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Citation: Sone, Akira et al. "Nonclassical correlations for quantum metrology in thermal equilibrium." *Physical Review A* 99, 5 (May, 2019): 052318. © 2019 American Physical Society.

As Published: 10.1103/PHYSREVA.99.052318

Publisher: American Physical Society (APS)

Persistent URL: <https://hdl.handle.net/1721.1/124336>

Version: Final published version: final published article, as it appeared in a journal, conference proceedings, or other formally published context

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Nonclassical correlations for quantum metrology in thermal equilibriumAkira Sone,^{1,2} Quntao Zhuang,^{1,3,4} Changhao Li,^{1,2} Yi-Xiang Liu,^{1,2} and Paola Cappellaro^{1,2,*}¹*Research Laboratory of Electronics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA*²*Department of Nuclear Science and Engineering, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA*³*Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, USA*⁴*Department of Physics, University of California Berkeley, Berkeley, California 94720, USA*

(Received 10 October 2018; published 13 May 2019)

Nonclassical correlations beyond entanglement might provide a resource in quantum information tasks, such as quantum computation or quantum metrology. Quantum discord is a measure of nonclassical correlations to which entanglement belongs as a subset. Exploring the operational meaning of quantum discord as a resource in quantum information processing tasks, such as quantum metrology, is of essential importance to our understanding of nonclassical correlations. In our recent work [*Phys. Rev. A* **98**, 012115 (2018)], we considered a protocol—which we call the greedy local thermometry protocol—for estimating the temperature of thermal equilibrium states from local measurements, elucidating the role of diagonal discord in enhancing the protocol sensitivity in the high-temperature limit. In this paper, we extend our results to a general greedy local parameter estimation scenario. In particular, we introduce a quantum discord—which we call discord for local metrology—to quantify the nonclassical correlations induced by the local optimal measurement on the subsystem. We demonstrate explicitly that discord for local metrology plays a role in sensitivity enhancement in the high-temperature limit by showing its relation to loss in quantum Fisher information. In particular, it coincides with diagonal discord for estimating a linear coupling parameter.

DOI: [10.1103/PhysRevA.99.052318](https://doi.org/10.1103/PhysRevA.99.052318)**I. INTRODUCTION**

Although the ability of entanglement to enhance quantum metrology has been well explored in ideal scenarios [1,2], experimental constraints, such as noise, mixed states, and restriction to local measurements, usually make reaching the ultimate quantum limit impossible. In this context, a more general study of the role of nonclassical correlations in quantum metrology is critical as it can lead to more general measurement schemes, such as quantum illumination [3], that take advantage of nonclassical properties [4]. Nonclassical correlations described by the quantum discord are of particular relevance as they quantify loss of information as a result of measuring a local subsystem [5,6] and can be applied to mixed states. The role of discordlike correlations has thus been recently studied in the context of parameter estimation [7], such as the geometric discord in phase estimation [8], quantum discord in the global phase estimation with mixed states [9–12], in local phase estimation assisted by interferometry [13,14], and the diagonal discord in quantum thermometry [15].

Most of these works have analyzed the usual scenario for quantum parameter estimation where a quantum (entangled) probe evolves under the action of an Hamiltonian that depends on the external parameter to be measured, before a measurement is performed on the final state [1,2,16]. Although the optimal measurement does not require global measurements on the total system for schemes without entanglement [1],

for the entanglement-enhanced schemes described above, a global measurement is usually needed to achieve the optimal performance [17]. Since performing a global measurement is usually a demanding task, and one has to rely on local adaptive measurements, it is important to study whether this restriction degrades the achievable estimation performance in the case of nonclassical correlations more general than entanglement. To better focus on this question, we consider a different metrology scenario where the parameter is not encoded during the evolution but in the equilibrium state. We show that, for a local detection protocol, nonclassical correlations in the state can be detrimental, in contrast to the dynamic scenario where they help in the estimation. In particular, we consider a “greedy” local measurement scheme [15] in which each subsystem is measured sequentially with a local optimal measurement for estimating a general parameter (see Fig. 1). This protocol belongs to the class of local operations and classical communication (LOCC) [18]. In addition, we focus on systems at thermal equilibrium in the Gibbs state and consider the high-temperature limit, which is a practical scenario in various systems, such as a room-temperature NMR system or biological system, and where only nonclassical correlations beyond entanglement are typically found. Even in this regime, we find a precision loss when considering only local measurements, and we bound it by considering the discord present in the system.

Hamiltonian parameter estimation at thermal equilibrium has been considered before by Mehboudi *et al.* [19] in which they considered a special Hamiltonian consisting of two commuting operators to which temperature-independent parameters are linearly coupled. For this special case, they

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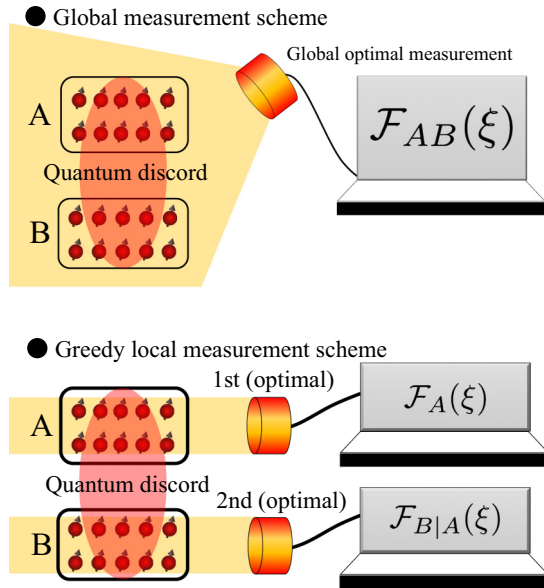


FIG. 1. Global measurement and greedy local measurement scheme: One first measures a subsystem A with the local optimal measurement in the sense of the local QFI and then measures the other subsystem B in order to estimate an unknown parameter ξ . The constrained QFI is given as $\mathcal{F}_{A \rightarrow B}(\xi) = \mathcal{F}_A(\xi) + \mathcal{F}_{B|A}(\xi)$. We explore the relation between the quantum discord $D_{A \rightarrow B}(\xi)$ and the precision loss $\Delta\mathcal{F}(\xi) = \mathcal{F}_{AB}(\xi) - \mathcal{F}_{A \rightarrow B}(\xi)$.

proved that the quantum Fisher information (QFI) for estimating either parameter can be characterized as a curvature of the Helmholtz free energy at an arbitrary temperature. However, for a general Hamiltonian H_λ parametrized by a temperature-independent parameter λ , this is not always the case because of the noncommutativity of the Hamiltonian and the generator of parameter λ . Still, in the high-temperature limit, the QFI can be well approximated by the susceptibility as discussed in Sec. II, and we can apply the relation provided in Ref. [19].

The paper is organized as follow. In Sec. II A, we review the QFI for estimating a single parameter and discuss the QFI in the global measurement scheme, namely, global QFI, in Sec. II B, and the constrained QFI in the greedy local measurement scheme, namely, LOCC QFI, in Sec. II C. Based on the definition of quantum discord [6], we introduce a quantum discord induced by local optimal measurements by considering the greedy local measurement scheme, namely, *discord for local metrology* in Sec. III. Then, we show the relation between the discord for local metrology and precision loss quantified by the difference between global QFI and LOCC QFI at high temperatures in Sec. IV and demonstrate that discord for local metrology coincides with diagonal discord when the parameter to be estimated is linearly coupled. Before concluding, we also provide examples to further illustrate our results.

II. GLOBAL AND GREEDY LOCAL MEASUREMENT SCHEME

We first review the definition of QFI for estimating a single parameter and discuss QFI for global and local measurement

schemes. In particular, we devise an optimal measurement protocol that only exploits local measurements and define an associated QFI metric to evaluate its performance.

A. QFI for estimating a single parameter

The ultimate precision of parameter estimation is quantified by the QFI. Let ξ be the parameter to be estimated, which could be a temperature-independent parameter λ in the Hamiltonian H_λ or the temperature T itself, i.e., $\xi \in \{\lambda, T\}$. Although often ξ is estimated from a state ρ_ξ that arises after interacting with the external field to be measured for a given time, here we consider a different scenario, where ρ_ξ is an equilibrium state that is determined by the parameter-dependent Hamiltonian. The variance $(\delta\xi)^2$ quantifies the estimate precision. Its lower bound, which is the ultimate precision limit achievable, is bounded by the quantum Cramér-Rao bound $(\delta\xi)^2 \geq 1/\mathcal{F}(\xi, \rho_\xi)$ [20–22]. Here, $\mathcal{F}(\xi, \rho_\xi)$ is the QFI, defined as $\mathcal{F}(\xi, \rho_\xi) = -2 \lim_{\epsilon \rightarrow 0} \partial^2_\epsilon \mathbb{F}[\rho_\xi, \rho_{\xi+\epsilon}]$, where $\mathbb{F}[\rho, \sigma]$ denotes the fidelity between states ρ and σ [23].

B. Global QFI

Consider a finite-dimensional system described by a Hamiltonian H_λ parametrized by a single temperature-independent parameter λ at temperature T . We assume the state to be in a Gibbs state $\rho_\xi = e^{-H_\lambda/T} / \mathcal{Z}$ where we set the Boltzmann constant to be unit $k_B = 1$, and $\mathcal{Z} = \text{Tr}[e^{-H_\lambda/T}]$ is the partition function.

We first consider a global measurement scheme for a finite-dimensional system and derive the relation between the global QFI, $\mathcal{F}(\xi, \rho_\xi)$ and the entropy of the global system, $S(\rho_\xi)$ in the high-temperature limit. We have obtained the following lemma.

Lemma 1. Consider a finite-dimensional system in the Gibbs state at temperature T with its Hamiltonian parametrized by a temperature-independent parameter λ to be estimated. Then, the global QFI for estimating λ and the total system entropy $S(\lambda; T)$ are related as

$$\partial_T(T\mathcal{F}(\lambda; T)) = \partial_\lambda^2 S(\lambda; T) + O(T^{-3}). \quad (1)$$

The full proof is in Appendix A; here, we explain the basic idea of the proof. In the high-temperature limit, the QFI for estimating λ can be quantified by the susceptibility $\chi(\lambda; T)$ to leading order: $\mathcal{F}(\lambda; T) = \chi(\lambda; T)/T + O(T^{-3})$. From the relation between the general susceptibility and entropy $\partial_T \chi(\lambda; T) = \partial_\lambda^2 S(\lambda; T)$, we can obtain Eq. (1). Furthermore, let $A(\lambda; T)$ be the Helmholtz free energy. Then, from the relation between the Helmholtz free energy and entropy $\partial_T A(\lambda; T) = -S(\lambda; T)$, we can obtain

$$\mathcal{F}(\lambda; T) = -\frac{1}{T} \partial_\lambda^2 A(\lambda; T) + O(T^{-3}).$$

This recovers the result of Ref. [19] to leading order, which demonstrates that, in the high-temperature limit, the QFI can be characterized as the curvature of the Helmholtz free energy. If the parameter to be estimated is the temperature $\xi = T$, the relation becomes *exact*:

Corollary 1. For a system in the Gibbs state, we have

$$\partial_T(T\mathcal{F}(\lambda; T)) = \partial_T^2 S(T). \quad (2)$$

In the classical case, Eq. (1) becomes exact as it can also be derived from properties of the classical Fisher information in the linear exponential family [24–26].

Let us define the optimal measurement in the high-temperature limit as the measurement which achieves the ultimate precision up to order $O(T^{-2})$ of the QFI (for thermometry $O(T^{-4})$ of the QFI [15]). Different from the thermometry case ($\xi = T$), whereas to estimate a generic parameter λ , the optimal measurement is generally not the projection measurement onto energy eigenstates, this is instead the case for thermometry or if λ is linearly coupled to the Hamiltonian. Formally, we have the following lemma (see Appendix B for proof):

Lemma 2. Consider a finite-dimensional system in the Gibbs state at temperature T with its Hamiltonian parametrized by a temperature-independent parameter λ to be estimated. If the Hamiltonian depends only linearly on λ , i.e., $\partial_\lambda^2 H_\lambda = 0$, projection measurements on the energy eigenstates are optimal to estimate λ .

Corollary 2. Since the temperature multiplies the Hamiltonian in the Gibbs state, projection on the energy eigenstates is also optimal for thermometry.

Here, we note that, for a generic Hamiltonian H_λ , the susceptibility with respect to λ is given by

$$\begin{aligned} \chi(\lambda; T) &= \frac{\langle G_\lambda^2 \rangle - \langle G_\lambda \rangle^2}{T} - \langle \partial_\lambda G_\lambda \rangle \\ &= \frac{(\delta G_\lambda)^2}{T} - \langle \partial_\lambda G_\lambda \rangle, \end{aligned}$$

where $G_\lambda = \partial_\lambda H_\lambda$. From Eq. (A1), the QFI becomes

$$\mathcal{F}(\lambda; T) = \frac{(\delta G_\lambda)^2}{T^2} - \frac{\langle \partial_\lambda G_\lambda \rangle}{T} + O(T^{-3}). \quad (3)$$

If λ is linearly coupled to the Hamiltonian, i.e., $\partial_\lambda G_\lambda = \partial_\lambda^2 H_\lambda = 0$, the projection measurements on the energy eigenstate are optimal since measuring G_λ corresponds to projection measurements on the energy eigenstates and the sensitivity of measuring G_λ saturates the Fisher information as follows:

$$(\delta\lambda)^2 = \frac{(\delta G_\lambda)^2}{(\partial_\lambda \langle G_\lambda \rangle)^2} = \frac{(\delta G_\lambda)^2}{\chi(\lambda; T)^2} = \frac{T^2}{(\delta G_\lambda)^2} \approx \frac{1}{\mathcal{F}(\lambda; T)}. \quad (4)$$

For a general parameter λ , we usually have $\langle \partial_\lambda G_\lambda \rangle \neq 0$, and from Eq. (3), the projection measurements on the energy eigenstate are not optimal. However, there still exists a set of observables that achieves the optimal measurement.

C. LOCC QFI

Global measurements on a composite system are generally required to achieve the optimal QFI but are usually difficult to implement. If only local measurements are available, even the best measurement protocol might not reach optimality. Here, we consider a local measurement scheme with sequential local optimal measurements on subsystems that we call the greedy local measurement scheme [15]. This scheme belongs to the

class of LOCCs, and thus, we call the constrained QFI of this scheme the LOCC QFI.

Consider an arbitrary bipartite system in state $\rho_{AB, \xi}$. In the greedy local measurement scheme, we first perform a local optimal projection measurement $\tilde{\Pi}_j^A$ on the first subsystem where we use the notation $\tilde{\Pi}$ in order to emphasize that the measurement is optimal. After the measurement, the state of subsystem B is a conditional state based on the measurement result of $\tilde{\Pi}_j^A$, $\rho_{B|\tilde{\Pi}_j^A, \xi} = \text{Tr}_A[(\tilde{\Pi}_j^A \otimes \mathbb{1}^B)\rho_{AB, \xi}(\tilde{\Pi}_j^{A\dagger} \otimes \mathbb{1}^B)]/p_j(\xi)$ with $p_j(\xi) = \text{Tr}[(\tilde{\Pi}_j^A \otimes \mathbb{1}^B)\rho_{AB, \xi}(\tilde{\Pi}_j^{A\dagger} \otimes \mathbb{1}^B)]$ as the measurement probability. Given the conditional QFI for outcome j , $\mathcal{F}_{B|\tilde{\Pi}_j^A}(\xi) = \mathcal{F}(\xi, \rho_{B|\tilde{\Pi}_j^A, \xi})$, the unconditional local QFI for subsystem B is given by

$$\mathcal{F}_{B|A}(\xi) = \sum_j p_j(\xi) \mathcal{F}_{B|\tilde{\Pi}_j^A}(\xi).$$

Note that feedforward is required as the optimal measurement on B depends on the outcome of $\tilde{\Pi}_j^A$. From the additivity of the Fisher information, the LOCC QFI $\mathcal{F}_{A \rightarrow B}(\xi)$ is given by

$$\mathcal{F}_{A \rightarrow B}(\xi) = \mathcal{F}_A(\xi) + \mathcal{F}_{B|A}(\xi),$$

where $\mathcal{F}_A(\xi) = \mathcal{F}(\xi, \rho_{A, \xi})$ is the local QFI for subsystem A [15,27,28]. Note that the LOCC QFI has been originally proposed by Ref. [27] from an information-theoretic perspective and by Ref. [28] from a quantum metrology perspective. By definition, the global QFI $\mathcal{F}_{AB}(\xi)$ always satisfies $\mathcal{F}_{AB}(\xi) \geq \mathcal{F}_{A \rightarrow B}(\xi)$ [15,28]. Here, we are interested in relating the precision loss,

$$\Delta\mathcal{F}(\xi) = \mathcal{F}_{AB}(\xi) - \mathcal{F}_{A \rightarrow B}(\xi),$$

due to local measurements to the presence of nonclassical correlations in $\rho_{AB, \xi}$.

III. DISCORD FOR LOCAL METROLOGY

Nonclassical correlations associated with the loss of quantum certainty in local measurements have been quantified by quantum discord [5,6,29]. For a bipartite system (AB) , the quantum discord [6] upon measuring subsystem A is defined as

$$D_{A \rightarrow B} = -S_{AB} + S_A + \min_{\{\Pi_j^A\}} S_{B|(\Pi_j^A)},$$

where $\{\Pi_j^A\}$'s are the set of projection measurements on subsystem A and $S_i = -\text{Tr}[\rho_i \ln \rho_i]$ is the entropy of state ρ_i . Here, $S_{B|(\Pi_j^A)}$ is defined as

$$S_{B|(\Pi_j^A)} = \sum_j p_j S_{B|\Pi_j^A},$$

with $p_j = \text{Tr}[(\Pi_j^A \otimes \mathbb{1}_B)\rho_{AB}(\Pi_j^{A\dagger} \otimes \mathbb{1}_B)]$ as the probability associated with the projection measurement Π_j^A . The minimization over all sets of projection measurements on subsystem A is required in order for quantum discord to be basis independent and for extracting maximum information about subsystem B .

In order to connect nonclassical correlations to the precision loss in metrology, we need to define a related metric, that

we call *discord for local metrology* where the minimization is restricted to projectors achieving optimal estimate of ξ :

Definition 1. Let $\{\tilde{\Pi}_j^A\}$ be a set of optimal projection measurements on subsystem A so that there exists an observable $\tilde{\Gamma}^A = \sum_j c_j \tilde{\Pi}_j^A$ ($c_j \in \mathbb{C}$), which can achieve the ultimate precision of estimating ξ , i.e.,

$$(\delta\xi)^2 = \frac{(\delta\tilde{\Gamma}^A)^2}{(\partial_\xi \langle \tilde{\Gamma}^A \rangle)^2} = \frac{1}{\mathcal{F}_A(\xi)},$$

where $\mathcal{F}_A(\xi)$ is the local QFI for estimating ξ from $\rho_{A,\xi}$. Then, discord for local metrology $\tilde{D}_{A \rightarrow B}(\xi)$ is defined as

$$\tilde{D}_{A \rightarrow B}(\xi) = -S_{AB}(\xi) + S_A(\xi) + \min_{\{\tilde{\Pi}_j^A\}} S_{B|(\tilde{\Pi}_j^A)}(\xi),$$

which is minimized over all the possible sets of projection measurements that are optimal for estimating the parameter ξ .

The minimization indicates that discord for local metrology is independent of the choice of the optimum basis for estimating ξ . Because the measurement basis is chosen according to the optimal parameter estimation, discord for local metrology is an upper bound of the discord, i.e., $\tilde{D}_{A \rightarrow B}(\xi) \geq D_{A \rightarrow B}$. Also, the minimization is required to avoid the ambiguity when multiple projection bases are optimal. Note that the discord for local metrology is a function of a state and a parameter; therefore, it is not a typical correlation measure for the state. Discord for local metrology has the following properties:

- (1) $\tilde{D}_{A \rightarrow B} \geq 0$ (non-negative);
- (2) $\tilde{D}_{A \rightarrow B} \neq \tilde{D}_{B \rightarrow A}$ (asymmetric);
- (3) If the total system is in the product state, i.e., $\rho_{AB} = \rho_A \otimes \rho_B$, then $\tilde{D}_{A \rightarrow B} = 0$. If $\tilde{D}_{A \rightarrow B} = 0$, then the total system is in a classical-quantum state, i.e., $\rho_{AB} = \sum_j p_j |j\rangle \langle j| \otimes \rho_B^{(j)}$, for some set of orthonormal basis vectors $\{|j\rangle\}$, probability distribution $\{p_j\}$ and states $\{\rho_B^{(j)}\}$.
- (4) $\tilde{D}_{A \rightarrow B}$ is invariant under local unitary operations.

Properties (1) and (2) are trivial. The first half of Property (3) is straightforward, and the second part follows from the fact that $\tilde{D}_{A \rightarrow B}(\xi) \geq D_{A \rightarrow B}$; thus $\tilde{D}_{A \rightarrow B}(\xi) = 0$ leads to zero discord, and the state must be classical quantum. Property (4) is due to the state dependence of the local measurement basis, which makes the quantity only a function of the state and parameter choice. Local unitary operations change the state, but the optimal basis also changes accordingly, thus leaving invariant the discord for local metrology. Note that one does not expect invariance under more general local operations since discord can increase under local noise [30]. Property (4) distinguishes our metric from the family of basis-dependent discord [31–33] with which it otherwise shares many commonalities.

Since discord for local metrology satisfies the conditions of non-negativity and invariance under local unitary operations, we can regard it as a *good* measure of correlations [34]. Although it can be nonzero for some specific classical-quantum state, an unpleasant property for a discord metric, it is a practical quantity to measure correlations in terms of local optimal measurement for metrology.

For a general bipartite system, it is a demanding task to find $\{\tilde{\Pi}_j^A\}$. However, when $\xi = T$ or $\xi = \lambda$ is a linear coupling parameter for a Gibbs state in the high-temperature limit,

$\tilde{\Pi}_j^A$ becomes the eigenbasis of ρ_A , i.e., $\rho_A = \sum_j r_j \tilde{\Pi}_j^A$, as shown in Sec. II B. Therefore, $\tilde{D}_{A \rightarrow B}(\xi)$ becomes the so-called diagonal discord $\mathcal{D}_{A \rightarrow B}(\xi)$ [35].

IV. QUANTIFYING $\Delta\mathcal{F}(\xi; T)$ VIA $\tilde{D}_{A \rightarrow B}(\xi; T)$

In this section, we prove our main result Theorem 1 stating the relation between the discord for local metrology and the precision loss quantified by the difference between global QFI and LOCC QFI.

Theorem 1. Consider a finite-dimensional system in a Gibbs state with its Hamiltonian H_λ parametrized by a temperature-independent parameter λ at temperature T . Let $\xi \in \{\lambda, T\}$ denote an unknown parameter to be estimated. If $\mathcal{F}_{AB}(\xi; T)$ is the global QFI and $\mathcal{F}_{A \rightarrow B}(\xi; T)$ is the LOCC QFI for estimating ξ , in the high-temperature limit, we have

$$-\partial_\xi^2 \tilde{D}_{A \rightarrow B}(\xi; T) = \partial_T(T \Delta\mathcal{F}(\xi; T)) + O(T^{-\alpha_\xi}), \quad (5)$$

where $\alpha_\lambda = 3$ and $\alpha_T = 5$. Particularly, for thermometry ($\xi = T$), $\tilde{D}_{A \rightarrow B}(T)$ becomes the diagonal discord $\mathcal{D}_{A \rightarrow B}(T)$, which obeys

$$-\partial_T^2 \mathcal{D}_{A \rightarrow B}(T) = \partial_T(T \Delta\mathcal{F}(T)) + O(T^{-5}), \quad (6)$$

Proof. First, let us prove the case for $\xi = \lambda$. For a general finite-dimensional system in the high-temperature limit, the state of the total system $\rho_{AB,\lambda}$ can be written as

$$\rho_{AB,\lambda} = \frac{1}{d_{AB}} \left[\mathbb{1}_{AB} - \frac{1}{T} \left(H_\lambda - \frac{\text{Tr}[H_\lambda]}{d_{AB}} \right) \right] + O(T^{-2}),$$

where d_{AB} is the dimension of the system. The reduced state of subsystem A is given by $\rho_{A,\lambda} = \text{Tr}_B[\rho_{AB,\lambda}]$, and within the same approximation, we have $\rho_{A,\lambda} \propto \mathbb{1}_A - \frac{1}{T}(H_{A,\lambda} + \Omega_{A,\lambda}) + O(T^{-2})$, where $\Omega_{A,\lambda} = \text{const} + \frac{1}{d_B} \sum_k \langle E_k^{(B)} | H_{AB,\lambda} | E_k^{(B)} \rangle$, which is independent of temperature. In the high-temperature limit, $\rho_{A,\lambda}$ can be approximated by a Gibbs state $\rho_{A,\lambda} \simeq \mathcal{Z}_{A,\lambda}^{-1} e^{-H_{A,\lambda}^{\text{eff}}/T}$ with the effective Hamiltonian $H_{A,\lambda}^{\text{eff}} = H_{A,\lambda} + \Omega_{A,\lambda}$ and the normalization factor $\mathcal{Z}_{A,\lambda} = \text{Tr}[e^{-H_{A,\lambda}^{\text{eff}}/T}]$. Then, the local QFI follows Eq. (1), i.e.,

$$\partial_T(T \mathcal{F}_A(\lambda; T)) = \partial_\lambda^2 S_A(\lambda; T) + O(T^{-3}). \quad (7)$$

Suppose that projectors $\tilde{\Pi}_j^A$ are the local optimal projection measurements for estimating λ from state $\rho_{A,\lambda}$. Then, the conditional state $\rho_{B|\tilde{\Pi}_j^A,\lambda}$ after measuring subsystem A can also be approximated as a Gibbs state in the high-temperature limit with the effective Hamiltonian $H_{B|\tilde{\Pi}_j^A,\lambda} = H_{B,\lambda} + \Omega_{B|\tilde{\Pi}_j^A,\lambda}$, where $\Omega_{B|\tilde{\Pi}_j^A,\lambda} = \text{const} + \text{Tr}[H_{AB,\lambda} \tilde{\Pi}_j^A]$. Then, the local QFI obeys Lemma 1, i.e.,

$$\partial_T(T \mathcal{F}_{B|\tilde{\Pi}_j^A}(\lambda; T)) = \partial_\lambda^2 S_{B|\tilde{\Pi}_j^A}(\lambda; T) + O(T^{-3}). \quad (8)$$

Let us select $\tilde{\Pi}_{j^*}^A$ such that $\sum_{j^*} p_{j^*}(\lambda; T) S_{B|\tilde{\Pi}_{j^*}^A}(\lambda; T) = \min_{\{\tilde{\Pi}_j^A\}} \sum_j p_j(\lambda; T) S_{B|\tilde{\Pi}_j^A}(\lambda; T)$. Then,

$$\begin{aligned} \partial_\lambda^2 \tilde{D}_{A \rightarrow B}(\lambda; T) &= \left(\partial_\lambda^2 S_A + \sum_{j^*} p_{j^*} \partial_\lambda^2 S_{B|\tilde{\Pi}_{j^*}^A} - S_{AB} \right) \\ &\quad + \sum_{j^*} \left(\partial_\lambda^2 p_{j^*} S_{B|\tilde{\Pi}_{j^*}^A} + 2 \partial_\lambda p_{j^*} \partial_\lambda S_{B|\tilde{\Pi}_{j^*}^A} \right). \end{aligned}$$

From Eqs. (1), (7), and (8), we can obtain

$$-\partial_\lambda^2 \tilde{D}_{A \rightarrow B}(\lambda; T) = \partial_T(T \Delta \mathcal{F}(\lambda; T)) - \sum_{j^*} (\partial_\lambda^2 p_{j^*} S_{B|\tilde{\Pi}_{j^*}^A} + 2 \partial_\lambda p_{j^*} \partial_\lambda S_{B|\tilde{\Pi}_{j^*}^A}).$$

In the high-temperature limit, the entropy has the order of $S_{B|\tilde{\Pi}_{j^*}^A}(\lambda; T) = \ln(d_B) + O(T^{-2})$, and the measurement probability is

$$\begin{aligned} p_{j^*}(\lambda; T) &= \text{Tr}[(\tilde{\Pi}_{j^*}^A \otimes \mathbb{1}_B) \rho_{AB, \lambda} (\tilde{\Pi}_{j^*}^{A\dagger} \otimes \mathbb{1}_B)] \\ &= \frac{1}{d_A} + O(T^{-1}). \end{aligned} \quad (9)$$

In the high-temperature limit, we have

$$\partial_\lambda^2 S_{B|\tilde{\Pi}_{j^*}^A}(\lambda; T) = O(T^{-2}).$$

By using the fact that $\sum_{j^*} p_{j^*}(\lambda; T) = 1$, we can write

$$\begin{aligned} \sum_{j^*} \partial_\lambda^2 p_{j^*}(\lambda; T) S_{B|\tilde{\Pi}_{j^*}^A}(\lambda; T) &= O(T^{-1}) O(T^{-2}) \\ &= O(T^{-3}) \\ \sum_{j^*} \partial_\lambda p_{j^*}(\lambda; T) \partial_\lambda S_{B|\tilde{\Pi}_{j^*}^A}(\lambda; T) &= O(T^{-1}) O(T^{-2}) \\ &= O(T^{-3}). \end{aligned}$$

Therefore, we can write

$$-\partial_\lambda^2 \tilde{D}_{A \rightarrow B}(\lambda; T) = \partial_T(T \Delta \mathcal{F}(\lambda; T)) + O(T^{-3}).$$

Second, for thermometry, from Lemma 2 and Definition 1, the optimal measurement basis is the diagonal basis of $\rho_{A, T}$. Therefore, discord for local metrology $\tilde{D}_{A \rightarrow B}(T)$ becomes diagonal discord $\mathcal{D}_{A \rightarrow B}(T)$. From our previous result in Ref. [15] since we have already known that

$$-\frac{1}{T} \partial_T \mathcal{D}_{A \rightarrow B}(T) = \Delta \mathcal{F}(T) + O(T^{-5}),$$

we can obtain

$$-\partial_T^2 \mathcal{D}_{A \rightarrow B}(T) = \partial_T(T \Delta \mathcal{F}(T)) + O(T^{-5}),$$

■

Therefore, for any parameter ξ , in the high-temperature limit, we can approximately write

$$\partial_\xi^2 \tilde{D}_{A \rightarrow B}(\xi; T) \simeq -\partial_T(T \Delta \mathcal{F}(\xi; T)), \quad (10)$$

which demonstrates that $\partial_T(T \Delta \mathcal{F}(\lambda; T))$ is the curvature of $\tilde{D}_{A \rightarrow B}$. Even if the curvature of the discord for local metrology is not directly related to the amount of nonclassical correlations, Eq. (10) still describes the role of nonclassical correlations in the greedy local measurement scheme in the LOCC regime. Although we derived Theorem 1 for a bipartite system, the results in the high-temperature limit can be extended to the case of multipartite systems (see Appendix D).

When the parameter λ is linearly coupled in the Hamiltonian, discord for local metrology becomes diagonal discord. From Theorem 1 and Lemma 2, we can obtain the following corollary:

Corollary 3. Consider a finite-dimensional system in a Gibbs state at temperature T with its Hamiltonian parametrized by a temperature-independent parameter λ .

When λ is linearly coupled to the Hamiltonian H_λ , i.e., $\partial_\lambda^2 H_\lambda = 0$, we have

$$\partial_\lambda^2 \mathcal{D}_{A \rightarrow B}(\lambda; T) = -\partial_T(T \Delta \mathcal{F}(\lambda; T)) + O(T^{-3}), \quad (11)$$

where $\mathcal{D}_{A \rightarrow B}(\lambda; T)$ is the diagonal discord.

In addition, let us note the case of estimating a parameter linearly coupled to the single-body term. For this case, we can obtain the following corollary (see Appendix C for proof):

Corollary 4. For a finite-dimensional system in a Gibbs state at temperature T , when λ is a parameter linearly coupled to the single-body term as

$$H_\lambda = \lambda H_A + \lambda H_B + H_{AB},$$

where H_A and H_B are the system Hamiltonians and H_{AB} is the interaction Hamiltonian, then, we have

$$\begin{aligned} -\partial_\lambda^2 \mathcal{D}_{A \rightarrow B}(\lambda; T) &= O(T^{-3}) \\ \partial_T(T \Delta \mathcal{F}(\lambda; T)) &= O(T^{-3}). \end{aligned} \quad (12)$$

To this order, the local measurements are optimal. Here, note that the leading term that Theorem 1 cares about is $O(T^{-2})$ and this is 0 in this case.

In the following section, we show some examples that verify Theorem 1, Corollaries 3, and 4.

V. EXAMPLES

In this section, we verify the relation in Eqs. (5), (6) and (11), (12) by providing several examples of two-qubit Heisenberg interaction, whose Hamiltonian can be written as

$$H = \frac{B_1}{2} Z_A + \frac{B_2}{2} Z_B + \frac{J_x}{2} X_A X_B + \frac{J_y}{2} Y_A Y_B + \frac{J_z}{2} Z_A Z_B,$$

where X_j, Y_j , and Z_j ($j = A, B$) are the Pauli matrices acting on the j th spin.

A. Thermometry

First, let us discuss the case of thermometry. From our recent result [15], we have

$$\begin{aligned} \Delta \mathcal{F}(T) &= \frac{J_x^2 + J_y^2}{4T^4} + O(T^{-5}) - \frac{1}{T} \partial_T \mathcal{D}_{A \rightarrow B}(T) \\ &= \frac{J_x^2 + J_y^2}{4T^4} + O(T^{-5}), \end{aligned}$$

which directly yields

$$\begin{aligned} \partial_T(T \Delta \mathcal{F}(T)) &= -\frac{3(J_x^2 + J_y^2)}{4T^4} + O(T^{-5}) \\ -\partial_T^2 \mathcal{D}_{A \rightarrow B}(T) &= -\frac{3(J_x^2 + J_y^2)}{4T^4} + O(T^{-5}). \end{aligned}$$

Therefore, Eq. (6) is valid.

B. Coupling strength

Next, let us consider the case of estimating the coupling strength J when $J_x = J_y = J$. Then, we have

$$\begin{aligned} \Delta \mathcal{F}(J; T) &= \frac{1}{2T^2} + O(T^{-3}) \\ \mathcal{D}_{A \rightarrow B}(J; T) &= \frac{J^2}{4T^2} + O(T^{-3}), \end{aligned}$$

which directly yields

$$\begin{aligned}\partial_T(T \Delta \mathcal{F}(J; T)) &= -\frac{1}{2T^2} + O(T^{-3}) \\ -\partial_J^2 \mathcal{D}_{A \rightarrow B}(J; T) &= -\frac{1}{2T^2} + O(T^{-3}),\end{aligned}$$

Therefore, Eq. (11) is valid.

C. Magnetometry

Finally, let us consider magnetometry, which demonstrates Eq. (12). We consider the case of $B_1 = B_2 = B$, where B is the parameter to be estimated. In this case, we can find that

$$\begin{aligned}\partial_T(T \Delta \mathcal{F}(B; T)) &= -\frac{(J_x - J_y)^2}{8T^4} + O(T^{-5}), \\ -\partial_B^2 \mathcal{D}_{A \rightarrow B}(B; T) &= -\frac{J_x^2 + J_x J_y + J_y^2}{24T^4} + O(T^{-5}).\end{aligned}$$

From Eq. (12), the leading term should be $O(T^{-3})$; therefore, we can say that Eq. (12) is valid, but the term to the corresponding order $O(T^{-3})$ is 0.

VI. CONCLUSION

In conclusion, we introduced a metric for nonclassical correlations, the discord for local metrology, which is defined as a quantum discord in the greedy local measurement scheme, and we derived a relation between the discord for local metrology and the difference between the QFI of the global optimal scheme and the greedy local measurement scheme in the high-temperature limit. We demonstrated that the curvature of the discord for local metrology quantifies the precision loss in the estimation of a general parameter due to availability of local measurements only (Theorem 1). This also indicates that variations in nonclassical correlations at thermal equilibrium, quantified by discord for local metrology, are related to the ability of the greedy local measurement scheme to achieve the ultimate estimation precision limit, quantified by the global QFI. We also showed that discord for local metrology coincides with diagonal discord when one estimates a linear coupling parameter (Corollaries 3 and 4).

Although we focused on finite-dimensional systems in the high-temperature limit, it would be interesting to extend the relation between the discord for local metrology and the QFI for more general Gibbs states, especially in the low-temperature limit where one could search for connections to phase-transition phenomena or for infinite-dimensional systems, such as bosonic gases [36,37].

The relation between the curvature of the discord for local metrology and the difference in the QFI explicitly demonstrates the role of nonclassical correlations in quantum metrology based on the original definition of quantum discord. This provides insight on the role of nonclassicality in quantum metrology and motivates further exploration in more general settings, which can potentially inspire experimentalists to design measurement and control protocols to utilize quantum discord as a resource to achieve precise sensing and imaging, e.g., in the context of room-temperature nuclear magnetic resonance or bioimaging with defect spins [38–41].

ACKNOWLEDGMENTS

This work was supported, in part, by the U.S. Army Research Office through Grants No. W911NF-11-1-0400 and No. W911NF-15-1-0548 and by the NSF Grant No. PHY0551153. A.S. acknowledges a Thomas G. Stockham, Jr. Fellowship. Q.Z. acknowledges the U.S. Department of Energy through Grant No. PH-COMPHEP-KA24 and the Claude E. Shannon Research Assistantship. We thank B. Yadin, K. Modi, and R. Takagi for helpful discussions.

APPENDIX A: PROOF OF LEMMA 1

First, let us prove the case of $\xi = \lambda$.

Let ϵ be an error in our estimation. Then, the Hamiltonian with the error becomes

$$H_{\lambda+\epsilon} = H_\lambda + \epsilon G_\lambda + O(\epsilon^2),$$

where

$$G_\lambda = \partial_\lambda H_\lambda.$$

The fidelity between ρ_λ and $\rho_{\lambda+\epsilon}$ is defined as

$$\mathbb{F}[\rho_\lambda, \rho_{\lambda+\epsilon}] = \left(\text{Tr}[\sqrt{\rho_\lambda^{1/2} \rho_{\lambda+\epsilon} \rho_\lambda^{1/2}}] \right)^2.$$

Since

$$e^{-(H_\lambda/2T)} e^{-(H_\lambda+\epsilon G_\lambda)/T} e^{-(H_\lambda/2T)} = e^{-(2H_\lambda+\epsilon G_\lambda)/T+O(T^{-3})},$$

we can write

$$\begin{aligned}\mathbb{F}[\rho_\lambda, \rho_{\lambda+\epsilon}] &= \frac{1}{\mathcal{Z}_\lambda \mathcal{Z}_{\lambda+\epsilon}} \left(\text{Tr}[e^{-[H_\lambda+(\epsilon/2)G_\lambda]/T+O(T^{-3})}] \right)^2 \\ &= \frac{1}{\mathcal{Z}_\lambda \mathcal{Z}_{\lambda+\epsilon}} \left(\text{Tr}[e^{-[H_\lambda+(\epsilon/2)G_\lambda]/T}] \right)^2 + O(T^{-3}).\end{aligned}$$

In the high-temperature limit, the fidelity between ρ_λ and $\rho_{\lambda+\epsilon}$ becomes

$$\mathbb{F}[\rho_\lambda, \rho_{\lambda+\epsilon}] = \frac{\mathcal{Z}_{\lambda+(\epsilon/2)}^2}{\mathcal{Z}_\lambda \mathcal{Z}_{\lambda+\epsilon}} + O(T^{-3}),$$

where $\mathcal{Z}_{\lambda+(\epsilon/2)} = \text{Tr}[e^{-H_{\lambda+(\epsilon/2)}/T}]$ and from the definition of the QFI, we can obtain

$$\mathcal{F}(\lambda; T) = \frac{\mathcal{Z}_\lambda \partial_\lambda^2 \mathcal{Z}_\lambda - (\partial_\lambda \mathcal{Z}_\lambda)^2}{\mathcal{Z}_\lambda^2} + O(T^{-3}).$$

Here, for the Gibbs state, $\langle G_\lambda \rangle = \text{Tr}[G_\lambda \rho_\lambda]$ is always

$$\langle G_\lambda \rangle = -T \partial_\lambda \ln \mathcal{Z}_\lambda.$$

Then, the susceptibility with respect to a temperature-independent parameter λ can be defined as

$$\chi(\lambda; T) = -\partial_\lambda \langle G_\lambda \rangle.$$

so that we have

$$\mathcal{F}(\lambda; T) = \frac{\chi(\lambda; T)}{T} + O(T^{-3}). \quad (\text{A1})$$

Since the entropy of the bipartite system $S(\lambda; T) = -\text{Tr}[\rho_\lambda \ln \rho_\lambda]$ satisfies the following relation:

$$\partial_T \langle G_\lambda \rangle = -\partial_\lambda S(\lambda; T), \quad (\text{A2})$$

from Eqs. (A1) and (A2), we can obtain

$$\partial_T(T \mathcal{F}(\lambda; T)) = \partial_\lambda^2 S(\lambda; T) + O(T^{-3}).$$

Second, for the thermometry case $\xi = T$, the global QFI is $\mathcal{F}(T) = C(T)/T^2$ [42] for finite temperature, where $C(T)$ is the heat capacity so that $C(T) = T \partial_T S(T)$. Therefore, we can obtain an exact relation,

$$\partial_T^2 S(T) = \partial_T (T \mathcal{F}(T)).$$

Therefore, Lemma 1 is valid.

APPENDIX B: PROOF OF LEMMA 2

First, let us prove the case $\xi = \lambda$. When $\partial_\lambda G_\lambda = 0$, the QFI becomes

$$\mathcal{F}(\lambda, T) = \frac{(\delta G_\lambda)^2}{T^2} + O(T^{-3}). \quad (\text{B1})$$

Let $E_k(\lambda)$ be the eigenvalues of the Hamiltonian H_λ . Then, H_λ can be diagonalized as $H_\lambda = P_\lambda K_\lambda P_\lambda^\dagger$, where P_λ is a unitary operator $P_\lambda^\dagger P_\lambda = P_\lambda P_\lambda^\dagger = \mathbb{1}$ and $K_\lambda = \text{diag}[E_1(\lambda), E_2(\lambda), \dots, E_d(\lambda)] = \sum_{k=1}^d E_k(\lambda) |k\rangle\langle k|$ and $|k\rangle$'s form a complete basis independent of λ , and d is the dimension of the system. Thus,

$$\partial_\lambda K_\lambda = \sum_{k=1}^d \partial_\lambda E_k(\lambda) |k\rangle\langle k|.$$

Then, the Gibbs state becomes

$$\rho_\lambda = \frac{1}{\mathcal{Z}_\lambda} P_\lambda e^{-K_\lambda/T} P_\lambda^\dagger = \frac{1}{\mathcal{Z}_\lambda} \sum_{k=1}^d e^{-E_k(\lambda)/T} P_\lambda |k\rangle\langle k| P_\lambda^\dagger.$$

Let us calculate the expectation value of $G_\lambda = \partial_\lambda H_\lambda$. Since

$$G_\lambda = \partial_\lambda P_\lambda K_\lambda P_\lambda^\dagger + P_\lambda \partial_\lambda K_\lambda P_\lambda^\dagger + P_\lambda K_\lambda \partial_\lambda (P_\lambda^\dagger),$$

we have

$$\begin{aligned} \langle G_\lambda \rangle &= \text{Tr}[\rho_\lambda G_\lambda] \\ &= \frac{1}{\mathcal{Z}_\lambda} \text{Tr}\{(P_\lambda e^{-K_\lambda/T} P_\lambda^\dagger)[\partial_\lambda P_\lambda K_\lambda P_\lambda^\dagger + P_\lambda \partial_\lambda K_\lambda P_\lambda^\dagger \\ &\quad + P_\lambda K_\lambda \partial_\lambda (P_\lambda^\dagger)]\} \\ &= \text{Tr}\left[\frac{e^{-K_\lambda/T}}{\mathcal{Z}_\lambda} \partial_\lambda K_\lambda\right] + \frac{1}{\mathcal{Z}_\lambda} \text{Tr}\{K_\lambda e^{-K_\lambda/T} (P_\lambda^\dagger \partial_\lambda P_\lambda) \\ &\quad + [\partial_\lambda (P_\lambda^\dagger) P_\lambda] e^{-K_\lambda/T} K_\lambda\} \\ &= \text{Tr}\left[\frac{e^{-K_\lambda/T}}{\mathcal{Z}_\lambda} \partial_\lambda K_\lambda\right] + \frac{1}{\mathcal{Z}_\lambda} \text{Tr}[e^{-K_\lambda/T} K_\lambda \partial_\lambda (P_\lambda^\dagger P_\lambda)] \\ &= \text{Tr}\left[\frac{e^{-K_\lambda/T}}{\mathcal{Z}_\lambda} \partial_\lambda K_\lambda\right] = \text{Tr}[\rho_\lambda P_\lambda \partial_\lambda K_\lambda P_\lambda^\dagger], \end{aligned}$$

where we used the cyclic property of trace operation and the fact that $[e^{-K_\lambda/T}, K_\lambda] = 0$. Therefore,

$$\langle G_\lambda \rangle = \langle P_\lambda \partial_\lambda K_\lambda P_\lambda^\dagger \rangle,$$

and $P_\lambda \partial_\lambda K_\lambda P_\lambda^\dagger$ has same diagonal basis of ρ_λ , which is $\{P_\lambda |k\rangle\langle k| P_\lambda^\dagger\}_{k=1}^d$. This means that the optimal measurement for estimating the linear coupling parameter is the projection measurement to the diagonal basis of ρ_λ .

Second, for the case of $\xi = T$, the QFI is given as

$$\mathcal{F}(T) = \frac{C(T)}{T^2},$$

where $C(T)$ is the heat capacity [42]. Because of $C(T) = \partial_T \langle H_\lambda \rangle = (\delta H_\lambda)^2/T^2$, the temperature variance $(\delta T)^2$ becomes

$$(\delta T)^2 = \frac{(\delta H_\lambda)^2}{(\partial_T \langle H_\lambda \rangle)^2} = \frac{T^2}{C(T)} = \frac{1}{\mathcal{F}(T)}.$$

Therefore, for thermometry, the projection measurements on a diagonal basis are optimal.

APPENDIX C: PROOF OF COROLLARY 4

Let us consider the following Hamiltonian:

$$H_\lambda = \lambda H_A + \lambda H_B + H_{AB},$$

where H_A and H_B are the system Hamiltonians, i.e., $[H_A, H_B] = 0$ and H_{AB} is the interaction Hamiltonian and generally $[H_A + H_B, H_{AB}] \neq 0$. Here, λ is the parameter to be estimated. In this case, $H_{\lambda+\epsilon} = H_\lambda + \epsilon G_\lambda + O(\epsilon^2)$, where $G_\lambda = H_A + H_B$, which is independent of $\xi = \{\lambda, T\}$. Here, we just simply write G_λ as G in order to emphasize its independence of λ .

We already know that for the Gibbs state, we have $\langle G \rangle = -T \partial_\lambda \ln \mathcal{Z}_\lambda$. In this case, we can immediately obtain

$$\langle G \rangle = \langle H_A \rangle + \langle H_B \rangle = O(T^{-1})$$

because the entropy is $S_{AB}(\lambda; T) = \ln(d_{AB}) + O(T^{-2})$ and the relation between the entropy and $\langle G \rangle$ is

$$\partial_T \langle G \rangle = -\partial_\lambda S_{AB}(\lambda; T) = O(T^{-2}).$$

By defining a general susceptibility with respect to λ as

$$\chi(\lambda; T) = -\partial_\lambda \langle G \rangle = O(T^{-1}),$$

the QFI can be given as

$$\mathcal{F}_{AB}(\lambda; T) = -\frac{1}{T} \partial_\lambda \langle H_A \rangle - \frac{1}{T} \partial_\lambda \langle H_B \rangle = O(T^{-2}). \quad (\text{C1})$$

Now, let us consider the subsystem A . The effective Hamiltonian $H_{A,\lambda}^{\text{eff}}$ can be written as $H_{A,\lambda}^{\text{eff}} = \lambda(H_A + \text{const}) + \Omega_A$. Therefore,

$$\mathcal{F}_A(\lambda; T) = -\frac{1}{T} \partial_\lambda \langle H_A \rangle + O(T^{-3}).$$

Similarly, for $\rho_{B|\hat{\Pi}_{j^*}^A, \xi}$, we have

$$\mathcal{F}_{B|\hat{\Pi}_{j^*}^A}(\lambda; T) = -\frac{1}{T} \partial_\lambda \langle H_B \rangle + O(T^{-3}).$$

Therefore, by using Eq. (9), we have

$$\mathcal{F}_{A \rightarrow B}(\lambda; T) = -\frac{1}{T} (\partial_\lambda \langle H_A \rangle + \partial_\lambda \langle H_B \rangle) + O(T^{-3}).$$

From Eqs. (C1) and (11), we can obtain

$$-\partial_\lambda^2 \mathcal{D}_{A \rightarrow B}(\lambda; T) = O(T^{-3})$$

$$\partial_T (T \Delta \mathcal{F}(\lambda; T)) = O(T^{-3}).$$

APPENDIX D: GENERALIZATION TO THE MULTIPARTITE CASE

Let us consider a finite-dimensional system composed of N subsystems indexed by integers $1 \leq k \leq N$. In the multipartite case, each subsystem is measured with local

optimal measurement sequentially, and we demonstrate that the difference in global QFI and LOCC QFI can be quantified via the curvature of the discord for local metrology in the high-temperature limit, in parallel to Ref. [15].

We denote the order of measurement in a greedy local measurement scheme by $\sigma_{1:N} \equiv (\sigma_1, \sigma_2, \dots, \sigma_N)$, where $\sigma_k = \{1, 2, \dots, N\}$. Let us write $\mathcal{H}(\sigma_k)$ as the Hilbert space of the system on which we perform local optimal measurement $\tilde{\Pi}_{\sigma_k}$ and $\mathcal{H}(\sigma_{k+1:N})$ as the Hilbert space of the rest of system on which we perform the local optimal measurement $\tilde{\Pi}_{\sigma_{k+1:N}}$. Therefore, the total system can be decomposed sequentially into

$$\begin{aligned} \mathcal{H}(\sigma_{1:N}) &= \mathcal{H}(\sigma_1) \otimes \mathcal{H}(\sigma_{2:N}) \\ &= \mathcal{H}(\sigma_1) \otimes \mathcal{H}(\sigma_2) \otimes \mathcal{H}(\sigma_{3:N}) \\ &\vdots \\ &= \mathcal{H}(\sigma_1) \otimes \mathcal{H}(\sigma_2) \otimes \dots \otimes \mathcal{H}(\sigma_k) \otimes \mathcal{H}(\sigma_{k+1:N}), \end{aligned}$$

where $2 \leq k \leq N-1$.

In the first step ($k=1$), we first perform the local optimal measurement $\tilde{\Pi}_{\sigma_1}$. Then conditioned on the measurement result of $\tilde{\Pi}_{\sigma_1}$, we perform the other local optimal measurement $\tilde{\Pi}_{\sigma_{2:N}}$ on the rest of system. Let us write the global QFI as $\mathcal{F}_{\sigma_{1:N}}$ and LOCC QFI as $\mathcal{F}_{\sigma_1 \rightarrow \sigma_{2:N}}$. Then, in the high-temperature limit, from Eq. (10), we have

$$\partial_T(T(\mathcal{F}_{\sigma_{1:N}} - \mathcal{F}_{\sigma_1 \rightarrow \sigma_{2:N}})) \simeq -\partial_\xi^2 \tilde{D}_{\sigma_1 \rightarrow \sigma_{2:N}}.$$

For the $2 \leq k \leq N-1$ steps, the measurement $\tilde{\Pi}_{\sigma_k}$ is conditioned on the results of the previous sequence of local optimal measurements $\tilde{\Pi}_{1:k-1} \equiv (\tilde{\Pi}_{\sigma_1}, \tilde{\Pi}_{\sigma_2}, \dots, \tilde{\Pi}_{\sigma_{k-1}})$. We treat the rest of system as a bipartite system composed of $\mathcal{H}(\sigma_k)$ and

$\mathcal{H}(\sigma_{k+1:N})$. Then, from Eq. (10), we have $\partial_T(T(\mathcal{F}_{\sigma_{k:N}|\tilde{\Pi}_{\sigma_{1:k-1}}} - \mathcal{F}_{\sigma_k \rightarrow \sigma_{k+1:N}|\tilde{\Pi}_{\sigma_{1:k-1}}})) \simeq -\partial_\xi^2 \tilde{D}_{\sigma_k \rightarrow \sigma_{k+1:N}|\tilde{\Pi}_{\sigma_{1:k-1}}}$. Here, we have $\mathcal{F}_{\sigma_k \rightarrow \sigma_{k+1:N}|\tilde{\Pi}_{\sigma_{1:k-1}}} = \mathcal{F}_{\sigma_k|\tilde{\Pi}_{\sigma_{1:k-1}}} + \mathcal{F}_{\sigma_{k+1:N}|\tilde{\Pi}_{\sigma_{1:k-1}}}$.

The unconditional QFI is given by the average over measurement outcome distribution $p(\tilde{\Pi}_{\sigma_{1:k-1}})$ as

$$\mathcal{F}_{\sigma_k \rightarrow \sigma_{k+1:N}|\sigma_{1:k-1}} \equiv \sum_{\tilde{\Pi}_{\sigma_{1:k-1}}} p(\tilde{\Pi}_{\sigma_{1:k-1}}) \mathcal{F}_{\sigma_k \rightarrow \sigma_{k+1:N}|\tilde{\Pi}_{\sigma_{1:k-1}}}.$$

Then, one can define an unconditional version of discord,

$$\tilde{D}_{\sigma_k \rightarrow \sigma_{k+1:N}|\sigma_{1:k-1}} = \sum_{\tilde{\Pi}_{\sigma_{1:k-1}}} p(\tilde{\Pi}_{\sigma_{1:k-1}}) \tilde{D}_{\sigma_k \rightarrow \sigma_{k+1:N}|\tilde{\Pi}_{\sigma_{1:k-1}}},$$

which is related to the average measurement precision difference,

$$\partial_T(T(\mathcal{F}_{\sigma_{k:N}|\sigma_{1:k-1}} - \mathcal{F}_{\sigma_k \rightarrow \sigma_{k+1}|\sigma_{1:k-1}})) \simeq -\partial_\xi^2 \tilde{D}_{\sigma_k \rightarrow \sigma_{k+1:N}|\sigma_{1:k-1}},$$

where $\mathcal{F}_{\sigma_k \rightarrow \sigma_{k+1}|\sigma_{1:k-1}} = \mathcal{F}_{\sigma_k|\sigma_{1:k-1}} + \mathcal{F}_{\sigma_{k+1:N}|\sigma_{1:k-1}}$. Therefore, by adding the equation above from $k=1$ and $k=N$, the difference in the QFI can be written as

$$\Delta \mathcal{F}_{\sigma_{1:N}} = \mathcal{F}_{\sigma_{1:N}} - \sum_{k=1}^N \mathcal{F}_{\sigma_{\sigma_k}|\sigma_{1:k-1}}$$

so that we can obtain

$$\partial_T(T \Delta \mathcal{F}_{\sigma_{1:N}}) \simeq -\partial_\xi^2 \tilde{D}_{\sigma_{1:N}}, \quad (\text{D1})$$

where

$$\tilde{D}_{\sigma_{1:N}} = \sum_{k=1}^N \tilde{D}_{\sigma_k \rightarrow \sigma_{k+1:N}|\sigma_{1:k-1}}.$$

Equation (D1) is the generalization of Eq. (10) for the multipartite case.

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