Essays in Financial Economics

by

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Submitted to the Sloan School of Management
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy in Management

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

June 2019

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Abstract

This dissertation consists of three chapters.

Chapter 1 proposes a dynamic general equilibrium model to study jointly (i) the pace of technological progress and (ii) asset pricing properties of investment in innovation. Risk is a key characteristic that links the two together. Both empirically and in my theory innovation activity is associated with elevated levels of idiosyncratic risk. In the model, idiosyncratic risk is driven by uncertain productivity improvements and disruption that emerge in the process of innovation. Thus idiosyncratic risk is an instrumental determinant of the rate of technological progress and expected returns on investment in innovation. A calibrated version of the model provides an accurate quantitative description of the venture capital cycles both in terms of investment flows and financial returns.

Joint study with Hui Chen and Jiang Wang in Chapter 2 investigates the effects of market-wide trading halts, also called circuit breakers, on stock prices and trading behavior. We develop a model to examine how circuit breakers impact the market when investors trade to share risk. We show that a downside circuit breaker tends to lower the stock price, increase its volatility and raise the likelihood of reaching the triggering price. The volatility amplification effect becomes stronger when the wealth share of the relatively pessimistic agent is small.

In Chapter 3 I develop a theory that shows how search frictions in the labor market shape asset pricing properties of stock returns. My theory reconciles and links together two empirical facts: (i) economic downturns are associated with higher pace of job reallocation and (ii) rapidly growing firms on average yield lower returns on their stocks compared to shrinking firms. In the model, firms with more growth opportunities benefit from recessions due to more slack in the labor market, since in recessions growing firms can hire more and expand quicker. This feature makes growth firms' value less cyclical. A calibrated version of the model successfully replicates predictability of returns in the cross-section by a range of growth indicators.

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Acknowledgments

I would like to express my deep gratitude to Hui Chen, Leonid Kogan and Jonathan Parker, my research supervisors, for their generous support, patient guidance and enthusiastic encouragement throughout the course of my PhD studies. My supervisors helped me to grow as a researcher, sharpened my economic intuition and research rigor.

I am in debt to the entire MIT Sloan Finance faculty for the lessons and knowledge they passed to me. Among many others, I am grateful to Daniel Greenwald, Andrey Malenko, Egor Matveyev, Paul Mende, Jun Pan, Gita Rao, Larry Schmidt, Antoinette Schoar, Alp Simsek, David Thesmar, Adrien Verdelhan, Jiang Wang and Haoxiang Zhu.

I would also like to thank my peers at MIT Sloan and friends who helped me learn and made my years at MIT joyful. Special thanks goes to Yixin Chen, Bill Goulding, Daniel Green, Vikram Jambulapati, Stepan Ochodek, Yury Olshanskiy, Gleb Romanyuk and Anton Tsoy. I would like to thank Millie Huang for her patient support and wise advice that helped me navigate through my PhD journey, maintain confidence in difficult times and find the time to stop and smell the rose along the way.

Finally, I would like to thank my parents Tatiana and Vladimir, and my brother Maxim. Maxim has always been an inexhaustible source of inspiration and advice for me. Unconditional love, trust and support of my parents motivated me to persist along this journey and made completion of this thesis possible.

Contents

Chapter 1

Non-Diversifiable Risk and Endogenous Innovation

1.1 Introduction

The process of technological innovation is inherently risky. On the one hand, chances of success in innovation are low. On the other hand, technological innovation poses a risk of disruption to operating businesses through the Schumpeterian force of creative destruction. In this paper, I study the role financial risks play in the process of innovation. I propose a general equilibrium model in which agents' exposures to the risk of disruption and hedging motives shape jointly the nature of technological progress and asset prices.

The starting point to this study is a set of stylized facts associated with innovation and risk at the aggregate economy level. First, innovation activity follows moderately persistent medium-term cycles.[1] Second, aggregate returns on investment in innovation, as measured by performance of the venture capital sector, are highly risky. Third and most important, I show that innovation activity at the aggregate economy level tightly

[1]See Comin and Gertler (2006), Barlevy (2007) among others.

comoves with idiosyncratic risk measured both at the firm level and investor portfolio level. At the same time, there is only a weak relationship between the volatility of the market portfolio and the innovation activity.

I propose a model that mirrors these empirical facts. My model highlights the importance of unpredictable reallocative consequences of innovation that give rise to idiosyncratic risk of disruption, in addition to more commonly studied uncertain improvements to productivity resulting from innovation. For investors holding the fully diversified market portfolio, the process of reallocation might not pose a substantial risk. The risk of disruption is, however, acute for investors, who hold concentrated positions in incumbent firms. Among others, this group of investors includes executives, entrepreneurs, firm founders and family members who own significant undiversified equity stakes in their businesses. Jointly, this group of investors owns a large share of wealth in the population and their portfolio choices have significant consequences for the aggregate economy.

I find that agents' non-diversified exposure to incumbent firms has crucial effects on the equilibrium amount of innovation activity and its nature. In particular, it magnifies the risk of disruption and exacerbates the business-stealing effect of radical innovation. Agents fail to coordinate and invest excessive amount of resources in disruptive technologies. In the dynamic context, this effect leads investment in innovation to sharply respond to changes in the aggregate economic conditions. As a result, consistent with the data, investment in innovation is highly volatile. The risk premium on investments in innovation is not constant but time-varying and can switch signs. When the economy is in a phase of expansion, investment in radical innovation is high and the risk premium on innovation is low. The opposite is true in periods of contraction.

I evaluate quantitative performance of the model by calibrating it to reflect the interplay between the market of publicly traded firms and the venture capital sector. The model reproduces both average returns and volatility of returns in VC; however, it slightly understates volatility of fund flows. In addition, endogenous variation in risk premium helps explain the stylized facts linking aggregate VC flows with the sector's financial performance, that are well documented in the empirical literature. First, the model replicates strong negative relationship between fund inflows and subsequent returns. Second, qualitatively the model generates the positive relationship between realized returns and following inflows to the venture capital sector.

I conduct additional tests of the asset pricing implications of the model using the cross-section of stocks. I form portfolios based on the number of years since IPO. Firms with more recent IPO are on average younger, and are more likely to benefit from technological developments. Consistent with the theory, I show that returns on the "young-minus-old" portfolio are positively loaded on shocks to idiosyncratic risk. In addition, the level of innovation activity significantly negatively forecasts returns on the "young-minus-old" portfolio. One standard deviation increase in R&D-to-GDP ratio is associated with five percentage point decrease in return on that portfolio in the following year.

My setup builds upon the classical quality ladder framework of Aghion and Howitt (1992) and Grossman and Helpman (1991). In the model, technological progress is volatile and uncertain both at the aggregate and individual firm level. Most of the time the economy grows in a steady manner. Yet occasionally, aggregate innovation waves arrive, causing radical changes in production technologies with heterogeneous impact on individual firms. Some firms stay unaffected by the wave, some succeed in disruptive innovation and enjoy an increase in productivity, while the other are left behind in the technological race and are displaced by new entrants. Historical examples of such waves include inventions of steam and combustion engines, breakthroughs in space exploration, the discovery of semiconductors, or more recently — the development of mobile internet and breakthroughs in machine learning, among many others. The size of each wave depends endogenously on the amount of investment in innovation, and the economy-wide impact of innovation waves is a source of systematic risk that is associated with a significant risk premium.

A crucial departure of this framework from traditional models of growth with quality ladders is that the economy features heterogeneous agents. Agents of the first type, which I call households, can freely choose allocations of wealth across investment opportunities. In contrast, the second type, which I call managers, bears a non-diversifiable background risk. In particular, each manager is matched with an incumbent firm and is obliged to hold a fixed share of her wealth in the stocks of the managed firm. This assumption captures in a reduced form the moral hazard problem. Hence the manager faces a risk of disruption by an entering firm, which can be realized in case of a wave arrival. The manager can (imperfectly) hedge this risk, first, by investing in disruptive innovation on behalf of the managed firm to outcompete entrants in the technological race and, second, by allocating a bigger share of the portfolio into an innovation sector. The innovation sector is an intermediary that allocates investors' contributions among startups. Payoffs on investments in the innovation sector positively correlate with successful entry of new firms, and thus such investments offer a good hedge for managers.

Aggregate allocations to the innovation sector determine the amount of disruptive innovation in the economy. At the same time aggregate amount of innovation impacts investors' hedging behavior. The two-way feedback loop between agents' portfolio choice and the amount of innovation gives rise to the *herding* in the innovation effect. Higher innovation activity implies higher probability of disruption of any given firm. Hence, in an attempt to hedge, each incumbent manager contributes even more to the innovation sector. Collectively, through portfolio choice investors increase probability of disruption of any given firm. In my setup, the business-stealing effect of disruptive innovation unambiguously hurts managers' welfare as a group. Herding in the innovation amplifies this negative effect. This externality, induced by the optimal portfolio choice, is reminiscent of DeMarzo et al. (2004, 2007, 2008). The underlying mechanism is, however, different. In DeMarzo et al. (2004, 2007, 2008) agents compete for limited resources in the incomplete market setup, which leads to relative wealth concerns and under-diversification. In the current setup, agents attempt to diversify their own risk, but do not take into account the higher risk imposed on other agents, which leads to the prisoner's dilemma scenario.

The sign and the magnitude of the risk premium on innovation depend on the relative strength of two channels. First, improvements to productivity of incumbent firms, that come with arrival of innovation waves, push the premium up. Second, the downside risk, that managers face, drives it down. In my calibration, the risk premium is positive most of the time. It can turn negative, however, in periods of very intensive innovation activity with a high potential for disruption, akin to the one observed during the internet boom of the late 90s.

Both the background risk and lumpy arrival of disruptive innovations are critical ingredients for my results. To show this, first, I gradually reduce the size of stakes that managers have to keep in their firms. Smaller stake sizes reduce the amount of background risk faced by the agents. Disruptive effects of innovation become less important for pricing the risk of innovation. Both risk premium and flows into the innovation sector become less responsive to the state of the economy. Equilibrium amount of investment in innovation goes down and risk premium goes up.

Second, I vary the rate of arrival of innovation waves while keeping intensity of disruptive innovation arrival for any given firm unchanged. Higher rates of arrival make each wave smaller in size. Disruptive innovations become less lumpy and fluctuations of the price of risk become less pronounced. In the limiting case of infinite intensity, waves arrive at any given instant. The amount of disruptive innovations becomes fully predictable and the innovation sector loses its hedging properties. The price of innovation risk converges to zero.

Related Literature This paper fits into the growing literature that studies the interplay of economic growth and technological change with asset prices and returns.

Greenwood and Jovanovic (1999), Bond et al. (2000), Laitner and Stolyarov (2003), Comin et al. (2009) relate evolution of the stock market valuations to technological shifts. Pástor and Veronesi (2009) explore the effect of learning and uncertainty on asset prices caused by arrival of new technologies. Garleanu et al. (2012) study how adoption of technologies shapes consumption and risk premia on the stock market. Andrei and Carlin (2018) explore the effects of competition on equilibrium innovation and stock prices. Closer to this paper, a set of studies endogenize the process of economic growth. Kung and Schmid (2015), Bena et al. (2015) investigate how endogenous innovation can give rise to the medium and long-run risk (Bansal and Yaron, 2004). Loualiche (2014), Corhay et al. (2017) model the interplay between risk premia on stocks and firm entry and exit decisions. Different from this strand of literature, the focus of my paper is not only implications of innovation for the stock market but also the asset pricing properties of investments in innovation.

In this respect, the current study contributes to the emerging literature that ties the macroeconomy with the venture capital industry. Opp (2016) and Greenwood et al. (2018), pioneers of this literature, carefully model micro frictions of the VC sector and investigate the impact of venture capital industry on the stock market, growth rate and welfare in the economy. Complementary to these studies, I focus on the endogenous risk properties of the innovation sector and how this risk shapes the process of innovation and returns on investment in innovation. In a contemporaneous and independent paper, Jovanovic and Rousseau (2018) show that fluctuations in the price of capital, as measured by Tobin's Q, can help explain behavior of returns on VC and buyout funds. My approach differs as it focuses on the sources and pricing of risk in an equilibrium framework with no arbitrage.

The key risk channel associated with the process of innovation in my model builds upon the notion of displacement introduced by Gârleanu et al. (2012). Kogan et al. (2016), Garleanu et al. (2015), Gârleanu and Panageas (2017) study implications of displacement for the pricing of aggregate stock market, cross-section of stocks and alternative assets. Distinct from these studies, the displacement effect of innovation in my setup is critical only for a subset of population while the majority of agents benefit from the process of innovation. As a result, the risk premium associated with innovation can be both positive and negative depending on the state of the economy. In this respect my mechanism is tightly linked to the literature on the asset pricing effects of idiosyncratic risk (Constantinides and Duffie, 1996; Heaton and Lucas, 2000; Storesletten et al., 2007; Herskovic et al., 2016; Schmidt, 2016; Constantinides and Ghosh, 2017).

A number of papers study implications of innovation on the evolution of inequality. Aghion et al. (2018) find an empirical link between innovation and top income shares. Jones and Kim (2018) in a Schumpeterian framework show that disruptive innovation gives rise to the Pareto distribution of income. Gomez (2018) focuses on idiosyncratic risk as a source of wealth inequality dynamics. The model, I develop in the current paper, has implications for the wealth inequality driven by technological innovation as well. More importantly, I show the reverse mechanism: wealth concentrated in the hands of non-diversified agents can be a driver of innovation on its own.

From a methodological perspective, my study is closely related to the vast literature on endogenous economic growth, in particular models of innovation by incumbents and entrants (Comin and Mulani, 2009; Acemoglu et al., 2013; Acemoglu and Cao, 2015; Garcia-Macia et al., 2015). In addition, I rely on techniques developed in macroeconomic models with agent heterogeneity (He and Krishnamurthy, 2013; Brunnermeier and Sannikov, 2014; Di Tella, 2017).

1.2 Stylized Facts

Numerous studies have documented that innovation activity at the economy level, measured by R&D expenditures, follows cyclical patterns.[2] Campbell et al. (2001), Comin and Philippon (2005) document that firm-level volatility was trending up during 1950–2000. Comin and Philippon (2005) associate it with increasing R&D-based competition over the time period. In this section I confirm that innovation activity follows medium term cycles. In addition, I show new evidence on comovement patterns between innovation and (i) idiosyncratic risk at the firm level, (ii) idiosyncratic risk at the individual investor level,[3] and (iii) aggregate market risk.

Firm-level Idiosyncratic Risk My preferred measure of the innovation activity is R&D expenses normalized by the GDP. To measure the economy level R&D expenditures I use annual data collected by the National Science Foundation on R&D funded by US domestic businesses. The R&D-to-GDP ratio has been trending up since the beginning of the NSF data sample, hence I extract cyclical component by removing a linear trend and work with the obtained series from now on.[4]

Solid line on Figure 1-1 plots a rescaled version of the measure of innovation activity IA_t. The graph shows that innovation activity follows moderately persistent medium-term fluctuations with several distinct peaks. It was at the highest levels during the technological boom in late 1990s and early 2000s. These medium-term fluctuations seem to be not related to more traditional higher frequency business cycles: innovation activity was declining during three recessionary episodes identified by the NBER and was on the rise during five recessionary episodes in my sample.

The dash line on Figure 1-1 denotes a market capitalization based measure of

[2]See Griliches (1990), Francois and Lloyd-Ellis (2003), Comin and Gertler (2006), Barlevy (2007), among others.

[3]In an independent paper Gomez (2018) finds a similar relationship between individual investor risk and innovation.

[4]See Griliches (1994) or more recent Bloom et al. (2017) for the expansive discussion and more references on the trends in R&D volume and productivity.

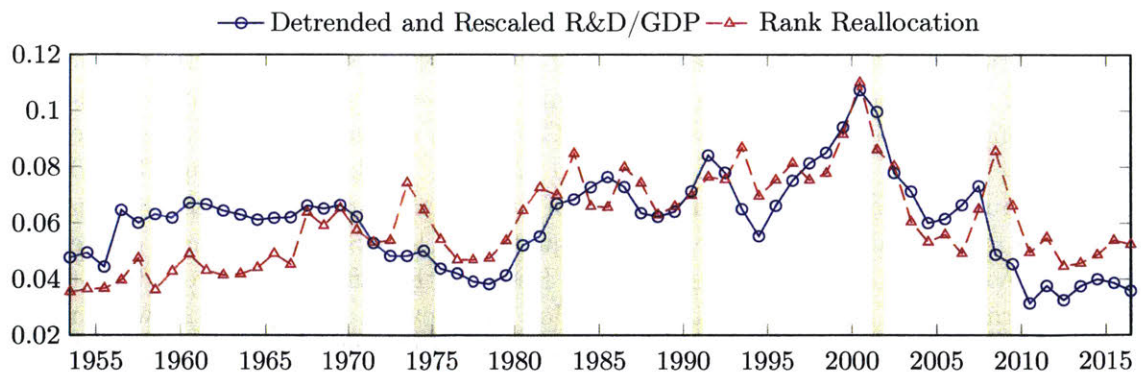

Figure 1-1: **Detrended private business R&D expenses to GDP ratio and CRSP market capitalization based reallocation.** Reallocation is defined by equation (1.1). R&D/GDP has linear trend removed and normalized to have the same mean and standard deviation as reallocation. Shaded bars are NBER recessions.

reallocation R_t. I construct this measure using the CRSP sample of publicly traded firms headquartered in the US. In year t it is defined by

$$R_t = \frac{\sum_{i \in \text{CRSP}} \left| Q_{i,t}^{\text{MC}} - Q_{i,t-1}^{\text{MC}} \right|}{N}, \tag{1.1}$$

where $Q_{i,t}^{\text{MC}}$ is firm's i quantile in the market capitalization ranking on the last trading day of year t; the sum is calculated over firms for which market capitalization and, hence, the corresponding quantile are observed in both years t and $t-1$; N is the number of firms in the summation.[5] In a hypothetical scenario, when firms' capitalization changes in the same proportion across firms from year t to year $t-1$, this measure is zero. In presence of idiosyncratic risk, firms' capitalization ranking changes from one year to another and, hence, R_t is positive. As we see on Figure 1-1, R_t follows a pattern very similar to the innovation activity and strongly positively comoves with it. Correlation between the two series is 0.62.

[5]Due to year-to-year changes in the CRSP sample there are two possible ways to compute quantiles $Q_{i,t}^{\text{MC}}$. I report results for the case when each quantile $Q_{i,t}^{\text{MC}}$ is calculated on the set of all firms with available market capitalization in the end of year t. The results are virtually identical when I define R_t using $Q_{i,t}^{\text{MC}}$ and $Q_{i,t-1}^{\text{MC}}$ calculated on a subset of firms with observed market capitalization both in years t and $t-1$, or when the sum in (1.1) is calculated over the subset of 10% largest firms as of year $t-1$.

Table 1.1: **Innovation activity and firm-level idiosyncratic risk.** The table presents results of linear regressions of measures Y_t of firm-level idiosyncratic risk on innovation activity IA_t: $Y_t = \text{Const} + \beta IA_t + \gamma_1 \text{Output Gap}_t + \gamma_2 t + \epsilon_t$. Innovation activity is measured as the detrended ratio of economy-wide R&D/GDP ratio normalized to have zero mean and unit variance. Output gap is normalized to unit variance. Rank Reallocation, CS StDev and Quantile Range are calculated in the sample of public firms present in CRSP, annual data: 1953–2016. Rank Reallocation is defined by equation (1.1), CS StDev is the cross-sectional standard deviation of annual returns, Quantile Range is the difference between 95%-tile and 5%-tile of firm annual returns in a given year. Job Reallocation is the economy-wide measure reported by Business Dynamics Statistics, annual data 1975–2015. HAC standard errors in parentheses. One, two and three stars denote significance at the 10, 5 and 1% level.

	Dependent variable:			
	Rank Reallocation	CS StDev	Quantile Range	Job Reallocation
IA_t	0.010***	0.106***	0.209***	0.014***
	(0.002)	(0.030)	(0.069)	(0.002)
Output Gap	−0.001	−0.045	−0.058	−0.0004
	(0.002)	(0.042)	(0.066)	(0.004)
Mean(Y_t)	0.061	0.559	1.431	0.286
R^2	0.599	0.368	0.504	0.749

The first column in Table 1.1 reports results of the regression of R_t on the innovation activity, output gap and a linear trend. I rescale IA_t and the Output Gap to have unit variance for easier interpretation. The slope coefficient of IA_t is both statistically and economically significant: one standard deviation increase in IA_t corresponds to 0.01 increase in reallocation or about 16% increase relative to its mean value (the mean value is reported in the second to last row). This observation confirms the comovement between the two series that we saw on Figure 1-1. Inclusion of the output gap in the regression does not have any significant impact on the slope coefficient of IA_t which indicates that the observed comovement is not directly driven by the regular business cycle phenomenon.

The second and third columns of Table 1.1 report regressions of two additional measures of idiosyncratic risk of public companies on the innovation activity. The first one, denoted by CS StDev, is the cross-sectional standard deviation of annual firm returns. The second, quantile range, is the difference between 95%-tile and 5%-tile of annual returns in the cross-section of firms in a given year. These two regressions confirm the strong positive relationship between idiosyncratic risk and the innovation activity. The slope coefficients for IA_t are positive and significant. Magnitudes of the effect are similar to the case of rank reallocation R_t: one standard deviation increase in innovation activity is associated with 18% increase in CS StDev and 15% increase in quantile range relative to the respective means.

Finally, the last column reports regressions of the job reallocation on the innovation activity, output gap and a linear trend. Job reallocation is from the Business Dynamics Statistics. This measure of reallocation covers the set of both private and publicly traded firms in the US. The slope on innovation activity is economically and statistically significant, which indicates that the comovent between idiosyncratic risk and innovation is not restricted to publicly traded firms but is a broad economy phenomenon.

One explanation for this comovement is based on Joseph Schumpeter's notion of creative destruction. In times of high R&D spending new firms challenge incumbents by attempting to develop new, better quality, goods. Incumbent firms increase R&D spending in search of new ideas to preserve or improve their position in the market space. During this process it is more likely that any given firm will experience a change in its market position, which can either improve or deteriorate. Hence, increased economy-wide innovation activity will be associated with more reallocation between firms.

Idiosyncratic Risk to Investors' Wealth Most businesses, both private and public, have stakeholders for whom the share owned constitutes a large portion of their personal wealth. Owners of privately held firms naturally have limited options for diversification. Hence value of their undiversified stakes often comprises a large part of their total net worth. Moskowitz and Vissing-Jørgensen (2002) report that households, conditional on owning a stake in a private business, on average have more than 40% of wealth invested in it. In the case of public firms, firm founders or founders' family members are often heavily invested in the company's stock and are highly non-diversified. For example heirs of Samuel Walton, the founder of Walmart Inc, own about 50% of Walmart's common stock. This investment corresponds to more than 80% of their net worth.[6] Apart from founders and families, almost all public firms tie chief executives' and key partners' compensation to the firms' performance. For these employees compensation in the current firm constitutes a large part of their lifetime income.

Given the evidence on firm-level risk presented above, we should expect heavily non-diversified business owners to face higher idiosyncratic risk in times of high innovation activity. Using data on wealth of individuals on the Forbes 400 list I confirm this conjecture.

Since 1982 Forbes magazine has been tracking and publishing on the annual basis the list of 400 wealthiest Americans along with their estimated net worth. Two important features make this data attractive for use in my study in comparison with other few data sources that contain information on personal wealth. First, most individuals stay on the Forbes 400 list for multiple years. This allows me to track individuals in time and calculate returns on their wealth. Second, most members on the Forbes list have non-diversified and, often, active ownership in an operating business, private or public.

I define logarithmic return on wealth according to

$$r_{i,t}^W = \log\left(\frac{W_{i,t}}{W_{i,t-1}}\right) , \tag{1.2}$$

[6]As of August 2018 based on Forbes net worth estimates and Walmart Inc 2018 proxy statement.

where $W_{i,t}$ is the net worth of individual i in year t reported by Forbes.[7,8] I use logarithmic returns in this setting to avoid the influence of extremely high realizations when calculating measures of idiosyncratic risk. Every year a subset of individuals exits the Forbes 400 list, some of these exits are caused by large drops in wealth of these individuals. Performance related exits from the list introduce a downward bias in the measures of idiosyncratic risk. To mitigate this issue I consider the subset of one hundred wealthiest, as of year $t-1$, individuals from the Forbes 400 list when calculating measures of risk in year t. Performance related exits for this subset are extremely rate.[9]

For each year in the sample I calculate two measures of idiosyncratic risk. The first one is the cross-sectional dispersion of the return on wealth. The second measure is the quantile range. As in the case of the firm-level analysis, the quantile range is defined as the difference between 95%-tile and 5%-tile of annual returns on wealth of individuals $r_{i,t}^W$ in year t. This definition of idiosyncratic risk is robust to possible outliers but at the same time serves as a good measure of tail risk.

In the first and second column of Table 1.2 I report the results of OLS regressions of the cross-sectional standard deviation and the quantile range on the innovation activity IA_t, output gap and a linear trend. As before, IA_t is the detrended R&D-to-GDP ratio normalized to unit variance. Regression results for both measures are consistent with each other: slope coefficients are statistically significant and positive. The mean value of CS StDev is 0.270 and one standard deviation increase in IA_t corresponds to $0.063/0.270 \approx 23\%$ increase in its value relative to the mean. The mean value of the quantile range is 0.805 and one standard deviation increase in IA_t is associated

[7]The ideal definition of return on wealth involves a measure of consumption of individual i, $C_{i,t}$: $r_{i,t}^W = \log\left((C_{i,t} + W_{i,t})/W_{i,t-1}\right)$. Forbes does not report data on consumption so I define the return on wealth according to equation (1.2). This simplification should not have any significant impact on the results since the ideal measure perfectly correlates with (1.2) under the assumption $C_{i,t} = \text{const} \times W_{i,t}$. Deviations from this condition are small in magnitude compared to fluctuations in wealth.

[8]I use the data collected by Capehart (2014).

[9]In the empirical appendix I repeat the exercise for the full sample of individuals on the Forbes 400 list. All conclusions remain unchanged.

Table 1.2: **Innovation activity and portfolio-level idiosyncratic risk.** The table presents results of linear regressions of measures Y_t of idiosyncratic risk on innovation activity IA_t: $Y_t = \text{Const} + \beta IA_t + \gamma_1 \text{Output Gap}_t + \gamma_2 t + \epsilon_t$. Innovation activity is measured as the detrended ratio of economy-wide R&D/GDP ratio normalized to have zero mean and unit variance. Output gap is normalized to unit variance. CS StDev is the cross-sectional standard deviation of annual returns on wealth, Quantile Range is the difference between 95%-tile and 5%-tile of annual return on wealth, CS StDev^U /CS StDev^D are upside/downside standard deviations of return on wealth. Dependent variables are calculated using the data on 100 wealthiest, as of year $t-1$, members of the Forbes 400 list. HAC standard errors in parentheses. One, two and three stars denote significance at the 10, 5 and 1% level. Annual observations, 1983–2013.

	Dependent variable:			
	CS StDev	Quantile Range	CS StDev^U	CS StDev^D
IA_t	0.063***	0.203***	0.041***	0.096**
	(0.017)	(0.060)	(0.013)	(0.045)
Output Gap	−0.020*	−0.056	−0.001	−0.043
	(0.011)	(0.047)	(0.011)	(0.036)
Mean(Y_t)	0.270	0.805	0.264	0.279
R^2	0.461	0.448	0.396	0.306

with about 25% increase in its value relative to the mean. The relationship is both statistically and economically significant. Quantitatively this pattern is similar to the one observed for the firm-level risk.

Theoretically, increased intensity of innovation activity can have asymmetric impact on the left and right tails of the return on wealth distribution. To investigate this question I define upside and downside versions of idiosyncratic risk measures:

$$\text{CS StDev}_t^U = \sqrt{\frac{\sum_{i, r_{i,t}^W \geq \mu_t} \left(r_{i,t}^W - \mu_t\right)^2}{N^U}}, \tag{1.3}$$

$$\text{CS StDev}_t^D = \sqrt{\frac{\sum_{i, r_{i,t}^W < \mu_t} \left(r_{i,t}^W - \mu_t\right)^2}{N^D}}, \tag{1.4}$$

$$\mu_t = \frac{\sum_i r_{i,t}^W}{N}, \tag{1.5}$$

where N^U and N^D stand for the number of observations with $r_{i,t}^W \geq \mu_t$ and $r_{i,t}^W < \mu_t$

Table 1.3: **Innovation activity and market portfolio volatility.** The table presents results of linear regressions of measures the market portfolio annual realized volatility RV_t on innovation activity IA_t: $RV_t = \text{Const} + \beta IA_t + \gamma_1 \text{Output Gap}_t + \gamma_2 t + \epsilon_t$. Innovation activity is measured as the detrended ratio of economy-wide R&D/GDP ratio normalized to have zero mean and unit variance. Output gap is normalized to unit variance. Realized volatility is defined as the square root of the annual realized variance of CRSP value-weighted returns calculated using daily data. HAC standard errors in parentheses. One, two and three stars denote significance at the 10, 5 and 1% level. Annual observations, 1953–2016.

	Dependent variable:			
	Market Volatility			
IA_t	0.006			
	(0.010)			
IA_{t-1}		0.013*		
		(0.007)		
IA_{t-2}			0.012	
			(0.008)	
IA_{t-3}				0.006
				(0.009)
Output Gap	−0.006	−0.006	−0.006	−0.007
	(0.006)	(0.007)	(0.007)	(0.007)
Mean(Y_t)	0.137	0.137	0.138	0.138
R^2	0.234	0.260	0.241	0.217

correspondingly. I report results of the regressions of these variables on the innovation activity in columns 3 and 4 of Table 1.2. On average distribution of log returns on wealth appears to be not skewed: mean values of measures of upside and downside risk are very close to each other. Slope coefficients, however, tell us a different story if one conditions on the amount of innovation activity. Distribution of returns is skewed relatively more to the left when innovation activity is high. I can not, however, statistically reject the null hypothesis that slope coefficients are equal for upside and downside risk against the one sided alternative that downside risk slope is higher than upside at the 5% level. The p-value of such test is equal to 0.12 (not reported in the table). This exercise thus provides mild evidence that owners of incumbent businesses face higher left tail risks when the economy undergoes high innovation activity periods.

Aggregate Market Volatility The first column of Table 1.3 reports results of regressions of realized volatility of the CRSP value-weighted returns on innovation activity, output gap and linear trend. The slope coefficient on IA_t is not significant and economically small. Columns two through four investigate potential comovent of lagged innovation activity and market volatility. In regressions with lags all slope coefficients on innovation activity are positive and are either insignificant or marginally significant. There is only weak positive correlation in innovation activity and stock market volatility.

From the standpoint of an undiversified investor these stylized facts suggest relative importance of the redistributive effects of innovation. The model formulated in the next section captures these observations.

1.3 Model Setup

1.3.1 Firms

The economy is populated by a set of firms operating in a variety of product lines. Overall there is a continuum of measure one of product lines indexed by i, $i \in [0, 1]$. Each product line is characterized by the latest technology or the latest quality $q_{i,t}$ at time t, $q_{i,t} > 0$. At any point in time there is exactly one incumbent firm in a given product line i. The incumbent firm owns production rights on the good of the highest quality within its product line.

Each incumbent firm generates a flow operating profit of

$$\Pi_{i,t} = a \times q_{i,t} \,. \tag{1.6}$$

Here a is a positive constant which I normalize to be equal to one. This setup emerges as an outcome of monopolistic competition in a class of quality ladder type models of growth (Acemoglu, 2008).

Operating profit $\Pi_{i,t}$ can be either distributed to the firm's shareholders or invested in innovation inside the firm. The quality of variety i evolves according to

$$\frac{dq_{i,t}}{q_{i,t}} = (g(\iota_{i,t}) + y_t)\, dt + (\lambda - 1)dJ_{i,t} + \sigma dB_t\,. \tag{1.7}$$

Here $g(\iota_{i,t})$ is expected growth through internal incremental innovation given the flow of investment per unit of quality $\iota_{i,t}$. Function $g(\cdot)$ is deterministic, increasing and concave. Term y_t is a small, mean-zero and persistent component of expected growth

$$dy_t = -\kappa_y y_t dt + \sigma_y^y dB_t^y, \tag{1.8}$$

where dB_t^y is an exogenous Brownian shock. Exogenous growth y_t in a reduced form proxies for time-varying economic conditions that are important for firms' and agents' policies but are not modeled explicitly. I follow Bansal and Yaron (2004) to calibrate parameters κ_y and σ_y^y in my quantitative analysis .

The second term in equation (1.7), $(\lambda - 1)dJ_{i,t}$, is the central element of the dynamics of quality $q_{i,t}$ and the main focus of the model. It represents the process of radical or disruptive innovation in variety i. The symbol $J_{i,t}$ denotes the Poisson counting process, $J_{i,t} \in \mathbb{N}_0$. Constant λ denotes a proportional quality increase in a given variety conditional on a successful radical innovation, $\lambda > 1$. In the next section I describe the process of radical innovation and the structure of shock $dJ_{i,t}$ in more detail. The term dB_t is the aggregate Brownian shock. This shock uniformly affects productivity of all firms in the economy. All three aggregate shocks, dB_t^y, dJ_t, and dB_t are not correlated. This model does not feature any fully exogenous idiosyncratic shocks (although it is straightforward to add them). Innovation shock $dJ_{i,t}$ is the sole source of idiosyncratic risk that I am focusing on in this paper. Equation (1.7) can be rewritten in the integrated form as

$$q_{i,t} = q_{i,0} \times e^{\int_0^t (g(\iota_{i,u})+y_u)du} \times \lambda^{J_{i,t}} \times e^{\sigma B_t - \frac{\sigma^2}{2}t}\,. \tag{1.9}$$

According to this formula the current quality of variety i is the result of accumulation of (i) gradual improvements, (ii) radical improvements and (iii) aggregate productivity shocks.

1.3.2 Radical Innovation and the Innovation Sector

Radical innovation is the source of creative destruction in the economy. This type of innovation can be pursued both by incumbent firms and by new entrants. Active firms invest in radical innovation to improve their profitability and protect their current business from a potential disruption. The goal of an entrant is to achieve a radical improvement in one of the varieties and displace the incumbent.

Entrants' R&D efforts are fully financed by the innovation sector. The innovation sector is a financial intermediary that accepts contributions from outside investors and allocates the raised funds optimally across new business ventures. I interpret the innovation sector very broadly and its real world counterpart would include the majority of the sources through which new businesses attract outside money. Among others, these sources include venture capital, angel investing, equity crowdfunding and initial public offerings of young high-growth companies.

Radical innovations happen in aggregate waves. Each wave comes with a constant and exogenous Poisson intensity $\overline{h}$. I denote by J_t the Poisson counting process that governs their arrival. Conditional on arrival of a wave at time t, $dJ_t = 1$, three distinct scenarios are possible for a given variety i:

1. With probability $\overline{\omega}^I_{i,t-}$ the incumbent firm achieves radical improvement of the product line it owns, so $q_{i,t} = \lambda q_{i,t-}$ and $dJ_{i,t} = 1$.

2. With probability $\overline{\omega}^E_{i,t-}$ a new firm succeeds in radical innovation, replacing the incumbent and starting production of variety i of quality $q_{i,t} = \lambda q_{i,t-}$, $dJ_{i,t} = 1$.

3. With probability $(1 - \overline{\omega}^I_{i,t-} - \overline{\omega}^E_{i,t-})$ neither the incumbent nor the entrant

succeed in radical innovation; the incumbent keeps producing variety i of quality $q_{i,t} = q_{i,t-}$ and $dJ_{i,t} = 0$.

Conditional on $dJ_t = 1$, realizations of these scenarios are independent across i. Probabilities $\varpi^I_{i,t}$ and $\varpi^E_{i,t}$ are determined by the flows of investment into radical innovation made by the incumbent $z^I_{i,t}$ and allocated by the innovation sector $z^E_{i,t}$ to variety i, according to the innovation technology

$$\omega^E_{i,t} = \left(1 - \exp\left(-\chi^E \frac{z^E_{i,t}}{q_{i,t}}\right)\right)^{\alpha^E}, \tag{1.10a}$$

$$\omega^I_{i,t} = \left(1 - \exp\left(-\chi^I \frac{z^I_{i,t}}{q_{i,t}}\right)\right)^{\alpha^I}, \tag{1.10b}$$

$$\varpi^E_{i,t} = \omega^E_{i,t}\left(1 - \omega^I_{i,t} s\right), \tag{1.10c}$$

$$\varpi^I_{i,t} = \omega^I_{i,t}\left(1 - \omega^E_{i,t}(1-s)\right). \tag{1.10d}$$

Parameters in the above equations satisfy the constraints: χ^E, $\chi^I > 0$, α^E, $\alpha^I \in (0,1)$, $s \in [0,1]$. For fixed t optimal investment decisions in this setup are homothetic in $q_{i,t}$ across firms. Hence for brevity, I introduce additional notation $\hat{z}^E_{i,t} = z^E_{i,t}/q_{i,t}$, $\hat{z}^I_{i,t} = z^I_{i,t}/q_{i,t}$, and in many cases omit subscripts i and t.

The functional specification of the radical innovation technology has several important properties. First, the parametric form in equations (1.10a–1.10b) satisfies the Inada conditions. Second, it insures that, conditional on arrival of a wave, probability of successful innovation by the incumbent or an entrant does not exceed one. Equations (1.10c–1.10d) insure that total conditional probability of innovation by the incumbent and an entrant does not exceed one $\varpi^I_{i,t} + \varpi^E_{i,t} \leq 1$. In addition, the last two equations capture the notion of competition between incumbents and entrants in the technological race. Incumbent's (entrant's) probability of success in radical innovation decreases with the amount of investment made by the innovation

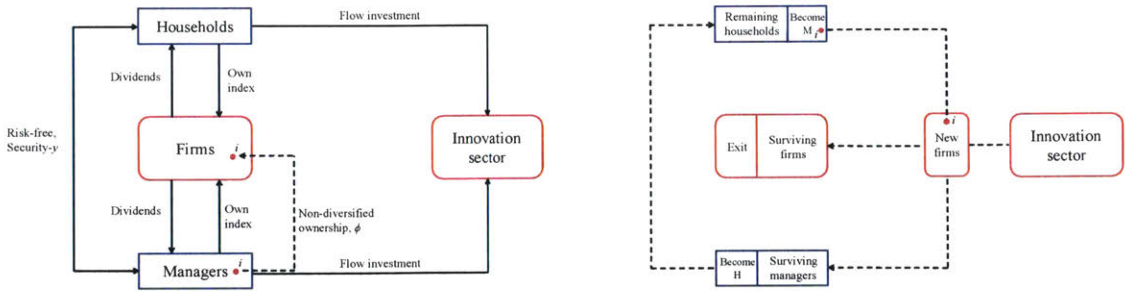

Figure 1-2: **Structure of the model economy.** The left panel depicts the general structure of the economy. The right panel illustrates the arrival of an innovation wave.

sector (the incumbent). I assume that, when making a choice of $z^I_{i,t}$, the incumbent firm takes investment flow from the innovation sector $z^E_{i,t}$ as given. The investment flow into the innovation sector is optimally allocated between different varieties to maximize the return on investment, taking innovation intensity by incumbents $z^I_{i,t}$ as given.

The assumption of lumpy arrival of radical innovations in a novel way unites the models by Aghion and Howitt (1992) and Grossman and Helpman (1991). It allows the technological progress to have heterogeneous impact on firms and at the same time allows me to study the risk premium on innovation. Pástor and Veronesi (2009), Garleanu et al. (2012) develop models with aggregate technological waves and asset pricing effects as well, but focus on the process of learning and adoption of new technologies correspondingly.

1.3.3 Agents and Capital Markets

The left panel of Figure 1-2 summarizes the structure of the model. The economy is populated by capitalists who can be of two types: households and managers. The type of each agent is not permanent and can change with the arrival of an innovation wave as will be described below. Households consume out of their wealth and save through investments in the stock market (operating firms), the risk-free asset, the innovation sector and an additional technical security, which I call security-y. By definiton, return on security-y is perfectly correlated with shock dB_t^y. This setup insures that financial markets are dynamically complete with respect to the three aggregate shocks dB_t^y, dJ_t and dB_t.[10] The risk-free asset and security-y are in zero net supply.

As households, managers consume out of their savings and have access to the same investible securities. Crucially, there is a one-to-one mapping between managers and incumbent firms. The manager matched to firm i makes dividend payout and investment decisions $\iota_{i,t}$ and $z_{i,t}^I$ on behalf of the firm. On top of that, the manager is forced to keep at least share ϕ of her net worth invested in the firm's stock, $\phi \geq 0$. Parameter ϕ in a reduced form captures necessary incentive provisions for the manager. This type of constraint on manager's ownership can be rationalized in the setup with moral hazard and multiplicative utility, akin to Edmans et al. (2008). Managers in my model should be interpreted broadly and include individuals who have non-diversified ownership of an operating business. These include business founders, family owners, company officers, board members and other kinds of non-diversified individual investors.

The assumption that managers and households invest their liquid wealth in the broad stock market index, as opposed to a subset of stocks, is without loss of generality. One could allow agents to invest in single stocks, but due to the risk aversion each agent will pick the fully diversified stock market index. When investing in the innovation sector, however, neither managers nor households have control over the allocation of their investments into the exact varieties. In cross-section, payouts to investors made by the innovation sector are returned in proportion to investors' original contributions.

For tractability I assume that households constitute a double-continuum set, so that for every firm or manager in the economy there is a continuum of households. Collectively aggregate wealth of households is comparable to the aggregate wealth of managers. Each manager, however, is substantially more wealthy than a typical

[10]Presence of security-y makes the model more tractable but is not essential for the results.

household $w^M \gg w^H$. This setup can be thought of as an economy with N firms, N managers and $\sim N^2$ households in the limiting case $N \to +\infty$. This technical assumption allows characterizing the evolution of the aggregate state of the economy without keeping track of complete distribution of wealth among agents.

Both managers and households have stochastic differential utility introduced by Duffie and Epstein (1992):[11]

$$U_t = \mathbb{E}_t \int_t^{\infty} f(c_u, U_u)\, du\,, \tag{1.11}$$

$$f(c_u, U_u) = \frac{\rho}{1-\psi}\left(\frac{c^{1-\psi}}{((1-\gamma)U)^{\frac{\gamma-\psi}{1-\gamma}}} - (1-\gamma)U\right). \tag{1.12}$$

The right panel of Figure 1-2 illustrates the arrival of an innovation wave $dJ_t = 1$. Upon arrival of the wave share $\int \overline{\omega}^I_{i,t} di$ of firms succeeds in radical innovation and improves the quality of the product by a factor of λ. Share $\int \overline{\omega}^E_{i,t} di$ of firms gets disrupted by new entrants who improve the quality of corresponding products by λ as well. Disrupted firms exit the market and corresponding managers become households and consume out of their remaining net worth. Each entering firm is managed by a new manager chosen randomly from the set of households. The new manager receives share β of the firm value. Share $(1-\beta)$ of the entering firm value constitutes part of the innovation sector's profits and is paid back to investors.

1.4 Solving the Model

The economy features three aggregate shocks: Brownian shocks dB_t and dB^y_t and the jump shock dJ_t. The first two shocks affect uniformly all firms in the economy; the last benefits some firms but hurts the other. Financial markets are complete with

[11] A continuous time extension of the recursive preferences studied by Kreps and Porteus (1978), Epstein and Zin (1989), Weil (1990).

respect to aggregate shocks, hence the stochastic discount factor η_t can be written as a jump-diffusion process

$$\frac{d\eta_t}{\eta_t} = -r_t\,dt - \pi_t\,dB_t - \pi_t^y\,dB_t^y - \pi_{t-}^J\,\left(dJ_t - \overline{h}\,dt\right). \tag{1.13}$$

Here r_t is the risk-free rate, π_t and π_t^y are the market prices of risk to aggregate productivity and expected growth, π_t^J is the market price of jump (or innovation wave) risk. The price of jump risk has the following economic interpretation: an agent willing to receive a payoff of Δ in case of jump arrival $dJ_t = 1$ has to pay a flow $\Delta \times \overline{h}(1-\pi_t^J)$ in exchange, where $\overline{h}$ is the intensity of jump arrival. When the price of jump risk is equal to zero, $\pi_t^J = 0$, the money flow paid by the agent in an instant of time $\Delta \times \overline{h}(1-\pi_t^J)\,dt$ is equal to the expected payoff $\Delta \times \overline{h}\,dt$. When the price of jump risk is positive (negative) the money flow paid in an instant of time is lower (higher) than the expected payoff.

In a symmetric equilibrium all operating firms will have the same stock price per unit of quality p_t. I postulate the process for p_t

$$\frac{dp_t}{p_t} = \mu_{p,t}\,dt + \sigma_{p,t}\,dB_t + \sigma_{p,t}^y\,dB_t^y + \sigma_{p,t-}^J\,dJ_t\,. \tag{1.14}$$

1.4.1 Problem of the Innovation Sector

The innovation sector maximizes the payoff to the investors conditional on wave arrival

$$\max_{z_{i,t}^E} \lambda(1-\beta)p_t\left(1+\sigma_{p,t}^J\right)\int q_{i,t}\overline{\omega}_{i,t}^E\left(z_{i,t}^E, z_{i,t}^I\right)\,di \tag{1.15}$$

subject to the constraint

$$\int z_{i,t}^E\,di = Z_t^E\,, \tag{1.16}$$

where by Z_t^E I denote the aggregate investment flow into the innovation sector. The term $\lambda(1-\beta)p_t(1+\sigma_{p,t})$ in equation (1.15) does not impact the allocation decision, but I include it for clarity. The innovation sector takes as given the aggregate inflow Z_t^E and investment in radical innovation by incumbent firms $z_{i,t}^I$. The inflow Z_t^E is determined by optimal portfolio choices of households and managers.

1.4.2 Problem of Households

Households choose optimal consumption and optimal investments in the risk-free asset, the stock market and the innovation sector. Since available investment opportunities span all aggregate shocks, the households' portfolio choice problem is equivalent to the choice of exposure to these shocks. Households choose optimal consumption c_t^H, optimal exposures to the aggregate productivity shock $\sigma_{w,t}^H$ and shock to the expected growth $\sigma_{w,t}^{y,H}$, and finally exposure to the jump risk $\sigma_{w,t}^{J,H}$ to solve

$$\max_{\substack{c_t^H \geq 0,\, \sigma_{w,t}^H \\ \sigma_{w,t}^{y,H},\, \sigma_{w,t}^{J,H}}} \mathbb{E}_t \int_t^{\infty} f(c_u^H, U_u^H)\, du \tag{1.17}$$

subject to the budget constraint

$$\frac{dw_t^H}{w_t^H} = -\hat{c}_t^H\, dt + \left(r_t + \sigma_{w,t}^H \pi_t + \sigma_{w,t}^{y,H} \pi_t^y + \sigma_{w,t}^{J,H} (\pi_t^J - 1)\, \bar{h}\right) dt \\ + \sigma_{w,t}^H\, dB_t + \sigma_{w,t}^{y,H}\, dB_t^y + \sigma_{w,t-}^{J,H}\, dJ_t\,. \tag{1.18}$$

Here w_t^H is agent's wealth, $\hat{c}_t^H$ is her consumption normalized by wealth. In equilibrium all households will choose the same optimal investment portfolio and consumption rate.

1.4.3 Problem of Managers

Managers choose consumption c_t^M and solve the portfolio allocation problem by choosing exposures $\{\tilde{\sigma}_{w,t}^M, \tilde{\sigma}_{t,w}^{y,M}, \tilde{\sigma}_{t,w}^{J,M}\}$ of their *liquid* wealth to aggregate shocks. Tildes in the notation indicate association with the liquid share of wealth. In addition, managers make decisions on the investment policies $\{\iota_{i,t},\ \hat{z}^I_{i,t}\}$ of the firms they run. Hence the problem of a manager is written as

$$\max_{\substack{c_t^M \geq 0,\, \tilde{\sigma}^M_{w,t},\, \tilde{\sigma}^{y,M}_{w,t} \\ \tilde{\sigma}^{J,M}_{w,t},\, \iota_{i,t},\, \hat{z}^I_{i,t}}} \mathbb{E}_t \int_t^{\infty} f(c_u^M, U_u^M)\, du \tag{1.19}$$

subject to the budget constraint

$$\frac{dw_t^M}{w_t^M} = -\hat{c}_t^M\, dt + \phi\, dR_{i,t} + (1-\phi)\, d\tilde{R}_t\,, \tag{1.20}$$

where $dR_{i,t}$ is return on the stock of the firm she controls, and $d\tilde{R}_t$ is return on the liquid part of her wealth. In equation (1.20) I implicitly take into account the fact that a risk averse manager chooses minimal possible allocation of her wealth in her own firm ϕ. As in the case of households, the return on liquid wealth is defined by

$$\begin{aligned} d\tilde{R}_t = \Big(r_t + \tilde{\sigma}^M_{w,t}\, \pi_t + \tilde{\sigma}^{y,M}_{w,t}\, \pi_t^y + \tilde{\sigma}^{J,M}_{w,t}\, (\pi_t^J - 1)\bar{h}\Big)\, dt & \\ + \tilde{\sigma}^M_{w,t}\, dB_t + \tilde{\sigma}^{y,M}_{w,t}\, dB_t^y + \tilde{\sigma}^{J,M}_{w,t-}\, dJ_t\,. & \end{aligned} \tag{1.21}$$

The return on firm's i stock consists of the dividend yield and capital gains

$$dR_{i,t} = \underbrace{\left(\frac{a - \iota_{i,t} - \hat{z}^I_{i,t}}{p_t}\right)}_{\text{dividend yield}} + \underbrace{\left(\frac{d(p_t\, q_{i,t})}{p_t\, q_{i,t}}\right)}_{\text{capital gains}}, \tag{1.22}$$

where I took into account that firm's i dividend is equal to the difference between the operating profit $aq_{i,t}$ and total spending on gradual and radical innovation $q_{i,t}(\iota_{i,t} + \hat{z}^I_{i,t})$. Applying Ito's lemma to the second term in the equation above, and taking into account dynamics of $q_{i,t}$ and p_t defined in equations (1.7) and (1.14), I obtain

$$dR_{i,t} = \left(\frac{a - \iota(g_{i,t}) - \hat{z}^I_{i,t}}{p_t} + g_{i,t} + \mu_{p,t} + \sigma\sigma_{p,t} \right) dt + (\sigma + \sigma_{p,t})dB_t + \sigma^y_{p,t} dB^y_t$$
$$+ \sigma^J_{p,t-} \left(dJ_t - dJ^I_{i,t} - dJ^E_{i,t} \right) - dJ^E_{i,t} + \left(\lambda(1 + \sigma^J_{p,t-}) - 1 \right) dJ^I_{i,t} \,, \quad (1.23)$$

where by $dJ^I_{i,t}$ and $dJ^E_{i,t}$ I denote successful radical innovations in variety i by the incumbent and an entrant correspondingly. By substituting definitions of $d\tilde{R}_t$ and $dR_{i,t}$ we can rewrite the budget constraint (1.20) as

$$\frac{dw^M_t}{w^M_t} = -\hat{c}^M_t \, dt + \mu^M_{w,t} + \sigma^M_{w,t} \, dB_t + \sigma^{y,M}_{w,t} \, dB^y_t + (1 - \phi)\sigma^{J,M}_{w,t-} \, dJ_t$$
$$+ \phi \left(\sigma^J_{p,t-} \left(dJ_t - dJ^I_{i,t} - dJ^E_{i,t} \right) - dJ^E_{i,t} + \left(\lambda(1 + \sigma^J_{p,t-}) - 1 \right) dJ^I_{i,t} \right) \,, \quad (1.24)$$

where I introduced additional notation

$$\mu^M_{w,t} = (1 - \phi)\mu_{\tilde{R},t} + \phi\mu_{R,t} \,, \quad (1.25)$$
$$\sigma^M_{w,t} = (1 - \phi)\tilde{\sigma}^M_{w,t} + \phi(\sigma + \sigma_{p,t}) \,, \quad (1.26)$$
$$\sigma^{y,M}_{w,t} = (1 - \phi)\tilde{\sigma}^{y,M}_{w,t} + \phi\sigma^y_{p,t} \,, \quad (1.27)$$

and $\mu_{\tilde{R},t}$ and $\mu_{R,t}$ are drifts of the continuous components of $d\tilde{R}_t$ and dR_t given by (see equations 1.21, 1.23)

$$\mu_{\tilde{R},t} = r_t + \tilde{\sigma}^M_{w,t} \, \pi_t + \tilde{\sigma}^{y,M}_{w,t} \, \pi^y_t + \tilde{\sigma}^{J,M}_{w,t} \, (\pi^J_t - 1)\bar{h} \,, \quad (1.28)$$
$$\mu_{R,t} = \frac{a - \iota(g_{i,t}) - \hat{z}^I_{i,t}}{p_t} + g_{i,t} + \mu_{p,t} + \sigma\sigma_{p,t} \,. \quad (1.29)$$

Equilibrium The definition of equilibrium involves aggregation of policies and other equilibrium variables across households, managers, and productive firms. Hence I introduce notation $\int_H f_j \, dj$ and $\int_M f_i \, di$ to represent a quantity f aggregated across the set of households and managers correspondingly. Since there is a one-to-one mapping between managers and product lines, $\int_M f_i \, di$ also denotes the value of f aggregated across product lines.

Definition 1. *An equilibrium is a set of stochastic processes for the price per unit of quality $\{p_t\}$, the stochastic discount factor $\{\eta_t\}$; for each product line i the processes for quality $\{q_{i,t}\}$, optimal investment policies $\{\iota_{i,t}, z^I_{i,t}, z^E_{i,t}\}$ by incumbents and the innovation sector; for each manager i and household j wealth $\{w^M_{i,t}, w^H_{j,t}\}$, portfolio loadings $\{\tilde{\sigma}^M_{i,w,t}, \tilde{\sigma}^{y,M}_{i,w,t}, \tilde{\sigma}^{J,M}_{i,w,t}, \sigma^H_{j,w,t}, \sigma^{y,H}_{j,w,t}, \sigma^{J,H}_{j,w,t}\}$ and consumption $\{c^M_{i,t}, c^H_{j,t}\}$ such that:*

1. *Taking aggregate conditions and investment of the innovation sector $z^E_{i,t}$ as given households and managers maximize their respective utilities subject to their budget constraints,*

2. *Taking investment in radical innovation by incumbents $z^I_{i,t}$ as given the innovation sector solves its optimal allocation problem,*

3. *Market for consumption goods clears*

$$\int_M c^M_{i,t} di + \int_H c^H_{j,t} dj = \int_M \left((1-\iota_{i,t})q_{i,t} - z^I_{i,t}\right) di - \int_M z^E_{i,t} di \,, \tag{1.30}$$

4. *Financial markets clear*

$$(1-\phi)\int_M \tilde{\sigma}^M_{i,w,t} w^M_{i,t} di + \int_H \sigma^H_{j,w,t} w^H_{j,t} dj = (\sigma + \sigma_{p,t}) \int_M \left(p_t q_{i,t} - \phi w^M_{i,t}\right) di \,, \tag{1.31}$$

$$(1-\phi)\int_M \tilde{\sigma}^{y,M}_{i,w,t} w^M_{i,t} di + \int_H \sigma^{y,H}_{j,w,t} w^H_{j,t} dj = \sigma^y_{p,t} \int_M \left(p_t q_{i,t} - \phi w^M_{i,t}\right) di \,, \tag{1.32}$$

$$
\begin{aligned}
(1-\phi)\int_M \tilde{\sigma}^{J,M}_{i,w,t} w^M_{i,t} di + \int_H \sigma^{J,H}_{j,w,t} w^H_{i,t} di = &\\
& (1-\beta)\lambda\left(1+\sigma^J_{p,t}\right)\int_M p_t q_{i,t}\overline{\omega}^E_{i,t} di \\
& + \sigma^J_{p,t}\int_M \left(1-\overline{\omega}^E_{i,t}-\overline{\omega}^I_{i,t}\right)\left(p_t q_{i,t}-\phi w^M_{i,t}\right) di \\
& + \left(\lambda(1+\sigma^J_{p,t})-1\right)\int_M \overline{\omega}^I_{i,t}\left(p_t q_{i,t}-\phi w^M_{i,t}\right) di \\
& - \int_M \overline{\omega}^E_{i,t}\left(p_t q_{i,t}-\phi w^M_{i,t}\right) di\,. \qquad (1.33)
\end{aligned}
$$

In the definition above conditions 1–3 are standard. Condition 4 and equations (1.31, 1.32, 1.33) define market clearing for the aggregate productivity, growth and jump risks correspondingly. The market for the risk-free asset clears by Walras' law. The left-hand side of each of the three equations in condition 4 corresponds to the aggregate demand for a given type of risk, the right-hand side — to aggregate supply. For each type of risk aggregate demand has the same structure and consists of two parts: aggregated across managers exposure of liquid wealth and aggregated exposure of households' wealth. Aggregate supply of the exposure to shock dB_t (right-hand side of equation 1.31) is equal to the product of the aggregate stock market return exposure to shock dB_t, $(\sigma+\sigma_{p,t})$, times the value of the free-float equity $\int\left(p_t q_{i,t}-\phi w^M_{i,t}\right) di$. The free-float is defined as the aggregate value of the stock market excluding the shares that managers have to keep in their own firms. Supply of exposure to shock dB^y_t has the same structure. Supply of exposure to the jump risk consists of four parts. The first line on the right-hand side of equation (1.33) calculates the total payoff to the investors in the innovation sector conditional on wave arrival. The second line aggregates the change in value of firms that are not directly affected by the innovation wave. The third line accounts for the firms that succeed in radical innovation. The fourth accounts for the firms that get displaced conditional on an arrival of a wave.

1.4.4 Solution

The general solution strategy follows Di Tella (2017). The value function of a household has the form

$$U_t^H(w_t^H) = \frac{(\xi_t^H w_t^H)^{1-\gamma}}{1-\gamma}, \tag{1.34}$$

where ξ_t^H is a stochastic process that reflects household's investment opportunities. It depends on the aggregate state of the economy and does not depend on wealth w_t^H. Homogeneity of the value function $U_t^H(w_t^H)$ in wealth follows from three observations. First, preferences defined in (1.11) are homogeneous in consumption of degree $(1-\gamma)$. Second, households' budget constraint (1.18) is linear in wealth. Third, probability for a household to become a manager conditional on wave arrival is infinitesimal.

I focus on the equilibrium with symmetric investment policies pursued by managers. In such equilibrium investment in gradual growth $\iota_{i,t}$, investment in radical innovation normalized by quality $\hat{z}_{i,t}^I$ by incumbent, and $\hat{z}_{i,t}^E$ by the innovation sector do not depend on i. Probability of each manager to become a household conditional on a wave arrival $\varpi_{i,t}^E$ does not depend on i as well. Hence homogeneity of preferences and households' value function (equation 1.34) imply that the value function of a manager is homogeneous in wealth as well

$$U_t^M(w_t^M) = \frac{(\xi_t^M w_t^M)^{1-\gamma}}{1-\gamma}. \tag{1.35}$$

Processes for ξ_t^H and ξ_t^M follow jump-diffusions

$$\frac{d\xi_t^i}{\xi_t^i} = \mu_{\xi,t}^i dt + \sigma_{\xi,t}^i dB_t + \sigma_{\xi,t}^{y,i} dB_t^y + \sigma_{t-}^{J,i} dJ_t, \quad i \in \{H, M\}, \tag{1.36}$$

where $\mu_{\xi,t}^i$, $\sigma_{\xi,t}^i$, $\sigma_{\xi,t}^{y,i}$, $\sigma^{J,i}$ are to be determined in equilibrium. Appendix derives the HJB equations for households and managers. The following lemma formulates the optimal policies (except for optimal loadings on jump risk and investment in radical innovation which I describe in the next paragraph):

Lemma 1. *Optimal consumption policies of households and managers are given by*

$$\hat{c}_t^i = \rho^{\frac{1}{\psi}} \left(\xi_t^i\right)^{\frac{\psi-1}{\psi}}, \quad i \in \{H, M\}. \tag{1.37}$$

Optimal loadings on aggregate productivity shock dB_t and shock to exogenous growth dB_t^y are given by

$$\sigma_{w,t}^i = \frac{\pi_t + (1-\gamma)\sigma_{\xi,t}^i}{\gamma}, \quad \sigma_{w,t}^{y,i} = \frac{\pi_t + (1-\gamma)\sigma_{\xi,t}^{y,i}}{\gamma}, \quad i \in \{H, M\}. \tag{1.38}$$

Investment in gradual innovation ι_t is given by

$$g'(\iota_t) = 1/p_t. \tag{1.39}$$

Radical Innovation and Portfolio Loadings on Jump Now I turn to investigating agents' optimal loadings on the jump risk. Households choose optimal exposure to jump risk by solving the problem[12]

$$\max_{\sigma_{w,t}^{J,H}} \left\{ \sigma_{w,t}^{J,H}(\pi_t^J - 1) + \frac{1}{1-\gamma} \times \left(\left[\left(1 + \sigma_{\xi,t}^{J,H}\right) \left(1 + \sigma_{w,t}^{J,H}\right) \right]^{1-\gamma} - 1 \right) \right\}. \tag{1.40}$$

In absence of arbitrage the price of jump risk π_t^J is less than one. When choosing $\sigma_{w,t}^{J,H}$ households weight the tradeoff between the change in wealth of $\sigma_{w,t}^{J,H}(\pi_t^J - 1)\overline{h}$ per unit of time and change in utility conditional on wave arrival captured by the second term in equation (1.40). The utility change depends on the jump in investment opportunities $\sigma_{\xi,t}^{J,H}$ and a proportional change in wealth $\sigma_{w,t}^{J,H}$ chosen by the agent. The first order condition to problem (1.40) determines the optimal loading on jump risk $\sigma_{w,t}^{J,H}$

$$\sigma_{w,t}^{J,H} = \left(1 + \sigma_{\xi,t}^{J,H}\right)^{\frac{1-\gamma}{\gamma}} \left(1 - \pi_t^J\right)^{-\frac{1}{\gamma}} - 1. \tag{1.41}$$

---

[12] This equation follows from collecting the terms that depend on $\sigma_w^{J,H}$ in the households' HJB. See appendix.

The managers' problem is more complex,[13]

$$
\begin{aligned}
\max_{\tilde{\sigma}_{w,t}^{J,M},\,\hat{z}_t^I} \Bigg\{ & -\phi \frac{\hat{z}_t^I}{p_t} + (1-\phi)\tilde{\sigma}_{w,t}^{J,M}(\pi_t^J - 1)\overline{h} \\
& + \frac{\overline{h}\,(1-\varpi_t^I-\varpi_t^E)}{1-\gamma} \times \left(\left[\left(1+\sigma_{\xi,t}^{J,M}\right)\left(1+(1-\phi)\tilde{\sigma}_{w,t}^{J,M}+\phi\sigma_{p,t}^J\right)\right]^{1-\gamma} - 1 \right) \\
& + \frac{\overline{h}\,\varpi_t^I}{1-\gamma} \times \left(\left[\left(1+\sigma_{\xi,t}^{J,M}\right)\left(1+(1-\phi)\tilde{\sigma}_t^{J,M}+\phi(\lambda(1+\sigma_{p,t}^J)-1)\right)\right]^{1-\gamma} - 1 \right) \\
& + \frac{\overline{h}\,\varpi_t^E}{1-\gamma} \times \left(\left[\frac{\xi_t^H}{\xi_t^M}\left(1+\sigma_{\xi,t}^{J,H}\right)\left(1+(1-\phi)\tilde{\sigma}_t^{J,M}-\phi\right)\right]^{1-\gamma} - 1 \right) \Bigg\}, \quad (1.42)
\end{aligned}
$$

where probabilities ϖ_t^I and ϖ_t^E depend on $\hat{z}_t^I$ and $\hat{z}_t^E$ according to equations (1.10a–1.10d). A manager jointly chooses the optimal exposure of her liquid wealth to jump $\tilde{\sigma}_{w,t}^{J,M}$ and spending on radical innovation inside the firm $\hat{z}_t^I$. The first line in equation (1.42) accounts for the decrease in dividend due to investment in radical innovation and continuous change in wealth due to exposure to jump of the liquid wealth. Lines two through four account for the three alternatives that might happen upon wave arrival. First, conditional on a wave arrival with probability $(1-\varpi_t^I-\varpi_t^E)$ the managed firm is not affected by the innovation wave. The manager's investment opportunities jump by a factor of $\sigma_{\xi,t}^{J,M}$ and the firm's market capitalization changes by a factor of $\sigma_{p,t}^J$ due to the jump in the stock price per unit of quality. Second, with probability ϖ_t^I the firm succeeds in the radical innovation and its market capitalization jumps up by $\lambda(1+\sigma_{p,t}^J)$. Third, with probability ϖ_t^E the managed firm is displaced by an entrant. As a result, the manager loses share ϕ of her wealth. In addition, the manager joins the set of households and investment opportunities change by a factor of $(\xi_t^H/\xi_t^M)(1+\sigma_{\xi,t}^{J,H})$.

Financial loss associated with disruption is the highest threat that a manager faces. She can reduce the conditional probability ϖ_t^E of this outcome by investing more in radical innovation $\hat{z}^I$, but she can't drive it all the way to zero: as a function of $\hat{z}^E$, probability of disruption ϖ_t^E satisfies the Inada conditions for any value of $\hat{z}^I$.

[13]This equation follows from collecting the terms that depend on $\sigma_w^{J,M}$ and $\hat{z}^I$ in the managers' HJB. See appendix.

Hence, the manager will tend choose higher loadings on jump $\tilde{\sigma}_t^{J,M}$ to hedge against a potential financial loss.

Equilibrium Inflow into the Innovation Sector Aggregate flow into the innovation sector Z_t^E equals $\hat{z}_t^E \times Q_t$, where Q_t is the aggregated quality of goods in the economy $Q_t = \int q_{i,t} di$. The quantity $\hat{z}_t^E$ is determined from the absence of arbitrage condition

$$\hat{z}_t^E Q_t = \underbrace{(1-\beta)\lambda Q_t p_t \left(1 + \sigma_{p,t}^J\right) \overline{\omega}^E(\hat{z}_t^E, \hat{z}_t^I) \overline{h}}_{\text{Expected payoff}} \times \underbrace{\left(1 - \pi_t^J\right)}_{\text{Risk Adjustment}} . \tag{1.43}$$

The right-hand side of this equation consists of two components. The first one corresponds to the expected payoff to investors. It is equal to the share of new firms going to investors $(1-\beta)$, times the value of new firms $\lambda Q_t p_t \left(1 + \sigma_{p,t}^J\right)$, times intensity of innovation arrival $\overline{h}$. The second part is the adjustment for the jump risk $(1 - \pi_t^J)$. When the jump risk is not priced, $\pi_t^J \equiv 0$, equation (1.43) coincides with the free-entry in innovation condition, common to models of endogenous growth.

Rearranging equation (1.43) we obtain

$$\frac{\hat{z}_t^E}{\overline{\omega}^E(\hat{z}_t^E, \hat{z}_t^I)} = (1-\beta)\lambda p_t \left(1 + \sigma_{p,t}^J\right) \overline{h} \times \left(1 - \pi_t^J\right) . \tag{1.44}$$

The left-hand side of (1.44) is an increasing function of $\hat{z}_t^E$ taking investment in radical innovation by the incumbent $\hat{z}_t^I$ fixed.[14] Hence, investment in radical innovation by entrants $\hat{z}_t^E$ is increasing in the price of stock conditional on success $p_t \left(1 + \sigma_{p,t}^J\right)$ and decreasing in the price of jump risk π_t^J. The price of jump risk becomes an important determinant of the amount of radical innovation.

[14] Indeed, denote $g(z) = \overline{\omega}^E(z, \hat{z}_t^I)$. Function $g(z)$ is strictly concave and $g(0) = 0$, see equation (1.10c). Take any two points z_1 and z_2, so that $z_2 > z_1 > 0$. Then $g(z_2)/z_2 - g(z_1)/z_1 = [(g(z_2) \times z_1/z_2 + g(0) \times (z_2 - z_1)/z_2) - g(z_1))]/z_1$. The nominator of the last expression is less than zero by the definition of a strictly concave function, hence $g(z_2)/z_2 < g(z_1)/z_1$ or equivalently $z_2/g(z_2) > z_1/g(z_1)$.

1.4.5 State Variables

I focus on equilibrium that is Markov in three state variables: (i) exogenous growth component y_t, (ii) scale of the economy as measured by the average quality Q_t, (iii) share of wealth owned by the managers x_t. I don't need to keep track of the distribution of wealth within each class of agents and distribution of qualities across product lines due to homogeneity of policy functions.

Share of wealth owned by the managers x_t is defined by

$$x_t = \frac{\int_M w_{j,t}^M dj}{\int_M w_{j,t}^M dj + \int_H w_{j,t}^H dj}, \tag{1.45}$$

and follows a jump-diffusion process

$$dx_t = \mu_{x,t}dt + \sigma_{x,t}dB_t + \sigma_{x,t}^y dB_t^y + \sigma_{x,t-}^J dJ_t\,. \tag{1.46}$$

Coefficients in equation (1.46) depend on agents' optimal policies and are specified in the appendix in Lemma 2. Here I just note that in equilibrium $\sigma_{x,t} \equiv 0$, so x_t is driven only by two shock.

Markov equilibrium of the model is fully characterized once we know the functions $\xi^H(x,y)$, $\xi^M(x,y)$, $p(x,y)$, $\pi(x,y)$, $\pi^y(x,y)$, $\pi^J(x,y)$, $\mu_x(x,y)$, $\sigma_x^y(x,y)$ and $\sigma_x^J(x,y)$. Appendix 1.9.4 shows how these functions are characterized and solved for numerically.

1.5 Quantitative Results

Now I proceed to the numerical solution and properties of the economy. Table 1.4 summarizes the set of parameters along with their values used in the main calibration. For a subset of standard parameters I use the values commonly used in the literature. Time t is measured in years. I set the parameter of relative risk aversion γ to 10 and elasticity of intertemporal substitution ψ^{-1} to 2, values frequently used in the

Table 1.4: Calibration

Preferences		
Risk aversion	γ	10
EIS	$1/\psi$	2
Time discounting	ρ	0.067
Innovation Technology		
Innovation productivity, incumbent	χ^I	7.44
Innovation productivity, entrant	χ^E	0.61
Elasticity of innovation, incumbent	α^I	0.89
Elasticity of innovation, entrant	α^E	0.89
Radical quality improvement	λ	1.1
Jump intensity	$\bar{h}$	0.4
Innovation productivity	s	0.5
Share going to new managers	β	0.5
Shocks		
Aggregate volatility	σ	10%
Mean reversion of exogenous growth	κ_y	0.225
Shock to exogenous growth volatility	σ_y^y	0.0143
Non-diversifiable share	ϕ	0.4
Incremental Growth: $\iota = Ag^2$		
Productivity (inverse)	A	266

asset pricing literature (see e.g. Bansal and Yaron, 2004). Using recursive preferences helps to achieve a stable risk-free rate. The EIS above one implies that the stock price responds positively to shocks to exogenous growth y. Time discount ρ is set to 0.067 to keep the risk-free rate at the average level of 1%. Volatility of the aggregate productivity shock σ is set to 10% to roughly match the standard deviation of aggregate dividend growth. Parameters of exogenous growth process κ_y and σ_y^y are set to match discrete-time counterparts reported in Bansal and Yaron (2004), Table 1.

Fraction of net worth that managers invest in their own firms ϕ plays an important role in the model. When ϕ equals zero managers are effectively identical to households. Positive values of ϕ make them exposed to non-diversified risk which impacts optimal portfolio choice and the amount of innovation in the economy. In the baseline calibration I use $\phi = 0.4$. This value corresponds to the average share of net worth invested in private equity by households with active business ownership in the Survey of Consumer Finances (see Moskowitz and Vissing-Jørgensen, 2002). I do a comparative statics analysis with respect to the value of ϕ to illustrate the effect of non-diversifiable risk on equilibrium. According to the data reported in Hall and Woodward (2010), Table 1, on average 50% of the value of successful start ups, backed by VC, remains in the hands of the founders. The rest goes to outside investors, so I set $\beta = 0.5$.

I set gradual innovation technology to have quadratic functional form $\iota = Ag^2$, which makes numerical algorithm tractable and provides sufficient flexibility to match the moments described below. Parameters α^I and α^E are set to 0.89. This value implies that elasticities of radical innovation for both incumbents and entrants are within the range of estimates reported by Griliches (1990).[15] Parameter s is set to 0.5. It gives entrants and incumbents equal chances to succeed conditional on a preliminary success in radical innovation. I set parameter λ to be equal to 1.1. Garcia-Macia et al. (2015) estimate average increase in quality of existing varieties due to drastic

[15]Griliches (1990) surveys the studies that estimate elasticity of R&D output, measured by patents, in response to changes in R&D.

innovation to be around 7.5%. Acemoglu et al. (2013) estimate this parameter to be around 14.8%. The value I use lies near the middle of this range. Remaining technological parameters of innovation (χ^I, χ^E, A, $\overline{h}$) are set jointly to match a set of moments in the data. To keep the matter concrete, I consider the set of public firms as the real-world counterpart of incumbents, and venture capital industry as the counterpart of the innovation sector.[16] I choose χ^I, χ^E, A to match average incumbents' R&D to incumbents' operating profit ratio, average entrants' R&D to incumbents' operating profit ratio and average incumbents' exit rate.[17] In the baseline case innovation waves arrive on average once every two and a half years years $\overline{h} = 0.4$. The frequency of wave arrival determines lumpiness of the innovation process which is one of the key assumptions that differentiate the present model from the previous literature. Lower values of $\overline{h}$ make investments in the innovation sector more risky. The value I use in the baseline calibration delivers a good match for the volatility of returns in the innovation sector and returns in venture capital observed in the data. In the analysis below I also do comparative statics with respect to this parameter to show its impact on solution properties.

Properties of the Equilibrium The upper panel of Figure 1-3 shows equilibrium amount of investment in radical innovation by entrants on the left and by incumbents on the right. Here I plot all quantities as functions of managers' share x. State variable y is set to be equal to 0 (solid), -0.041 (dash) and 0.041 (dash-dot) which correspond to 0.025, 0.5 and 0.975 quantiles of its stationary distribution. Investment rate in radical innovation by both incumbents and entrants increases both with x and y. The bottom panel shows probabilities of successful innovation in a given product

[16]A substantial part of the economic activity is conducted by firms which are note publicly traded. At the same time venture capital constitutes only a part of the innovation sector. Given lack of data for broader definitions of the productive and innovation sectors I limit my calibration to these economically important definitions. The data on public firms comes from Compustat, data on VC flows is from VentureXpert. I limit the sample to 1993–2016, period when flows to VC have stabilized to exhibit stationary behavior.

[17]Exits are not directly observable in Compustat or CRSP. I proxy exits as a loss of 90% of market capitalization during one year.

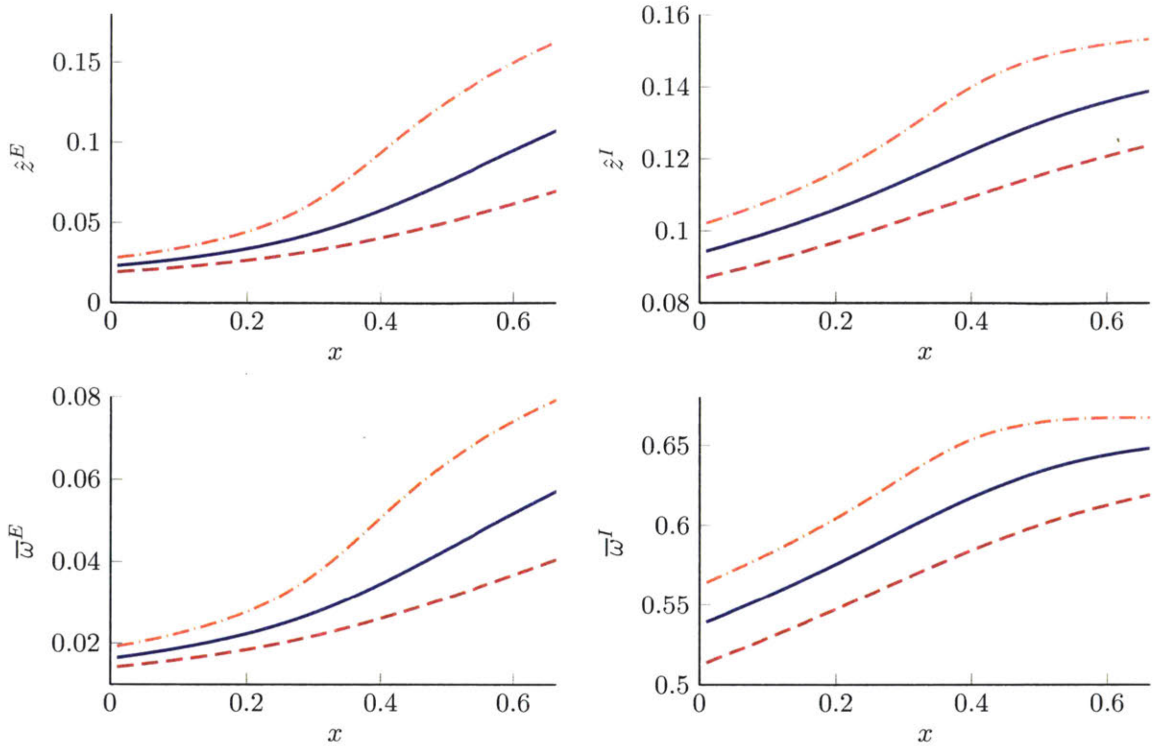

Figure 1-3: **Investment in and probabilities of radical innovation.** Top: investment in radical innovation by entrants $\hat{z}^E$ and incumbents $\hat{z}^I$. Bottom: probabilities of successful radical innovation by an entrant $\overline{\omega}^E$ and incumbent $\overline{\omega}^I$. Solid line is for y being equal to 0 or its median value, dash and dash-dot are for y being equal to 0.025 and 0.975 quantiles of its stationary distribution.

line conditional on arrival of a wave $\overline{\omega}^I$ and $\overline{\omega}^E$. Alternatively, $\overline{\omega}^I$ represents the share of firms that succeed in radical innovation and $\overline{\omega}^E$ represents the share of firms that get displaced. Increasing pattern of investment by the innovation sector $\hat{z}^E$ is driven by the different portfolio allocations chosen by managers and households. Incumbent firms increase investment in radical innovation with x in response to more aggressive activity by entrants.

In the upper panel of Figure 1-4 I plot agents' exposure to the jump risk as functions of managers' share x (left) and y (right). Exposure to the jump risk is defined as a proportional change in wealth upon arrival of the innovation wave. Dash line denotes households' exposure. For managers I plot two measures of exposure to

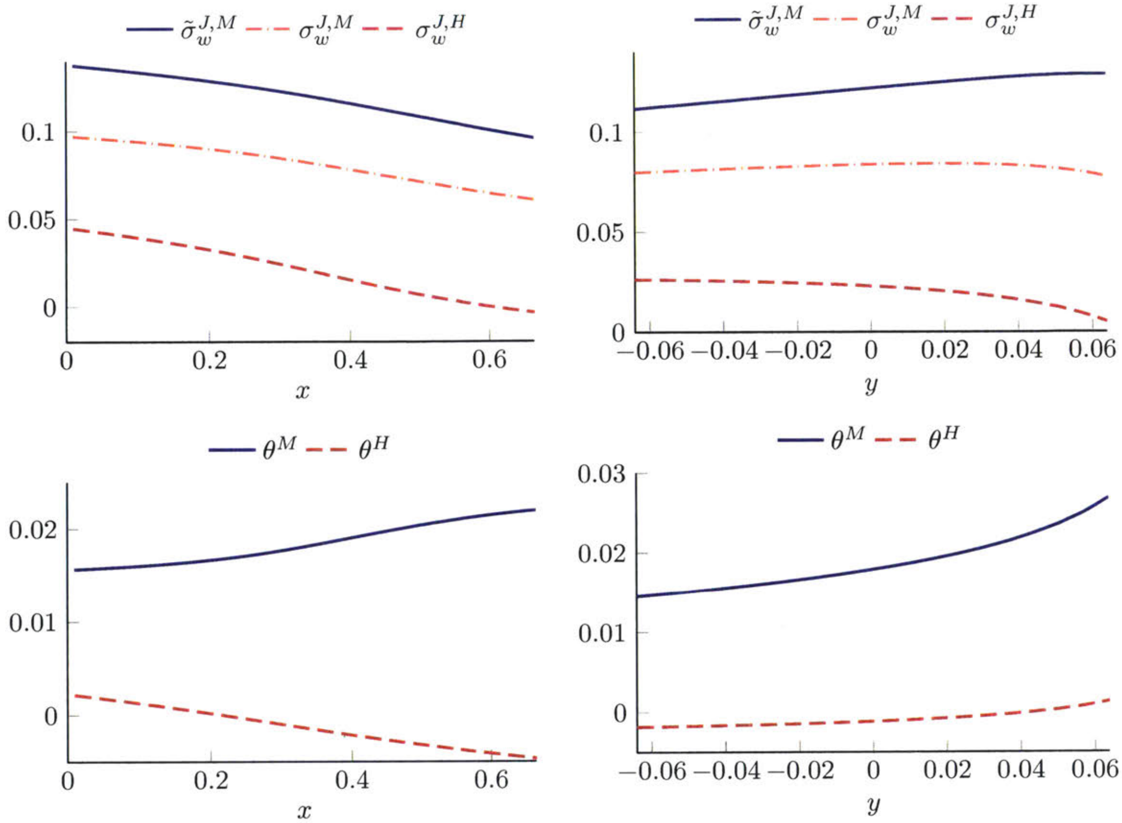

Figure 1-4: **Portfolio loadings on jump risk and flow investment in the innovation sector.** Upper panel: managers' liquid wealth loading on jump risk $\tilde{\sigma}^{J,M}$, managers' total wealth loading on jump risk $\sigma^{J,M}$, households' loading on jump risk $\sigma^{J,H}$. Lower panel: investments in the innovation sector by managers and households normalized by personal wealth, θ^M and θ^H correspondingly. Left (right) column keeps state variable x (y) constant at its median value.

the jump risk. Denoted by the solid line, $\tilde{\sigma}_w^{J,M}$ is exposure to jump of the liquid part of manager's portfolio. Dash-dot line plots average across managers exposure to jump of the total net worth $\sigma_w^{J,M}$

$$\sigma_w^{J,M} = (1-\phi)\,\tilde{\sigma}_w^{J,M} + \phi\,\sigma_S^J\,. \tag{1.47}$$

Here σ_S^J is the expected instantaneous return on a single stock conditional on the arrival of a wave (which coincides with the aggregate stock market exposure to the jump risk) defined by

$$\sigma_S^J = \left(1 - \overline{\omega}^I - \overline{\omega}^E\right) \times \sigma_p^J + \overline{\omega}^I \times \left(\lambda(1 + \sigma_p^J) - 1\right) - \overline{\omega}^E . \tag{1.48}$$

The first term reflects the change in market value of firms that are not directly influenced by the innovation wave, the second corresponds to the firms that succeed in radical innovation and the third accounts for the firms that exit. Both managers' liquid and total wealth are significantly more exposed to jump risk relative to households' wealth. Each manager faces a considerable probability of her business being disrupted by an entrant conditional on arrival of the innovation wave. To hedge this risk managers expose their liquid wealth to more of the jump risk.

A different way to look at agents' portfolio choice is to replicate risk exposures using the set of tradable securities: the stock market, the innovation sector, security-y and the risk-free asset. Among them, the stock market is the only security that provides exposure to the aggregate productivity shock dB_t. So managers' portfolio allocation in the stock market is simply given by $(1 - \phi)\tilde{\sigma}_w^M/\sigma_S$, and households' is given by σ_w^H/σ_S. Investment flows, normalized by net worth, into the innovation sector for households and managers are defined respectively by

$$\theta^H = \overline{h}(1 - \pi^J)\left(\sigma_w^{J,H} - \sigma_S^J \times \frac{\sigma_w^H}{\sigma_S}\right) , \tag{1.49a}$$

$$\theta^M = (1 - \phi)\overline{h}(1 - \pi^J)\left(\tilde{\sigma}_w^{J,M} - \sigma_S^J \times \frac{\tilde{\sigma}_w^M}{\sigma_S}\right) . \tag{1.49b}$$

The bottom panel of Figure 1-4 plots investment flows into the innovation sector by managers and households. Each manager invests significantly more in the innovation sector relative to her net worth compared to households. Interestingly each manager's flow investment in the innovation sector θ^M is increasing with the state variables x and y under this calibration. This implies that increasing in x pattern of aggregate investment in the innovation sector observed on Figure 1-3 is driven by two forces.

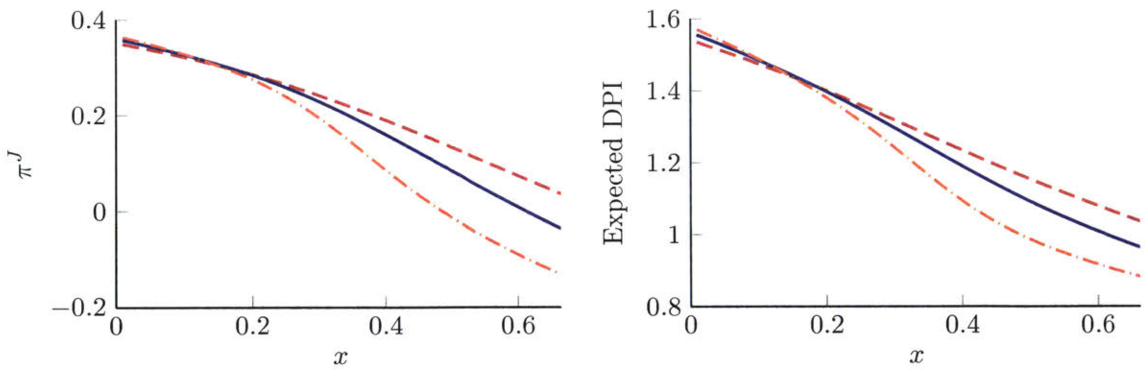

Figure 1-5: **Price of jump risk and expected distributions to paid-in capital ratio.** Solid lines are for y being equal to 0 or its median value, dash and dash-dot are for y being equal to 0.025 and 0.975 quantiles of its stationary distribution.

First, managers' allocation to the innovation sector is higher than households' per unit of wealth. Second, each manager faces higher risk of disruption and invests even higher share of own wealth in the innovation sector when x is higher. The second channel illustrates the feed back loop between the amount of innovation activity and agents' portfolio choice.

As it follows from Figure 1-4 households' flow investment in the innovation sector gradually declines with x. This effect is driven by the decline in price of jump risk plotted on the left panel of Figure 1-5. The price of jump risk equilibrates demand and supply for it. Higher x leads to more demand for positive exposure to jump risk stemming from the managers willing to hedge against disruption. At the same time net supply of the jump risk is more likely to decline with x due to displacing effect of the innovation sector. The right hand side of Figure 1-5 plots the expected distributions to paid-in capital multiple (DPI) for the innovation sector in the model. DPI is one of the most popular and important performance measures in the Venture Capital industry. Realized DPI represents the ratio of total distributions that investors in VC receive to the capital invested. In my model expected DPI is related to the price of

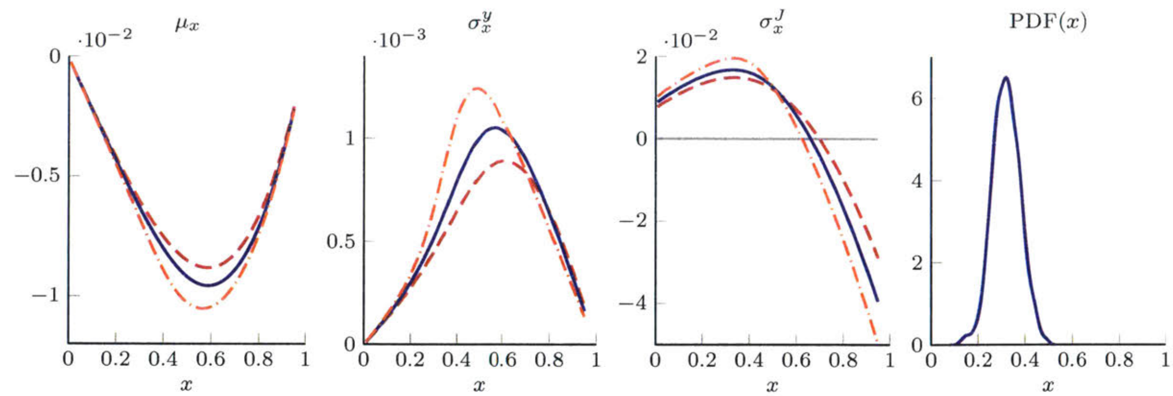

Figure 1-6: **Dynamics of x.** State variable x follows a jump-diffusion process $dx_t = \mu_x\, dt + \sigma_x^y\, dB_t^y + \sigma_x^J\, dJ_t$. Right panel plots kernel density of x obtained using a 10000 years simulation of the model. Solid lines are for y being equal to 0 or its median value, dash and dash-dot are for y being equal to 0.025 and 0.975 quantiles of its stationary distribution.

jump risk via a simple equation

$$\text{Expected DPI} = \frac{1}{1-\pi^J}. \tag{1.50}$$

Lower price of risk results in lower expected return on investments in the innovation sector. When the price of risk equals zero expected DPI equals one. In other words on average for every dollar invested agents receive one dollar back. In the majority of expanding variety or Schumpeterian growth models, that do not feature aggregate uncertainty, the DPI ratio is deterministic and always equals one. In the current setup expected DPI is a decreasing function of x. Innovation sector provides a hedge against disruption, hence expected returns fall when the risk of disruption is higher.

Dynamics of the Economy Now I explore the dynamic properties of the model. The state of the economy is described by one exogenous state variable, growth y, and one endogenous state variable, share of wealth owned by the managers x. Managers' share follows the process

$$dx_t = \mu_{x,t}dt + \sigma^y_{x,t}dB^y_t + \sigma^J_{x,t}dJ_t\,. \tag{1.51}$$

The drift component $\mu_{x,t}dt$ determines expected change of x in absence of innovation waves. The second term represents response of x to shocks in growth y_t. The third term, $\sigma^J_{x,t}dJ_t$, governs change in x upon arrival of the innovation wave. Note that in the above equation I omitted the term $\sigma_{x,t}dB_t$ due to the fact that $\sigma_{x,t} \equiv 0$. The first three panels of Figure 1-6 plot each component of the right hand side of equation (1.51). The rightmost panel plots the probability density function of x obtained by simulating the economy for 10000 years. Quantitatively the drift $\mu_{x,t}dt$ and jump $\sigma^J_{x,t}dJ_t$ components are the key drivers of x_t.

The magnitude of drift $\mu_{x,t}$ is affected, first, by the difference in consumption rates by managers and households and, second, by the difference in investment rates in the innovation sector. The drift is consistently negative mostly due to managers' higher investment in innovation $\theta^M > \theta^H$ that we saw on Figure 1-4 and a higher rate of consumption.

The sign of $\sigma^y_{x,t}$ is consistently positive. It is driven by the fact that managers choose higher loading on dB^y_t shock in their investment portfolio relative to households. As we saw on Figure 1-3 higher y_t corresponds to a higher rate of creative destruction — a state relatively more harmful for managers compared to households. Hence managers hedge against the increase in y_t. Quantitatively this effect is small. One standard deviation shock to dB^y_t leads to increase in x_t by no more than 15 basis points.

The magnitude of jump $\sigma^J_{x,t}$ is influenced, first, by exit of displaced managers and entry of the new ones, and, second, by the difference in wealth exposures to the innovation risk between households and managers. As it follows from Figure 1-4 jump in x is positive when x is below a certain threshold for a given value of y. In this region managers as a group gain wealth relative to households due to taking larger

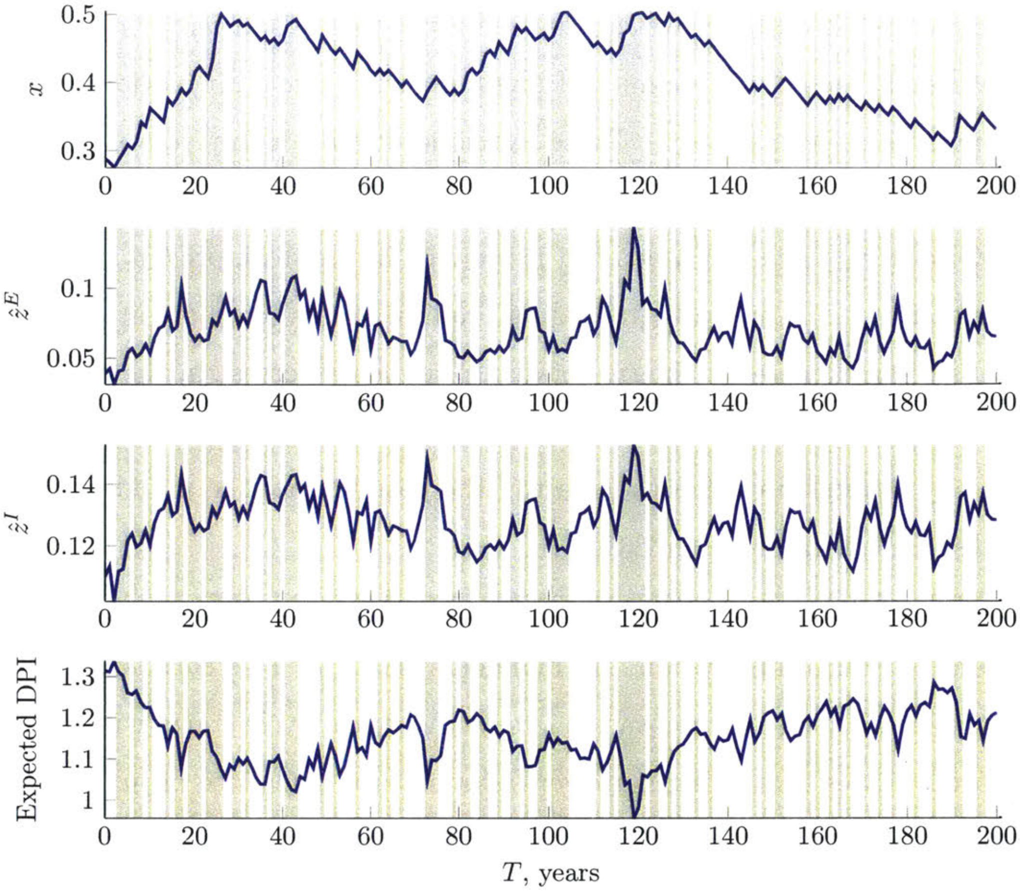

Figure 1-7: **Simulation of the economy for 200 years.** From top to bottom: share of wealth owned by managers, aggregate investment in the innovation sector, investment in radical innovation by incumbents, Expected DPI in the innovation sector. Shaded bars are years with at least on wave arrival.

exposure to jump risk (Figure 1-4). The effect of entry and exit on x is positive for smaller values of x and negative for larger values of x. When x is sufficiently high, $\sigma^J_{x,t}$ is negative and managers as a group own so much wealth that arrival of a wave reduces their share due to exit of wealthy managers and entry of more poor ones. As it follows from the rightmost panel, in equilibrium the economy spends most of the time in the region $x \in (0.1, 0.6)$, where $\sigma^J_{x,t} > 0$.

Figure 1-7 illustrates a simulation of the economy for 200 years. Shaded bars correspond to years with one or more wave arrivals. The first panel shows dynamics of the state variable x. Consistent with the discussion above, managers' share goes up in periods, when innovation waves arrive, and tends to go downwards in their absence. Panels two and three plot dynamics of investment in radical innovation by entrants and incumbents correspondingly. These two series exhibit highly volatile behavior at the frequencies of up to ten years. This volatility is driven by the exogenous shock to growth y. Positive shock to y leads to the increase in the stock price p_t. Higher stock prices make investment in both gradual and radical innovation more attractive. This effect leads to an increase in the risk of disruption and decrease in the price of jump risk π^J and expected return on innovation, which I plot in the last panel. Drop in the price of jump risk amplifies the effect of the positive shock to y and leads to even more investment in radical innovation.

At lower frequencies, of the order of decades, there is a clear comovent between the amount of radical innovation and managers' share x. This correlation gives rise to the endogenous long-run innovation cycles. A typical cycle starts with a sequence of innovation waves that bring managers' share x up. The price of jump risk drops leading to more investment in radical innovation.

Table 1.5: **Innovation and the stock market.** The table presents results of linear regressions of risk measures Y_t on innovation activity IA_t: $Y_t = \text{Const} + \beta\, IA_t + \epsilon_t$. Regressions are based on a 10000 years simulation of the model. Simulation is at the monthly frequency and all variables are aggregated to the annual level. Innovation activity is defined as $\iota + \hat{z}^I + \hat{z}^E$. CS StDev is the cross-sectional standard deviation of annual firm-level stock returns; Exit Rate is the share of firms that go out of business in a given year; Market Volatility is the realized volatility on aggregate stock market in year t.

	Dependent variable:		
	CS StDev	Exit Rate	Market Volatility
IA_t	0.0111	0.0035	0.0004
Mean(Y_t)	0.0677	0.0119	0.1055

Innovation and the Stock Market Time-varying innovation activity in the model economy generates time-varying volatility of both aggregate and individual stock returns. Volatility of the aggregate stock market varies due to time-varying exposure to aggregate shocks. At the individual firm level innovation waves create extra risk

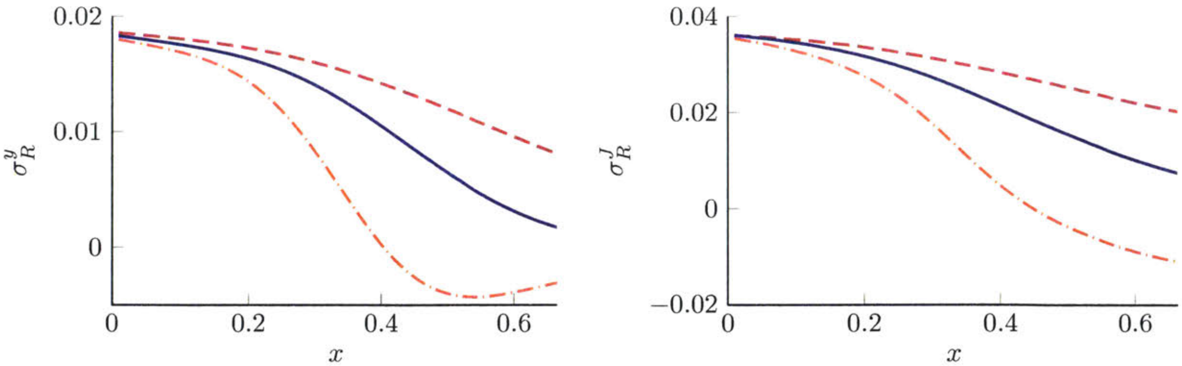

Figure 1-8: **Aggregate stock market exposure to shocks.** Return on the aggregate stock market is given by the jump diffusion $dR_t = \mu_{R,t}dt + \sigma_{R,t}dB_t + \sigma^y_{R,t}dB^y_t + \sigma^J_{R,t}dJ_t$, where $\sigma_{R,t}$ is constant in equilibrium and is equal to the aggregate productivity shock volatility σ. Solid lines are for y being equal to 0 or its median value, dash and dash-dot are for y being equal to 0.025 and 0.975 quantiles of its stationary distribution.

due to their unpredictable heterogeneous effect across firms. In Table 1.5 I present linear regressions of measures of risk on aggregate innovation activity. Regressions are based on a single 10000 years simulation of the economy. I define innovation activity as the sum of all R&D spendings in the economy $IA_t = \iota_t + \hat{z}^I + \hat{z}^E$ and normalize it to have zero mean and unit standard deviation in the sample. Along with the slope coefficients I report mean values of dependent variables.

In the first two columns I report results for two measures of idiosyncratic risk, namely cross-sectional standard deviation of annual returns and share of firms exiting in a given year. One standard deviation increase in IA_t is associated with a large increase in idiosyncratic risk. Cross-sectional standard deviation goes up by 1.1 percentage point or roughly 16% relative to its average value.[18] Exit rate goes up by 35 basis points or 29% of its mean value. The mechanism behind this result is clearly seen on Figure 1-3: both the exit rate $\overline{\omega}^E$ and radical innovation spendings $\hat{z}^E$ and $\hat{z}^I$ increase in each of the state variables x and y. This gives rise to the positive

[18]Increase in cross-sectional standard deviation by 1.1 percentage point might seem to be very mild relative to the empirical counterpart. This is however an artifact of the stylized nature of the model that features a very simple structure of idiosyncratic risk.

correlation between innovation and idiosyncratic risk.

The last column in Table 1.5 shows regression of the aggregate stock market volatility on IA_t. Aggregate volatility is measured as realized volatility of the aggregate market portfolio in a given year. The slope coefficient is very small in absolute value (and statistically insignificant in this simulation), so quantitatively there is virtually no relationship between the two variables. In addition, mean realized volatility is only slightly higher than the volatility of aggregate productivity shock σ. To understand this result better, recall that aggregate stock market return is given by $dR_t = \mu_{R,t}dt + \sigma_{R,t}dB_t + \sigma^y_{R,t}dB^y_t + \sigma^J_{R,t}dJ_t$. In equilibrium it turns out that $\sigma_{R,t}$ does not vary in time and is equal to σ. Exposure to the second Brownian shock $\sigma^y_{R,t}$ is plotted on the left panel of Figure 1-8. Its magnitude relative to σ (which equals 0.1 in my calibration) and time-variation are small. The last contributor to the aggregate stock market volatility is $\sigma^J_R dJ_t$. I plot σ^J_R on the right panel of Figure 1-8. The graph implies that the stock market almost always appreciates with an arrival of the innovation wave. Quantitatively it adds little to the aggregate stock market volatility though: $\overline{h} \times \sigma^J_R$ rarely exceeds 1.5 percentage points. Hence consistent with empirical observations, times of high innovation activity in a substantial manner redistribute market value between the firms while keeping aggregate stock market returns only mildly affected.

Strategic Complementarity The sharp reaction of the investment in the innovation sector to changes in state variables to large extent is driven by the strategic complementarity of managers' decisions to invest in the innovation sector. To abstract from general equilibrium effects I demonstrate this feature by solving the manager's problem (1.42) in a partial equilibrium setting. In particular, I fix the quantities x_t, y_t, p_t, $\sigma^J_{p,t}$, $\sigma^J_{S,t}$, π^J_t, ξ^M_t, ξ^H_t, $\sigma^{J,M}_{\xi,t}$, $\sigma^{J,M}_{\xi,t}$ and solve the individual manager's problem (1.42) for varying values of aggregated investment in the innovation sector by managers as a

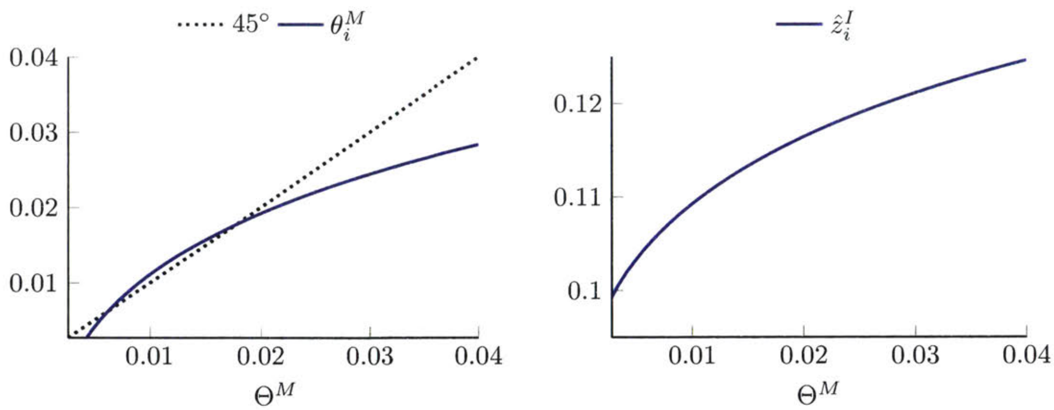

Figure 1-9: **Individual manager's best response to aggregate investment rate in the innovation sector by managers Θ^M.**

group normalized by their wealth.[19] I denote this quantity by Θ_t^M. Investment in the innovation sector by managers feeds into the problem (1.42) through probabilities of successful radical innovation by the incumbent $\overline{\omega}^I(\hat{z}_t^I, \hat{z}_t^E)$ or an entrant $\overline{\omega}^E(\hat{z}_t^I, \hat{z}_t^E)$, where

$$\hat{z}_t^E = \frac{x_t p_t Q_t \Theta_t^M + \int_j \theta_{j,t}^H w_{j,t}^H dj}{Q_t}. \tag{1.52}$$

Note that households' investment rates in the innovation sector $\theta_{j,t}^H$ do not vary with Θ_t^M since they depend on π_t^J and $\sigma_{\xi,t}^J$ only (see formula 1.41). Figure 1-9 plots best responses of an individual manager to the value of Θ_t^M. Both $\theta_{i,t}^M$ and $\hat{z}_{i,t}^I$ increase in Θ^M. Higher Θ^M implies a higher probability of disruption. Hence the manager allocates more resources both within the firm and through portfolio choice to hedge this risk. Positive response of $\theta_{i,t}^M$ to Θ_t^M represents a strategic complementarity. This relationship gives rise to the high volatility of the innovation activity in the economy. For example, increase in expected growth y_t leads to higher stock prices. Higher stock prices attract more investment in innovation. As a result, managers increase portfolio allocations to the innovation sector to both take advantage of the higher price and in attempt to hedge the higher risk of disruption.

[19] I set x_t and y_t to their corresponding median values, the other variables are set to the values obtained in the general equilibrium given median values of x_t and y_t.

Innovation by entrants unambiguously reduces managers' welfare. Hence, apart from generating high volatility, this complementarity exacerbates the business stealing effect of disruptive innovation. Managers hedging behavior as a group accelerates the process of creative destruction and reduces their welfare.

The Effect of Lumpiness of the Innovation Process Unlike the majority of economic growth models with expanding varieties or quality ladders, this model features lumpy arrival of innovations that make investments in radical innovation risky at the aggregate. This feature allows me to study expected returns on investment in the innovation sector and has significant impact on the intensity and time-variation of the innovation process. In this section I do comparative statics with respect to the frequency of innovation wave arrivals $\overline{h}$. To isolate the effect of wave arrival intensity from the productivity of innovation technology I modify the investment technology (1.10c–1.10d) according to:

$$\overline{\omega}^E_{i,t} = \frac{h_0}{\overline{h}} \times \omega^E_{i,t} \left(1 - \omega^I_{i,t} s\right) \ , \tag{1.53a}$$

$$\overline{\omega}^I_{i,t} = \frac{h_0}{\overline{h}} \times \omega^I_{i,t} \left(1 - \omega^E_{i,t}(1-s)\right) \ , \tag{1.53b}$$

where $\omega^I_{i,t}$ and $\omega^E_{i,t}$ are still defined by (1.10a–1.10b). Here h_0 is a fixed constant that I set to be equal to 0.4 (on average waves arrive every 2.5 years), the value of $\overline{h}$ in the baseline calibration. This specification possesses a couple of properties which are important for the comparative statics exercise. First, when $\overline{h} = h_0$ this technology coincides with the original one, specified by equations (1.10a–1.10d). Second, keeping h_0 constant and increasing $\overline{h}$ leads to an increase in the frequency of aggregate wave arrivals. However, probability of successful innovation in an instant of time dt by either incumbent or an entrant remains unchanged, provided investment rates $\hat{z}^I$ and $\hat{z}^E$ are kept constant. This means that higher $\overline{h}$ increases intensity of aggregate innovation arrivals and proportionally reduces share of firms that succeed in radical innovation

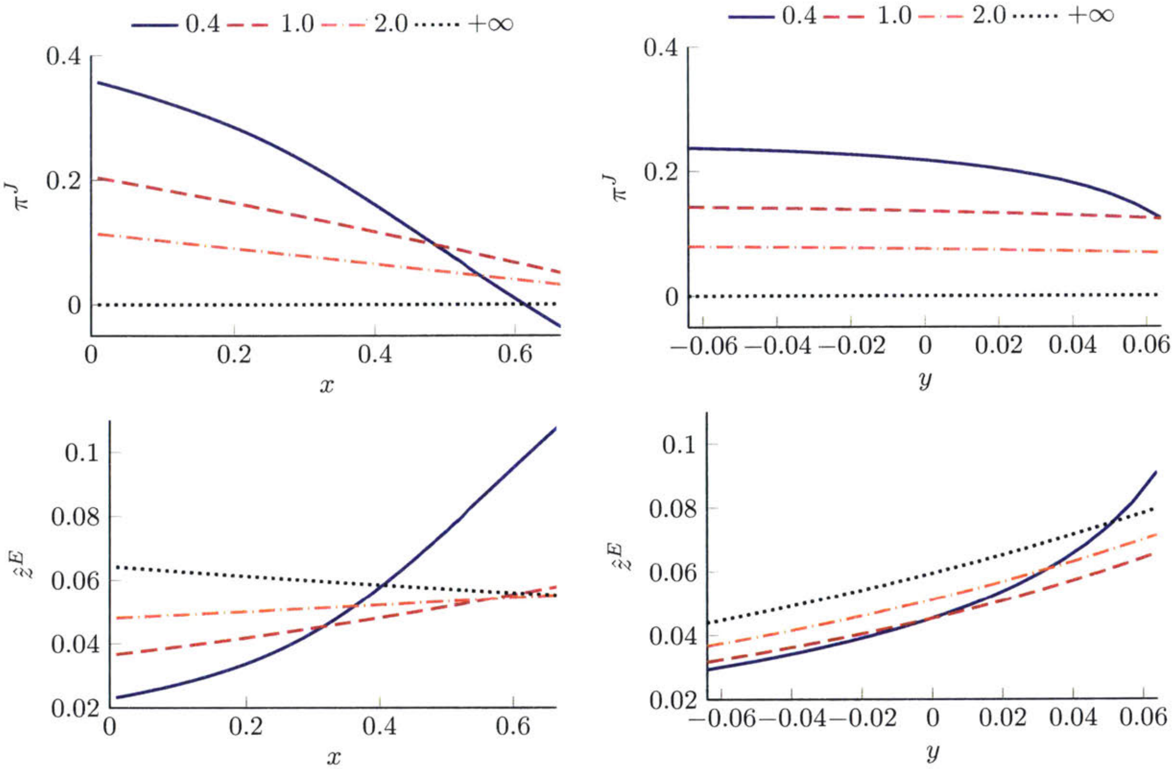

Figure 1-10: **Comparative statics with respect to jump intensity $\overline{h}$.** Upper row: price of jump risk π^J; lower row: investment in the radical innovation by entrants $\hat{z}^E$. Left column plots variables as functions of managers' share x keeping state variable y constant and equal to 0, its median value. Right column plots variables as functions of exogenous component in expected growth y keeping x constant and equal to its median value in the baseline calibration. Solid line is for $\overline{h} = 0.4$, dashed for $\overline{h} = 1$, dash-dot for $\overline{h} = 2$, dotted for $\overline{h} = +\infty$.

$\overline{\omega}^E_{i,t}$ or are replaced by entrants $\overline{\omega}^I_{i,t}$. In the limiting case $\overline{h} \to +\infty$ this technology effectively converges to the one with idiosyncratic arrivals of radical innovations at the firm level, a common assumption in the models of economic growth.[20]

Figure 1-10 plots equilibrium quantities in (i) the baseline economy with $\overline{h} = 0.4$ (solid line), (ii) economy with waves arriving on average once a year (dashed line), (iii) waves arriving on average twice a year (dash-dot line) and (iv) infinite arrival intensity (dotted line). The upper panel plots the price of jump risk π^J. The profile of π^J has

[20]Chapters 13 and 14 in Acemoglu (2008) provide a good overview.

the steepest (negative) slope for $\overline{h} = 0.4$ both as a function of managers' share x and as a function of exogenous component of expected growth y. The slope gradually declines with $\overline{h}$. With lower rate of wave arrival the innovation sector provides a good hedge for managers in case of disruption to their business. When waves are more frequent share of firms disrupted in each wave goes down. This makes aggregate jumps less correlated with displacement of any given manager. Hence exposure to the jump risk becomes a worse hedge, and managers' demand for jump exposure goes down. The bottom row plots investment in R&D by entrants $\hat{z}^E$. Behavior of $\hat{z}^E$ effectively mirrors the profiles of π^J: the slope flattens out for higher values of $\overline{h}$. In the $\overline{h} \to +\infty$ case, when the price of jump risk equals zero, the slope of $\hat{z}^E$ as a function of x even turns negative. This effect is driven by the positive dependence of the risk-free rate on x.[21]

The Effect of Non-diversifiable Risk On Figure 1-11 I compare solutions of the model for different values of non-diversifiable share of managers' wealth ϕ. I plot the price of jump risk in the upper panel and investment in R&D by entrants in the lower one. In both panels I plot variables as functions of each state variable x and y while keeping the remaining state variable at its median value in the baseline calibration with $\phi = 0.4$. The price of jump risk π^J is a decreasing function of managers' share x. The absolute value of its slope, however, sharply goes down for lower ϕ. This effect is reflected in a weaker response of $\hat{z}^E$ to changes in x observed in the lower panel. For lower values of ϕ managers are less worried about the risk of disruption. Hence as a group, they demand lower amount of insurance against displacement. This effect makes their portfolio look more like the portfolio chosen by households and dependence of π^J and $\hat{z}^E$ on x becomes milder.

As can be seen on the upper left graph of Figure 1-11, in the baseline calibration π^J is a decreasing function of managers' share. Higher expected growth y implies

[21]When EIS is higher than one, presence of idiosyncratic risk makes managers less patient relative to households. Hence risk-free rate is an increasing function of x

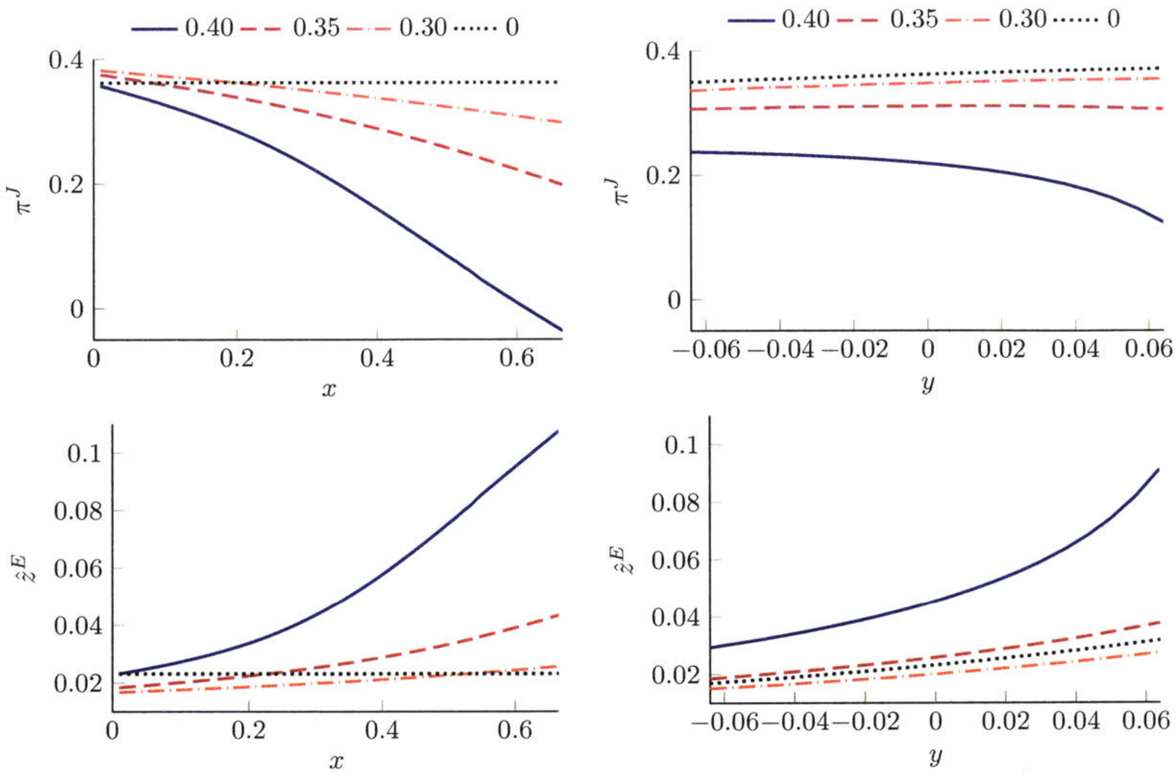

Figure 1-11: **Comparative statics with respect to non-diversifiable share ϕ.** Upper row: price of jump risk π^J; lower row: investment in the radical innovation by entrants $\hat{z}^E$. Left column plots variables as functions of managers' share x keeping state variable y constant and equal to 0, its median value. Right column plots variables as functions of exogenous component in expected growth y keeping x constant and equal to its median value in the baseline calibration. Solid line is for $\phi = 0.4$, dashed for $\phi = 0.35$, dash-dot for $\phi = 3$, dotted for $\phi = 0$.

higher prices on stocks. High prices stimulate entry of new firms and more demand for insurance against disruption as a result. For lower values of ϕ the dependence of π^J on y turns from negative to positive. The easiest way to understand this result is to consider $\phi = 0$ case, when managers are effectively identical to households. Arrival of an innovation wave is good news for all agents in the economy, since the wave brings more value than destroys. Higher y implies higher stock prices in the EIS > 1 case and arrival of a wave brings proportionally more value, so the price of jump risk becomes even higher.

In summary, comparative statics with respect to $\overline{h}$ and ϕ have shown that both lumpy arrival of innovations and lack of diversification are responsible for the economy's sharpe response to changes in state variables.

1.6 Puzzles in Venture Capital

In this section I contrast predictions of the model with the data on the venture capital industry, a close empirical counterpart of the innovation sector in the model. The data on fund flows and financial performance of venture capital funds allows me to investigate the model's ability to explain jointly the asset pricing properties and the volumes of innovation activity.

Empirical literature has documented a set of stylized facts related to the venture capital activity.[22] Some of these facts remain to be puzzling for classical theories. I will look at these empirical observations through the lense of my model to both test the model's implications and give potential explanations to the pervasive patterns.

Measuring Financial Performance in VC Longer investment horizons and unobservability of portfolio values in intermediate periods complicate measuring performance of investments in venture capital. Hence approaches to measuring performance in VC differ from the methods used for publicly traded securities. Below I give a brief overview and definitions of the measures I use in my analysis.

In VC, the three most intuitive and popular among researchers and practitioners measures of investment performance are (i) internal rate of return or IRR, (ii) public market equivalent or PME, and (iii) investment multiples, such as distributions to paid-in capital (DPI) and the total value to paid-in capital (TVPI). I focus on TVPI and the PME measure of Kaplan and Schoar (2005). These two measures can be

[22] Empirical studies have uncovered stylized facts and patterns both at the aggregate industry and individual fund level. Since my model treats the innovation sector as a centralized intermediary I will focus on the aggregate industry level evidence only.

defined in my model with minimal additional assumptions which I describe in detail below.[23]

For a VC fund raised in a given year TVPI is defined by

$$\text{TVPI} = \frac{\text{Distributions} + \text{Residual value}}{\text{Paid-in capital}}. \tag{1.54}$$

Paid-in capital is the total amount of funds that have been called and invested by general partners during the fund's life. Distributions is the total amount of funds that has been paid back to investors. Residual value is the value of assets still in the fund's portfolio. By the end of fund's life residual value equals zero and all equity is paid back to limited partners. A typical lifetime of a VC fund is ten years. Capital is usually called during the first five years, followed by five years of divestment. To define TVPI on VC sector in the model I assume, first, that capital is invested uniformly in time during the first five years of funds' life. Second, conditional on succesful innovation, new firms are held in the portfolio for five more years. For a cohort of funds of vintage t the final pooled TVPI multiple thus equals

$$\text{TVPI}_t = \frac{\text{Distributions}}{\text{Paid-in capital}} = \frac{\int_t^{t+5} P_u \times \frac{R_{u+5}}{R_u} \times dJ_u}{\int_t^{t+5} 1 \times du}. \tag{1.55}$$

In the formula above, P_u is the payoff on investment in the innovation sector at a rate of one dollar a year, at time u, conditional on wave arrival

$$P_u = \frac{1}{(1 - \pi_u^J)\,\overline{h}}. \tag{1.56}$$

The multiple R_{u+5}/R_u is the five-year return on the stock market. It accounts for the five year holding period of the portfolio companies. Note that TVPI multiple ignores an important notion of the time value of money. PME measure suggests a way to

[23] For fully divested funds DPI coincides with TVPI. IRR measure is not defined for some realizations of the cashflow stream. One such scenario is a cashflow with positive capital calls and zero realized distributions, a scenario that occasionally happens in the model.

address this issue.

For a given fund Kaplan and Schoar (2005) define PME as the ratio of the sum of discounted distributions to the sum of discounted capital calls. The discount rate between dates u and s is the gross realized return on a benchmark portfolio between dates u and s. PME for an investment in the benchmark at any horizon always equals one. In practice most popular benchmarks are portfolios of broad equity indices such as S&P500 or Russell indices. A natural benchmark index in my model is the aggregate portfolio of incumbent firms. Hence to define PME in the model I discount cash flows in the nominator and denominator of formula (1.55) to date t by the stock market return

$$\mathrm{PME}_t = \frac{\text{Discounted distributions}}{\text{Discounted paid-in capital}} = \frac{\int_t^{t+5} P_u \frac{R_{u+5}}{R_u} \frac{R_t}{R_{u+5}} dJ_u}{\int_t^{t+5} 1 \times \frac{R_t}{R_u} du} = \frac{\int_t^{t+5} P_u \frac{R_t}{R_u} dJ_u}{\int_t^{t+5} \frac{R_t}{R_u} du}. \tag{1.57}$$

PME has an intuitive interpretation: $(\mathrm{PME}_t - 1) \times 100$ corresponds to the percentage point outperformance of the VC funds of vintage t relative to the identically timed investment in the stock market.

Fund Flows and Performance: Level and Volatility In the first three rows of Table 1.6 I report summary statistics of annual flows into venture capital in the US normalized by aggregate earnings of public corporations. The data on flows comes from VentureXpert by Thomson and data on earnings is from Compustat. First, the table shows that annual flows into VC are sizable: in 1993–2010 on average they corresponded to about 0.07 of corporate earnings.[24] Second, fund flows into venture capital are highly volatile. This feature of venture capital has been widely documented in the empirical literature,[25] I report two measures of volatility: the standard deviation

[24]Data on VC flows and performance in my sample is available starting from 1984. Due to rapid growth of the industry and data collection efforts VC flows series exhibits highly non-stationary behavior in 80s and early 90s.

[25]See Gompers and Lerner (2003), Gompers and Lerner (2004), Kaplan and Schoar (2005) among others.

Table 1.6: **Venture capital flows and performance summary.** The table reports summary statistics of fund flows into VC, TVPI multiple and PME. Mean is average value, SD is standard deviation, IQR is interquartile range. In the data flows are defined as aggregate VC funds commitments in the US in a given year, source: VentureXpert by Thomson. Earnings are the total annual earnings of public firms, source: Compustat. TVPI is total value to paid-in multiple of the funds raised in a given year, PME is public market equivalent of Kaplan and Schoar (2005). Both performance measures are calculated by Brown et al. (2015) using Burgiss data. Standard errors for mean and standard deviation are Newey-West and standard errors for IQR are based on block bootstrap. The data is annual, 1993–2010. The model counterparts are obtained in a 18000 year simulation of the model.

	Data		Model
	Estimate	SE	
Mean(Flows/Earnings)	0.072	(0.020)	0.060
SD(Flows/Earnings)	0.057	(0.015)	0.017
IQR(Flows/Earnings)	0.039	(0.035)	0.019
Mean(TVPI)	2.623	(0.808)	2.301
SD(TVPI)	2.105	(0.498)	1.731
IQR(TVPI)	1.940	(1.564)	2.171
Mean(PME)	1.641	(0.414)	1.254
SD(PME)	1.069	(0.232)	0.889
IQR(PME)	1.330	(0.756)	1.220

and the interquartile range to make the picture more complete given small sample size. Large ratio of the standard deviation to IQR in the data[26] indicates a significant impact of extremely high fund flows observed in 1999–2001.

The third column in Table 1.6 reports the model counterparts of the summary statistics. These values are obtained in a 18000 years simulation of the model. The simulation is at the monthly frequency. In the model I define flows by the instantaneous value of $\hat{z}_t^E$ and earnings by the instantaneous value $(a - \iota_t - \hat{z}_t^I)$.[27] By virtue of the choice of parameter χ^E the model matches well the average amount of flows into the

[26]For a normally distributed random variable this ratio is 0.74.

[27]Note that instantaneous value of $\hat{z}_t^E$ does not exactly represent fund commitments, which in practice are drawn down and invested over the course of several years. However, in my analysis I find this measure to be a better proxy for commitments compared to, for example, realized investments in the innovation sector over five years. Similar to fund commitments, the former measure is not forward looking (while the latter is) which is very important in forecasting regressions I conduct below.

innovation sector. The model seems to understate volatility of fund inflows which is not targeted directly in the calibration. The standard deviation of the fund flows is 0.017 in the model compared to 0.057 in the data. The IQR is 0.019 compared to 0.039 observed in the data.

First two columns in the middle and bottom panels of Table 1.6 report summary statistics of returns on investment in VC. TVPI and PME are the capital-weighted performance measures for North Americal venture funds calculated and reported by Brown et al. (2015). Mean value of TVPI in the sample equals 2.6 with the standard deviation of 2.1 and the IQR of 1.9. These numbers show that even at the aggregate industry level investments in VC are extremely risky. Recall that these numbers correspond to an investment at the ten year horizon. The mean value of PME equals 1.64. This indicates that on average a VC fund outperformed the stock market by 64 percentage points throughout its lifetime. Average value of PME is lower than TVPI since it makes an adjustment for the time value of money and a crude adjustment for the market risk.

The third column in Table 1.6 reports the model implied counterparts of TVPI and PME. The model suggests slightly lower average returns on VC as measured by PME and TVPI, although the measures lie well within the range of one standard error of empirical estimates. Remarkably, the model matches well the volatility of VC returns. Both standard deviation and IQR are very close to empirical estimates. This result implies that even a stylized way of modelling the aggregate risk of innovation, via pure jumps, delivers rich and realistic properties of returns.

Fund Flows and Performance: Predictability. A range of empirical studies find strong relationship between capital commitments and financial returns in the VC. Gompers and Lerner (2000), Kaplan and Schoar (2005), Harris et al. (2014), Robinson and Sensoy (2016), among others, find that aggregate volume of VC commitments in a given year negatively correlate with performance of the VC funds raised in that year.

Table 1.7: **Fund flows and subsequent performance.** The table reports slope coefficients in linear regressions of VC performance measures Y_t on fund flows: $Y_t = \text{Const} + \beta \log(\text{Flows/Earnings})_t + \gamma t + \epsilon_t$. Standard errors are Newey-West. The data is annual, 1993–2010. The model counterparts are obtained in a 18000 year simulation of the model.

	Data		Model
Dependent variable	Estimate	SE	
TVPI	−1.950	(0.421)	−1.565
PME	−0.977	(0.154)	−0.888

Gompers and Lerner (1998), Kaplan and Schoar (2005), Harris et al. (2014) show that high aggregate returns in VC are followed by more capital inflows.

In the first two columns of Table 1.7 I report slope coefficients in OLS regressions of performance measures on the log of commitments to corporate earnings ratio. I include a linear time trend in these regressions to focus on cyclical fluctuations of investment activity. Consistent with previous studies I find a strong negative relationship between vintage fund flows and performance of funds of that vintage. A ten percent increase in fund flows corresponds to an average decrease in PME by 0.1, a 15% percent drop in excess return of VC over the stock market.[28]

I run the same regressions in the simulated data (without a time trend) and report the results in the last column of Table 1.7. The model reproduces the data very well. PME and TVPI are forecasted with the correct sign and magnitudes lie very close to the empirical estimates. In the model, the price of innovation risk π^J is the key determinant of expected returns on innovation. Times of high innovation activity are associated with a higher rate of disruption. In these circumstances managers are willing to pay more for a financial hedge against it and drive the risk premium on the innovation sector down. This effect directly translates into a decrease in expected returns on innovation and lower average TVPI and PME. The described mechanism is consistent with Gompers and Lerner (2000) who find that *"inflows of capital into*

[28]Average PME of 1.641 implies an excess return of 64.1 percentage points, $0.0977/0.641 \approx 15\%$.

venture funds increase the valuation of these fundsâĂŹ new investments... Changes in valuations do not appear related to the ultimate success of these firms. The findings are consistent with competition for a limited number of attractive investments being responsible for rising prices."

Table 1.8: **VC performance and subsequent flows.** The table reports slope coefficients in linear regressions of VC fund flows on lagged performance measured by PME: $\log(\text{Flows}/\text{Earnings})_t = \text{Const} + \beta\,\text{PME}_{t-k} + \gamma t + \epsilon_t$ for k up to ten years. Standard errors are Newey-West. The data is annual, 1984–2010. The model counterparts are obtained in a 18000 year simulation of the model.

	Data		Model
	Estimate	SE	
Lag = 1	0.024	(0.095)	−0.016
Lag = 2	0.216	(0.101)	−0.002
Lag = 3	0.413	(0.100)	0.009
Lag = 4	0.582	(0.065)	0.020
Lag = 5	0.584	(0.084)	0.029
Lag = 6	0.471	(0.159)	0.020
Lag = 7	0.310	(0.169)	0.014
Lag = 8	0.166	(0.121)	0.013
Lag = 9	0.097	(0.110)	0.004
Lag = 10	0.090	(0.094)	0.000

Table 1.8 reports results of the regressions of fund flows on lagged aggregated VC performance. I report results for PME only to save space, but the conclusions are robust to using TVPI or other measures of financial performance. I extend the sample backward up to 1984 to have enough data points to conduct a meaningful analysis for the lags of up to ten years. Estimates in the first column exhibit a clear hump-shaped dependence between performance of funds of past vintages and capital commitments to VC. Quantitatively the relationship is strong: a ten percentage point increase in PME of vintage $t-5$ corresponds to the increase in inflows by about six per cent. I repeat the analysis using the data simulated from the model and report the results in the last column of Table 1.8. The model generates a similar hump-shaped pattern.

In the model past performance is related to the fund flows largely due to the time variation in managers' wealth share x. High returns on the innovation sector are driven by a sequence of innovation wave arrivals. Each arrival leads to an upward jump in the state variable x. As we saw in the previous sections demand for exposure to jumps is higher for managers. Hence flows into the innovation sector increase with x. This effect generates a positive response of $\hat{z}^E$ observed in the last column of Table 1.8. Quantitatively this effect appears to be not strong enough to match the data. Alternative theories, based on additional assumptions (e.g learning, slowly-moving capital or sentiment) have a potential to better match the pattern quantitatively.

Overall the model shows a good ability to replicate the empirical stylized facts qualitatively and in most cases quantitatively. Given the stylized structure of the risk of innovation the model delivers rich and realistic predictions on the distribution of aggregate returns in venture capital.

1.7 Evidence from the Cross-Section of Stocks

In this section I test asset pricing predictions of the model using the panel of publicly traded stocks. Using CRSP data I form portfolios that are likely to be heterogeneously exposed to the technological progress and test two predictions of the model. First, the model predicts a positive comovement between unexpected changes of idiosyncratic risk (driven by technological innovation) and returns on stocks that are likely to benefit from innovation. Second, I test if returns on these portfolios are predictable by the level of innovation activity in the economy and if predictability varies across the portfolios. The model implies that returns on stocks, that are more likely to appreciate with the arrival of a technological breakthrough, are lower in times of high innovation activity.

Every month I sort firms into five portfolios based on the number of years the firms have been publicly traded. Portfolio 1 includes stocks of the youngest firms and

Table 1.9: **Comovement of returns and idiosyncratic risk.** The table presents results of linear regressions of value-weighted excess returns on shocks to idiosyncratic volatility controlling for market return. Portfolios are formed monthly based on the number of years since a firm has been publicly listed. Shocks to idiosyncratic risk are defined as the difference between the cross-sectional standard deviation of returns and one-month lagged 12-month moving average value. Monthly data, July 1951 – December 2017. HAC standard errors in parentheses. One, two and three stars denote significance at the 10, 5 and 1% level.

	Portfolio:					
	1	2	3	4	5	$1-5$
CS StDev_t	0.008***	0.005***	0.004***	0.003***	−0.003***	0.011***
	(0.002)	(0.001)	(0.001)	(0.001)	(0.001)	(0.003)
$R_t^M - r_t^f$	1.183***	1.146***	1.142***	1.079***	0.915***	0.268***
	(0.043)	(0.032)	(0.019)	(0.018)	(0.017)	(0.059)
R^2	0.840	0.879	0.904	0.922	0.939	0.266

portfolio 5 — stocks of the oldest.[29] Younger firms have higher valuations, higher R&D-to-revenue ratios, and often have high growth prospects. Older firms generally have more established business strategy with limited growth potential and higher exposure to the negative effects of disruption. To test the first hypothesis, outlined in the previous paragraph, I run a regression of monthly value-weighted excess returns on these portfolios against the shock to idiosyncratic risk. I measure idiosyncratic risk by the cross-sectional standard deviation of returns. The shock is defined as the difference between the cross-sectional standard deviation and its one-month lagged 12-month average value. For easier interpretation, I standardize the shock to have unit variance. I control for the contemporaneous market excess return to ensure the results are not driven by heterogeneity in market beta and the comovement between idiosyncratic risk with the aggregate market.

Table 1.9 reports the results. Consistent with the predictions of the model, portfolios exhibit heterogenous comovement with shocks to idiosyncratic volatility. Portfolio 1, that contains the youngest firms, is significantly positively loaded on shocks to idiosyncratic risk. The slope coefficient monotonically declines from portfolio 1 to

[29] The actual firm age is difficult to measure. Years since IPO is a proxy for firm age. IPO date is defined either by the date reported in Compustat or the date of first observation in CRSP.

Table 1.10: **Forecasting returns by innovation activity.** The table presents results of forecasting regressions of one-year value-weighted excess returns by intensity of innovation activity. Portfolios are formed based on the number of years since a firm has been publicly listed. Innovation activity is defined as the difference between the R&D expenses to GDP ratio and its one-year lagged 10-year moving average value. Annual data, 1964 - 2017. HAC standard errors in parentheses. One, two and three stars denote significance at the 10, 5 and 1% level.

	Portfolio:					
	1	2	3	4	5	$1-5$
IA_{t-1}	-0.071^{**}	-0.066^{***}	-0.044^{***}	-0.021	-0.023	-0.048^{***}
	(0.028)	(0.025)	(0.016)	(0.023)	(0.019)	(0.017)
R^2	0.071	0.083	0.040	0.010	0.022	0.084

portfolio 5. For the "1–5" long-short portfolio the loading on the shock to idiosyncratic risk is both statistically and economically significant: one standard deviation shock to idiosyncratic volatility is associated with 1.1% monthly return spread between portfolios one and five.

Table 1.10 reports results of forecasting regressions of portfolios' excess returns by innovation activity. In every year t monthly portfolio returns are aggregated to the annual level. I define the forecasting variable for returns realized in year t as the difference in R&D-to-GDP ratio in year $t-1$ and the one-year lag of its 10-year moving average value. The forecasting variable is standardized to have unit variance. According to Table 1.10, innovation activity significantly negatively forecasts returns of portfolios of young firms. One standard deviation increase in innovation activity (relative to the 10-year moving average value) is associated with about seven percentage point decline in subsequent one-year return on portfolio 1. The predictability pattern almost monotonically declines from portfolio 1 to portfolio 5. The "1–5" spread is significantly negatively related to the lagged innovation activity. This result is consistent with the model's prediction that the risk-premium on assets, positively correlated with innovation shocks, is lower in times of higher innovation activity.

1.8 Conclusion

In this paper I proposed a model of endogenous innovation with an important role for risk and asset prices. I showed that risky nature of returns on innovation can be responsible for fluctuations in spendings on innovation. When contrasted with the data on venture capital the model showed a good ability to replicate the stylized facts on VC performance. These results demonstrate that disruption risk considerations can have far reaching consequences for the VC sector and innovation more broadly. Exploring the implications of other salient features of the investments in innovation, such as lack of liquidity, longer investment horizons or learning about new technologies, in an equilibrium framework, can be a fruitful direction of further research in an attempt to better understand the asset pricing properties of innovation.

1.9 Appendix

1.9.1 Empirical Appendix

Table 1.11: **Innovation activity and portfolio-level idiosyncratic risk.** The table presents results of linear regressions of measures Y_t of idiosyncratic risk on innovation activity IA_t: $Y_t = \text{Const} + \beta IA_t + \gamma_1 \text{Output Gap}_t + \gamma_2 t + \epsilon_t$. Innovation activity is measured as the detrended ratio of economy-wide R&D-to-GDP ratio normalized to have zero mean and unit variance. Output gap is normalized to unit variance. CS StDev is cross-sectional standard deviation of annual returns on wealth, Quantile Range is the difference between 95%-tile and 5%-tile of annual return on wealth, CS StDev^U /CS StDev^D are upside/downside standard deviations of return on wealth. Dependent variables are calculated using the data on 400 members of the Forbes 400 list. HAC standard errors in parentheses. One, two and three stars denote significance at the 10, 5 and 1% level. Annual observations, 1983–2013.

	Dependent variable:			
	CS StDev	Quantile Range	CS StDev^U	CS StDev^D
IA_t	0.042***	0.141***	0.018	0.074**
	(0.013)	(0.042)	(0.012)	(0.037)
Output Gap	−0.012	−0.042	0.009	−0.036
	(0.012)	(0.027)	(0.012)	(0.027)
Mean(Y_t)	0.247	0.743	0.273	0.228
R^2	0.468	0.484	0.203	0.362

Data Sources and Definitions Data on VC fund flows comes from VentureXpert by Thomson. I focus on domestic VC funds. Annual fund inflows are measured by the amount raised in a given year. Data on fund performance, PME and TVPI, comes from Brown et al. (2015). I use capital-weighted series calculated by Brown et al. (2015) using Burgiss data for North American funds.

Data on corporate earnings and corporate R&D spending comes from Compustat. I restrict my sample to US firms with non-missing observations of R&D spending and EBIT (earnings before interest and tax). When calculating aggregate R&D/Earnings and Flows/Earnings ratio I use EBIT measure for Earnings. This measure is more stable compared to earnings after tax and interest, which can be close to zero or even negative, which makes empirical moments of ratios R&D/Earnings and Flows/Earnings meaningless.

In section "Evidence from the Cross-Section of Stocks" I consider portfolios of firms that are both in Compustat and CRSP. Years since IPO are determined using the IPO date reported in Compustat, or the earliest observation of a given firm in CRSP.

1.9.2 Hamilton-Jacobi-Bellman Equations

Households I will suppress the subscript for time. Recall that a household's value function is given by $U = (\xi^H w^H)^{1-\gamma}/(1-\gamma)$. Ito's lemma applied to dU and equation (1.18) imply

$$\begin{aligned} dU = {} & U'_\xi \xi^H (\mu_\xi^H dt + \sigma_\xi^H dB + \sigma_\xi^{y,H} dB^y) + U'_w w^H \left((\mu_w^H - \hat{c}^H)dt + \sigma_w^H dB + \sigma_w^{y,H} dB^y\right) \\ & + \frac{1}{2} U''_{\xi\xi} \left(\xi^H\right)^2 \left(\left(\sigma_\xi^H\right)^2 + \left(\sigma_\xi^{y,H}\right)^2\right) dt + \frac{1}{2} U''_{ww} \left(w^H\right)^2 \left(\left(\sigma_w^H\right)^2 + (\sigma_w^{y,H})^2\right) dt \\ & + U''_{\xi w} \xi^H w^H \left(\sigma_\xi^H \sigma_w^H + \sigma_\xi^{y,H} \sigma_w^{y,H}\right) dt \\ & + \frac{(\xi^H w^H)^{1-\gamma}}{1-\gamma} \times \left(\left[\left(1 + \sigma_\xi^{J,H}\right)\left(1 + \sigma_w^{J,H}\right)\right]^{1-\gamma} - 1\right) dJ, \end{aligned} \tag{1.58}$$

where $\mu_w^H = r + \sigma_w^H \pi + \sigma_w^{y,H} \pi^y + \sigma_w^{J,H} (\pi^J - 1)\overline{h}$. The HJB is then given by

$$\begin{aligned} 0 = \max_{\substack{c^H \geq 0,\, \sigma_w^H \\ \sigma_w^{y,H},\, \sigma_w^{J,H}}} \Bigg\{ & f(c^H, U) + U'_\xi \xi^H \mu_\xi^H + U'_w w^H \left(r + \sigma_w^H \pi + \sigma_w^{y,H} \pi^y + \sigma_w^{J,H} (\pi^J - 1)\overline{h} - \hat{c}^H\right) \\ & + \frac{1}{2} U''_{\xi\xi} \left(\xi^H\right)^2 \left(\left(\sigma_\xi^H\right)^2 + \left(\sigma_\xi^{y,H}\right)^2\right) + \frac{1}{2} U''_{ww} \left(w^H\right)^2 \left(\left(\sigma_w^H\right)^2 + (\sigma_w^{y,H})^2\right) \\ & + U''_{\xi w} \xi^H w^H \left(\sigma_\xi^H \sigma_w^H + \sigma_\xi^{y,H} \sigma_w^{y,H}\right) \\ & + \frac{\left(\xi^H w^H\right)^{1-\gamma}}{1-\gamma} \times \left(\left[\left(1 + \sigma_\xi^{J,H}\right)\left(1 + \sigma_w^{J,H}\right)\right]^{1-\gamma} - 1\right) \overline{h} \Bigg\}. \end{aligned} \tag{1.59}$$

Substituting functional forms for $f(c,U)$ and partial derivatives of U in the above equation yields

$$0 = \max_{\substack{\hat{c}^H \geq 0, \sigma_w^H \\ \sigma_w^{y,H}, \sigma_w^{J,H}}} \left\{ \frac{\rho}{1-\psi} \left[\left(\frac{\hat{c}^H}{\xi^H} \right)^{1-\psi} - 1 \right] + \mu_\xi^H + \left(r + \sigma_w^H \pi + \sigma_w^{y,H} \pi^y + \sigma_w^{J,H} (\pi^J - 1) \overline{h} - \hat{c}^H \right) \right.$$
$$- \frac{\gamma}{2} \left(\left(\sigma_\xi^H \right)^2 + \left(\sigma_\xi^{y,H} \right)^2 + \left(\sigma_w^H \right)^2 + \left(\sigma_w^{y,H} \right)^2 - 2 \frac{1-\gamma}{\gamma} \left(\sigma_\xi^H \sigma_w^H + \sigma_\xi^{y,H} \sigma_w^{y,H} \right) \right)$$
$$\left. + \frac{\overline{h}}{1-\gamma} \times \left(\left[\left(1 + \sigma_\xi^{J,H} \right) \left(1 + \sigma_w^{J,H} \right) \right]^{1-\gamma} - 1 \right) \right\}. \quad (1.60)$$

Managers As before I will suppress the subscript for time. Following the same steps as for households I obtain

$$0 = \max_{\substack{\hat{c}^M \geq 0, \tilde{\sigma}_w^M, \tilde{\sigma}_w^{y,M} \\ \tilde{\sigma}_w^{J,M}, \iota_i, \hat{z}_i^I}} \left\{ \frac{\rho}{1-\psi} \left[\left(\frac{\hat{c}^M}{\xi^M} \right)^{1-\psi} - 1 \right] + \mu_\xi^M \right.$$
$$+ \left(r + (1-\phi) \left(\tilde{\sigma}_w^M \pi + \tilde{\sigma}_w^{y,M} \pi^y + \tilde{\sigma}_w^{J,M} (\pi^J - 1) \overline{h} \right) + \phi(\mu_R - r) - \hat{c}^M \right)$$
$$- \frac{\gamma}{2} \left(\left(\sigma_\xi^M \right)^2 + \left(\sigma_\xi^{y,M} \right)^2 + \left(\sigma_w^M \right)^2 + \left(\sigma_w^{y,M} \right)^2 - 2 \frac{1-\gamma}{\gamma} \left(\sigma_\xi^M \sigma_w^M + \sigma_\xi^{y,M} \sigma_w^{y,M} \right) \right)$$
$$+ \frac{\overline{h}(1 - \overline{\omega}^I - \overline{\omega}^E)}{1-\gamma} \times \left(\left[\left(1 + \sigma_\xi^{J,M} \right) \left(1 + (1-\phi) \tilde{\sigma}_w^{J,M} + \phi \sigma_p^J \right) \right]^{1-\gamma} - 1 \right)$$
$$+ \frac{\overline{h} \overline{\omega}^E}{1-\gamma} \times \left(\left[\frac{\xi^H}{\xi^M} \left(1 + \sigma_\xi^{J,H} \right) (1-\phi)(1 + \tilde{\sigma}^{J,M}) \right]^{1-\gamma} - 1 \right)$$
$$\left. + \frac{\overline{h} \overline{\omega}^I}{1-\gamma} \times \left(\left[\left(1 + \sigma_\xi^{J,M} \right) \left(1 + (1-\phi) \tilde{\sigma}^{J,M} + \phi(\lambda(1 + \sigma_p^J) - 1) \right) \right]^{1-\gamma} - 1 \right) \right\}. \quad (1.61)$$

In the equation above μ_R stands for the expected change of the continuous component of stock return (see equation 1.23)

$$\mu_R = \frac{a - \iota(g_{i,t}) - \hat{z}_{i,t}^I}{p_t} + g_{i,t} + \mu_{p,t} + \sigma \sigma_{p,t}\,; \quad (1.62)$$

symbols σ_w^M and $\sigma_w^{y,M}$ denote manager's total exposure to shocks dB and dB^y

$$\sigma_w^M = (1-\phi)\tilde{\sigma}^M + \phi(\sigma + \sigma_p)\,, \tag{1.63}$$

$$\sigma_w^{y,M} = (1-\phi)\tilde{\sigma}^{y,M} + \phi\sigma_p^y\,. \tag{1.64}$$

1.9.3 Proofs

Proof of Lemma 1 Optimal consumption and investment policies are determined by HJB equations (1.60, 1.61). Taking the first order condition with respect to consumption $\hat{c}^H$ in equation (1.60) and $\hat{c}^M$ in equation (1.61) one obtains equation (1.37). Analogously, first order conditions of households' HJB with respect to σ_w^H and $\sigma_w^{y,H}$ imply equation (1.38) for $i = H$. The first order condition of managers' HJB with respect to $\tilde{\sigma}_w^M$ implies

$$(1-\phi)\pi - \gamma(1-\phi)\left((1-\phi)\tilde{\sigma}_w^M + \phi(\sigma+\sigma_p)\right) - (1-\gamma)\sigma_\xi^M(1-\phi)\,. \tag{1.65}$$

By rearranging this equation and using the definition of σ_w^M (1.26) we obtain

$$\sigma_w^M = (1-\phi)\tilde{\sigma}_w^M + \phi(\sigma+\sigma_p) = \frac{\pi + (1-\gamma)\sigma_\xi^M}{\gamma}\,. \tag{1.66}$$

Analogously, the first order condition with respect to $\tilde{\sigma}_w^{y,M}$ and equation (1.27) imply

$$\sigma_w^{y,M} = (1-\phi)\tilde{\sigma}_w^{y,M} + \phi\sigma_p^y = \frac{\pi^y + (1-\gamma)\sigma_\xi^{y,M}}{\gamma}\,. \tag{1.67}$$

The first order condition with respect to ι in equations (1.61, 1.62) implies equation (1.39). □

1.9.4 Characterizing Equilibrium

Lemma 2. *Managers' wealth share x follows a jump diffusion*

$$dx = \mu_x dt + \sigma_x dB + \sigma_x^y dB^y + \sigma_x^J dJ\,, \tag{1.68}$$

with coefficients defined by

$$\mu_x = x\left(\mu_w^M - \hat{c}^M - \mu_p - g - y + (\sigma+\sigma_p)^2 - \sigma_w^M(\sigma_p+\sigma) - \sigma_p\sigma + \left(\sigma_p^y\right)^2 - \sigma_p^y\sigma_w^{y,M}\right), \tag{1.69}$$

$$\sigma_x = x(\sigma_w^M - \sigma - \sigma_p), \quad \sigma_x^y = x\left(\sigma_w^{y,M} - \sigma_p^y\right), \tag{1.70}$$

$$\sigma_x^J = x\frac{\left(1-\overline{\omega}^I-\overline{\omega}^E\right)\left(1+(1-\phi)\tilde{\sigma}_w^{J,M}+\phi\sigma_p^J\right)+\overline{\omega}^I\left(1+(1-\phi)\tilde{\sigma}_w^{J,M}+\phi(\lambda(1+\sigma_p^J)-1)\right)}{\left(1+\sigma_p^J\right)\left(1+(\lambda-1)(\overline{\omega}^I+\overline{\omega}^E)\right)} + \frac{\lambda\beta\overline{\omega}^E}{1+(\lambda-1)(\overline{\omega}^E+\overline{\omega}^I)} - x\,. \tag{1.71}$$

Proof of Lemma 2 Managers' wealth share can be written as

$$x = \frac{W^M}{pQ}, \tag{1.72}$$

where $W^M = \int_M w_i^M di$ is aggregated managers' wealth and pQ is the aggregate wealth in the economy, which equals the product of the price per unit of quality times the aggregate quality. All three components, W^M, p and Q, in this equation follow jump-diffusions. Managers' wealth W^M follows a jump-diffusion

$$\frac{dW^M}{W^M} = -\hat{c}^M\,dt + \mu_w^M + \sigma_w^M\,dB + \sigma_w^{y,M}\,dB^y + \sigma_W^{J,M}\,dJ\,. \tag{1.73}$$

The equation above follows directly from the evolution of wealth of an individual manager defined in equation (1.24), where by $\sigma_W^{J,M}$ I denote the exposure to jump. Stock price per unit of quality follows a jump-diffusion specified in equation (1.14). Aggregate quality follows the process

$$\frac{dQ}{Q} = (g+y)\,dt + \sigma\,dB + (\lambda-1)\left(\overline{\omega}^I + \omega^E\right)dJ\,. \tag{1.74}$$

This expression follows from equation (1.7) describing the evolution of individual variety taking into account the increase in quality of product lines conditional on arrival of a wave. Using the processes for W^M, p and Q and applying Ito's formula to equation (1.72) we obtain the drift μ_x and diffusion σ_x and σ_x^y coefficients specified in the lemma.

I obtain the jump coefficient σ_x^J by calculating the value x_t given x_{t-} conditional on jump arrival at time t. Aggregate managers' wealth conditional on jump arrival is given by

$$\begin{aligned} W_t^M = N_{t-}\left(1-\overline{\omega}_{t-}^I-\overline{\omega}_{t-}^E\right)&\left(1+(1-\phi)\tilde{\sigma}_{w,t-}^{J,M}+\phi\sigma_{p,t-}^J\right) \\ &+N_{t-}\overline{\omega}_{t-}^I\left(1+(1-\phi)\tilde{\sigma}_{w,t-}^{J,M}+\phi(\lambda(1+\sigma_{p,t-}^J)-1)\right) \\ &\qquad\qquad+\beta\,\overline{\omega}_{t-}^E\lambda p_{t-}(1+\sigma_{p,t-}^J)Q_{t-}\,. \end{aligned} \tag{1.75}$$

The first line on the right-hand side accounts for the wealth of managers whose firms were not directly affected by the wave. The second line accounts for the managers of firms that succeeded in radical innovation, the third line reflects the wealth of the new managers that govern the successful entrants.[30] Aggregate quality conditional on wave arrival is given by

$$Q_t = Q_{t-}\left(1+(\lambda-1)\left(\overline{\omega}_{t-}^I+\overline{\omega}_{t-}^E\right)\right)\,, \tag{1.76}$$

Price per unit of quality is given by

$$p_t = p_{t-}\left(1+\sigma_{p,t-}^J\right)\,. \tag{1.77}$$

Jump coefficient $\sigma_{x,t-}^J$ equals to the change in managers' share $x_t - x_{t-} = W_t^M/\left(p_tQ_t\right) - W_{t-}^M/\left(p_{t-}Q_{t-}\right)$. Substituting the expressions for W_t^M, Q_t, p_t outlined above into this formula delivers the expression for σ_x^J in the statement of the lemma. $\square$

Market Clearing Conditions Now I rewrite market clearing conditions formulated in Definition 1 taking into account symmetric policies chosen by firms and agents within each class. Market clearing for consumption goods (1.30) becomes

$$\hat{c}^M p\,x+\hat{c}^H p\,(1-x) = a-\iota-\hat{z}^I-\hat{z}^E\,. \tag{1.78}$$

[30]New managers are chosen at random from the set of households. Aggregate wealth of these households prior to jump arrival is negligible due to the assumption of extreme wealth differences between a firm's market capitalization and a typical household's wealth. Hence, wealth of these households, prior to wave arrival, is not reflected in equation (1.75).

Market clearing conditions for exposures to diffusion risk are

$$\sigma_w^M x + \sigma_w^H (1-x) = \sigma + \sigma_p \,, \tag{1.79}$$

$$\sigma_w^{y,M} x + \sigma_w^{y,H} (1-x) = \sigma_p^y \,. \tag{1.80}$$

Market clearing condition for jump risk is

$$x(1-\phi)\tilde{\sigma}_w^{J,M} + (1-x)\sigma_w^{J,H} = (1-x\phi)\left((1-\varpi^E-\varpi^I)\sigma_p^J + \varpi^I\left(\lambda\left(1+\sigma_p^J\right)-1\right)-\varpi^E\right)\right) \\ + (1-\beta)\lambda\left(1+\sigma_p^J\right)\varpi^E \,. \tag{1.81}$$

Solving for Equilibrium Quantities Given Functions $\xi^H(x,y)$, $\xi^M(x,y)$ This section characterizes all equilibrium quantities given functions $\xi^H(x,y)$ and $\xi^M(x,y)$ and results we established so far. First, I characterize coefficients of the process for managers' share x. From Lemmas 1 and 2 and market clearing (1.79) it follows that

$$\sigma_x = x\left(\sigma_w^M - \sigma_p - \sigma\right) = x\left(\sigma_w^M - \sigma_w^M x - \sigma_w^H(1-x)\right) = \\ x(1-x)(\sigma_w^M - \sigma_w^H) = x(1-x)\frac{1-\gamma}{\gamma}\left(\sigma_\xi^M - \sigma_\xi^H\right) = \\ x(1-x)\frac{1-\gamma}{\gamma}\left(\frac{\xi_x^{M\prime}}{\xi^M} - \frac{\xi_x^{H\prime}}{\xi^H}\right)\sigma_x \,. \tag{1.82}$$

In the last transition I used Ito's lemma and denoted partial derivatives with primes. In addition, I used the fact that state variable y is not correlated with the aggregate productivity shock dB. The last equation implies $\sigma_x \equiv 0$. The aggregate productivity shock dB does not have any effect on the managers' share. It's only impact is on the scale of the economy. From Ito's lemma it also follows

$$\sigma_p = \frac{p_x'}{p}\sigma_x \equiv 0 \,. \tag{1.83}$$

Following the same steps, i.e. using Lemmas 1 and 2 and market clearing condition 1.80 I obtain the expression for managers' share loading on shock dB^y

$$\sigma_x^y = x\left(\sigma_w^{y,M} - \sigma_p^y\right) = x(1-x)\left(\sigma_w^{y,M} - \sigma_w^{y,H}\right) = x(1-x)\frac{1-\gamma}{\gamma}\left(\sigma_\xi^{y,M} - \sigma_\xi^{y,H}\right) =$$

$$x(1-x)\frac{1-\gamma}{\gamma}\left[\left(\frac{\xi_x^{M\prime}}{\xi^M} - \frac{\xi_x^{H\prime}}{\xi^H}\right)\sigma_x^y + \left(\frac{\xi_y^{M\prime}}{\xi^M} - \frac{\xi_y^{H\prime}}{\xi^H}\right)\sigma_y^y\right]. \quad (1.84)$$

From the last equation we can solve for σ_x^y

$$\sigma_x^y = \frac{\left(\frac{\xi_y^{M\prime}}{\xi^M} - \frac{\xi_y^{H\prime}}{\xi^H}\right)\sigma_y^y}{\frac{\gamma}{x(1-x)(1-\gamma)} - \left(\frac{\xi_x^{M\prime}}{\xi^M} - \frac{\xi_x^{H\prime}}{\xi^H}\right)}. \quad (1.85)$$

A set of equilibrium quantities, namely price $p(x,y)$, investment in radical innovation by incumbents $\hat{z}^I(x,y)$ and entrants $\hat{z}^E(x,y)$, price of jump risk $\pi^J(x,y)$ and managers' exposure of liquid wealth to jump $\tilde{\sigma}_w^{J,M}(x,y)$ are determined by the following set of non-linear equations

1. Market clearing of goods (1.78), where $\hat{c}^H$, $\hat{c}^M$ are expressed through ξ^H and ξ^M correspondingly, and ι is expressed as a function of p implied by the equation $g'(\iota) = p$.

2. Equilibrium inflow in the innovation sector (1.44).

3. First order conditions of the managers' HJB with respect to jump exposure $\tilde{\sigma}_w^{J,M}(x,y)$ and optimal investment in radical innovation $\hat{z}^I(x,y)$.

4. Market clearing for jump risk (1.81) with $\sigma_w^{J,H}$ defined by equation (1.41).

5. Loading on jump σ_p^J is determined by $\sigma_p^J(x,y) = \left(p(x(1+\sigma_x^J),y) - p(x,y)\right)/p$; loadings on jump $\sigma_\xi^{J,H}$, $\sigma_\xi^{J,M}$ are defined in the same manner, where σ_x^J is specified in Lemma 2.

The system of the above listed equations is solved numerically iteratively.

Having $p(x,y)$ in hand we obtain

$$\sigma_p^y = \frac{p_x'}{p}\sigma_x^y + \frac{p_y'}{p}\sigma_y^y. \quad (1.86)$$

By rearranging equations (1.70) we obtain managers' portfolio loadings on the two types of diffusion risk

$$\sigma_w^M = \sigma_p + \sigma + \frac{\sigma_x}{x} = \sigma\,, \quad \sigma_w^{y,M} = \sigma_p^y + \frac{\sigma_x^y}{x}\,. \tag{1.87}$$

Substituting these values into the optimal portfolio choice, specified in Lemma 1, I solve for the prices of diffusion risk

$$\pi = \gamma\sigma_w^M + (\gamma-1)\sigma_\xi\,, \quad \pi^y = \gamma\sigma_w^{y,M} + (\gamma-1)\sigma_\xi^y\,. \tag{1.88}$$

Numerical Algorithm I modify HJBs of households and managers to allow for time-dependence in ξ^M and ξ^H. In addition, I differentiate the market clearing condition (1.78) with respect to time allowing for time dependence in p, in addition to ξ^M and ξ^H. The obtained system of partial differential equations for ξ^M, ξ^H, p is iterated backward in time using the explicit scheme on a grid for (x, y) starting from an initial guess for functions $\xi^M(x, y)$ and $\xi^H(x, y)$. At every iteration, given functions ξ^M and ξ^H, I solve for all other equilibrium quantities following the procedure described above. The backward iterations are conducted till updates to functions $\xi^M(x, y)$ and $\xi^H(x, y)$ do not exceed a pre-specified threshold.

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Chapter 2

The Dark Side of Circuit Breakers

2.1 Introduction

Large stock market swings in the absence of significant macroeconomic shocks often raise questions about the confidence in the financial market from market participants, policy makers and general public alike. While the cause of these swings are still not well understood, various measures have been adopted to intervene in the normal trading process during these extreme times in the hope to stabilize prices and maintain proper functioning of the market. These measures, sometimes referred to as throwing sand in the gears, range from market-wide trading halts, price limits on the whole market or individual assets, to limits on order flows, positions and margins, even transaction taxes, just to name a few.[2] They have grown to be an important part of the broad market architecture. Yet, the merits of these measures, either from

This chapter is joint work with Hui Chen and Jiang Wang.

[2]It is worth noting that contingent trading halts and price limits are part of the normal trading process for individual stocks and futures contracts. However, their presence there have quite different motivations. For example, the trading halt of an individual stock prior to major corporate announcements is motivated by the desire for fair information disclosure, and daily price limits on futures are motivated by the desire to guarantee the proper implementation of the mark-to-market mechanism as well as to deter market manipulation. In this paper, we focus on market-wide trading interventions in underlying markets such as stocks as well as their derivatives, which have very different motivations and implications.

Table 2.1: Adoption of market-wide circuit breakers among leading stock markets. The table also reports markets with price limits on individual stocks. Markets with market-wide circuit breakers as well as individual stock price limits are denoted by Y and N otherwise. Y/N for China denotes its adaptation of circuit breakers and then the abandonment.

	2017 Market Cap (Tn $)	Rank	Circuit Breaker	Price Limit
Developed markets				
United States	32.1	1	Y	Y
Japan	6.2	3	N	Y
Hong Kong	4.4	4	N	N
France	2.7	5	Y	N
Canada	2.4	6	Y	N
Developing markets				
China	8.7	2	Y/N	Y
India	2.3	7	Y	Y
Brazil	1.0	14	Y	N
Russia	0.6	17	Y	N

a theoretical or an empirical perspective, remain largely unclear (see, for example, Grossman, 1990).

Probably one of the most prominent of these measures is the market-wide circuit breaker in the U.S., which was advocated by the Brady Commission (Presidential Task Force on Market Mechanisms, 1988) following the Black Monday of 1987 and subsequently implemented in 1988. It temporarily halts trading in all stocks and related derivatives when a designated market index drops by a significant amount. Following this lead, circuit breakers of various forms have been widely adopted by equity and derivative exchanges around the globe.[3] Table 2.1 shows the adoption of market-wide circuit breakers among the leading stock markets in both the developed and developing world.

Since its introduction, the U.S. circuit breaker was triggered only once on October 27, 1997 (see, e.g., Figure 2-1, left panel). At that time, the threshold was based

[3]According to a 2016 report, "Global Circuit Breaker Guide" by ITG, over 30 countries around the world have rules of trading halts in the form of circuit breakers, price limits and volatility auctions.

on points movement of the DJIA index. At 2:36 p.m., a 350-point (4.54%) decline in the DJIA led to a 30-minute trading halt on stocks, equity options, and index futures. After trading resumed at 3:06 p.m., prices fell rapidly to reach the second-level 550-point circuit breaker at 3:30 p.m., leading to the early market closure for the day.[4] But the market stabilized the next day. This event led to the redesign of the circuit breaker rules, moving from the change in level of DJIA to percentage drop of S&P 500, with a considerably wider bandwidth.[5]

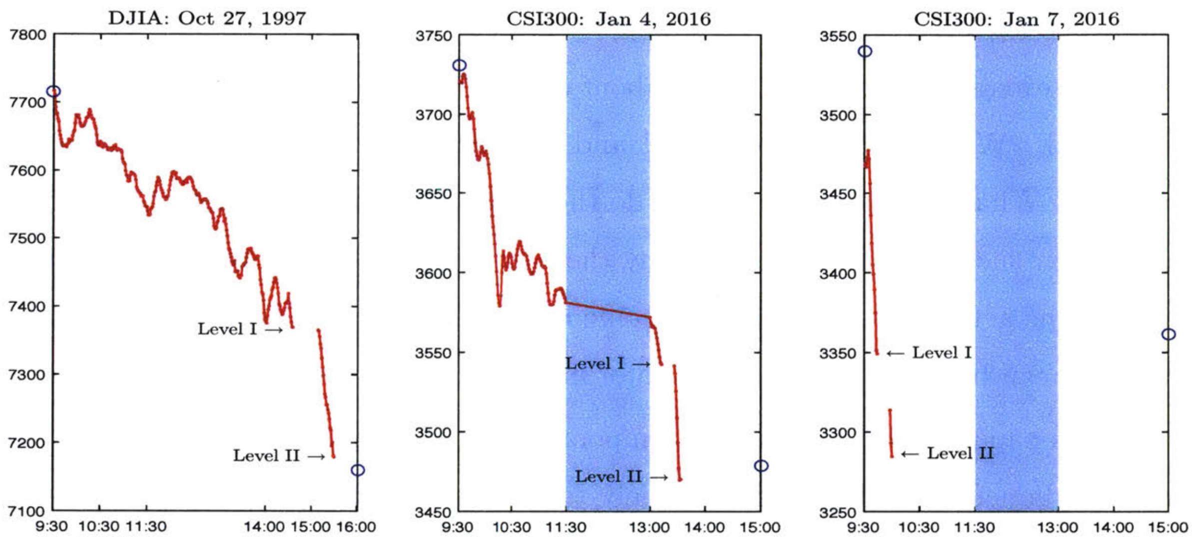

Figure 2-1: **Circuit breakers in the U.S. and Chinese stock market.** The left panel plots the DJIA index on Oct 27, 1997, when the market-wide circuit breaker was triggered, first at 2:36 p.m., and then at 3:30 p.m. The middle and right panels plot the CSI300 index on January 4 and January 7 of 2016. Trading hours for the Chinese stock market are 9:30-11:30 and 13:00-15:00 (the shaded interval in the panels marks the lunch break). Level 1 (2) circuit breaker is triggered after a 5% (7%) drop in price from the previous day's close. The blue circles on the left (right) vertical axes mark the price on the previous day's close (following day's open).

After the Chinese stock market experienced extreme price declines in 2015, a market-wide circuit breaker was introduced in January 2016, with a 15-minute trading

[4]For a detailed review of this event, see Securities and Exchange Commission (1998).

[5]In its current form, the market-wide circuit breaker can be triggered at three thresholds: 7% (Level 1), 13% (Level 2), both of which will halt market-wide trading for 15 minutes when the decline occurs between 9:30 a.m. and 3:25 p.m. Eastern time, and 20% (Level 3), which halts market-wide trading for the remainder of the trading day; these triggers are based on the prior day's closing price of the S&P 500 Index.

halt when the CSI 300 Index falls by 5% (Level 1) from previous day's close, and market closure for the day after a 7% decline (Level 2).[6] On January 4, 2016, the first trading day after the circuit breaker was put in place, both thresholds were reached (Figure 2-1, middle panel), and it took only 7 minutes from the re-opening of the markets following the 15-minute halt for the index to reach the 7% threshold. Three days later, on January 7, both circuit breakers were triggered again (Figure 2-1, right panel), and the entire trading session lasted just 30 minutes. On the same day, the circuit breaker was suspended indefinitely.

These events have revived debates about circuit breakers and market interventions in general. What are the theoretical and empirical basis for introducing circuit breakers? What are their goals? How do they actually impact the market? How to assess their success or failure? How may their effectiveness depend on the particular market, the actual design, and the specific market conditions? More broadly, these questions can be raised about any form of interventions in the trading process.

In this paper, we develop an intertemporal equilibrium model to capture investors' most fundamental trading needs, namely to share risk. We then examine how the introduction of a downside circuit breaker affects investors' trading behavior and the equilibrium price dynamics. In addition to welfare loss by reduced risk sharing, which can be substantial, we show that a circuit breaker also lowers price levels, increases conditional and realized volatility, and increases the likelihood of triggering trading halts. These consequences run contrary to the often stated goal of circuit breakers, which is to calm the markets. Our model not only demonstrates the potential cost of circuit breakers, but also provides a basic setting to further incorporate market imperfections to fully examine their costs and benefits.

In our model, two (classes of) investors have log preferences over terminal wealth and have heterogeneous beliefs about the dividend growth rate, which generates

[6]The CSI 300 index is a market-cap weighted index of 300 major stocks listed on the Shanghai Stock Exchange and the Shenzhen Stock Exchange, compiled by the China Securities Index Company, Ltd.

trading. For simplicity, one investor's belief is set to be the same as the objective belief while another investor's belief is different, who will also be referred to as the irrational investor.[7] Without the circuit breaker, the stock price is a weighted average of the prices under the two investors' beliefs, with the weights being their respective shares of total wealth.

The introduction of a downside circuit breaker in the market, however, makes the equilibrium stock price disproportionately reflect the beliefs of the relatively pessimistic investor, especially when the stock price approaches the circuit breaker limit. To understand this result, first consider the scenario when the stock price has just reached the circuit breaker threshold. Immediate market closure is an extreme form of illiquidity, which forces the relatively optimistic investor to refrain from taking on leverage due to the inability to rebalance his portfolio during closure and the risk of default it may entail. As the optimistic investor faces binding leverage constraints, the pessimistic investor becomes the marginal investor, and the equilibrium stock price upon market closure entirely reflects his belief, regardless of his wealth share.

Next, the threat of market closure also affects trading and prices before the circuit breaker is triggered. Compared to the case without circuit breaker, the relatively optimistic investor will preemptively reduce his leverage as the price approaches the circuit breaker limit. For a downside circuit breaker, the price-dividend ratios become lower throughout the trading interval. Thus, a downside circuit breaker tends to drive down the overall asset price levels.

In addition, in the presence of a downside circuit breaker, the conditional volatilities of stock returns can become significantly higher. These effects are stronger when the price is closer to the circuit breaker threshold, when it is earlier during a trading session. Surprisingly, the volatility amplification effect of downside circuit breakers is stronger when the initial wealth share for the irrational investor (who tends to be pessimistic

[7]Interpreting the investor heterogeneity as difference in beliefs is solely for convenience. We can also interpret it as difference in preferences or endowments. See the discussion on the model in Section 2.2.

at the triggering point) is smaller, because the gap between the wealth-weighted belief of the representative investor and the belief of the pessimist is larger in such cases.

Our model shows that circuit breakers have multifaceted effects on intra-day price variability. On the one hand, almost mechanically, a (tighter) downside circuit breaker eliminates a possibility of very large downward price movements. Such effects could be beneficial, for example, in reducing inefficient liquidations due to intra-day mark-to-market. On the other hand, a (tighter) downside circuit breaker tends to raise the probabilities of intermediate and large price ranges, and can significantly increase the median of daily realized volatilities as well as the probabilities of very large conditional and realized volatilities. These effects could exacerbate market instability in the presence of imperfections.

Furthermore, our model demonstrates a "magnet effect." The very presence of downside circuit breakers makes it more likely for the stock price to reach the threshold in a given amount of time than when there are no circuit breakers (the opposite is true for upside circuit breakers). The difference between the probabilities is negligible when the stock price is sufficiently far away from the threshold, but it gets bigger as the stock price gets closer to the threshold. Eventually, when the price is sufficiently close to the threshold, the gap converges to zero as both probabilities converge to one.

This "magnet effect" is important for the design of circuit breakers. It suggests that using the historical data from a period when circuit breakers were not implemented can substantially underestimate the likelihood of future circuit breaker triggers, which might result in picking a downside circuit breaker threshold that is excessively tight.

Prior theoretical work on circuit breakers focuses on their role in reducing excess volatility and restore orderly trading in the presence of market imperfections such as limited participation, information asymmetry and market power. For example, Greenwald and Stein (1991) argue that, in a market with limited participation and the resulting execution risk, circuit breakers can help to better synchronize trading for market participants and improve the efficiency of allocations (see also Greenwald and

Stein, 1988).[8] On the other hand, Subrahmanyam (1994) shows that in the presence of asymmetric information, circuit breakers can increase price volatility by causing investors to shift their trades to earlier periods with lower liquidity supply (see also Subrahmanyam, 1995).

By developing a model to capture investors' first-order trading needs, our work complements the studies above in three important dimensions. First, it captures the cost of circuit breakers, in welfare, price distortion, and excess volatility, in a benchmark setting without imperfections. For instance, we show that the excess volatility effect that Subrahmanyam (1994) demonstrates in his model is quite general and is present even in the absence of asymmetric information. This is relevant because various forms of market imperfections such as information asymmetry and strategic behavior could be less important for deep markets, such as the aggregate stock market, than for shallow markets, such as markets for individual securities.

Second, our model provides a basis to further include different forms of market imperfections, if suitable, such as asymmetric information, strategic behavior, cost of participation and failure of coordination, which are needed to justify and quantify the benefits of circuit breakers. Such imperfections are undoubtedly important in providing the basis for interventions. However, focusing solely on their influence may understate the fundamental merits of the market mechanism itself.

Third, our model sheds light on the importance of properly capturing the most fundamental trading needs of investors, i.e., risk sharing or liquidity, in analyzing, understanding and managing financial markets. In models of information asymmetry, these trading needs are represented by the liquidity demands of "noise traders," which are treated as exogenous. Our results suggest that the behavior of these liquidity traders can be significantly affected by circuit breakers, which should be carefully

[8]In Greenwald and Stein (1991), limited participation takes several forms. In particular, value traders, who act as price stabilizers, enter the market at different times with uncertainty. This uncertainty in their participation, which is assumed to be exogenous, gives rise to the additional risk in execution prices. Also, these value traders can only rely on market orders or simple limit orders, rather than limit order schedules, in their trading.

taken into account.

In this spirit, our paper is closely related to Hong and Wang (2000), who study the effects of periodic market closures in the presence of asymmetric information. The liquidity effect caused by market closures as we see here is qualitatively similar to what they find. By modeling the stochastic nature of a circuit breaker, we are able to fully capture its impact on market dynamics, such as volatility and conditional distributions.

While our model focuses on circuit breakers, our theoretical results about the dynamic impact of disappearing market liquidity in the presence of levered investors is more broadly applicable. Besides exchange-implemented trading halts, other types of market interventions such as price limits, short-sale bans, trading speed restrictions (e.g., penalties for HFT), and other forms of market freezes could also have similar effects on the willingness of highly-levered investors to continue to hold their risky positions. As these investors preemptively delever, it will depress prices, amplify volatility, and raise the chances of market freeze. In fact, the setup we have developed here can be extended to examine the impact of these interventions.

The empirical work on the impact of market-wide circuit breakers is scarce due to the fact that their likelihood to be approached, not to mention triggered, is very small by design. Goldstein and Kavajecz (2004) provide a detailed analysis on the behavior of market participants in the period around October 27, 1997, the only time the circuit breaker has been triggered in the U.S. since its introduction. They find that leading up to the trading halt, market participants accelerated their trades, which is consistent with the magnet effect. In addition, they show that sellers' behavior is less influenced when approaching circuit breaker than the buyers', who are withdrawing from the market by canceling their buy limit orders. This is consistent with what our model predicts: sellers are becoming marginal traders when approaching a circuit breaker.[9]

[9] Ackert et al. (2001) study the impact of market-wide circuit breakers through experiments. They

There is, however, more extensive empirical work on the impact of conditional trading restrictions on individual assets including futures. For example, Bertero and Mayer (1990) and Lauterbach and Ben-Zion (1993) study the effects of trading halts based on price limits imposed on individual stocks around the 1987 stock market crash. Bertero and Mayer (1990) find that stock indices of countries with price limits imposed on individual stocks experienced declines in magnitude of up to 9% lower compared to aggregate indices in countries without circuit breakers. Lauterbach and Ben-Zion (1993) find that stocks with trading halts experienced smaller decline on the day of the crash, but trading halts did not have any effect on long run performance. Lee et al. (1994) explore behavior of individual stock prices traded at NYSE around times of trading halts. They find that individual stock volatility and trading volume both increase on the days following a trading halt. On the other hand, Christie et al. (2002) find that for news related trading halts of individual NASDAQ stocks, longer halts tend to reduce post-halt uncertainty.[10] Although the focus of our paper is on market-wide circuit breakers, the results we obtain are broadly compatible with the empirical findings on the impact of trading halts for individual assets. But the results from individual assets are in general richer and less robust. In many ways, this is expected given the relative importance of various type of imperfections in these markets.

The rest of the paper is organized as follows. Section 2.2 describes the basic model for our analysis. Section 2.3 provides the solution to the model. In Section 2.4, we examine the impact of a downside circuit breaker on investor behavior and equilibrium prices. Section 2.5 discusses the robustness of our results with respect to some of

find that circuit breakers do not impact prices significantly but alter market participants' trading behavior substantially by accelerating trading when the breakers are approaching.

[10]Chen (1993), Santoni and Liu (1993), Kim and Rhee (1997), Corwin and Lipson (2000), Jiang et al. (2009), Gomber et al. (2012), Brugler and Linton (2014), among others, study the effects of trading halts and price limits on the market behavior individual stocks. Chen et al. (2017) examine the impact of daily price limits on trading patterns and price dynamics in the Chinese stock market. Kuserk et al. (1992), Berkman et al. (1998), Coursey and Dyl (1990), Ma et al. (1989b), Ma et al. (1989a), Chen and Jeng (1996) study effects of trading restrictions related to price fluctuations in futures markets.

our modeling choices such as continuous-time trading and no default. In Section 2.6, we consider several extensions of the basic model to different types of trading halts. Section 2.7 concludes. All proofs are given in the appendix.

2.2 The Model

We consider a continuous-time endowment economy over the finite time interval $[0, T]$. Uncertainty is described by a one-dimensional standard Brownian motion Z, defined on a filtered complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$, where $\{\mathcal{F}_t\}$ is the augmented filtration generated by Z.

There is a single share of an aggregate stock, which pays a terminal dividend of D_T at time T. The process for D is exogenous and publicly observable, given by:

$$dD_t = \mu D_t dt + \sigma D_t dZ_t, \quad D_0 = 1, \tag{2.1}$$

where μ and $\sigma > 0$ are the expected growth rate and volatility of D_t, respectively.[11] Besides the stock, there is also a riskless bond with total net supply $\Delta \geq 0$. Each unit of the bond yields a terminal pays off of one at time T.

There are two competitive agents A and B, who are initially endowed with ω and $1 - \omega$ shares of the aggregate stock and $\omega\Delta$ and $(1 - \omega)\Delta$ units of the riskless bond, respectively, with $0 \leq \omega \leq 1$ determining the initial wealth distribution between the two agents. Both agents have logarithmic preferences over their terminal wealth at time T:

$$u_i(W_T^i) = \ln(W_T^i), \quad i = \{A, B\}. \tag{2.2}$$

There is no intermediate consumption.

The two agents have heterogeneous beliefs about the terminal dividend, and they

[11] For brevity, throughout the paper we will refer to D_t as "dividend" and S_t/D_t as the "price-dividend ratio," even though dividend will only be realized at time T.

"agree to disagree" (i.e., they do not learn from each other or from prices). Agent A has the objective beliefs in the sense that his probability measure is consistent with $\mathbb{P}$ (in particular, $\mu^A = \mu$). Agent B's probability measure, denoted by $\mathbb{P}^B$, is different from but equivalent to $\mathbb{P}$.[12] In particular, he believes that the dividend growth rate at time t is:

$$\mu_t^B = \mu + \delta_t, \tag{2.3}$$

where the difference in beliefs δ_t follows an Ornstein-Uhlenbeck process:

$$d\delta_t = -\kappa(\delta_t - \bar{\delta})dt + \nu dZ_t, \tag{2.4}$$

with $\kappa \geq 0$ and $\nu \geq 0$. Equation (2.4) describes the dynamics of the gap in beliefs from the perspective of agent A (the physical probability measure). Notice that δ_t is driven by the same Brownian motion as the aggregate dividend. With $\nu > 0$, agent B becomes more optimistic (pessimistic) following positive (negative) shocks to the aggregate dividend, and the impact of these shocks on his belief decays exponentially at the rate κ. Thus, the parameter ν controls how sensitive B's conditional belief is to realized dividend shocks, while κ determines the relative importance of shocks from recent past vs. distant past. The average long-run disagreement between the two agents is $\bar{\delta}$. In the special case with $\nu = 0$ and $\delta_0 = \bar{\delta}$, the disagreement between the two agents remains constant over time. In another special case where $\kappa = 0$, δ_t follows a random walk.

Heterogeneous beliefs are a simple way to introduce heterogeneity among agents, which is necessary to generate trading. The heterogeneity in beliefs can easily be interpreted as heterogeneity in utility, which can be state dependent. For example, time-varying beliefs could represent behavioral biases ("representativeness") or a form of path-dependent utility that makes agent Beffectively more (less) risk averse

[12]More precisely, $\mathbb{P}$ and $\mathbb{P}^B$ are equivalent when restricted to any σ-field $\mathcal{F}_T = \sigma(\{D_t\}_{0 \leq t \leq T})$. Two probability measures are equivalent if they agree on zero probability events. Agents beliefs should be equivalent to prevent seemingly arbitrage opportunities under any agents' beliefs.

following negative (positive) shocks to fundamentals. Alternatively, we could introduce heterogeneous endowment shocks to generate trading (see, e.g., Wang (1995)). In all these cases, trading allows agents to share risk.

Let the Radon-Nikodym derivative of the probability measure $\mathbb{P}^B$ with respect to $\mathbb{P}$ be η. Then from Girsanov's theorem, we get

$$\eta_t = \exp\left(\frac{1}{\sigma}\int_0^t \delta_s dZ_s - \frac{1}{2}\frac{1}{\sigma^2}\int_0^t \delta_s^2 ds\right). \tag{2.5}$$

Intuitively, since agent B will be more optimistic than A when $\delta_t > 0$, those paths with high realized values for $\int_0^t \delta_s dZ_s$ will be assigned higher probabilities under $\mathbb{P}^B$ than under $\mathbb{P}$.

Because there is no intermediate consumption, we use the riskless bond as the numeraire. Thus, the price of the bond is always 1. Let S_t denote the price of the stock at t.

Circuit Breaker. To capture the essence of a circuit breaker rule, we assume that the stock market will be closed whenever the stock price S_t hits a threshold $(1-\alpha)S_0$, where S_0 is the endogenous initial price of the stock, and $\alpha \in [0,1]$ is a constant parameter determining the floor of downside price fluctuations during the interval $[0,T]$. Later in Section 2.6, we extend the model to allow for market closures for both downside and upside price movements, which represent price limit rules. The closing price for the stock is determined such that both the stock market and bond market clear when the circuit breaker is triggered. After that, the stock market will remain closed until time T.

In practice, the circuit breaker threshold is often based on the closing price from the previous trading session instead of the opening price of the current trading session. For example, in the U.S., a cross-market trading halt can be triggered at three circuit breaker thresholds (7%, 13%, and 20%) based on the prior day's closing price of the

S&P 500 Index. However, the distinction between today's opening price and the prior day's closing price is not crucial for our analysis. The circuit breaker not only depends on but also endogenously affects the initial stock price, just like it does for prior day's closing price in practice.[13]

Finally, we impose usual restrictions on trading strategies to rule out arbitrage.

2.3 The Equilibrium

2.3.1 Benchmark Case: No Circuit Breaker

In this section, we solve for the equilibrium when there is no circuit breaker. To distinguish from the case with circuit breakers, we use the symbol "$\widehat{\ }$" to denote variables in the case without circuit breakers.

In the absence of circuit breakers, markets are dynamically complete. The equilibrium allocation in this case can be characterized as the solution to the following planner's problem:

$$\max_{\widehat{W}_T^A, \widehat{W}_T^B} \mathbb{E}_0 \left[\lambda \ln \left(\widehat{W}_T^A \right) + (1-\lambda) \eta_T \ln \left(\widehat{W}_T^B \right) \right], \tag{2.6}$$

subject to the resource constraint:

$$\widehat{W}_T^A + \widehat{W}_T^B = D_T + \Delta. \tag{2.7}$$

From the agents' first-order conditions and the budget constraints, we then obtain

[13]Other realistic features of the circuit breaker in practice is to close the market for m minutes and reopen (Level 1 and 2), or close the market until the end of the day (Level 3). In our model, we can think of T as one day. The fact that the price of the stock reverts back to the fundamental value D_T at T resembles the rationale of CB to "restore order" in the market.

$\lambda = \omega$, and

$$\widehat{W}_T^A = \frac{\omega}{\omega + (1-\omega)\eta_T}(D_T + \Delta), \tag{2.8a}$$

$$\widehat{W}_T^B = \frac{(1-\omega)\eta_T}{\omega + (1-\omega)\eta_T}(D_T + \Delta). \tag{2.8b}$$

As it follows from the equations above agent B will be allocated a bigger share of the aggregate dividend when realized value of the Radon-Nikodym derivative η_T is higher, i.e., under those paths that agent B considers to be more likely.

The state price density under agent A's beliefs, which corresponds the objective probability measure $\mathbb{P}$, is given by:

$$\widehat{\pi}_t^A = \mathbb{E}_t\left[\xi u'(\widehat{W}_T^A)\right] = \mathbb{E}_t\left[\xi(\widehat{W}_T^A)^{-1}\right], \quad 0 \leq t \leq T \tag{2.9}$$

for some constant ξ. Then, from the budget constraint for agent A we see that the planner's weights are equal to the shares of endowment, $\lambda = \omega$. Using the state price density, one can then derive the price of the stock and individual investors' portfolio holdings.

In the limiting case with bond supply $\Delta \to 0$, the complete markets equilibrium can be characterized in closed form. We focus on this limiting case in the rest of this section. First, the following proposition summarizes the pricing results.

Proposition 1. *When there are no circuit breakers, the price of the stock in the limiting case with bond supply $\Delta \to 0$ is:*

$$\widehat{S}_t = \frac{\omega + (1-\omega)\eta_t}{\omega + (1-\omega)\,\eta_t\, e^{a(t,T)+b(t,T)\delta_t}} D_t e^{\left(\mu-\sigma^2\right)(T-t)}, \tag{2.10}$$

where

$$a(t,T) = \left[\frac{\kappa\overline{\delta} - \sigma\nu}{\frac{\nu}{\sigma} - \kappa} + \frac{1}{2}\frac{\nu^2}{\left(\frac{\nu}{\sigma} - \kappa\right)^2}\right](T-t) - \frac{1|}{4}\frac{\nu^2}{\left(\frac{\nu}{\sigma} - \kappa\right)^3}\left[1 - e^{2\left(\frac{\nu}{\sigma}-\kappa\right)(T-t)}\right]$$

$$+ \left[\frac{\kappa\overline{\delta} - \sigma\nu}{\left(\frac{\nu}{\sigma} - \kappa\right)^2} + \frac{\nu^2}{\left(\frac{\nu}{\sigma} - \kappa\right)^3}\right]\left[1 - e^{\left(\frac{\nu}{\sigma}-\kappa\right)(T-t)}\right], \tag{2.11a}$$

$$b(t,T) = \frac{1 - e^{\left(\frac{\nu}{\sigma}-\kappa\right)(T-t)}}{\frac{\nu}{\sigma} - \kappa}. \tag{2.11b}$$

From Equation (2.10), we can derive the conditional volatility of the stock $\widehat{\sigma}_{S,t}$ in closed form, which is available in the appendix.

Next, we turn to the wealth distribution and portfolio holdings of individual agents. At time $t \leq T$, the shares of total wealth of the two agents are:

$$\widehat{\omega}_t^A = \frac{\omega}{\omega + (1-\omega)\eta_t}, \quad \widehat{\omega}_t^B = 1 - \widehat{\omega}_t^A. \tag{2.12}$$

The number of shares of stock $\widehat{\theta}_t^A$ and units of riskless bonds $\widehat{\phi}_t^A$ held by agent A are:

$$\widehat{\theta}_t^A = \frac{\omega}{\omega + (1-\omega)\eta_t} - \frac{\omega(1-\omega)\eta_t}{[\omega + (1-\omega)\eta_t]^2}\frac{\delta_t}{\sigma\widehat{\sigma}_{S,t}} = \widehat{\omega}_t^A\left(1 - \widehat{\omega}_t^B\frac{\delta_t}{\sigma\widehat{\sigma}_{S,t}}\right), \tag{2.13}$$

$$\widehat{\phi}_t^A = \widehat{\omega}_t^A\widehat{\omega}_t^B\frac{\delta_t}{\sigma\widehat{\sigma}_{S,t}}\widehat{S}_t, \tag{2.14}$$

and the corresponding values for agent B are $\theta_t^B = 1 - \theta_t^A$ and $\phi_t^B = -\phi_t^A$.

As Equation (2.13) shows, there are several forces affecting the agents' portfolio positions. First, all else equal, agent A owns fewer shares of the stock when B has more optimistic beliefs (larger δ_t). This effect becomes weaker when the volatility of stock return $\widehat{\sigma}_{S,t}$ is high. Second, changes in the wealth distribution (as indicated by (2.12)) also affect the portfolio holdings, as the richer agent will tend to hold more shares of the stock.

We can gain more intuition on the stock price by rewriting Equation (2.10) as follows:

$$\widehat{S}_t = \frac{1}{\frac{\omega}{\omega+(1-\omega)\eta_t}\mathbb{E}_t\left[D_T^{-1}\right] + \frac{(1-\omega)\eta_t}{\omega+(1-\omega)\eta_t}\mathbb{E}_t^B\left[D_T^{-1}\right]} = \left(\frac{\widehat{\omega}_t^A}{\widehat{S}_t^A} + \frac{\widehat{\omega}_t^B}{\widehat{S}_t^B}\right)^{-1}, \tag{2.15}$$

which states that the stock price is a weighted harmonic average of the prices of the stock in two single-agent economies with agent A and B being the representative agent, respectively, denoted by $\widehat{S}_t^A$ and $\widehat{S}_t^B$, where

$$\widehat{S}_t^A = e^{\left(\mu-\sigma^2\right)(T-t)} D_t, \tag{2.16a}$$

$$\widehat{S}_t^B = e^{\left(\mu-\sigma^2\right)(T-t)-a(t,T)-b(t,T)\delta_t} D_t, \tag{2.16b}$$

and the weights $(\widehat{\omega}_t^A, \widehat{\omega}_t^B)$ are the two agents' shares of total wealth. Controlling for the wealth distribution, the equilibrium stock price is higher when agent B has more optimistic beliefs (larger δ_t).

One special case of the above result is when the amount of disagreement between the two agents is the zero, i.e., $\delta_t = 0$ for all $t \in [0, T]$. The stock price then becomes:

$$\widehat{S}_t = \widehat{S}_t^A = \frac{1}{\mathbb{E}_t[D_T^{-1}]} = e^{\left(\mu-\sigma^2\right)(T-t)} D_t, \tag{2.17}$$

which is a version of the Gordon growth formula, with σ^2 being the risk premium for the stock. The instantaneous volatility of stock returns becomes the same as the volatility of dividend growth, $\widehat{\sigma}_{S,t} = \sigma$. The shares of the stock held by the two agents will remain constant and be equal to the their endowments, $\widehat{\theta}_t^A = \omega$, $\widehat{\theta}_t^B = 1 - \omega$.

Another special case is when the amount of disagreement is constant over time ($\delta_t = \delta$ for all t). The results for this case are obtained by setting $\nu = 0$ and $\delta_0 = \bar{\delta} = \delta$ in Proposition 1. Equation (2.10) then simplifies to:

$$\widehat{S}_t = \frac{\omega + (1-\omega)\eta_t}{\omega + (1-\omega)\eta_t e^{-\delta(T-t)}} e^{\left(\mu-\sigma^2\right)(T-t)} D_t. \tag{2.18}$$

As expected, $\widehat{S}_t$ increases with δ, which reflects agent B's optimism on dividend growth.

2.3.2 Circuit Breaker

We start this section by introducing some notation. By θ_t^i, ϕ_t^i, and W_t^i we denote stock holdings, bond holdings, and wealth of agent i at time t, respectively, in the market with a circuit breaker. Let τ denote the time when the circuit breaker is triggered. It follows from the definition of the circuit breaker and the continuity of stock prices that τ satisfies

$$\tau = \inf\{t \geq 0 : S_t = (1-\alpha)S_0\}. \tag{2.19}$$

We use the expression $\tau \wedge T$ to denote $\min\{\tau, T\}$. Next, we define the equilibrium with a circuit breaker.

Definition 2. *The equilibrium with circuit breaker is defined by an $\mathcal{F}_t$-stopping time τ, trading strategies $\{\theta_t^i, \phi_t^i\}$ $(i = A, B)$, and a continuous stock price process S defined on the interval $[0, \tau \wedge T]$ such that:*

1. *Taking stock price process S as given, the two agents' trading strategies maximize their expected utilities under their respective beliefs and budget constraints.*

2. *For any $t \in [0, T]$, both the stock and bond markets clear,*

$$\theta_t^A + \theta_t^B = 1, \quad \phi_t^A + \phi_t^B = \Delta. \tag{2.20}$$

3. *The stopping time τ is consistent with the circuit breaker rule in* (2.19).

One crucial feature of the model is that markets remain dynamically complete until the circuit breaker is triggered. Hence, we solve for the equilibrium with the following three steps. First, consider an economy in which trading stops when the stock price reaches any given triggering price $\underline{S} \geq 0$. By examining the equilibrium conditions upon market closure, we can characterize the $\mathcal{F}_t$-stopping time τ that is consistent with $S_\tau = \underline{S}$. Next, we solve for the optimal allocation at $\tau \wedge T$ through the planner's problem as a function of $\underline{S}$, as well as the stock price prior to $\tau \wedge T$, also as a function of $\underline{S}$. Finally, the equilibrium is the fixed point whereby the triggering price $\underline{S}$ is consistent with the initial price, $\underline{S} = (1-\alpha)S_0$. We describe these steps in detail below.

Suppose the circuit breaker is triggered before the end of the trading session, i.e., $\tau < T$. We start by deriving the agents' indirect utility functions at the time of market closure. Agent i has wealth W^i_τ at time τ. Since the two agents behave competitively, they take the stock price S_τ as given and choose the shares of stock θ^i_τ and bonds ϕ^i_τ to maximize their expected utility over terminal wealth, subject to the budget constraint:

$$\begin{aligned} V^i(W^i_\tau, \tau) \quad = \quad & \max_{\theta^i_\tau,\, \phi^i_\tau} \quad \mathbb{E}^i_\tau \left[\ln(\theta^i_\tau D_T + \phi^i_\tau)\right], \\ & s.t. \quad \theta^i_\tau S_\tau + \phi^i_\tau = W^i_\tau, \end{aligned} \tag{2.21}$$

where $V^i(W^i_\tau, \tau)$ is the indirect utility function for agent i at time $\tau < T$.

The market clearing conditions at time τ are:

$$\theta^A_\tau + \theta^B_\tau = 1, \quad \phi^A_\tau + \phi^B_\tau = \Delta. \tag{2.22}$$

For any $\tau < T$, the Inada condition implies that terminal wealth for both agents needs to stay non-negative, which implies $\theta^i_\tau \geq 0$ and $\phi^i_\tau \geq 0$. That is, neither agent will take short or levered positions in the stock. This is a direct result of the inability to rebalance one's portfolio after market closure, which is an extreme version of illiquidity.

Solving the problem (2.21) – (2.22) gives us the indirect utility functions $V^i(W^i_\tau, \tau)$. It also gives us the stock price at the time of market closure, S_τ, as a function of the dividend D_τ, the gap in beliefs δ_τ, and the wealth distribution at time τ (which is determined by the Radon-Nikodym derivative η_τ). Thus, the condition $S_\tau = \underline{S}$ translates into a condition on D_τ, δ_τ, and η_τ, which in turn characterizes the stopping time τ as a function of exogenous state variables. As we will see later, in the limiting case with bond supply $\Delta \to 0$, the stopping rule satisfying this condition can be expressed in closed form. When $\Delta > 0$, the solution can be obtained numerically.

Next, the indirect utility for agent i at $\tau \wedge T$ is given by:

$$V^i(W^i_{\tau\wedge T}, \tau \wedge T) = \begin{cases} \ln(W^i_T), & \text{if} \quad \tau \geq T \\ V^i(W^i_\tau, \tau), & \text{if} \quad \tau < T. \end{cases} \tag{2.23}$$

These indirect utility functions make it convenient to solve for the equilibrium wealth allocations in the economy at time $\tau \wedge T$ through the following planner problem:

$$\max_{W^A_{\tau\wedge T}, W^B_{\tau\wedge T}} \mathbb{E}_0 \left[\lambda V^A(W^A_{\tau\wedge T}, \tau \wedge T) + (1-\lambda)\eta_{\tau\wedge T} V^B(W^B_{\tau\wedge T}, \tau \wedge T)\right], \tag{2.24}$$

subject to the resource constraint:

$$W^A_{\tau\wedge T} + W^B_{\tau\wedge T} = S_{\tau\wedge T} + \Delta, \tag{2.25}$$

where

$$S_{\tau\wedge T} = \begin{cases} D_T, & \text{if} \quad \tau \geq T \\ \underline{S}, & \text{if} \quad \tau < T. \end{cases} \tag{2.26}$$

Taking the equilibrium allocation $W^A_{\tau\wedge T}$ from the planner's problem, the state price density for agent A at time $\tau \wedge T$ can be expressed as his marginal utility of wealth times a constant ξ:

$$\pi^A_{\tau\wedge T} = \xi \left.\frac{\partial V^A(W, \tau \wedge T)}{\partial W}\right|_{W = W^A_{\tau\wedge T}}. \tag{2.27}$$

The price of the stock at any time $t \leq \tau \wedge T$ is then given by:

$$S_t = \mathbb{E}_t \left[\frac{\pi^A_{\tau\wedge T}}{\pi^A_t} S_{\tau\wedge T}\right], \tag{2.28}$$

where like in Equation (2.9),

$$\pi_t^A = \mathbb{E}_t \left[\pi_{\tau \wedge T}^A \right]. \tag{2.29}$$

The expectations above are straightforward to evaluate, at least numerically.

Having obtained the solution for S_t as a function of $\underline{S}$, we can finally solve for the equilibrium triggering price $\underline{S}$ through the following fixed point problem,

$$\underline{S} = (1 - \alpha) S_0. \tag{2.30}$$

Proposition 2. *There exists a solution to the fixed-point problem in* (2.30) *for any* $\alpha \in [0, 1]$.

To see why Proposition 2 holds, consider S_0 as a function of $\underline{S}$, $S_0 = f(\underline{S})$. First notice that when $\underline{S} = 0$, there is essentially no circuit breaker, and $f(0)$ will be the same as the initial stock price in the complete markets case. Next, there exists $s^* > 0$ such that $s^* = f(s^*)$, which is the initial price when the market closes immediately after opening. The fact that f is continuous ensures that there exists at least one crossing between the function $f(s)$ and $s/(1 - \alpha)$, which will be a solution for (2.30).

Below we will show how these steps can be neatly solved in the special case when riskless bonds are in zero net supply.

Because neither agent will take levered or short positions during market closure, there cannot be any lending or borrowing in that period. Thus, in the limiting case with net bond supply $\Delta \to 0$, all the wealth of the two agents will be invested in the stock upon market closure. Consequently the leverage constraint will always bind for the relatively optimistic investor in the presence of heterogeneous beliefs. The result is that the relatively pessimistic investor becomes the marginal investor, as summarized in the following proposition.

Proposition 3. *Suppose the stock market closes at time* $\tau < T$. *In the limiting case with bond supply* $\Delta \to 0$, *at* τ *both agents will hold all of their wealth in the stock,*

$\theta^i_\tau = \frac{W^i_\tau}{S_\tau}$, *and hold no bonds,* $\phi^i_\tau = 0$. *The market clearing price is:*

$$S_\tau = \min\left\{\widehat{S}^A_\tau, \widehat{S}^B_\tau\right\} = \begin{cases} e^{\left(\mu-\sigma^2\right)(T-\tau)} D_\tau, & \text{if} \quad \delta_\tau > \underline{\delta}(\tau) \\ e^{\left(\mu-\sigma^2\right)(T-\tau)-a(\tau,T)-b(\tau,T)\delta_\tau} D_\tau, & \text{if} \quad \delta_\tau \leq \underline{\delta}(\tau) \end{cases} \tag{2.31}$$

where $\widehat{S}^i_\tau$ *denotes the stock price in a single-agent economy populated by agent* i, *as given in* (2.16a)-(2.16b)*:*

$$\underline{\delta}(t) = -\frac{a(t,T)}{b(t,T)}, \tag{2.32}$$

and $a(t,T)$, $b(t,T)$ *are given in Proposition 1.*

Clearly, the market clearing price S_τ only depends on the belief of the relatively pessimistic agent. This result is qualitatively different from the complete markets case, where the stock price is a wealth-weighted average of the prices under the two agents' beliefs. It is a crucial result: the lower stock valuation upon market closure affects both the stock price level and dynamics before market closure, which we analyze in Section 2.4. Notice that having the lower expectation of the growth rate at the current instant is not sufficient to make the agent marginal. One also needs to take into account the agents' future beliefs and the risk premium associated with future fluctuations in the beliefs, which are summarized by $\underline{\delta}(t)$.[14]

Equation (2.31) implies that we can characterize the stopping time τ using a stochastic threshold for dividend D_t, as summarized below.

Lemma 3. *Take the triggering price* $\underline{S}$ *as given. Define a stopping time:*

$$\tau = \inf\{t \geq 0 : D_t = \underline{D}(t, \delta_t)\}, \tag{2.33}$$

[14] Technically, there is a difference between the limiting case with $\Delta \to 0$ and the case with $\Delta = 0$. When $\Delta = 0$, any price equal or below S_τ in (2.31) will clear the market. At such prices, both agents would prefer to invest more than 100% of their wealth in the stock, but both will face binding leverage constraints, which is why the stock market clears at these prices. However, these alternative equilibria are ruled out by considering a sequence of economies with bond supply $\Delta \to 0$. In each of these economies where $\Delta > 0$, the relatively pessimistic agent needs to hold the bond in equilibrium, which means his leverage constraint cannot not be binding.

where

$$\underline{D}(t,\delta_t) = \begin{cases} \underline{S}e^{-\left(\mu-\sigma^2\right)(T-t)}, & \textit{if} \quad \delta_t > \underline{\delta}(t) \\ \underline{S}e^{-\left(\mu-\sigma^2\right)(T-t)+a(t,T)+b(t,T)\delta_t}, & \textit{if} \quad \delta_t \leq \underline{\delta}(t). \end{cases} \tag{2.34}$$

Then, in the limiting case with bond supply $\Delta \to 0$, the circuit breaker is triggered at time τ whenever $\tau < T$.

Having characterized the equilibrium at time $\tau < T$, we plug the equilibrium portfolio holdings into (2.21) to derive the indirect utility of the two agents at τ:

$$V^i(W^i_\tau,\tau) = \mathbb{E}^i_\tau\left[\ln\left(\frac{W^i_\tau}{S_\tau}D_T\right)\right] = \ln(W^i_\tau) - \ln(S_\tau) + \mathbb{E}^i_\tau[\ln(D_T)]. \tag{2.35}$$

The indirect utility for agent i at $\tau \wedge T$ is then given by:

$$V^i(W^i_{\tau\wedge T},\tau\wedge T) = \begin{cases} \ln(W^i_T), & \text{if} \quad \tau \geq T \\ \ln(W^i_\tau) - \ln(S_\tau) + \mathbb{E}^i_\tau[\ln(D_T)], & \text{if} \quad \tau < T. \end{cases} \tag{2.36}$$

Substituting these indirect utility functions into the planner's problem (2.24) and taking the first order condition, we get the wealth of agent A at time $\tau \wedge T$:

$$W^A_{\tau\wedge T} = \frac{\omega S_{\tau\wedge T}}{\omega + (1-\omega)\,\eta_{\tau\wedge T}}, \tag{2.37}$$

where $S_{\tau\wedge T}$ is given in (2.26). Then, we obtain the state price density for agent A and the price of the stock at time $t \leq \tau\wedge T$ as in (2.27) and (2.28), respectively. In particular,

$$S_t = \left(\omega^A_t \mathbb{E}_t\left[S^{-1}_{\tau\wedge T}\right] + \omega^B_t \mathbb{E}^B_t\left[S^{-1}_{\tau\wedge T}\right]\right)^{-1}. \tag{2.38}$$

Here, ω^i_t is the share of total wealth owned by agent i, which, in the limiting case with $\Delta \to 0$, is identical to $\widehat{\omega}^i_t$ in (2.12) before market closure. Equation (2.38) is reminiscent of its complete markets counterpart (2.15). Unlike in the case of complete markets, the expectations in (2.38) are no longer the inverse of the stock prices from the respective representative agent economies.

From the equilibrium stock price, we can then compute the conditional mean $\mu_{S,t}$ and volatility $\sigma_{S,t}$ of stock returns, which are given by:

$$dS_t = \mu_{S,t} S_t dt + \sigma_{S,t} S_t dZ_t. \tag{2.39}$$

In Appendix 2.8.1, we provide the closed-form solution for S_t in the special case with constant disagreements ($\delta_t \equiv \delta$).

Finally, by evaluating S_t at time $t = 0$, we can solve for $\underline{S} = (1 - \alpha)S_0$ from the fixed point problem (2.30). Beyond the existence result of Proposition 2, one can further show that the fixed point is unique when the riskless bond is in zero net supply.[15]

The case of positive bond supply. When the riskless bond is in positive net supply, there are four possible scenarios upon market closure: the relatively optimistic agent faces binding leverage constraint, while the relatively pessimistic agent is either unconstrained (Scenario i) or faces binding short-sale constraint (Scenario ii); the relatively optimistic agent is unconstrained, while the relatively pessimistic agent is either unconstrained (Scenario iii) or faces binding short-sale constraint (Scenario iv). In contrast, only Scenario (i) is possible when the riskless bond is in zero net supply. The three new scenarios originate from the fact that when $\Delta > 0$ the two agents can hold different portfolios without borrowing and lending; furthermore, when the relatively optimistic agent is sufficiently wealthy, he could potentially hold the entire stock market without having to take on any leverage.

In particular, Scenario (iv) is the opposite of Scenario (i) in that the relatively optimistic agent, instead of the pessimistic one, becomes the marginal investor. As a result, the price level can become higher and volatility lower in the economy with a circuit breaker. Under Scenarios (ii) and (iii), the equilibrium stock price upon market

[15]The uniqueness is due to the fact that S_0 will be monotonically decreasing in the triggering price $\underline{S}$ in the limiting case when $\Delta \to 0$, which is not necessarily true when $\Delta > 0$.

closure is somewhere in between the two agents' valuations.

In Section 2.5.1, we examine the conditions (wealth distribution, size of bond supply, and amount of disagreement upon market closure) that determine which of the four scenarios occur in equilibrium. As we show later, which of the scenarios is realized has important implications for the equilibrium price process.

Circuit breaker and wealth distribution. We conclude this section by examining the impact of circuit breakers on the wealth distribution. As explained earlier, the wealth shares of the two agents before market closure (at time $t \leq \tau \wedge T$) will be the same as in the economy without circuit breakers, and take the form in (2.12) when the riskless bonds are in zero net supply.

However, the wealth shares at the end of the trading day (time T) will be affected by the presence of the circuit breaker. This is because if the circuit breaker is triggered at $\tau < T$, the wealth distribution after τ will remain fixed due to the absence of trading. Since irrational traders on average lose money over time, market closure at $\tau < T$ will raise their average wealth share at time T. This "mean effect" implies that circuit breakers will help "protecting" the irrational investors in this model. How strong this effect is depends on the amount of disagreement and the distribution of τ. In addition, circuit breakers will also make the tail of the wealth share distribution thinner as they put a limit on the amount of wealth that the relatively optimistic investor can lose over time along those paths with low realizations of D_t.

2.4 Impact of Circuit Breakers on Market Dynamics

We now turn to the quantitative implications of the model. In Section 2.4.1 we examine the special case of constant disagreement, $\delta_t \equiv \delta$. This case helps to demonstrate the main mechanism through which circuit breakers affect trading and asset prices. Then, in Section 2.4.2, we examine the general case with time-varying disagreements.

Throughout this section we focus on the case where riskless bonds are in zero net supply ($\Delta \to 0$). We examine the robustness of these results in Section 2.5.

2.4.1 Constant Disagreement

For calibration, we normalize $T = 1$ to denote one trading day. We set the expected value of the dividend growth $\mu = 10\%/250 = 0.04\%$ (implying an annual dividend growth rate of 10%) and its (daily) volatility $\sigma = 3\%$. The downside circuit breaker threshold is set at $\alpha = 5\%$. For the initial wealth distribution, we assume agent A (with rational beliefs) owns 90% of total wealth ($\omega = 0.9$) at $t = 0$. For the amount of disagreement, we set $\delta = -2\%$. This means agent B is relatively pessimistic about dividend growth, and his valuation of the stock at $t = 0$, $\widehat{S}_0^B$, will be 2% lower than that of agent A, $\widehat{S}_0^A$, which is fairly modest.

In Figure 2-2, we plot the equilibrium price-dividend ratio S_t/D_t (left column), the conditional volatility of returns (middle column), and the stock holding for agent A (right column). The stock holding for agent B can be inferred from that of agent A, as $\theta_t^B = 1 - \theta_t^A$. In each panel, the solid line denotes the solution for the case with circuit breaker, while the dotted line denotes the case without circuit breaker. To examine the time-of-the-day effect, we plot the solutions at two different points in time, $t = 0.25$ and 0.75, respectively.

Let's start with the price-dividend ratio. As discussed in Section 2.3.1, the price of the stock in the case without circuit breaker is the weighted (harmonic) average of the prices of the stock from the two representative-agent economies populated by agent A and B, respectively, with the weights given by the two agents' shares of total wealth (see equation (2.15)). Under our calibration, the price-dividend ratio is close to one for any $t \in [0, T]$ under agent A's beliefs ($\widehat{S}_t^A/D_t$), and it is approximately equal to $e^{\delta(T-t)} \leq 1$ under agent B's beliefs ($\widehat{S}_t^B/D_t$). These two values are denoted by the upper and lower horizontal dash lines in the left column of Figure 2-2.

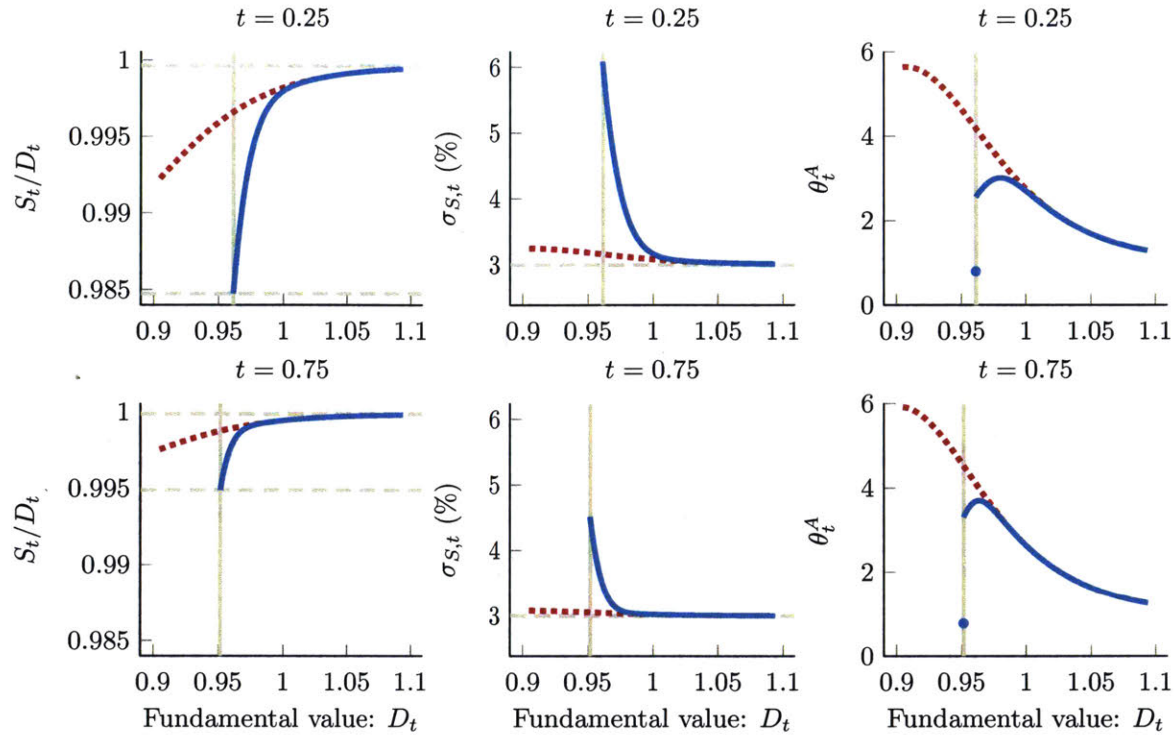

Figure 2-2: **Price-dividend ratio, conditional return volatility, and agent A's (rational optimist) portfolio holding.** Blue solid lines are for the case with circuit breaker. Red dotted lines are for the case without circuit breaker. The grey vertical bars denote the circuit breaker threshold $\underline{D}(t)$.

The price-dividend ratio in the economy without circuit breaker (red dotted line) indeed lies between $\widehat{S}_t^A/D_t$ and $\widehat{S}_t^B/D_t$. Since agent A is relatively more optimistic, he will hold levered position in the stock (see the red dotted line in the middle column), and his share of total wealth will become higher following positive shocks to the dividend. Thus, as dividend value D_t rises (falls), the share of total wealth owned by agent A increases (decreases), which makes the equilibrium price-dividend ratio approach the value $\widehat{S}_t^A/D_t$ ($\widehat{S}_t^B/D_t$).

In the case with circuit breaker, the price-dividend ratio (blue solid line) still lies between the price-dividend ratios from the two representative agent economies, but it is always below the price-dividend ratio without circuit breaker for any given level of dividend. The gap between the two price-dividend ratios is negligible when D_t is sufficiently high, but it widens as D_t approaches the circuit breaker threshold $\underline{D}(t)$.

The reason that stock price declines more rapidly with dividend in the presence of a circuit breaker can be traced to how the stock price is determined upon market closure. As explained in Section 2.3.2, at the instant when the circuit breaker is triggered, neither agent will be willing to take on levered position in the stock due to the inability to rebalance the portfolio. With bonds in zero net supply, the leverage constraint always binds for the relatively optimistic agent (agent A), and the market clearing stock price has to be such that agent B is willing to hold all of his wealth in the stock, *regardless of his share of total wealth.* Indeed, we see the price-dividend ratio with circuit breaker converging to $\widehat{S}_t^B/D_t$ when D_t approaches $\underline{D}(t)$, instead of the wealth-weighted average of $\widehat{S}_t^A/D_t$ and $\widehat{S}_t^B/D_t$. The lower stock price at the circuit breaker threshold also drives the stock price lower before market closure, with the effect becoming stronger as D_t moves closer to the threshold $\underline{D}(t)$. This explains the accelerated decline in stock price as D_t drops.

The higher sensitivity of the price-dividend ratio to dividend shocks due to the circuit breaker manifests itself in elevated conditional return volatility, as shown in the middle column of Figure 2-2. Quantitatively, the impact of the circuit breaker on the conditional volatility of stock returns can be quite sizable. Without circuit breaker, the conditional volatility of returns (red dotted lines) peaks at about 3.2%, only slightly higher than the fundamental volatility of $\sigma = 3\%$. This small amount of excess volatility comes from the time variation in the wealth distribution between the two agents. With circuit breaker, the conditional volatility (blue solid lines) becomes substantially higher as D_t approaches $\underline{D}(t)$. For example, when $t = 0.25$, the conditional volatility reaches 6% at the circuit breaker threshold, almost twice as high as the return volatility without circuit breaker.

We can also analyze the impact of the circuit breaker on the equilibrium stock price by connecting it to how the circuit breaker influences the equilibrium portfolio holdings of the two agents. Let us again start with the case without circuit breaker (red dotted lines in right column of Figure 2-2). The stock holding of agent A, $\widehat{\theta}_t^A$, continues to rise as D_t falls to $\underline{D}(t)$ and beyond. This is the result of two effects: (i) with lower D_t, the stock price is lower, implying higher expected return under agent A's beliefs; (ii) lower D_t also makes agent B (who is shorting the stock) wealthier and thus more capable of lending to agent A, who then takes on a more levered position.

With circuit breaker, while the stock holding θ_t^A takes on similar values as $\widehat{\theta}_t^A$, its counterpart in the case without circuit breaker, for large values of D_t, it becomes visibly lower than $\widehat{\theta}_t^A$ as D_t approaches the circuit breaker threshold, and it eventually starts to decrease as D_t continues to drop.

This is because agent A becomes increasingly concerned with the rising return volatility at lower D_t, which eventually dominates the effect of higher expected stock return. Finally, θ_t^A takes a discrete drop when $D_t = \underline{D}(t)$. With the leverage constraint binding, agent A will hold all of his wealth in the stock, which means θ_t^A will be equal to his wealth share ω_t^A. The preemptive deleveraging by agent A can be interpreted as a form of "self-predatory" trading. The stock price in equilibrium has to fall enough such that agent A has no incentive to sell more of his stock holding.

Time-of-the-day effect. Comparing the cases with $t = 0.25$ and $t = 0.75$, we see that the impact of circuit breaker on the price-dividend ratio and return volatility weakens as t approaches T. For example, at $t = 0.25$, the price-dividend ratio with circuit breaker can be as much as 1.2% lower than the level without circuit breaker, and the conditional return volatility peaks 6%. In contrast, at $t = 0.75$, the gap in price-dividend ratio is at most 0.3%, and the peak return volatility is 4.5%.

The reason behind this result is quite straightforward: A shorter remaining horizon reduces the potential impact of agent B's pessimistic beliefs on the equilibrium stock price, as reflected in the shrinking gap between $\widehat{S}_t^A/D_t$ and $\widehat{S}_t^B/D_t$ (the two horizontal dash lines) from the top left panel to the bottom left panel in Figure 2-2. Thus, this "time-of-the-day" effect really reflects the fact that the potential impact of circuit breaker is larger when there is more disagreement.

Notice also that the circuit breaker threshold $\underline{D}(t)$ becomes lower as t increases. That is, the dividend needs to drop more to trigger the circuit breaker later in the day. This is because the price-dividend ratio for any given D_t becomes higher as t increases.

Circuit breaker vs. pre-scheduled trading halt. Like the price-based circuit breaker, pre-scheduled trading halts, such as daily market closures, will also prevent investors from rebalancing their portfolios for an extended period of time. However, the implications of such pre-scheduled trading halts on trading behavior and price dynamics are quite different from those of circuit breakers. The key difference is that, in the case of a circuit breaker, the trigger of trading halt endogenously depends on the dividend. A negative shock to fundamentals not only reduces the price-dividend ratio through its impact on the wealth distribution (as in the case without circuit breakers), but also drives the price-dividend ratio closer to the level based on the pessimist's beliefs by moving the markets closer to the trading halt threshold.

This second effect is absent in the case of a pre-scheduled trading halt. As t approaches the pre-scheduled time of market closure $\mathcal{T}$, the price-dividend ratio converges to the pessimist valuation for all levels of dividend (which only occurs when D_t approaches $\underline{D}(t)$ in the case with circuit breaker). Away from $\mathcal{T}$, the price level is lower for all levels of dividend due to the expectation of trading halt, but there is no additional sensitivity of the price-dividend ratio to fundamental shocks, hence no volatility amplification.

To illustrate these differences, Figure 2-3 plots the price-dividend ratio, the conditional return volatility, and agent A's portfolio holding in the case when the market is scheduled to close at $\mathcal{T} = 0.5$ and remains closed until $T = 1$ (red dotted lines). We then compare these results against the case with a 5% circuit breaker (blue solid lines) as well as the case without any trading halts (grey dotted lines).

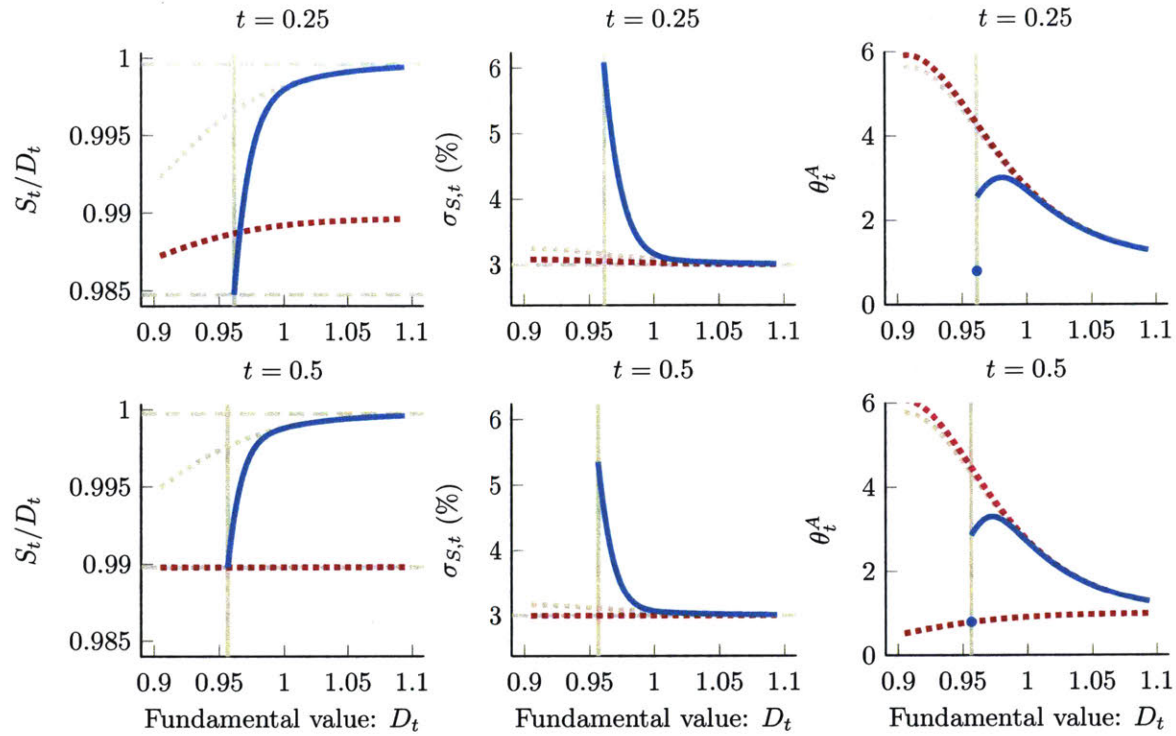

Figure 2-3: **Circuit breaker vs. pre-scheduled trading halt.** Blue solid lines are for the case with circuit breaker. Red dotted lines are for the case with pre-scheduled trading halt at $\mathcal{T} = 0.5$. Grey dotted lines are for the case without trading halts. The grey vertical bars denote the circuit breaker threshold $\underline{D}(t)$. The purple dotted line in the last panel denotes θ_t^A for $t = 0.49$.

Among the three cases, the price-dividend ratio has the most sensitivity to changes in dividend in the case of a circuit breaker; consequently, the conditional return volatility is the highest in that case. Interestingly, the conditional return volatility is the lowest with the pre-scheduled trading halt, and it almost does not change with the dividend. Moreover, unlike in the circuit breaker case, there is no preemptive deleveraging with pre-scheduled trading halt – agent A continues to take levered positions in the stock market as t approaches $\mathcal{T}$, and only delevers at the instant of market closure (see the purple dotted line – agent A's stock holding at $t = 0.49$ – and the red dotted line – stock holding at $t = 0.5$ – in the bottom right panel).

2.4.2 Time-varying Disagreement

In the previous section, we use the special case of constant disagreement to illustrate the impact of circuit breakers on trading and price dynamics. We now turn to the full model with time-varying disagreement, where the difference in beliefs δ_t follows a random walk. We do so by setting $\kappa = 0$, $\nu = \sigma$, and $\delta_0 = \bar{\delta} = 0$. Thus, there is neither initial nor long-term bias in agent B's belief.

Under this specification, Agent B's beliefs resemble the "representativeness" bias in behavioral economics. As a form of non-Bayesian updating, he extrapolates his belief about future dividend growth from the realized path of dividend.[16] As a result, he becomes overly optimistic following large positive dividend shocks and overly pessimistic following large negative dividend shocks. An alternative interpretation of such beliefs is that they capture in reduced form the behavior of constrained investors, who effectively become more (less) pessimistic or risk averse as the constraint tightens (loosens).

In Figure 2-4, we plot the price-dividend ratio, conditional return volatility, conditional expected returns under the objective probability measure, and agent A's stock holding. Unlike the constant disagreement case, dividend D_t and time of the day t are no longer sufficient to determine the state of the economy. Thus, we plot the average values of the variables conditional on t and D_t.[17]

Let's start with the price-dividend ratio, shown in the first column of Figure 2-4. Since agent A's belief about the dividend growth rate is constant over time, the price-dividend ratio under his beliefs is constant over different values of D_t (the horizontal grey dash line). However, due to the variation in δ_t which is perfectly correlated

[16]Specifically, $\delta_t = \ln\left[\frac{D_t}{D_0}e^{-(\mu-\sigma^2/2)t}\right]$, which is a mean-adjusted nonannualized realized growth rate.

[17]Given our calibration of δ_t process as a random walk, the one additional state variable besides t and D_t is the Radon-Nikodym derivative η_t, or equivalently, $\int_0^t Z_s^2 ds$ (which together with D_t determines η_t). There is no need to keep track of δ_t separately because of the one-to-one mapping between δ_t and D_t. Thus, we plot the variables of interest while setting the integral $\int_0^t Z_s^2 ds$ equal to its expected value conditional on D_t.

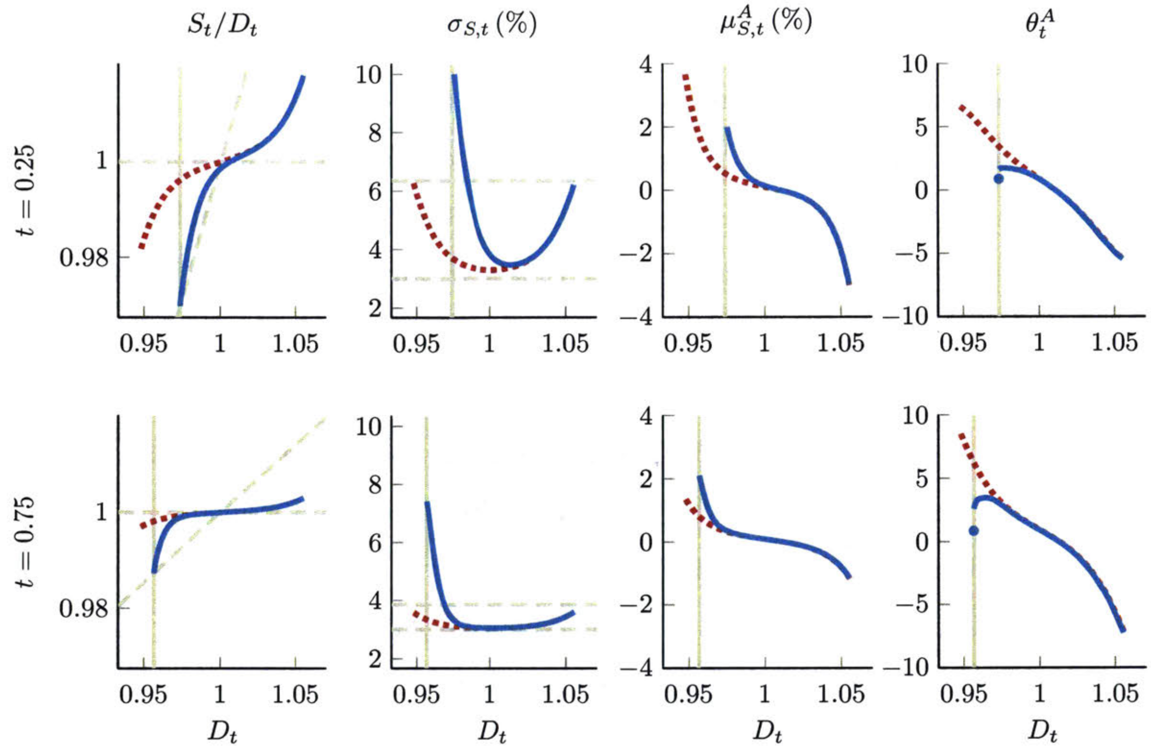

Figure 2-4: **Price-dividend ratio, conditional volatility, return drift and agent A's portfolio in the case of time-varying disagreements.** Blue solid lines are for the case with circuit breaker. Red dotted lines are for the case without circuit breaker. The grey vertical bars denote the circuit breaker threshold $\underline{D}(t)$.

with D_t, the price-dividend ratio under agent B's beliefs now increases with D_t (the upward-sloping grey dash line). The price-dividend ratio in the equilibrium without circuit breaker (red dotted line) is still a wealth-weighted average of the price-dividend ratios under the two agents' beliefs. In the presence of a circuit breaker, for any given level of dividend D_t above the circuit breaker threshold, the price-dividend ratio is lower than the value without circuit breaker, and the difference becomes more pronounced as D_t approaches the threshold $\underline{D}(t)$.[18] These properties are qualitatively the same as in the case of constant disagreement.

The circuit breaker does rule out extreme low values for the price-dividend ratio

[18]In general cases, the threshold $\underline{D}(t, \delta_t)$ depends on both t and δ_t. Since our calibration of the δ process implies a one-to-one mapping between δ_t and D_t, the threshold becomes unique for any t.

during the trading session, which could have occurred at extreme low dividend values had trading continued. This could be one of the benefits of circuit breakers. When there are intra-day mark-to-market requirements for some of the market participants, a narrower range for the price-dividend ratio can help reduce the chances of inefficient liquidations that could further destabilize the market. Formally modeling such frictions will be an interesting direction for future research.

However, the circuit breaker generates significant volatility amplification when D_t is close to $\underline{D}(t)$ (see Figure 2-4, second column). The conditional return volatility with time-varying disagreement can reach as high as 10% at $t = 0.25$, compared to the peak volatility of 6% in the constant disagreement case and the fundamental volatility of 3%. Like in the constant disagreement case, the volatility amplification effect weakens as t approaches T ("time-of-the-day effect"), which is because the effective amount of disagreement between the two agents is falling with t.

The third column of Figure 2-4 plot the conditional expected returns under the agent A's (objective) beliefs. Even when there is no circuit breaker, the conditional expected return rises as dividend falls. This is because the irrational agent B is both gaining wealth share and becoming more pessimistic as D_t falls, driving prices lower and expected returns higher for agent A higher. The presence of the circuit breaker accelerates the increase in the conditional expected return as D_t approaches the threshold $\underline{D}(t)$. Despite the higher expected returns, agent A still becomes more and more conservative when investing in the stock (see Figure 2-4, last column) due to the concern of market closure. In fact, the preemptive deleveraging by agent A is again evident as D_t approaches $\underline{D}(t)$.

Unconditional distributions of price and volatility. So far we have been analyzing the conditional effects of the circuit breaker on prices, volatility, and portfolio holdings. Next, in Figure 2-5, we examine the impact of circuit breakers on the distribution of daily average price-dividend ratios, daily price ranges, and daily return volatility. Daily price range is defined as daily high minus low prices, while daily return volatility is defined as the square root of the quadratic variation of $\log(S_t)$ over the period $[0, \tau \wedge T]$ and scaled back to daily value.

The top panel of Figure 2-5 shows that the distribution of daily average price-dividend ratio is shifted to the left in the presence of a circuit breaker, and the left tail of the distribution becomes fatter. The magnitude of the price distortion is small on average, because the large price distortions (when D_t approaches $\underline{D}(t)$) occur infrequently.

The middle and bottom panels illustrate the impact of the circuit breaker on volatility. Theoretically, when it comes to the daily price range, a commonly used measure of volatility in market microstructure studies, the circuit breaker, by limiting stock price from below, has a potential to reduce daily price ranges for certain paths of the dividend process. As the middle panel shows, however, statistically this effect is dominated by an increased price ranges for other realizations of the dividend process. So the presence of the circuit breaker shifts the whole distribution of the daily price range to the right. One gets a similar message when it comes to daily realized volatility. The presence of a circuit breaker generates a significantly fatter right tail for the distribution of daily realized volatility.

The "magnet effect". The "magnet effect" is a popular term among practitioners that refers to the changes in price dynamics as the price moves closer to the limit. While there is no formal definition of this effect, we try to formalize this notion in our model by computing the conditional probability that the stock price, currently at S_t, will reach the circuit breaker threshold $(1-\alpha)S_0$ within a given period of time h, which we refer to as conditional hitting probability, and comparing these probabilities to their counterparts in absence of the circuit breaker.

In Figure 2-6, we plot the conditional hitting probabilities for the horizon of $h = 10$ minutes. When S_t is sufficiently far from $(1-\alpha)S_0$ (say $S_t > 0.98$), the conditional

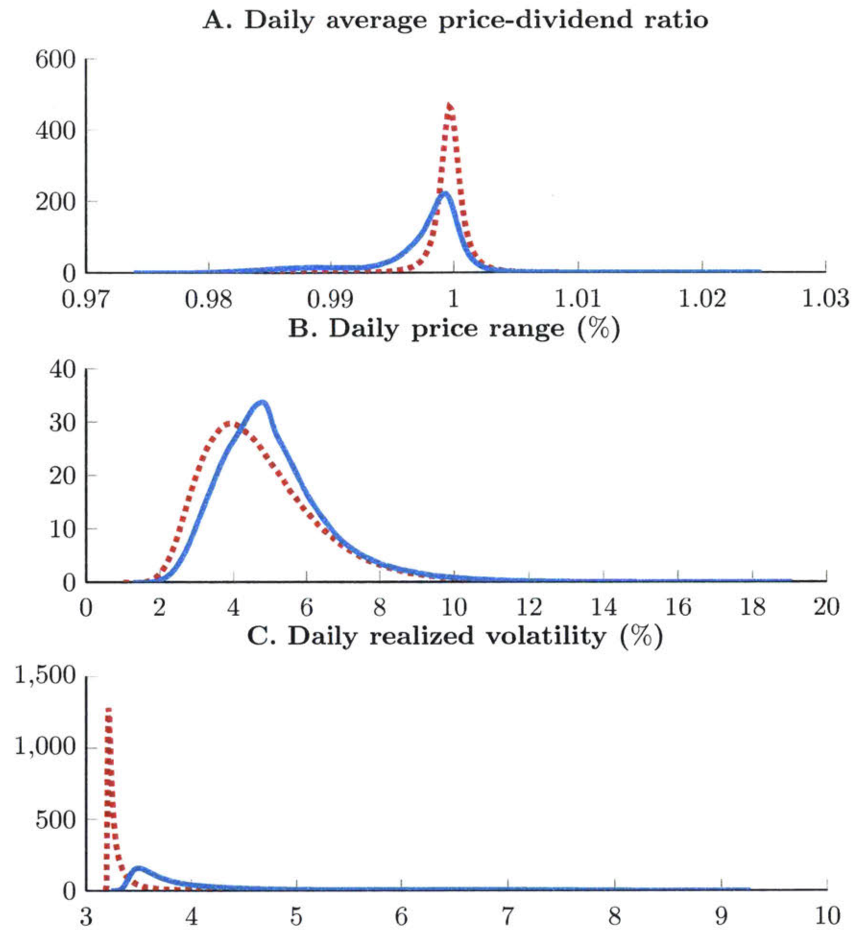

Figure 2-5: **Distributions of price-dividend ratio, daily price range, and realized volatility.** Blue solid lines are for the case with circuit breaker. Red dotted lines are for the case without circuit breaker.

hitting probabilities with and without circuit breaker are both essentially zero. The gap between the two hitting probabilities quickly widens as the stock price moves closer to the threshold. By the time S_t reaches 0.96, the conditional hitting probability with circuit breaker has risen above 20%, while the hitting probability without circuit breaker is still close to 0. The gap eventually narrows as both hitting probabilities will converge to 1 as S_t reaches $(1-\alpha)S_0$.

This is our version of the "magnet effect": the very presence of a circuit breaker raises the probability of the stock price reaching the threshold. Moreover, the pace at which the hitting probability increases as the stock price moves closer to the threshold

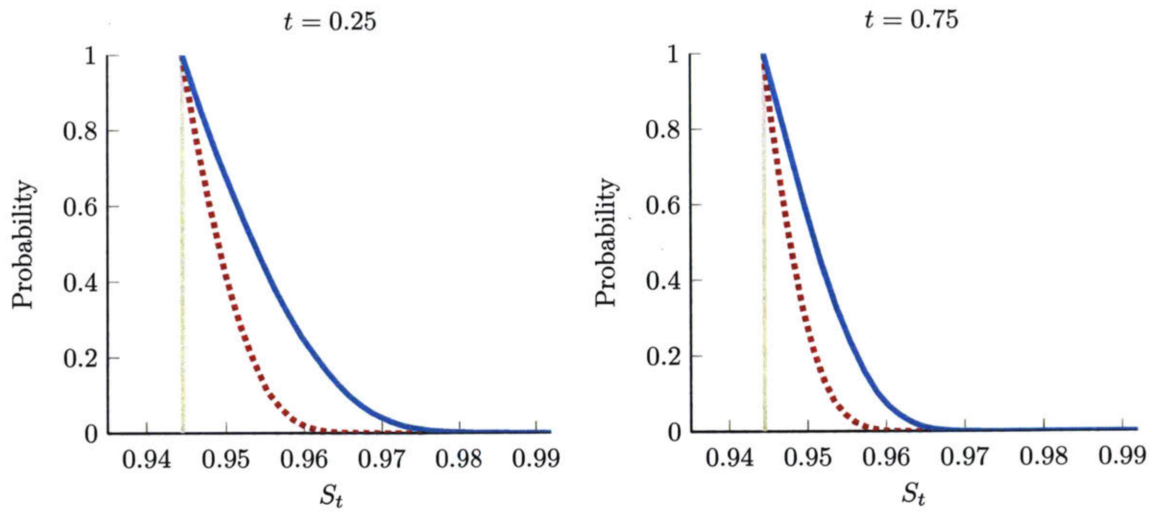

Figure 2-6: **The "magnet effect".** Conditional probabilities for the stock price to reach the circuit breaker limit within the next 10 minutes. Blue solid lines are for the case with circuit breaker. Red dotted lines are for the case without circuit breaker. The grey vertical bars denote the circuit breaker threshold $\underline{D}(t)$.

will be much faster with a circuit breaker than what a "normal price process" would imply. The "magnet effect" is caused by the significant increase in conditional return volatility in the presence of a circuit breaker. Combined with the "time-of-the-day" effect, it is not surprising to see that the "magnet effect" is stronger earlier during the trading day.

Welfare implications In the absence of other frictions, trading halts would reduce investors' abilities to share risk. When the reason to trade is heterogeneous beliefs and the social planner respects the beliefs of individual investors, then any such trading halts will inevitably reduce welfare. The amount of welfare loss depends on the initial wealth distribution (it will be higher when wealth is more evenly distributed). Based on our calibration, the certainty equivalent loss in wealth peaks at close to 3%.

Alternatively, if the planner takes a paternalistic view by evaluating welfare under the correct probability measure, then circuit breakers could protect those investors with irrational beliefs from hurting themselves by trading too much. Such calculations yield certainty equivalent gain in wealth that peaks at about 1%. While this result might appear to provide some justification for implementing circuit breaker rules, its implication would be to prevent trading by irrational investors altogether.

While our framework provides a neoclassical benchmark that highlights some of the negative effects of circuit breakers, it may well be incomplete in providing a full analysis on welfare or the optimal design of circuit breaker rules. To do so requires one to properly take into account the actual frictions such as coordination and information problems, which we leave for future research.

2.5 Robustness

Our analysis in Section 2.4 has focused on the case where riskless bonds are in zero net supply ($\Delta \to 0$). In this section, we examine the robustness of these results when riskless bonds are in positive supply. In addition, we also discuss the differences between the continuous-time and discrete-time settings.

2.5.1 Positive Bond Supply

In the model with positive riskless bond supply, we first consider the problem at the instant before market closure, which will provide us with much of the intuition about the effect of positive bond supply. Suppose the stock market will close at some arbitrary time τ with fundamental value D_τ. There is still trading at time τ, but the agents have to hold onto their portfolios thereafter until time T. The equilibrium conditions are already given in Section 2.3.2.

For illustration, in Figure 2-7 we plot the equilibrium stock price as a function of the wealth share of agent A for $\tau = 0.25$ and $D_\tau = 0.97$ (blue solid line), and compare it to the stock price under complete markets (red dotted line). The bond supply is assumed to be $\Delta = 0.17$. Notice that because of the one-to-one mapping between D_t

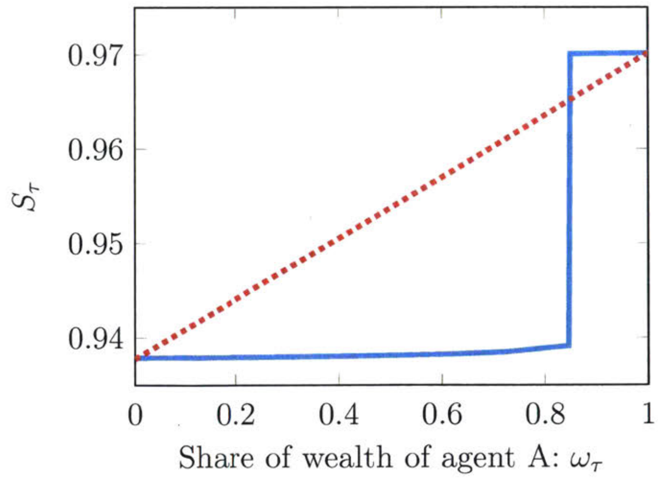

Figure 2-7: **Stock price upon market closure: positive bond supply.** Blue solid lines are for the case with circuit breaker. Red dotted lines are for the case without circuit breaker.

and δ_t, we know that agent A is relatively more optimistic for this value of D_τ.

As before, the stock price without circuit breaker is a weighted average of the optimist and pessimist valuations in the case with positive bond supply, with the weight depending on their respective wealth shares. Since agent A is more optimistic, the stock price without circuit breaker is linearly increasing in her wealth share.

We have seen that, when riskless bonds are in zero net supply, the stock price with a circuit breaker will always be equal to the pessimistic valuation at the time of market closure. However, this is no longer the case with $\Delta > 0$. As Figure 2-7 shows, when agent A's wealth share is not too high, the stock price with circuit breaker is lower than its complete markets counterpart, but the opposite occurs when agent A's wealth share is sufficiently high.

The intuition is as follows. When agent A's wealth share ω_τ^A is not too high, he invests all of his wealth into the stock, but that is still not enough to clear the stock market. In this case, agent A's leverage constraint will be binding, agent B will hold all the riskless bonds and the remaining stock not held by agent A, and the market clearing price has to agree with agent B's (the pessimist) valuation.

This scenario is similar to the case where riskless bonds are in zero net supply, where the pessimist is also the marginal investor. One difference is that the pessimist's valuation here increases with ω_τ^A instead of remaining constant. This is because as agent A gets wealthier, agent B's portfolio will become less risky (he is required to hold less stock relative to the riskless bonds), which makes him value the stock more. However, this effect is quantitatively small.

When agent A's wealth share becomes sufficiently high, he will be able to hold the entire stock market without borrowing. This is possible because not all of the wealth in the economy is in the stock. In such cases, agent A will hold all of the stock and potentially some bonds, while agent B will invest all of his wealth in the bonds. As long as the stock price is above agent B's private valuation, he would want to short the stock, but the short-sales constraint would be binding (an arbitrarily small short position can lead to negative wealth). Consequently, agent A (the optimist) becomes the marginal investor, and the market clearing price has to agree with his valuation.[19] This equilibrium is qualitatively different. The switch of the marginal investor from pessimist to optimist means that the price-dividend ratio will be higher and conditional return volatility lower with circuit breaker.

The above analysis highlights the key differences between the economies with positive and zero bond supply. For a given amount of bond supply Δ, the circuit breaker equilibrium will be similar to what we have seen in the zero bond supply case as long as agent A's wealth share is not too high. However, if agent A's wealth share becomes sufficiently high, the property of the equilibrium changes drastically. Price level will become higher and volatility lower with circuit breaker. Moreover, the region for which this alternative scenario occurs will become wider as bond supply Δ increases (relative to the net supply of the stock).

Figure 2-8 shows a heat map for the ratio of average daily return volatility in

[19]There is also a knife-edge case where agent A does not hold any bonds, the stock price is below his valuation, but he faces binding leverage constraint. In this case, the stock price and the wealth distribution have to satisfy the condition $\omega_\tau^A = \frac{S_\tau}{S_\tau+\Delta}$.

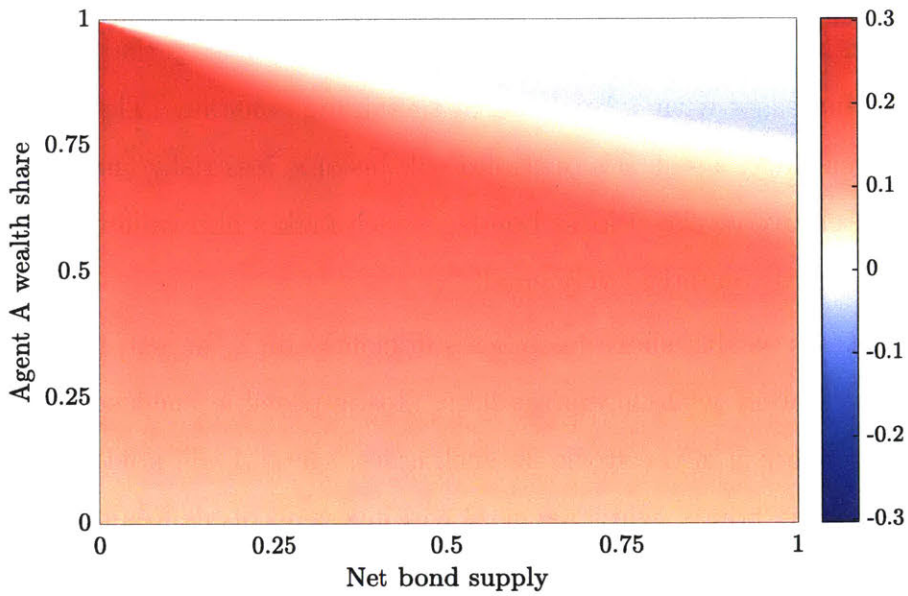

Figure 2-8: **Heat map: ratio of realized volatilities.** Heat map for the ratio of average daily volatilities in the economies with and without circuit breakers.

the economies with and without circuit breaker. It is done for a wide range of net bond supply (Δ) and initial wealth share for agent A (ω). The red region indicates volatility amplification by the circuit breaker, while the blue region indicates the opposite. Quantitatively, the volatility amplification effect is stronger when net bond supply is small, and when agent A's initial wealth share is not too low or too high.

In the data, the net supply of riskless bonds relative to the stock market is likely small. For example, the total size of the U.S. corporate bond market is about \$8 trillion in 2016, while the market for equity is about \$23 trillion. If we assume a recovery rate of 50%, the relative size of the market for riskless bonds would be 0.17. Even if one counts the total size of the U.S. market for Treasuries, federal agency securities, and money market instruments (about \$16.8 trillion in 2016) together with corporate bonds, the relative size of riskless bonds will be less than 1, and we still get volatility amplification for most values of ω.

2.5.2 Bounded Stock Prices

Our model is set in continuous time with the dividend following a geometric Brownian motion. In a finite time interval $(t, t+s)$, the dividend can in theory take any value on the interval $(0, \infty)$, as does the stock price, no matter how small s is. This feature together with a utility function that satisfies the Inada condition implies that the agents in our model cannot take any levered or short positions in the stock upon market closure. This is why the pessimist has to become the marginal investor upon market closure when bonds are in zero net supply.

In the previous section, we have already seen how the equilibrium could change when riskless bonds are in positive supply. Similar changes could occur if the stock price has a non-zero lower bound during the period of market closure. With bounded prices, the optimistic agent will be able to maintain some leverage when the market closes. Like in the case with positive bond supply, this agent will then be able to hold the entire stock market by himself if his wealth share is sufficiently large, and he might become the marginal investor when the pessimistic agent faces binding short-sales constraint.

Why might the stock price be bounded from below? One reason could be government bailout. It would also effectively capture the fact that bankruptcy is not infinitely costly. In unreported results, we study a discrete-time version of the model where the dividend process is modeled as a binomial tree. We find that when the share of wealth owned by the optimist is not too high, the presence of a circuit breaker lowers prices and increases conditional volatility. The magnitude of the effects is also similar to what we see in the continuous-time model.

2.6 Extensions

In this section, we extend the model to two-sided circuit breakers. We also discuss how the model can be extended to allow for multi-tier circuit breakers as well as circuit

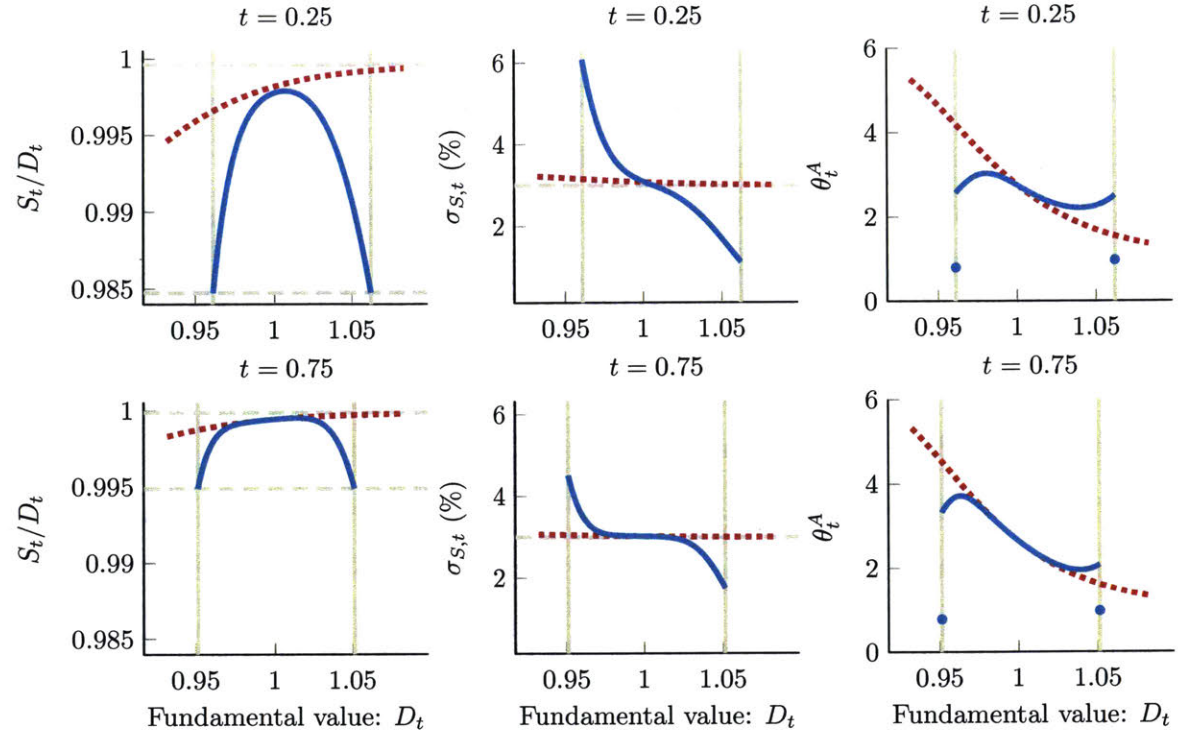

Figure 2-9: **Price-dividend ratio, conditional return volatility, and agent A's (rational optimist) portfolio holdings in the two-sided circuit breaker case.** Blue solid lines are for the case with circuit breaker. Red dotted lines are for the case without circuit breaker. The grey vertical bars denote the downside threshold $\underline{D}(t)$ and upside threshold $\overline{D}(t)$.

breakers triggered by variables other than price levels.

Two-sided circuit breakers. In some markets, the circuit breaker is triggered when the stock price reaches either the lower bound $(1-\alpha_D)\,S_0$ or the upper bound $(1+\alpha_U)\,S_0$, whichever happens first. This is straightforward to model in our framework. For simplicity, we consider the two-sided circuit breakers in the constant disagreement case, with $\alpha_D = \alpha_U = 5\%$.

Figure 2-9 shows the results. With the two-sided circuit breaker, the price-dividend ratio is no longer monotonically increasing in D_t; instead, it takes an inverse U-shape. As in the downside circuit breaker case, the price-dividend ratio converges to the pessimist valuation as D_t approaches the downside threshold $\underline{D}(t)$. As D_t rises, the price-dividend ratio also rises and approaches the level in the economy without circuit breaker. The key difference is that as D_t keeps increasing, the price-dividend ratio starts to decline, and eventually converges to the pessimist valuation again as D_t approaches the upside threshold $\overline{D}(t)$.

The reason that the price-dividend ratio converges to the pessimist valuation at the upside circuit breaker threshold $\overline{D}(t)$ is the same as at the downside threshold $\underline{D}(t)$. Since the leverage constraint starts to bind for the optimistic agent upon market closure, the pessimistic agent has to become the marginal investor regardless of whether it is the upside or downside circuit breaker.

While the price-dividend ratio converges to the same value for downside and upside circuit breakers, the implications for conditional return volatility are quite different. As Figure 2-9 shows, while the conditional return volatility is amplified (relative to the case without circuit breaker) when D_t is low, the opposite is true when D_t is high, with the volatility dropping to as low as 1% (compared to the fundamental volatility of 3%) at $t = 0.25$. Intuitively, when D_t is high, the direct impact of a positive fundamental shock on the stock price is partially offset by the negative impact due to the increase in the probability of market closure. Thus, the stock price becomes less sensitive to fundamental shocks, which results in lower volatility.

In summary, like the downside circuit breaker, the upside circuit breaker also causes the leverage constraint to bind for the optimistic investor and the pessimistic investor to become the marginal investor. Their impacts on volatility are quite different: conditional return volatility is amplified near the downside circuit breaker threshold but reduced near the upside threshold.

Multi-tier circuit breakers. Circuit breakers implemented on exchanges often have more than one trigger threshold. For example, market-wide circuit breaker in the U.S. stock market can be triggered at three different downside thresholds: 7%, 13%, and 20%. One can solve the model with multi-tier circuit breakers using backward induction, with the starting point being the state when only the highest threshold remains un-triggered (this is identical to the single-tier circuit breaker problem).

To gain some intuition on the effects of multi-tier circuit breakers, consider an example with two downside thresholds, α_1 and α_2, with $\alpha_1 < \alpha_2$, and assume bonds are in zero net supply. Whenever the stock price reaches $(1-\alpha_1)S_0$ during the trading day, trading is suspended for a period s (e.g., 10 minutes) if there is more than s remaining in the trading day; otherwise the market is closed for the rest of the day. After reopening, if the price reaches $(1-\alpha_2)S_0$, the market will be closed till the end of the trading day.

When market reopens following the first trading halt, the price dynamics are isomorphic to the case of single-tier circuit breaker. Suppose the first threshold is reached at τ_1 (and we will focus on the interesting case where $\tau_1 + s < T$). Even though the duration of trading halt will be relatively short, both agents will still avoid taking on levered or short positions upon the first trading halt. This again means that the pessimist (agent B) has to become the marginal investor at τ_1. However, a key difference between multi-tier and single-tier circuit breakers is that agent B's valuation of the stock at time τ_1 now depends on his beliefs about the stock price at the time when the market reopens, S_{τ_1+s}.

There are two possible scenarios here. First, if agent B's private valuation at the time of reopening (which depends on D_{τ_1+s}) is higher than the second circuit breaker threshold, $\widehat{S}^B_{\tau_1+s} > (1-\alpha_2)S_0$, then the market will reopen, and the participation of the optimistic agent will raise the market price above agent B's private valuation, $S_{\tau_1+s} > \widehat{S}^B_{\tau_1+s}$. Second, if agent B's private valuation at the time of reopening is lower than the second circuit breaker threshold, the second-tier circuit breaker will be triggered immediately when market reopens. In this case, $S_{\tau_1+s} = \widehat{S}^B_{\tau_1+s}$. Thus, the stock price at the time of reopening following the first trading halt will be on average higher than agent B's private valuation, which means the closing price upon the first trading halt will also be higher than agent B's private valuation, $S_{\tau_1} > \widehat{S}^B_{\tau_1}$.

The above result suggests that the dynamics of the two agents' portfolio holdings will be similar as the market approaches earlier versus later trading halts. However, the impact of trading halt on the price level and return volatility will be weaker near earlier trading halts due to the expectation of market reopening, more so when the duration of trading halt s is shorter.

Circuit breakers based on non-price variables. An appealing property of the solution strategy presented in Section 2.3.2 is that so long as market closure is characterized by an $\mathcal{F}_t$-stopping time τ, we can determine the stock price upon and prior to market closure in the same way. This means we can use the same solution strategy to study any other types of circuit breakers where the circuit breaker trigger criterion is determined by the history of (D_t, δ_t, η_t) (we will need to search for the stopping rule that is consistent with the circuit breaker criterion, a fixed point problem). For example, we can use this method to solve for models where trading halts are based on conditional return volatility or measures of trading volume (see e.g., Xiong and Yan, 2010).

2.7 Conclusion

In this paper, we build a dynamic model to examine the mechanism through which market-wide circuit breakers affect trading and price dynamics in the stock market. As we show, a downside circuit breaker tends to lower the price-dividend ratio, reduce daily price ranges, but increase conditional and realized volatility. It also raises the probability of the stock price reaching the circuit breaker limit as the price approaches the threshold (the "magnet effect"). The effects of circuit breakers can be further amplified when some agents' willingness to hold the stock is sensitive to recent shocks to fundamentals, which can be due to behavioral biases, institutional constraints, etc.

Our results demonstrate some of the negative impacts of circuit breakers even without any other market frictions, and they highlight the source of these effects, namely the tightening of leverage constraint when levered investors cannot rebalance their portfolios during trading halts. These results also shed light on the design of circuit breaker rules. Using historical price data from a period when circuit breakers have not been implemented can lead one to significantly underestimate the likelihood of triggering a circuit breaker, especially when the threshold is relatively tight.

2.8 Appendix

2.8.1 Proofs

Proof of Proposition 1

When there are no circuit breakers, the stock price is

$$\widehat{S}_t = \mathbb{E}_t\left[\frac{\widehat{\pi}_T^A D_T}{\mathbb{E}_t\left[\widehat{\pi}_T^A\right]}\right] = \frac{\mathbb{E}_t\left[\theta + (1-\theta)\,\eta_T\right]}{\mathbb{E}_t\left[D_T^{-1}\left(\theta + (1-\theta)\,\eta_T\right)\right]} = \frac{\theta + (1-\theta)\,\eta_t}{\theta \mathbb{E}_t\left[D_T^{-1}\right] + (1-\theta)\,\mathbb{E}_t\left[D_T^{-1}\eta_T\right]}, \tag{2.40}$$

where

$$\mathbb{E}_t\left[D_T^{-1}\right] = D_t^{-1} e^{-(\mu-\sigma^2)(T-t)}, \tag{2.41}$$

and

$$\mathbb{E}_t\left[D_T^{-1}\eta_T\right] = \eta_t \mathbb{E}_t\left[D_T^{-1}\frac{\eta_T}{\eta_t}\right] = \eta_t \mathbb{E}_t^B\left[D_T^{-1}\right]. \tag{2.42}$$

Define the log dividend $x_t = \log D_t$. Under measure $\mathbb{P}^B$, the processes for x_t and δ_t are

$$dx_t = \left(\mu - \frac{\sigma^2}{2} + \delta_t\right) dt + \sigma dZ_t^B, \tag{2.43a}$$

$$d\delta_t = \left(\kappa\bar{\delta} + \left(\frac{\nu}{\sigma} - \kappa\right)\delta_t\right) dt + \nu dZ_t^B. \tag{2.43b}$$

Define $X_t = [x_t \ \delta_t]'$, then X_t follows an affine process,

$$dX_t = (K_0 + K_1 X_t)\, dt + \sigma_X dZ_t^B, \tag{2.44}$$

with

$$K_0 = \begin{bmatrix} \mu - \frac{\sigma^2}{2} \\ \kappa\bar{\delta} \end{bmatrix}, \; K_1 = \begin{bmatrix} 0 & 1 \\ 0 & \frac{\nu}{\sigma} - \kappa \end{bmatrix}, \; \sigma_X = \begin{bmatrix} \sigma \\ \nu \end{bmatrix}. \tag{2.45}$$

We are interested in computing

$$g\,(t, X_t) = \mathbb{E}_t^B\left[e^{\rho_1' X_T}\right], \quad \text{with } \rho_1 = [-1 \ \ 0]'. \tag{2.46}$$

By applying standard results for the conditional moment-generating functions of affine processes (see, .e.g., Singleton 2006), we get

$$g\left(t, X_t\right) = \exp\left(A\left(t, T\right) + B\left(t, T\right)' X_t\right), \tag{2.47}$$

where

$$\begin{aligned} 0 &= \dot{B} + K_1' B, \quad B\left(T, T\right) = \rho_1 && (2.48a) \\ 0 &= \dot{A} + B' K_0 + \frac{1}{2} tr\left(B B' \sigma_X \sigma_X'\right), \quad A\left(T, T\right) = 0 && (2.48b) \end{aligned}$$

Solving for the ODEs gives:

$$B(t, T) = \begin{bmatrix} -1 & \dfrac{1 - e^{\left(\frac{\nu}{\sigma} - \kappa\right)(T-t)}}{\frac{\nu}{\sigma} - \kappa} \end{bmatrix}', \tag{2.49}$$

and

$$\begin{aligned} A(t, T) = &\left[\mu - \sigma^2 - \frac{\kappa\bar{\delta} - \sigma\nu}{\frac{\nu}{\sigma} - \kappa} - \frac{\nu^2}{2\left(\frac{\nu}{\sigma} - \kappa\right)^2}\right](t - T) - \frac{\nu^2}{4\left(\frac{\nu}{\sigma} - \kappa\right)^3}\left[1 - e^{2\left(\frac{\nu}{\sigma} - \kappa\right)(T-t)}\right] \\ &+ \left[\frac{\kappa\bar{\delta} - \sigma\nu}{\left(\frac{\nu}{\sigma} - \kappa\right)^2} + \frac{\nu^2}{\left(\frac{\nu}{\sigma} - \kappa\right)^3}\right]\left[1 - e^{\left(\frac{\nu}{\sigma} - \kappa\right)(T-t)}\right]. \end{aligned} \tag{2.50}$$

After plugging the above results back into (2.40) and reorganizing the terms, we get

$$S_t = \frac{\theta + (1-\theta)\eta_t}{\theta + (1-\theta)\eta_t H(t, \delta_t)} D_t e^{\left(\mu - \sigma^2\right)(T-t)}, \tag{2.51}$$

where

$$\begin{aligned} H(t, \delta_t) &= e^{a(t,T) + b(t,T)\delta_t}, && (2.52a) \\ a(t, T) &= \left[\frac{\kappa\bar{\delta} - \sigma\nu}{\frac{\nu}{\sigma} - \kappa} + \frac{\nu^2}{2\left(\frac{\nu}{\sigma} - \kappa\right)^2}\right](T - t) - \frac{\nu^2}{4\left(\frac{\nu}{\sigma} - \kappa\right)^3}\left[1 - e^{2\left(\frac{\nu}{\sigma} - \kappa\right)(T-t)}\right] \\ &\quad + \left[\frac{\kappa\bar{\delta} - \sigma\nu}{\left(\frac{\nu}{\sigma} - \kappa\right)^2} + \frac{\nu^2}{\left(\frac{\nu}{\sigma} - \kappa\right)^3}\right]\left[1 - e^{\left(\frac{\nu}{\sigma} - \kappa\right)(T-t)}\right], && (2.52b) \\ b(t, T) &= \frac{1 - e^{\left(\frac{\nu}{\sigma} - \kappa\right)(T-t)}}{\frac{\nu}{\sigma} - \kappa}. && (2.52c) \end{aligned}$$

Finally, to compute the conditional volatility of stock returns, we have

$$\begin{aligned}
d\widehat{S}_t &= \widehat{\mu}_{S,t}\widehat{S}_t dt + \widehat{\sigma}_{S,t}\widehat{S} dZ_t \\
&= o(dt) + \widehat{S}_t \frac{dD_t}{D_t} + \eta_t D_t e^{\left(\mu-\sigma^2\right)(T-t)} \frac{\theta(1-\theta)[1-H(t,\delta_t)]}{[\theta+(1-\theta)\eta_t H(t,\delta_t)]^2} \frac{d\eta_t}{\eta_t} \\
&\quad - D_t e^{\left(\mu-\sigma^2\right)(T-t)} \frac{[\theta+(1-\theta)\eta_t](1-\theta)\eta_t H(t,\delta_t) b(t,T)}{[\theta+(1-\theta)\eta_t H(t,\delta_t)]^2} d\delta_t.
\end{aligned}$$

After collecting the diffusion terms, we get

$$\begin{aligned}
\widehat{\sigma}_{S,t} = \sigma + \frac{D_t e^{\left(\mu-\sigma^2\right)(T-t)}}{\widehat{S}_t} &\left\{ \frac{\theta(1-\theta)\left[1-H(t,\delta_t)\right]}{[\theta+(1-\theta)\eta_t H(t,\delta_t)]^2} \frac{\delta_t \eta_t}{\sigma} \right. \\
&\left. - \frac{\left[\theta+(1-\theta)\eta_t\right](1-\theta)\eta_t b(t,T) H(t,\delta_t)}{[\theta+(1-\theta)\eta_t H(t,\delta_t)]^2} \nu \right\}. \qquad (2.53)
\end{aligned}$$

Proof of Proposition 3

Suppose market closes at time τ. Since bonds are zero net supply,

$$W_{1,\tau} + W_{2,\tau} = S_\tau.$$

Then the agents' problems at time τ are:

$$\begin{aligned}
V^1\left(W_{1,\tau}\right) \quad = \quad & \max_{\theta_1,b_1} \quad E_\tau^1\left[\ln\left(\theta_1 D_T + b_1\right)\right] \\
& s.t. \quad \theta_1 S_\tau + b_1 \leq W_{1,\tau} \\
& \qquad \theta_1 \geq 0,\ b_1 \geq 0
\end{aligned}$$

and

$$\begin{aligned}
V^2\left(W_{2,\tau}\right) \quad = \quad & \max_{\theta_2,b_2} \quad E_\tau^2\left[\ln\left(\theta_2 D_T + b_2\right)\right] \\
& s.t. \quad \theta_2 S_\tau + b_2 \leq W_{2,\tau} \\
& \qquad \theta_2 \geq 0,\ b_2 \geq 0.
\end{aligned}$$

The Lagrangian:

$$L = E_\tau^1 \left[\ln\left(\theta_1 D_T + b_1\right)\right] + \zeta^1 \left(W_{1,\tau} - \theta_1 S_\tau - b_1\right) + \xi_a^1 \theta_1 + \xi_b^1 b_1.$$

FOC:

$$\begin{aligned} 0 &= E_\tau^i \left[\frac{D_T}{\theta_i D_T + b_i}\right] - \zeta^i S_\tau + \xi_a^i \\ 0 &= E_\tau^i \left[\frac{1}{\theta_i D_T + b_i}\right] - \zeta^i + \xi_b^i. \end{aligned}$$

Suppose agent 1 is less optimistic than agent 2. Then we can quickly examine following three cases:

1. It could be an equilibrium when the price is sufficiently low such that both agents want to take levered positions (putting more than 100% of their wealth in the stock) but are constrained from borrowing. In this case, both agents submit demands proportional to their wealth:

$$\begin{aligned} \theta_i^* &= \frac{W_{i,\tau}}{S_\tau} \\ \xi_a^1 &= \xi_a^2 = 0 \\ \xi_b^1 &> 0, \;\; \xi_b^2 > 0 \end{aligned}$$

   and the market for the stock clears. In this case,

$$S_\tau = \frac{E_\tau^1 \left[\frac{D_T}{\theta_1 D_T + b_1}\right]}{E_\tau^1 \left[\frac{1}{\theta_1 D_T + b_1}\right] + \xi_b^1} = \frac{1}{E_\tau^1 \left[\frac{1}{D_T}\right] + \xi_b^1 \theta_1} < \frac{1}{E_\tau^1 \left[\frac{1}{D_T}\right]} = S_\tau^*.$$

2. It could be an equilibrium when agent 1 (pessimist) finds it optimal to hold all his

wealth in the stock, while agent 2 (optimist) is constrained from borrowing:

$$\begin{aligned}\theta_i^* &= \frac{W_{i,\tau}}{S_\tau}\\ \xi_a^1 &= \xi_a^2 = 0\\ \xi_b^1 &= 0, \;\; \xi_b^2 > 0.\end{aligned}$$

Then from agent 1,

$$\begin{aligned}0 &= E_\tau^1\left[\frac{D_T}{\theta_1 D_T + b_1}\right] - \zeta^1 S_\tau\\ 0 &= E_\tau^1\left[\frac{1}{\theta_1 D_T + b_1}\right] - \zeta^1.\end{aligned}$$

This implies:

$$S_\tau = \frac{E_\tau^1\left[\frac{D_T}{\theta_1^* D_T + b_1^*}\right]}{E_\tau^1\left[\frac{1}{\theta_1^* D_T + b_1^*}\right]} = \frac{E_\tau^1\left[\frac{1}{\theta_1^*}\right]}{E_\tau^1\left[\frac{1}{\theta_1^* D_T}\right]} = \frac{1}{E_\tau^1\left[\frac{1}{D_T}\right]} = D_\tau e^{(\mu-\sigma^2)(T-\tau)} = S_\tau^*.$$

The latter follows from the fact that market clearing for bonds implies that $b_1^* = b_2^* = 0$, and $\theta_1^* > 0$ as long as $W_{1,\tau} > 0$. Let's check whether this is this consistent with agent 2:

$$\begin{aligned}0 &= E_\tau^2\left[\frac{D_T}{\theta_2 D_T + b_1}\right] - \zeta^2 S_\tau\\ 0 &= E_\tau^2\left[\frac{1}{\theta_2 D_T + b_2}\right] - \zeta^2 + \xi_b^2\\ S_\tau &= \frac{E_\tau^2\left[\frac{D_T}{\theta_2^* D_T + b_2^*}\right]}{E_\tau^2\left[\frac{1}{\theta_2^* D_T + b_2^*}\right] + \xi_b^2} = \frac{E_\tau^2\left[\frac{1}{\theta_2^*}\right]}{E_\tau^2\left[\frac{1}{\theta_2^* D_T}\right] + \xi_b^2} = \frac{1}{E_\tau^2\left[\frac{1}{D_T}\right] + \xi_b^2\theta_2^*}.\end{aligned}$$

Since agent 2 is more optimistic, we have:

$$E_\tau^1\left[\frac{1}{D_T}\right] > E_\tau^2\left[\frac{1}{D_T}\right],$$

which implies:

$$\xi_b^2 = \frac{E_\tau^1\left[\frac{1}{D_T}\right] - E_\tau^2\left[\frac{1}{D_T}\right]}{\theta_2^*} > 0.$$

3. For any $S_\tau > S_\tau^*$, agent 1 will prefer to hold less than 100% of the wealth in the stock. This would require agent 2 to take levered position, which cannot be an equilibrium.

We will restrict our attention to equilibriums of type 2.

Special Case: Constant Disagreement

The stock price can be computed in closed form in the case of constant disagreement, $\delta_t = \delta$. Without loss of generality, we focus on the case where agent B is relatively more optimistic, $\delta \geq 0$. The results are summarized below.

Proposition 4. *Take S_0 as given. With $\delta \geq 0$, the stock price time $t \leq \tau \wedge T$ is*

$$S_t = \left(\omega_t^A \mathbb{E}_t[S_{\tau\wedge T}^{-1}] + \omega_t^B \mathbb{E}_t^B[S_{\tau\wedge T}^{-1}]\right)^{-1}, \tag{2.54}$$

where

$$\begin{aligned}
\mathbb{E}_t[S_{\tau\wedge T}^{-1}] &= \frac{1}{\alpha S_0}\left\{N\left[\frac{d_t - \frac{\sigma(T-t)}{2}}{\sqrt{T-t}}\right] + e^{\sigma d_t} N\left[\frac{d_t + \frac{\sigma(T-t)}{2}}{\sqrt{T-t}}\right]\right\} \\
&\quad + D_t^{-1} e^{-(\mu-\sigma^2)(T-t)}\left\{N\left[-\frac{d_t + \frac{\sigma(T-t)}{2}}{\sqrt{T-t}}\right] - e^{-\sigma d_t} N\left[\frac{d_t - \frac{\sigma(T-t)}{2}}{\sqrt{T-t}}\right]\right\}, &(2.55)\\
\mathbb{E}_t^B[S_{\tau\wedge T}^{-1}] &= \frac{1}{\alpha S_0}\left\{N\left[\frac{d_t - \left(\frac{\delta}{\sigma} + \frac{\sigma}{2}\right)(T-t)}{\sqrt{T-t}}\right] + e^{\left(\sigma + \frac{2\delta}{\sigma}\right)d_t} N\left[\frac{d_t + \left(\frac{\delta}{\sigma} + \frac{\sigma}{2}\right)(T-t)}{\sqrt{T-t}}\right]\right\} \\
&\quad + D_t^{-1} e^{-(\mu-\sigma^2+\delta)(T-t)}\left\{N\left[-\frac{d_t - \left(\frac{\delta}{\sigma} - \frac{\sigma}{2}\right)(T-t)}{\sqrt{T-t}}\right]\right. \\
&\quad \left. - e^{\left(\frac{2\delta}{\sigma} - \sigma\right)d_t} N\left[\frac{d_t + \left(\frac{\delta}{\sigma} - \frac{\sigma}{2}\right)(T-t)}{\sqrt{T-t}}\right]\right\}, &(2.56)
\end{aligned}$$

and

$$d_t = \frac{1}{\sigma}\left[\log\left(\frac{\alpha S_0}{D_t}\right) - \left(\mu - \sigma^2\right)(T-t)\right]. \tag{2.57}$$

Proof. As show in Section 2.3.2, the stock price at time $t \leq \tau \wedge T$ is

$$
\begin{aligned}
S_t = \frac{\mathbb{E}_t\left[\pi^A_{\tau\wedge T} S_{\tau\wedge T}\right]}{\pi^A_t} &= \frac{\theta + (1-\theta)\eta_t}{\mathbb{E}_t\left[\frac{\theta+(1-\theta)\eta_{\tau\wedge T}}{S_{\tau\wedge T}}\right]} \\
&= \frac{1}{\frac{\theta}{\theta+(1-\theta)\eta_t}\mathbb{E}_t\left[S^{-1}_{\tau\wedge T}\right] + \frac{(1-\theta)\eta_t}{\theta+(1-\theta)\eta_t}\mathbb{E}_t\left[\frac{\eta_{\tau\wedge T}}{\eta_t} S^{-1}_{\tau\wedge T}\right]} \\
&= \frac{1}{\frac{\theta}{\theta+(1-\theta)\eta_t}\mathbb{E}_t\left[S^{-1}_{\tau\wedge T}\right] + \frac{(1-\theta)\eta_t}{\theta+(1-\theta)\eta_t}\mathbb{E}^B_t\left[S^{-1}_{\tau\wedge T}\right]}.
\end{aligned}
\tag{2.58}
$$

The second equality follows from Doob's Optional Sampling Theorem, while the last equality follows from Girsanov's Theorem.

Now consider the case when $\delta_t = \delta$. Taking S_0 as given and imposing the condition for stock price at the circuit breaker trigger, we have

$$
\mathbb{E}_t\left[S^{-1}_{\tau\wedge T}\right] = \frac{1}{\alpha S_0} P_t(\tau \leq T) + \mathbb{E}_t\left[D_T^{-1}\mathbb{1}_{\{\tau > T\}}\right], \tag{2.59}
$$

$$
\mathbb{E}^B_t\left[S^{-1}_{\tau\wedge T}\right] = \frac{1}{\alpha S_0 \eta_t}\mathbb{E}_t\left[\eta_\tau \mathbb{1}_{\{\tau\leq T\}}\right] + \frac{1}{\eta_t}\mathbb{E}_t\left[\eta_T D_T^{-1}\mathbb{1}_{\{\tau > T\}}\right]. \tag{2.60}
$$

The following standard results about hitting times of Brownian motions are helpful for deriving the expressions for the expectations in (2.59)-(2.60) (see e.g., Jeanblanc et al., 2009, chap 3). Let Z^μ denote a drifted Brownian motion, $Z^\mu_t = \mu t + Z_t$, with $Z^\mu_0 = 0$. Let $\mathcal{T}^\mu_y = \inf\{t \geq 0 : Z^\mu_t = y\}$ for $y < 0$. Then:

$$
\Pr\left(T^\mu_y \leq t\right) = N\left(\frac{y-\mu t}{\sqrt{t}}\right) + e^{2\mu y} N\left(\frac{y+\mu t}{\sqrt{t}}\right), \tag{2.61}
$$

$$
E\left[e^{-\lambda T^\mu_y}\mathbb{1}_{\{T^\mu_y \leq t\}}\right] = e^{(\mu-\gamma)y} N\left(\frac{y-\gamma t}{\sqrt{t}}\right) + e^{(\mu+\gamma)y} N\left(\frac{y+\gamma t}{\sqrt{t}}\right), \tag{2.62}
$$

where $\gamma = \sqrt{2\lambda + \mu^2}$.

Recall the definition of the stopping time τ in Equation (2.33), which simplifies in the case with constant disagreement,

$$
\tau = \inf\left\{t \geq 0 : D_t = \alpha S_0 e^{-\left(\mu-\sigma^2\right)(T-t)}\right\}. \tag{2.63}
$$

Through a change of variables, we can redefine τ as the first hitting time of a drifted Brownian motion for a constant threshold. Specifically, define:

$$y_t = \frac{1}{\sigma} \log \left(e^{-(\mu-\sigma^2)t} D_t \right), \tag{2.64}$$

then $y_0 = 0$, and

$$y_t = Z_t^{\frac{\sigma}{2}} = \frac{\sigma}{2} t + Z_t. \tag{2.65}$$

Moreover,

$$\mathcal{T}_d^{\frac{\sigma}{2}} = \inf \{ t \geq 0 : y_t = d \} \overset{a.s.}{=} \tau, \tag{2.66}$$

where the threshold is constant over time,

$$d = \frac{1}{\sigma} \log \left(\alpha S_0 e^{-(\mu-\sigma^2)T} \right). \tag{2.67}$$

Conditional on y_t and the fact that the circuit breaker has not been triggered up to time t, the result from (2.61) implies

$$P_t (\tau \leq T) = P_t \left(\mathcal{T}_{d_t}^{\frac{\sigma}{2}} \leq T - t \right) = N \left[\frac{d_t - \frac{\sigma(T-t)}{2}}{\sqrt{T-t}} \right] + e^{\sigma d_t} N \left[\frac{d_t + \frac{\sigma(T-t)}{2}}{\sqrt{T-t}} \right], \tag{2.68}$$

where

$$d_t = d - y_t = \frac{1}{\sigma} \left[\log \left(\frac{\alpha S_0}{D_t} \right) - \left(\mu - \sigma^2 \right) (T - t) \right]. \tag{2.69}$$

The threshold d_t is normalized with respect to y_t so as to start the drifted Brownian motion $Z^{\frac{\sigma^2}{2}}$ from 0 at time t.

Next,

$$
\begin{aligned}
&\mathbb{E}_t\left[D_T^{-1}\mathbb{1}_{\{\tau>T\}}\right] \\
&= D_t^{-1}e^{-\left(\mu-\frac{\sigma^2}{2}\right)(T-t)}\mathbb{E}_t\left[e^{-\sigma(Z_T-Z_t)}\mathbb{1}_{\{\tau>T\}}\right] \\
&= D_t^{-1}e^{-\left(\mu-\sigma^2\right)(T-t)}\mathbb{E}_t\left[e^{-\sigma(Z_T-Z_t)-\frac{\sigma^2}{2}(T-t)}\mathbb{1}_{\{\tau>T\}}\right] \\
&= D_t^{-1}e^{-\left(\mu-\sigma^2\right)(T-t)}\mathbb{E}_t^{\mathbb{Q}}\left[\mathbb{1}_{\{\tau>T\}}\right] \\
&= D_t^{-1}e^{-\left(\mu-\sigma^2\right)(T-t)}\left\{N\left[-\frac{d_t+\frac{\sigma(T-t)}{2}}{\sqrt{T-t}}\right]-e^{-\sigma d_t}N\left[\frac{d_t-\frac{\sigma(T-t)}{2}}{\sqrt{T-t}}\right]\right\}.
\end{aligned}
\tag{2.70}
$$

The third equality follows from Girsanov's Theorem, and the fourth equality again follows from (2.61). Under $\mathbb{Q}$, $Z_t^{\sigma}=Z_t+\sigma t$ is a standard Brownian motion, and

$$y_t=-\frac{\sigma}{2}t+Z_t^{\sigma}. \tag{2.71}$$

Next, it follows from (2.64) and the definition of τ that

$$y_\tau=y_t+\frac{\sigma}{2}(\tau-t)+(Z_\tau-Z_t)=d. \tag{2.72}$$

Thus,

$$Z_\tau-Z_t=d_t-\frac{\sigma}{2}(\tau-t). \tag{2.73}$$

With these results, we can evaluate the following expectation:

$$
\begin{aligned}
\mathbb{E}_t\left[\eta_\tau\mathbb{1}_{\{\tau\le T\}}\right] &= \mathbb{E}_t\left[\eta_t e^{\frac{\delta}{\sigma}(Z_\tau-Z_t)-\frac{\delta^2}{2\sigma^2}(\tau-t)}\mathbb{1}_{\{\tau\le T\}}\right] \\
&= \eta_t e^{\frac{\delta d_t}{\sigma}}\mathbb{E}_t\left[\exp\left(-\left(\frac{\delta}{2}+\frac{\delta^2}{2\sigma^2}\right)(\tau-t)\right)\mathbb{1}_{\{\tau\le T\}}\right] \\
&= \eta_t\left\{N\left[\frac{d_t-\left(\frac{\delta}{\sigma}+\frac{\sigma}{2}\right)(T-t)}{\sqrt{T-t}}\right]+e^{\left(\sigma+\frac{2\delta}{\sigma}\right)d_t}N\left[\frac{d_t+\left(\frac{\delta}{\sigma}+\frac{\sigma}{2}\right)(T-t)}{\sqrt{T-t}}\right]\right\},
\end{aligned}
$$

where the last equality follows from an application of (2.62).

Finally,

$$\begin{aligned}
\mathbb{E}_t\left[\eta_T D_T^{-1}\mathbb{1}_{\{\tau>T\}}\right] &= \mathbb{E}_t\left[\eta_t e^{\frac{\delta}{\sigma}(Z_T-Z_t)-\frac{\delta^2}{2\sigma^2}(T-t)} D_t^{-1} e^{-\left(\mu-\frac{\sigma^2}{2}\right)(T-t)-\sigma(Z_T-Z_t)}\mathbb{1}_{\{\tau>T\}}\right] \\
&= \eta_t D_t^{-1} e^{-\left(\mu-\sigma^2+\delta\right)(T-t)}\mathbb{E}_t\left[e^{\left(\frac{\delta}{\sigma}-\sigma\right)(Z_T-Z_t)-\frac{\left(\frac{\delta}{\sigma}-\sigma\right)^2}{2}(T-t)}\mathbb{1}_{\{\tau>T\}}\right] \\
&= \eta_t D_t^{-1} e^{-\left(\mu-\sigma^2+\delta\right)(T-t)}\mathbb{E}_t^{\widetilde{\mathbb{Q}}}\left[\mathbb{1}_{\{\tau>T\}}\right] \\
&= \eta_t D_t^{-1} e^{-\left(\mu-\sigma^2+\delta\right)(T-t)}\left\{N\left[-\frac{d_t-\left(\frac{\delta}{\sigma}-\frac{\sigma}{2}\right)(T-t)}{\sqrt{T-t}}\right]\right. \\
&\quad \left. -e^{\left(\frac{2\delta}{\sigma}-\sigma\right)d_t} N\left[\frac{d_t+\left(\frac{\delta}{\sigma}-\frac{\sigma}{2}\right)(T-t)}{\sqrt{T-t}}\right]\right\}.
\end{aligned}$$

The third equality follows from Girsanov's Theorem, and the fourth equality follows from (2.61). Under $\widetilde{\mathbb{Q}}$, $Z_t^{\sigma-\frac{\delta}{\sigma}} = Z_t + \left(\sigma - \frac{\delta}{\sigma}\right)t$ is a standard Brownian motion, and

$$y_t = \left(\frac{\delta}{\sigma}-\frac{\sigma}{2}\right)t + Z_t^{\sigma-\frac{\delta}{\sigma}}. \tag{2.74}$$

□

2.8.2 Numerical Solution

Now we outline the numerical algorithm used to solve the model for the $\Delta > 0$ case. Time interval $[0, T]$ is discretized using a grid $\{t_0, t_1, ..., t_n\}$, where $t_0 = 0$ and $t_n = T$. For every point t_i on the time grid we construct grids for a set of state variables which uniquely determine fundamental value D_{t_i}, Radon-Nikodym derivative η_{t_i} and disagreement δ_{t_i} [20]. We will denote by $\Theta_{i,k} = (t_i, \Xi_k)$ a tuple (which we will call a "node") that summarizes the state of the economy at time t_i and k here indexes a particular point of the discretized state-space. We assign probabilities of the transition $\Theta_{i,k} \to \Theta_{i+1,j}$ for all i, j, k to match expected value and dispersion of one-step changes in state variables with their continuous time counterparts, namely, drifts and diffusions.

[20]One obvious potential choice of state variables is D_t, η_t and δ_t. However, depending on specific assumptions of the model the state-space dimensionality can be reduced, e.g. in the case of constant disagreement one need to keep track of t and D_t only since they uniquely determine disagreement δ_t and the Radon-Nikodym derivative.

Using this structure we can solve for the equilibrium in complete markets in the following way.

1. For time point $t_n = T$: use equations (2.8a)—(2.9) to calculate consumption allocations and state-price density in all nodes $\Theta_{n,k}$; set stock price equal to D_T.

2. For time point t_{n-1}: use transition probabilities and the fact that $\pi_t S_t$, $\pi_t \widehat{W}_t^A$, π_t are martingales to calculate the state-price density, stock price and wealth of agent A in nodes $\Theta_{n-1,k}$ for all k.

3. Proceding backwards repeat the above step $n-1$ times to obtain the state-price density, stock price and wealth of agent A in every point $\Theta_{i,k}$.

4. Using transition probabilities calculate drift and diffusion of the stock price process and portfolio holdings of each agent.

In the case with circuit breakers we want to use the algorithm similar to the one above initialized it at time $[\tau \wedge T]$ instead of time T. The problem is that τ is endogenous itself. The following steps show how we solve the problem:

1. In every node $\Theta_{i,k}$: pick a grid for $\omega_{t_i} = W_{t_i}^A/(W_{t_i}^A + W_{t_i}^B)$ spanning the interval $[0,1]$ and solve the problem (2.21)—(2.22) for every point of the grid[21]. Solution to this problem yields "stop" prices $\underline{S}_{i,k,j}$ and marginal utilities of wealth of agents which we will denote by $V_{i,k,j}^{A\prime}$ and $V_{i,k,j}^{B\prime}$, where j indexes grid points for ω_{t_i}. Using the planner's problem (2.24) first order condtion we can define,

$$\lambda_{i,k,j} = \frac{\eta V_{i,k,j}^{B\prime}}{V_{i,k,j}^{A\prime} + \eta V_{i,k,j}^{B\prime}}.$$

2. Now pick an initial guess for the planner's weight λ_g and price threshold $\underline{S}$. In every node $\Theta_{i,k}$ using the values $\lambda_{i,k,j}$ and $S_{i,k,j}$ from the previous step find the "stop" price $\underline{S}_{i,k}$ in point λ_g by interpolation. If $\underline{S}_{i,k} \leq \underline{S}$ then the node $\Theta_{i,k}$ will be either a "stop"

[21]In equilibrium given initial wealth distribution the value $W_{t_i}^A/(W_{t_i}^A + W_{t_i}^B)$ is uniquely pinned down by the Radon-Nikodym derivative η_{t_i} and fundamental value D_{t_i}. Since this relationship is endogenous and is not known before the model is solved our algorithm requires solving the problem for a wide range of wealth distributions.

node or a node that will never be reached in the equilibrium with circuit breakers. For "stop" nodes we define stock price to be equal to $\underline{S}_{i,k}$ and state-price density to be proportional to $V^{A\prime}_{i,k}$ (which is also obtained by interpolation of $V^{A\prime}_{i,k,j}$ in point λ).

The above procedure effectively defines the stopping time rule τ corresponding to the economy with threshold $\underline{S}$ and planner's weight λ_g. Now we can use the backward procedure described for the case of complete markets to obtain the state-price density, stock price and wealth of every agent in every node $\Theta_{i,k}$. Note that initial wealth share of agent A in this economy will be different from both λ_g and ω. The final step is to find λ_g and $\underline{S}$ so that initial wealth share in the resulting economy is equal to ω and $\underline{S} = (1-\alpha)S_0$. This can be done using the standard bisection method.

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Chapter 3

Business Cycle, Reallocation of Labor and Asset Prices

3.1 Introduction

In the seminal paper Davis and Haltiwanger (1992) propose several measures of job flows and describe their empirical properties for the manufacturing sector data in the U.S. One of the striking patterns that they find in the data is that job destruction rate is very volatile and highly countercyclical while job creation rate is much less volatile and almost acyclical. The measure of job reallocation proposed by Davis and Haltiwanger (1992) is highly countercyclical in the data. Literature on job reallocation grew very fast since then with additional empirical evidence both in favor and against the phenomenon and with theories attempting to explain it[1].

In this paper I study potential implications of countercyclical reallocation for asset pricing. In particular I construct a theoretical model that features search and

[1]Barlevy (2003), Blanchard and Diamond (1989, 1990), Burda and Wyplosz (1994), Caballero and Hammour (1994, 1996), Davis and Haltiwanger (1990, 1992), Davis et al. (2006, 2013), Davis et al. (1998), Foote (1998), Foster et al. (2001), Foster et al. (2014), Fujita and Ramey (2009), Greenwood et al. (1996), Mortensen (1994), Mortensen and Pissarides (1994), Moscarini and Postel-Vinay (2009a,b), Roys (2016), Schuh et al. (1998), Shimer (2005, 2012) constitute an incomplete list of theoretical and empirical literature studying labor reallocation.

match frictions in the labor market along the lines of Mortensen and Pissarides (1994). The aggregate uncertainty in the economy is driven by the aggregate productivity shock. Firms in the economy differ by the composition of their assets: the amount of growth opportunities relative to the amount of assets in place. In the model bad times are times of high unemployment and hence hiring is much easier. This provides an opportunity for firms with higher growth potential to expand quicker. Thus during downturns labor is reallocated from mature to growing firms. This mechanism makes growth opportunities safer relative to assets in place giving rise to cross-sectional heterogeneity in stock returns. In accordance with previous empirical studies the model generates heterogeneity in returns of portfolios formed based on different growth indicators such as earnings-to-price and book-to-market ratio (see e.g. Fama and French 1992) or hiring rate (Belo et al., 2014). Empirically I find that book-to-market sorted portfolios have different exposure to the shock to unemployment rate. In particular, loadings of returns on unemployment shock decrease monotonically from low book-to-market to high book-to-market portfolios. This result is in line with the intuition that growth firms can benefit from more slack in the labor market in times of higher unemployment.

Theoretical literature attempting to explain cross-sectional heterogeneity in stock returns through firms' economic activity is large[2]. Here I find it most important to contrast my model with the theoretical models involving only one source of aggregate risk. Most of these models relate stock returns to firms' investment into physical capital decisions. In these models firms value is comprised from the value of assets in place and the value of growth opportunities. The difference between firms' risk may come from the difference in riskiness of their assets in place, the difference in riskiness of their growth opportunities and from difference of the share of growth opportunities versus assets in place in firms value. For example, in Berk et al. (1999)[3] all firms have

[2]Kogan and Papanikolaou (2012) contains a recent survey of the literature.

[3]Apart from aggregate productivity shock the model of Berk et al. (1999) features also shocks to the risk-free rate.

identical growth opportunities and controlling for market capitalization of the firm, high book-to-market signals higher riskiness of the assets in place giving rise to value premium. Similar mechanism is used in Gomes et al. (2003). In their model assets that have higher market value have lower conditional beta. Hence firms with higher book-to-market ratio have higher risk, which generates value premium. In both of these models value premium arises due to differences in risks between firms' assets in place and it is worth noting, that growth opportunities in these models are riskier than assets in place due to option-like payoff of the growth opportunities. Carlson et al. (2004) take a different approach. They introduce operational leverage and fixed adjustment costs into the model to make assets in place riskier than growth opportunities. In their model high book-to-market firms are firms with relatively less growth options in their value which makes these firms riskier giving rise to value premium. Zhang (2005) utilizes similar mechanism but puts more accent on asymmetric adjustment costs and countercyclical risk premia. In this paper, as in Carlson et al. (2004) and Zhang (2005) heterogeneity in returns arises due to difference in risk between growth opportunities and assets in place, however this heterogeneity follows from a different mechanism. In my model due to procyclical cashflows the value of assets in place is procyclical as well. Value of growth opportunities, in contrast, is countercyclical. Booms are associated with low unemployment and tight labor market, whereas in downturns unemployment is high and labor market is loose. This generates countercyclical probability of growth opportunities turning into profitable assets in place and hence countercyclical variation in the value of growth opportunities.

This paper fits into the growing literature that studies the connection of the labor market with stock returns[4]. Danthine and Donaldson (2002) observe that smoother than aggregate productivity wage payments can generate operating leverage giving

[4]An incomplete list of this literature includes Belo et al. (2014), Belo et al. (2015), Danthine and Donaldson (2002), Donangelo (2014), Eisfeldt and Papanikolaou (2013), Favilukis and Lin (2015), Gourio (2007), Merz and Yashiv (2007), Moscarini and Postel-Vinay (2010), Petrosky-Nadeau et al. (2018), Simutin et al. (2014).

rise to equity premium and excess volatility. Gourio (2007) and Favilukis and Lin (2015) argue that the same mechanism can explain heterogeneity in the cross-section of stock returns. Merz and Yashiv (2007) introduce labor and capital adjustment costs into firm's profit maximization problem and study jointly investment, hiring decisions and stock returns in the cross-section. Belo et al. (2014) and Belo et al. (2015) find empirically that higher hiring rates predict lower stock returns in cross-section. They develop a theory involving adjustment costs shock that reproduces the results of their empirical study. As will be shown in Section 3.4, the model with single aggregate productivity shock proposed in this paper is capable to generate this predictability. Eisfeldt and Papanikolaou (2013) empirically find that higher organizational capital predicts lower stock returns and associate this finding with outside options of firms' key employees. Current work is most closely related to several papers that explicitly introduce labor market search frictions to study the connection between labor market dynamics and stock returns. Moscarini and Postel-Vinay (2010) introduce labor search frictions to connect cyclical hiring behaviour of large and small employers with profitability. Petrosky-Nadeau et al. (2018) study the effect of labor market search frictions on the aggregate stock market. In their model search frictions amplify aggregate shocks and generate disaster risks with properties similar to those observed in the data. From the modeling prospective my paper is closest to Simutin et al. (2014). Simutin et al. (2014) find that after controlling for conventional risk-factors firms with higher loadings on labor market tightness have lower future returns. To explain the observed pattern they construct a model with search frictions and shocks to matching technology bearing time-varying price of risk.

The rest of the paper is organized as follows. Section 3.2 reviews key stylized facts and empirical results that motivate this study. Section 3.3 presents the model. Section 3.4 contains calibration strategy and numerical results. Section 3.5 concludes.

3.2 Empirical Motivation

Davis and Haltiwanger (1992) define gross job reallocation as a sum of job destruction and job creation within a period of time, where job destruction is the absolute value of the total change of the number of employees in all shrinking establishments and job creation is the absolute value of the total change of the number of employees in all expanding establishments. On Figure 3-1 I plot gross job reallocation in the manufacturing sector as a percentage of the total employment in the sector. The data comes from Davis et al. (2006) and BLS Business Employment Dynamics (see notes to Figure 3-1 for details)[5]. One of the striking features of this plot is that gross job reallocation spikes in recessions. This pattern is driven by sharp increases in job destruction rates and much less pronounced decreases in job creation during economic downturns. On Figure 3-2 I present the logarithm of unemployment outflow as a share of labor force. This series is constructed using the job finding rate series constructed by Shimer (2012) and data on unemployment-to-employment worker flows from the Bureau of Labor Statistics (BLS). A remarkable pattern observed on this figure is that unemployment-to-employment flows tend to increase sharply during recessions[6]. These two patterns suggest that recessions are periods of increased turnover of workers through the unemployment pool and are periods of increased pace of hiring from the unemployment pool. Abundance of workers in the unemployment pool makes it easier for expanding firms to hire more people and find better match for available positions. Hence increased worker reallocation during recessions can be beneficial for these firms.

Vast empirical asset pricing literature documents that firms having more growth opportunities (as measured by different observable variables such as book-to-market equity ratio) have lower average stock returns. The evidence presented in the previous

[5] I present job reallocation for the manufacturing sector only due to data availability. Existing series of job reallocation in the whole private sector are available from 1990 only.

[6] Employment-to-unemployment worker flows increase during recessions too and this increase typically leads the increase of unemployment-to-employment flows, see e.g. Blanchard and Diamond (1990).

paragraph suggests that increased reallocation observed during economic downturns can act as a hedge for the "growth" firms, that get access to large pool of unemployed people and a big pool of talent with lack of employment opportunities. To test if fluctuations in reallocation have any relation to the cross-section of stock returns I run regressions of book-to-market sorted portfolios quarterly returns against changes in unemployment rate during the same period:

$$R_t^i - R_t^f = \alpha + \gamma \Delta U_t + \varepsilon_t, \tag{3.1}$$

here R_t^i is the return on portfolio i in quarter t, R_t^f is the risk-free rate and ΔU_t is the change in unemployment rate from quarter $t-1$ to t. In this analysis I use five value-weighted book-to-market sorted portfolios published by professor Kenneth R. French on his website and unemployment rate from BLS. Data sample starts in 1948Q2 and ends in 2014Q4. Table 3.1 reports the estimates of γ for the five portfolios and $5-1$ long/short portfolio. All five estimates for long portfolios are negative reflecting intuition that times of higher unemployment are also times of lower stock market valuations. More interesting, the estimates decrease monotonically from low book-to-market to high book-to-market portfolio. On top of that $5-1$ spread loading on unemployment change is both statistically and economically significant: one percentage point change in unemployment rate on average corresponds to 2% spread in realized returns. One can not give a causal interpretation to the results of this regression. However, these results go in line with the hypothesis that increased unemployment can act as a hedge for "growth" firms making them less risky.

The evidence presented motivates the theoretical model developed in the following section.

3.3 The Model

Time is discrete. The economy consists of a continuum of firms and a continuum of workers of measure 1. Every firm i in the beginning of period t is characterized by the number of active projects $N^a_{i,t}$ and the number of not active projects $N^n_{i,t}$. The structure of the labor market is close to Mortensen and Pissarides (1994). An active project is matched with a worker, whereas a non-active project is not. Every period an active project generates profit z_t, where z_t is aggregate state dependent productivity which follows a first order Markov chain. I assume that profits are the same for every project of every firm[7] and depend on the aggregate state only. Thus a firm's i profit is $\pi_{i,t} = z_t\, N^a_{i,t}$. Every period after production has happened an active project can disappear with exogenous state dependent probability $s^a(z_t)$, in which case the worker that was matched with that project becomes unemployed.

Non-active projects disappear with probability $s^n(z_t)$. A non-active project does not generate profit, but becomes active if matched with a worker. Matching happens with aggregate state dependent probability

$$q(u_t, v_t) = \mu \frac{g(u_t, v_t)}{v_t}.$$

Here by u_t I denote an unemployment level in the economy and by v_t an aggregate number of vacancies in the economy, μ is a constant parameter of the matching efficiency, $g(u_t, v_t)$ is constant returns to scale matching technology. More detailed definition of u_t and v_t will be given below.

Entry and exit

Every period a firm exits the market with constant exogenous probability δ, in which case all of its projects disappear and employees become unemployed. To keep the mass

[7] This assumption is for simplicity and is not crucial for my results. The model can be easily extended by introducing idiosyncratic productivity shocks with similar results.

of the firms constant every period a measure δ of new firms enters the market. A new firm owns no active projects, but owns a state dependent number $\lambda(z_t)$ of non-active projects.

Timing

The timing within every period is the following: i) Aggregate productivity z_t is revealed; ii) Production; iii) Share $s^n(z_t)$ of non-active projects disappears, $s^a(z_t)$ share of active projects disappears (separated employees join labor force next period only), share $\delta(z_t)$ of firms exit; iv) δ mass of new firms enter the market with mean stock of non-active projects $\lambda(z_t)$, entering firms participate in labor market in this period; v) Matching of non-active projects with unemployed workers happens (matched pairs become productive in the next period).

Matching and aggregation

To make non-active projects profitable firms post vacancies. A non-active project corresponds to one vacancy. Denoting by n_t^n total number of non-active projects in the beginning of period t:

$$n_t^n = \int N_{i,t}^n \, d\, i \, , \tag{3.2}$$

aggregate number of vacancies is determined by

$$v_t = n_t^n (1 - \delta)(1 - s^n(z_t)) + \delta \lambda(z_t). \tag{3.3}$$

This equation states that number of vacancies in period t equals number of non-active projects in the beginning of the period corrected for exit and destruction plus number of non-active projects of newly entering firms. Total labor force in the economy is normalized to 1. So unemployment is given by:

$$u_t = 1 - n_t^a = 1 - \int N_{i,t}^a \, d\,i\,, \tag{3.4}$$

where n_t^a – total number of projects in the economy that are active in the beginning of period t.

Number of new matches arising in the economy is:

$$G(u_t, v_t) = \mu\, g(u_t, v_t). \tag{3.5}$$

Formulation of this model allows complete aggregation using the law of large numbers. Dynamics of aggregate variables in the economy is completely described by equations (3.3), (3.4), (3.5) and:

$$\begin{aligned} n_{t+1}^a &= n_t^a(1-\delta)(1-s^a(z_t)) + G(u_t, v_t)\,, \\ n_{t+1}^n &= n_t^n(1-\delta)(1-s^n(z_t)) + \delta\lambda(z_t) - G(u_t, v_t)\,. \end{aligned} \tag{3.6}$$

Here the first equation describes the evolution of the number of active projects. Number of active projects in period $t+1$ equals the number of active projects in period t corrected for exits and job destruction plus number of new matches in period t. The second equation describes the evolution of non-active projects. Taking into account equation (3.3), number of non-active projects in the beginning of period $t+1$ equals number of vacancies minus number of new matches in period t.

Stochastic discount factor

I postulate an exogenous stochastic discount factor dependent on realization of the aggregate productivity shock z_t. Stochastic discount between periods t and $t+1$ is defined by:

$$M_{t+1} = \frac{\exp\left(-\bar{r} - \gamma[z_{t+1} - \mathbb{E}_t z_{t+1}]\right)}{\mathbb{E}_t\left[\exp\left(-\gamma[z_{t+1} - \mathbb{E}_t z_{t+1}]\right)\right]} \tag{3.7}$$

This definition of the stochastic discount factor is close to Carlson et al. (2004). For positive γ it has the feature that states corresponding to positive unexpected changes in aggregate shock are discounted relatively more compared to the states corresponding negative shocks. The difference from the stochastic discount factor used in Carlson et al. (2004) is the normalization term in the denominator of expression (3.7). This normalization insures that risk-free rate is constant in every state of the world and is equal to $\bar{r}$.[8]

Pricing

All firms are financed by equity only, and I assume that dividend payments coincide with firms' profits. I normalize number of shares of each firm to be 1. A firm i end of period t value,[9] $V_{i,t}$, consists of value of its active projects, $V_{i,t}^a$ and value of non-active $V_{i,t}^n$ projects

$$V_{i,t} = V_{i,t}^a + V_{i,t}^n. \tag{3.8}$$

Since all active projects in the economy have the same productivity and all non-active projects have the same probability of being matched with a worker $V_{i,t}^a$ and $V_{i,t}^n$ can be written as

$$\begin{aligned} V_{i,t}^a &= \overline{V_t^a} N_{i,t+1}^a, \\ V_{i,t}^n &= \overline{V_t^n} N_{i,t+1}^n, \end{aligned} \tag{3.9}$$

[8]Constant risk-free rate is counterfactual. I utilize this specification of the SDF to highlight heterogeneity in the cross-section of stock returns generated by the mechanism studied in this paper. Time-varying risk-free rate introduces additional effects in the cross-section of expected returns.

[9]Here and after I simplify notation and omit dependence of variables on aggregate state of the economy by introducing subscript t, e.g. $V_{i,t} = V_i(z_t, n_{t+1}^a, n_{t+1}^n)$.

where $\overline{V_t^a}$ and $\overline{V_t^n}$ are per unit values of active and non-active projects correspondingly. Values $\overline{V_t^a}$ and $\overline{V_t^n}$ are defined recursively by

$$\overline{V_t^a} = \mathbb{E}_t[M_{t+1}(z_{t+1} + (1 - \delta)(1 - s^a(z_{t+1}))\overline{V_{t+1}^a})], \quad (3.10)$$

$$\overline{V_t^n} = \mathbb{E}_t[M_{t+1}((1-\delta)(1-s^n(z_{t+1}))(q(u_{t+1},v_{t+1})\overline{V_{t+1}^a} + (1-q(u_{t+1},v_{t+1}))\overline{V_{t+1}^n}))]. \quad (3.11)$$

Note that $\overline{V_t^a}$ is a function of z_t only and can be computed exactly using matrix inversion in case when z_t follows a finite state Markov chain. Probability of a vacancy being matched in period $t+1$, $q(u_{t+1}, v_{t+1})$, depends on unemployment and number of vacancies in period $t+1$. Using equations (3.3) and (3.4) these variables can be expressed through z_{t+1}, n_{t+1}^a and n_{t+1}^n. Since values n_{t+1}^a and n_{t+1}^n are known in the end of period t, value $\overline{V_t^n}$ can be solved numerically on a grid for z_t, n_{t+1}^a, n_{t+1}^n.

Intuition

Equations (3.10) and (3.11) allow us to compare risks of assets in place and growth opportunities qualitatively. As it follows from equation (3.10) fluctuations of the value of active projects are driven by fluctuations of cashflows through aggregate productivity z_{t+1}, by fluctuations of destruction rate $s^a(z_{t+1})$ and by comovement of aggregate shock with the stochastic discount factor. Cashflow is equal to z_{t+1} and hence procyclical, in my calibration $s^a(z_{t+1})$ is countercyclical, so both of these effects make $\overline{V_t^a}$ more volatile and procyclical, which increases the risk of assets in place.

From equation (3.11) it follows that fluctuations in the value of non-active projects are driven by fluctuations of $\overline{V_t^a}$, by fluctuations of $s^n(z_{t+1})$ and by fluctuations in the vacancy match probability $q(u_{t+1}, v_{t+1})$. The value of an active project, $\overline{V_t^a}$, is procyclical. In my calibration destruction rate $s^n(z_{t+1})$ is proportional to $s^a(z_{t+1})$ and hence countercyclical. These two effects make the value of growth opportunities riskier. However, note that since cashflows from active projects are non-negative, it must be the case that $\overline{V^a_{t+1}} \geq \overline{V^n_{t+1}}$ in every state of the world. Since $q(u_{t+1}, v_{t+1})$ is increasing in unemployment and decreasing in v_t, the term $(q(u_{t+1}, v_{t+1})\overline{V^a_{t+1}} + (1 - q(u_{t+1}, v_{t+1}))\overline{V^n_{t+1}})$ is less procyclical than $\overline{V^a_{t+1}}$ and can be even countercyclical if $q(u_{t+1}, v_{t+1})$ varies significantly with the business cycle. This observation makes growth opportunities safer than assets in place driving heterogeneity in returns across firms with different ratio of growth opportunities to assets in place. I confirm this intuition in the simulations of a calibrated version of the model in the next section.

3.4 Numerical Analysis

Calibration

The model is calibrated at the monthly frequency. Values of parameters chosen in my baseline calibration are summarized in Table 3.2. I assume that logarithm of the aggregate shock z_t follows AR(1) process:

$$\ln z_{t+1} = \rho \ln z_t + \sigma_z \varepsilon_{t+1}\,, \tag{3.12}$$

where ε_{t+1} is iid with zero mean and unit variance. I discretize the process with $n_z = 3$ aggregate states using the procedure described in Rouwenhorst (1995). I set $\rho = 0.92$ to match average duration of the aggregate shock being in its lowest state with the average duration of NBER recessions after 1927, which is about 13 months. I set $\sigma_z = 0.32$ jointly with the sensitivity of the stochastic discount factor to aggregate shock parameter $\gamma = 0.4$ to match equity market premium 7.3%, and market volatility 17.3%. I set $\bar{r} = 0.002$ to roughly match level of the risk-free rate in the economy, this value translates into annual rate of 2.4%. I follow Corhay et al. (2015) and set monthly firm exit rate to $\delta = 0.01/3$ which translates into 4% annual exit rate.

Rate of active project destruction, s_a, rate of non-active project destruction, s_n, and the number of non-active projects an entering firm owns depend on the aggregate productivity. In calibrating these variables I pursue three goals. First, I want to restrict the dependence of these parameters on aggregate state to avoid proliferation of parameters. Second, I want to generate dynamics of the labor market that is consistent with the dynamics observed in the data. Third, I am trying to connect these values to previous studies as much as possible. To achieve these goals I postulate the following functional dependence

$$\begin{aligned}
\lambda(z_t) &= \lambda_0 z_t^{\kappa}, \\
s^a(z_t) &= s_0^a z_t^{-\kappa}, \\
s^n(z_t) &= s_0^n z_t^{-\kappa}.
\end{aligned} \tag{3.13}$$

Here I assume $\kappa > 0$. Consistent with the business cycle stylized facts and common intuition, these functional forms and the sign constraint insure that number of new projects appearing in the economy is procyclical, project destruction rate is countercyclical and thus more jobs/active projects are destroyed in downturns and less in booms. Destruction rate of non-active projects has the same dependence on the aggregate shock as destruction rate of active projects to ensure that difference in risk between active and non-active projects is not driven by exogenously postulated destruction rate. Note that when aggregate shock takes its median value $z = 1$ we have $s^a(z_t) = s_0^a$, $s^n(z_t) = s_0^n$. Using this observation I set $s_0^a = 0.02/3$, which is close to the average job destruction rate of 0.026/3 per month for firms in private sector with more than 1000 employees[10]. This value is also consistent with the common value for depreciation rate of capital used in the business cycle literature. I set $s_0^n = 0.0375/3$, this value translates into annual depreciation rate of non-active projects of 15%. Kung and Schmid (2015) uses this value for the depreciation rate of R&D stock. I pick values

[10] Average job destruction rate for firms in private sector is computed for the sample from 1992 to 2015, the data on job destruction is available at the Bureau of Labor Statistics website.

of λ_0 and κ to match the mean value and standard deviation of monthly unemployment rate which are 5.8% and 1.7% correspondingly in my sample. Setting $\lambda_0 = 7.82$ and $\kappa = 0.59$ achieves this goal.

Finally, I define values of parameters of the matching function. I use the constant return to scale matching function introduced by Ramey et al. (2000):

$$g(u, v) = \frac{uv}{(u^{\psi} + v^{\psi})^{1/\psi}}, \tag{3.14}$$

where I set elasticity of the matching technology $\psi = 1.25$. This value is used by Petrosky-Nadeau et al. (2018) and is very close to the value suggested by Ramey et al. (2000). I set the efficiency of the matching technology parameter μ equal to 0.19. In the model of Ramey et al. (2000) this parameter is effectively set to 1. My model features different assumptions about project arrival process and hence different dynamics of vacancies. This difference requires introduction of the matching efficiency parameter to obtain plausible values of the first two moments of unemployment and plausible mean value of worker flow from unemployment to employment (UE flow). In this calibration average monthly UE flow is 1% of the labor force. This value is slightly lower than 1.6%, the number obtained using data reported in Blanchard and Diamond (1990)[11] study of the worker flows.

Portfolio Sorts

Empirical literature establishing connection between growth opportunities of firms and expected stock returns is very rich. Regardless of the measure associated with growth opportunities most of the studies reveal a similar pattern: higher growth opportunities forecast lower stock returns. Fama and French (1992) document that high book-to-market and earnings-to-price ratios positively forecast returns in cross section, Titman

[11] Average stock of employed ppopulation in Blanchard and Diamond (1990) sample is 93.2 million, average stock of unemployed is 6.5 and average monthly UE flow is 1.6 million, this gives the UE transition rate of $1.6/(6.5 + 93.2) \approx 1.6\%$.

et al. (2004) find that higher investment rate forecasts lower stock returns, Cooper et al. (2008) document that a firm's total asset growth negatively predicts stock returns, in a recent paper Belo et al. (2014) find that higher employment growth rates forecast lower stock returns. Even though signals used to construct portfolios in cited above papers are different, all of them are likely to represent some common feature between firms. Firms having high book-to-market ratio, asset growth or employment growth are very likely to be the firms that have more growth opportunities.

In this section I evaluate the model's ability to generate negative relation between growth opportunities and expected returns. In the model a firm is represented by a collection of matched and unmatched projects. A project becomes active and production happens when it is matched with a worker. Hence number of the firm's active projects equals the number of workers employed by a firm. At the same time, one can view every project as unit of assets owned by a firm. In the model I abstract from investment spending, however if every matched project required some investment outlay then investment spending on projects the firm owns could be treated as book value of equity or physical capital, in this case ratio of book equity to market value of the firm would be proportional to the ratio of the number of active projects to the market value of the firm. Hence, within this model I relate the number of active projects-to-market equity ratio to the conventional book-to-market equity ratio.

To evaluate the performance of the model in its ability to generate cross sectional difference in expected returns I perform sorts of stocks in simulated data. I simulate 200 panels of 2500 firms of length 1500. For every simulated panel I discard the first 500 observations to let the model converge to its stationary regime. Remaining sample length of 1000 month is comparable to empirical studies utilizing data starting from 1927. In all of the experiments I focus on value-weighted portfolios. Equal-weighted portfolios yield the same qualitative results with slight differences in quantities.

In Table 3.3 I present summary statistics of portfolios sorted based on book-to-market equity value. The first panel shows summary statistics for empirical portfolios.

To compute these statistics I use the data downloaded from Kenneth R. French website. I use the longest available sample which starts in January 1927 and ends in July 2015. The second panel reports mean values of summary statistics across 200 simulated panels. To make comparison of the simulated and empirical data fair I follow the timing conventions used by Fama and French (1992) when constructing portfolios in the simulated data. As can be seen the model generates a sizable spread between portfolios that is comparable to the data.

In Table 3.4 I present summary statistics for portfolios sorted by the earnings-to-price ratio. Again empirical data comes from professor Kenneth R. French website. To construct earnings-to-price ratio in simulated data I calculate the sum of firm's monthly earnings in year $y-1$ and divide it by stock price in the end of December of year $y-1$. The portfolios are formed in the end of June of year y. Again model implied average returns have a substantial heterogeneity across portfolios, which roughly matches heterogeneity observed in the data. Fama and French (1992) argues that in the data heterogeneity in returns of stocks with different E/P ratio is captured by book-to-market equity. In the model developed in this paper there is a very tight relation between E/P and book-to-market ratio. Firms that have low book-to-market ratio are firms with a lot of growth opportunities and few active projects. Since only active projects generate earnings, low book-to-market firms have also low earnings-to-price ratios.

As was mentioned above other indicators of growth, such as employment growth, investment-to-capital ratio, asset growth were found to predict lower stock returns in previous empirical studies. I check if a similar pattern holds in the model. I perform employment growth rate sort (which can essentially be related to capital growth sort as well). I define yearly employment growth using the formula $(N^a_{i,t+12} - N^a_{i,t})/(0.5(N^a_{i,t+12} + N^a_{i,t}))$. This definition of employment growth is used in Belo et al. (2014). I form portfolios in the end of June of year y using employment growth observed in year $y-1$. The results of this sort are presented in Table 3.5. For comparison in this table I also present empirical portfolios sorted based hiring rate constructed following Belo et al. (2014) methodology. Again in accordance with empirical studies faster growing firms yield lower stock returns. The intuition behind this result in the model is similar to the intuition behind sorts based on book-to-market and earnings-to-price. Faster growing firms are the firms that have bigger stock of non-active projects relative to active project. Hence, employment growth is essentially inversely related to the book-to-market and earnings-to-price ratios.

Inspecting the mechanism

I the model active projects and non-active projects have different levels of risk. Cross-sectional difference in returns is driven by the difference in composition of firms' sets of projects. Non-active projects are much safer than active projects because of the countercyclical fluctuations in unemployment. In bad times, when realization of the stochastic discount factor is high, unemployment tends to increase. Higher unemployment translates into higher probability for a non-active project to get matched with a worker, which increases the value of non-active projects. This mechanism makes non-active projects or growth opportunities much safer relative to active projects or assets in place. However, within the model non-active projects can potentially be safer than active projects even without frictions in the labor market. In this section I conduct a comparative statics experiment to evaluate to which extent cross-sectional difference in average returns in the model is driven by the labor market frictions.

For this purpose, I simulate the analog of the model abstracting from the labor market. More precisely, I solve the model and simulate panels of firms in the case when all of the parameters take the same values as presented in Table 3.2, except that now I assume that probability for a non-active project to be matched with a worker is constant and is equal to the average value of this probability in the baseline calibration. As before I simulate 200 panels and do sorts based on book-to-market, earnings-to-price and employment growth rates. The results of these simulations are presented in Table

3.6. One can see that heterogeneity in returns is still observed among ten portfolios for all three sorts. However, the magnitude of dispersion is much lower. For example, in the full version of the model 10–1 spread for book-to-market portfolios is 6.5%, in the version without matching frictions this spread is 1.8% only. This leads to the conclusion that labor market frictions are potentially an important mechanism that can drive significant heterogeneity in the cross section of stock returns.

3.5 Conclusion

In this paper I formulate the hypothesis and provide intuition of how fluctuations in employment can generate heterogeneity in stock returns. I find empirical evidence that book-to-market sorted portfolios returns comove with unemployment in a different way: value stocks on average depreciate relatively more than growth stocks when unemployment goes up. This intuition and empirical findings motivate the theoretical model developed in this paper. The model links labor market search frictions with risk of firms' equity. Simulations of the model show that labor market frictions can be a powerful mechanism in generating heterogeneity in the cross-section of stock returns. In particular, it is shown than the mechanism is able to generate empirically documented patterns such as heterogeneity in returns on portfolios sorted based on earnings-to-price ratio, book-to-market equity, hiring rate. The main insight behind these results in the model is simple: growth opportunities gain in value when matched with workers, since probability of a non-active project to be matched with a worker is higher in recession, growth opportunities are less prone to lose value in recession, which makes them safer.

The model is consistent with aggregate patterns of labor market dynamics and delivers a rich set of predictions regarding hiring decision of firms with different characteristics. Empirical tests of these implications seem to be a logical continuation of this line of research.

3.6 Tables and Figures

Table 3.1: Book-to-Market Portfolios Returns and Unemployment

	Portfolio					
	1	2	3	4	5	5 − 1
γ	−0.20	−0.38	−1.13	−1.50	−2.30	−2.09
	(1.44)	(1.38)	(1.46)	(1.70)	(1.58)	(0.94)

Notes: Here I present results of the regressions of book-to-market equity sorted value weighted portfolios returns on quarterly change in unemployment rate: $R_t^i - R_t^f = \alpha + \gamma \Delta U_t + \varepsilon_t$. Newey-West autocorrealtion robust standard errors in parentheses. Data on portfolio returns is downloaded from Kenneth R. French web site: `http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html`. Data sample is from 1948Q2 to 2014Q4.

Table 3.2: Baseline Calibration Parameters

Parameter	Symbol	Value
Persistence of aggregate productivity	ρ	0.92
Volatility of aggregate productivity	σ_z	0.32
Monthly risk-free interest rate	$\bar{r}$	0.002
Loading of the SDF on productivity shock	γ	0.4
Firm exit rate	δ	0.01/3
Destruction rate of active projects in median state	s_0^a	0.02/3
Destruction rate of non-active projects in median state	s_0^n	0.0375/3
Number of non-active projects owned by a new firm in median state	λ_0	7.82
Loading of destruction rates on aggregate productivity shock	κ	0.59
Elasticity of the matching technology	ψ	1.25
Coefficient of the matching efficiency	μ	0.19

Notes: These are parameter values used in the benchmark model. Detailed description of the choice of numerical values see in the text.

Table 3.3: Properties of Book-to-Market Sorted Portfolios

	Portfolio									
	1	2	3	4	5	6	7	8	9	10
	Data									
$\mathbb{E}[R - R_f]$	6.93	7.62	7.68	7.95	8.67	8.85	8.63	10.19	10.66	12.09
β	1.07	1.00	0.98	0.98	0.90	0.93	0.90	0.92	0.96	1.09
σ	17.01	15.53	15.19	15.59	14.67	14.95	15.02	15.63	16.35	19.82
	Model									
$\mathbb{E}[R - R_f]$	4.34	7.49	9.09	9.95	10.42	10.65	10.78	10.80	10.84	10.84
β	0.65	1.02	1.21	1.32	1.37	1.40	1.41	1.42	1.42	1.42
σ	11.56	17.83	21.02	22.80	23.77	24.28	24.52	24.61	24.65	24.67

Notes: The table reports annualized mean excess return (%), market beta and annualized volatility (%) of portfolios sorted based on book-to-market equity ratio, firms in postfolio 1 have the lowest book-to-market ratio. Data on portfolio returns is downloaded from Kenneth R. French web site: `http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html`. Data sample is from January 1948 to December 2014.

Table 3.4: Properties of Earnings-to-Price Sorted Portfolios

	Portfolio									
	1	2	3	4	5	6	7	8	9	10
	Data									
$\mathbb{E}[R - R_f]$	6.29	5.77	7.21	7.04	7.75	9.22	9.86	10.73	11.48	12.51
β	1.17	1.01	0.95	0.91	0.93	0.90	0.89	0.91	0.96	1.04
σ	18.95	15.82	15.18	14.66	14.96	14.87	14.72	15.42	16.38	17.89
	Model									
$\mathbb{E}[R - R_f]$	5.71	8.11	9.39	10.09	10.48	10.70	10.78	10.81	10.83	10.84
β	0.81	1.10	1.25	1.33	1.38	1.40	1.41	1.42	1.42	1.42
σ	14.30	19.04	21.62	23.09	23.90	24.34	24.54	24.62	24.65	24.67

Notes: The table reports annualized mean excess return (%), market beta and annualized volatility (%) of portfolios sorted based on earnings-to-price ratio, firms in postfolio 1 have the lowest earnings-to-price ratio. Data on portfolio returns is downloaded from Kenneth R. French web site: `http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html`. Data sample is from July 1951 to December 2014.

Table 3.5: Properties of Employment Growth Sorted Portfolios

	Portfolio									
	1	2	3	4	5	6	7	8	9	10
	Data									
$\mathbb{E}[R - R_f]$	7.58	7.38	5.75	6.97	4.64	6.29	5.56	5.20	6.07	4.38
β	1.00	0.90	0.85	0.89	0.91	0.96	1.00	1.09	1.12	1.15
σ	17.10	15.74	15.10	15.39	15.63	16.30	17.17	18.56	18.86	19.67
	Model									
$\mathbb{E}[R - R_f]$	10.83	10.83	10.81	10.79	10.69	10.49	10.10	9.41	8.14	5.73
β	1.42	1.42	1.42	1.41	1.40	1.38	1.33	1.25	1.10	0.82
σ	24.68	24.65	24.62	24.54	24.35	23.91	23.11	21.65	19.10	14.35

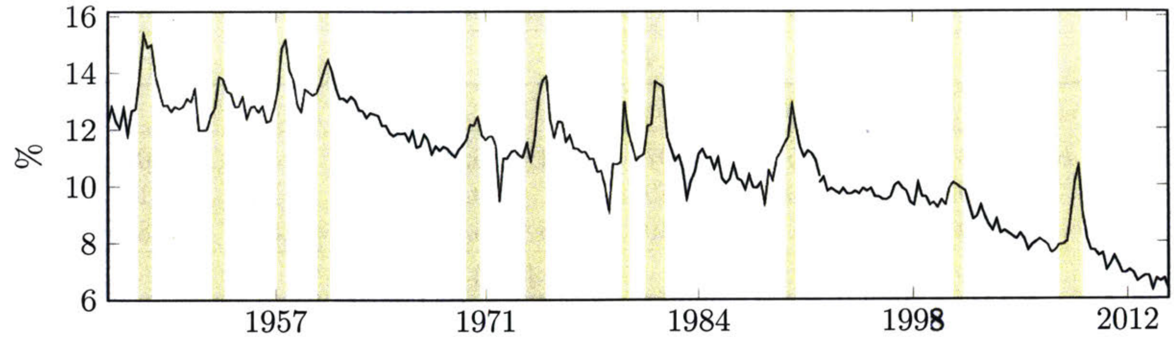
Notes: The table reports annualized mean excess return (%), market beta and annualized volatility (%) of simulated portfolios sorted based on employment growth defined as $(N^a_{i,t+12} - N^a_{i,t})/(0.5(N^a_{i,t+12} + N^a_{i,t}))$, firms in portfolio 1 have the lowest employment growth. Data on portfolio returns is constructed following methodology of Belo et al. (2014). Data sample is from July 1964 to December 2014.

Table 3.6: Properties of Simulated Portfolios with Constant Match Probability

	Portfolio									
	1	2	3	4	5	6	7	8	9	10
	Book-to-Market									
$\mathbb{E}[R - R_f]$	9.04	10.06	10.49	10.67	10.77	10.78	10.82	10.83	10.81	10.81
β	0.91	1.02	1.06	1.08	1.09	1.10	1.10	1.10	1.10	1.10
σ	20.31	22.72	23.77	24.25	24.47	24.56	24.58	24.60	24.60	24.62
	Earnings-to-Price									
$\mathbb{E}[R - R_f]$	9.51	10.22	10.55	10.72	10.77	10.80	10.82	10.82	10.81	10.80
β	0.96	1.03	1.07	1.09	1.09	1.10	1.10	1.10	1.10	1.10
σ	21.39	23.14	23.95	24.33	24.50	24.56	24.59	24.60	24.60	24.62
	Hiring Rate									
$\mathbb{E}[R - R_f]$	10.80	10.82	10.82	10.81	10.80	10.77	10.72	10.55	10.22	9.52
β	1.10	1.10	1.10	1.10	1.10	1.09	1.09	1.07	1.03	0.96
σ	24.62	24.60	24.60	24.59	24.56	24.50	24.33	23.96	23.16	21.41

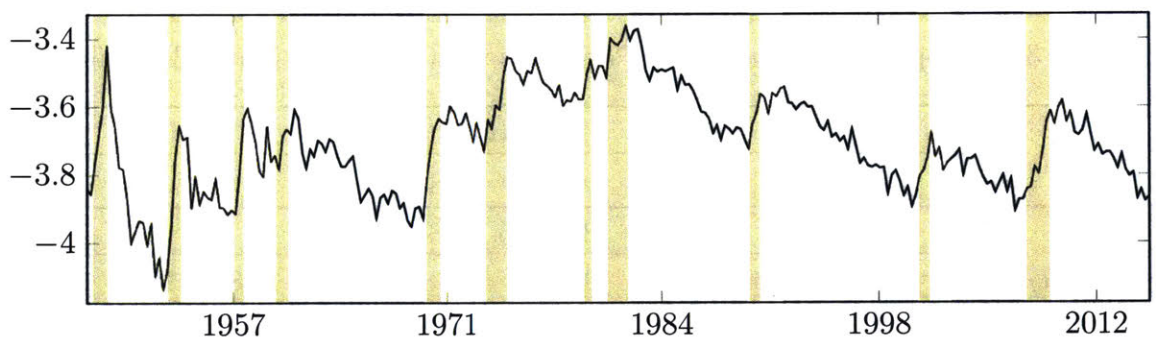
Notes: Each panel reports annualized mean excess return (%), market beta and annualized volatility (%) of simulated portfolios sorted based on: book-to-market ratio, earnings-to-price ratio and employment growth defined as $(N^a_{i,t+12} - N^a_{i,t})/(0.5(N^a_{i,t+12} + N^a_{i,t}))$.

Figure 3-1: Gross Job Reallocation in Manufacturing



Notes: Gross job reallocation is defined as sum of job creation and job destruction. Numbers are percentages of employment. Source: 1947Q1 – 1992Q4 Davis et al. (2006), 1993Q1 – 2015Q1 BLS Business Employment Dynamics.

Figure 3-2: Log Unemployment Outflow Rate



Notes: Logarithm of unemployment outflow as share of labor force. Source: July 1948 – March 1990 constructed using job finding rate from Shimer (2012), April 1990 – Oct 2015 BLS; second series is shifted to obtain continuous graph.

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