finite size**.

Edelman-Greene algorithm on a finite tableau. Let λ be a YD of Δ_∞ of finite size and $T : \lambda \to \mathbb{R}_{\geq 0}$ a tableau such that all its entries are distinct. The Edelman-Greene map EG takes as input T and outputs a triple $(x, t, \hat{T})$, where $x \in \mathbb{Z}$, $t \in \mathbb{R}_{\geq 0}$ and $\hat{T}$ is a sub-tableau.

The sliding path of T is a directed path $c_0, c_1, \ldots, c_k$ defined by

(1) $c_0 = \operatorname{argmin}\{\, T(x, y) : (x, y) \in \lambda\}$.

(2) $c_{i+1} = \operatorname{argmin}\{\, T(c_i + (-1, 1)), T(c_i + (1, 1))\,\}$.

(3) c_k = peak of λ obtained when both $c_k + (\pm 1, 1)$ belong to $\partial\lambda$.

Let $\hat{\lambda} = \lambda \setminus \{c_k\}$ and define $\hat{T} : \hat{\lambda} \to \mathbb{R}_{\geq 0}$ by

$$\hat{T}(x,y) = \begin{cases} T(x,y), & \text{if } (x,y) \in \hat{\lambda} \setminus \{c_0, \ldots, c_{k-1}\}; \\ T(c_{i+1}), & \text{if } (x,y) = c_i \text{ for some } 0 \leq i \leq k-1. \end{cases}$$

The cell c_0 must be on the bottom level of Δ_∞ and has the form $(2x, 0)$ for some $x \in \mathbb{Z}$. Set $t = T(c_0)$. The output is $\mathrm{EG}(T) = (x, t, \hat{T})$, and empty if T is the empty tableau.

The Edelman-Greene algorithm on T outputs a discrete subset $S(T) \subset \mathbb{Z} \times \mathbb{R}_{\geq 0}$, denoted the *swaps* of T. Let $(x_j, t_j, \hat{T}_j)$, for $1 \leq j \leq |\lambda|$, be defined iteratively by $(x_1, t_1, \hat{T}_1) = \mathrm{EG}(T)$ and $(x_j, t_j, \hat{T}_j) = \mathrm{EG}(\hat{T}_{j-1})$ for $2 \leq j \leq |\lambda|$. Then,

$$S(T) = \{(x_j, t_j) : 1 \leq j \leq |\lambda|\}. \tag{6.3}$$

If the cell $(x, y) \in \lambda$ contains the k-th smallest entry of T then its entry is removed during the k-th iteration of the algorithm. We will say that the entry at (x, y) *exits* at time t_k from row x_k. We will also say that (x_k, t_k) *originates* from cell (x, y).

Edelman-Greene algorithm on an admissible tableau. Let $T_1, T_2, \ldots$ be the clusters of an EG-admissible tableau T. Observe that for $i \neq j$, the swaps of T_i and T_j exit from mutually disjoint rows. Thus, the swap sets $S(T_1), S(T_2), \ldots$ are row-wise mutually disjoint. The swaps of T are defined as

$$S(T) = \bigcup_i S(T_i).$$

The local tableau $\mathcal{T}_{\text{edge}}$ is not EG-admissible. In order to define swaps for the local tableau we cut off large entries so that it becomes EG-admissible, and then process the tableau in a graded manner. For this to be successful, the EG algorithm ought to be consistent in the sense that running it on a tableau, and then restricting to swaps that originate from a sub-tableau, must produce the same outcome as the algorithm applied to the sub-tableau. This is not always the case and the following explains when it may be so.

Given two tableaux T_{small} and T_{big}, we say $T_{\text{small}} \leq T_{\text{big}}$ if the following criteria hold.

(1) $T_{\text{small}}(x, y) = T_{\text{big}}(x, y)$ for every $(x, y) \in \text{support}(T_{\text{small}})$.

(2) For every $(x, y) \in \text{support}(T_{\text{small}})$, and $(x', y') \in \text{support}(T_{\text{big}}) \setminus \text{support}(T_{\text{small}})$, if (x, y) belongs to the same cluster of T_{big} as (x', y') then $T_{\text{small}}(x, y) < T_{\text{big}}(x', y')$.

Lemma 6.1. *Let $T_{\text{small}} \leq T_{\text{big}}$, and suppose that T_{big} is EG-admissible. Then, applying the EG algorithm to T_{big} and restricting to the swaps that originate from the cells of T_{small} produces the same outcome as applying the EG algorithm to T_{small}. In particular, $S(T_{\text{small}}) \subset S(T_{\text{big}})$.*

Proof. Observe that the clusters of T_{small} are contained within the clusters of T_{big}. The EG algorithm acts independently on each cluster of T_{big} is a row-wise disjoint manner. Fix a particular cluster T of T_{big}, and suppose that the clusters of T_{small} that are contained inside T are $T_1, \ldots, T_k$. It suffices to prove that the EG algorithm applied to T, and then restricted to the swaps that originate from $T_1, \ldots, T_k$, produces the same outcome as the algorithm applied to each individual T_i.

Let $\lambda = \text{support}(T)$ and $\lambda_i = \text{support}(T_i)$. The assumption is that each entry of $\lambda \setminus (\cup_i \lambda_i)$ is larger than every entry of $\cup_i \lambda_i$. Therefore, the EG algorithm applied to T will process every entry of $\cup_i \lambda_i$ before it ever processes an entry from the complement. When some entry from $\lambda \setminus (\cup_i \lambda_i)$ enters a cell of some λ_i during the first $\sum_i |\lambda_i|$ steps, the algorithm treats that entry as if it were ∞. Since T_{small} agrees with T_{big} on $\cup_i \lambda_i$, the EG algorithm will output the swaps of $T_1, \ldots, T_k$ during the first $\sum_i |\lambda_i|$ steps, and then output the remaining swaps of $T \setminus (\cup_i T_i)$. This is what was claimed. □

A tableau T is *graded EG-admissible* if all of its finite-valued entries are distinct and, if for every $t > 0$, the sub-tableau

$$T^{\leq t} = \{T(x, y) : T(x, y) \leq t\} \text{ is EG-admissible.}$$

Observe that $T^{\leq t_1} \leq T^{\leq t_2}$ whenever $t_1 \leq t_2$. Lemma 6.1 thus implies that $S(T^{\leq t_1}) \subset S(T^{\leq t_2})$. Therefore, for a graded EG-admissible tableau T, we may define

$$S(T) = \bigcup_{t \geq 0} S(T^{\leq t}). \tag{6.4}$$

Lemma 6.2. *Suppose a sequence of tableaux $T_n \to T_\infty$, and also that every T_n and T_∞ are graded EG-admissible. Then for every integer x and $t \geq 0$, there is a finite YD λ that contains the cluster of $(2x, 0)$ in $T_n^{\leq t}$ for every n.*

Proof. This follows from a diagonalization argument, more precisely, König's infinity lemma, which states that every infinite connected graph with finite vertex degrees contains an infinite path.

Suppose for the sake of a contradiction that the conclusion of the lemma fails. Let $T_{n,x,t}$ denote the cluster of $(2x, 0)$ in $T_n^{\leq t}$. Call a cell $(x', y') \in \Delta_\infty$ bad if there is a undirected path in Δ_∞ from $(2x, 0)$ to (x', y') that is contained in infinitely many of the clusters $T_{n,x,t}$. Consider the connected component of $(2x, 0)$ in Δ_∞ that is spanned by the subgraph of bad vertices. If the component is finite then there is a finite YD λ that contains the component. This implies that for all sufficiently large n, every cell of $\partial\lambda$ lies outside $T_{n,x,t}$ because any path from $(2x, 0)$ to a cell outside λ must pass through $\partial\lambda$. Therefore, $T_{n,x,t} \subset \lambda$ for all large n. Since every T_n is graded EG-admissible, this means that there is a finite YD that contains every $T_{n,x,t}$, which is a contradiction.

Therefore, the connected component of $(2x, 0)$ spanned by the bad vertices is infinite. Since every vertex of Δ_∞ has degree at most 4, König's lemma provides an infinite path of (distinct) bad vertices $(x_0, y_0), (x_1, y_1), \ldots$ starting from $(x_0, y_0) = (2x, 0)$. By definition of being bad, for every m, there is a path from $(2x, 0)$ to (x_m, y_m) that is contained in some infinite subsequence of the clusters $T_{n_i^m,x,t}$ with $n_i^m \to \infty$ as $i \to \infty$. Let ℓ_m be the length of this path. Observe that $\ell_m \to \infty$ with m because the distance from $(2x, 0)$ to (x_m, y_m) in Δ_∞ must tend to infinity due to every vertex having degree at most 4.

The YD λ_m formed by the cells of Δ_∞ that are at or below the cells on the path from $(2x, 0)$ to (x_m, y_m) must be contained in every cluster $T_{n_i^m,x,t}$. Since T_n converges to T_∞, this implies that $\lambda_m \subset T_{\infty,x,t}$ for every m. Since $|\lambda_m| \geq \ell_m \to \infty$, we deduce that $T_{\infty,x,t}$ is infinite. However, this is a contradiction to T_∞ being graded EG-admissible. □

Theorem 6.3. *Suppose a sequence of tableaux $T_n \to T_\infty$, and that every T_n as well as T_∞ is graded EG-admissible. Then $S(T_n) \to S(T_\infty)$ as discrete subsets of $\mathbb{Z} \times \mathbb{R}_{\geq 0}$.*

Proof. A compact subset of $\mathbb{Z} \times \mathbb{R}_{\geq 0}$ is a finite, disjoint union of sets of the from $\{x\} \times C$ for $x \in \mathbb{Z}$ and compact $C \subset \mathbb{R}_{\geq 0}$. Therefore, we must show that for every such x and C,

$$\limsup_n \# \left[S(T_n) \cap (\{x\} \times C)\right] \leq \# \left[S(T_\infty) \cap (\{x\} \times C)\right].$$

Fix a $t > 0$ such that $C \subset [0, t]$.

Suppose T is a graded EG-admissible tableau. The swaps of T on $\{x\} \times [0, t]$ are the entries of $T^{\leq t}$ that exit from row x. Let $T_{x,t}$ denote the cluster of $(2x, 0)$ in $T^{\leq t}$. By Lemma 6.1, the swaps of T on $\{x\} \times [0, t]$ are completely determined by running the EG algorithm on $T_{x,t}$. We deduce from Lemma 6.2 that there is a finite YD λ such that

$$\text{support}(T_{n,x,t}) \subset \lambda \text{ for every } n \text{ and } \text{support}(T_{\infty,x,t}) \subset \lambda.$$

Since $\sup_{(x',y')\in\lambda} |T_n(x', y') - T_\infty(x', y')| \to 0$, we conclude that the following must occur for all sufficiently large n.

(1) The order of the entries of T_n on λ stabilizes to the order of the entries of T_∞ on λ.

(2) For every $(x', y') \in \lambda$, if $T_\infty(x', y') \notin C$ then $T_n(x', y') \notin C$.

Once condition (1) holds then, due to $T_{n,x,t} \subset \lambda$, a swap from $S(T_n)$ lies on $\{x\} \times C$ if and only if there is a cell $(x', y') \in \lambda$ such that $T_n(x', y') \in C$ and, when the EG algorithm is applied to T_∞ restricted to λ, the entry at cell (x', y') exits from row x. The same conclusion holds for swaps of $S(T_\infty)$ on $\{x\} \times C$. This property along with condition (2) implies that

$$S(T_n) \cap (\{x\} \times C) \subset S(T_\infty) \cap (\{x\} \times C) \text{ for all large } n.$$

This completes the proof. □

6.4. **Completing the proof of Theorem 1.2.** Theorem 1.2 will follow from Theorem 6.3 once we prove that the local tableau $\mathcal{T}_{\text{edge}}$ is graded EG-admissible almost surely. To this end, first observe that the entries of $\mathcal{T}_{\text{edge}}$ are finite and distinct by part (1) of Proposition 5.4. We must show that, almost surely, the clusters of $\mathcal{T}_{\text{edge}}^{\leq t}$ are finite for every t.

By part (4) of Proposition 5.4, the local tableau satisfies the following almost surely: for every t and x, there are integers $a, b \geq 0$ such that $\mathcal{T}_{\text{edge}}(2x - 2a, 0) > t$ and $\mathcal{T}_{\text{edge}}(2x + 2b, 0) > t$. When this property holds the tableau constraints imply that the cluster of $\mathcal{T}_{\text{edge}}^{\leq t}$

containing $(2x, 0)$ must be contained within cells whose row and column indices are both between $2x - 2a$ and $2x + 2b$. The set of such cells is finite, and so the cluster of every bottom level cell in $\mathcal{T}_{\text{edge}}^{\leq t}$ is finite. Now if $\mathcal{T}_{\text{edge}}(2x-k, k) \leq t$ then cell $(2x-k, k)$ belongs to the same cluster as $(2x, 0)$ in $\mathcal{T}_{\text{edge}}^{\leq t}$ since the row entries are non-decreasing. This implies that, almost surely, $\mathcal{T}_{\text{edge}}^{\leq t}$ is EG-admissible for every t, as required.

Finally, we complete the proof. The law of $S_{\alpha,n}$ is that of the Edelman-Greene algorithm applied to the rescaled uniformly random staircase shaped tableau $\mathbf{T}_{\Delta_n}^{\text{rsc}}$ from (5.7). Theorem 5.3 asserts that $\mathbf{T}_{\Delta_n}^{\text{rsc}}$ converges weakly to $\mathcal{T}_{\text{edge}}$ as a tableau embedded in Δ_∞. By Skorokhod's representation theorem, there exists random tableaux $\mathcal{T}_n$ and $\mathcal{T}_\infty$ defined on a common probability space such that $\mathcal{T}_n$ has the law of $\mathbf{T}_{\Delta_n}^{\text{rsc}}$, $\mathcal{T}_\infty$ has the law of $\mathcal{T}_{\text{edge}}$, and $\mathcal{T}_n \to \mathcal{T}_\infty$ almost surely.

The tableaux $\mathcal{T}_n$ and $\mathcal{T}_\infty$ are graded EG-admissible almost surely. Theorem 6.3 then implies that $S(\mathcal{T}_n)$ converges to $S(\mathcal{T}_\infty)$ almost surely. This means that $S_{\alpha,n}$, which has the law of $S(\mathcal{T}_n)$, converges weakly to $S(\mathcal{T}_{\text{edge}})$, which is the law of $S(\mathcal{T}_\infty)$. □

We conclude with some statistical properties of the local swap process.

Proposition 6.4. *The process S_{local} has the following properties.*

(1) S_{local} *is invariant under translations and reflection of the $\mathbb{Z}$-coordinate.*

(2) S_{local} *is stationary in time in that for every $t \geq 0$, the process $S_{\text{local}} \cap (\mathbb{Z} \times \mathbb{R}_{\geq t})$ has the same law as (shifted) S_{local}.*

(3) S_{local} *is ergodic under translations of the $\mathbb{Z}$-coordinate in that the sigma-algebra*

$$\mathscr{F}_{\text{inv}} = \{\textit{Events of } S_{\text{local}} \textit{ that are invariant under every translation}\} \quad \textit{is trivial.}$$

Remark 6.5. We believe that S_{local} is also ergodic in the time coordinate. However, the proof of this is more challenging and, therefore, we leave it as a conjecture.

Proof. We have that $S_{\text{local}} = S(\mathcal{T}_{\text{edge}})$ in law. Applying a $\mathbb{Z}$-automorphism to S_{local} is the same as first applying its analogue to $\mathcal{T}_{\text{edge}}$ (the maps ϕ_h and ϕ_-), and then applying the EG algorithm to the resulting tableau. Thus, the invariance of S_{local} under $\mathbb{Z}$-automorphisms follows from the corresponding invariance of $\mathcal{T}_{\text{edge}}$ stated in Proposition 5.4.

Time stationarity of S_{local} is a consequence of the stationarity of finite random sorting networks [AHRV, Theorem 1(i)], as we explain. If $(s_1, \ldots, s_N)$ is the sequence of swaps of a random sorting network of $\mathfrak{S}_n$, then $(s_1, \ldots, s_{N-1})$ has the same law as $(s_2, \ldots, s_N)$.

The ergodicity of S_{local} under $\mathbb{Z}$-translations is a consequence of the ergodicty of $\mathcal{T}_{\text{edge}}$ under translations (part 3 of Proposition 5.4). Indeed, a translation invariant event for S_{local} is the image of a translation invariant event of $\mathcal{T}_{\text{edge}}$ under the EG algorithm. □

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