DIFFERENTIAL GEOMETRIC
AND
SYMPLECTIC INTERPRETATIONS
OF
STABILITY IN THE SENSE OF GIESEKER

by
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CHAPTER ONE
INTRODUCTION

In this paper, we will find a geometric interpretation of Gieseker stability of a holomorphic vector bundle $E$ of rank $r$ over a compact Kahler manifold $X$ of dimension $n$ with Kahler form $\omega$. We also give a new form of symplectic quotient in this setting. The main result is:

**THEOREM 1 (MAIN THEOREM)** Let $E$ be an irreducible sufficiently smooth holomorphic vector bundle over a compact Kahler manifold $X$. Then $E$ is Gieseker stable if and only if there exists an almost Hermitian Einstein metric on $E$.

**EQUATIONS**

From studies of stability of holomorphic vector bundles we are naturally leaded to some new geometry for vector bundles. We would be interested in equations of the following type:

$$[e^{\frac{i}{2\pi}Rtd_X}]^{TOP} = (\text{const})I_E\frac{\omega^n}{n!}$$

where $Td_X$ is the harmonic representative (with respect to $\omega$) of the Todd class. In these equations, the constant term - (const) - is determined by the topology of the bundle $E$. Taking trace over $E$ and integrate it over $X$. We have

$$\int_X Tr e^{\frac{i}{2\pi}Rtd_X} = (\text{const}) \cdot r \cdot vol(M)$$

The left hand side of this equation is equal to $\int_X ch(E) \cdot Td_X$. By the Hirzebruch Riemann Roch Theorem (for $X$ being projective algebraic) or Atiyah Singer Index Theorem (for general complex manifold $X$), it is the index for $\bar{\partial}$ operator on $E$. We also call it the Euler characteristic of $E$, $\chi(X, E)$.

$$\chi(X, E) = \sum_{i=0}^{n}(-1)^i dim H^i(X, E)$$
Therefore, the constant in the equation will be

\[(\text{const}) = \frac{1}{r \cdot \text{vol}(X)} \chi(X, E) =: \chi_E.\]

We will also require that the equations to be elliptic (ellipticity will be explain in chapter three). Therefore, we will restrict ourselves to bundles which have certain kind of positivity (which is similar to the notion of Nakano positivity).

We can also couple the above equations with a Higgs field $\theta \in H^0(X, \Omega^1(EndE))$. Then the equation will look like

\[\left[e^{\frac{i}{2\pi}(R + [\theta, \theta^*])} Td_X\right]^{TOP} = (\text{const}) I_E \frac{\omega^n}{n!}\]

If we forget the Todd form in the equation, since $\frac{i}{2\pi} R$ is positive, $\frac{i}{2\pi} Tr R$ can be regarded as a Kahler form on $X$. If we couple $R$ and $\omega$ together, then we will obtain an equation independent of the Kahler form (or even the Kahler class) of $X$:

\[R^n = (\text{const})(Tr R)^n \cdot I_E\]

This equation is the same as:

\[\frac{R^n}{\int ch_n(E)} = \frac{(Tr R)^n}{r \cdot \int c_1(E)^n} I_E\]

One can also generalize it to the case where Higgs field is included.

Some of these equations should be considered as analog of Ricci flatness condition in a new category.

**ALMOST HERMITIAN EINSTEIN BUNDLES**

For a general bundle $E$, we take the tensor product of $E$ with a high power of an ample(positive) line bundle $L$. (We now assume $\omega$ is an integral form and therefore defines a line bundle $L$ on $X$. However, such an assumption is not necessary). $E \otimes L^k$ is always positive for $k$ sufficiently large and we are going to look at the equation on $E \otimes L^k$. The equation will become
\[ e^{(\omega_I + \frac{i}{2\pi} R)} T d\chi \] TOP = \chi_{1/k} I_E \frac{\omega^n}{n!} \\

where \( \chi_{1/k} = \frac{1}{r \cdot Vol(X)} \chi(X, E \otimes L^k) \) is the normalized Hilbert polynomial.

We let \( \eta = 1/k \) and rewrite the above equation in terms of \( \eta \) and get \( (aHE)_n \).

\[ e^{(\omega_I + \eta \frac{i}{2\pi} R)} T d\chi \] TOP = \chi_{\eta} I_E \frac{\omega^n}{n!} \\

where \( \chi_\eta \) and \( \eta = 1/k \) is different by a multiple of \( k^n \).

As \( \eta \) goes to zero, this equation will reduce to the Hermitian Einstein Equation (or Hermitian Yang Mills Equation) on \( E \).

\[ \Lambda R = \mu_E I_E. \]

By perturbation, we can prove that existence of Hermitian Einstein connection will imply existence of solution to \( (aHE)_n \) for all sufficiently small positive \( \eta \). Moreover, for small \( \eta \), their curvatures are close to the Hermitian Einstein Curvature.

**DEFINITION 1 (ALMOST HERMITIAN EINSTEIN BUNDLES)**

Let \( E \) be a holomorphic vector bundle over \( X \). If there exists a smooth family of solution of \( (aHE)_n \) for sufficiently small positive \( \eta \) on \( X \) such that its curvature is bounded (in \( C^{k,\alpha}\)-norm, \( k > 0 \)) independent of \( \eta \). Then \( E \) is called an almost Hermitian Einstein bundle.

Therefore a Hermitian Einstein holomorphic bundle is also almost Hermitian Einstein. By the theorem of Narashiham-Seshadri [NS](for \( n=1 \), Donaldson [Do1](for \( n=2 \), Uhlenbeck and Yau [UY](for all \( n \)), an irreducible holomorphic bundle is Hermitian Einstein if and only if \( E \) is Mumford stable. Under a natural smoothness assumption on \( E \), we proved in this paper that \( E \) is almost Hermitian Einstein if and only if \( E \) is Gieseker stable. For an explanation of the different notions of stability in algebraic geometry, one can see the next chapter of this paper.
LAGRANGIAN FUNCTIONAL

Using secondary characteristic classes, we construct a lagrangian generalizing Donaldson's lagrangian such that the Euler-Lagrange equation for this lagrangian is precisely the almost Hermitian Einstein equation. It reads as follow:

\[ \mathcal{L}(h, h_0) = \int_X (BC(h, h_0) T d_X - \chi_E BC_1(h, h_0)) e^{k \omega}. \]

Stability of \( E \) should be related to asymptotic boundedness of \( \mathcal{L} \).

EXTENDED MODULI

Next, we will consider the moduli problem. Taking the union of moduli space of solutions to \((aHE)_n\) for varying \( \eta \), we form an extended moduli space. One end of this extended moduli will be the space of almost Hermitian Einstein connections. In case, \( c_1(E) = 0 \), it is plausible that this extended moduli is essentially compact except with boundary component corresponding to the moduli space of Gieseker stable bundles. If this is the case, then we can define generalized Donaldson polynomials using this bigger moduli space.

We will also see an unexpected relationship to String Theory where the following equation appeared:

\[ R_{ij} + \eta[R^3]_{ij} + \ldots = 0 \]

Moreover, we would like to ask that if the almost Hermitian-Einstein equation related to the vanishing of \( \beta \) function for renormalizability up to higher loops order for Gauge theory path integral (see chapter three more details).

SYMPLECTIC PICTURE

The almost Hermitian Einstein equations can be interpreted as the zeros of moment map. We define a one parameter family of gauge invariant symplectic structure on the space of connection on \( E \) as follow:

\[ \Omega_k(B, C) = \int_X Tr_E B \wedge [e^{(k\omega I + \frac{i}{2\pi} R)} T d_X] \wedge C : \]

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These symplectic forms arised naturally as first Chern form of certain determinant line bundles. Then we define a new notion of symplectic quotient corresponding to the moduli space of Gieseker stable bundles. It is roughly a limit of a family of symplectic quotients on a symplectic space with varying symplectic forms on it. We call this limit as Large Symplectic Quotient. Further studies of this large symplectic quotient would be of interested. Moreover, the extended moduli space can also be interpreted as certain symplectic quotient.

HISTORY

Before we go to the main course, let us first recall some history and motivation for this problem. By the work of Uhlenbeck, Taubes and Donaldson, Donaldson (\cite{Do3}) introduce a series of differentiable invariants for compact four dimensional manifolds $X$. Using the intersection theory on some compactification of the moduli space of Anti-Self-Dual connections on a $SU(2)$ bundle $E$ over $X$, he defined Donaldson polynomials for a large class of four dimensional manifolds. It turns out that these polynomial invariants are very difficult to compute.

Then Donaldson (\cite{Do1}) proved that in case $X$ is an Kahler surface, this moduli space of Anti-Self-Dual connections can be identified with the moduli space of Mumford (poly) stable holomorphic vector bundles on $X$. Since then, Donaldson, Friedman, Morgan and others use powerful tools in algebraic geometry to compute some of these invariants. They derived striking results concerning the differential topology for four dimensional manifolds and complex surfaces.

It should be noted that the Anti-Self-Dual equation is equivalent to a Hermitian connection which satisfies (i) $F^{0,2} = 0$ and (ii) $\Lambda F = 0$. The first equation means that the connection defines a holomorphic structure on $E$ by the result of Atiyah, Singer and Hitchin (\cite{AHS}). The second equation is called the Hermitian Einstein equation or the Hermitian Yang Mills equation on $E$. What Donaldson, Uhlenbeck and Yau proved is that the existence of Hermitian Einstein connection on an irreducible holomorphic bundle $E$ is guaranteed by the Mumford stability of $E$. In case $X$ is a projective manifold, Donaldson also proved the general case using a different approach (\cite{Do2}).
Stability is a notion introduced by Mumford in trying to construct moduli space of certain algebraic geometric structures. In case of vector bundles, Mumford stability is defined using the first Chern class. Seshedra proved that, in case \(X\) is a curve, the space of stable vector bundles forms a nice moduli space with a canonical compactification by equivalent classes of Mumford semi-stable vector bundles.

If dimension of \(X\) is greater than one, Gieseker ([Ge]) introduced another notion of stability using the Hilbert polynomial and proved that moduli space of Gieseker stable vector bundles over a surface exists. (Maruyama ([Ma]) also proved this result). In case \(X\) is a curve, these two notions of stability coincides. However, for higher dimensions, Gieseker stability is the correct notion for geometric invariant theory which captures informations for higher codimensions inside \(X\). Mumford stability should be regarded as a linearization of it.

It has been asked to find the differential geometric interpretation of Gieseker stability. This paper answers this question and it also opens up a new geometry for vector bundles over a Kahler manifold.

In chapter two, we shall recall definitions and known results that we need about stability in algebraic geometry. Then, in chapter three, we shall discuss the new geometry initiated by studying of Gieseker stability. In chapter four, we shall look at this picture from a symplectic viewpoint. Chapter five and six are devoted to the proof of the relationship between Gieseker stability and almost Hermitian Einstein connection on an irreducible sufficiently smooth holomorphic vector bundle. The main arguments are contained in chapter six.

Let me briefly explain the ideas in the proof here. We will solve the system of equations there by a carefully arranged singular perturbation argument. We start with a Hermitian-Einstein metric at infinity, which means that the holomorphic structure on \(E\) has been changed. Then we try to deform that metric back to our gauge orbit which defines the holomorphic structure on \(E\). Suppose that \(E\) is stable up to order \(m\) (which will become clear in the proof). When we try to perturbed the metric at infinity order
by order to solve our equation, the first \( \frac{m}{2} \) order will still lie in the orbit at infinite. Then at the next order, we start to bring it back to our gauge orbit (infinitesimally). After we have solved it up to the \( m^{th} \) order, then another perturbation tells us that we can solve it up to all order. Although the assumption of Gieseker stability has been used throughout the proof, but the author think that this last step is the most crucial place where Gieseker stability is needed. Moreover, in the proof, we have to normalized the volume of the metric at each point suitably. This normalization does not come for free. In order to do this, we introduce a scalar function \( t \) at the last step whose existence is proved by implicit function theorem.

In fact, from the proof, we also know the differential geometric analog for semi-stability in the sense of Gieseker.
CHAPTER TWO
STABILITY OF VECTOR BUNDLES

In this chapter, we are going to review the basic concepts in Geometric Invariant Theory and the crucial notions of stability of a holomorphic vector bundle over a compact Kahler manifold.

We start with a smooth projective manifold $X$ with an ample line bundle $L$ on $X$. Recall that an ample line bundle has the defining property that global sections of high enough power of it gives a projective embedding of $X$. The isomorphism class of $L$ is also called a polarization of $X$.

Choose any Hermitian metric on $L$ and let $\omega = \frac{i}{2\pi} R$ (where $R$ is the curvature of the Hermitian metric), then $\omega$ is a positive real $(1,1)$ form on $X$. It defines a Kahler metric on $X$. We also have $[\omega] = c_1(L)$. Therefore, the Kahler form $\omega$ defines an integral cohomology class on $X$. Conversely, if $X$ is a Kahler manifold such that its Kahler form is integral, then $[\omega] = c_1(L)$ for some ample line bundle $L$ on $X$ by the Kodaira embedding theorem.

Now, suppose that a compact Lie group $G$ acts on both $X$ and $L$ such that its complexification $G^\mathbb{C}$ acts holomorphically on them. Mumford ([Mu]) defines a Zariski open subset $X_s$ of $X$ consisting of 'stable points' such that the $G^\mathbb{C}$ orbits in $X_s$ are all closed and $X_s/G^\mathbb{C}$ is a well-defined smooth quasi-projective variety.

Moreover, there is a canonical compactification of this geometric quotient. For this purpose, Mumford introduced the notion of semi-stable points. Under certain equivalent relations, by adding semi-stable points, the quotient will exist as a projective variety.

We want to apply Geometric Invariant Theory to construct moduli space of holomorphic vector bundles over $X$. First of all, we know that for any holomorphic bundle $E$, there exists a constant $k_0$ (depending on $E$), such that, for all $k > k_0$, we have

(i). $H^i(X, E \otimes L^k) = 0$ for all $i > 0$. 

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(ii). \( E \otimes L^k \) is generated by global sections.

Suppose that we have a family \( \mathcal{F} \) of holomorphic vector bundles (with fix Hilbert polynomial, \( \chi(X, E \otimes L^k) \)) such that one can find a uniform \( k_0 \), so that for any \( k > k_0 \), (i) and (ii) work for each member \( E \in \mathcal{F} \). Then, after tensoring with high power of \( L \), each \( E \) can be determined by the behavior of its global sections. In other words, we hope to get a 'holomorphic' embedding of \( \mathcal{F} \) to the Grassmannian. We denote \( r \) to be the rank of these bundles and consider the following natural morphism:

\[
\bigwedge^r H^0(X, E \otimes L^k) \to H^0(X, \det E \otimes L^{rk})
\]

For simplicity, we consider those bundles with a fix the determinant line bundle \( M, \bigwedge^r E = \det E \cong M \) for some fix line bundle \( M \) on \( X \). Then \( H^0(X, \det E \otimes L^{rk}) \) becomes a fix vector space, we call it \( W \). By (i) and the Riemann Roch theorem, the dimension of \( H^0(X, E \otimes L^k) \) is also constant. If we 'pick' an identification of \( H^0(X, E \otimes L^k) \) to some fix vector space \( V \) of that dimension, then we have a morphism corresponding to each \( E \)

\[
\bigwedge^r V \to W
\]

The upshot is this map determines the holomorphic bundle \( E \) up to a choice of the base of \( V \). So \( C^g = SL(V) \) acts on \( \text{Hom}(\bigwedge^r V, W) \) and the quotient, if exists, would be the moduli space for the family \( \mathcal{F} \).

In order to interpret stable point in \( \text{Hom}(\bigwedge^r V, W) \) in terms of the bundle informations, Gieseker ([Ge]) introduced his definition of stability. He proved that a holomorphic bundle is Gieseker stable is the same as the corresponding homomorphism \( \bigwedge^r V \to W \) being a stable point under \( G^C = SL(V) \) action. He also proved that in case of a projective surface, such bundles forms a bounded family which implies the existence of a uniform \( k_0 \) that makes (i) and (ii) works for all Gieseker semi-stable bundles (with a fix Hilbert polynomial).

**DEFINITION 2 (GIESEKER STABILITY) ([Ge])**

Let \( E \) be a rank \( r \) holomorphic vector bundle (or coherent torsion-free sheaf in general) over a projective variety \( X \) with ample line bundle \( L \), \( E \) is called Gieseker stable if for any nontrivial coherent subsheaf \( S \) of \( E \), we have
\[
\frac{\chi(X, S \otimes L^k)}{\text{rank} S} < \frac{\chi(X, E \otimes L^k)}{\text{rank} E}
\]

for large enough \(k\).

\(E\) is called Gieseker semi-stable if

\[
\frac{\chi(X, S \otimes L^k)}{\text{rank} S} \leq \frac{\chi(X, E \otimes L^k)}{\text{rank} E}
\]

for large enough \(k\).

Notice that a nontrivial subsheaf \(S\) here is assumed to have rank strictly between 0 and \(r\).

**Theorem 2 (Gieseker)** ([Ge]) Let \(X\) be a projective surface, then the moduli space of Gieseker stable torsion-free coherent sheaves with a fix Hilbert polynomial exists as a quasi-projective variety.

Moreover, the moduli space of equivalent classes of Gieseker semi-stable torsion-free coherent sheaves with a fix Hilbert polynomial exists as a projective variety.

We will explain the equivalent relation later in this chapter.

Before works of Gieseker, there has been a lot of works by many people on trying to construct the moduli space for vector bundles over a curve or complex projective spaces. Among them are Weil, Mumford, Narasimhan, Ramanan, Seshadri, Barth, Horrocks, Hartshorne and many others. For the surface case, Maruyama also proved the above theorem.

In case the base manifold is a curve, Mumford defined stability and generalized it to higher dimensional case as follow (before Gieseker):

**Definition 3 (Mumford Stability)** Let \(E\) be a rank \(r\) holomorphic vector bundle over a projective variety \(X\) with ample line bundle \(L\), \(E\) is called Mumford stable if for any nontrivial coherent subsheaf \(S\) of \(E\), we have
\[ \mu_S \leq \mu_E \]

\[ E \text{ is called Mumford semi-stable if} \]

\[ \mu_S \leq \mu_E \]

where \( \mu_E \) is called the slope of \( E \) and defined as

\[ \mu_E = \frac{(c_1(E) \cdot c_1(L)^{n-1}) \cdot [X]}{\text{rank}E} \]

To define \( \mu_S \) for a torsion free coherent sheaf which is not necessarily locally free, we need to make sense of \( c_1(S) \). One way to do this is to define \( c_1(S) \) as the first Chern class of the double dual of the determinant of \( S \).

Remark: Using Mumford's Geometric Invariant Theory, Seshadri proved that over a curve, Mumford stable bundles form a quasi-projective moduli space with a canonical compactification by equivalent classes of Mumford semi-stable bundles. It can be checked that Mumford stability and Gieseker stability are the same over a curve. For higher dimensional base manifold, Mumford stable implies Gieseker stable and Gieseker semi-stable implies Mumford semi-stable.

The following standard theorem tells us that each Mumford semi-stable bundle is built from Mumford stable bundle in an essentially unique way. In particular, a Gieseker stable bundle can also be decomposed into Mumford stable bundles. This theorem is important in our proof for the existence of almost Hermitian Einstein metric on a Gieseker stable bundle. It is also useful in defining the equivalent relation for semi-stable bundles.

**THEOREM 3 (JORDAN-HOLDER THEOREM)** If \( E \) is a Mumford semi-stable torsion-free coherent sheaf over \( X \), there exist a filtration of \( E \) by torsion free subsheaves \( E_i \)'s

\[ E = E_0 \supset E_1 \supset E_2 \supset \ldots \supset E_{k+1} = 0 \]

such that \( Q_j = E_j/E_{j+1} \) is Mumford stable and \( \mu(Q_j) = \mu(E) \) for each \( j \). Moreover,
\[ Gr(E) = Q_1 \oplus Q_2 \oplus \ldots \oplus Q_k \]
is uniquely determined by \( E \) up to isomorphism.

A semi-stable torsion-free coherent sheaf is called sufficiently smooth if \( Gr(E) \) is locally free. In the case \( E \) is Mumford stable, then sufficiently smoothness is the same as \( E \) being a smooth vector bundle since in that case the Jordan-Holder filtration consists of only one term, namely \( E \) itself. But in the semi-stable category, sufficiently smoothness is the natural analog for smoothness for stable object.

Now, we can explain the equivalent relation among semi-stable bundles. Two semi-stable bundles are said to be equivalent if their corresponding graded sheaves \( Gr(E) \) in their Jordan-Holder theorem are isomorphic. Notice that this equivalent relationship on the set of Mumford stable bundles is simply the equivalent relation of isomorphism classes. The second part of Gieseker's theorem says that the set of equivalent classes of Gieseker semi-stable torsion-free coherent sheaves carries a natural projective-algebraic structure.

Now, we would like to define these stabilities over a compact Kähler manifold \( X \) which is not necessarily projective. For Mumford stability, we shall replace \( c_1(L) \) by the cohomology class of \( \omega \) in the defining inequality. For Gieseker stability, we have used Hilbert polynomial in its definition. By Riemann-Roch theorem and Chern-Weil theory, it can be expressed in terms of characteristic classes of \( E \) and \( L \) which, in turn, can be computed by curvature of \( E \) and \( L \). Therefore, only the curvature of \( L \) is used instead of \( L \) itself. Up to a constant, the curvature of \( L \) is our Kähler class, therefore, we can define the notion of stability even though the Kähler metric \( \omega \) may not be an integral form. For simplicity, we shall continue to write \( \omega \) as the first Chern form of a positive line bundle. However, when \( E \) is singular, we would have problem in defining its curvature and we should continue to use the Chern classes expression for \( \chi \). When \( E \) is a torsion-free coherent sheaf on \( X \), its Chern character is defined to be the alternating sum of Chern characters of a locally free resolution of it. Notice that the Chern character defines in this way is independent of the choice of the resolution of \( E \).
Let us now look at a example of Gieseker stable bundle which is not Mumford stable. The following example is given by Maruyama. Let $F$ be any Mumford stable bundle over $\mathbb{CP}^2$ of rank two with vanishing first Chern class. Suppose that $H^1(\mathbb{CP}^2, F)$ is non-trivial, then any point in $\mathbb{P}(H^1(\mathbb{CP}^2, F))$ defines a non-trivial extension of the trivial bundle by $F$:

$$0 \rightarrow F \rightarrow E \rightarrow O \rightarrow 0$$

Then $E$ is a rank three vector bundle over $\mathbb{CP}^2$ with vanishing first Chern class. Therefore, $F$ is already a Mumford destabilizing subsheaf of $E$. On the other hand, it can be checked that $E$ is in fact Gieseker stable.

There is another notion of stability defined by Bogomolov, his definition is independent of the choice of the ample line bundle. Using his notion of stability, he proved the Chern numbers inequality for surfaces of general type, $c_1^2 \leq 4c_2$. The technique he used becomes very useful in later development. It would be interesting to find a differential geometric characterization of this notion of stability.

From an algebraic geometric point of view, Mumford stability, which is defined using first Chern class, is closely related to codimension one behaviors. On the other hand, the whole geometric picture should capture other codimension too, which is related to the higher Chern classes of the sheaf. It turns out that the normalized Hilbert polynomial is the correct notion to study.

As we shall explain in later chapters, Mumford stability is related to a system of linear elliptic equations and Gieseker stability is related to a system to Fully-Nonlinear elliptic equations of Monge-Ampere type. In some sense, Mumford stability should be interpreted as (quasi-) linearization of stability in the sense of Geometric Invariant Theory (that is Gieseker stability).
CHAPTER THREE
NEW EQUATIONS AND GEOMETRY

NEW EQUATIONS

Motivated by the geometric interpretation of Gieseker stability, we are naturally led to a new kind of system of elliptic differential equations on holomorphic vector bundles. If $E$ is a holomorphic vector bundle of rank $r$ over a compact Kahler manifold $X$ with Kahler form $\omega$ and of dimension $n$. For any Hermitian connection $A$, let $R$ be the curvature of $A$; $R = dA + A^2$, which is an $\text{End}(E)$ valued (1,1) form on $X$. Consider

$$e^{\frac{i}{2\pi}R} = I + \frac{i}{2\pi}R - \frac{1}{8\pi^2}R \wedge R + \ldots$$

where $e^{\frac{i}{2\pi}R}$ is a sum of $\text{End}(E)$ valued $(p,p)$ forms on $X$. Since $R \wedge R \ldots \wedge R$ ($n+1$) times is zero by dimension counting, $e^{\frac{i}{2\pi}R}$ is only a finite sum. Notice that $\text{Tr} e^{\frac{i}{2\pi}R}$ is a closed differential form on $X$. By the Chern-Weil theory, the cohomology class it represents is the Chern character for the bundle $E$, $\text{ch}(E)$.

Using the Kahler metric on $X$, we write $Td_X$ to be the harmonic differential form representing the Todd class of $X$. We call the $(n,n)$ form part of the product of $e^{\frac{i}{2\pi}R}$ and $Td_X$ as the Atiyah-Singer Volume Form and denote it by $\text{AS}(E, A)$.

DEFINITION 4 (ATIYAH SINGER VOLUME FORM) Let $E$ be a holomorphic vector bundle over $(X, \omega)$ as before. For any Hermitian connection $A$ on $E$. We define the Atiyah Singer Volume Form by the following

$$\text{AS}(E, A) = [e^{\frac{i}{2\pi}R}Td_X]^{\text{TOP}}$$

where $Td_X$ is the harmonic representative of the Todd class of $X$. $\text{AS}(E, A)$ is an $\text{End}(E)$ valued $(n,n)$ form on $X$.

For later purposes, we also introduce the following definitions:

$$\text{AS}^k(E, A) = [e^{k\omega I + \frac{i}{2\pi}R}Td_X]^{\text{TOP}}$$

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and

$$AS_n(E, A) = \eta^n \left[e^{\frac{1}{n}R T d_X} \right]^{TOP}$$

$$= \left[e^{\omega l + \frac{i}{2\pi} R T d_X^n} \right]^{TOP}$$

Notice that we have

$$AS_n(E, A) = \frac{\omega^n}{n!} + \eta \left[ \frac{i}{2\pi} R \wedge \frac{\omega^{n-1}}{(n-1)!} + \frac{n}{2} \mu_X \frac{\omega^n}{n!} \right] + O(\eta^2)$$

where $\mu_X$ is the slope of $X$ which depends only on the topology of $X$ and the Kahler class of $\omega$, namely

$$\mu_X = \frac{1}{n} \int c_1(X) \wedge \frac{\omega^{n-1}}{(n-1)!}$$

Therefore, the condition that $AS_n(E, A)$ being constant up to first order in $\eta$ is $\frac{i}{2\pi} R \wedge \frac{\omega^{n-1}}{(n-1)!} = C \omega^n I_E$. or $\Lambda R = C I_E$. This is the Hermitian Einstein equation on $E$. It is also called the Hermitian Yang Mills equation on $E$.

Remark: In general, Atiyah Singer Volume Form can be defined for complex vector bundles over a symplectic manifold. In that case, we needs to choose a Todd form. A simple way to achieve this is to choose a compatible almost complex structure on $X$ so that $X$ becomes an almost Kahler manifold. Then $T d_X$ will again be the harmonic representative of the Todd class of $X$.

By the Hirzebruch Riemann Roch theorem (when $X$ is a projective manifold), or Atiyah Singer Index theorem (for a general complex manifold $X$), if we take the trace of the Atiyah Singer Volume Form and integrate it over $X$. We shall get the Euler characteristic of $E$, $\chi(X, E) = \sum_{i=0}^{n} (-1)^i dim H^i(X, E)$.

$$\chi(X, E) = \int_X Tr AS(E, A).$$

We want to find a Hermitian connection $A$ on $E$ such that the Atiyah Singer Volume form is constant. More precisely, we have equation (1):

$$AS(E, A) = \frac{\chi(X, E)}{r \cdot Vol(X)} \cdot \frac{\omega^n}{n!} I_E$$
We also require that the equation is elliptic at $A$, which is an assumption on the positivity of $R$. In case $X$ is a curve, no extra assumption is needed. When $X$ is a surface, then the ellipticity assumption will be equivalent to the positivity of $\frac{i}{2\pi}R + \frac{c_1(X)}{2}I_E$ being an endomorphism of $T_X \otimes E$.

To write down the precise condition for ellipticity, we should linearize the equation (1). We need to consider the 'derivative' of $AS(E, A)$ with respect to $R$. A simple computation shows that

$$\frac{d}{dR} AS(E, A) = [e^{\frac{i}{2\pi}R}Tdx]^{n-1,n-1}$$

which is an $\text{End}(E)$ valued $(n-1,n-1)$ form on $X$. It defines a Hermitian bilinear form on $\Gamma(T_X^* \otimes E^*)$ as follows:

$$\Theta(B, C) = \int_X Tr_E B \wedge : [\frac{d}{dR} AS(E, A)] \wedge C^* :$$

which is the same as

$$\Theta(B, C) = \int_X Tr_E B \wedge : e^{\frac{i}{2\pi}R}Tdx \wedge C^* :$$

The normalized product $\wedge :$ means that in each term of the expansion of $\text{exp}$, the $C^*$ should be put in each of the slot between the $R$'s. For example,

$$e^R \wedge C := \Sigma_{l=1} (\Sigma_{m=0}^l R^m C R^{l-m-1})/n!$$

If the gauge group $G$ is Abelian, then the normalized product defined above is the same as the usual product. However, for non-Abelian theory, this is a better product which captures the non-commutativity of the product. Now, the positivity of $E$ that we used here is related to the above bilinear form being positive definite.

**DEFINITION 5 (POSITIVITY)** Let $E$ be a holomorphic vector bundle with a Hermitian connection $A$, we say that $A$ is positive (or simply $E$ is positive) if the Hermitian bilinear form $\Theta$ is positive definite for some Hermitian metric on $E$.

It is easy to see that the above definition is equivalent to the positivity of the following pointwise Hermitian bilinear form:
\ast (Tr_E B \wedge \left[ \frac{d}{dR} AS(E, A) \right] \wedge C^*)

where the first \ast is the Hodge star operator of \Omega and therefore the above expression is an endomorphism on \Omega. Although an arbitrary bundle may not be positive, by tensoring with high power of an ample line bundle, any bundle is positive in this sense and the proof is easy.

**Lemma 1** Let \( L \) be a fix ample line bundle on \( X \) and \( E \) be any bundle. There exists a positive integer \( k_0 \) depending on \( E \) such that for any \( k > k_0 \), \( E \otimes L^k \) is positive.

The following proposition speaks out the importance of the positivity of \( E \). It tells us that ellipticity at \( A \) is equivalent to positivity of \( A \). Let us first explain the notion of ellipticity for a nonlinear differential equation. For a linear differential equation, if the symbol of the highest order term is invertible, we call it an elliptic equation. However, for a fully-nonlinear equation, the symbol itself depends on the solution itself. Hence, we can only discuss ellipticity at a particular solution for it. In that case, we say the equation is elliptic at that particular solution if when we linearize the equation at that point, it becomes a linear elliptic equation. We are going to see that ellipticity of our equation is the same at positivity of the solution connection.

**Proposition 1** Suppose that \( A \) is a Hermitian connection on \( E \) over a Kahler satisfying equation (1)

\[
AS(E, A) = \frac{\chi(X, E)}{r \cdot Vol(X)} \cdot \frac{\omega^n}{n!} I_E
\]

Then the nonlinear equation is elliptic at \( A \) if and only if \( A \) is positive.

**Proof of Proposition:**

Suppose that \( h_0 \) is the Hermitian metric defined by \( A \), then any other Hermitian metric on \( E \) can be expressed as \( h \cdot h_0 \) for some self-adjoint positive endomorphism \( h \) of \( E \). Let \( u = h^{-1} \delta h \) be any infinitesimal change of \( h \), then the linearized equation at \( h_0 \) becomes

\[
[ : e^{\frac{\delta}{2} R} \wedge \bar{\partial} \partial u : Tdx]^{TOP} = 0
\]
where \( R \) is the curvature tensor for the chosen metric \( h_0 \).

Therefore the symbol becomes a homomorphism on sections of \( T^* \otimes \text{End}(E) \) as follow:

\[
C \rightarrow (\ast[: e^{\frac{i}{2\pi} R} \wedge C : Td_X])^{0,1}
\]

The positivity for this homomorphism is equivalent to the one we constructed above.

QED

The positivity defined above is very similar to the definition of positivity by Nakano. Nakano positivity is used by Nakano to prove vanishing of certain cohomology group associated with \( E \). We will recall the Nakano positivity in a way parallel to our previous discussion.

**DEFINITION 6 (NAKANO POSITIVITY)** Let \( E \) be a holomorphic vector bundle with a Hermitian connection \( A \), we say that \( A \) is Nakano positive (or simply saying that \( E \) is Nakano positive) if the Hermitian bilinear form \( \Theta_N \) is positive definite.

\[
\Theta_N(B, C) = \int_X Tr_E B \wedge \ast \frac{i}{2\pi} R \wedge C^*
\]

**RELATION TO KAHLER EINSTEIN EQUATIONS**

Suppose that there exists certain Hermitian connection on \( E \) such that its curvature is a positive operator. Denote the space of all such connections by \( \mathfrak{A}^h_0(E) \) or simply \( \mathfrak{A}_0 \). The curvature \( R \) for such a connection will be regarded as an analog of a Kahler form on a Kahler manifold. There are the following similarities between them. First, by the Bianchi identity, we have \( DF = 0 \), which is analog to the Kahler condition that \( d\omega = 0 \). Second, any two curvatures \( F \) are different by a term of the following form \( \overline{\partial}(h^{-1}\partial h) \), for some positive self-adjoint endomorphism of the background metric (which is one of the two Hermitian metric on \( E \)). On a compact Kahler manifold, any two Kahler forms in the same Kahler class are different by \( \overline{\partial}\partial \varphi = \overline{\partial}(e^{-\varphi}\partial e^\varphi) \) for some real valued function \( \varphi \) on \( X \).
What should be the ‘Ricci Curvature’ for such a ‘Kahler form’ \( R \)? Let us consider the top exterior power of \( R, (\frac{i}{2\pi} R)^n \), which is an \( \text{End}(E) \) valued top form on \( X \). Dividing it by \( \omega^n \) (which is equivalent to take the Hodge star operator acting on it).

Look at the following expression:

\[
C_\omega(E) := \frac{i}{2\pi} \partial \bar{\partial} \left( \left( \frac{(\frac{i}{2\pi} R)^n}{\omega^n} \right)^{-1} \partial \left( \frac{(\frac{i}{2\pi} R)^n}{\omega^n} \right) \right)
\]

which can be written symbolically as

\[
C_\omega(E) = \frac{i}{2\pi} \partial \bar{\partial} \log \left( \frac{(\frac{i}{2\pi} R)^n}{\omega^n} \right)
\]

This should be considered as ‘Ricci Curvature’ for \( R \) as compared to the Ricci curvature for a usual Kahler metric.

\( C_\omega(E) \) represents the zero element in \( H^{1,1}(X, \text{End}(E)) \) because \( \partial \bar{\partial} \log \left( \frac{(\frac{i}{2\pi} R)^n}{\omega^n} \right) \) is a well-defined element in \( \Omega^{1,0}_X(\text{End}(E)) \). Therefore, the Ricci flatness condition could be rewritten as

\[
\frac{(\frac{i}{2\pi} R)^n}{\omega^n} = (\text{const}) I_E
\]

The constant involved can be determined by the topology of \( E \) as before, which is equal to \( \frac{1}{r \cdot n! \cdot \text{Vol}(X)} \int_X \text{ch}_n(E) \). Therefore, the ‘Kahler Einstein’ equation becomes

\[
\left( \frac{i}{2\pi} \right)^n = \left( \frac{1}{r \cdot n! \cdot \text{Vol}(X)} \int_X \text{ch}_n(E) \right) \cdot \frac{\omega^n}{n! I_E}.
\]

Notice that this is a fully nonlinear elliptic system of Monge-Ampere type. In the above discussion, the ‘Ricci Curvature’ \( C_\omega(E) \) depends not only on \( R \), but it also depends on \( \omega \). In fact,

\[
C(E) := \frac{i}{2\pi} \partial \bar{\partial} \left( \left( \frac{i}{2\pi} R \right)^{-n} \partial \left( \frac{i}{2\pi} R \right)^n \right)
\]

is a well-defined \( \text{End}(E) \) valued form with no dependence on \( \omega \). It also defines a cohomology class in \( H^{1,1}(X, \text{End}(E)) \) which is equal to the image of \( c_1(X) \) under the natural map.
\[ H^{1,1}(X) \rightarrow H^{1,1}(X, \text{End}(E)) \]

That is \([C(E)] = [c_1(X) \cdot I_E]\)

We can study a more intrinsic equation by coupling the 'Kahler Metric' \(R\) on \(E \rightarrow X\) and the ordinary Kahler metric \(\omega\) on \(X\). We impose \(\omega = Tr(\frac{i}{2\pi} R)\) as a constraint. Because \(R\) is assumed to be positive, \(Tr \frac{i}{2\pi} R\) is always a Kahler form on \(X\) and the Kahler class for it is the first Chern class for \(E\). That is \(\omega = Tr(\frac{i}{2\pi} R) \in c_1(E)\). The volume of \(X\) becomes \(\frac{1}{n!} \int c_1(E)^n\) and the coupled equation becomes:

\[
\left(\frac{i}{2\pi} R\right)^n = \frac{\int ch_n(E)}{r \cdot \int c_1(E)^n} (Tr \frac{i}{2\pi} R)^n I_E
\]

or

\[
R^n = \left(\frac{\int R^n}{r \cdot \int (Tr R)^n}\right)(Tr R)^n I_E
\]

or simply,

\[
R^n = (\text{const}) \cdot (Tr R)^n I_E
\]

As we mentioned before, in order for the equation to be elliptic, we need to have certain positivity for the bundle \(E\). However, a general bundle is not (Nakano) positive. There exists topological obstruction for the positivity of a bundle. For instance, \(c_1(E)\) must be a positive \((1,1)\) class on \(X\). Nevertheless, we can achieve this by taking tensor product of it with a sufficiently ample (positive) line bundle on \(X\) by the previous lemma. The main idea is to make \(k \omega I_E + \frac{i}{2\pi} R\) to be positive enough.

The existence of an ample line bundle on \(X\) is the same as \(X\) being a projective algebraic manifold. However, as we have seen, what we really need is just a positive (real) \((1,1)\) form \(\omega\) on \(X\). Whether \(\omega\) is integral or not is not important. Therefore, assuming \(X\) to be Kahler is already enough. We are going to study the behaviors for the solutions as \(k\) goes to infinity.
ALMOST HERMITIAN EINSTEIN BUNDLES

For a fixed (irreducible) rank $r$ holomorphic bundle $E$ over the compact Kahler manifold $X$ with Kahler form $\omega$. We rewrite our first equation for small $\eta = 1/k$ (or equivalently for large $k$). $(aHE)_\eta$:

$$[e^{(\omega I + \eta \frac{i}{2\pi} R)}Tc^n_X]TOR_{\eta} = \chi_{n}I_E\frac{\omega^n}{n!}$$

where $Tc^n_X$ is the harmonic Todd polynomial for $X$ with variable $\eta$. We have

$$Tc^n_X = 1 + \eta \frac{c_1}{2} + \eta^2 \frac{c_1^2 + c_2}{12} + ...$$

Here, $c_i$ means the harmonic form representing the $i$-th Chern class of $X$. As explaining in the beginning of this chapter that $\chi_E$ is topological and is a polynomial in $\eta$. By the Atiyah-Singer Index theorem, it is equal to the Hilbert polynomial of $E \to M$ divided by the rank of $E$. That is,

$$\chi_{\eta} = \frac{1}{r} \chi(X, E \boxtimes L^{1/n})$$

Notice that even though $L$ may not exist (since $X$ is only assumed to be Kahler) and $1/\eta$ may not be integral, $\chi_E$ is still well-defined. The linear term in the $\eta$ variable in $\chi_E$ is the slope of $E$, $\mu_E$. Recall that Mumford used $\mu_E$ to define stability and Gieseker used $\chi_E$ to define stability. Now we are going to use the above equation to define almost Hermitian Einstein connection.

DEFINITION 7 (ALMOST HERMITIAN EINSTEIN BUNDLES)
Let $E$ be a holomorphic vector bundle over $X$. If there exists a smooth family of solutions of $(aHE)_\eta$ for sufficiently small positive $\eta$ on (some blown up of) $X$ such that its curvature is bounded (in $C^{k,\alpha}$-norm, $k > 0$) independent of $\eta$. Then $E$ is called an almost Hermitian Einstein bundle.

If we expand the equation $(aHE)_\eta$ in $\eta$, we have

$$\frac{\omega^n}{n!} I_E + \eta \left[ \frac{i}{2\pi} R \wedge \frac{\omega^{n-1}}{(n-1)!} + \frac{n}{2} \mu_X \frac{\omega^n}{n!} \right] + O(\eta^2) = \frac{\omega^n}{n!} I_E + \eta [\mu_E I_E \frac{\omega^n}{n!} + \frac{n}{2} \mu_X \frac{\omega^n}{n!}] + O(\eta^2)$$

Notice that we have used the assumption that $c_1(X)$ is harmonic to obtain the above equation. Simplifying it and we shall get
\[ \frac{i}{2\pi} R \wedge \frac{\omega^{n-1}}{(n-1)!} = \mu_E \frac{\omega^n}{n!} I_E + O(\eta). \]

Therefore, the principal term for the equation \((aHE)_\eta\) for small positive \(\eta\) is the Hermitian Einstein equation or the Hermitian Yang Mills equation on \(E\), which can also be written in the following familiar form:

\[ \Lambda R = \mu_E I_E \]

In terms of indexes, it reads \(g^{\alpha\beta} R^{i}_{\alpha j \beta} = \mu_E \delta^i_j\), where \(i, j\ldots\) are indexes for the bundle \(E\) and \(\alpha, \beta, \ldots\) are indexes for the base manifold \(X\).

**PROPOSITION 2** If \(E\) is a holomorphic bundle on \(X\) which admits a Hermitian Einstein connection, then it also admits an almost Hermitian Einstein connection.

The proof for this proposition is a simple application of implicit function theorem for elliptic operators between Banach spaces. One should notice that the corresponding statement for stability is also true, namely, Mumford stable bundle is always Gieseker stable.

There is another occasion where a holomorphic vector bundle admits an almost Hermitian Einstein connection. Suppose that \(E\) has a connection such that all powers of its curvature are harmonic. By Bianchi identity, it is the same as requiring \(D^* R^l = 0\), for all \(l\). In that case, \(*[e^{\omega l + \eta \pi} R T d_X]\)^{TOP}\ would be a holomorphic endomorphism of \(E\). If \(E\) is simple, then it must be a constant multiple of the identity. As a result, it satisfies the equation \((aHE)_\eta\) for all \(\eta\). Notice that, in this case, the same connection will fulfill the equations for varies \(\eta\). Examples of such bundles come from irreducible projective representation of the fundamental group of \(X\). In particular, flat bundles are in this category.

Narasimhan and Seshadri ([NS] for \(n=1\), Donaldson ([Do1] for \(n=2\), Uhlenbeck and Yau ([UY] for all \(n\)) proved that the solvability of the Hermitian Einstein equation is equivalent to the Mumford stability of \(E\) provided that \(E\) is irreducible. The corresponding statement for reducible bundles can be
similarly obtained. In the case of Mumford stability, he used the first Chern class to define it which is a 'linear' object. In Gieseker's definition of stability, he used all Chern classes and the theory becomes more 'nonlinear' in nature. The main purpose of this paper is to interpret Gieseker stable (sufficiently smooth) holomorphic bundles in term of a new differential geometric object: almost Hermitian Einstein connections.

VANISHING THEOREM

In Geometric Invariant Theory, the first step to prove the existence of a good moduli is to prove the 'boundedness' of the family of structures. Roughly speaking, it says that the family of algebraic structures has a finite dimensional (multi-valued) parametrization.

In the case of vector bundles over a fix manifold \( X \), we need the following two consequences of the boundedness of the family of vector bundles, \( \mathcal{F} \): Let \( L \) be an ample line bundle on \( X \). There exists a positive integer \( k_0 \) such that for any \( k > k_0 \) and for any \( E \in \mathcal{F} \), we have (i) \( H^i(X, E \otimes L^k) = 0 \), for and \( i > 0 \). (ii) \( E \otimes L^k \) is generated by global sections.

The importance of these two conditions is the choice of \( k_0 \) being independent of the individual member in the family. Vanishing of these higher cohomology is usually related to certain positivity conditions on the vector bundle \( E \). By introducing Nakano positivity, he proved the following vanishing theorem:

**THEOREM 4 (NAKANO VANISHING THEOREM)** *If \( E \otimes K_X^{*} \) is Nakano positive, then*

\[
H^i(X, E) = 0, \text{ for all } i > 0
\]

When \( X \) is a surface, \( E \) being positive is equivalent to \( E \otimes (K_X^{*})^{1/2} \) being Nakano positive. This is similar to the condition for the vanishing of \( H^{>0}(X, E) \).

On the other hand, if \( E \) has a positive solution to equation (1)

\[
[e^{\tau^R Td_X}]^{TOP} = (\text{const}) I_{E} \frac{\omega^n}{n!}
\]
then $E$ is likely to have some stable properties. As in our discussion, vanishing of its higher cohomology is somehow related to forming the moduli, it would be interesting to see if there is any deep relationship between these. For instance, whether $E$ has a positive solution to (1) will imply certain vanishing of cohomology group or not.

**EXTENDED MODULI**

Now, for each $\eta$, we can try to solve $(aH E)_\eta$ and denote the space of elliptic solutions on a fix complex vector bundle by $\mathcal{M}_\eta$. Notice that we do not fix the holomorphic structure on the bundle. Then $\mathcal{M}_0$ is the moduli space of Hermitian Einstein bundles. It is also equal to the moduli space of poly-stable holomorphic vector bundles on $X$. Then $\mathcal{M}_{0^+} = \lim_{\eta \to 0^+} \mathcal{M}_\eta$ would be the moduli space of almost Hermitian Einstein bundles of $X$. Notice that $\mathcal{M}_0$ is an open subset of $\mathcal{M}_{0^+}$.

We let $\mathfrak{M} = \bigcup_{\eta > 0} \mathcal{M}_\eta$.

Notice that when $\eta$ goes to infinity, the equation becomes the first equation that we study, equation (1):

$$AS(E, A) = \frac{\chi(X, E)}{r \cdot Vol(X)} \cdot \frac{\omega^n}{n!} I_E$$

Therefore, $\mathcal{M}_\infty$ is the moduli space for elliptic solutions for equation (1). There is certain topological obstruction for $\mathcal{M}_\infty$ to be non-empty. For ex-
ample, if $E$ is a $SU(r)$ bundle, then $\mathcal{M}_\infty$ is always empty. Therefore, the extended moduli space might have a chance to be essentially compact. However, the extended moduli space $\mathcal{M}$ may have very complicated structure.

We can also extend the Donaldson polynomial to this situation. Recall that the Donaldson polynomials was defined using the intersection theory on some compactification of $\mathcal{M}_0$, which is one side of the boundary of our extended moduli space $\mathcal{M}$. We would like to compactify $\mathcal{M}$. Then analog of Donaldson polynomials can be defined using $\overline{\mathcal{M}}$.

Remark: Although Donaldson polynomials were defined using a differential geometric compactification of $\mathcal{M}_0$. The relation of this compactification and Gieseker compactification has been recently studied by J. Li ([Li]). Therefore, one could hope to compute the Donaldson polynomial using this Gieseker compactification.

**ACTION FUNCTIONAL**

In ([Do1]), Donaldson introduced a remarkable Lagrangian functional on the space of Hermitian metric of $E$ using Bott-Chern secondary characteristic classes. Unlike Yang-Mills action, the Euler-Lagrangian equation for this action coincides with almost Hermitian-Einstein equations on $E$. With the help of the action, he proved that irreducible Mumford stable bundle on a Kahler surface is equivalent to a holomorphic bundle with a Hermitian-Einstein metric on it. Nowadays, we find this functional is useful in many other areas and is usually referred to as Donaldson functional or Donaldson Lagrangian.

Let us first recall the definition of Donaldson Lagrangian, $\mathcal{L}_D$. Suppose that $E$ is a holomorphic vector bundle over $X$ and $h, h_0$ are two Hermitian metrics on $E$, then he defines,

$$
\mathcal{L}_D(h, h_0) = \int_X BC_2(h, h_0) \wedge \frac{\omega^{n-1}}{(n-1)!} - \mu_E BC_1(h, h_0) \wedge \frac{\omega^n}{n!}
$$

where $BC_j$ is the Bott-Chern form for the $j^{th}$ Chern character of $E$ which is a $(j-1,j-1)$ form on $X$. Fixing $h_0$ as background metric and letting $h$ varies,
he shows that $h$ is a critical point for $\mathcal{L}_D$ if and only if $\Lambda R(h) = \mu_E I_E$, that is $h$ is a Hermitian-Einstein metric on $E$. Now, let $BC$ be the Bott-Chern form for the total Chern character of $E$, then we define a new Lagrangian:

$$\mathcal{L}(h, h_0) = \int_X (BC(h, h_0) TdX - \chi_k BC_1(h, h_0)) e^{k\omega}$$

Notice that the second term inside the integral is a normalization term for the volume on the fiber (Recall that $BC_1(h, h_0) = \log \det (h_0^{-1} h)$) which is indeed crucial.

**PROPOSITION 3** If we fix a Hermitian metric $h_0$ on $E$, then $\mathcal{L}(h, h_0)$ defines a functional on the space of Hermitian metric on $E$. The Euler-Lagrange equation for it is equal to the almost Hermitian Einstein equation

$$[e^{k\omega I + \frac{1}{2\pi} R TdX}]^{TOP} = \chi_k \frac{\omega^n}{n!} I_E$$

**PROOF OF PROPOSITION:**

Choose any smooth one parameter family of Hermitian metric $h_t$ on $E$, then by standard secondary construction, we have

$$BC(h_b, h_0) - BC(h_a, h_0) = \int_a^b Tr u_t e^{\frac{1}{2\pi} R_t}$$

modulo $Im\delta + Im\bar{\delta}$ where $R_t$ is the curvature for the metric $h_t$ and $u_t = h_t^{-1} \frac{d}{dt} h_t$. We also have similar formula for $BC_1$. Hence, by differentiating $BC$, we have

$$\delta \mathcal{L} = \int Tr (ue^{\frac{1}{2\pi} R} \wedge TdX - \chi_k u) e^{k\omega}$$

where $u$ is the infinitesimal variation for $h$.

Therefore, the Euler-Lagrange equation is our almost Hermitian Einstein equation.

QED

**STRING THEORY**
In String theory, the following equation comes up naturally \((aKE)_\eta\):

\[ R_{ijj} + \eta \ast [R^2]_{ijj} + \eta^2 (\ldots)_{ijj} = 0 \]

This equation is a close analog of \((aHE)_\eta\) as we shall explain in the following. In string theory, we need to 'compactify' a ten dimensional space time to a four dimension one by assuming the extra six dimensions to be very tiny. We assume the ten dimensional space to be of the form \(X^6 \cdot \mathbb{R}^{3,1}\). By the phenomenology consideration ([W]), \(X\) has to be a complex three dimensional compact Kahler manifold with vanishing first Chern class, the so-called Calabi-Yau manifold. By the famous solution of Yau on Calabi’s conjecture, there exists a unique Kahler Einstein metric on each Kahler class of \(X\). Which means that the Ricci curvature is zero for such a metric:

\[ R_{ijj} = 0 \]

However, in String theory, the equation that one needs to solve on \(X\) is not exactly the Kahler-Einstein equation. Instead, it is a perturbed equation, \((aKE)_\eta\). Here,

\[ \ast [R^2] = R_{ij} k l R_{kl mn} R_{mn ij} - R_{ij} k l R_{kl mn} R_{mn i j} \]

where \(R_{ijkl}\) is the curvature tensor of \(X\). Notice that the integral of \([R^2]\) over \(X\) is proportional to \(ch_3(E)[X]\). Now, if we express the Ricci curvature using the Kahler form \(\omega\) by the Ricci identity:

\[ R_{ij} = \partial_i \partial_j \log \omega^3 \]

Then, this equation will looks like

\[ \partial \overline{\partial} (\log \omega^3 + \eta \ast [R^2] + \eta^2 (\ldots)) = 0 \]

Notice that \(\log \omega^3\) is not a well-defined function on \(X\). Only \(\partial \overline{\partial} \log \omega^3\) is well-defined. We choose any background Kahler metric \(\omega_0\), then \(\log \omega^3 - \log \omega_0^3 = \log \frac{\omega^3}{\omega_0^3}\) becomes a well-defined function on \(X\). However, the first Chern class of \(X\) is zero, therefore, \(Rc_X(\omega_0)\) can be written as \(R_{ij}(\omega_0) = \partial_i \partial_j \log \omega_0^3 = \partial_i \partial_j F\) for some function \(F\) on \(X\).

Then, up to first order in \(\eta\), the equation will become
$$\omega^3 + \eta [R]^3 + \eta^2 (...) = e^F \cdot \omega_0^3$$

Now, it becomes clear that why this perturbed Kahler Einstein equation is a close relative of almost Hermitian Einstein equation. However, there are some differences between them. First, this is a scalar equation. Second, we only consider the first order term in this equation. In fact, there are infinite number of correction terms in this perturbed Kahler Einstein equation.

According to Witten, there is strong evidence that there exists solution to \((aKE)_n\), not just order by order in \(\eta\), there should be exact solution for it. Moreover, there is a version of it when it couples with a stable holomorphic vector bundle \(E\) on \(X\), where \(c_1(E) = c_1(X) = 0\) and \(c_2(E) = c_2(X)\). He also argues that the modified equation (with \(E\) included) also has solution up to all order in \(\eta\). However, for the exact solution to it, it require the bundle \(E\) to be trivial over any rational curves (i.e. image of \(\mathbb{CP}^1\) under a holomorphic map) on \(X\). It would be very interesting to understand the role of rational curves in these perturbed type equations.

**HIGHER LOOPS CORRECTIONS**

The Ricci flat metric for a manifold is the condition for certain path integral in string theory to be renormalizable up to first order in perturbation theory - first loop level. The above perturbed equation is the condition which includes higher loops corrections as well. In physics term, it means the vanishing of \(\beta\) function. A similar situation is in gauge theory, where the Hermitian-Einstein equation is only the condition for renormalizability up to first loop. The full equation for renormalizability (which includes higher loops corrections) in perturbation theory would be a perturbed equation from Hermitian-Einstein equation, we wonder if that equation is related to almost Hermitian-Einstein which comes up naturally from algebraic geometry.
CHAPTER FOUR
SYMPLECTIC CONSIDERATIONS

In this chapter, we shall introduce a symplectic quotient in a generalized sense, which is larger than the usual symplectic quotient. It is motivated by the Gieseker stable bundle being a generalization of the Mumford stable bundle.

The moduli space of flat bundles over a Riemann surface $\Sigma$ can be considered as a symplectic quotient (in an infinite dimensional setting). We shall recall the construction of moment map and symplectic quotient later in this chapter. The symplectic space in question is the space of connections $\mathfrak{A}$ on a bundle $E$ over $\Sigma$ with structure group $SU(r)$ and the symplectic form on it is defined as follows: Let $D_A \in \mathfrak{A}$ be a connection on $E$ and $B, C$ are two tangent vectors of $\mathfrak{A}$ at $D_A$. By the natural identification of the tangent space of $\mathfrak{A}$, $T\mathfrak{A} = \Omega^1(\Sigma, adE)$, $B$ and $C$ can be regarded as endomorphism valued one form on $\Omega$. Then the symplectic form at $D_A$ evaluated at $B$ and $C$ is

$$\Omega(B, C) = - \int_{\Sigma} Tr B \wedge C$$

The Gauge group $\mathfrak{G}$ acts symplectically on it and the symplectic quotient is the moduli space of flat bundle. Although both $\mathfrak{A}$ and $\mathfrak{G}$ are of infinite dimensional, the symplectic quotient is of finite dimensional. It can also be identified as the space of representations for the fundamental group of $\Sigma$ in $SU(r)$ up to conjugation. Notice that the moment map in this case is simply the curvature of the connection $D_A$.

A similar situation occurs when the base manifold $X$ is a symplectic manifold of $\dim_{\mathbb{C}} X = n$. Then the symplectic form on the space of Hermitian connections would be

$$\Omega(B, C) = - \int_X Tr B \wedge C \wedge \frac{\omega^{n-1}}{(n-1)!}$$

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where $\omega$ is the symplectic form on $X$.

By dimension counting, the symplectic quotient (by the Gauge group action) would be of infinite dimensional. In fact, the moment map in this situation is simply

$$\mu(D_A) = F_A \wedge \frac{\omega^{n-1}}{(n-1)!} \in \Omega^{n,n}(X, End(E)).$$

If we further assume that $X$ has a compatible complex structure such that $X$ becomes a Kahler manifold with Kahler form $\omega$ and we restrict our attention to only those connections whose $(0,\cdot)$ part of its curvature vanishes, $F^{0,2} = 0$. By ([AHS]), these connections will define holomorphic structures on the bundle $E$, and conversely every holomorphic bundle over $X$ possesses such a connection. It can be checked that the Gauge group also acts on this space $\mathfrak{A}$ symplectically. The symplectic quotient would be the moduli space of Hermitian-Einstein connections on $E$. Let us denote it by $\mathcal{M}_\omega$.

In fact, there (essentially) exists a one parameter family of symplectic structure $\Omega$ on $\mathfrak{A}$ such that $\Omega\omega = \Omega$. This family of symplectic structure is defined by the degree $(n-1, n-1)$ term in the Atiyah-Singer integrand (in contrast to our equation which used the degree $(n, n)$ term of it):

$$\Omega_{1/k}(B, C) = \int_X Tr_E B \wedge [e^{(k\omega I + \frac{1}{k} R)} TdX] \wedge C :$$

Recall that $:\wedge$ is the normalized product which we defined in chapter three. The Gauge group acts symplectically on $\mathfrak{A}$ with respect to any one of these symplectic structures. One can show that the moment map in this case is indeed given by our almost Hermitian-Einstein equation.

**Proposition 4** The zeros of the moment map for the gauge group action on $\mathfrak{A}$ with (possibly degenerate) symplectic form $\Omega_{1/k}$ is equivalent to almost Hermitian Einstein connections.

**Proof of Proposition:**

Let us denote the operator in the almost Hermitian Einstein equation as a map $\mu$ from $\mathfrak{A}$ to $\Omega^{n,n}(X, End(E)) = Lie(\mathfrak{g})^*$. For any $u \in \Omega^0(X, End(E)) =$
$\text{Lie}(\mathcal{G})$, we have a functional $\mu(u) : \mathfrak{A} \to \mathbb{R}$. We want to show that

\[
d(\mu(u))|_{A}(C') = -\Omega_A(B, C')
\]

where $B$ is the vector at $A$ defined by the flow on $\mathfrak{A}$ generated by $u$. In our gauge theory setting, we have $B = D_A u$. We have,

\[
\mu(u) = \int Tr u \cdot e^{kwl + \frac{i}{2} R_A} T d_{X}.
\]

For any $C \in \Omega^1(X, \text{End}(E)) = T_A \mathfrak{A}$, we have

\[
d(\mu(u))|_{A}(C') = \int Tr u \delta C \cdot (e^{kwl + \frac{i}{2} R_A}) \wedge T d_X
\]

\[
= \int Tr u : e^{kwl + \frac{i}{2} R_A} \wedge D_A C : \wedge T d_X
\]

\[
= -\int Tr D_A u : e^{kwl + \frac{i}{2} R_A} \wedge C : \wedge T d_X
\]

\[
= -\Omega_A(B, C').
\]

QED

Now, we denote their corresponding symplectic quotient by $\mathcal{M}$. That is $\mathcal{M} = \mu^{-1}(0)/\mathcal{G}$. Let me draw a simplified picture for what happens as $k$ goes to infinity.

![Diagram](image)

The big circle is the infinity of $\mathfrak{A}/\mathcal{G}$ (although $\mathfrak{A}/\mathcal{G}$ does not have a good compactification). We would like to consider $\lim_{\eta \to 0} \mathcal{M}_\eta =: \mathcal{M}_0$, where $\eta = 1/k$. We should consider $\mathcal{M}_0$ and those points in the "boundary" (if the boundary exists) of $\mathcal{M}_0$ which are limits of points in $\mathcal{M}_\eta$. $\mathcal{M}_0$ is still a symplectic manifold.
In our case, $\mathcal{M}_{0}$ would be the moduli space of almost Hermitian-Einstein connections (which is essentially equivalent to the moduli space of (poly) Gieseker stable bundles over $X$. We will come back and give more details of this situation. Let us first recall some general result in symplectic geometry and relations to Geometric Invariant Theory of Mumford. Then we define Large Symplectic Quotient.

Let $A$ be a symplectic manifold (not necessary compact), and $\Omega$ be its symplectic form. If $G$ is a compact Lie group acting on $A$ symplectically and $\mu$ is the corresponding moment map.

$$\mu : A \rightarrow g^*$$

where $g^*$ is the dual of the Lie algebra of $G$. To define $\mu$, for any $\phi \in g$, $\phi$ generates a (Hamiltonian) vector field $X_\phi$ on $A$ via the group action. Using the symplectic form, this vector field will correspond to certain one from on $A$. Under certain mild condition (say $H^1(A) = 0$), this one form can be written as $df_\phi$ for some function $f_\phi : A \rightarrow \mathbb{R}$. That is $df_\phi(v) = \omega(X_\phi, v)$. This $f_\phi$ defines the moment map $\mu$ as follows:

$$< \mu(A), \phi > = f_\phi(A)$$

$f_\phi$ is called the Hamiltonian function. It is not hard to check that $G$ also acts on the zero set of the moment map and this quotient space $\mu^{-1}(0)/G$ can be proved to admit a natural symplectic structure, we denoted it by $A/\!/G$. $A/\!/G$ is called the symplectic quotient of $(A, \Omega)$ by $G$. In general, we expect

$$\text{dim}A/\!/G = \text{dim}A - 2 \text{dim}G$$

More generally, for each co-adjoint orbit $\Lambda \subset g^*$, its inversed image in $A$ is also acted by $G$ and the quotient $\mu^{-1}(\Lambda)/G$ inherits a natural symplectic structure from $A$.

A simple example is $U(1)$ action on $\mathbb{C}^n$ by scalar multiplication, where the symplectic structure is the standard one given by the (constant) Kahler form on $\mathbb{C}^n$. Then the moment map is is given by $\mu(z) = i(|z|^2 - C')$ where $C$ is some constant. If we choose $C'$ equals to one, then $\mathbb{C}^n//U(1)$ would be $S^{2n-1}/U(1)$, which is $\mathbb{C}P^{n-1}$.
Now, we suppose that we have a decreasing family of open subset of $A$. That is, for each small $\eta \geq 0$, $A_\eta$ is an open subset of $A$ such that

1. $A_0 = A$
2. $A_\eta \subset A_{\eta'}$ if $\eta > \eta'$
3. $A = \bigcup_{\eta > 0} A_\eta$

On each $A_\eta$, there is a symplectic form $\Omega_\eta$ such that they vary smoothly in $\eta$. If $G$ is a compact Lie group acting on $A$ leaving each $A_\eta$ invariant and acting symplectically with respect to each $\Omega_\eta$. Then we would like to form the space $\lim_{\eta \to 0}(A_\eta, \Omega_\eta)//G$. Let us denote $M_\eta = (A_\eta, \Omega_\eta)//G$ and $M_{0+} = \lim_{\eta \to 0^+} M_\eta$. To define this limit, we need to choose a almost complex structure $J_\eta$ on $A_\eta$ for every $\eta$ (including $\eta = 0$) such that they vary smoothly in $\eta$. By averaging over the compact group $G$ if necessary, we can assume each $J_\eta$ is $G$ invariant. Using $\Omega_\eta$ and $J_\eta$ on $A_\eta$, we obtain a $G$ invariant metric $g_\eta$ on it and hence $A_\eta$ becomes an almost Kahler manifold. Using $\frac{d\mu_\eta}{d\eta}$ and the almost complex structure, we can construct a vector field $V_\eta$ on $A_\eta$ for each $\eta$. At a point $D_A$ in $A_\eta$, we denote the tangent space of the $G$ orbit at $D_A$ by $S$. Then $V_\eta$ is defined to be the vector field such that at each point $D_A$, it lies inside $J_\eta S$ and such that for any $\psi \in \text{Lie}(G)$,

$$\Omega_{\eta, D_A}(X_\psi, V_\eta) = \frac{d\mu_\eta}{d\eta}(D_A)(\psi)$$

One can show that this vector field $V$ defines a ‘flow’ from $M_\eta$ to $M_{\eta'}$ with $\eta' < \eta$. Now, we can use this flow to define the limit $M_{0+}$ of $M_\eta$'s and a limiting symplectic structure on $M_{0+}$. In general, this symplectic structure may degenerate. However, if there is control of the symplectic structure along the flow, this would not happen and this is what happens in our gauge theory situation (curvature is bounded along the flow line).

Although $M_{0+}$ is defined using the almost complex structure, it is in fact independent of it. More is also true, the limiting symplectic structure on it is independent of the choice of almost complex structure too. It is clear that $M_0 = (A_0, \Omega_0)//G$ is a (possibly empty) open subset of $M_{0+}$ and the restriction of the limiting symplectic structure to $M_0$ coincide with the symplectic structure coming from the symplectic quotient construction. Since
$M_{0^+}$ contains $M_0$ as an open subset, we will call $M_{0^+}$ the Large Symplectic Quotient. In case $A$ is a compact finite dimensional symplectic manifold, these two spaces are the same.

In our gauge theory situation, $A$ is the space of holomorphic connections on a (topological) vector bundle over a compact Kahler manifold $X$ and $\Omega_\eta$ defined as before. The reason to restrict to an open subset $A_\eta$ of $A$ is because for each $D_A$, $\Omega_\eta$ is non-degenerate whenever $\eta$ is small enough. In this situation, we have a natural gauge invariant complex structure on $A$ inherited from the complex structure on $X$. Therefore, even though the gauge group is not compact, we can still have an invariant almost complex structure to define the limit. In fact, we have a complex structure and $A$ is an infinite dimensional Kahler manifold. The symplectic quotient $M_0$ should be regarded as the moduli space of Mumford stable bundles and the large symplectic quotient $M_{0^+}$ should be regarded as the moduli space of Gieseker stable bundles.

Now, we want to show that the symplectic form $\Omega_k$ comes naturally from family index theorem by Bismut and Freed ([BF]) and this line of ideas in holomorphic gauge theory appeared in Donaldson’s paper ([Do2]). Let $\mathfrak{A}$ be the space of all connections on (topological bundle) $E \otimes L^k$ and $\mathfrak{A}^{1,1}_k$ be the subspace consisting of those connections whose $(0,2)$ part curvature vanish. Consider the projection $\pi : X \times \mathfrak{A} \to \mathfrak{A}$ and the universal bundle $E(k) = \pi^*(E \otimes L^k)$. There is a canonical universal connection $A$ on $E(k)$ such that its curvature $F$ satisfies

$$
F^{2,0} = F_A \text{ at } x,
F^{1,1} = (x, A)(v, B) = B(v) \text{ at } x,
F^{0,2} = 0.
$$

where $(x, A) \in X \times \mathfrak{A}$ and $v \in T_xX, B \in \Omega^1(X, \text{End}(E))$. (These objects are introduced by Atiyah and Singer ([AS]) in studying anomaly for gauge theory). We restrict our attention to $\mathfrak{A}^{1,1}_k$ where each connection defines a holomorphic structure on $E \otimes L^k$. Over $\mathfrak{A}^{1,1}_k$, we can form the determinant line bundle for this family of $\overline{\partial}$ operators, $DET = \text{Det} R^* \pi_* E(k) \to \mathfrak{A}^{1,1}_k$. (Notice that if we are only in differential category, we can still defines a determinant line bundle over $\mathfrak{A}$ provided that $X$ is spin, in that case, the
family of operators comes from twisted Dirac operators.) The first Chern form of $\text{DET}$ with respect to Quillen metric is the same as $\Omega_k$ (up to constant multiple). Recall that Quillen metric is defined by $L^2$ metric on harmonic forms multiple by Ray-Singer analytic torsion.

**PROPOSITION 5** The symplectic form $\Omega_k$ is the first Chern form of the determinant line bundle $\text{DET}$ over the space of connections with respect to the Quillen metric.

**PROOF OF PROPOSITION:**

This will be derived from family index theorem of Bismut and Freed as follows:

$$c_1(\text{DET}, h_Q) = \left( \int_X Tr \ e^{\frac{i}{4\pi} \bar{\partial}} \cdot Td_X \right)^{(2)}$$

Let $B, C \in T_\mathcal{A} = \Omega(X, End(E))$,

$$c_1(\text{DET}, h_Q)(A)(B, C') = \int_X Tr \ e^{\frac{i}{4\pi} \bar{\partial}}(B, C') \wedge Td_X$$

$$= \int_X Tr \ e^{\frac{i}{4\pi} (F^{2,0} + F^{1,1})}(B, C') \wedge Td_X$$

However, only two of $F^{1,1}$ can survive in the integral since we only need the two form part on $\mathcal{A}$ of the universal Chern character form. By using the property of $Tr$ to permut $B$ inside the integral to reach the left most position, we see that $c_1(\text{DET}, h_Q)(A)(B, C')$ is then given by $\frac{1}{4\pi} \int_X Tr_B \wedge : [e^{(k\omega I + \frac{i}{2\pi} R)} Td_X] \wedge C$; since in our case $\frac{i}{2\pi} F_A = k\omega I + \frac{i}{2\pi} R$. This is the same as $\Omega_k(B, C')$ up to a constant multiple.

QED

Moreover, instead of using Bott-Chern secondary characteristic classes to define our functional $\mathcal{L}$, we can also use Ray-Singer analytic torsion for $E \otimes L^k$ to do it in a way similar to the one Donaldson used in his paper ([Do2]).

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Remark: We can study this whole picture from another point of view. We consider all connections on $E \otimes L^k$ for all $k \in \mathbb{C}$. Since we have fix a background curvature on $L$, namely the Kahler form on $X$, this total space of connections can be identified as $\mathfrak{a} \times \mathbb{C}$. One can define a degenerate symplectic form on it (which is in fact non-degenerate in the large $k$ limit). The gauge group still acts symplectically on it and the moment map $\mu$ is given by as follows:

$$\mu(D_A, \eta) := \mu_\eta(D_A)$$

The whole picture that we discussed before can be rephrased in this setting.
CHAPTER FIVE
EXISTENCE OF SOLUTION IMPLIES STABILITY

In this chapter, we will prove Gieseker stability on holomorphic vector bundles assuming almost Hermitian Einstein connection exists. We shall see that an almost Hermitian-Einstein bundle is always Mumford semi-stable and hence it has a Jordan-Holder filtration. $E$ is called sufficiently smooth if the associated graded sheaf $Gr(E)$ is locally free. This is the natural smoothness assumption on holomorphic vector bundle analog to the usual smoothness in Hermitian-Einstein metric situation.

PROPOSITION 6 Let $E$ be an irreducible sufficiently smooth holomorphic vector bundle of rank $r$ over a compact Kahler manifold of dimension $n$. Suppose that $E$ is almost Hermitian Einstein, then $E$ is Gieseker stable.

Let us first recall the almost Hermitian Einstein equation $(aHE)_\eta$,

$$[e^{(\omega l + \eta \frac{i}{2\pi} R)} Td_X^\eta]_{TOP} = \chi_E I_E \omega^n \frac{n!}{n!}$$

where $Td_X^\eta$ is the harmonic Todd polynomial for $X$ with variable $\eta$.

For an almost Hermitain Einstein bundle, we mean the above can be solved by a connection for all sufficiently small positive real number $\eta$ such that the $(C^k)$ norm of the curvature is bounded independent of $\eta$.

PROOF OF PROPOSITION :

For simplicity, we assume that the volume of $X$ is one. That is $\int_X \frac{\omega^n}{n!} = 1$. We shall first see that $E$ is Mumford semi-stable. From the equation and the boundedness of the curvature, we have for any positive constant $\delta$, a Hermitian metric $h$ on $E$ such that its curvature $R$ satisfies the following estimate :

$$|\Lambda R - \mu_E I_E| < \delta.$$ 

According to ([Ko]), such a bundle is said to have an approximate Hermitian Einstein Structure. It is not hard to show that an approximate Hermitian Einstein bundles is Mumford semi-stable. The proof goes essentially
the same as the one for Mumford stability for a Hermitian Einstein bundle.

Recall that a Mumford semi-stable bundle has a Jordan-Holder filtration

$$E = E_0 \supset E_1 \supset E_2 \supset \ldots \supset E_{k+1} = 0$$

By assumption, each $E_i$ is a holomorphic vector bundle. In the following, we are going to show that for each $i$,

$$\frac{\chi(X, E \otimes L^k)}{\text{rank}E} \geq \frac{\chi(X, E_i \otimes L^k)}{\text{rank}E_i}$$

for large enough $k$.

In general, suppose that we are given a subbundle $S$ of $E$ with $\text{rank}S = s$. Let $h = h(\eta)$ be the Hermitian metric that solves equation $(aHE)_\eta$. Then we decompose $E$ orthogonally with respect to $h$:

$$E = S \bigoplus Q$$

where $Q$ is the quotient bundle for $S \to E$. In general we have a holomorphic exact sequence:

$$0 \to S \to E \to Q \to 0$$

With respect to this orthogonal decomposition, we can also decompose the Hermitian connection $D$ on $E$ and its curvature $R$ as follow:

$$D = \begin{pmatrix} D_S & A^* \\ A & D_Q \end{pmatrix}$$

and

$$R = \begin{pmatrix} R_S - A^*A & \partial A^* \\ -\overline{\partial} A & R_Q - AA^* \end{pmatrix}$$

Here, we have $D = \partial + \overline{\partial}$ as the decomposition of $D$ into sum of its $(1,0)$-part and $(0,1)$-part. Since $S$ is holomorphic, $A$ is a $(1,0)$ form with valued in $S^* \otimes Q$ and $\partial A = 0$ (which is equivalent to $\overline{\partial} A^* = 0$.) From $\overline{\partial} A^* = 0$, $A^*$ represents certain cohomology class. In fact, $A^*$ represents a cohomology class which is the extension class for the above holomorphic exact sequence.

$$[A^*] \in \text{Ext}^1_{\Omega_X}(Q, S) = H^1(X, Q^* \otimes S)$$

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Use $R|_S = R_S - A^* A$. We take the trace of it over $S$ and integrate it over $X$ to obtain
\[
\int_X |A|^2 < C \cdot B + s \cdot \mu_S
\]
Since $\lim_{\eta \to 0} \eta |R|_{C^1} = 0$ for some family of Hermitian metric depending on $\eta$, one can prove by elliptic estimate that $\lim_{\eta \to 0} \eta |A|^2 = 0$.

Now, from the equation and the decomposition of the curvature, we have the following

\[
\frac{\chi_E - \chi_S}{\eta} = \frac{i}{2\pi} \int_X \text{Tr} A (\omega + \eta (\frac{i}{2\pi} R + T d_1)) A^* \wedge \frac{\omega^{n-2}}{(n-2)!} - \frac{i}{2\pi} \int_X \text{Tr} A^* (\omega + \eta (\frac{i}{2\pi} R + T d_1)) A \wedge \frac{\omega^{n-2}}{(n-2)!} - \eta T_1 - \eta^2 T_2 - \ldots.
\]

Choose a small $\eta$ such that $|\eta (\frac{i}{2\pi} R + T d_1)| \leq 1/2$, then the two integrals on the right side are greater than or equal to $\int |A|^2$. To deal with those $T_i$ terms. Notice that each $T_i$ is an integral of product of terms like $A^* A$, $\partial A$, $\partial A^*$, $R_S$, $R_Q$ and some closed even forms of $X$. The number of these terms in the product of $T_i$ is less than or equals to $l + 1$. We claim that each $\eta^l T_i$ is not greater than a small fraction of $\int |A|^2$. If this is the case, then we have proved the right hand side of the above equation is positive since $A$ cannot be trivial (otherwise it will violate the irreducibility assumption of $E$).

Therefore, we have to prove the above estimates of $\eta^l T_i$. First, we observe that $\eta$ times any one of $A^* A$, $\partial A$, $\partial A^*$, $R_S$, $R_Q$ is very small by our assumption $\lim_{\eta \to 0} \eta |R|_{C^1} = 0$ and the previous lemmas. Then we see that if there is a term of $A^* A$ in the expression of $T_i$, we are done already. It is because there are now at most $l$ terms left in the integral and when they are multiplied by $\eta^l$, they become very small and we can control them by a small fraction of the $L^2$ norm of $A$. (small) $\cdot \int |A|^2$.

Now, if there is no $A^* A$ term in $T_i$ but there is a $\partial A$ (or $\partial A^*$). Then we integrate by part that derivative. If the derivative hits a $R_S$ or $R_Q$ term, it will be zero by Bianchi identity. If it hit the closed form of $X$, it is also zero.
So, the only possibility is when it hits $\partial A^*$ (or $\overline{\partial} A$). Then we interchange $\partial$ and $\overline{\partial}$. Using $\overline{\partial} A = 0$, it will only produce terms which are products of curvature and $A^* A$ or $AA^*$. Then using previous argument, we are done in this case.

The final case is there is no $A$ term and only curvature terms appeared. Then we can change them to have only $R$ appeared but not $R_N$ or $R_Q$ since their differences are terms involving $A^* A$ or $AA^*$. Those terms involving integral of $R$'s will cancel out automatically. Therefore, we have proved that if $S$ is a subbundle of $E$, then $\chi_S < \chi_E$.

Suppose $S$ is any coherent subsheaf of $E$ (of $s = \text{rank } S < \text{rank } E$) and $S$ is a Gieseker destabilizing subsheaf for $E$. Since $E$ is Mumford semi-stable, we have $\mu_S$ is not larger than $\mu_E$. In order for $S$ to be a candidate as a Gieseker destabilizing subsheaf of $E$, $\mu_S$ must equal to $\mu_E$. Without loss of generality, we can assume that $S$ is Gieseker stable and $S$ is a Gieseker destability subsheaf for $E$. Consider the following exact sequence coming from the first step in the Jordan-Holder filtration,

$$0 \rightarrow E_1 \rightarrow E \rightarrow Q_0 \rightarrow 0$$

Consider the composition map from $S$ to $E$ to $Q_0$. Since $S$ is Gieseker stable, $Q_0$ is Mumford stable and they have the same slope (which is the slope of $E$), this map must be either zero of an isomorphism. However, by the irreducibility assumption of $E$, this cannot be an isomorphism. Therefore, we can lift the subsheaf $S$ of $E$ to a coherent subsheaf of $E_1$.

We are going to show that the rank of $S$ is strictly less than the rank of $E_1$. Suppose not, then the quotient sheaf $T_1$ of $E_1$ by $S$ is a torsion sheaf.

$$0 \rightarrow S \rightarrow E_1 \rightarrow T_1 \rightarrow 0$$

For large enough positive integer $k$, we have
\[
\chi(X, E_1 \otimes L^k) = \dim H^0(E_1 \otimes L^k) \\
\geq \dim H^0(S \otimes L^k) \\
= \chi(X, S \otimes L^k)
\]

Since rank \(E = \) rank \(S\) by assumption, we therefore have

\[
\frac{\chi(X, S \otimes L^k)}{\text{rank } E} \leq \frac{\chi(X, E_1 \otimes L^k)}{\text{rank } E_i}
\]

By our previous argument, this is smaller than \(\frac{\chi(X, E \otimes L^k)}{\text{rank } E_i}\) for \(k\) sufficiently large which contradicts to our assumption that \(S\) is a Gieseker destabilizing subsheaf for \(E\). Therefore, rank of \(S\) is strictly smaller than rank of \(E_1\).

We now look at the next step in the Jordan-Hölder filtration

\[
0 \rightarrow E_2 \rightarrow E_1 \rightarrow Q_1 \rightarrow 0
\]

and now \(S\) is a sheaf of \(E_1\) of smaller rank and with the same slope. Repeating the previous argument, we see that \(S\) is in fact a subsheaf of \(E_2\) of strictly smaller rank. Inductively, we can conclude that \(S = E_{k+1}\) which is zero sheaf. As a result, \(S\) does not exist and hence \(E\) is a Gieseker stable bundle.
CHAPTER SIX
PROOF OF MAIN THEOREM

In this chapter, we shall prove the existence of almost Hermitian Einstein metric on a Gieseker stable sufficiently smooth holomorphic vector bundle $E$ over a compact Kahler manifold $X$. The equations involved is a fully nonlinear elliptic system of partial differential equation.

The method that we are going to use is singular perturbation arguments. By using Jordan-Holder filtration of $E$, we can decompose $E$ as Mumford stable bundles and their extension classes. On a Mumford stable bundle, there is a unique Hermitian Einstein metric by the theorem of Uhlenbeck and Yau. If we try to use their method to construct an almost Hermitian Einstein metric on $E$. Then we expect the metric to blow up as $\eta$ goes to zero. In fact, after suitable rescaling, these metric will 'converge' to the direct sum of Hermitian Einstein metric on each component of Mumford stable bundles and 'forget' the extension classes.

Our proof will confirm the about observation and in fact reverse the blowing up. We will perturb from the direct sum of Hermitian Einstein metric (which is at the infinity of the complex gauge orbit for $E$). However, the linearized opertor will have a large kernel. The situation is similar to Taubes' ([Ta]) proof of existence of anti-self-dual connection on $SU(2)$ bundles over compact four dimensional manifolds if the second Chern class of $E$ is big enough. In his proof, he was gluing in concentrated instantons and try to perturb them, first he need to use the conformal symmetry of Yang Mills theory to rescale the neighborhood of those points where concentrated instanton is added. Then to perturb the equation, there is still a finite dimensional obstruction which can be killed by adding enough instantons and therefore the second Chern class of the bundle has to be big enough.

Instead of the conformal symmetry, we have a complexified Gauge group as our symmetry group. We need to rescale the equation order by order using this complexified Gauge group. Then we can identify the obstruction for perturbation is exactly the coefficients of the Hilbert polynomail of $E$. As a result, if $E$ is Gieseker stable, we will prove that there exists an almost...
Hermitian Einstein metric on it.

Let us first recall the Jordan-Holder filtration for a Mumford semi-stable bundle, in particular, a Gieseker stable bundle.

**THEOREM 5 (JORDAN-HOLDER THEOREM)** If $E$ is a Mumford semi-stable bundle over $X$, there exist a filtration of $E$ by torsion free subsheaves $E_i$'s

$$E = E_0 \supset E_1 \supset E_2 \supset \ldots \supset E_{k+1} = 0$$

such that $Q_j = E_j/E_{j+1}$ is Mumford stable and $\mu(Q_j) = \mu(E)$ for each $j$. Moreover,

$$Gr(E) = Q_0 \bigoplus Q_1 \bigoplus \ldots \bigoplus Q_k$$

is uniquely determined by $E$ up to isomorphism.

Let us first proof the existence result in a very particular case. We assume that only two components appears in the filtration and they are both locally free.

**PROPOSITION 7** Let $E$ be a Gieseker stable bundle over a compact Kahler manifold $X$. Suppose that

$$0 \rightarrow S \rightarrow E \rightarrow Q \rightarrow 0$$

is a Jordan-Holder filtration of $E$ and both $S$ and $Q$ are Mumford stable bundles over $X$.

Then $E$ is an almost Hermitian Einstein bundle.

**PROOF**: Let us denote the $i^{th}$ coefficient of $\chi_E$ by $\chi_{E}^i$. That is

$$\chi_E = 1 + \eta\chi_E^1 + \eta^2\chi_E^2 + \eta^3\chi_E^3 + \ldots$$

As a warm up, we will first assume that $\chi_S^2 < \chi_E^2$. Notice that since $\mu(S) = \mu(E) = \mu(Q)$, we have $\chi_S^1 = \chi_E^1 = \chi_Q^1$.

Now, the almost Hermitian Einstein equation on $E$ is $(aH E)$:
\[ \frac{i}{2\pi} R_E \wedge \frac{\omega^{n-1}}{(n-1)!} = \mu_E \frac{\omega^n}{n!} I_E + \sum_{j=1}^{n-1} \eta^j T_{j+1} \]

for sufficiently small positive \( \eta \) where

\[ T_j = \chi_E^j \frac{\omega^n}{n!} - \sum_{k=0}^{j} \left( \frac{i}{2\pi} R_E \right)^k t d_X^{n-k} \frac{\omega^{n-j}}{(n-j)!} \]

Since both \( S \) and \( Q \) are Mumford stable bundles, there exists unique Hermitian Einstein metric \( h_{OS} \) and \( h_{OQ} \) on them, and with connections \( D_{OS}, D_{OQ} \) and curvatures \( R_{OS}, R_{OQ} \), by the theorem of Uhlenbeck and Yau. Therefore, by using \( \mu(S) = \mu(E) = \mu(Q) \), we have

\[ \wedge R_{OS} = \mu_E I_S \]

and

\[ \wedge R_{OQ} = \mu_E I_Q \]

Let \( B \) be a holomorphic \((0,1)\) form on \( X \) with values in \( Hom(Q, S) \) such that its cohomology class represents the extension class of \( E \) being an extension of \( Q \) by \( S \).

\[ 0 \rightarrow S \rightarrow E \rightarrow Q \rightarrow 0 \]

Notice that the extension class of \( E \) in \( Ext^1(Q, S) \) is unique determined by the above exact sequence up to scalar multiple. Therefore, there exists a unique harmonic \( B \) which satisfies

\[ \int |B|^2 = (\chi_E^2 - \chi_S^2) \cdot (rkS) \]

The existence of such a \( B \) is equivalent to \( \chi_S^2 < \chi_E^2 \), which is from the assumption of \( E \) being Gieseker stable. We shall denote \(-B^*\) by \( A \). Now, we want to construct a Hermitian connection \( D_E \) on \( E \) of the following form:

\[ D_E = \begin{pmatrix} D_{OS} + h_S^{-1} \partial h_S & -A^* - \bar{\partial} \phi^* \\ A + \partial \phi & D_{OQ} + h_Q^{-1} \partial h_Q \end{pmatrix} \]
where $h_S$ (resp. $h_Q$) is a positive self-adjoint endomorphism of $S$ (resp. $Q$) with respect to the background metrics and $\phi$ is a homomorphism from $S$ to $Q$. Therefore, the curvature of $D_E$ is of the form:

$$R_E = \begin{pmatrix}
R_{0S} + \bar{\partial}(h_S^{-1}\partial h_S) - (A^* + \bar{\partial}\phi^*)(A + \partial\phi) & \partial\nu^e(A^* + \bar{\partial}\phi^*) \\
\bar{\partial}(A + \partial\phi) & R_{0Q} + \bar{\partial}(h_Q^{-1}\partial h_Q) - (A + \partial\phi)(A^* + \bar{\partial}\phi^*)
\end{pmatrix}$$

Let us now denote (see [UY])

$$u_S = \log h_S$$

and

$$u_Q = \log h_Q$$

Now, we will regard $u_S, u_Q$ and $\phi$ as the variable for the almost Hermitian Einstein equation for a fix small $\eta$. It is clear that the equation is solvable for $\eta$ equals to zero. However, the linearized operator at $\eta = 0$ is not invertible. Therefore, we make the following rescaling:

$$u_S \rightarrow \eta u_S$$

$$u_Q \rightarrow \eta u_Q$$

$$A \rightarrow \eta^{1/2} A$$

$$\phi \rightarrow \eta \phi$$

We just rescale each variable by multiplying by $\eta$ in this case. However, in the general case, the rescaling is more complicated as we will soon see. Now, we rewrite the equation in terms of these new variables. That is

$$D_E = \begin{pmatrix}
D_{0S} + \eta h_S^{-1}\partial h_S - \eta^{1/2} A^* - \eta \bar{\partial}\phi^* \\
\eta^{1/2} A + \eta \partial\phi & D_{0Q} + \eta h_Q^{-1}\partial h_Q
\end{pmatrix}$$

Because both background metrics on $S$ and $Q$ are Hermitian Einstein, the constant term in $\eta$ of both sides of the equation are equal. After subtracting $\mu_E \frac{\omega^n}{n!} I_E$ from both sides of the equation and divides both sides by $\eta$, the almost Hermitian Einstein equation will become the following form
(we divide the equation into three equations according to different blocks in the decomposition):

\[ (1)_n : \]
\[ \frac{i}{2\pi} \left( \overline{\partial} (h^{-1}_S \partial h_s) - A^* A \right) \wedge \frac{\omega^{n-1}}{(n-1)!} \]
\[ = \chi^2 \omega^n \frac{i}{n!} I_S - \left[ \left( \frac{i}{2\pi} R_{0S} \right)^2 + \frac{i}{2\pi} R_{0S} \cdot td_1^X + td_2^X \right] \wedge \frac{\omega^{n-2}}{(n-2)!} + \mathcal{O}(\eta^{1/2}) \]

\[ (2)_n : \]
\[ \frac{i}{2\pi} \left( \overline{\partial} (h^{-1}_Q \partial h_Q) - AA^* \right) \wedge \frac{\omega^{n-1}}{(n-1)!} \]
\[ = \chi^2 \omega^n \frac{i}{n!} I_Q - \left[ \left( \frac{i}{2\pi} R_{0Q} \right)^2 + \frac{i}{2\pi} R_{0Q} \cdot td_1^X + td_2^X \right] \wedge \frac{\omega^{n-2}}{(n-2)!} + \mathcal{O}(\eta^{1/2}) \]

\[ (3)_n : \]
\[ \frac{i}{2\pi} \overline{\partial} \phi \wedge \frac{\omega^{n-1}}{(n-1)!} = \mathcal{O}(\eta^{1/2}). \]

Notice that in equation three, we should have a term \( \overline{\partial} A \wedge \frac{\omega^{n-1}}{(n-1)!} \eta^{1/2} \). However, using the fact that \( B \) is harmonic, this term is equal to zero identically.

Next, we claim that these three equations have a unique solution at \( \eta = 0 \) and the corresponding linearized operator is indeed invertible. For this purpose, we need to use a refined statement from Uhlenbeck and Yau. Although they did not write it down as a theorem, the proof is already inside their works.

**THEOREM 6 (UHLENBECK+YAU)** Let \( E \) be a Mumford stable bundle over a compact Kahler manifold \( X \) and \( h_0 \) be any Hermitian metric on \( E \). Suppose \( H_0 \) is an endomorphism of \( E \) such that \( \int Tr_E H_0 = 0 \). Then there exists a unique positive self-adjoint endomorphism \( h \) of \( E \) with \( det h = 1 \) which solves

\[ \overline{\partial} (h^{-1}_0 \partial h) = H_0 \]

Moreover, the linearized operator from \( \mathcal{B}^{k+2, \alpha} \) to \( \mathcal{B}^{k, \alpha} \) at \( log h \) is invertible.
Here $\mathcal{B}^{k,\alpha}$ denotes the Sobolev space of $C^{k,\alpha}$ self-adjoint endomorphism $u$ of $E$ with $\int Tr \ u = 0$.

Now, in equation (1), we shall take $\ast H_0$ to be

$$\chi^E \frac{\omega^n}{n!} I_S - \left[ \left( \frac{i}{2\pi} R_{0S} \right)^2 + \frac{i}{2\pi} R_{0S} \cdot td^*_N + td^*_N \right] \wedge \frac{\omega^{n-2}}{(n-2)!} + \frac{i}{2\pi} A^* A \wedge \frac{\omega^{n-1}}{(n-1)!}$$

By the choice of $B$, we have $\int Tr H_0 = 0$. Therefore, by the above theorem and the fact that $S$ is Mumford stable, $(1)_{\eta=0}$ always has a solution and the corresponding linearized equation at the solution is invertible provided that we restricts to the space of $u_S$ such that $\int Tr_S u_S = 0$. By the same reason, equation two can be solved unique in the same way. For $(3)_{\eta=0}$, it is

$$\frac{i}{2\pi} \bar{\partial} \partial \phi \wedge \frac{\omega^{n-1}}{(n-1)!} = 0$$

If $\phi$ is a solution of it, then we have $\bar{\partial} \phi^* = 0$. That is $\phi^*$ defines a holomorphic morphism from $S$ to $Q$. However, both $S$ and $Q$ are Mumford stable bundles of the same slope. By the general properties of Mumford stable bundles (see chapter two), $\phi$ is either zero or an isomorphism. But $\phi$ cannot be an isomorphism, otherwise, we have $\chi_S = \chi_Q = \chi_E$ which violates our assumption that $E$ is Gieseker stable. Therefore, the last equation has zero as its only solution and it is also a linear equation. That is, it is invertible too.

Now, we are almost ready to perturb these solution to obtain the almost Hermitian Einstein metric on $E$. However, there is one dimensional kernel for $u_S$ and one dimensional kernel for $u_Q$ that we still need to take care of. Let me first introduce the suitable Banach spaces and bounded operator that we are going to do perturbation. Let $\mathcal{B}^{k,\alpha}$ be the space of triples $(u_S, u_Q, \phi)$, where $u_S$ is a self-adjoint endomorphism of $S$ with $\int Tr_S u_S = 0$, and $u_Q$ is similarly defined. $\phi$ is a homomorphism from $S$ to $Q$ and all these homomorphisms are assumed to be of the class $C^{k,\alpha}$. The above three equations at $\eta = 0$ are therefore uniquely solved in these space. However, the almost Hermitian Einstein equation does not define an operator on these space for $\eta$ is not zero. For the equation

$$\left[ e^{(\omega + \eta^2 R)} T e^{\omega} \right]^{TOP} - \chi_E I_E \frac{\omega^n}{n!} = 0$$
if we take the trace of the left side and integrate it over $X$, then we will get zero. However, if we only do it on the upper left block, the part corresponding to the bundle $S$, then it is only zero up to second order in $\eta$. This is true because of our choice of $B$ and the rescaling. The same is true for the lower right block, that corresponding to the bundle $Q$ part. So, if we try to define a nonlinear differential operator $\mathcal{L}$ from $\mathfrak{B}^{k+2,\alpha}$ to $\mathfrak{B}^{k,\alpha}$, then we need to carefully rescale $A$ again. Instead of looking at $\eta^{1/2}A$, we should look at $\eta^{1/2} \cdot t \cdot A$, where $t$ is a function depending on $(uS, uQ, \phi)$ and $\eta$ such that it goes to one as $\eta$ goes to zero. Now, the Hermitian connection looks like:

$$D_E = \begin{pmatrix} D_{0S} + h^{-1}_S \partial h_S & -\eta^{1/2} t A^* - \overline{\partial} \phi^* \\ \eta^{1/2} t A + \partial \phi & D_{0Q} + h^{-1}_Q \partial h_Q \end{pmatrix}$$

The insertion of $t$ is so that the integral of the trace of the equation OVER $S$ is zero. The existence of $t$ can be proved by implicit function theorem and $t$ can be written down in terms of the variables explicitly. Now, we automatically get the same result for the $Q$ part because integrating the trace of the equation over the whole bundle $E$ is always zero.

As a result, we can define the nonlinear differential operator $\mathcal{L}$ as follow:

$$\mathcal{L} : \mathfrak{B}^{k+2,\alpha} \rightarrow \mathfrak{B}^{k,\alpha}$$

$$\mathcal{L} \cdot \begin{pmatrix} uS \\ \phi \\ \phi^* \\ uQ \end{pmatrix} = \frac{1}{\eta^2} \left( [e^{(\omega l + n_{\alpha}^{1/2})} T d_{\chi}^{\nu}]^{TOP} - \chi_E I e^{\omega n} \right)$$

where $R$ is the curvature for the above rescaled connection. Notice that the operator $\mathcal{L}$ is well-defined even for $\eta = 0$ because $D_{0S}$ and $D_{0Q}$ are both Hermitian Einstein of the same slope.

From the above discussion, we know that $\mathcal{L} = 0$ can be solved when $\eta$ equals to zero and the linearized operator is invertible there. By the Implicit Function Theorem in Banach space, the equation $\mathcal{L} = 0$ would therefore have solution for all small enough $\eta$ which depends smoothly in $\eta$ positive. Moreover, the $(C^{k,\alpha})$ norm of their curvatures are bounded. Therefore, $E$ is
almost Hermitian Einstein in this case.

Next, we will only assume $\chi_S < \chi_E$ for small enough $\eta$. There exists an integer $m > 1$ such that

$$\chi_S^j = \chi_E^j, \text{ for } j = 1, 2, \ldots m$$

and

$$\chi_S^{m+1} < \chi_E^{m+1}$$

Now, the representative $B$ for the extension class of $E$ will be chosen to be the unique harmonic form such that

$$\int |B|^2 = (\chi_E^{m+1} - \chi_S^{m+1}) \cdot (r k S)$$

which is possible by the above assumption. As before, $A = -B^*$. Before the rescaling, the connection on $E$ looks like:

$$D_E = \begin{pmatrix} D_{0S} + h_s^{-1} \partial h_s & -\eta^{m/2} A^* - \overline{\partial} \phi^* \\ \eta^{m/2} A + \partial \phi & D_{0Q} + h_Q^{-1} \partial h_Q \end{pmatrix}$$

We need to rescale the variables order by order in $\eta$. First, we note that the connection $D_{0E} = \begin{pmatrix} D_{0S} & -\eta^{m/2} A^* \\ \eta^{m/2} A & D_{0Q} \end{pmatrix}$ solves the equation in zeroth order in $\eta$. Consider

$$D_{1E} = \begin{pmatrix} D_{0S} + \eta h_s^{-1} \partial h_{1S} & -\eta^{m/2} A^* \\ \eta^{m/2} A & D_{0Q} + \eta h_Q^{-1} \partial h_{1Q} \end{pmatrix}$$

We claim that there exist unique pair $(h_{1S}, h_{1Q})$ for each small $\eta$ which solves the almost Hermitian Einstein equation up to order one (where the uniqueness is under the normalisation that $\int Tr_S \log h_{1S} = 0$ and $\int Tr_Q \log h_{1Q} = 0$). Since these variables has no contribution to the zeroth order (in $\eta$) part of the equation, we only need to look at the first order term, which are:

$$\frac{i}{2\pi} \big( \overline{\partial} (h_s^{-1} \partial h_{1S}) \big) \wedge \frac{\omega^{n-1}}{(n - 1)!} = \chi_E^2 \frac{\omega^n}{n!} I_S - \sum_{k=0}^{2} (\frac{i}{2\pi} R_0 s)^k \cdot \frac{\omega^{n-2}}{(n - 2)!}$$

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\[
\frac{i}{2\pi} (\bar{\partial}(h_{1Q}^{-1}\partial h_{1Q})) \wedge \frac{\omega^{n-1}}{(n-1)!} = \chi_{\bar{E}}^{2} \frac{\omega^{n}}{n!} I_{Q} - \sum_{k=0}^{2} \left(\frac{i}{2\pi} R_{0Q}^{k}\right) t d_{X}^{3-k} \frac{\omega^{n-2}}{(n-2)!}
\]

By our assumption that \(\chi_{S}^{2} = \chi_{\bar{E}}^{2} = \chi_{Q}^{2}\), \(S\) and \(Q\) and Mumford stable and the theorem of Uhlenbeck and Yau, there exist a unique (normalized) solution pair \((h_{1S}\), \(h_{1Q}\)).

In order to solve the equation up to order two, we consider the following connection:

\[
D_{2E} = \begin{pmatrix}
D_{1S} + \eta^{2}(h_{2S}^{-1}\partial h_{2S}) & -\eta^{m/2} A^{*} \\
\eta^{m/2} A & D_{1Q} + \eta^{2}(h_{2Q}^{-1}\partial h_{2Q})
\end{pmatrix}
\]

We should notice that the background metric has been changed to a new one, which is \(h_{0}(\frac{h_{1S}^{\eta}}{h_{1Q}^{\eta}})\). For example, the operator \(\partial\) is with respect to this new background metric, the \(A\) is also rescaled such that its \(L^{2}\)-norm satisfies the previous equality under this new metric. However, we are only perturbing everything in first order of \(\eta\), there is no change for the lower order parts of the equation.

As before, by adding terms involving \((h_{2S}\), \(h_{2Q}\)) will not affect the equation up to order one. Therefore, the almost Hermitian Einstein equation up to second order in \(\eta\) will be:

(1)
\[
\frac{i}{2\pi} \bar{\partial}(h_{2S}^{-1}\partial h_{2S}) \wedge \frac{\omega^{n-1}}{(n-1)!} = \chi_{3}^{3} \frac{\omega^{n}}{n!} I_{S} - \sum_{k=0}^{3} \left(\frac{i}{2\pi} R_{1S}^{k}\right) t d_{X}^{3-k} \frac{\omega^{n-3}}{(n-3)!}
\]

(2)
\[
\frac{i}{2\pi} \bar{\partial}(h_{2Q}^{-1}\partial h_{2Q}) \wedge \frac{\omega^{n-1}}{(n-1)!} = \chi_{3}^{3} \frac{\omega^{n}}{n!} I_{Q} - \sum_{k=0}^{3} \left(\frac{i}{2\pi} R_{1Q}^{k}\right) t d_{X}^{3-k} \frac{\omega^{n-3}}{(n-3)!}
\]
By the same reasoning as before, there exists a unique (normalized) pair \((h_{2S}, h_{2Q})\) such that the connection \(D_{2E}\) solves the almost Hermitian Einstein equation up to second order in \(\eta\).

Repeat this process to get \((h_{1S}, h_{1Q}), (h_{2S}, h_{2Q}), \ldots.\) When we arrive to the \(\left(\frac{m}{2}\right)^{th}\) step, we will have a off diagonal term \(\eta^{m/2}A\) that we need to deal with (for simplicity, we have assumed \(m\) is an even integer). However, its contribution to the equation would be zero because of the harmonicity of \(A\). But if we are in \((j = m/2 + 1)^{th}\) order, we will need to add a term \(\eta^{m/2+1}\phi_{m/2+1}\) to the variables. Then in the \(\left(\frac{m}{2} + 1\right)^{th}\) order in \(\eta\), there will be a third equation:

\[
\frac{i}{2\pi} \partial \phi_{m/2+1} \wedge \frac{\omega^{n-1}}{(n-1)!} = T_{m/2+1}
\]

where \(T_{m/2+1}\) is some expression in \(A\) and the background curvature. Since \(S\) and \(Q\) are Mumford stable of the same slope, we have \(H^0(X, \text{Hom}(Q, S)) = 0\). By the standard Hodge theory, equation (3) has a unique solution \(\phi_{m/2+1}\) and the equation (3) is invertible there. But we also have to solve the equations (1) and (2). In general, if we try to solve the equation for all order \(j > m/2\), then there will be contribution from the \(A\) and \(\phi_l\) \(l < j\). To be precise, we look at

\[
D_{jE} = \begin{pmatrix}
D_{(j-1)S} + \eta^j h^{-1}_{jS} \partial h_{jS} & -\eta^{m/2} A^* - \eta^{m/2+1} \overline{\phi}_{m/2+1} - \ldots \eta^j \overline{\phi}_j \\
\eta^{m/2} A + \eta^{m/2+1} \partial \phi_{m/2+1} + \ldots + \eta^j \partial \phi_j & D_{(j-1)Q} + \eta^j h^{-1}_{jQ} \partial h_{jQ}
\end{pmatrix}
\]

We can see that in order \(j > m/2\), there will be terms in the almost Hermitian Einstein equation coming from products of \(A\) or \(\phi_l\)'s with the background. However, these extra terms are all in the off-diagonal part as long as \(j < m\). Therefore, in the diagonal parts (the \(S\) part and \(Q\) part) of the equation, we still face the same equation as before:

\[
\frac{i}{2\pi} (\overline{\partial}(h^{-1}_{jS} \partial h_{jS})) \wedge \frac{\omega^{n-1}}{(n-1)!} = \chi_E^{j+1} \frac{\omega^n}{n!} I_S - \sum_{k=0}^{j+1} (\frac{i}{2\pi} R_{jS})^k \cdot t d^{2-k} \frac{\omega^{n-j-1(2)}}{(n-j-1)!}
\]
\[
\frac{i}{2\pi} \left( \overline{\partial} (h_{jQ}^{-1} \partial h_{jQ}) \right) \wedge \frac{\omega^{n-1}}{(n-1)!} = \chi_E^{j+1} \frac{\omega^n}{n!} I_Q - \sum_{k=0}^{j+1} \left( \frac{i}{2\pi} R_{jQ} \right)^k \cdot td_X^{d-k} \frac{\omega^{n-j-1}}{(n-j-1)!}
\]

(Notice, if \( j > m \), then there will be terms like \( A^* A \) in the diagonal part which would make the trace to be nonzero.) Therefore, there always exists a unique normalized solution pair \((h_{jS}, h_{jQ})\) for equation (1) and (2). Now, for the off-diagonal term, although there might be contribution from the \( A \) and \( \phi \), but the equation three can always be solved without any traceless assumption by the Hodge theory as in the previous case. As a result, the almost Hermitian Einstein equation can be solved up to order \( j \) for all \( j < m \).

When \( j = m \), we do have a extra term \( A^* A \) in the diagonal whose integral of its trace over \( S \) is not zero. Nevertheless, this just compensates with those coming from the curvature of \( S \), namely, \((\chi_E^{m+1} - \chi_S^{m+1})(rkS') > 0\), by the choice of \( A \) (or the same as for \( B \)). Hence, we can still solve the almost Hermitian Einstein equation uniquely (after normalization) up to order \( m \).

For higher order, we shall introduce \( t \) and do a perturbation argument. By replacing \( \eta^{m/2} A \) by \( \eta^{m/2} t A \) for some suitable function \( t \) (with the property that \( t \) goes to one as \( \eta \) goes to zero), we can manage to make the two diagonals corresponding to the \( S \) part and \( Q \) part of the equation to have the property that taking the trace over \( S \) (or \( Q \)) and integrate it over \( X \) will get zero. First of all, we only need to do it for the \( S \) part, and the \( Q \) part will follow. For the \( S \) part, it is because, we already have this property for the equation up to order \( m \), and for the even higher order terms, we can perturb the \( L^2 \) norm of \( A \) a little bit to adjust them. To be more precise, the existence of \( t \) can be proved by implicit function theorem easily.

After all these careful arrangement, we can finally define the nonlinear differential operator \( \mathcal{L} \) as follow:

\[
\mathcal{L} : \mathfrak{B}^{k+2,\alpha} \longrightarrow \mathfrak{B}^{k,\alpha}
\]

\[
\mathcal{L} \cdot \begin{pmatrix} u_S & \phi \\ \phi^* & u_Q \end{pmatrix} = \frac{1}{\eta^{m+1}} \left[ (\omega^{\eta} + \eta \frac{i}{2\pi} R_{jQ}) \cdot \mathcal{T} \right]^{TOP} \mathcal{L}_E \frac{\omega^n}{n!}
\]

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where $R$ is the curvature for the connection:

$$D = \begin{pmatrix} D_{mS} + \eta^m h_S^{-1}(\partial h_S) & -\eta^{m/2}tA^* - \overline{\partial} \Phi^* \\ \eta^{m/2}tA + \partial \Phi & D_{mQ} + \eta^m h_Q^{-1}(\partial h_Q) \end{pmatrix}$$

where $u_S = \log h_S$, $u_Q = \log h_Q$ and

$$\Phi = \eta^{m/2+1} \phi_{m/2+1} + \ldots + \eta^m \phi_m + \eta^m \phi$$

From the above discussion, we know that $\mathcal{L} = 0$ can be solved by the triple $(u_S, u_Q, \phi) = (0, 0, 0)$ when $\eta$ equals to zero and the linearized operator is invertible there. By the Implicit Function Theorem in Banach space, the equation $\mathcal{L} = 0$ would therefore have solution for all small enough $\eta$ which depends smoothly in $\eta$ positive. Moreover, the $(C^{k,a})$ norm of their curvatures are bounded. Therefore, $E$ is almost Hermitian Einstein in this case.

We have therefore proved the proposition.

Next, we are going to move forward to study the case when there are more than two components in the Jordan-Holder filtration for the Gieseker stable bundle $E$.

**PROPOSITION 8** Let $E$ be a Gieseker stable bundle over a compact Kahler manifold $X$. Suppose that

$$E = E_0 \supset E_1 \supset E_2 \supset \ldots \supset E_{k+1} = 0$$

denote its Jordan-Holder filtration as a Mumford semi-stable bundle. If each $E_j$ is a vector bundle, then $E$ is an almost Hermitian Einstein bundle.

**PROOF:** For simplicity, we will assume that $\chi_{E_j}^2 < \chi_E^2$ for all $j > 0$. The more general case can be treated using the same method as in the proof of
the previous proposition. Notice that since these \( E_j \)'s are the components of the Jordan-Holder filtration of \( E \), they are have the same slope as \( E \) does, it implies that \( \chi_{E_j}^1 = \chi_E^1 \). Denote \( Q_j = E_j/E_{j+1} \), then they are all Mumford stable bundles and also have the same slope as \( E \).

By the Uhlenbeck and Yau's theorem, there is a unique Hermitian Einstein metric on each of the \( Q_j \)'s. Denote the Hermitian Einstein connection on \( Q_j \) by \( D_j \). If we write the connection on \( \Gamma r(E) = Q_0 \oplus Q_1 \oplus \ldots \oplus Q_k \) as follow:

\[
D = \begin{pmatrix}
D_k & 0 & \ldots & 0 \\
0 & D_{k-1} & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & D_0
\end{pmatrix}
\]

then it solve the almost Hermitian Einstein equation up to zeroth order in \( \eta \) with the same reason as in the proof of the previous proposition. Next, we are going to choose the second fundamental forms for the successive extension one by one. We will start from the bottom of the filtration. Consider the last extension sequence:

\[
0 \to E_k \to E_{k-1} \to Q_{k-1} \to 0
\]

But we have \( E_k = Q_k \) and therefore this exact sequence becomes

\[
0 \to Q_k \to E_{k-1} \to Q_{k-1} \to 0
\]

On those \( Q_j \)'s, we already have a background (Hermitian Einstein) metric, now we shall choose a harmonic second fundamental form \( A_{k-1,k}^* \) (with respect to these metrics) to represent this extension class such that its \( L^2 \)-norm is normalized so that

\[
\int |A_{k-1,k}|^2 = (\chi_{E_k}^2 - \chi_{Q_k}^2)(r^k Q_k)
\]

This is possible because of our assumption that \( \chi_{Q_k}^2 = \chi_{E_k}^2 < \chi_E^2 \). Now, we have a Hermitian connection on \( E_{k-1} \), namely, \( \begin{pmatrix} D_k & -A_{k-1,k}^* \\ A_{k-1,k} & D_{k-1} \end{pmatrix} \).
Suppose we have chosen the second fundamental form and get a Hermitian metric on $E_j$, then the next step, we will be looking at the following exact sequence:

$$0 \rightarrow E_j \rightarrow E_{j-1} \rightarrow Q_{j-1} \rightarrow 0$$

then with respect to the newly formed Hermitian metric on $E_j$ and the Hermitian Einstein metric on $Q_{k-1}$, we pick the harmonic second fundamental form $A_{j-1,k}^* + A_{j-1,k-1}^* + \ldots + A_{j-1,j}^*$, where $A_{j-1,j}^*$ is its $E_l$ component (that is $A_{j-1,l}^*$ is a $(0,1)$ form with valued in $Hom (Q_{j-1}, Q_k)$). The suitable normalization turns out to be $\int |A_{j-1,j}^*|^2 = (\chi_E^2 - \chi_{E_j}^2)(rkE_j)$, the positivity of the right hand side is guaranteed by the stability assumption of $E$. Notice that, this equality is equivalent to $\int |A_{j-1,j}^*|^2 - \int |A_{j,j+1}^*|^2 = (\chi_E^2 - \chi_{Q_j}^2)(rkQ_j)$ which we will be using in the followings.

At this stage, we can introduce the rescaling:

$$A_{j-1,j} \rightarrow \eta^{1/2} A_{j-1,j}$$

$$A_{j,l} \rightarrow \eta A_{j,l} \quad \text{if } |j - l| > 1$$

Now, the domain Banach space would be the $\mathcal{B}^{k+2,\alpha}$ such that an element of it would be like $(u_j, \phi_{i,j}, i, j = 0, 1, 2, \ldots, k; i < j)$, where $u_j$ is a trace-free endomorphism of $Q_j$ and $\phi_{i,j}$ is a homomorphism from $Q_i$ to $Q_j$. Let us write $h_j = e^{u_j}$. The connection on $E$ would be:

$$\begin{pmatrix}
D_k + \eta h_k^{-1} \partial h_k & -\eta^{1/2} A_{k-1,k}^* - \eta \bar{\partial} \phi_{k-1,k}^* & -\eta A_{k-2,k}^* - \eta \bar{\partial} \phi_{k-2,k}^* & \ldots \\
\eta^{1/2} A_{k-1,k}^* + \eta \partial \phi_{k-1,k} & D_{k-1} + \eta h_{k-1}^{-1} \partial h_{k-1} & -\eta^{1/2} A_{k-2,k-1}^* - \eta \bar{\partial} \phi_{k-2,k-1}^* & \ldots \\
\eta A_{k-2,k} + \eta \partial \phi_{k-2,k} & \eta^{1/2} A_{k-2,k-1} + \eta \partial \phi_{k-2,k-1} & D_k + \eta h_{k-2}^{-1} \partial h_{k-2} & \ldots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}$$

Notice that the power $\eta^{1/2}$ only attaches to $A_{j-1,j}$ and all the others have a $\eta$ attaches to them. By the harmonicity of the successive second fundamental forms and the carefully chosen normalization of them, it can be proved in a similar fashion as before that up to first order in $\eta$, the almost Hermitian
Einstein equation can be solved by a unique element in $\mathfrak{B}^{k+2,\alpha}$.

In order to apply the implicit function theorem, we have to introduce the functions $t_k, t_{k-1}, \ldots, t_1$, each of them is a function of $\eta$ and the variables such that $t_j$ goes to one as $\eta$ goes to zero for each $j$. By replacing $\eta^{1/2}A_{j-1,j}$ by $\eta^{1/2} t_j A_{j-1,j}$, the function $t_j$ is so chosen (uniquely) such that on each diagonal block (corresponding to the $Q_{j-1}$ part), taking the trace of the equation over that block and integrate it over $X$ will give zero. After this procedure, we can then define our nonlinear differential operator $\mathcal{L}$ as in the previous proposition which has the property that when $\eta > 0$, the equation $\mathcal{L} = 0$ is the almost Hermitian Einstein equation and the equation at $\eta = 0$ can be solved such that the corresponding linearized operator is invertible there. Hence, by the implicit function theorem, $\mathcal{L} = 0$ can be solved for small $\eta$. Moreover, the resulting curvature is bounded in $\mathfrak{B}^{k,\alpha}$ independent of $\eta$. That is, we have obtained an almost Hermitian Einstein bundle $E$ and the proposition is proved.

QED

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