NUMERICAL ANALYSIS OF CONSOLIDATION PROBLEMS

by

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Graduate Students

Abstract
of a thesis entitled

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For some time it has been realized that the analytical approach to solutions of many engineering problems necessarily limits the problems considerably in scope, and in some cases prejudices the practical significance of the conclusions drawn from the idealized analytical solutions. One of the earliest issues in soil mechanics to be examined mathematically was that of the settlement of clay soils under load, and since the solution to a simplified problem of this nature was obtained, little further practical advance has been made.

With the increasing interest of the engineer in approximate methods of analysis, these methods and their application to the consolidation of clay have been examined in this investigation. The methods of "iteration" have been developed and applied to a number of common problems of consolidation, some of which have not been critically examined previously because of their complexity, and approaches have been indicated to more advanced situations. Some examples have been investigated with a view to clarifying the text, but no exhaustive evaluation of the many problems outstanding has been attempted. One-, two-, and three-dimensional consolidation with the addition of radial drainage has been discussed, including various aspects of the lack of isotropy, and the presence of soils possessing differing properties.

It was concluded that in spite of the labor involved, dependable solutions to many practical problems could be obtained by these methods of successive approximation.

Thesis Supervisor: Harl P. Aldrich, Jr.
Title: Assistant Professor of Soil Mechanics

Cambridge, Massachusetts
May 25, 1953

Dr. Earl Bowman Millard
Secretary of the Faculty
Massachusetts Institute of Technology
Cambridge, Massachusetts

Dear Sir:

In partial fulfillment of the requirements for the degree of Master of Science in Civil Engineering at the Massachusetts Institute of Technology, this thesis entitled "Numerical Analysis of Consolidation Problems" is submitted.

Respectfully yours,

Ronald F. Scott

ACKNOWLEDGMENTS

The writer wishes to acknowledge the helpful advice and criticism which he received in the course of this investigation from Professors H. P. Aldrich, H. M. Paynter, and S. H. Crandall.

His thanks are also due to Miss E. T. Parker for her patience in the preparation of the typescript.

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NUMERICAL ANALYSIS OF CONSOLIDATION PROBLEMS.

INTRODUCTION.

The governing equation of one of the fundamental problems in soil mechanics, that for the variation of hydrostatic excess pressure with time, in the pore water of a saturated clay stratum under vertical load, was first formulated by K. von Terzaghi in 1925 (32)*, after certain simplifying assumptions had been made. The solution of this equation, which is a diffusion equation common to fluid, heat, and electrical flow problems, was obtained by the rigorous analytical method of mathematics. Apart from the soil properties, the assumptions involved one-dimensional pore water movement, drainage at the top and bottom surfaces of the clay layer, and instantaneous application of the load, which is also assumed to be initially distributed throughout the pore water alone, the pressure in the pore water due to the superimposed load being termed the hydrostatic excess pressure.

The assumptions appear very much idealised, but it has been found that in many cases in actual practice the time-settlement curve of the soil may be roughly predicted by the appropriate use of curves obtained from the analytical solution (31).

For cases other than those involving one-dimensional pore water flow, a few analytical solutions have been obtained, after the exercise of considerable mathematical dexterity and a great deal of computational labor. These solutions are limited to two-dimensional consolidation,

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* Numbers in parentheses refer to the Bibliography, Appendix I

of which M. A. Biot (4) and R. A. Barron (2) give examples. By an extension of the simple one-dimensional case solution, A. B. Newman (20) originally, and, later N. Carrillo (6) show the possibility of obtaining results for more complex two and three-dimensional problems. The complexity of Barron's solutions for cases of variable soil properties well illustrates the disadvantages of an analytical approach.

K. von Terzaghi (33) mentions that the consolidation curve for loading other than instantaneous can be drawn by the methods of superposition, but this cannot be applied to situations where release of load, or excavation, occurs, unless very approximate assumptions regarding the soil properties are made.

However, the very nature of compressible soil, with its inherent heterogeneity, tends to discourage the widespread development and application of rigorous analytical solutions. This was recognised by K. von Terzaghi (33) in suggesting the application of "Methods of Successive Approximation" (Southwell (29)) to problems in Soil Mechanics. Until 1949, however, when S. T. Yang completed a thesis (34) at Harvard University, little or no work was done along these lines, to the author's knowledge. Yang's study was solely related to seepage problems, and the solution of consolidation problems by approximate methods was first carried out by K. V. Helenelund (15) in 1951. Since then one other paper on consolidation problems has appeared in the Journal of the Institution of Civil Engineers in Great Britain, written by Gibson and Lumb in March 1953. (12)

These last two papers did not become known to the writer until the majority of his work on the numerical analysis of consolidation problems had been completed.

The following sections present the development and use of the equations and methods used in numerical analysis. It is demonstrated how the basic procedures can be extended to consider variable soil properties and stratification in one-and two-dimensional consolidation problems, together with the treatment of a variety of loading situations. With more labor than the simpler cases, the solution of three-dimensional consolidation problems can also be accomplished. A special case of two-dimensional consolidation, that of radially symmetrical pore water flow, will also be considered.

PART I

ONE-DIMENSIONAL CONSOLIDATION

(a) Derivation of Equations

The general equation (32) of three-dimensional consolidation of a clay stratum is

$$\frac{\partial u}{\partial t} = c_x \frac{\partial^2 u}{\partial x^2} + c_y \frac{\partial^2 u}{\partial y^2} + c_z \frac{\partial^2 u}{\partial z^2} \qquad \text{I-1}$$

It will be convenient in this part to treat only the case of one-dimensional consolidation as certain important aspects of the expressions and equations derived can be observed more clearly in the simpler form.

Thus the equation

$$\frac{\partial u}{\partial t} = c_z \frac{\partial^2 u}{\partial z^2} \qquad \text{I-2}$$

will be considered.

This equation represents the conditions of one-dimensional flow of pore water in a saturated clay soil under stress, and as such the equation may be considered to hold over a certain "region", or "domain", of the stratum being considered. A solution of this equation implies a knowledge of the hydrostatic excess pore-water pressure, u, at any point in the clay layer at an instant of time. The expression involves partial derivatives, and thus defines the movement of water in infinitesimally small elements of the soil-water medium. If, now, elements other than infinitesimal are considered, then the equation will no longer hold <u>exactly</u>, but any solution obtained may be good enough for soil engineering purposes in view of the nature of the assumptions involved in the derivation of the governing equation. This will be done in the following sections and such a solution will be termed a "numerical" solution, as distinct from an "analytical" solution.

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* A list of symbols will be found in Appendix II.

A small section of the region being considered is shown in Fig. I-1 and the meaning of the differential equation is investigated.

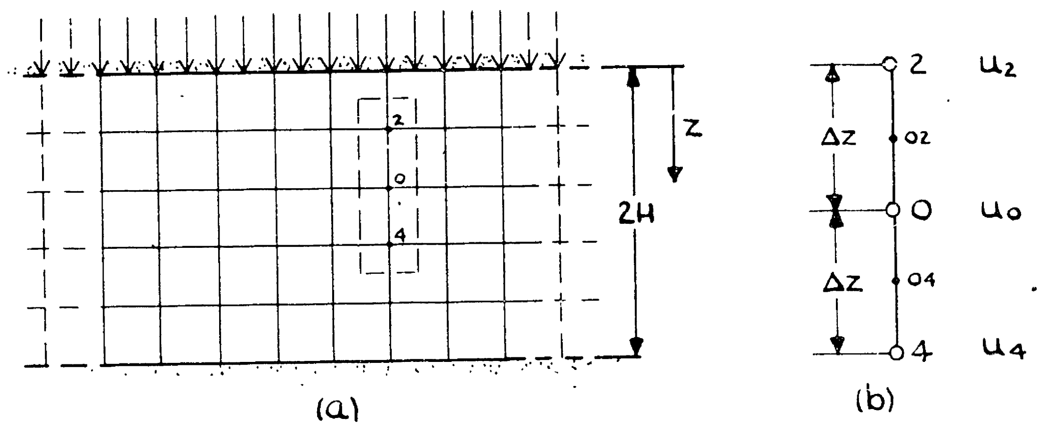


FIG. I-1

Fig. 1(a) demonstrates a homogeneous isotropic clay stratum subjected to a uniform surface load, and drained at top and bottom surfaces. The stratum is divided up into a number of horizontal layers of thickness Δz. At various points in the stratum the hydrostatic excess pore water pressure will be investigated and to this end Fig. 1(b) is given illustrating a section including any three such points from Fig. 1(a). Thus the domain or region is covered with a "mesh" or "grid" of points.

Then the hydrostatic excess pore pressure gradient $\left(\frac{\partial u}{\partial z}\right)_{02}$ at point 02, may be represented approximately by $\frac{u_2 - u_0}{\Delta z}$.

Similarly

$$\left(\frac{\partial u}{\partial z}\right)_{04} \approx \frac{u_0 - u_4}{\Delta z}$$

It follows that

$$\frac{\partial^2 u}{\partial z^2} \approx \frac{\left(\frac{\partial u}{\partial z}\right)_{02} - \left(\frac{\partial u}{\partial z}\right)_{04}}{\Delta z}$$

$$\approx \frac{\frac{u_2 - u_0}{\Delta z} - \frac{u_0 - u_4}{\Delta z}}{\Delta z}$$

$$\approx \frac{u_2 - 2u_0 + u_4}{\Delta z^2} \qquad \text{I - 3.}$$

When equation I-2 is considered it is seen that time is involved. Then it will be necessary to associate an hydrostatic excess pore-water pressure with a time, as well as a space co-ordinate, and to this end the notation is used that $u_{n,t}$ indicates the hydrostatic excess pore-water pressure at point n and at time t .

Then equation I-3 may be rewritten

$$\left(\frac{\partial^2 u}{\partial z^2}\right)_{0,t} \approx \frac{u_{2,t} - 2u_{0,t} + u_{4,t}}{\Delta z^2} \qquad \text{I - 3.}$$

Also $\frac{\partial u}{\partial t}$ may be written in finite difference (14) form as

$$\frac{\partial u}{\partial t} \approx \frac{u_{0,t+\Delta t} - u_{0,t}}{\Delta t} \qquad \text{I - 4.}$$

Combining I-3 and I-4 as indicated in equation I-2

$$\frac{u_{0,t+\Delta t} - u_{0,t}}{\Delta t} \approx c_z \frac{u_{2,t} - 2u_{0,t} + u_{4,t}}{\Delta z^2}$$

or

$$u_{0,t+\Delta t} \approx \frac{c_z \Delta t}{\Delta z^2}\left[u_{2,t} - 2u_{0,t} + u_{4,t}\right] + u_{0,t}$$

I - 5.

Thus the hydrostatic excess pore water pressure at any point at a time $t + \Delta t$, can be obtained from the pressures existing at the point and its two neighbouring points, at time t. Thus a numerical method is given for tracing the distribution of hydrostatic excess pressure in the pore-water, with time.

It is desirable for purposes of comparison with analytical solutions to introduce dimensionless parameters into equation I-5, and it will be convenient to consider these parameters in Part I-(b) following, together with some aspects of the equation.

(b) Stability Considerations

$$T = \frac{c \cdot t}{H^2}$$

whence $$\Delta t = \frac{H^2 \Delta T}{c} \qquad \text{I - 6}$$

and if $$z = sH$$

then $$\Delta z = \Delta s \cdot H \qquad \text{I - 7}$$

Now consider the factor

$$\frac{c_z \Delta t}{\Delta z^2} = A$$, say, in equation I-5

Substitute from I-6 and I-7 to get

$$A = \frac{\Delta T}{\Delta s^2}$$ in terms of dimensionless parameters. Then also the original partial differential equation I-2 can be written

$$\frac{\partial u}{\partial T} = \frac{\partial^2 u}{\partial s^2} \qquad \text{I - 8}$$

Equation I-3 is a partial differential equation of parabolic form, and as such, is common to problems in many fields in which diffusion occurs, such as heat transfer and electrical problems. For the derivation and discussion of equation I-5, a study of References (9), (10), (14), (24) will be found useful.

It will be observed that the factor A is dimensionless, and, in order to make use of equation I-5, some value must be assigned to A. There are limitations to the values which may be assigned, and these limitations are not obvious. If a value of 1/2 or greater is given to A, the computed values of excess hydrostatic pressure will be found to oscillate -- converging in the case of A = 1/2, diverging in the case of $A > 1/2$. A full mathematical discussion of the oscillation, convergence, and divergence is given in Refs. (17) (21). Dusinberre (9) also gives examples.

However, values of A = 1/4 or A = 1/3 are immediately seen to be useful, and give results in good agreement with analytical solutions. Dusinberre (9) and Gibson and Lumb (12) demonstrate the variance of the numerical solution from the analytical one for various examples in which analytical solutions have been obtained.

(c) Effect of A Value on Computational Labor

Substituting A = 1/4 and 1/3 respectively, the stable expressions

$$u_{o,t+\Delta t} = \frac{1}{4}\left[u_{2,t} + 2u_{o,t} + u_{4,t}\right] \qquad \text{I - 9}$$

and

$$u_{o,t+\Delta t} = \frac{1}{3}\left[u_{2,t} + u_{o,t} + u_{4,t}\right] \qquad \text{I - 10}$$

are given. Equation I-9 may be used for a graphical solution of the hydrostatic excess pore-water pressures, and I-10 is more convenient for hand calculation.

When a value for A is chosen in a particular case, then the relationship of ΔT to Δs is defined and the time interval between two successive calculations is decided by the coarseness or fineness of the mesh adopted to cover the region. To achieve a solution up to a particular, wanted, time factor T, requires the iteration of equation I-5 for each mesh point a number of times, the number being given by $\frac{T}{\Delta T}$. It is obvious, then, that the labor required to achieve a desired solution depends both on the value chosen for the factor A, and on the size of mesh adopted. That is, if A is taken equal to 1/4 then more steps are required for a solution up to a given time factor than for an A = 1/3, when the mesh size in each case is the same. Generally, it has been found that values of Δs = 1/3 to 1/5 are adequate for a degree of accuracy good enough for practical purposes, although the initial points may vary considerably from an exact solution. Numerical solutions plotted against analytical solutions are given by Gibson and Lumb (12), Dusinberre (9), etc. for the values of A and Δs mentioned. As an example, if both A and Δs are taken equal to 1/4 then

$$\Delta T = 1/4(1/4)^2 \qquad - I - 5(a)$$

$$= 1/64 = 0.0156$$

An example of a calculation based on the above figures may be given here for clarity. A homogeneous, isotropic clay layer of thickness 2H is taken, with free drainage at top and bottom surfaces. Initially the pore-water is subjected to a hydrostatic excess pressure of 100 units everywhere in the clay, by a superimposed, external load. One-dimensional drainage and consolidation ensures, and the reduction in excess hydrostatic pore-water pressure can be traced by means of equation I-9 as applied to the example in Fig. I-2.

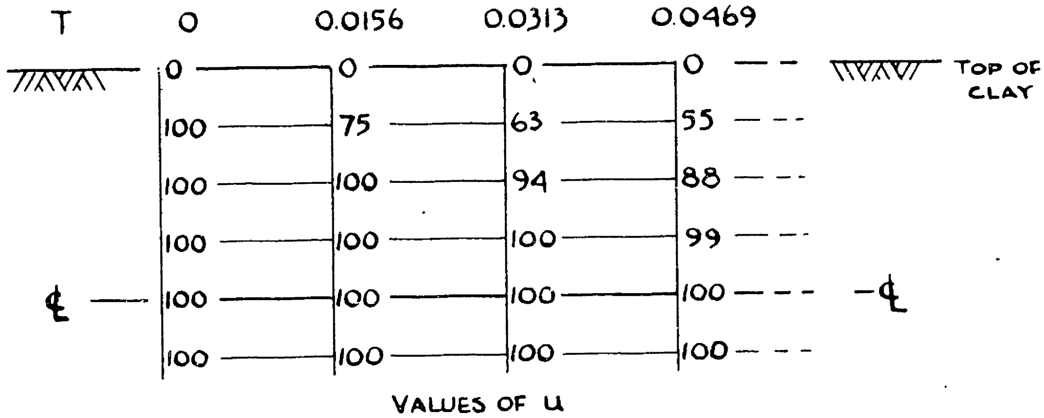


FIG. I-2

The pore-water pressure becomes zero at the drainage surface instantaneously, creating the initial gradient which begins the process.

(d) Observations on Example

Several important points can be observed from the simple example of Figure I-2.

(1) Since $U = 1 - u/u_i$ it followes that U, the degree of consolidation also satisfies the equation $\frac{\partial U}{\partial t} = c_z \frac{\partial^2 U}{\partial z^2}$ I - 11

and that, therefore, values of U could have been used at the mesh points initially, giving a calculation in terms of the degree of consolidation at any mesh point.

(2) The solutions are of a type to which is given the name "marching" solutions, as opposed to "jury" solutions (1) in which all the boundary conditions are prescribed (the ultimate time boundary is not defined in consolidation problems) and the methods of "Relaxation" (29) are used. The Laplacian equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \qquad \text{I - 12}$$

which represents a "steady state" condition such as occurs in seepage problems may be solved in particular cases by relaxation techniques, such as have been demonstrated by Shaw (30) and Yang (34).

In a marching solution it is necessary to keep developing the solution until the desired time is obtained; that is, the boundary at the right of Fig. I-2 is an open one. Allen and Severn (1) demonstrated the possibility of converting a marching problem into a jury one, but the procedure involves the solution of a more complex equation giving rise to more complicated relaxation expressions and doubt has been expressed concerning the advantage of such procedures. They may also involve larger errors in the solution.

(3) If the average degree of consolidation $\bar{U}$, is required at any time, or if the curve of average degree of consolidation versus time is required, the hydrostatic excess pore water pressure must be integrated over the depth of the layer. The average consolidation

$$\bar{U} = 1 - \frac{\int_0^{2H} u\,dz}{\int_0^{2H} u_i\,dz} \qquad \text{I - 13}$$

In practice this will be accomplished by plotting the curve of hydrostatic excess pore-water pressure versus depth, for the time concerned, and finding the area under the curve by means of a planimeter.

(4) A case in which the loading is not accomplished instantaneously, can be considered merely by adding the desired load to the hydrostatic excess pore water pressure at the appropriate time, since the consolidation due to any one increment proceeds independently of the consolidation existing previously. Such a requirement can be fulfilled in two ways which can best be observed by means of examples shown in Fig. I-3. For both cases, a load increasing uniformly with time will be assumed. In practice this is approximated by a series of step-load increments.

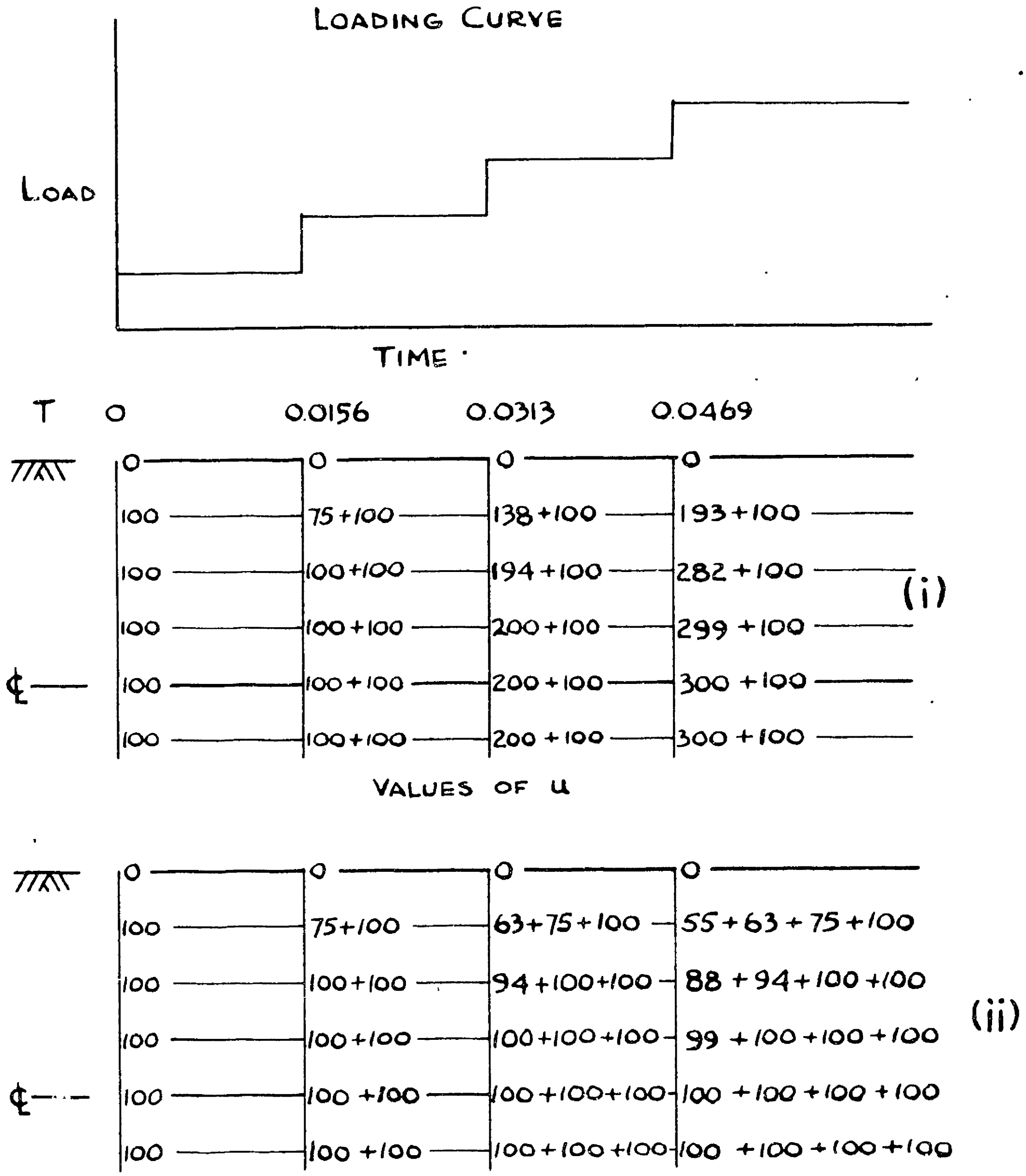


FIG. I-3

Fig. I-3(i) Here it is seen that, after each iteration, 100 units is added to each u value as the load curve indicates and the computation proceeds using the new value.

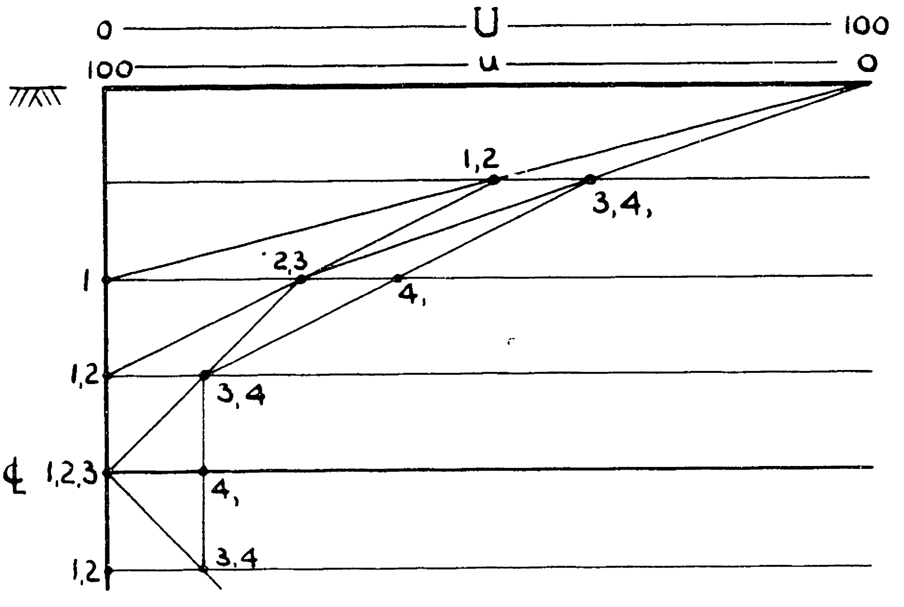
Fig. I-3(ii) In this example, the usefulness of superposition is demonstrated, in that only the values obtained from the simple instantaneous loading case are utilised, and these values at each point are merely added to the preceding value. Thus, for any case of a complicated loading-time curve, it is only necessary to have the values from the instantaneous case at hand. It is, of course, necessary at any particular time, to divide the value of hydrostatic excess pressure by the total load carried by the layer up to that time, in order to find the amount of pore-water pressure dissipation.

(e) Graphical Solutions

In the study of unsteady-state diffusion and heat flow problems Schmidt (25) showed how solutions could be obtained graphically. This work was developed and extended later by Nessi and Nissole (19) and treatments of graphical methods can be found in Dusinberre (9) and Sherwood and Reed (26). They will be demonstrated here for completeness, and although Dusinberre points out the disadvantages of graphical methods in the heat problem, it is possible that the immediate results in the form of curves would prove of some value in a situation in soil mechanics where the curves must be integrated, in general, for the complete solution.

A graphical solution may be carried out where A is chosen equal to 1/2 or 1/4. In the case of A = 1/2, a simple averaging procedure only is required; when A = 1/4 more complex drafting work is concerned, but better results, as mentioned previously, are obtained. Fig. I-4 (i) and (ii) shows the two procedures in practice, for the simple case, taken previously, of instantaneous loading.

The procedure for constructing Fig. I-4(ii) together with the proof will be found on the figure.

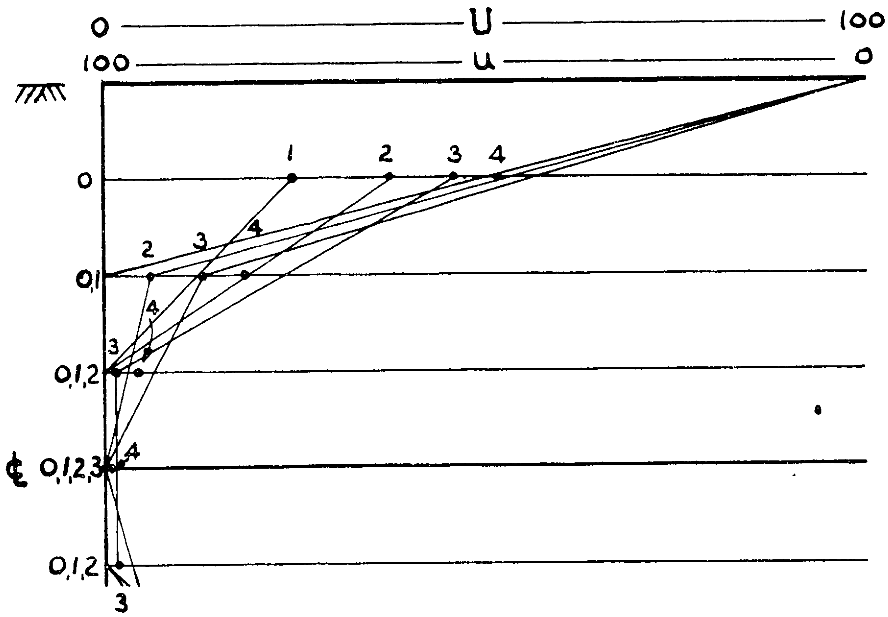


EQUATION: $u_{0,t+\Delta t} = \frac{1}{2}\left[u_{2,t} + u_{4,t}\right]$

$$\Delta T = \frac{1}{2}\left(\frac{1}{4}\right)^2 = 0.0313$$

KEY: Points with same numbers exist at same times.

FIG. I-4(i)

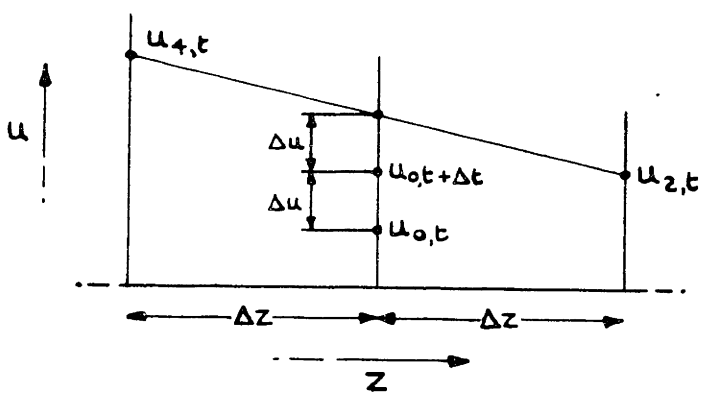


EQUATION: $u_{o,t+\Delta t} = \frac{1}{4}\left[u_{2,t} + 2u_{o,t} + u_{4,t}\right]$

$$\Delta T = \tfrac{1}{4}\left(\tfrac{1}{4}\right)^2 = 0.0156$$

KEY: Points with same numbers exist at same times.

Construction: Join points $u_{2,t}$ and $u_{4,t}$ with a straight line, which intersects the intervening layer on which $u_{o,t}$ lies to give an average value, $\frac{u_{2,t} + u_{4,t}}{2}$. The point bisecting the distance between this average value and the value of $u_{o,t}$ is the new, required point $u_{o,t} + \Delta t$.



PROOF: $u_{o,t+\Delta t} = u_{o,t} + \Delta u$

$$= u_{o,t} + \tfrac{1}{2}\left[\frac{u_{4,t} + u_{2,t}}{2} - u_{o,t}\right]$$

$$= \tfrac{1}{4}\left[u_{2,t} + 2u_{o,t} + u_{4,t}\right] \quad \text{—— Equation I-9}$$

FIG. I-4(ii)

(f) Variations In Homogeneity

In the analytical solution of the diffusion equation, one of the most questionable assumptions, when the solution is correlated with results experienced in practice, is that concerning the homogeneity of the soil.

It becomes possible, when numerical methods of solution are used, to consider the soil variation and so-called soil "constants" by computations adhering more closely to the results observed in either field or laboratory tests on the soil. It will be found that solutions can be obtained in this manner with very little more labor than is required for the homogeneous case. Two cases of variation in homogeneity will be considered.

(1) Stratification considered

If stratification is encountered in a compressible soil, it will generally be found possible to represent the boundary between two types of soil by a horizontal line. Thus, when one-dimensional consolidation is considered, the pore-water flow will be normal to the line of stratification. The difference in behavior of the soils is represented by differing values of c_z. Each layer, as before, may be covered by a mesh of points, but certain conditions have to be satisfied, before calculations can be made. It will be found desirable to have the dimensionless factors, A, of the same value in the two adjacent layers, as this renders the individual calculations identical, and, in order to plot satisfactory curves, the time interval, ΔT, between any two operations should be the same in both soils.

Thus $$\frac{c_{z_1}\Delta t}{\Delta z_1^2} = \frac{c_{z_2}\Delta t}{\Delta z_2^2} \qquad \text{I - 14}$$

and $$\therefore \frac{\Delta z_1}{\Delta z_2} = \sqrt{\frac{c_{z_1}}{c_{z_2}}} \qquad \text{I - 15}$$

So that the size of the mesh in each case varies as the square root of the coefficient of consolidation. It would also be possible to transform the thickness of one of the layers by the ratio of the square roots of the coefficients of consolidation, and in this case, the mesh intervals would be the same in both layers. The results could then be retransformed to the original dimensions before plotting.

In this way it is possible to demonstrate the effect of remolding at the surface layer of a laboratory consolidation test sample. Helenelund (15) shows the effect on the consolidation of samples, of having layers of differing coefficients of consolidation in the sample. It is necessary, if the isochrones are to be drawn, to include the effect of refraction of the isochrone at the interface between two layers, because of the differing permeabilities of the two adjacent materials. For any one isochrone, the slope of the isochrone at a point represents the hydraulic gradient at that point. Then, at any interface, the velocity of flow entering the interface must equal the velocity of flow leaving. Consequently, for two materials having permeabilities given by k_1 and k_2, the condition to be satisfied at the interface is that

$$V_1 = V_2$$

$$\text{or} \quad k_1 i_1 = k_2 i_2 \qquad \text{I - 16.}$$

where i_1, i_2 are the hydraulic gradients existing in each material at the interface. If the isochrone meets the interface at angles α_1 and α_2 to the normal to the interface respectively, then the condition I-16 becomes

$$k_1 \tan \alpha_1 = k_2 \tan \alpha_2 \qquad \text{I - 17.}$$

This equation is satisfied by a simple construction demonstrated by Helenelund (15), in Fig. 14 of his book which is given here for reference.

Point A represents the hydrostatic excess pressure existing at time t in zone 1. at a distance of Δz_1 from the interface. Similarly point C is the pressure in zone 2. at a distance of Δz_2 from the interface. Draw AB perpendicular to the interface, meeting it in B. Draw line H-H (the "help-line") parallel to interface and at a distance of $\frac{k_1}{k_2} \cdot \Delta z_1$ from it in zone 1 to intersect AB in D. Join CD, intersecting the interface in E. Join AE.

Then the gradient of AE = $\left(\frac{\partial u}{\partial z}\right)_{Zone\,1} = \tan \alpha_1$

and the gradient of CE = $\left(\frac{\partial u}{\partial z}\right)_{zone\,2} = \tan \alpha_2$

$$\therefore \frac{\tan \alpha_1}{\tan \alpha_2} = \frac{k_2}{k_1}$$

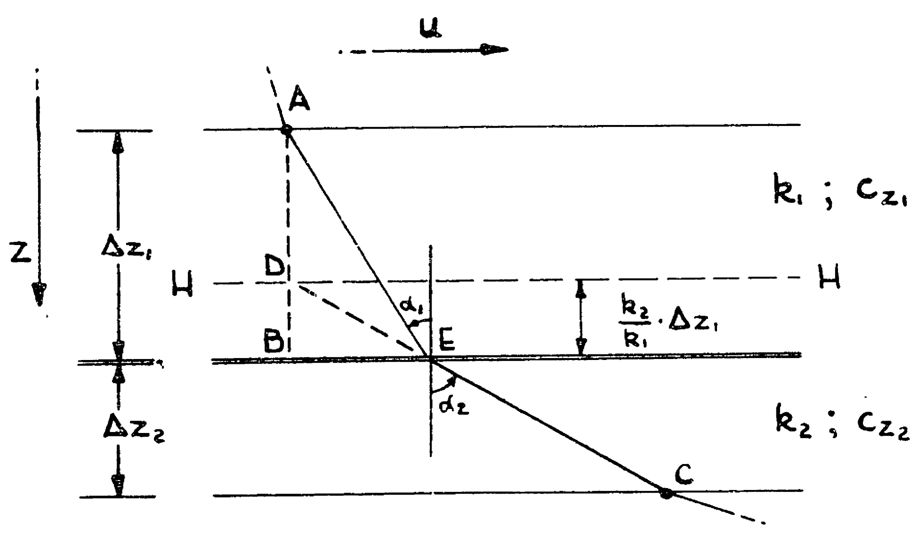


FIG. 14, REF. (15)

(2) Variation of C_V with Intergranular pressure

Conventionally, when the results of a laboratory consolidation test are shown, a plot of coefficient of consolidation versus intergranular pressure or log (intergranular pressure) is also given (Fig. I-5). Then,

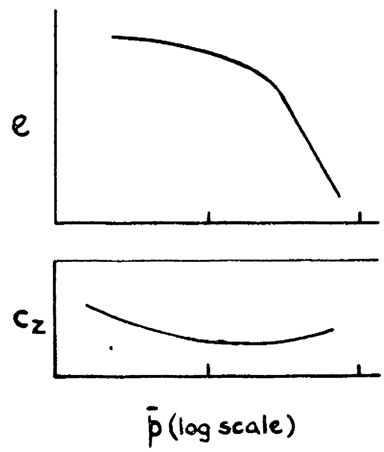


FIG. I-5

if the virgin, straight line, portion of the curve is considered, it will be seen that the coefficient of consolidation varies in some fashion with intergranular pressure. As a first approximation it could be assumed that the coefficient of consolidation varied linearly with pressure over this part of the curve, or, if an actual case were to be considered, some average set of values could be taken from typical consolidation curves of the soil being encountered.

In order to apply this information, it is necessary to consider the iteration process once more.

It has been observed that the intergranular pressure (or the hydrostatic excess pore-water pressure) at a point after a small time increment bears a relation to the pressure at that point and the two points surrounding it, at the previous time. The relation depends on the value chosen for the factor A which contains the coefficient of consolidation. Hitherto, once chosen, the factor A has been constant all through an iteration. Suppose now, that the coefficient of consolidation varies, depending on the intergranular pressure existing at the point, and that this variation could be expressed in some function of the intergranular pressure. Then, once the initial value of A had been selected, there would be a predictable variation in A, the value for any stage of the iteration calculation depending in some known way on the value of intergranular pressure existing at the point at the previous instant of time.

For example, assume that the coefficient of consolidation varies linearly with intergranular pressure or a natural scale, Fig. I-6.

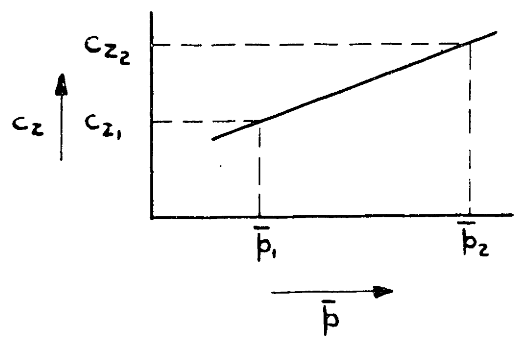


FIG. I-6

Then with soil being considered initially at the intergranular pressure $\bar{p}_1$, when a load giving a pressure of $\bar{p}_2$ at the surface, and, therefore, throughout the pore water, is applied, the soil begins to consolidate, so that the degree of consolidation may be considered to be 0% at points where an intergranular pressure of $\bar{p}_1$, exists and 100% where a pressure of $\bar{p}_2$ exists. Thus, the revised diagram, Fig. I-7, can be drawn to represent the case being considered,

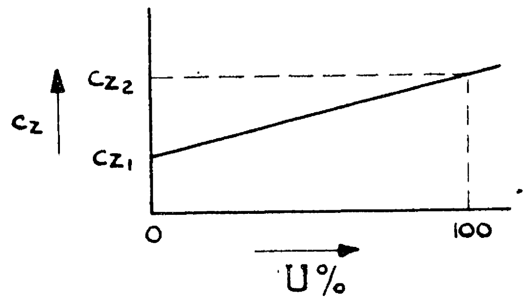


FIG. I - 7

in which c_z varies linearly with U.

Then the initial conditions are:

$$U = 0 \,, c_z = c_{z_1}$$

and finally $$U = 100 \,, c_z = c_{z_2}$$

The equation giving the coefficient of consolidation, c_z, at any degree of consolidation, U, will be

$$c_z = c_{z_1} + (c_{z_2} - c_{z_1})\frac{U}{100} \qquad \text{I - 18.}$$

The iteration expression I-5 can now be rewritten

$$U_{o,t+\Delta t} = \frac{c_z \Delta t}{\Delta z^2}\left[U_{2,t} - 2U_{o,t} + U_{4,t}\right] + U_{o,t} \qquad \text{I - 19.}$$

where U is the degree of consolidation existing at a point due to the dissipation of hydrostatic excess pore-water pressure resulting from the initial applied load.

In I-19 c_z is no longer a constant, but is given by expression I-18. This will obviously result in an extremely tedious computation, wherein c_z varies with each time increment and also spatially in the mesh. However, such computations can be greatly facilitated by the construction of a set of curves for each particular example considered, in which curves of $U_{o,t+\Delta t}$ are plotted against an ordinate of $(U_{2,t}+U_{4,t})$ and an abscissa of $U_{o,t}$ using an equation similar to I-19 which will depend on the initial value assumed for A. For the assumptions indicated, appropriate calculations together with sets of curves are given in Appendix III, Chart I, and a plot of the results obtained by using the curves in an iteration process is given in Chart IV. These results were integrated, as indicated by equation I-13, to give the average consolidation curve in Chart V compared with an analytical solution assuming c_z constant, and the numerical solution, assuming c_z constant, with the same initial value of A.

Chart II in Appendix III gives curves used in obtaining the numerical solutions of the consolidation process, when c_z is considered constant.

Some interesting details about the curves are more fully discussed in Appendix III.

(3) Coefficient of Swelling

Previously (4) very few cases have been examined, in which gradual increase of hydrostatic excess pore water pressure, or swelling of the soil, has taken place. The problems which have been treated, have primarily been subject to the assumption that the coefficient of swelling of the soil equals the coefficient of consolidation. This is seldom, if ever, true, and Terzaghi (33) mentions that to assume the coefficient of swell equal to infinity would be a more valid representation of actual conditions.

However, in the case of a particular soil, it is possible, in laboratory tests, to evaluate, by a method similar to that used to determine the coefficient of consolidation, a coefficient of swelling. The values obtained are neither equal to the coefficient of consolidation, nor infinity. If swelling in a particular case is to be expected, it should be possible to run tests to find the coefficient of swelling, which could be related to the coefficient of consolidation by simple ratios. Use of numerical analysis could then give some guide to the behavior of the soil.

The same iteration expression used previously, equation I-5, will hold when swelling is taking place. If no consolidation at all is to be encountered, but only swelling, then a procedure identical to that outlined previously may be utilised.

In cases where load is applied to the soil, and then some portion of it is released, or where some load is first released, and then additional stress applied, as occurs frequently in construction work, more care must be taken. For, here, two iteration expressions are involved, one including a term for the coefficient of consolidation, the other, a term for the coefficient of swelling.

If c_{z_s} denotes coefficient of swelling, then

$$A = \frac{c_z \Delta t}{\Delta z^2}$$

and $$A_s = \frac{c_{z_s} \Delta t}{\Delta z^2}$$

It is desirable to have the intervals Δz and Δt equal, consequently two values of A must be used. If c_{z_s} is larger than c_z, as will generally be the case, a value of A must be chosen so that A_s does not exceed 1/2, as indicated in Part I-(b). In general, in cases of combined consolidation and swelling, some layers of the soil will be consolidating, while other, <u>at the same time value</u>, are expanding. Thus, two different equations will be in operation on the same group of mesh points (points at the same time) and each point must be examined in the light of its previous behavior, and the loading curve, in order to determine which equation must be utilised.

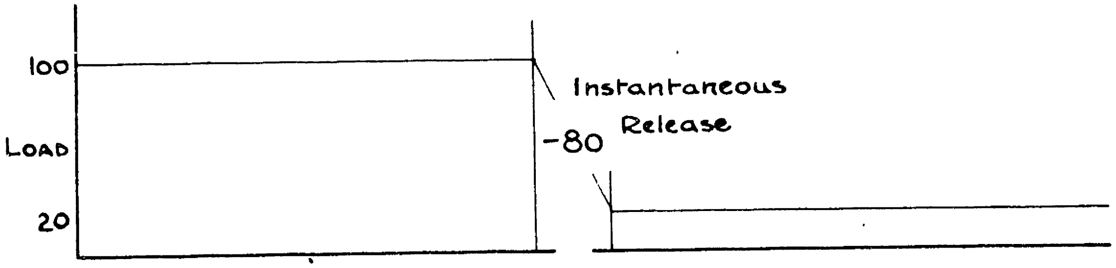
Such a procedure is best illustrated by an example, given in Fig. I-8. Similar considerations will prevail when swelling is encountered in cases of two-dimensional consolidation.

(g) Initial Times, and More Exact Solutions

Some consideration must be given to the way in which a numerical solution is initiated. For instance, the initial conditions, i.e., when T = 0, in the example given in Part I(c) are hydrostatic excess pore water pressures of 0, 100, 100, 100, etc., at mesh points beginning at the surface of the compressing layer, and the example proceeds from there to pressures at ΔT, $2\Delta T$, and so on. It has been demonstrated, however, by Crandall (8) that the closeness of the numerical solution to the analytical one is affected by the method in which the iteration is begun. It has been found in the case shown that initial, T = 0 conditions of 50, 100, 100, 100--give a more exact solution. In this case, one iteration is carried out giving the distribution at $T = \Delta T$ and the boundary value is reduced at this stage to 0, whereupon the cycles of iteration are carried out as before. However, the method of initiating the problem depends upon the nature and initial conditions of each problem, and some work still needs to be done in this line.

The accuracy of a solution (that is, the closeness to which the numerical solution approximates the analytical one) is also very sensitive to the value chosen for A. The value giving the best solution in a particular case within a range of different mesh spacings is termed the "optimum" value. Crandall (8) shows that the solution obtained using the "optimum" value of A for one particular problem demonstrates smaller errors than solutions obtained from smaller A's and finer nets. Again this varies from problem to problem and no definite guide can be given.

Loading Curve



Time Factor

	0	0.0185	0.0371	0.0556	0.0556	0.0741	0.0927	0.1113 ....
Surface	0	0	0	0	0	0	0	0
	100	83.3	72.2	64.3	(-15.7)	(-0.7)	(+2.9)	(+3.5)
	100	100	97.2	93.5	+13.5	+9.4	+7.5	(+8.3)
¢	100	100	100	99.1	+19.1	+17.2	+14.6	+12.2

Figures in parentheses indicate swelling.

Let $c_{z_s} = c_z \times 2$

Then

$$A = \frac{c_z \Delta t}{\Delta z^2} = \frac{1}{6}, \text{ say.}$$

$$\therefore A_s = \frac{c_{z_s} \Delta t}{\Delta z^2} = \frac{1}{3}$$

$$\text{and } \Delta T = \frac{1}{6}(\Delta s)^2 = 0.0185$$

Equations:

Consolidation $u_{o,t+\Delta t} = \frac{1}{6}\left[u_{2,t} + 4u_{o,t} + u_{4,t}\right]$

Swelling $u_{o,t+\Delta t} = \frac{1}{3}\left[u_{2,t} + u_{o,t} + u_{4,t}\right]$

Key:

Hydrostatic excess pressure

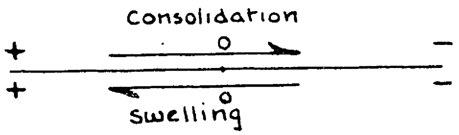


Fig. I-8

More exact solutions can be obtained by numerical analysis, with the application of less labor, if a method of extrapolation is used. It will be seen, in the derivation of equation I-5 from Taylor's Series in Appendix IV that an error depending on $(\Delta x)^2$ is inherent in the iteration expression. It was demonstrated by Richardson (22) that a so-called "h^2-(or Δx^2) -extrapolation" could be made from two solutions to a given problem using two different mesh sizes to give a result having a smaller error than either solution individually. In this method, two different mesh sizes are chosen, and the solutions obtained in the desired time range. Then for the same point at the same value of time factor, two values of hydrostatic excess pore water pressure are given from the two solutions. It is known that both of these values are in error by an amount depending on the square of the mesh size chosen.

Then, when u_1 is the solution obtained at a point using mesh size Δx_1, u_2 from Δx_2, and E_1 and E_2 are the errors in each solution, respectively,

$$E_1 = k(\Delta x_1)^2 \qquad E_2 = k(\Delta x_2)^2 \qquad \text{I - 20.}$$

where k is a proportionality constant

Then a more exact solution for the hydrostatic excess pressure at the point is

$$u = u_1 + k(\Delta x_1)^2 = u_2 + k(\Delta x_2)^2 \qquad \text{I - 21.}$$

Eliminating u from I-21, and considering I-20 gives a solution for k whence

$$u = u_1 + \frac{u_2 - u_1}{1 - \left(\frac{\Delta x_2}{\Delta x_1}\right)^2} \qquad \text{I - 22.}$$

It must be borne in mind however, that this solution may be applied to some problems only, and that in certain other cases a two-point extrapolation is of little value, or may even lead to a solution with a greater error than obtained from either of the first two solutions. Salvadori (23) has studied this aspect and gives examples.

PART II

TWO-DIMENSIONAL CONSOLIDATION

The case of two-dimensional consolidation is one which occurs frequently in practice, in the consolidation of hydraulically placed cores for earth dams, and the settlement of long dams, dikes, levees and embankments founded on compressible soils. Two-dimensional problems cannot be treated readily by analytical methods and it follows, then, that a study of the solution of various cases by numerical methods will be found valuable.

(a) Derivation of equations.

The general equation of two-dimensional consolidation is

$$\frac{\partial u}{\partial t} = c_x \frac{\partial^2 u}{\partial x^2} + c_z \frac{\partial^2 u}{\partial z^2} \qquad \text{II - 1.}$$

and a solution of this in terms of finite differences may be made available by the methods and results of Part I.

The situation now being considered is shown in Fig. II-1(a) and a small element of finite dimensions is demonstrated with the appropriate notation in Fig. II-1(b).

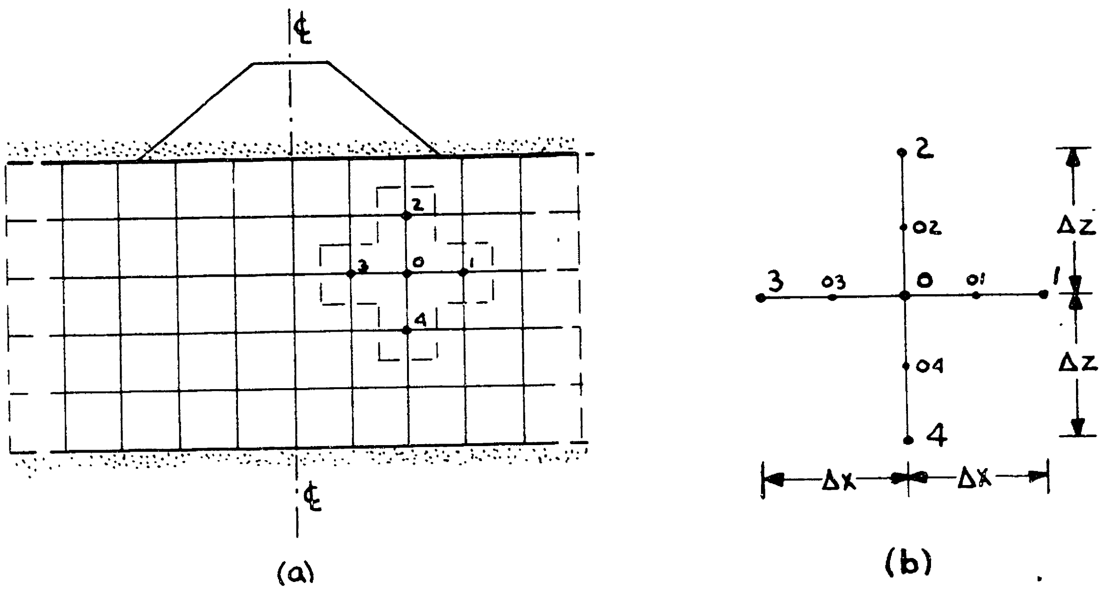


FIG II- 1

With the same assumptions as in Part I, it is noted that, in equation I-3

$$\left(\frac{\partial^2 u}{\partial z^2}\right)_o \approx \frac{u_2 - 2u_o + u_4}{\Delta z^2} \qquad \text{I - 3.}$$

By similar reasoning

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_o \approx \frac{u_1 - 2u_o + u_3}{\Delta x^2}$$

is obtained.

Combining these two equations as indicated by equation II-1, an expression is found in finite differences for the hydrostatic excess pore pressure at point o

$$\frac{u_{o,t+\Delta t} - u_{o,t}}{\Delta t} = \frac{c_x}{\Delta x^2}\left[u_{1,t} - 2u_{o,t} + u_{3,t}\right] + \frac{c_z}{\Delta z^2}\left[u_{2,t} - 2u_{o,t} + u_{4,t}\right] \qquad \text{II - 2.}$$

Making the assumption that the soil is homogeneous and unstratified, $c_x = c_z$, and for convenience Δx can be taken equal to Δz.

Then

$$u_{o,t+\Delta t} = \frac{c_z \Delta t}{\Delta z^2}\left[u_{1,t} + u_{2,t} + u_{3,t} + u_{4,t} - 4u_{o,t}\right] + u_{o,t} \qquad \text{II - 3.}$$

As in the case of one-dimensional consolidation, the use of this equation is dependent upon the choice made for the value of $\frac{c_z \Delta t}{\Delta z^2}$. Obviously, in this case, some multiple of 1/4 will give solutions most readily. If 1/4 itself is chosen, the equation reduces to the simple expression

$$u_{o,t+\Delta t} = \frac{1}{4}\left[u_{1,t} + u_{2,t} + u_{3,t} + u_{4,t}\right] \qquad \text{II - 4.}$$

that is, the hydrostatic excess pore pressure at a point at a time $(t + \Delta t)$ is equal to the mean of the values at the four surrounding points at time t.

In most cases of two-dimensional consolidation, the initial values of hydrostatic excess pore water pressure are not easily defined, or of simple geometric distribution as in the one-dimensional situation, but must be calculated from one of the stress distribution theories; for example, see Skempton (28). Then it is a natural consequence that the accuracy or validity of the numerical solution is dependent on the assumptions involved in the stress transmission theory utilized. Because of the complexity of the initial distribution of hydrostatic excess water pressure it will generally be found necessary to solve the problem over the entire net, i.e., the center-line of the compressible soil layer is no longer an axis of symmetry, although there may exist an axis of symmetry in the actual case studied. One such axis of symmetry is demonstrated in Fig. II-1(a) in the case of an earth embankment with equal angles of slope on each side.

As time, t, enters into the solution of these problems, a three-dimensional situation actually exists in two-dimensional spatial solutions, and it will probably be found most convenient to write the successive values of hydrostatic pore water pressure at a point as found by iteration, one below the other, in a column, each value being obtained for a particular time.

The boundary conditions are not as fully prescribed in certain types of two-dimensional consolidation as in one-dimensional problems. It will be noticed that in Fig. II-1(a) the boundaries to left and right in the compressible layer are not sharply defined. The termination of the calculations in these directions will be a consideration which depends on the nature of the problem, the precision required, and the judgment of the computer. However, it will generally be a fairly simple matter to choose the number of significant figures desired, and to terminate the calculations where values of less than half the last significant figure are encountered. In certain cases, care must be taken, as the values will tend to spread outwards. This will occur where one of the boundary layers is impervious, so that free drainage through its surface is prevented. Thus the dissipation of hydrostatic excess pressure in the high pressure regions can only be accomplished by the raising of pressure in the low pressure regions by flow of the pore water. This will result in swelling in those regions into which the water is flowing. If the coefficient of swelling is assumed equal to the coefficient of consolidation, no extra labor is involved in the calculations, but where a different value is assumed, or obtained from tests, the computations will be altered in regions where swelling is taking place as demonstrated in Part I-(f.) (3).

In the case of two-dimensional consolidation it is no longer advisable to work with degrees of consolidation rather than with values of hydrostatic excess pore water pressure, as the initial pressure at every point will generally be different, and the degree of consolidation at a point depends on the stress at the point. Consequently, although the ultimate stage will be reached where consolidation is 100% completed everywhere, and equilibrium has been reached under the applied load, the increase in intergranular stress at a point will be that due to the stress transmitted from the applied load, and strength increases must be calculated on this basis.

(b) Variations in Homogeneity of the Soil

As in the one-dimensional case, in a general solution, variations in the soil constants will have to be considered for two-dimensional consolidation problems. The derivations regarding variation of coefficient of consolidation will be discussed for the three cases usually encountered; horizontal c_x differing from vertical c_z, stratification of the soil, and variation of coefficient of consolidation with effective or intergranular pressure.

1. c_x differing from c_z

It is a common occurrence to find that the horizontal and vertical properties of the soil under consideration are not equal, and this situation can be dealt with by considering equation II-2 once more. It is given again here for clarity

$$u_{o,t+\Delta t} - u_{o,t} = \frac{c_x \Delta t}{\Delta x^2}\left[u_{1,t} - 2u_{o,t} + u_{3,t}\right] + \frac{c_z \Delta t}{\Delta z^2}\left[u_{2,t} - 2u_{o,t} + u_{4,t}\right] \qquad \text{II - 2.}$$

In this equation, for the homogeneous, isotropic case, c_x was taken equal to c_z, and Δx equal to Δz. Now c_x has a value differing from that of c_z. It is desirable, as before, to retain the value of Δt constant in all iterations, and to have the value of A fixed. Consequently, to satisfy these conditions

$$\frac{c_x \Delta t}{\Delta x^2} = \frac{c_z \Delta t}{\Delta z^2}$$

or

$$\frac{\Delta x}{\Delta z} = \sqrt{\frac{c_x}{c_z}} \qquad \text{II - 5.}$$

That is, the domain under consideration is divided up into convenient values of Δz and equation II-5 is then used to evaluate the length of the intervals in the x-direction. When a value is chosen for A, say 1/4, the iteration procedure may be carried out as before, using for A = 1/4, equation II-4 to calculate the hydrostatic excess pore water pressure. Thus it will be seen that the presence of aeolotropic soil adds little to the labor of the calculations.

2. Stratification

The presence of stratification in the soil presents a more difficult problem in two- than in one-dimensional cases. First of all, it is assumed that the soil layers which exist in the problem are homogeneous and isotropic and possess differing soil properties and that a horizontal boundary may be drawn between any two layers.

The situation under consideration is represented in Fig. II-2.

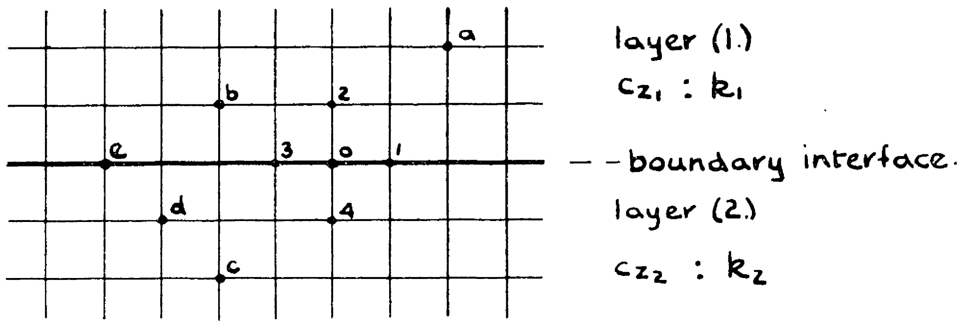


FIG. II - 2

It will be observed that no difficulty is encountered in the calculation of values of hydrostatic excess pore water pressure at successive times at points a and b; the iteration expression used is simply II-3, with a value of A_1 assumed. A similar procedure is carried out for points c and d with, in each case, the new value of hydrostatic excess pore water pressure depending solely on the four surrounding points. The value assumed for A_2 will depend on the value chosen for A_1 as it is obvious that the time increments, Δt, and the value given to the mesh spacings Δz, must be maintained equal in the two layers. That is, once Δz and A_1 are decided upon, Δt is fixed, and the value of A_2 depends upon Δt, Δz, and the relative values of c_{z_1} and c_{z_2}.

$$A_1 = \frac{c_{z_1} \Delta t}{\Delta z^2}$$

$$A_2 = \frac{c_{z_2} \Delta t}{\Delta z^2}$$

$$= \frac{D c_{z_1} \Delta t}{\Delta z^2}$$

$$= D \cdot A_1 \qquad \text{II - 6.}$$

Thus a value must be chosen for A_1 such that A_2 does not become unstable if $D > 1$. It will be seen that, in general, a value of Δz which gives a whole number of vertical mesh spacings in layer (1) will not give a whole number of spacings in layer (2). This gives rise to a situation at the base of stratum (2) such as is shown in the case shown in Fig. II-3(a) which is a particular example of the general case of Fig. II-3(b), in which the distances of the 4 mesh points from the central point are all different. An iteration expression can be derived for this situation and this derivation is given in Appendix IV.

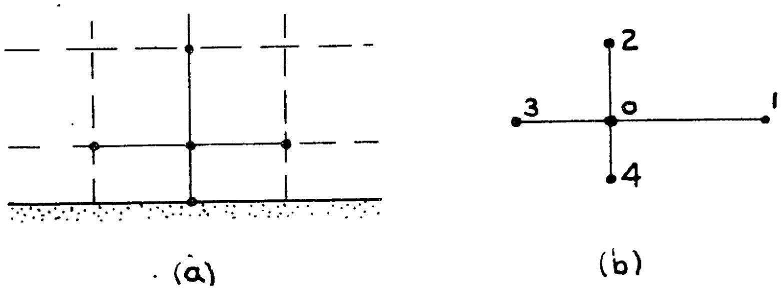


FIG. II-3

However, referring to Fig. II-2 once more, it will be seen that some difficulty will arise over the calculation of values arising at points of the mesh such as e on the boundary between the two layers. The value of hydrostatic excess pressure at such a point at any time will depend, as before, on the values of the four surrounding points at the previous time. But one of these points lies in another medium, and this requires some variation of the existing procedure.

In a discussion by F. S. Shaw in R. V. Southwell's book (30) the case of stratification is considered for the steady state seepage problem, and the methods used in that discussion can be adapted for the case of consolidation. For the sake of continuity only the results are given here and the relevant derivation occurs in Appendix V. The numbering of the points referred to is that occurring in Fig. II-2.

The equation derived is

$$u_{o,t+\Delta t} = \frac{1+\frac{k_1}{k_2}}{1+\frac{c_{z_2}}{c_{z_1}}\cdot\frac{k_1}{k_2}} \cdot \frac{c_{z_2}\Delta t}{\Delta z^2}\left[u_{1,t} + \frac{2k_1}{k_1+k_2}\cdot u_{2,t} + u_{3,t} + \frac{2k_2}{k_1+k_2}\cdot u_{4,t} - 4u_{o,t}\right] + u_{o,t}$$

II - 7.

This is a considerably more complicated expression than any utilised previously, and it will be observed that it also involves the permeabilities of the two layers. However, in any one case, the permeabilities and the coefficients of consolidation will be known, when the equation will reduce to the expression

$$u_{o,t+\Delta t} = E_1\cdot DA_1\left[u_{1,t} + E_2 u_{2,t} + u_{3,t} + E_3 u_{4,t} - 4u_{o,t}\right] + u_{o,t}$$

II - 8.

where E_1, E_2, E_3 are numerical constants and D is the same constant as in II-6. The use of this equation does not give rise to much more labor than previously, as there are only a few points on the boundary compared to the number inside.

If the two layers are stratified and possess properties differing in the vertical and horizontal directions, the problem becomes considerably more complex. If the ratio between the vertical and horizontal coefficients of consolidation and permeability is approximately the same for both layers then a horizontal transformation may be effected as demonstrated in Part II(b)(1), affecting the mesh dimensions in both layers in the same manner. If it is not possible to make this simplifying assumption, then the sole recourse is to cover the section with a mesh of squares and to use equation II-2 inside each layer as Δt and Δz are constant throughout. At the boundaries it will be necessary to derive an equation of the type of II-7, but in a much more general form. It is doubtful if such a computation, which would be exceedingly tedious, will ever be made, and no such general form of equation II-7 will be derived herein.

3. Variation of coefficient of consolidation with intergranular pressure.

In the two-dimensional case of consolidation the variation of the coefficient of consolidation with intergranular pressure may be considered in a manner similar to that developed in Part I. However, the labor involved will increase considerably. It will still be possible to construct a chart showing values of $u_{0,t+\Delta t}$ plotted for a given $u_{0,t}$ and a given sum of the terms $u_{1,t}$, $u_{2,t}$, $u_{3,t}$, $u_{4,t}$, when the assumption regarding the variation in the coefficient of consolidation is made. If both coefficients of consolidation c_x and c_z are considered to exist, and both to vary, the solution is made more laborious still, but still soluble. In this case, as the factor A is varying continuously, no advantage is to be gained from making the mesh dimensions Δx and Δz of different lengths, and a square mesh can be used conveniently.

(c) Superposition of solutions

Because of the tedium involved in the production of a solution to problems other than one-dimensional consolidation, it is convenient to consider here a method by which two-or three-dimensional problems can be considerably simplified.

It has been demonstrated by A. B. Newman (20) initially, and in work later independently developed by Carrillo (6) that the principle of super-position can be applied to the solution of the diffusion equation.

Carrillo shows that, in a three-dimensional domain, governed by the equation

$$\frac{\partial u}{\partial t} = c_x \frac{\partial^2 u}{\partial x^2} + c_y \frac{\partial^2 u}{\partial y^2} + c_z \frac{\partial^2 u}{\partial z^2}$$

the solution to the problem can be obtained by the consideration of three one-dimensional problems, namely

$$\frac{\partial u}{\partial t} = c_x \frac{\partial^2 u}{\partial x^2} \quad (i); \qquad \frac{\partial u}{\partial t} = c_y \frac{\partial^2 u}{\partial y^2} \quad (ii); \qquad \frac{\partial u}{\partial t} = c_z \frac{\partial^2 u}{\partial z^2} \quad (iii).$$

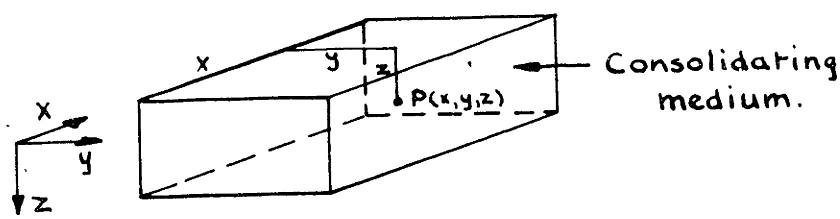


FIG. II-4

If a point, P, in Fig. II-4 is considered, with the coordinates (x,y,z) then only the x-coordinate, and the two boundaries perpendicular to the x-direction are considered in the solution of equation (i); only the y-coordinate and the two boundaries perpendicular to the y-direction in equation (ii); and the z-coordinate and two boundaries perpendicular to the z-direction in equation (iii); the flow in each case being taken parallel to the axis considered.

It is demonstrated that, if $u_{x,t}$ is the hydrostatic excess pore water pressure at point P at time t due to flow in the x-direction only, $u_{y,t}$ is similarly the excess pore water pressure due to y-direction, and $u_{z,t}$ due to z-direction flow only, then the hydrostatic excess pore water pressure due to flow in three dimensions is

$$\frac{u_{x,y,z,t}}{u_o} = \frac{u_{x,t}}{u_o} \cdot \frac{u_{y,t}}{u_o} \cdot \frac{u_{z,t}}{u_o} \qquad \text{II - 9.}$$

where u_o is the initial hydrostatic excess pressure.

Similarly, if $\bar{u}_{x,y,z,t}$ is the average excess pore water pressure throughout the mass at time t, then

$$\bar{u}_{x,y,z,t} = \bar{u}_{x,t} \cdot \bar{u}_{y,t} \cdot \bar{u}_{z,t} \qquad \text{II - 10}$$

where
$$\bar{u}_{x,t} = \frac{\int_{\text{bottom x-boundary}}^{\text{top x-boundary}} u\,dx}{\int_{\text{bottom x-boundary}}^{\text{top x-boundary}} u_o\,dx}$$

and the other terms follow similarly. Thus it will be observed, that in certain closed boundary problems, as in the consolidation of compressible fills in cellular cofferdams or in hydraulically placed cores in earth embankments, it is possible to break the problem into its component directions and to solve the new problems thus presented, individually. The final solution will be obtained by a combination of the results, using equations II-9 and II-10. Care must be taken that the mesh spacings and A-factors chosen in each direction will give compatible times, although this will not be necessary for solutions for average degrees of consolidation.

The solution of problems by this method will be returned to in Part III, where the case of radial flow is considered.

(d) Settlement

As the original purpose of many consolidation analyses is to predict the movement of the ground surface, it is desirable to consider a method by which this can be evaluated at any time from a knowledge of the hydrostatic excess pore water pressures existing in the compressible layer at that time.

For the analysis in a general, two-dimensional case, the layer is divided up into small, finite blocks, each block surrounding a point at which the pressure is being computed successively. In a uniform case these blocks will be square. The hydrostatic excess pressure existing at a particular time in the center of a block will be assumed to exist throughout the block, and this situation holds for every such block. Reference may be made to Fig. II-5, where a column of such blocks is shown.

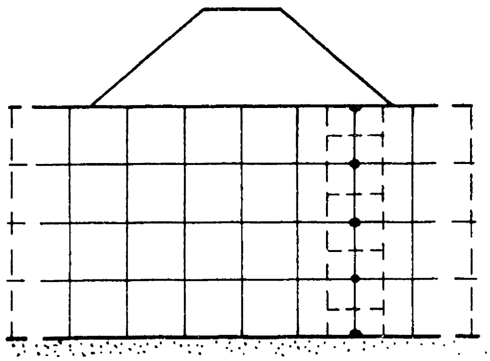


FIG. II-5

From the hydrostatic excess pressure existing at a point, and the total stress acting at the point calculated from one of the elasticity theories, due to the applied load causing consolidation, the intergranular pressure may be computed.

The settlement of $\Delta\rho$, or reduction in height of any particular block, can be calculated from the formula derived in standard texts: (31)

$$\Delta\rho = \frac{\text{thickness}}{1+e_1}(e_2 - e_1) \qquad \text{II - 11}$$

where e_1, e_2 are found from intergranular pressure-void ratio curves for the soil, or, more conveniently, as $e_1 - e_2$ is usually small, from the expression derived from equation II - 11:

$$\Delta\rho = \frac{\Delta z}{(1+e_1)} \cdot \frac{0.434 C_c(\bar{p}_2 - \bar{p}_1)}{\frac{(\bar{p}_2 + \bar{p}_1)}{2}}$$

where $\bar{p}_1$ is the intergranular pressure at the center of the block before consolidation begins, $\bar{p}_2$ the intergranular pressure at the time being considered, and C_c is the compression index, according to convention. As each block exists at a different initial intergranular pressure, and sustains a different incremental increase, the values of $e_1, \bar{p}_1, \bar{p}_2$ and C_c will differ for each block.

Summing up the settlements in each block in a column the expression

$$\rho = \sum \frac{\Delta z}{(1+e_1)} \cdot \frac{0.434 C_c (\bar{p}_2 - \bar{p}_1)}{\frac{\bar{p}_2 + \bar{p}_1}{2}} \qquad \text{II - 12.}$$

is obtained for the settlement at the surface.

It will be necessary, when any particular block has undergone both expansion and consolidation, to trace the history of the pore water pressures at the point since the load was applied, in order to evaluate properly the movements due to the variations in hydrostatic excess pressure. This investigation must be carried out with constant reference to the intergranular pressure - void ratio relationship for the soil.

PART III

RADIAL CONSOLIDATION

Consolidation of a soil in which drainage is taking place radially is a special case of one-dimensional consolidation. However, radial drainage usually occurs in conjunction with vertical drainage and will be considered in the form appropriate to such a situation. Two-dimensional radial drainage and consolidation take place when vertical sand drains are driven into compressible fills in order to accelerate settlement.

(a) Derivation of equations

The governing equation of two-dimensional consolidation when radial drainage is taking place is:

$$\frac{\partial u}{\partial t} = c_z \frac{\partial^2 u}{\partial z^2} + c_r \left[\frac{1}{r} \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial r^2} \right] \qquad \text{III - 1.}$$

The region being considered is shown in Fig. III-1(a) with a region of finite dimensions enlarged in Fig. III-1(b).

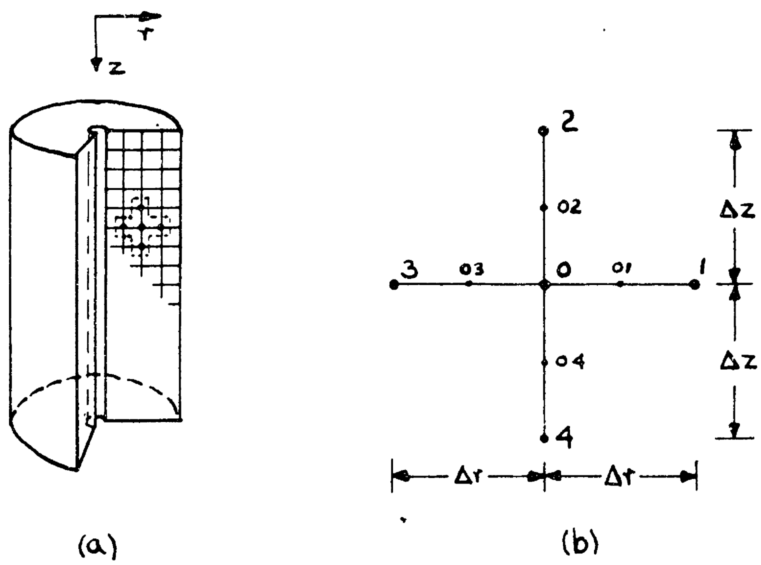


FIG. III-1

As in Part I $$\frac{\partial^2 u}{\partial z^2} = \frac{u_2 - 2u_0 + u_4}{\Delta z^2}$$ I - 3.

and hence $$\frac{\partial^2 u}{\partial r^2} = \frac{u_1 - 2u_0 + u_3}{\Delta r^2}$$

Also

$$\left(\frac{\partial u}{\partial r}\right)_{01} = \frac{u_1 - u_0}{\Delta r}$$

$$\left(\frac{\partial u}{\partial r}\right)_{03} = \frac{u_0 - u_3}{\Delta r}$$

whence $$\left(\frac{\partial u}{\partial r}\right)_{0} = \frac{u_1 - u_3}{2\Delta r}$$ III - 2.

Combining the three finite difference equations as indicated by equation II-1 gives:

$$\frac{u_{0,t+\Delta t} - u_{0,t}}{\Delta t} = \frac{c_z}{\Delta z^2}\left[u_{2,t} - 2u_{0,t} + u_{4,t}\right] + \frac{c_r}{\Delta r^2}\left[\frac{\Delta r}{2r_0}(u_{1,t} - u_{3,t}) + u_{1,t} - 2u_{0,t} + u_{3,t}\right]$$ III - 3.

Again assuming $c_r = c_z$, then Δr can be taken equal to Δz and equation III-3 reduces to

$$u_{0,t+\Delta t} = \frac{c_z \Delta t}{\Delta z^2}\left[\left(1 + \frac{\Delta z}{2r_0}\right)u_{1,t} + u_{2,t} + \left(1 - \frac{\Delta z}{2r_0}\right)u_{3,t} - 4u_{0,t}\right] + u_{0,t}$$ III - 4.

This equation, it will be observed, is similar to equation II-3, with the exception of a modifying factor for values of $u_{1,t}$ and $u_{3,t}$ due to the influence of radial flow. This modification has the greatest effect when r is small, i.e., on points near the axis.

After the mesh dimensions have been chosen, it will be found convenient to compute values of $(1 + \frac{\Delta z}{2r})$ and $(1 - \frac{\Delta z}{2r})$ for each column of mesh-points at radius r and to tabulate these values at the top of the relevant column for use in the calculations.

Use of the procedure outlined at the end of Part II, in which the directions of flow are considered separately, is of value in the solution of problems of combined radial and vertical consolidation. For radial drainage, in one dimension only, the iteration expression is:

$$u_{0,t+\Delta t} = \frac{c_r \Delta t}{\Delta r^2}\left[\left(1+\frac{\Delta r}{2r_0}\right)u_{1,t} + \left(1-\frac{\Delta r}{2r_0}\right)u_{3,t} - 2u_{0,t}\right] + u_{0,t}$$

III - 5.

The solution can be marched out using this equation, and the results combined with the desired vertical solution by means of equations II-9 and II-10.

In a practical case, the well considered is one of a number of wells of which the outer boundaries are adjacent (See Ref. (2)) and in which the "zones of influence," Fig. III-2(a) are hexagonal.

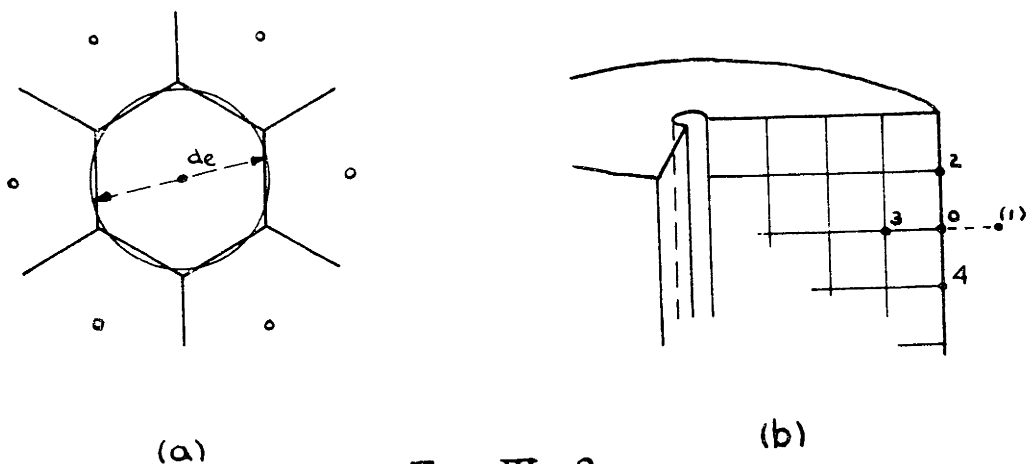


FIG. III - 2

Equivalent diameters are assumed as indicated, and an analysis by means of the methods outlined previously can be undertaken. There will be no flow across the common outer boundaries of the consolidating cylinders and these boundaries may be considered impervious. When an iteration process is carried out, it will be observed that a boundary point such as point 0 in Fig. III - 2(b) has two points, 2 and 4, surrounding it on the boundary and one, 3, inside the consolidating medium. In order to use expression III-4, four points are required, and it may be recognised that point 3 has an image point (1), beyond the boundary, in the adjacent consolidating cylinder, at the same value of hydrostatic excess pressure. Thus in the use of equation III-4 on the boundary, the value of hydrostatic excess pressure at point 3 is entered twice, giving the new value at point 0.

This situation will occur, in general, wherever a line of symmetry is encountered, and this fact may be used to eliminate needless calculation in many instances.

It has long been observed that the resistance of a pile driven in clay soils increases with time due, it is considered, to the consolidation of the clay round the pile, when the hydrostatic excess pressures caused by the driving of the pile decrease by movement of the pore water away from the pile. It will be possible, if the distribution of the hydrostatic excess pressures is known, to evaluate the time-consolidation characteristics of points adjacent to the pile, by numerical analysis, and from the results to calculate the shearing strength increase in the clay.

For the purposes of comparison with field tests, imperviousness, or varying degrees of porosity of the pile itself could be assumed in the calculations.

PART IV

THREE-DIMENSIONAL CONSOLIDATION

Although three-dimensional consolidation is that case which probably exists most frequently in nature, little or no attempt (3) to date has been made to tackle specific problems. As far as the numerical analysis of three-dimensional drainage cases is concerned, the ideas and procedures differ little from those already presented in preceding Parts, but the tedium of such procedures increases greatly. Consequently all available methods for abbreviating the computations should be investigated. To this end, the superposition methods outlined in Part II(c) will probably be found most useful.

(a) Derivation of equation

The derivation of the iteration expression for the three-dimensional consolidation case follows directly from the methods and results of Part I.

The governing equation is:

$$\frac{\partial u}{\partial t} = c_x \frac{\partial^2 u}{\partial x^2} + c_y \frac{\partial^2 u}{\partial y^2} + c_z \frac{\partial^2 u}{\partial z^2} \qquad \text{IV - 1.}$$

and the general iteration expression becomes

$$u_{0,t+\Delta t} = \frac{c_x \Delta t}{\Delta x^2}\left[u_{1,t} - 2u_{0,t} + u_{3,t}\right] + \frac{c_y \Delta t}{\Delta y^2}\left[u_{5,t} - 2u_{0,t} + u_{6,t}\right] + \frac{c_z \Delta t}{\Delta z^2}\left[u_{2,t} - 2u_{0,t} + u_{4,t}\right] \qquad \text{IV - 2.}$$

in which the points are shown in Fig. IV-1.

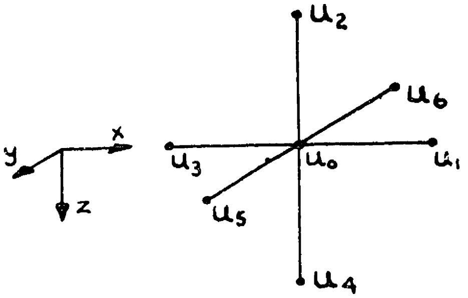


FIG. IV-1

And, as before, if homogeneous, isotropic soil is assumed to exist, then

$$u_{0,t+\Delta t} = \frac{c_z \Delta t}{\Delta z^2}\left[u_{1,t} + u_{2,t} + u_{3,t} + u_{4,t} + u_{5,t} + u_{6,t} - 6u_{0,t}\right] + u_{0,t} \quad \text{IV - 3.}$$

In practical computational work it is unlikely that this equation will be used however, necessitating, as it does, the use of various sheets of paper in order that the points may be plotted in a three-dimensional form. The analysis will most probably proceed from the consideration of the three directions of flow individually, with subsequent superposition.

PART V

TRANSFORMATION

In circumstances where certain boundary conditions prevail, a variation of the direct attack on the problem, as outlined previously, can be used.

For example, in the case of radial flow (10) it has already been pointed out that the modifications to the iteration expression have their greatest effect near the axis of the consolidating mass, and it is in this region that information is particularly desired regarding pressure conditions. Therefore, it would be advantageous to use a method which gave more information about the hydrostatic excess pore water pressures existing near the axis.

It has been shown in Part II that certain problems exist which have certain of their boundaries ill-defined, when the iteration procedure must be carried out in regions of little inherent interest, for the sake of the effect of the movement of the pore water in these regions, on consolidation adjacent to the loading area.

In the consolidation of hydraulically filled cores of earth dams(12) it will frequently be found that the mesh points do not intersect conveniently with the sloping boundaries, necessitating use of the cumbersome expressions derived in Appendix IV along these boundaries.

These situations can be considered by the use of transformation.

(a) Radial Flow

The equation

$$\frac{\partial u}{\partial t} = c_r\left[\frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial r^2}\right] \qquad V - 1.$$

has already been considered in Part III(a) and the direct iteration expression III-5 has been shown.

Now in equation V-1

let $\alpha = \log_e r$ V - 2.

Then $$\frac{\partial \alpha}{\partial r} = \frac{1}{r}$$

and $$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial \alpha} \cdot \frac{\partial \alpha}{\partial r} = \frac{1}{r} \cdot \frac{\partial u}{\partial \alpha}$$

Now

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial u}{\partial r}\right) = \frac{1}{r^2} \frac{\partial^2 u}{\partial \alpha^2}$$

Then $$\frac{\partial u}{\partial t} = \frac{c_r}{r^2} \cdot \frac{\partial^2 u}{\partial \alpha^2}$$ V - 3.

This demonstrates a simplified form of equation, with the α-axis in the same direction as the original r-axis. The iteration expression becomes

$$u_{o,t+\Delta t} = \frac{c_r \Delta t}{r_o^2 \Delta \alpha^2}\left[u_{1,t} - 2u_{o,t} + u_{3,t}\right] + u_{o,t}$$ V - 4.

For the sake of clarity, the untransformed and transformed sections of a sand drain well surrounded by a cylinder of compressible soil draining radially only toward the well are shown in Fig. V-1. The figure also shows the mesh spacing used in the calculation of values of hydrostatic excess pore water pressure.

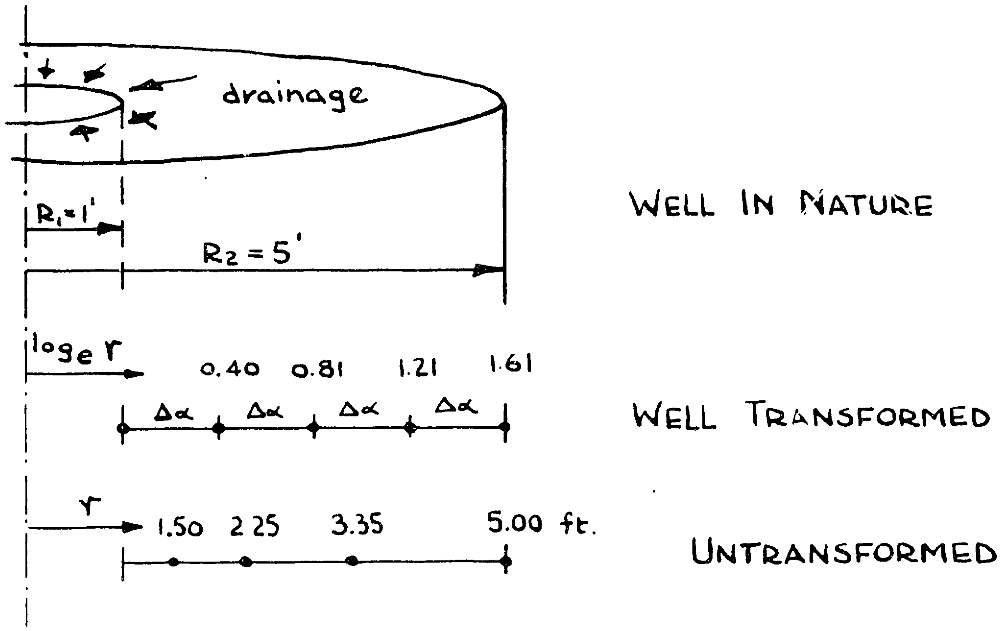


FIG. V-1

From the diagram it will be seen that the procedure consists in plotting from a suitable origin points with α -direction ordinates of $\log_e R_1$ and $\log_e R_2$. The distance between these points is divided into a convenient number of equal mesh spacings $\Delta\alpha$, and for the points thus obtained, the corresponding values of r are calculated using equation V-2. It will be convenient to have values of $\frac{1}{r^2}$ tabulated over the mesh points as this figure is utilised as indicated in expression V-4. The iteration procedure can be carried out as previously demonstrated until a desired time has been reached, when the values can be entered in the untransformed scale, and combined, if desired, with values obtained from the case of vertical drainage.

(b) Two-dimensional Flow With an Infinite Boundary

A case such as shown in Fig. II-1 will be considered here, with a transformation taking place only in the x-direction. For flow in the x-direction, the equation

$$\frac{\partial u}{\partial t} = c_x \frac{\partial^2 u}{\partial x^2} \qquad \text{V - 5.}$$

holds and an arbitrary axis as the origin of x-measurements can be chosen. This can conveniently be an axis of symmetry which Figure II-1 demonstrates.

If, again, the substitution

$$\alpha = \log_e x \qquad \text{V - 6.}$$

is made, equation V-5 becomes

$$\frac{\partial u}{\partial t} = \frac{c_x}{x^2}\left[\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial x}\right] \qquad \text{V - 7.}$$

and, consequently, the iteration equation developed is

$$u_{o,t+\Delta t} = \frac{c_x \Delta t}{x_o^2 \Delta\alpha^2}\left[u_{1,t} - 2u_{o,t} + u_{3,t} - \frac{\Delta\alpha}{2}(u_{1,t} - u_{3,t})\right] + u_{o,t}$$

V - 8.

In this case, values of $\frac{1}{x_o^2}$ and $\frac{\Delta\alpha}{2}$ must be plotted above each mesh point, to be used in the calculation. Equation V-8 is tedious to manipulate, but it will generally result in the computation proceeding with fewer mesh points.

(c) Hydraulically Filled Cores of Earth Dams.

Gibson and Lumb (12) in an analysis of hydraulically filled cores, point out that in general, the base angle of the core (see Fig. V-2) does not permit the mesh points to lie on the inclined faces of the core.

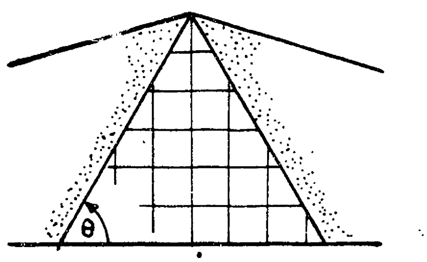


FIG. V-2

The equation governing the hydrostatic excess pore water pressure in this case is the two-dimensional one

$$\frac{\partial u}{\partial t} = c_x \frac{\partial^2 u}{\partial x^2} + c_z \frac{\partial^2 u}{\partial z^2} \qquad \text{V - 9.}$$

Because of the method of filling the material, involving thorough mixing of the soil in water in this case, it is likely that taking c_x equal to c_z is a reasonable assumption.

The substitution

$$z = \omega \tan\theta \qquad \text{V - 10.}$$

is then made, and equation V-9 is transformed into

$$\frac{\partial u}{\partial t} = c_z \left(\frac{\partial^2 u}{\partial x^2} + \cot^2\theta \frac{\partial^2 u}{\partial \omega^2} \right) \qquad \text{V - 11.}$$

By means of this substitution, all base angles become, in the x,w-plane, angles of 45°, and mesh points therefore lie on the inclined core faces. The finite difference equation in this case is

$$u_{o,t+\Delta t} = \frac{c_z \Delta t}{\Delta z^2} \left[u_{1,t} - 2u_{o,t} + u_{3,t} + \cot^2\theta \left(u_{2,t} - 2u_{o,t} + u_{4,t} \right) \right] + u_{o,t}$$

V - 12.

In this problem, the outer fill containing the core consists of pervious material, and in consequence of this, the hydrostatic excess pore water pressures become equal to zero on the inclined faces of the core. It should be pointed out that this case involves outward flow of pore water in appreciable quantities, affecting the unit weight of the core, and hence the total pressure at any point, so that care must be taken in the evaluation of intergranular pressures.

It is probable that other problems will be simplified or clarified by the use of substitutions, and each case should be examined on its own merits in order to see if such a simplification can be effected.

CONCLUSION

A method has been presented which can be used to attack various problems concerned with the diffusion of water in the pores of soil. No investigation of the accuracy of solutions has been entered into as various previous authors have examined the procedures from the mathematical point of view of convergence.

Emphasis must be laid on the fact that the equations and presentations of the previous Parts depend entirely on the validity of the partial differential equations developed to define the pore-water pressures. If these equations do not hold true, then there is little value in solving them by any method.

The phenomenon of secondary compression is one on which much thought has been expended, and it would seem a promising future course of research to investigate mathematical theories of the action of secondary compression by means of finite difference methods, as these, though tedious, will almost certainly demonstrate the validity of any hypothesis, with greater rapidity than an analytical approach. It may be possible to postulate several different formulations for the explanation of secondary compression, and by the exercise of numerical techniques, to choose those which give results most nearly approximating the behavior of laboratory or field specimens.

By the use of these methods, information derived from soil tests may be applied directly to the analysis of soil movements in actual practice.

The use of numerical analysis methods is not likely to be confined to the investigation of hypotheses regarding the mechanics and hydraulics of soil and water action, however, many other cases may be considered eventually. Among these cases, two are of immediate interest; the problem in which a frozen, supersaturated soil is melting, with subsequent consolidation and that of the differential settlement of a building on columns supported by a consolidating medium. In the first instance, the case of gradual non-linear melting of the ice lenses, with subsequent release of water under hydrostatic excess pressure into a soil which may or may not be homogeneous and isotropic, is one which can perhaps best be treated by iteration involving both heat transfer and consolidation in a similar manner.

The second problem, which, classed with all situations involving the foundation of elastic structures or settling media, has long interested mathematicians and engineers is concerned with the gradual movement of the foundation, with a subsequent redistribution of the loads upon it, which in turn influence the consolidation. A step by step procedure here, correlating the movement of the foundations with the redistribution of moments and forces in the structure above, would appear well suited to the problem.

Because of the tedium involved in solutions by numerical analysis it is to be hoped that the methods themselves may be considered as leading toward the development, where possible, of automatic computing machine solutions to the more complex soil problems.

In conclusion, it is felt that the attitude expressed by the numerical analysis of soil engineering problems is in keeping with the whole philosophy of engineering science, which is, at best, an empirical subject and in which knowledge itself has advanced step by step.

APPENDIX I

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APPENDIX II

List of Symbols

Units are given in parentheses: F = Force, L = Length, T = Time

$A_{x,y,z}$	dimensionless parameter = $\frac{c_{x,y,z} \cdot \Delta t}{(\Delta x,y,z)^2} = \frac{\Delta T}{(\Delta s)^2}$ (-)
α	substitution coordinate (L)
A, B, C, D, E, F, G, H	numerical constants (-)
c_x, c_y, c_z, c_r	coefficient of consolidation in x,y,z,r direction ($\frac{L^2}{T}$)
C_c	compression index (-)
e	void ratio (-)
E	error ($\frac{F}{L^2}$)
h	constant mesh dimension (L)
H	length of longest drainage path in one-dimensional consolidation (L)
θ	base angle of hydraulic fill core (-)
i	hydraulic gradient (-)
k	coefficient of permeability; proportionality constant ($\frac{L}{T}$)
$\bar{p}$	intergranular pressure ($\frac{F}{L^2}$)
r	radial coordinate (L)
r_o	radial coordinate at point 0 (L)
ρ	settlement (L)
r_1, r_2, s_1, s_2	dimensionless numbers (-)
s	ratio of z to H (-)
t	time (T)
T	time factor (-)
u	hydrostatic excess pore water pressure in the voids of the soil ($\frac{F}{L^2}$)

$\bar{u}$	average u $(\frac{F}{L^2})$
u_i	initial u $(\frac{F}{L^2})$
$u_{n,t}$	value of u at point n, at time t. $(\frac{F}{L^2})$
U	degree of consolidation = $1 - \frac{u}{u_i}$ (-)
$\bar{U}$	average U (-)
v	superficial velocity of pore water $(\frac{L}{T})$
w	substitution coordinate (L)
x,y,z	rectangular coordinates (L)
Δx, Δy, Δz, Δr	finite distances in x,y,z,r directions (L)

APPENDIX III

VARIABLE c_z

CALCULATIONS

Assume that c_z at final intergranular pressure is twice initial c_z, with a linear variation in between.

Then, from equation I-18

$$c_z = c_{z_1} + c_{z_1}\frac{U}{100} = c_{z_1}\left(1 + \frac{U}{100}\right)$$

Make $\dfrac{c_z \Delta t}{\Delta z^2} = \dfrac{1}{5}$, initially

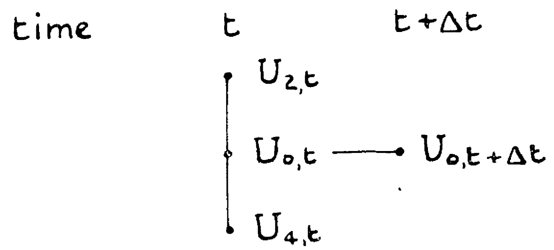
when $c_z = c_{z_1}$; $\bar{p} = \bar{p}_1$; $U = 0$.

Then, at any U,

$$A = \frac{1}{5c_{z_1}} \cdot c_{z_1}\left(1 + \frac{U}{100}\right)$$

$$= \frac{1}{5}\left(1 + \frac{U}{100}\right)$$

Now consider any four points as shown in the diagram.



We have $U_{0,t+\Delta t} = A\left[U_{2,t} - 2U_{0,t} + U_{4,t}\right] + U_{0,t}$ — I-19

$$\therefore\quad U_{0,t+\Delta t} = A\left[U_{2,t} + U_{4,t}\right] + (1-2A)\,U_{0,t}$$

$$= \frac{1}{5}\left(1 + \frac{U_{0,t}}{100}\right)\left[U_{2,t} + U_{4,t}\right] + \left[1 - \frac{2}{5}\left(1 + \frac{U_{0,t}}{100}\right)\right]U_{0,t}$$

Curves were then prepared as indicated in Part I-f(2)

For the case of constant c_z, the factor A was also chosen equal to 1/5 and Δs equal to 1/4.

Thus, from equation I-5(a) $\Delta T = \frac{1}{5}\left(\frac{1}{4}\right)^2 = \frac{1}{80}$

RESULTS

c_z constant

T				0.05				0.1	Z/H
100	100	100	100	100	100	100	100	100	0
0	20	32	40	45.8	50.5	54.1	56.9	59.3	1/4
0	0	4	8.8	15.4	18.9	22.3	25.6	28.5	1/2
0	0	0	0.8	2.2	4.5	6.7	9.0	11.3	3/4
0	0	0	0	0.3	1.1	2.5	4.2	6.1	1
T				0.15				0.2	
	100	100	100	100	100	100	100	100	0
	61.3	63.2	64.8	66.0	67.1	68.3	69.4	70.6	1/4
	31.4	33.9	36.1	38.2	40.3	42.4	44.3	45.9	1/2
	13.8	16.1	18.3	20.8	23.2	25.5	27.7	30.0	3/4
	8.0	10.3	12.7	14.9	17.3	19.7	22.1	24.4	1
T				0.25				0.3	
	100	100	100	100	100	100	100	100	0
	71.6	72.6	73.4	74.3	75.0	75.7	76.4	77.2	1/4
	47.6	49.3	51.0	52.6	54.0	55.3	56.6	57.8	1/2
	32.2	34.2	36.1	38.0	39.9	41.7	43.5	45.1	3/4
	26.6	28.8	31.0	33.0	34.9	36.8	38.7	40.6	1
T				0.35				0.4	
	100	100	100	100	100	100	100	100	0
	77.7	78.3	79.4	79.8	80.6	81.2	81.9	82.5	1/4
	59.0	60.3	61.4	62.7	63.9	64.9	66.1	67.2	1/2
	46.6	48.2	49.8	51.2	52.8	54.3	55.6	56.8	3/4
	42.5	44.2	45.6	47.4	49.0	50.4	51.9	53.4	1
T				0.45				0.5	
	100	100	100	100	100	100	100	100	0
	83.1	83.6	84.2	84.6	85.2	85.4	86.0	86.3	1/4
	68.2	69.4	70.1	71.2	71.8	73.0	73.6	74.7	1/2
	58.4	59.5	60.8	61.7	63.2	64.1	65.5	66.2	3/4
	54.6	56.3	57.4	58.8	59.7	61.3	62.3	63.9	1

c_z constant (cont.)

T				0.55				0.6
	100	100	100	100	100	100	100	100
	86.8	87	87.6	88.1	88.5	88.9	89.4	89.6
	75.2	76.1	76.7	77.7	78.3	79.2	79.8	80.7
	67.7	68.4	69.6	70.3	71.5	72.4	73.4	74.3
	64.6	66.0	66.7	68.1	69.0	70.0	71.0	72.1
T				0.65				0.7
	100	100	100	100	100	100	100	100
	90.1	90.3	90.9	91.1	91.6	91.8	92.2	92.5
	81.2	82.1	82.6	83.5	83.9	84.7	85.2	85.8
	75.3	76.1	77.3	77.8	79.0	79.6	80.6	81.3
	73.0	74.3	75.0	76.2	77.0	77.9	78.7	79.7
T				0.75				0.8
	100	100	100	100	100	100	100	100
	92.7	93.1	93.4	93.7	94.0	94.3	94.5	94.6
	86.3	87.0	87.5	88.2	88.7	89.3	89.6	90.2
	82.2	82.8	83.6	84.3	85.1	85.6	86.3	86.8
	80.6	81.6	82.2	83.0	83.7	84.5	85.1	85.7
T				0.85				0.9
	100	100	100	100	100	100	100	100
	94.9	95.1	95.3	95.6	95.7	96.0	96.2	96.4
	90.4	90.9	91.3	91.7	92.2	92.5	93.0	93.3
	87.4	87.9	88.6	89.1	89.6	90.2	90.6	91.3
	86.2	86.9	87.4	88.2	88.8	89.3	90.0	90.3
T				0.95				1.0
	100	100	100	100	100	100	100	100
	96.6	96.7	96.8	97.1	97.3	97.4	97.6	97.7
	93.7	94.0	94.5	94.7	95.0	95.3	95.6	95.8
	91.6	92.3	92.6	93.3	93.5	94.0	94.3	94.6
	91.2	91.4	92.2	92.5	93.2	93.4	93.8	94.3

RESULTS

c_z varying

T				0.05				0.1	z/H
100	100	100	100	100	100	100	100	100	0
0	20	34.3	43.7	50.0	54.4	57.9	60.6	62.8	1/4
0	0	4.0	9.2	14.7	20.0	24.5	28.4	31.9	1/2
0	0	0.0	0.8	2.3	4.6	7.3	10.0	12.7	3/4
0	0	0.0	0.0	0.3	1.2	2.6	4.7	7.0	1
T				0.15				0.2	
	100	100	100	100	100	100	100	100	0
	64.8	66.6	68.2	69.6	70.9	72.0	73.2	74.3	1/4
	35.0	37.9	40.5	42.9	45.3	47.6	49.9	51.9	1/2
	15.7	18.7	21.7	24.6	27.6	30.5	33.3	36.2	3/4
	9.4	11.9	14.7	18.2	21.2	24.3	27.4	30.4	1
T				0.25				0.3	
	100	100	100	100	100	100	100	100	0
	75.4	76.3	77.2	78.1	79.3	80.3	81.2	82.1	1/4
	53.9	55.7	57.5	59.5	61.3	63.0	64.8	66.6	1/2
	38.9	41.4	44.0	46.5	48.8	51.2	53.6	55.7	3/4
	33.4	36.3	39.2	41.7	44.4	47.0	49.6	52.0	1
T				0.35				0.4	
	100	100	100	100	100	100	100	100	0
	82.9	83.7	84.4	85.2	85.9	86.6	87.3	88.1	1/4
	68.2	69.7	71.1	72.5	73.9	75.2	76.6	77.7	1/2
	57.9	60.0	61.8	63.7	65.6	67.5	69.2	71.1	3/4
	54.2	56.4	58.6	60.5	62.6	64.6	66.7	68.5	1
T				0.45				0.5	
	100	100	100	100	100	100	100	100	0
	88.6	89.4	89.9	90.6	91.1	91.6	92.1	92.6	1/4
	79.3	80.2	81.4	82.4	83.4	84.4	85.3	86.1	1/2
	72.6	74.3	75.6	77.1	78.3	79.6	80.7	82.0	3/4
	70.5	72.1	73.7	75.1	76.6	77.7	79.3	80.3	1

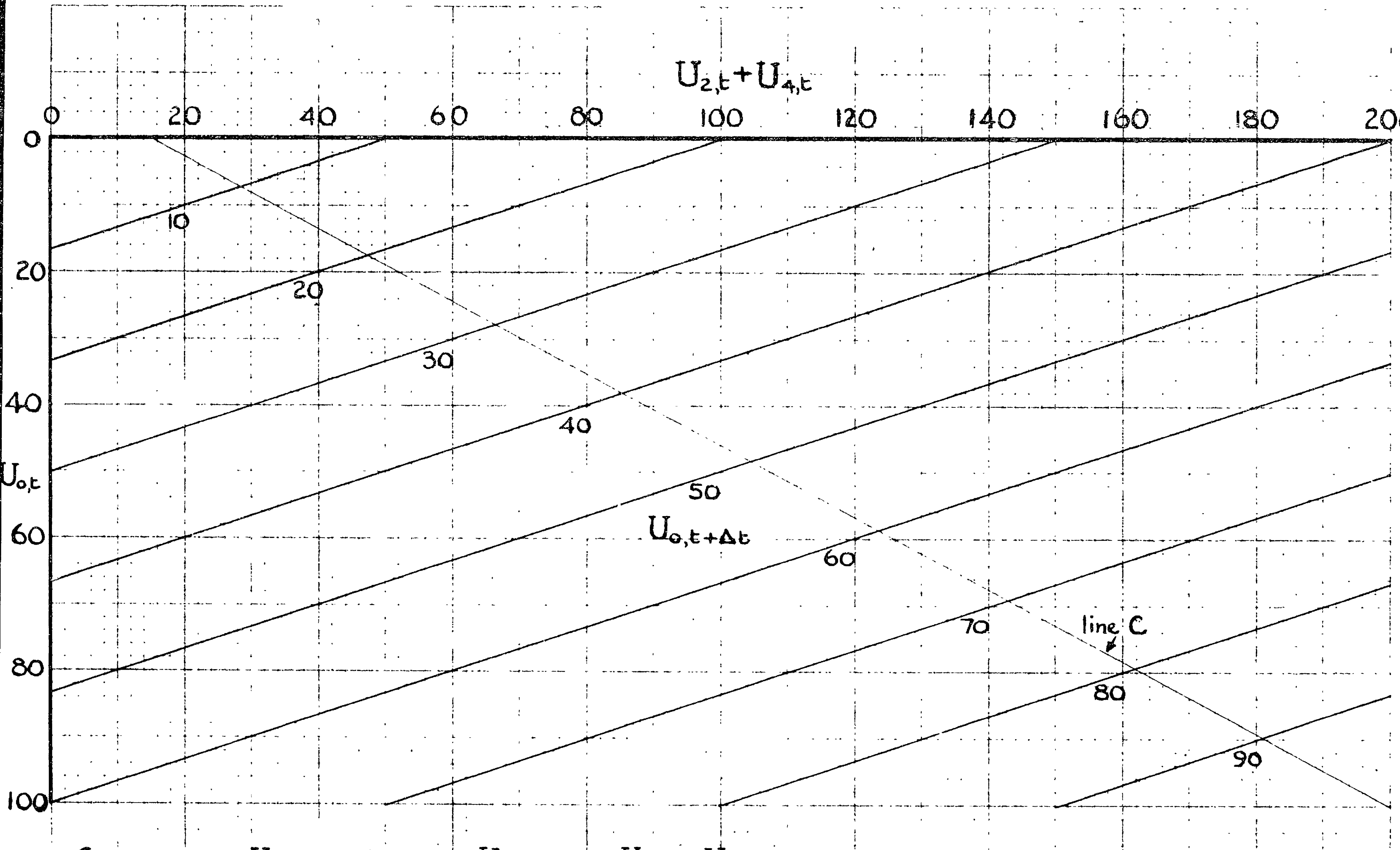
T			0.55				0.6
100	100	100	100	100	100	100	100
92.9	93.3	93.6	94.1	94.4	94.6	94.9	95.2
87.0	87.6	88.4	88.9	89.7	90.2	90.7	91.3
82.8	84.1	84.6	85.7	86.3	87.3	87.9	88.6
81.6	82.4	83.7	84.3	85.3	86.0	87.0	87.6

T			0.65				0.7
100	100	100	100	100	100	100	100
95.6	95.7	96.1	96.2	96.5	96.6	96.7	97.0
91.7	92.3	92.6	93.1	93.3	93.9	94.1	94.5
89.3	89.7	90.5	90.8	91.6	91.7	92.4	92.6
88.3	89.1	89.5	90.4	90.6	91.4	91.6	92.3

T			0.75				0.8
100	100	100	100	100	100	100	100
97.2	97.3	97.6	97.7	97.7	97.8	98.0	98.1
94.7	95.1	95.3	95.6	95.7	96.1	96.2	96.6
93.3	93.4	94.0	94.2	94.6	94.7	95.2	95.3
92.4	93.2	93.3	93.8	94.1	94.5	94.6	95.0

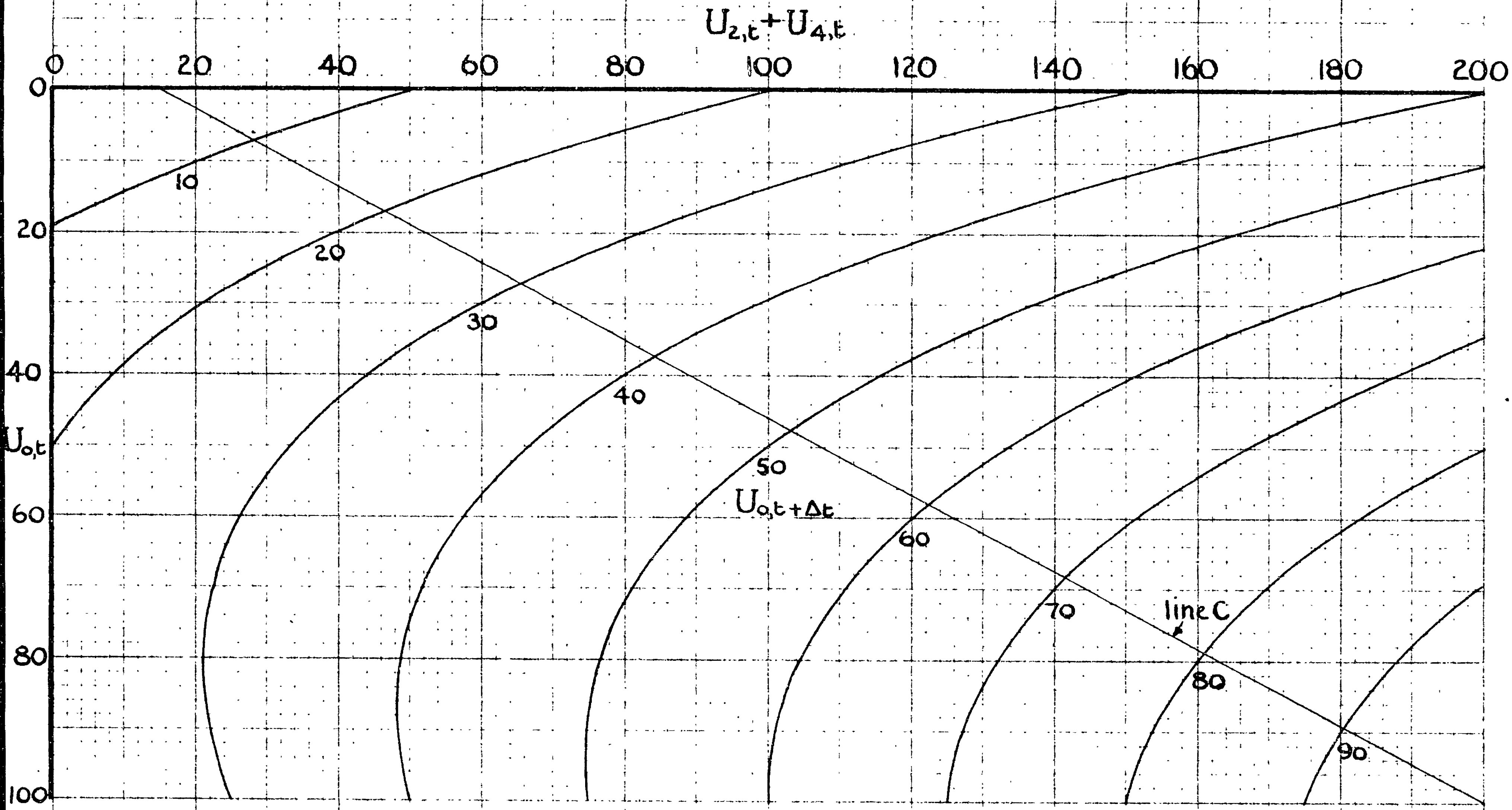
T			0.85				0.9
100	100	100	100	100	100	100	100
98.3	98.4	98.4	98.5	98.6	98.6	98.7	98.7
96.7	96.8	97.0	97.2	97.4	97.5	97.6	97.7
95.6	95.7	96.0	96.3	96.4	96.7	96.8	97.0
95.2	95.4	95.6	95.8	96.2	96.3	96.6	96.7

T			0.95				1.0
100	100	100	100	100	100	100	100
98.7	98.8	98.9	99.0	99.1	99.2	99.2	99.3
97.8	97.9	98.0	98.2	98.4	98.5	98.6	98.7
97.2	97.3	97.6	97.7	97.9	98.0	98.2	98.3
96.8	97.2	97.3	97.5	97.6	97.8	98.0	98.2



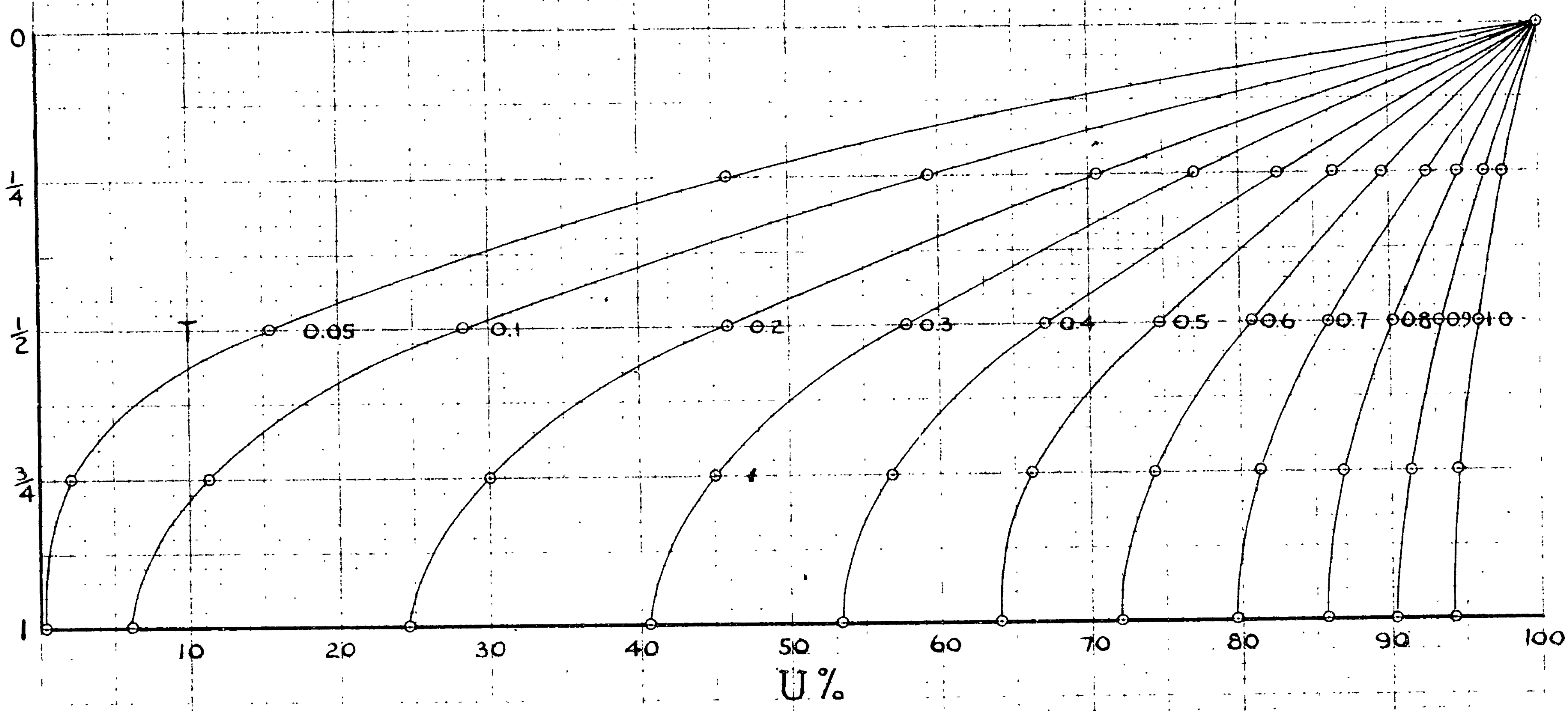
CURVES OF $U_{o,t+\Delta t}$ VERSUS $U_{o,t}$ AND $U_{2,t} + U_{4,t}$
FOR c_z CONSTANT AND $A = 1/5$.

CHART I



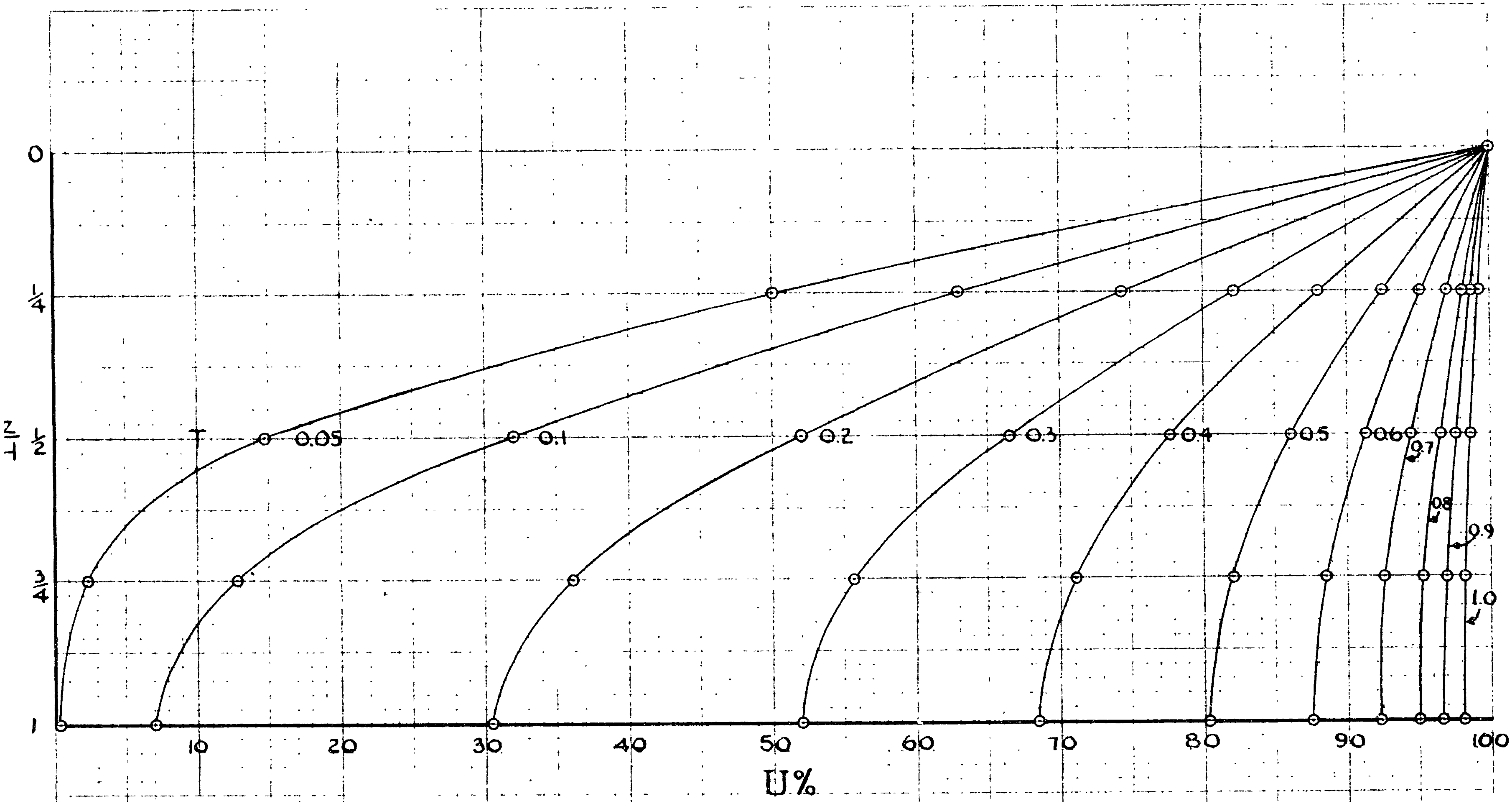
CURVES OF $U_{0,t+\Delta t}$ VERSUS $U_{0,t}$ AND $U_{2,t} + U_{4,t}$
FOR c_z VARYING LINEARLY AND
$A = 1/5$ INITIALLY.

CHART II



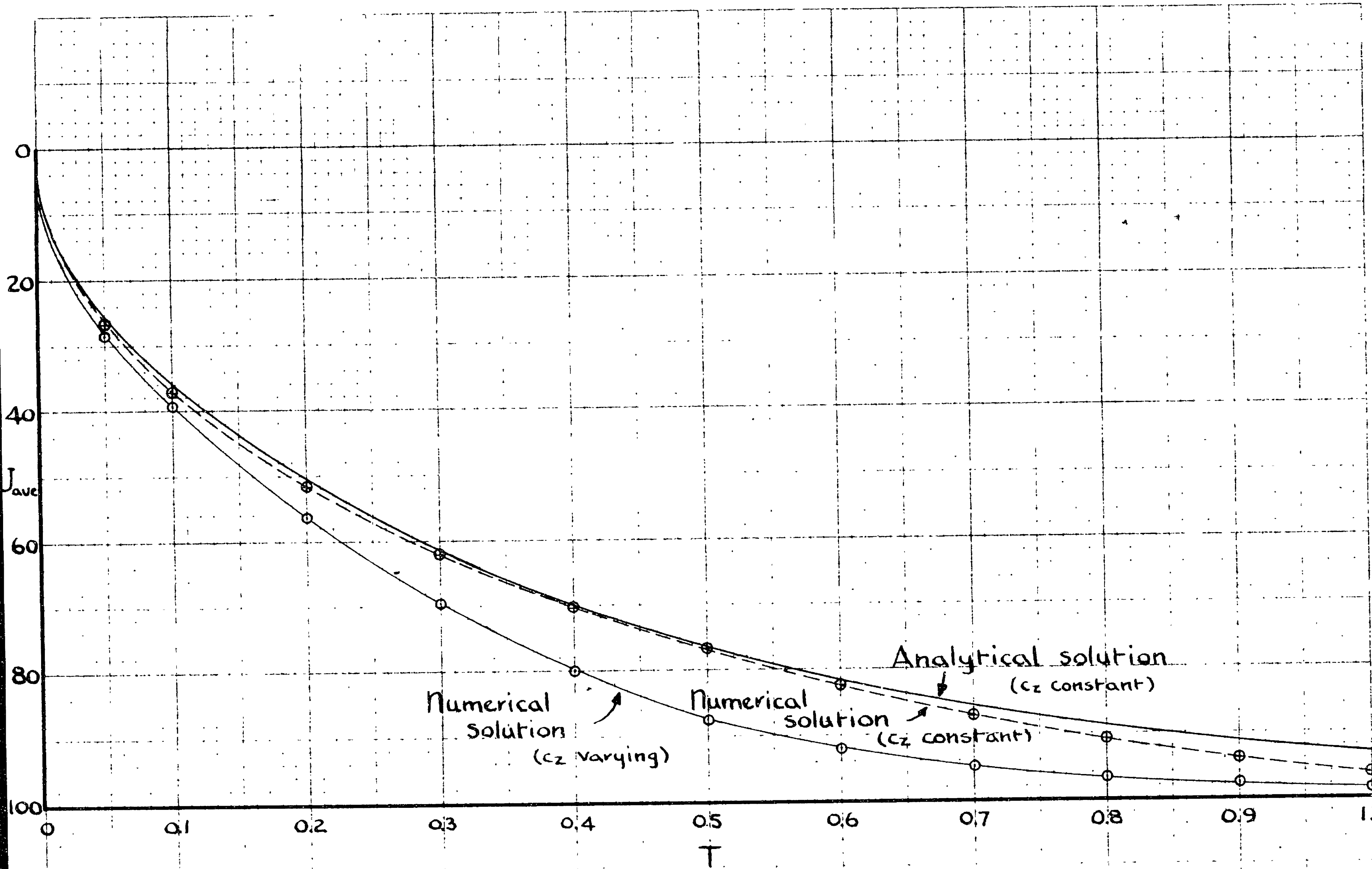
Curves of U versus $\frac{Z}{H}$ and T
For c_z Constant and $A = 1/5$

Chart III



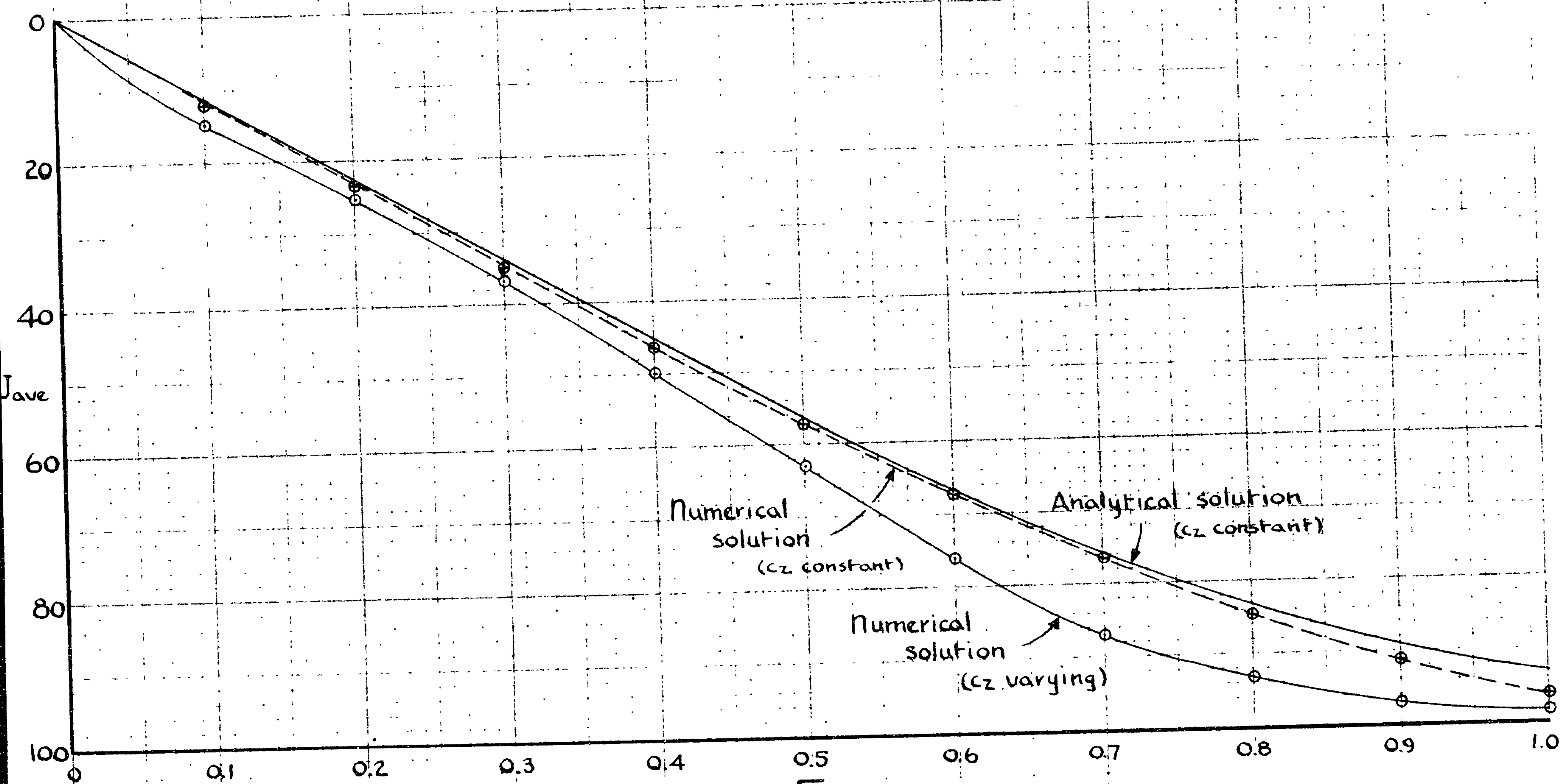
CURVES OF U VERSUS $\frac{Z}{H}$ AND T
FOR c_z VARYING

CHART IV



CURVES OF U_{ave} VERSUS T

CHART V



CURVES OF U_{ave} VERSUS $\sqrt{T}$

CHART VI

Some discussion of the methods and results obtained from the calculations in Appendix III is given in the following paragraphs.

(1) Using the computation curves.

For the case chosen, that of instantaneous loading of the soil surface, with subsequent one-dimensional pore water flow, it will be seen that the hydrostatic excess pressure at any one point, can be traced along a curve on Charts I and II, as time progresses. This curve eventually becomes a straight line, line C, on both charts, at different times for different points, which passes through the point $U_{0,t+\Delta t} = 100$, $U_{0,t} = 100$. By superposition of the two charts it is found that the straight lines are coincident on the two charts. It may be reasoned that eventually, the successive changes in hydrostatic excess pressure are so small in both the constant and variable c_z problems, that the effect of the variable c_z becomes negligible, although the same values of pressure are reached on the variable c_z curve at earlier times. The straight line is obtained for the reason that, as zero pore pressures are neared, the pressure at any point at time $t + \Delta t$ depends more on the value of the same point at t, than on the two neighboring points, as all points attain nearly the same value and consequently the later pressure becomes a linear function of the earlier one. This can be utilised to great advantage in speeding up the calculations, and relieving the tedium. However, this will not be possible if fluctuating loads are imposed on the boundary.

It would appear to be possible, when the straight line portion of the curve is reached to establish on it a time scale for each point in the adopted mesh, and by this means the pressure at any point at any time (on the linear portion) could be obtained without recourse to the intermediate steps. However, more work requires to be done on this aspect.

(2) Results

The curves shown on Charts III and IV were integrated, to give the average consolidation versus time factor curves shown on Charts V and VI. Together with the curves obtained by numerical analysis is displayed the analytical solution to the same problem, with the assumption of constant coefficient of consolidation, for purposes of comparison. It is seen that in the range in which most interest is centered, the numerical solution is not greatly in error, and this error may be decreased by the methods of Part I-(g). It is deduced, therefore, that, for the stated conditions, the numerical solution curve of variable coefficient of consolidation does not deviate from a correct, or analytical solution by more than the constant coefficient of consolidation curve does.

In this case, some noteworthy results are observed, which may be applied to the better interpretation of laboratory data, when more data is available regarding the variation of the coefficient of consolidation. For the sake of comparison with one standard laboratory procedure for determining the coefficient of consolidation Fig. VI is given showing a plot of average degree of consolidation versus the square root of time factor. No deductions will be made from the curve owing to the particularised nature of the assumptions leading to its production.

Appendix IV

Derivation of General, Two-Dimensional Iteration Equation From Taylor's Series.

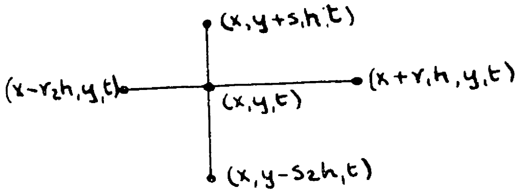
References (11) and (18)

If a function has continuous derivatives of high enough order then a Taylor's Series expansion of the function may be written:

If $f'_x(x,y,t) = \dfrac{\partial(x,y,t)}{\partial x}$, $f''_y(x,y,t) = \dfrac{\partial^2(x,y,t)}{\partial y^2}$

$f'''_{xxy}(x,y,t) = \dfrac{\partial^3(x,y,t)}{\partial x^2\,\partial y}$ and reference is made to Fig. 1

in which h is a constant length and r_1, r_2, s_1, and s_2 are dimensionless numbers, the following expressions may be derived:



$$f(x+r_1h,y,t) = f(x,y,t) + r_1hf'_x(x,y,t) + \tfrac{1}{2}r_1^2h^2f''_x(x,y,t) + \tfrac{1}{6}r_1^3h^3f'''_x(x,y,t) + \cdots \quad \text{— (a)}$$

$$f(x-r_2h,y,t) = f(x,y,t) - r_2hf'_x(x,y,t) + \tfrac{1}{2}r_2^2h^2f''_x(x,y,t) - \tfrac{1}{6}r_2^3h^3f'''_x(x,y,t) + \cdots \quad \text{— (b)}$$

$$f(x,y+s_1h,t) = f(x,y,t) + s_1hf'_y(x,y,t) + \text{etc.} \quad \text{— (c)}$$

$$f(x,y-s_2h,t) = f(x,y,t) - s_2hf'_y(x,y,t) + \text{etc.} \quad \text{— (d)}$$

Neglecting powers of h^5 and higher, multiply (a), (b), (c) and (d) by r_2, r_1, s_2 and s_1 respectively, and add (a) and (b), (c) and (d).

$$r_2f(x+r_1h,y,t) + r_1f(x-r_2h,y,t) = (r_1+r_2)f(x,y,t) + \tfrac{1}{2}h^2r_1r_2(r_1+r_2)f''_x(x,y,t) + \tfrac{1}{24}h^4r_1r_2(r_1^3+r_2^3)f^{iv}_x(x,y,t) + \cdots \quad \text{— (e)}$$

$$s_2f(x,y+s_1h,t) + s_1f(x,y-s_2h,t) = (s_1+s_2)f(x,y,t) + \text{etc.} \quad \text{— (f)}$$

Multiply (e) by $\dfrac{c_x}{r_1r_2(r_1+r_2)}$, (f) by $\dfrac{c_y}{s_1s_2(s_1+s_2)}$ and add

$$c_x\left[\frac{f(x+r_1h,y,t)}{r_1(r_1+r_2)} + \frac{f(x-r_2h,y,t)}{r_2(r_1+r_2)}\right] + c_y\left[\frac{f(x,y+s_1h,t)}{s_1(s_1+s_2)} + \frac{f(x,y-s_2h,t)}{s_2(s_1+s_2)}\right]$$

$$= \left[\frac{c_x}{r_1r_2} + \frac{c_y}{s_1s_2}\right]f(x,y,t) + \frac{c_xh^2}{2}f''_x(x,y,t) + \frac{c_yh^2}{2}f''_y(x,y,t) + \frac{c_xh^4}{24}\frac{(r_1^3+r_2^3)}{(r_1+r_2)}f^{iv}_x(x,y,t) + \frac{c_yh^4}{24}\frac{(s_1^3+s_2^3)}{(s_1+s_2)}f^{iv}_y(x,y,t) + \cdots \quad \text{— (g)}$$

Also $$f(x,y,t+\Delta t) = f(x,y,t) + \Delta t f'_t(x,y,t) + \frac{\Delta t^2}{2}f''_t(x,y,t) + \frac{\Delta t^3}{6}f'''_t(x,y,t) + \cdots \quad \text{(h)}$$

Multiply (g) by $\frac{2\Delta t}{h^2}$ and subtract from (h)

$$\therefore f(x,y,t+\Delta t) - \frac{2c_x\Delta t}{h^2}\left[\frac{f(x+r_1h,y,t)}{A} + \frac{f(x-r_2h,y,t)}{B}\right]$$

$$- \frac{2c_y\Delta t}{h^2}\left[\frac{f(x,y+s_1h,t)}{C} + \frac{f(x,y-s_2h,t)}{D}\right]$$

$$+ \left[\frac{2c_x\Delta t}{Eh^2} + \frac{2c_y\Delta t}{Fh^2} - 1\right]f(x,y,t)$$

$$= \Delta t f_t'(x,y,t) - \Delta t\left[c_x f_x''(x,y,t) + c_y f_y''(x,y,t)\right] + \frac{\Delta t^2}{2} f_t''(x,y,t)$$

$$- \frac{\Delta t h^2}{12}\left[c_x f_x^{IV}(x,y,t)G + c_y f_y^{IV}(x,y,t)H\right]$$

where $A = r_1(r_1+r_2)$; $B = r_2(r_1+r_2)$; $C = s_1(s_1+s_2)$; $D = s_2(s_1+s_2)$
$E = r_1 r_2$; $F = s_1 s_2$; $G = \frac{r_1^3+r_2^3}{r_1+r_2}$; $H = \frac{s_1^3+s_2^3}{s_1+s_2}$

From consolidation equation II-1

$$f_t'(x,y,t) = c_x f_x''(x,y,t) + c_y f_y''(x,y,t)$$

and $\therefore f_t''(x,y,t) = c_x^2 f_x^{IV}(x,y,t) + c_y^2 f_y^{IV}(x,y,t)$.

Then, using previous notation

$$u_{0,t+\Delta t} - \frac{2c_x\Delta t}{h^2}\left[\frac{u_{1,t}}{A} + \frac{u_{3,t}}{B}\right] - \frac{2c_y\Delta t}{h^2}\left[\frac{u_{2,t}}{C} + \frac{u_{4,t}}{D}\right] + \left[\frac{2\Delta t}{h^2}\left(\frac{c_x}{E} + \frac{c_y}{F}\right) - 1\right]u_{0,t}$$

$$= \frac{c_x\Delta t h^2}{12}\left[\frac{6c_x\Delta t}{h^2} - G\right]f_x^{IV}(x,y,t) + \frac{c_y\Delta t h^2}{12}\left[\frac{6c_y\Delta t}{h^2} - H\right]f_y^{IV}(x,y,t) + \quad \text{—— (i)}$$

This is the general iteration equation for two-dimensional consolidation.

Let $r_1 = r_2 = s_1 = s_2 = 1$; $c_x = c_y$

$$\therefore u_{0,t+\Delta t} - \frac{c_x\Delta t}{h^2}\left[u_{1,t} + u_{2,t} + u_{3,t} + u_{4,t} - 4u_{0,t}\right] - u_{0,t}$$

$$= \frac{c_x\Delta t h^2}{12}\left(\frac{6c_x\Delta t}{h^2} - 1\right)\left[f_x^{IV}(x,y,t) + f_y^{IV}(x,y,t)\right] + \cdots \quad \text{—— (j)}$$

This is the equation for the particular case of equal mesh dimensions and coefficients of consolidation in both directions. It is seen, then, that the principal error is dependent on h^2, the mesh-dimension, and this fact is utilised in Part I-(g) to effect more accurate solutions to the one-dimensional consolidation case. It is interesting to observe that a choice of $\frac{c_z \Delta t}{h^2} = 1/6$ will eliminate the error due to the term on the right, and it can therefore be deduced that solutions obtained using this value will be nearer the analytical (or exact) solution than solutions obtained with even smaller values, which hitherto tended to give better results.

APPENDIX V

BOUNDARY POINTS IN THE TWO-DIMENSIONAL STRATIFIED CASE.

Reference (30)

Consider mesh-points shown in Fig. 1(a).

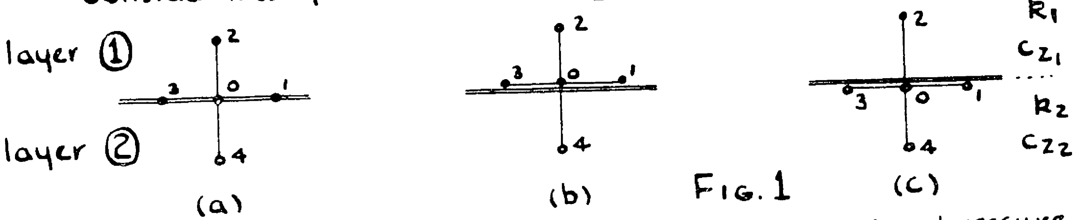


FIG. 1

The changing values of hydrostatic excess pore water pressure at point 0 depend on the actual values at points 1, 2, 3, and 4. Suppose, however, that points 0, 1, 2, and 3 exist wholly inside layer ① as in Fig. 1(b), and point 4 is inside layer ②.

Then equation II-3 gives

$$u_{o,t+\Delta t} - u_{o,t} = \frac{\Delta t\, c_{z_1}}{\Delta z^2}\left[u_{1,t} + u_{2,t} + u_{3,t} + u'_{4,t} - 4u_{o,t}\right] \quad \text{— (a)}$$

where $u'_{4,t}$ is some assumed value of hydrostatic excess pore water pressure existing at point 4 so that equation (a) is satisfied.

Similarly, if the situation existing in Fig. 1(c) holds, with points 0, 1, 3, and 4 inside layer ② and point 2 is in layer ①

Then

$$u_{o,t+\Delta t} - u_{o,t} = \frac{\Delta t\, c_{z_2}}{\Delta z^2}\left[u_{1,t} + u'_{2,t} + u_{3,t} + u_{4,t} - 4u_{o,t}\right] \quad \text{— (b)}$$

where $u_{o,t+\Delta t}$, $u_{o,t}$, $u_{1,t}$ and $u_{3,t}$ have the same significance in both equations (a) and (b) and $u_{2,t}$ in (a) and $u_{4,t}$ in (b) have real significance, but $u'_{2,t}$ and $u'_{4,t}$ can have other values.

Subtract (b) from (a) and represent $u_{1,t}$, $u_{2,t}$, etc., by u_1, u_2.

$$0 = A_1(u_1 + u_2 + u_3 + u'_4 - 4u_o) - A_2(u_1 + u'_2 + u_3 + u_4 - 4u_o) \quad \text{— (c)}$$

where $A_1 = \frac{\Delta t\, c_{z_1}}{\Delta z^2}$; $A_2 = \frac{\Delta t\, c_{z_2}}{\Delta z^2}$

If v_1 is the velocity of flow of pore water inside one layer at a point and v_2 inside other layer at a point, then at the boundary

$$v_1 = v_2$$

and

$$k_1\left(\frac{\partial u}{\partial n}\right)_1 = k_2\left(\frac{\partial u}{\partial n}\right)_2 \quad \text{— (d)}$$

where n is the direction of the normal to the interface

But from previous development in Part I

$$\left(\frac{\partial u}{\partial n}\right)_1 = \frac{u_2 - u_4'}{2\Delta z} \quad ; \quad \left(\frac{\partial u}{\partial n}\right)_2 = \frac{u_2' - u_4}{2\Delta z} \qquad \text{— (e)}$$

Then, substituting in (d),

$$u_4' = u_2 - \frac{k_2}{k_1}(u_2' - u_4) \qquad \text{— (f)}$$

Substituting (f) in (c),

$$\left(A_2 + A_1\frac{k_2}{k_1}\right)u_2' = A_1\left[u_1 + u_2 + u_3 + u_2 + \frac{k_2}{k_1}.u_4 - 4u_0\right] - A_2\left[u_1 + u_3 + u_4 - 4u_0\right] \qquad \text{— (g)}$$

Substituting in (b), and using expanded symbols,

$$u_{0,t+\Delta t} = \frac{\left(1+\frac{k_1}{k_2}\right)}{\left(1+\frac{A_2}{A_1}.\frac{k_1}{k_2}\right)} A_2\left[u_{1,t} + \frac{2k_1}{k_1+k_2}.u_{2,t} + u_{3,t} + \frac{2k_2}{k_1+k_2}.u_{4,t} - 4u_{0,t}\right] + u_{0,t} \qquad \text{— (h)}$$

or

$$u_{0,t+\Delta t} = \frac{\left(1+\frac{k_1}{k_2}\right)}{\left(1+\frac{c_{z_2}}{c_{z_1}}.\frac{k_1}{k_2}\right)} \frac{c_{z_2}\Delta t}{\Delta z^2}\left[u_{1,t} + \frac{2k_1}{k_1+k_2}.u_{2,t} + u_{3,t} + \frac{2k_2}{k_1+k_2}.u_{4,t} - 4u_{0,t}\right] + u_{0,t}. \qquad \text{— (i)}$$

(II-7)