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Citation: White, R.L., Hazeltine, R.D.. and Loureiro, N.F. "Symmetries of a reduced fluid-gyrokinetic system." *Journal of Plasma Physics* 84:2 (March 2018): 905840204 ©2018 Author(s)

As Published: 10.1017/S0022377818000247

Publisher: Cambridge University Press (CUP)

Persistent URL: <https://hdl.handle.net/1721.1/124330>

Version: Original manuscript: author's manuscript prior to formal peer review

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Symmetries of a reduced fluid-gyrokinetic system

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(Received xx; revised xx; accepted xx)

Symmetries of a fluid-gyrokinetic model are investigated using Lie group techniques. Specifically the nonlinear system constructed by Zocco and Schekochihin (Zocco & Schekochihin 2011), which combines nonlinear fluid equations with a drift-kinetic description of parallel electron dynamics, is studied. Significantly, this model is fully gyrokinetic, allowing for arbitrary $k_{\perp}\rho_i$, where k_{\perp} is the perpendicular wave vector of the fluctuations and ρ_i the ion gyroradius. The model includes integral operators corresponding to gyroaveraging as well as the moment equations relating fluid variables to the kinetic distribution function. A large variety of exact symmetries is uncovered, some of which have unexpected form. Using these results, new nonlinear solutions are constructed, including a helical generalization of the Chapman-Kendall solution for a collapsing current sheet.

1. Introduction

Symmetry transformations – changes in the dependent and independent variables of a physical model that leave the model equations unchanged – are revealing and useful throughout theoretical physics. The most general scheme for uncovering point symmetries of a system of equations is Lie group analysis (see, e.g., Olver 1993; Cantwell 2002). This scheme has been used extensively in plasma physics, including studies of the Vlasov-Maxwell model for an unmagnetized plasma (see Roberts 1985; Kovalev *et al.* 1996) and the Grad-Shafranov equation (White & Hazeltine 2009). A special case of Lie symmetry, scaling symmetry, was fruitfully employed by Connor & Taylor (1977). In this work we apply the Lie procedure to a particular nonlinear gyrokinetic fluid model used in magnetized plasma turbulence and magnetic reconnection studies.

The symmetries of any physical model have intrinsic interest, especially because one often uncovers unexpected symmetries – beyond the usual rotations, translations and so on which are obvious from physical considerations. Knowledge of the symmetries can simplify numerical calculations, while providing useful tests on their accuracy. When a variational principle is available, the symmetries can be used to identify dynamical constants. They can also be used to generate new solutions from old ones – in particular, physically interesting solutions can be constructed by applying the group operator to a trivial, less interesting solution. Finally, in many cases symmetries can be used to reduce the order of a differential equation system, in some cases leading to exact solutions.

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1.1. *Fluid-gyrokinetic model*

A magnetized plasma is one in which the ion gyro-radius, ρ_i , is small compared to all equilibrium gradient scale lengths. But scale lengths of *perturbed* quantities in a magnetized plasma, measured by the perpendicular wave length k_{\perp}^{-1} , can break this ordering: $k_{\perp}\rho_i \sim 1$. Theories allowing for such finite-Larmor-radius (FLR) effects increasingly dominate plasma physics research, entering both kinetic and fluid models of plasma dynamics.

There are two ways in which conventional fluid equations fall short in their description of magnetized plasma dynamics. First, they represent FLR effects crudely, retaining at most terms of second order in $k_{\perp}\rho_i$. Second, they entirely omit Landau resonances, which, in the magnetized context, enter through wave-particle interactions parallel to the field – effects conventionally treated by the drift-kinetic equation. *Gyrokinetics* (Rutherford & Frieman 1968; Taylor & Hastie 1968; Catto 1978; Catto *et al.* 1981; Frieman & Chen 1982; Dubin *et al.* 1983; Lee 1983, 1987; Hahn *et al.* 1988; Brizard 1992) addresses both shortcomings, providing in particular a full FLR treatment of the perturbed fields, with however the expense and complexity of computation (analytical and numerical) in five dimensions of phase space. *Gyro-fluid* models reduce this overhead by restricting the FLR physics to coordinate space (see, for example, Hammett & Perkins 1990; Hammett *et al.* 1992; Dorland & Hammett 1993; Hammett *et al.* 1993; Beer & Hammett 1996; Snyder & Hammett 2001; Waelbroeck *et al.* 2009; Bian & Kontar 2010). However, the validity of the approximations made in their derivation can be hard to ascertain, especially in nonlinear contexts (Dimits *et al.* 2000).

An alternative and conceptually straightforward approach combines a fluid treatment of the perpendicular physics with a drift-kinetic description, including resonances and collisions, of the parallel dynamics (Ramos 2010, 2011). Such a hybrid approach was proposed and applied as early as 1958 (Kruskal & Oberman 1958; Rosenbluth & Rostoker 1959). Called “kinetic MHD,” the early approach neglected most FLR effects, combining MHD with the drift-kinetic equation. However in other respects it resembles the gyrokinetic fluid hybrid considered here.

We study a particular representative of the fluid-kinetic approach: the reduced gyrokinetic model derived in Zocco & Schekochihin (2011), referred to below as ZS. The model uses five fields – five functions of five independent variables (including time). To make this work self-contained, and establish notation, we start by reviewing the physical assumptions built into the ZS model in Section 2.1, and then express the model equations in normalized variables in Section 2.2. The remainder of the paper uses exclusively normalized variables, so the reader who is already familiar with the model can skip Section 2.1. The symmetries obtained from our analysis are shown in Section 3. Section 4 presents new exact solutions of the reduced MHD (RMHD) equations—a limit of the ZS model—obtained using symmetry transformations. In Section 5 we display the Lie group generator and present the procedure used to derive the symmetries for the (integro-differential) ZS model. We do not attempt any full exegesis of the Lie procedure; readers unfamiliar with it may consult such texts as Olver (1993) or Cantwell (2002). Our conclusions are summarized and discussed in Section 6.

2. Model equations

2.1. Introduction

The detailed derivation of the ZS model from the gyrokinetic equations is given in Zocco & Schekochihin (2011). Here, we briefly survey the physical assumptions, summarize the resulting equations, and indicate the physical meaning of each of the fields.

The plasma, composed of electrons with charge $-e$ and ions with charge Ze , is assumed to have a uniform background magnetic field $B_0 \hat{\mathbf{z}}$, and the equilibrium electrons and ions are Maxwellian:

$$F_{0s} = \frac{n_{0s}}{(2\pi)^{3/2} v_{Ts}^3} \exp\left(-\frac{v^2}{2v_{Ts}^2}\right), \quad (2.1)$$

with $v_{Ts} = (T_{0s}/m_s)^{1/2}$. Here we deviate from the convention of Zocco & Schekochihin (2011), where the Maxwellian is characterized by its *most probable* speed $v_{\text{th},s} = \sqrt{2}v_{Ts}$. This translates to a slightly different definition of the Larmor radius, which for us is defined $\rho_s = v_{Ts}/|\Omega_s|$, with $\Omega_s = q_s B_0/(m_s c)$. This modification eliminates many factors of $\sqrt{2}$ in the final equations.

In accordance with the standard δf gyrokinetic ansatz, each field is split into its background value plus a small perturbation, with $\delta f_s/F_{0s} \sim \delta \mathbf{B}/\mathbf{B}_0 \sim k_{\parallel}/k_{\perp} \sim \omega/\Omega_s \sim \epsilon \ll 1$, and, additionally, $\beta_s \sim Zm_e/m_i$, with the mass ratio being treated as a second formal small parameter.

2.1.1. Electrostatic Ions

After ordering out electromagnetic effects and parallel streaming in the ion gyrokinetic equation, the ion distribution function is approximated by

$$\delta f_i = \frac{ZeF_{0i}}{\tau T_{0e}} (\langle \varphi \rangle_{\mathbf{R}_i} - \varphi), \quad (2.2)$$

where $\mathbf{R}_i(\mathbf{r}, \mathbf{v}) = \mathbf{r} - \Omega_i^{-1} \mathbf{v} \times \hat{\mathbf{z}}$, $\Omega_i = ZeB_0/(m_i c)$, $\tau = T_{0i}/T_{0e}$, and $\langle \cdots \rangle_{\mathbf{R}_i}$ denotes the gyroaverage at fixed \mathbf{R}_i .

It follows that the ion density perturbation δn_i and mean parallel flow $u_{\parallel,i}$ are given by

$$\frac{\delta n_i}{n_{0i}} = -\frac{Z}{\tau} (1 - \hat{\Gamma}_0) \frac{e\varphi}{T_{0e}}, \quad (2.3)$$

$$u_{\parallel,i} = 0, \quad (2.4)$$

where $\hat{\Gamma}_0$ is an ion gyroaveraging operator:

$$\hat{\Gamma}_0[\cdots] \equiv \frac{1}{n_{0i}} \int d^3\mathbf{v} \langle \cdots \rangle_{\mathbf{R}_i} F_{0i}(\mathbf{v}). \quad (2.5)$$

In Fourier space, Γ_0 has the closed-form expression

$$\Gamma_0 = I_0(\alpha_i) e^{-\alpha_i}, \quad (2.6)$$

where I_0 is the zeroth order modified Bessel function and $\alpha_i = k_{\perp}^2 \rho_i^2$.

2.1.2. Quasineutrality and Ampère's Law

Since $u_{\parallel,i} = 0$, we have $J_{\parallel} = -en_{0e}u_{\parallel,e}$, and thus the parallel component of Ampère's law becomes

$$u_{\parallel,e} = \frac{e}{m_e c} d_e^2 \nabla_{\perp}^2 A_{\parallel}. \quad (2.7)$$

According to Eq. (2.3), quasineutrality is expressed by

$$\frac{\delta n_e}{n_{0e}} = -\frac{Z}{\tau}(1 - \hat{T}_0) \frac{e\varphi}{T_{0e}}. \quad (2.8)$$

2.1.3. Drift-kinetic electrons

The electrons are described by a distribution function g_e from which the density and parallel flow terms have been extracted:

$$g_e = \delta f_e - \left(\frac{\delta n_e}{n_{0e}} - v_{\parallel} \frac{u_{\parallel,e}}{v_{Te}^2} \right) F_{0e}, \quad (2.9)$$

The electron dynamics is described by fluid equations for the explicit moments, together with a simplified drift kinetic equation:

$$\frac{d}{dt} \frac{\delta n_e}{n_{0e}} = -\hat{\mathbf{b}} \cdot \nabla u_{\parallel,e}, \quad (2.10)$$

$$\frac{d}{dt} \left(A_{\parallel} + \frac{cm_e}{e} u_{\parallel,e} \right) = -c \frac{\partial \varphi}{\partial z} + \frac{cT_{0e}}{e} \hat{\mathbf{b}} \cdot \nabla \left(\frac{\delta n_e}{n_{0e}} + \frac{\delta T_{\parallel,e}}{T_{0e}} \right), \quad (2.11)$$

where

$$\frac{\delta T_{\parallel,e}}{T_{0e}} = \frac{1}{n_{0e}} \int d^3\mathbf{v} \frac{v_{\parallel}^2}{v_{Te}^2} g_e, \quad (2.12)$$

is the electron temperature perturbation. We have also introduced the convective time derivative

$$\frac{df}{dt} \equiv \frac{\partial f}{\partial t} + \frac{c}{B_0} \{ \varphi, f \}, \quad (2.13)$$

with the Poisson bracket defined by

$$\{ f, g \} \equiv \frac{\partial f}{\partial x} \frac{\partial g}{\partial y} - \frac{\partial f}{\partial y} \frac{\partial g}{\partial x}, \quad (2.14)$$

and the parallel gradient operator

$$\hat{\mathbf{b}} \cdot \nabla f \equiv \frac{\partial f}{\partial z} - \frac{1}{B_0} \{ A_{\parallel}, f \}. \quad (2.15)$$

The remaining distribution g_e is determined by a simplified drift-kinetic equation:

$$\begin{aligned} \frac{dg_e}{dt} + v_{\parallel} \hat{\mathbf{b}} \cdot \nabla \left(g_e - \frac{\delta T_{\parallel,e}}{T_{0e}} F_{0e} \right) - C[g_e] \\ = \left(1 - \frac{v_{\parallel}^2}{v_{Te}^2} \right) F_{0e} \hat{\mathbf{b}} \cdot \nabla u_{\parallel,e}, \end{aligned} \quad (2.16)$$

in which electron FLR terms, as well as curvature drifts, are ordered out by the strong guide field. Finally,

$$C[g_e] = \nu_{ei} \left[v_{Te}^2 \frac{\partial}{\partial v_{\parallel}} \left(\frac{\partial}{\partial v_{\parallel}} + \frac{v_{\parallel}}{v_{Te}^2} \right) g_e - \left(1 - \frac{v_{\parallel}^2}{v_{Te}^2} \right) \frac{\delta T_{\parallel,e}}{T_{0e}} F_{0e} \right] \quad (2.17)$$

is a model collision operator that conserves particles, parallel momentum and parallel kinetic energy (Zocco & Schekochihin 2011). This model operator – a generalization of the so-called Lenard-Bernstein operator introduced by Rayleigh (1891) – also satisfies an H theorem.

Note that (2.9) requires the integral constraints

$$\int d^3\mathbf{v} \left(\frac{1}{v_{\parallel}} \right) g_e = 0. \quad (2.18)$$

2.1.4. Summary

Given a background characterized by B_0 and v_{Ts} , Eqs. (2.7), (2.8), (2.10), (2.11), and (2.16), are a closed system of equations governing small nonlinear perturbations of the fields φ , A_{\parallel} , $u_{\parallel e}$, $\delta n_e/n_{0e}$, and g_e . In the final formulation of the model presented in Eqs. (62)–(64) of Zocco & Schekochihin (2011), $u_{\parallel e}$ has been eliminated using (2.7). We will do the same in the remainder of the paper.

2.2. Normalization

For the purposes of obtaining symmetries, it is convenient to reduce the number of constants in the ZS model by normalizing all quantities. It turns out that the fields can be normalized in such a way that there are only two dimensionless constants: Z/τ and $\alpha \equiv \rho_i^2/d_e^2$, and these only appear in the integral closure relation relating the electrostatic potential to the density perturbation.

The dependent variables are normalized via

$$\begin{aligned} \delta n &= \frac{\delta n_e/n_{0e}}{\langle \delta n_e/n_{0e} \rangle}, & \psi &= \frac{A_{\parallel}}{\langle A_{\parallel} \rangle}, & \phi &= \frac{\varphi}{\langle \varphi \rangle}, \\ \delta T &= \frac{\delta T_{\parallel e}/T_{0e}}{\langle \delta T_{\parallel e}/T_{0e} \rangle}, & g &= \frac{g_e}{F_{0e} \langle g_e \rangle}, \end{aligned} \quad (2.19)$$

with

$$\langle A_{\parallel} \rangle = B_0 \nu_{ei} d_e^2 / v_{Te}, \quad (2.20)$$

$$\langle \varphi \rangle = B_0 \nu_{ei} d_e^2 / c, \quad (2.21)$$

$$\langle \delta n_e/n_{0e} \rangle = \langle \delta T_{\parallel e}/T_{0e} \rangle = \langle g_e \rangle = \langle \delta \rangle, \quad (2.22)$$

where

$$\langle \delta \rangle \equiv \frac{e B_0 \nu_{ei} d_e^2}{c T_{0e}} = \frac{\nu_{ei}}{\Omega_e} \sqrt{\beta_e}. \quad (2.23)$$

The independent variables are similarly normalized, with the following normalization scales:

$$\langle v_{\parallel} \rangle = v_{Te}, \quad \langle x_{\perp} \rangle = d_e, \quad \langle z \rangle = v_{Te} / \nu_{ei}, \quad \langle t \rangle = 1 / \nu_{ei}. \quad (2.24)$$

Defining the normalized convective time derivative, parallel gradient, and perpendicular Laplacian

$$d_t f \equiv \partial_t f + \{\phi, f\}, \quad \nabla_{\parallel} f \equiv \partial_z f - \{\psi, f\}, \quad \Delta \equiv \partial_x^2 + \partial_y^2, \quad (2.25)$$

and normalized gyrokinetic and collision operators

$$\hat{\mathcal{G}} \equiv - \frac{Z}{\tau} \frac{e \langle \varphi \rangle}{\langle \delta \rangle} \mathcal{G}, \quad (2.26)$$

$$\hat{\mathcal{C}} \equiv g_{vv} - v g_v - (1 - v^2) \delta T, \quad (2.27)$$

the normalized reduced fluid-kinetic model takes the form

$$d_t \delta n = -\nabla_{\parallel} \Delta \psi, \quad (2.28)$$

$$d_t \psi + \phi_z = \lambda [\Delta(\psi + d_t \psi) + \nabla_{\parallel}(\delta n + \delta T)], \quad (2.29)$$

$$d_t g + v \nabla_{\parallel}(g - \delta T) = \hat{C} + (1 - v^2) \nabla_{\parallel} \Delta \psi, \quad (2.30)$$

$$\delta T = \frac{1}{\sqrt{2\pi}} \int dv' v'^2 e^{-v'^2/2} g(v'), \quad (2.31)$$

Here, the brackets are the same as (2.14) except the perpendicular coordinates are now normalized, and $\lambda(= 1)$ is a tag for the terms that are dropped in the ideal reduced magnetohydrodynamic (RMHD) limit. These differential equations are to be solved subject to the integral constraints

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2\pi}} \int dv' \begin{pmatrix} 0 \\ v' \end{pmatrix} e^{-v'^2/2} g(v'), \quad (2.32)$$

together with

$$\delta n = \hat{\mathcal{G}} \phi. \quad (2.33)$$

We introduce a normalized Alfvén velocity,

$$v_A = \frac{1}{v_{Te}} \frac{B_0}{\sqrt{4\pi n_{0i} m_i}} \quad (2.34)$$

$$(=\sqrt{\tau \alpha / Z}), \quad (2.35)$$

and the normalized kernel,

$$\hat{K}(x) = \frac{-Z/\tau}{2\pi} \int dk_{\perp} k_{\perp} J_0(k_{\perp} x) \left[1 - I_0(\alpha k_{\perp}^2) e^{-\alpha k_{\perp}^2} \right] \quad (2.36)$$

obtained from (2.6). Then the operator $\hat{\mathcal{G}}$ becomes

$$\hat{\mathcal{G}} u = \int d^2 x'_{\perp} \hat{K}(|\mathbf{x}_{\perp} - \mathbf{x}'_{\perp}|) u(\mathbf{x}'_{\perp}), \quad (2.37)$$

$$= v_A^{-2} \Delta + \lambda \mathcal{O}(\alpha^2 \Delta^2). \quad (2.38)$$

Finally, to determine the symmetries of these equations, we must explicitly include the trivial relations

$$\partial_v \delta n = \partial_v \phi = \partial_v \psi = \partial_v \delta T = 0. \quad (2.39)$$

2.3. RMHD Limit

If we set $\lambda = 0$ in (2.29) and (2.38), the equations (2.28)–(2.29) become an autonomous subsystem for ϕ and ψ :

$$d_t \Delta \phi = -v_A^2 \nabla_{\parallel} \Delta \psi, \quad (2.40)$$

$$d_t \psi + \phi_z = 0, \quad (2.41)$$

while g becomes a decoupled scalar field, constrained to satisfy the driven integro-differential equation

$$d_t g + v \nabla_{\parallel}(g - \delta T[g]) = g_{vv} - v g_v + (1 - v^2)(\nabla_{\parallel} \Delta \psi - \delta T[g]). \quad (2.42)$$

Equations (2.40) and (2.41) define ideal RMHD (Kadomtsev & Pogutse 1974; Strauss 1976). There is no coupling to the kinetic equation.

3. Symmetries

Here, we present the symmetries of the system (2.28)–(2.33) in the form of transformations of known solutions, rather than in terms of the infinitesimal generators of the symmetries. The latter are obtained directly from the invariance criterion in section 5.

3.1. Gauge Transformation

Given a solution $(\phi, \psi, \delta n, \delta T, g)(v, \mathbf{x}_\perp, z, t)$, and an arbitrary function $H(z, t)$, one can generate a new solution $(\tilde{\phi}, \tilde{\psi}, \tilde{\delta n}, \tilde{\delta T}, \tilde{g})(v, \mathbf{x}_\perp, z, t)$ via

$$\begin{pmatrix} \tilde{\phi} \\ \tilde{\psi} \\ \tilde{\delta n} \\ \tilde{\delta T} \\ \tilde{g} \end{pmatrix} (v, \mathbf{x}_\perp, z, t) = \begin{pmatrix} \phi \\ \psi \\ \delta n \\ \delta T \\ g \end{pmatrix} (v, \mathbf{x}_\perp, z, t) + \begin{pmatrix} -\partial_t H \\ \partial_z H \\ 0 \\ 0 \\ 0 \end{pmatrix}. \quad (3.1)$$

It is not hard to see that this symmetry is expressing gauge invariance. After undoing the normalizations, (3.1) becomes

$$\begin{pmatrix} \tilde{\varphi} \\ \tilde{\mathbf{A}} \end{pmatrix} = \begin{pmatrix} \varphi \\ \mathbf{A} \end{pmatrix} + \begin{pmatrix} -\frac{1}{c} \frac{\partial \Lambda}{\partial t} \\ \nabla \Lambda \end{pmatrix}, \quad (3.2)$$

where $\Lambda = c \langle \varphi \rangle \langle t \rangle H$, and $\mathbf{A} = \hat{\mathbf{z}} A_\parallel + \mathcal{O}(\sqrt{\beta_s})$. Note that if H had \mathbf{x}_\perp dependence, then this gauge transformation would change \mathbf{A}_\perp as well. However, in the low- β limit, \mathbf{A}_\perp is ordered out of the model, so the gauge must be independent of \mathbf{x}_\perp .

Of course this symmetry also holds in the RMHD model – explaining the absence of λ in the transformation (3.1).

The appearance of gauge symmetry in the ZS model is not surprising, but also not without significance. In view of the many approximations involved in the construction of ZS and other reduced models, its emergence here gives confidence in the model's treatment of the electromagnetic field.

3.2. Perpendicular Translations

Let $\boldsymbol{\xi}(z, t)$ be an arbitrary displacement in the x - y plane. Then $\boldsymbol{\xi}$ produces the symmetry transformation

$$\begin{pmatrix} \tilde{\phi} \\ \tilde{\psi} \\ \tilde{\delta n} \\ \tilde{\delta T} \\ \tilde{g} \end{pmatrix} (v, x, y, z, t) = \begin{pmatrix} \phi \\ \psi \\ \delta n \\ \delta T \\ g \end{pmatrix} (v, \mathbf{x}_\perp + \boldsymbol{\xi}(z, t), z, t) + \begin{pmatrix} -\partial_t (\hat{\mathbf{z}} \cdot \boldsymbol{\xi} \times \mathbf{x}_\perp) \\ \partial_z (\hat{\mathbf{z}} \cdot \boldsymbol{\xi} \times \mathbf{x}_\perp) \\ 0 \\ 0 \\ 0 \end{pmatrix}. \quad (3.3)$$

In the case where $\boldsymbol{\xi}$ is a constant, we recover the obvious result that the model is translation invariant in the \mathbf{x}_\perp plane. In the more general case, the transformations of ϕ and ψ follow the same pattern as the gauge symmetry, but the overall transformation of these fields is not a gauge transformation: note the additional (z, t) -dependent translation of the initial fields, as well as the fact that the gradient of $\hat{\mathbf{z}} \cdot \boldsymbol{\xi} \times \mathbf{x}_\perp$ has nonzero $\hat{\mathbf{x}}_\perp$ components.

3.3. Alfvénic Rotations

Let $\Theta(z, t)$ be a solution to the one dimensional (Alfvén) wave equation

$$v_A^{-2} \Theta_{tt} = \Theta_{zz}. \quad (3.4)$$

This function will determine the z - t dependent rotation of the original solution about the z axis. After transforming to polar coordinates in the x - y plane, $\mathbf{x}_\perp = r\hat{\mathbf{r}}(\theta)$, the symmetry transformation takes the form

$$\begin{pmatrix} \tilde{\phi} \\ \tilde{\psi} \\ \tilde{\delta n} \\ \tilde{\delta T} \\ \tilde{g} \end{pmatrix}(v, r, \theta, z, t) = \begin{pmatrix} \phi \\ \psi \\ \delta n \\ \delta T \\ g \end{pmatrix}(v, r, \theta + \Theta, z, t) + \begin{pmatrix} -\partial_t(r^2\Theta/2) \\ \partial_z(r^2\Theta/2) \\ -2\partial_t\Theta/v_A^2 \\ 0 \\ G \end{pmatrix} + \lambda \begin{pmatrix} \mathcal{F} \\ 0 \\ 0 \\ \mathcal{T} \\ 0 \end{pmatrix}. \quad (3.5)$$

Here the function $G(v, z, t)$ appears as a displacement for the distribution function g :

$$g \rightarrow g + G. \quad (3.6)$$

A detailed discussion of G appears in the following subsection. We have also introduced

$$\mathcal{F} = 2(\Theta + \partial_t\Theta) - 2\partial_t\Theta/v_A^2 + \mathcal{T}, \quad (3.7)$$

and

$$\mathcal{T} = \frac{1}{\sqrt{2\pi}} \int dv' v'^2 e^{-v'^2/2} G(v'). \quad (3.8)$$

The first term on the right side of (3.7) is due to the combination of resistivity and electron inertia; the second term arises from the density contribution to the perturbed electron pressure; and the last term is due to the electron temperature perturbation.

Note that our symmetries apply to limiting case of RMHD, where the functions G and \mathcal{T} can be ignored.

3.4. The function G

3.4.1. Linear drift-kinetic equation

The function Θ determines G implicitly, through the kinetic equation

$$G_{vv} - vG_v - vG_z - G_t = -\text{He}_2(v)(\mathcal{T} - 2\Theta_{tt}/v_A^2) - \text{He}_1(v)\mathcal{T}_z, \quad (3.9)$$

with the constraints

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2\pi}} \int dv' \begin{pmatrix} 0 \\ v' \end{pmatrix} e^{-v'^2/2} G(v'). \quad (3.10)$$

In (3.9), the He_n are the “probabilist’s” Hermite polynomials.

Aside from the coefficients on its right-hand side, (3.9) is identical to the *linearized* version of the drift-kinetic equation (2.16), which has been previously studied in detail (see, for example, Zocco & Schekochihin 2011; Hatch *et al.* 2014; Schekochihin *et al.* 2016; White & Hazeltine 2017). In the symmetry context, the linearity of (3.9) does not result from approximation; here the linearity follows from the general structure of Lie groups. In particular, the infinitesimal generators of any Lie group form a vector space, so the determining equations for symmetry transformations are always linear. Similarly the absence of the electrostatic potential in (3.9) is not an approximation; it reflects exact Lie symmetries, such as (5.10), (5.11) and (5.16).

3.4.2. Closed form solution for $G(v, z, t)$

Using special choices for such functions as

$$\Theta(z, t) = \Theta_+(z + vAt) + \Theta_-(z - vAt), \quad (3.11)$$

it is not hard to find a closed-form solution for G . Here we are content to display a single example: the choice

$$\Theta = -\frac{T_0}{48} [(z + v_A t)^3 + (z - v_A t)^3], \quad (3.12)$$

where T_0 is a constant, together with

$$\mathcal{T} = \frac{T_0}{4v_A} [(z + v_A t)^2 - (z - v_A t)^2] = T_0 z t \quad (3.13)$$

allows an exact solution with

$$G(v, z, t) = \frac{T_0}{2} \left[t z \text{He}_2(v) - \frac{1}{3} \left(t - \frac{1}{3} + c e^{-3t} \right) \text{He}_3(v) \right], \quad (3.14)$$

where c is an arbitrary constant. It is easily verified that this function satisfies the differential equation as well as the integral constraints. In addition to furnishing an explicit symmetry, this relatively simple function is in fact an exact solution to the full nonlinear integrodifferential model, and thus can be used for benchmarking codes.

3.4.3. Fourier-Hermite expansion of G

A conventional approach to the drift-kinetic equation (Watanabe & Sugama 2004; Zocco & Schekochihin 2011; Hatch *et al.* 2014; Kanekar *et al.* 2015) expands the distribution function, in this case $G(v, z, t)$, as a series in Hermite polynomials. Here it is convenient to use “probabilists” Hermite polynomials, and to Fourier analyze the z - and t -dependent Hermite coefficient, thus expressing G as

$$G(v, z, t) = \sum_0^{\infty} \frac{\text{He}_n(v)}{\sqrt{n!}} \int \frac{d\omega dk}{(2\pi)^2} G_n(\omega, k) e^{i(kz - \omega t)}. \quad (3.15)$$

We note that this expansion restricts our consideration to solutions which have a Fourier transform; it would exclude, for example, (3.14).

The constraints (3.10) become

$$G_0 = G_1 = 0, \quad (3.16)$$

while (3.8) gives

$$G_2 = \frac{1}{\sqrt{2}} \mathcal{T}. \quad (3.17)$$

The remaining G_n are determined by the recursion relation

$$k(\sqrt{n+1}G_{n+1} + \sqrt{n}G_{n-1}) - inG_n - \omega G_n = 0, \quad n > 2. \quad (3.18)$$

Although (3.18) is a simple, linear recursion relation, solving it requires some care: there is a spurious divergent “solution” (White & Hazeltine 2017) that must be avoided by appropriate determination of initial data – in this case the ratio $\Delta \equiv G_3/G_2$. In numerical applications, one is only interested in calculating a finite subset $\{G_n\}_{n \leq N}$ because g is represented by a finite sum of Hermite polynomials. In this case, $\Delta(N)$ can be determined by the same closure scheme adopted by the numerical method to solve the full nonlinear model. For example, if one simply truncates by setting $G_{N+1} = 0$, then (3.18) can be iterated backward to determine $\{G_N/G_{N-1}, G_{N-1}/G_{N-2}, \dots, \Delta(N)\}$. See Zocco *et al.* (2015); Loureiro *et al.* (2016) for alternate closure schemes. The choice of Δ , as well as other approaches to solving the recursion relation, will be considered in a future publication.

Finally, \mathcal{T} is expressed in terms of the Θ and Δ via

$$\mathcal{T}(z, t) = \int \left(\frac{2ik}{\frac{\omega/k}{2} - \sqrt{3}\Delta(\omega, k)} \right) \Theta(\omega, k) e^{-ikz + i\omega t} \frac{d\omega dk}{(2\pi)^2}. \quad (3.19)$$

This expression, like (3.18), is obtained by direct Fourier-Hermite transformation of (3.15).

4. Sample Applications

The value of knowing the symmetries of a some mathematical description is appreciated in nearly all areas of physics. In addition to their relation to conservation laws (discussed below), symmetries can be used to test numerical solution schemes, to motivate approximation hypotheses, and to generate novel exact solutions. In an important sense, the symmetries of a system carry information about its deep structure. The following discussion, touching upon two samples of potential application, is merely intended to be suggestive.

4.1. Transforming the trivial solution

The most straightforward way to use symmetry transformations is to generate new solutions from known solutions. One obvious exact solution is the trivial solution, with all of the fields identically zero. In this case, by specifying the functions H , ξ , and Θ , one can generate nontrivial exact solutions by transforming the trivial solution using the symmetries presented in Section 3. In fact, one can directly verify that all of the nonlinear terms in the set of solutions obtained this way are exactly zero. In other words, by transforming the trivial solution, one obtains solutions to the linearized version of the model which happen to be exact solutions to the full nonlinear system.

4.2. Transformed Chapman-Kendall solution

As a second illustration of the use of the transformations presented in section 3, we consider the exact solution

$$\phi = \Gamma xy, \quad \psi = \frac{x^2}{a_0 e^{-2\Gamma\tau}} - \frac{y^2}{b_0 e^{2\Gamma\tau}} \quad (4.1)$$

of the RMHD equations (2.40) and (2.41) derived in Chapman & Kendall (1963). Here the arbitrary rate parameter Γ , which is set by boundary conditions, is assumed to be fast compared to any diffusion time scale. This solution corresponds to a thinning and elongating magnetic neutral line at $x = 0$, as would be found at the center of a localized collapsing current sheet Waelbroeck (1989, 1993); Loureiro *et al.* (2005).

This is a particularly relevant solution for the ZS model, as the orderings were constructed with magnetic reconnection studies in mind. For example, a prototypical model problem would be a localized thinning current sheet whose evolution is eventually disrupted by a reconnecting instability (Uzdensky & Loureiro 2016). Typically, in a high temperature plasma, the length scales associated with the reconnecting instability are much smaller than the width of the current sheet itself. In this circumstance, a localized model of the current sheet such as (4.1) can capture the salient features of the background which play a role in the physics of the instability, and subsequent nonlinear evolution.

Using very simple solutions of (3.4), one can generate more exotic versions of the Chapman-Kendall solution. For example, by choosing $\Theta = z/z_0$, the initial solution (4.1) transforms to

$$\tilde{\phi} = \Gamma \tilde{x} \tilde{y}, \quad \tilde{\psi} = \tilde{x}^2 \left(\frac{1}{a_0 e^{-2\Gamma\tau}} + \frac{1}{2z_0} \right) - \tilde{y}^2 \left(\frac{1}{b_0 e^{2\Gamma\tau}} - \frac{1}{2z_0} \right), \quad (4.2)$$

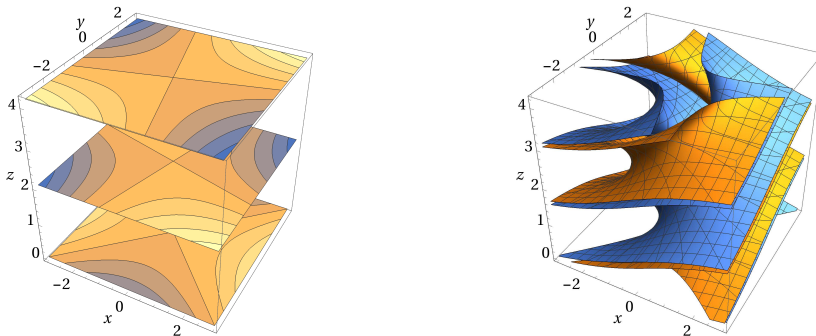


FIGURE 1. Flux surfaces near magnetic null for (4.2) with $t = 0$, $a = 1$, $b = 0.5$, and $z_0 = 1$.

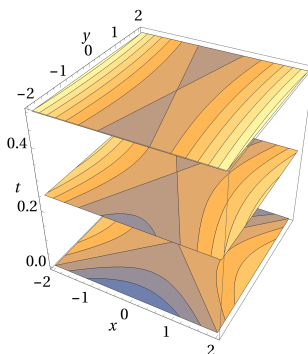


FIGURE 2. Flux surfaces near magnetic null for (4.2) with $z = 0$, $a = 1$, $b = 0.5$, $\Gamma = 1$ and $z_0 = 1$.

where

$$\bar{x} = x \cos(z/z_0) + y \sin(z/z_0), \quad \bar{y} = y \cos(z/z_0) - x \sin(z/z_0) \quad (4.3)$$

are the helically rotated coordinates. Physically, this transformation corresponds to a linear helical twisting of the original current sheet with a uniform current (amplitude proportional to the helical pitch) added. Note that if $b_0 < 2z_0$, then the flux surfaces will initially be hyperbolic, but at a later time t_c when $b_0 \exp(\Gamma t_c) = 2z_0$, they topologically transform to elliptic surfaces.

This three-dimensional magnetic structure is a simple, analytically-tractable model configuration of an evolving three dimensional magnetic structure that will eventually become unstable to reconnection-driven instabilities. The formation of helical three-dimensional magnetic fields – and the potential subsequent magnetic reconnection thought to occur in such structures – is highly relevant for solar flares (Janvier *et al.* 2013).

As a second example, by choosing $\Theta = (z - v_A t)/z_0$, we find a solution which, at time $t = 0$, is in the same helical configuration as (4.2), but now moves along the guide field at the Alfvén speed.

5. Symmetry Analysis

5.1. Infinitesimal Generators

The maps given in section 3 can be viewed geometrically as a continuous family of transformations of the graph of the solution, which is a set of points in the (10 dimen-

sional) space labeled by $(v, x, y, z, t; \phi, \psi, \delta n, \delta T, g)$. These subgroups of transformations can be characterized by their *infinitesimal generators* Ξ .

For example, consider the action of an infinitesimal gauge transformation $\delta H(z, t)$ on the graph of a solution $(\phi, \psi, \delta n, \delta T, g)$:

$$(\tilde{v}, \tilde{\mathbf{x}}_{\perp}, \tilde{z}, \tilde{t}, \tilde{\phi}, \tilde{\psi}, \tilde{\delta n}, \tilde{\delta T}, \tilde{g}) = [1 - \delta H_t \partial_{\phi} + \delta H_z \partial_{\psi} + \mathcal{O}(\delta^2)](v, \mathbf{x}, z, t; \phi, \psi, \delta n, \delta T, g). \quad (5.1)$$

Here, the infinitesimal generator is seen to be

$$\Xi_H = -H_t \partial_{\phi} + H_z \partial_{\psi}. \quad (5.2)$$

Similarly, the infinitesimal generators for (3.3) and (3.5) are

$$\Xi_{\xi} = -\xi \cdot \partial_{\mathbf{x}_{\perp}} + \partial_z \hat{\mathbf{z}} \cdot \xi \times \mathbf{x}_{\perp} \partial_{\psi} - \partial_t \hat{\mathbf{z}} \cdot \xi \times \mathbf{x}_{\perp} \partial_{\phi}, \quad (5.3)$$

and

$$\begin{aligned} \Xi_{\Theta} = & \Theta \partial_{\theta} + \partial_z \left(\frac{r^2}{2} \Theta \right) \partial_{\psi} + \left[-\partial_t \left(\frac{r^2}{2} \Theta \right) + \mathcal{F} \right] \partial_{\phi} \\ & + \left[-v_A^{-2} \Delta \partial_t \left(\frac{r^2}{2} \Theta \right) \right] \partial_{\delta n} + \mathcal{T} \partial_{\delta T} + G \partial_g, \end{aligned} \quad (5.4)$$

respectively.

There is a one-to-one correspondence between Lie-group transformations and their infinitesimal generators. For a rigorous but readable introduction to this formalism, see Chapter 1 of Olver (1993).

5.2. Generators acting on differential equations

As an extension to a standard graph, one can take a solution of the ZS model and produce a graph in the higher dimensional space consisting of the independent and dependent variables, as well as all higher derivatives up to second order (the highest order that appears in the model equations). In this higher dimensional *jet* space, labeled $(v, x, y, z, t; \phi, \psi, \delta n, \delta T, g; \phi_v, \dots, \phi_t, \dots) \equiv (x^i, u^{\alpha}, u_i^{\alpha}, u_{i,j}^{\alpha})$, the action of a symmetry transformation will also involve the coordinates associated with the derivatives:

$$\Xi_* = \underbrace{\xi^i(x, u) \partial_{x^i} + U^{\alpha}(x, u) \partial_{u^{\alpha}}}_{=\Xi} + \sum_{i,j} U^{\alpha;i,j}(x, u) \partial_{u_{i,j}^{\alpha}}.$$

In this higher dimensional space, the model equations, generically expressed in the form

$$F(x^i, u^{\alpha}, u_i^{\alpha}, u_{i,j}^{\alpha}) = 0 \text{ for all } x^i, \quad (5.5)$$

are formally algebraic equations.

A transformation generated by (5.2) is a symmetry of the model if the model equations themselves are invariant under the transformation while transforming a solution. That is

$$\Xi_* F = 0 \text{ whenever } F = 0 \Leftrightarrow \Xi \text{ is a symmetry of } F. \quad (5.6)$$

This *invariance criterion* is a fundamental theorem in symmetry analysis of differential equations.

The second key result is the *prolongation formula*, which expresses the functions $U^{\alpha;i,j}$ in terms of the U^{α} and ξ^i . In other words, given an infinitesimal generator Ξ in the space (x^i, u^{α}) , the prolongation formula provides an explicit expression of the form of the generator Ξ_* in the jet space $(x^i, u^{\alpha}, u_i^{\alpha}, u_{i,j}^{\alpha})$.

Using the prolongation formula, the invariance criterion (5.6) becomes a working

procedure to obtain symmetries. Starting from the general symmetry generator

$$\Xi = \xi^i(x, u)\partial_{x^i} + U^\alpha(x, u)\partial_{u^\alpha}, \quad (5.7)$$

with unknown coefficients ξ^i and U^α , one uses the prolongation formula to compute the form of this symmetry generator (denoted here as Ξ_*) in jet space. Once this is computed, the invariance criterion imposes conditions on the coefficients of Ξ_* which must be satisfied in order for (5.7) to correspond to a symmetry of the system. By solving these *determining equations*, the most general symmetry of the form (5.7) is obtained.

There are two characteristics of this procedure that will be leveraged to extend the procedure to the ZS model, which also has integral relations. First, in a model consisting of more than one equation, the invariance criterion can be applied to one equation at a time. The resulting symmetry group will be a subset of the full group; after all, any symmetry of the full model must leave each of its equations unchanged. Thus we can find the symmetries of the under-determined system (2.28)–(2.30) before considering the integral relations (2.31) and (2.33).

The second noteworthy point is that determining equations are usually straightforward to solve, even for highly complicated nonlinear models, provided there are no integral terms. This justifies the operation of deriving the (generally more complicated) symmetry group of the under-determined model first. Once the symmetry group of the under-determined model is obtained, this class of transformations is used to simplify application of the integral constraints.

5.3. Determining equations

We begin our analysis with a generator of the form

$$\Xi = c_1\partial_t + c_2\partial_z + X\partial_x + Y\partial_y + \Phi\partial_\phi + \Psi\partial_\psi + G\partial_g + \mathcal{T}\partial_{\delta T} + N\partial_{\delta n}, \quad (5.8)$$

where c_i are constants, and the remaining unknown functions depend only on the independent variables. This is not the most generic form. Our motivation for choosing this simpler but less general form is based on exploratory computational analysis, using software provided by Cantwell (2002). This exploration suggests that all of the symmetries are of the form (5.8).

5.3.1. Local determining equations

Following the procedure sketched in the preceding subsection, we obtain from (2.28) and (2.30) the following determining equations:

$$X_v = Y_v = \Phi_v = \Psi_v = T_v = N_v = 0, \quad (5.9)$$

$$X_x + Y_y = 0, \quad (5.10)$$

$$X_x = Y_y = 0, \quad (5.11)$$

$$\Psi_x - Y_z = \Psi_y - X_z = 0, \quad (5.12)$$

$$\Phi_y + X_t = \Phi_x - Y_t = 0, \quad (5.13)$$

$$N_x = N_y = 0, \quad (5.14)$$

$$\mathcal{T}_x = \mathcal{T}_y = 0, \quad (5.15)$$

$$G_x = G_y = 0, \quad (5.16)$$

$$N_t = \Delta\Psi_z, \quad (5.17)$$

$$\Psi_t - \Delta\Psi_t = \Delta\Psi - \Phi_z - N_z - \mathcal{T}_z, \quad (5.18)$$

$$\begin{aligned} G_t + v(G_z + \mathcal{T}) &= G_{vv} - vG_v \\ &+ (1 - v^2)(N_t + \mathcal{T}). \end{aligned} \quad (5.19)$$

These linear differential equations are sometimes called the “local” determining equations; symmetries of the non-local, integral relations (2.31) – (2.33) remain to be considered.

5.3.2. Integral determining equations

For integrodifferential equations, the notion of the jet space (itself a generalization of the graph space) can be extended to include the moments of independent variables which appear in the model. We denote the variables in this space generically as $(x^i, u^\alpha, u_i^\alpha, u_{i,j}^\alpha, m^\mu \equiv \int K_\alpha^\mu(x, x') u^\alpha(x'))$. For the ZS model, the two moment variables are δn and δT . What is needed for the invariance criterion (5.6) is the expression for Ξm^μ in terms of the generator coefficients U^α and X^i .

For this purpose, there is a very useful fact: one can re-express Ξ in *canonical form*, where it acts only on the dependent variables:

$$\Xi = \sum_{\alpha} Q^{\alpha} \partial_{u^{\alpha}}, \quad (5.20)$$

with

$$Q^{\alpha} = U^{\alpha} - \sum_i X^i u_i^{\alpha}. \quad (5.21)$$

It turns out (see e.g. Kovalev *et al.* 1996) that the action of the canonical generator (5.20) on an integral – a functional of the u^α – is obtained by replacing ordinary derivatives by functional derivatives in the canonical expression

$$\Xi m^\mu = \sum_{\alpha} Q^{\alpha} \frac{\delta m^\mu}{\delta u^{\alpha}}. \quad (5.22)$$

Using (5.22), the invariance criteria $\Xi(2.31) = 0$ and $\Xi(2.33) = 0$ give the *integral determining equations*

$$\mathcal{T} = \frac{1}{\sqrt{2\pi}} \int dv' v'^2 e^{-v'^2/2} G(v'), \quad (5.23)$$

and

$$N = \hat{\mathcal{G}}\Phi, \quad (5.24)$$

respectively. Similarly, the integral constraints (2.32) lead to the determining equations

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2\pi}} \int dv' \begin{pmatrix} 0 \\ v' \end{pmatrix} e^{-v'^2/2} G(v'). \quad (5.25)$$

Thus our full system of equations for the generator coefficients consist of the local equations (5.10)–(5.19) together with the integral relations (5.23)–(5.25). Note in particular that this system of equations also involves a gyroaveraging operator, as well as other integral relations which usually lead to analytical intractability. However in this case, the local determining equations form an autonomous subsystem; we are able to obtain their general solution before even deriving the remaining (integral) determining equations.

5.4. Solution of determining equations

The general solution of the local determining equations (5.9)–(5.19) is

$$X = \Theta y + \hat{\mathbf{x}} \cdot \boldsymbol{\xi}, \quad (5.26)$$

$$Y = -\Theta x + \hat{\mathbf{y}} \cdot \boldsymbol{\xi}, \quad (5.27)$$

$$\Psi = \partial_z \left(\Theta \frac{x^2 + y^2}{2} + \hat{\mathbf{z}} \cdot \boldsymbol{\xi} \times \mathbf{x}_\perp + H \right), \quad (5.28)$$

$$\Phi = -\partial_t \left(\Theta \frac{x^2 + y^2}{2} + \hat{\mathbf{z}} \cdot \boldsymbol{\xi} \times \mathbf{x}_\perp + H \right) + [2(\Theta_t + \Theta) + \mathcal{T} + N], \quad (5.29)$$

$$N = -2v_A^{-2} \Theta_t \quad (5.30)$$

$$\mathcal{T} = \mathcal{T}(z, t), \quad (5.31)$$

$$G_t + vG_z = -G_{vv} + vG_v - (1 - v^2)(N_t + \mathcal{T}) - vT_z, \quad (5.32)$$

Here $\hat{\mathbf{x}} \cdot \boldsymbol{\xi}(z, t)$, $\hat{\mathbf{y}} \cdot \boldsymbol{\xi}(z, t)$, and $H(z, t)$ are arbitrary functions, while $\Theta(z, t)$ is an arbitrary solution to the wave equation (3.4).

In obtaining this result, we have used

$$\hat{\mathcal{G}}\Phi = \frac{1}{v_A^2} \Delta\Phi, \quad (5.33)$$

$$= -\frac{2}{v_A^2} \Theta_t. \quad (5.34)$$

In other words, the leading FLR approximation to $\hat{\mathcal{G}}$, shown in (2.38), here becomes exact, since the Φ given in (5.29) is quadratic in the perpendicular coordinates.

The function G is determined implicitly by (5.32) and constrained to satisfy

$$0 = \int dv' G(v') e^{-v'^2/2}, \quad (5.35)$$

$$0 = \int dv' v' G(v') e^{-v'^2/2}. \quad (5.36)$$

The symmetries discussed in Section 3 follow from Eqs. (5.26)–(5.36).

6. Conclusion

We have found that the fluid-gyrokinetic ZS model has a rich symmetry group, with the full set of symmetries spanned by five arbitrary functions $H(z, t)$, $\hat{\mathbf{x}} \cdot \boldsymbol{\xi}(z, t)$, $\hat{\mathbf{y}} \cdot \boldsymbol{\xi}(z, t)$, and $\Theta^\pm(z \pm v_A t)$. These symmetries are discussed in Section 3, and summarized by Eqs. (5.8), with (5.26)–(5.36).

To our knowledge, this is the first time symmetry analysis has been applied to a model with a gyroaveraging operator. Gyroaveraging, viewed as a constitutive relation linking ϕ to δn , turns out to not pose a serious obstacle in our analysis, largely because the x_\perp dependence of the infinitesimal generator for ϕ allowed the exact gyroaverage to be expressed in closed form. Similarly, the generator G for displacement symmetry of the distribution function is found to satisfy exactly a *linear* drift-kinetic equation. The success of symmetry analysis for the ZS model suggests that a similar study for the full gyrokinetic equations might also be possible.

Our analysis assumes, based on computational exploration, a special form for the symmetries, so it is possible that additional symmetries remain undiscovered. In fact, even if we did begin our analysis with the most general possible transformation, the integral terms in ZS place it beyond the scope of the theorems that would prove completeness.

Because RMHD is a limit of the ZS model, our analysis also provides a large family of symmetries of RMHD. RMHD is a simpler and better studied model, so there are more exact solutions available to transform by our methods. In particular, the results obtained here can be used to generate new exact solutions to RMHD by transforming the Elsasser solutions (Elsasser 1950), which play an important role in MHD theories of turbulence (for a review, see, e.g., Biskamp 2003).

For illustrative purposes, the modified Chapman-Kendall solution obtained in Section 4.2 employed a very simple particular symmetry transformation. More generally, using the full set of transformations obtained here, the original two parameter Chapman-Kendall solution becomes a large family of solutions, spanned by the arbitrary functions H , ξ and Θ .

In the context of the full kinetic model, one can, for example, leverage simulation results that start from a Chapman-Kendall-like two-dimensional current configuration to infer the behavior of a whole family of (generally three dimensional) initial current profiles, such as the helical collapsing current sheet given in (4.2).

Noether's theorem applies to all of the symmetry transformations obtained here. If one is able to construct an action for this model (see Charidakos *et al.* 2014; Morrison *et al.* 2014; Burby 2017, for manifestly action-preserving derivations of reduced models), and if the action is invariant under any of these transformations, then one can use Noether's theorem to derive conserved quantities which, like the symmetries themselves, may not be obvious from physical considerations. Symmetry analysis can thus enhance the value of a reduced model by uncovering quantities which, while perhaps not exactly conserved in the full Maxwell-Boltzmann description, are approximately constant in particular regimes of interest. For the ZS model, this context would be nonlinear fluctuations in a high temperature strongly-magnetized plasma.

Acknowledgements

R.L.W. was supported by This research was supported by the U.S. Department of Energy Fusion Energy Sciences Postdoctoral Research Program administered by the Oak Ridge Institute for Science and Education (ORISE) for the DOE. ORISE is managed by Oak Ridge Associated Universities (ORAU) under DOE contract number de-sc0014664. All opinions expressed in this paper are the author's and do not necessarily reflect the policies and views of DOE, ORAU, or ORISE. The work of R.D.H. was funded by the U.S. Department of Energy under Contract No. DE-FG02-04ER-54742 and by The University of Texas at Austin. N.F.L. was partially funded by US Department of Energy Grant No. DE-FG02-91ER54109. We also acknowledge the use of Brian Cantwell's Lie-group software, implemented on Mathematica[®].

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