Effects of Mean-Reversion on the Valuation of Undeveloped Oil Reserves and the Results of the Optimal Investment Rules

by

Lead Wey

Submitted to the Department of Electrical Engineering and Computer Science
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Abstract

In this thesis, I re-establish the fact that conventional net present value method used to evaluate investment opportunities in the oil industry, is inaccurate. I present the random-walk model, employing stochastic processes, option pricing theory, and the contingent claims analysis as an alternative approach. At the same time, two mean-reversion models are developed to account for the inadequacies of the random walk model that is, of its failure to account for the mean reverting properties of oil prices. A study of the effects of mean reversion on investment decisions leads to the conclusion that mean reversion does affect both the valuation of undeveloped oil reserves and the optimal investment rules in significant ways.

Thesis Supervisor: Robert Pindyck
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Chapter 1

Introduction

Two important questions related to the oil industry's investment decisions are often asked. First, what is the value of a discovered but undeveloped oil and gas reserve? And second, when is the best time to begin its development?

These questions have been answered in the past through the use of a random walk model developed from option pricing theory, stochastic calculus, and contingent claims analysis ([5] Paddock, Siegel, Smith, and [6] Pindyck). However, oil prices, crucial to the valuation process seem to be mean-reverting in nature. The failure of the random walk model to account for this important mean-reverting property leads us to wonder if the results offered by the random walk representation would change when mean-reversion is taken into account. In this thesis, I propose two mean-reverting models as alternatives to the random walk model and attempt to show that mean-reversion is important to the valuation process and should not be disregarded.

Chapter One describes the inadequacies associated with the conventional method of evaluating petroleum properties.

Chapter Two discusses the derivation of the three different models used in analyzing investment projects – the geometric Brownian model, the first order mean-reverting model, and the second order mean-reverting model. The chapter begins with a detailed discussion of the various parameters necessary to the valuation process. This is followed by a test on the hypothesis that oil prices are mean-reverting.

Finally, the step by step derivation of the three models are presented.

Chapter Three presents the results of the three different models and discusses their general implications.

Chapter Four discusses the important issues related to mean-reversion. First, an estimate for the rate of mean-reversion is obtained. This is followed by a comparison between the two mean-reverting models. Finally, we study the significance of including mean-reversion in the evaluation of undeveloped oil reserves and in obtaining the optimal investment rules.

Chapter Five concludes with an identification of possible problems and suggestions for future research.

1.1 An Overview

The Net Present Value (NPV) approach to valuing petroleum properties makes use of discounted cash flow analysis or the capitalization of income. First, the future income from a property or project is forecasted. The value of the property is then calculated to be the net present value of the cash flow it generates over the lifetime of the project.

$$NPV = C_0 + \frac{C_1}{(1+r_1)} + \frac{C_2}{(1+r_1)(1+r_2)} + \frac{C_3}{(1+r_1)(1+r_2)(1+r_3)} + ... \qquad (1.1)$$

where C_n is the cash flow for the period n, and r_n is the discount rate associated with the period n.

This method presents many problems. First, it is inaccurate because its foundation lies in the forecasting of uncertain cash flow. At the same time, the choice of a discount rate is questionable since it is usually based on the capital asset pricing model (CAPM), which makes certain assumptions, such as a fixed scenario. Further discussion of the difficulties associated with the NPV approach can be found in many articles ([1] Lehman 1989, [2] Brealey and Myers, 1988). Secondly, the NPV method fails to take into consideration some very important characteristics of the investment problem. For instance, it usually ignores the value added to the project through a firm's ability to make operating decisions during the life of the project in order to adjust the investment to existing market conditions, or to delay investment decisions in exchange for additional market information.

For a firm to make investment decisions more effectively, an alternative tool, free from all the above difficulties, is necessary. To begin developing such a tool, I must identify the various characteristics of investment decisions made by oil companies.

A petroleum company operates in the following manner. Oil exploration projects are carried out in an effort to search for oil. About 50% of the time, dry wells are found. However, when a potential well is located, and an estimate of its production capabilities is obtained, the company must then decide when to develop the well. The NPV method seems to suggest that as long as value exceeds costs, the firm should invest, but the method offers no clue as to when the development should start.

When a firm decides to develop a discovered, but undeveloped reserve, the cost of developing the reserve, once spent cannot be retrieved should the well be abandoned. We must therefore assume that such a decision is *irreversible*. Although the development is irreversible, it can certainly be halted before completion should the economic conditions become unfavorable, and restarted again at a later date.

Uncertainty exists in all forms of investments. In the development of oil reserves, there are two types of uncertainty. The first is *technical uncertainty* that can only be resolved when the project has begun. This is especially relevant in the oil industry where the size of the oil reserve is only an estimate and no one is really sure how much time, effort or materials will be needed until the development of the reserve is well underway. Technical uncertainty can be minimized with experience, and also with the aid of technology. However, it will not be considered for the purpose of this thesis. The second type of uncertainty is *external uncertainty*, and as the name implies, external uncertainty is external to the firm's operations. The unpredictable nature of oil prices fluctuations due to unpredictable changes in world politics is an example. Uncertainty in oil prices can affect the results of the investment analysis process significantly.

The main concern of this thesis lies in the observation that any huge fluctuations in oil prices are only temporary. This observation lead us to suspect that oil prices may indeed be *mean – reverting* in nature. In the remaining chapters of this thesis, I will study the effects of including mean-reversion in the investment analysis process. By comparing the results of the different investment models, I will draw some conclusion as to the importance of taking mean-reversion into account.

We begin the next chapter by discussing about the parameters necessary for the investment analysis. This is followed by the derivation of three different investment models, and an explanation of the implications of their solution. Results from the different models are compared, and the thesis concludes by identifying and suggesting areas for future research.

Chapter 2

Valuation Model

2.1 Variables

Before I proceed with the model, it is important to briefly discuss the variables involved. In particular, how they are derived, and how they are related to each other.

2.1.1 Real Risk-Free Interest Rate (r)

The risk free interest rate, r, is a measure of the rate of return of an investment that is 'risk free'. The most 'risk free' investments are the U.S. Treasury bills. In the United States, a standard measure of the nominal interest rate is the yield on 3 months U.S. Treasury bills. The risk free interest rate, r, is the nominal interest rate less inflation. For the purpose of this thesis, we shall assume $r = 4\%$ or 0.04 unless otherwise noted.

2.1.2 Expected Rate of Return on Investments (μ)

Return on an investment in the oil industry comes in two ways – capital gain and dividend received. Capital gain is the result of a rise in the value of the investment, while dividends are cash payouts . The expected rate of return, μ, on any investment is thus the sum of the expected rate of capital gain, α, and the rate of dividend received, δ. For investments whose value, V, is mean-reverting, the rate of capital gain is given by some function of V.

2.1.3 Payout Rate (δ)

An investor who invests in developed petroleum reserves would demand a return on his/her investment equal to any other form of investments with the same risk. Developed reserves may offer some expected capital gain, which may not meet the rate of return expected by the investor. For a rational investor to hold developed petroleum reserves, there must be an additional return to make up for this difference. This difference is furnished by the payout rate, δ, given by $\delta = \mu - \alpha$.

δ can also be viewed as the opportunity cost associated with delaying the development of the reserves. When $\delta > 0$, there is an opportunity cost to keep the option alive, or to delay the development of the reserve. When $\delta = 0$, the value of the option is large and there would be no incentive to develop the reserve since $\alpha = \mu$. When $\delta \to \infty$, the value of the option becomes very small, and it costs too much to keep the option alive. Essentially, the bigger δ gets, the less longer we would want to keep the option open.

2.1.4 Value of Developed Reserve (V)

The value of any developed oil reserve is directly proportional to the current and expected future prices of oil. This relationship is expected since the value of any reserve depends directly on the amount of oil it can produce and on the price the oil can fetch. In this study, we shall refer to the value of a unit of developed reserve as V. Not much published, verifiable information on the value of developed reserves exists. The data for V used in this thesis have been obtained from [3] Adelman, a paper which deals with estimates of cost per barrel of reserves block.

Figure 2.1 shows the relationship between value and price. In general the value of any long-lived asset rises when the market expects an increase in the prices of the goods the assets will produce in the future. Thus V must reflect the expected rise or fall of P. For the 1946-1986 period, the ratio of V to P averages about 0.302, with a standard deviation of 0.037. Thus, the 95% confidence interval for $\frac{V}{P}$ is 0.302 $\pm$ 0.075.We must note however, that deviations from the mean can occur due to a

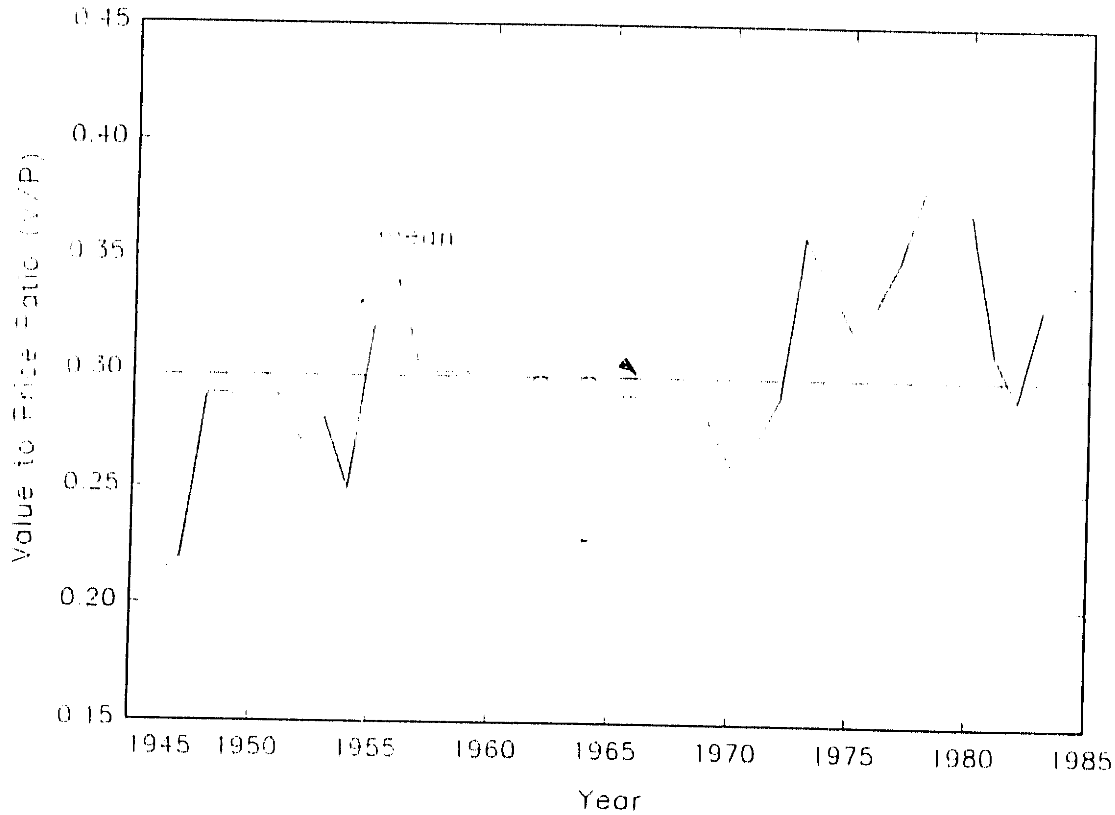

Figure 2-1: Plot of Crude Oil Value to Price Ratio ($\frac{V}{P}$)

change in the numerator caused by the expected rise or fall of oil prices, as well as a change in the denominator caused by the inclusion of commitments made years before. On the whole, the relationship $V = 0.3P$, is a good approximation, and will be assumed throughout this thesis.

2.1.5 Volatility (σ)

The measure of volatility, σ, related to the value of developed reserve is the measure of the standard deviation of the annual rate of change in oil prices, P. Figure 2.2 shows a plot of oil prices over the past 120 years.

For the period 1870 - 1990, the volatility is 21.2% (Please refer to Appendix A for a description on how the volatility is calculated). Another calculation using quarterly price data for the period 1985 to 1990, gives a volatility of 22.8%. Thus, volatility associated with oil investments is in the range of 20% . In this thesis, we shall assume $\sigma = 0.2$ unless otherwise stated.

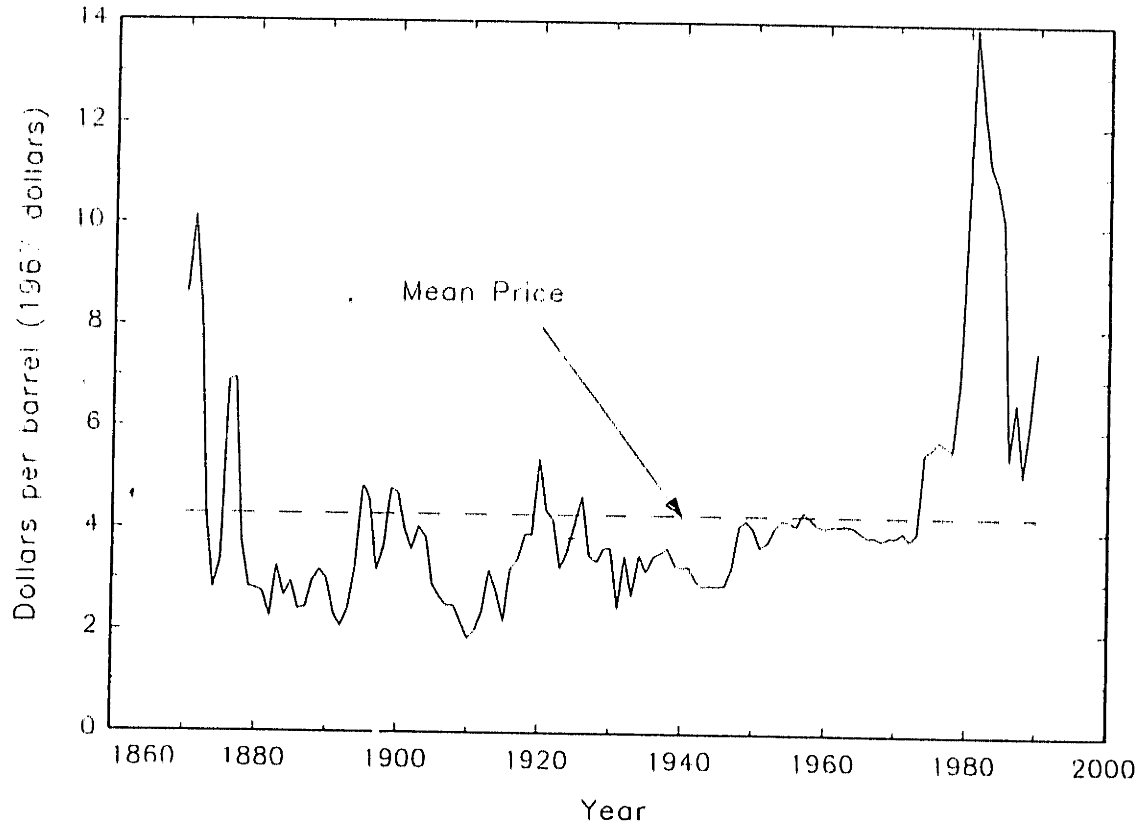

Figure 2-2: Plot of Oil Prices from 1870 to 1990 (1967 dollars)

2.1.6 Initial Cost of Investment (I)

The initial cost of investment, I, is the development cost of oil reserves measured in dollars per barrel ([7] Adelman 1988). Figure 2.3 shows the relation of value to investment of a typical oil reserve in the United States from the period 1959 to 1986. From 1959 to 1986, the ratio of value to investment cost per barrel averages 1.302 with a standard deviation of 0.472. This ratio may seem to be very high, since in the long run equilibrium, we expect the ratio to tend towards unity. This difference between value and development cost is the incentive to seek for new reserves.

2.2 Modeling Uncertainty

2.2.1 Random Walk Process

In the case of the oil industry, the value of an undeveloped reserve is most sensitive to uncertainty associated with oil prices. The price of oil, on the other hand, can be

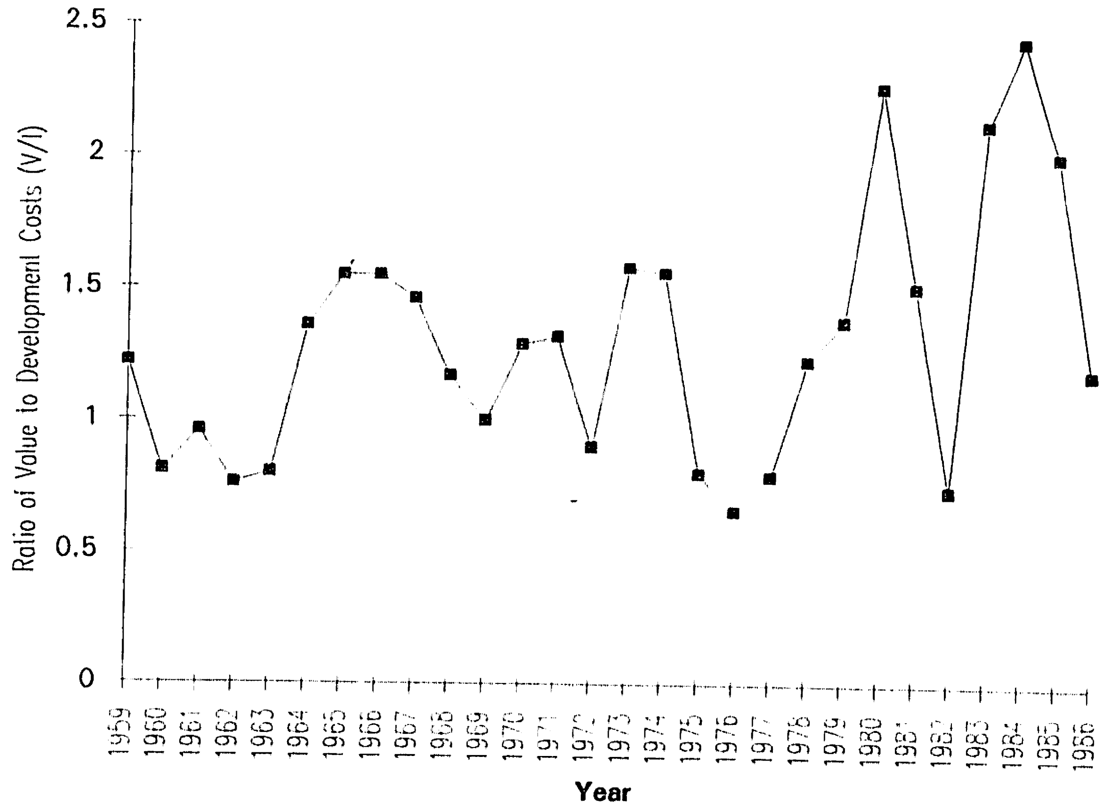

Figure 2-3: Ratio of Value to Cost of Investment per barrel of oil from 1959-1986

considered to follow an Ito process.

Ito process is a generalization of the Brownian motion, or Wiener process – a continuous-time stochastic process with three important properties. First, it is a Markov process. Thus future prices depend only on its current value and is independent of its past values. Second, changes in prices over any finite time interval are normally distributed. Third, the probability distribution for price changes over any two non-overlapping time interval are independent of each other.

Thus if z follows a Wiener process, we can relate a change in z, dz, over the time interval dt by

$$dP = \epsilon_t \sqrt{dt} \tag{2.1}$$

where ϵ_t is randomly distributed with 0 mean and unit standard deviation. We also note that $E[dz] = 0$ and $E[(dz)^2] = dt$.

Next, we model the price of oil, P, to follow a geometric Brownian motion with drift, as represented by the following equation

$$dP = \alpha f(P)dt + \sigma P dz \tag{2.2}$$

Although oil prices do fluctuate randomly in the short run (in response to war in oil producing countries, or due to activities associated with the OPEC cartel), in the long run, it tends to be drawn towards a value that is related to the marginal cost of producing oil. In such circumstances, the price of oil, P, would be best modeled as a mean-reverting process.

2.2.2 Mean-Reversion as an Alternative

In this thesis, we shali consider two such mean-reverting process. The first , which I shall refer to as the first order mean-reverting process, is given by

$$dP = \lambda(\bar{P} - P)dt + \sigma P dz \tag{2.3}$$

The second mean-reverting process, which I shall refer to as the second order mean-reverting process, is given by

$$dP = \eta(\bar{P} - P)P dt + \sigma P dz \tag{2.4}$$

Where $\bar{P}$ is the mean, and λ is the sensitivity constant, measuring the rate of reversion.

Figures 2.4 and 2.5, shows how fast the price, starting at P_0 at year 0, reverts to its mean value, $\bar{P}$ for various λ and η. When $P_0 > \bar{P}$, (in figure 2.4, $\bar{P} = 5P_0$) as expected, the second order mean-reverting process exerts a greater pull than the first order case. When $P_0 < \bar{P}$, (in figure 2.5, $\bar{P} = \frac{1}{2}P_0$) the first order case now exerts a greater pull, and it takes a shorter time for the price to revert to its mean.

However, we need to return to the question of whether the price of oil is best modeled by the above equations, and if so, which one. First, we need to answer the more fundamental question of whether we could reject the hypothesis that oil prices

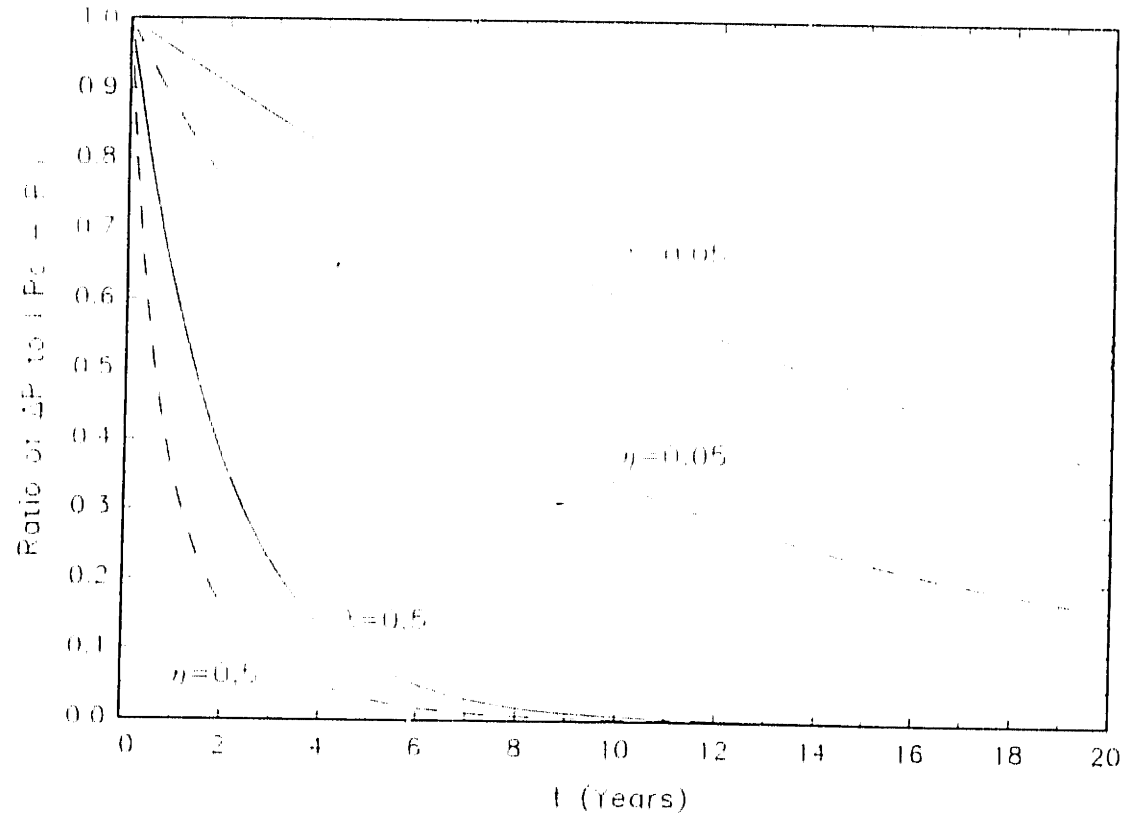

Figure 2-4: Behavior of prices when $\bar{P} = 5P_0$

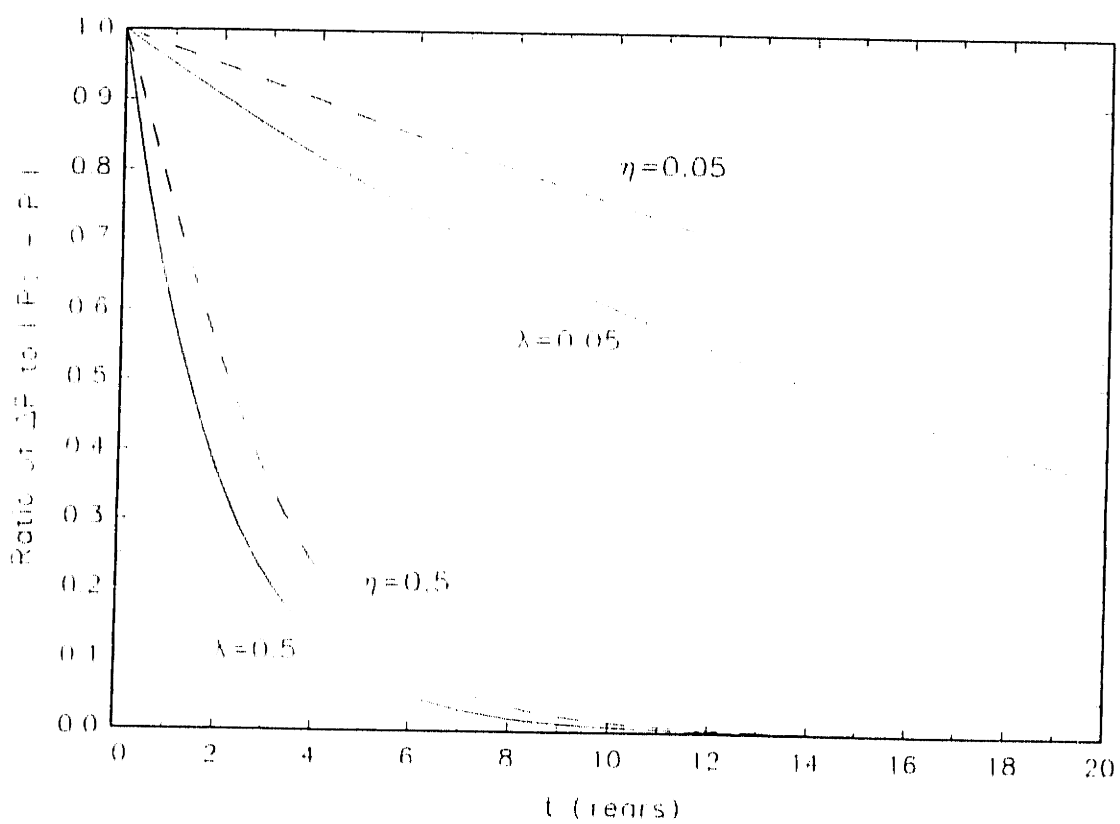

Figure 2-5: Behavior of prices when $2\bar{P} = P_0$

follows a random walk. To do so, we employ the unit root test (introduced by David Dickey and Wayne Fuller) for the price of crude oil over the last 120 years.

The test, as documented in appendix B, confirms that prices are mean reverting at a 99% confidence level. We note that this test requires many years of data so to give us some degree of confidence as to whether price is really mean-reverting. Thus, when the unit root test is performed for prices over the last 40 years, we fail to reject the hypothesis that oil prices follow a random walk, even at a 90% level of confidence. Although the latter test may be less accurate due to limited data, it suggests that oil prices are only slightly mean-reverting.

The decision of whether to model oil prices as a mean-reverting process is essentially the main theme of this thesis. To help in answering the question, we must study the process of valuing an oil development project and solving for the optimal investment rule. Would modeling oil prices as mean-reverting rather than the simpler geometric Brownian motion lead to a very different solution, and further complicate the computation process?

2.3 Solution through Contingent Claim Analysis

There are two ways of obtaining the solution to our problems. The dynamic programming approach, and the contingent claims analysis approach. Both techniques are closely related since they lead to the same results. However, they make different assumptions about the financial markets and about the discount rates that firms use to value future cash flow.

In this thesis, I will employ the contingent claims analysis to analyze the three different models – the geometric Brownian model, the first order mean-reverting, and the second order mean-reverting model.

The basis of the contingent claims analysis is built from ideas in financial economics. The technique starts by looking at the investment opportunity as described by its stream of costs and benefits through time, and on the unfolding of uncertain events. By treating the opportunity to invest as an asset, the option is given an implicit value. The value of the option to invest can be quantified by equating it to the total value of the combination of traded assets that exactly replicate the pattern of returns from the investment project. In doing so, the stochastic changes in the value of the firm's option to invest is completely spanned by existing assets in the economy. Once the value of the investment opportunity, $F(V)$, (in our case, the value of discovered but undeveloped oil reserves) is obtained, optimal investment policy in terms of the size, and timing of the investment can be found.

It is important to note that contingent claim analysis rests on the assumption that the uncertainty over the future value of the investment opportunity can be spanned and replicated by existing assets. Since oil prices (existing asset) is correlated to the value of the developed reserves, this assumption holds true. (Section 2.1.4)

2.3.1 Geometric Brownian Process

We restate the equation for the geometric Brownian process,

$$dP = \alpha P dt + \sigma P dz \tag{2.5}$$

where α is the rate of expected capital gain. Since $V \approx 0.3P$, we can rewrite the equation as,

$$dV = \alpha V dt + \sigma V dz \tag{2.6}$$

The next step requires the construction of a portfolio, which involves holding the option to invest (owning a discovered but undeveloped oil reserve), valued at $F(V)$, and going short on $\frac{dF}{dV} = F_V$ units of developed oil reserves, which has a value of V per unit. The value of this portfolio is thus given by

$$\Phi = F - VF_V \tag{2.7}$$

This portfolio is dynamic since V changes with time, leading to constant changes in F_V. To hold a short position in this portfolio, one has to make payments of δVF_V

per time period. The reason for this payment lies in that no rational investor would hold a long position in the developed oil reserves without getting the full risk adjusted expected return on the reserve, which is $\mu = \alpha + \rho$. Taking this payment into account, the instantaneous change in the value of this portfolio is given by,

$$d\Phi = dF - F_V dV - \delta V F_V dt \tag{2.8}$$

To obtain an expression for dF, we employ Ito's Lemma, which expands dF as a Taylor series. Dropping the insignificant terms of order higher than $(dV)^2$, we arrive at

$$dF = F_V dV + \frac{1}{2} F_{VV} (dV)^2 \tag{2.9}$$

Substituting this into equation (2.8) and using the relation $(dV)^2 = \sigma^2 V^2 dt$, which we can derive from equation (2.6), we obtain

$$d\Phi = \frac{1}{2}\sigma^2 V^2 F_{VV} dt - \delta V F_V dt \tag{2.10}$$

Since the return, $d\Phi$, is risk free, it must equal $r\Phi dt = r(F - VF_V)dt$ to prevent arbitrage opportunities from arising. Equating them,

$$r(F - VF_V)dt = \frac{1}{2}\sigma^2 V^2 F_{VV} dt - \delta V F_V dt \tag{2.11}$$

and simplifying, we arrive at the differential equation which $F(V)$ must satisfy.

$$\frac{1}{2}\sigma^2 V^2 F_{VV} + (r - \delta) V F_V - rF = 0 \tag{2.12}$$

In addition, $F(V)$ must satisfy the following boundary conditions:

$$F(0) = 0 \tag{2.13}$$

$$F(V^*) = V^* - I \tag{2.14}$$

$$F_V(V^*) = 1 \tag{2.15}$$

Condition (2.13) states that if V goes to zero, it will stay at zero, as required by equation (2.6). Since the value of developed reserve will always remain at zero, the option to invest has no value. Condition (2.14) states that upon investing at the optimal value, V^*, the firm will receive a net payoff of $V^* - I$. Finally, condition (2.15) requires that F(V) be continuous and smooth at the optimal investment point V^*, for if it were not so, one could do better by exercising at another point and V^* would no longer be optimal. The solution to equation (2.12) subjected to these boundary conditions are easy to find, and one can verify that it is given by

$$F(V) = aV^{\beta} \tag{2.16}$$

where a is a constant. Condition (1) requires that β be the positive root of the quadratic equation

$$\frac{1}{2}\sigma^2\beta(\beta - 1) + (r - \delta)\beta - r = 0 \tag{2.17}$$

Thus,

$$\beta = \frac{1}{2} - \frac{(r-\delta)}{\sigma^2} + \sqrt{[\frac{(r-\delta)}{\sigma^2} - \frac{1}{2}]^2 + \frac{2r}{\sigma^2}} \tag{2.18}$$

Using conditions (2) and (3), we obtain the expression for a and V^*,

$$V^* = \frac{\beta}{(\beta - 1)} I \tag{2.19}$$

$$a = \frac{V^* - I}{(V^*)^{\beta}} = \frac{(\beta - 1)^{\beta - 1}}{\beta^{\beta} I^{\beta - 1}} \tag{2.20}$$

Equation (2.16) gives us the value of the undeveloped oil reserve, while equation (2.19) gives us the optimal investment rule. The behavior of the solution will be discussed in the next chapter.

2.3.2 First Order Mean-Reverting Process

By modeling the price to follow a first order mean-reverting process, the unit value of developed oil reserves, V, exhibits the behavior as described by

$$dV = \lambda(\bar{V} - V)dt + \sigma V dz \tag{2.21}$$

As before, we proceed with the contingent claims analysis. The expected percentage rate of change of V,

$$\frac{1}{dt}E[\frac{dV}{V}] = \lambda(\frac{\bar{V} - V}{V}) \tag{2.22}$$

which leads to δ,

$$\delta = \mu - \lambda(\frac{\bar{V} - V}{V}) \tag{2.23}$$

By substituting this expression for delta into equation (2.12), we arrive at the differential equation which $F(V)$ must satisfy,

$$\frac{1}{2}\sigma^2 V^2 F_{VV} + [r - \mu - \lambda]VF_V + \lambda\bar{V}F_V - rF = 0 \tag{2.24}$$

and also the boundary conditions (2.14 - 2.15). The boundary condition (2.13) no longer applies. In fact $F(0) = 0$ only when $\lambda = 0$ since V always remain at zero as required by equation (2.21). When $\lambda > 0$, we expect V to revert to $\bar{V}$, and thus the option to invest will have some value, $F(0) > 0$. By substituting $V = 0$ into equation (2.24), we get the last two boundary conditions,

$$F(0) = 0 \tag{2.25}$$

when $\lambda = 0$, and

$$F_V(0) = \frac{rF(0)}{\lambda\bar{V}} \tag{2.26}$$

when $\lambda > 0$.

To solve equation(2.24), we assume that the solution takes the general form:

$$F(V) = V^s \sum_{n=0}^{\infty} C_n V^n \tag{2.27}$$

Substituting this into equation (2.24),

$$\tfrac{1}{2}\sigma^2[s(s+1)C_0V^s + s(s+1)C_1V^{s+1} + (s+1)(s+2)C_2V^{s+2} + ...]+$$

$$(r-\mu-\lambda)[sC_0V^s+(s+1)C_1Vs+1+(s+2)C_2Vs+2+....]+$$
$$\lambda\bar{V}[sC_0Vs-1+(s+1)C_1V^s+(s+2)C_2V^{s+1}+....]-$$
$$r[C_0V^s+C_1Vs+1+C_2Vs+2+....]=0$$

Summing the coefficients of the same order of V to zero, we obtain the following relations:

$$s=0 \tag{2.28}$$

$$C_1=\frac{rC_0}{\lambda\bar{V}} \tag{2.29}$$

$$C_2=\frac{(\mu+\lambda)}{2\lambda\bar{V}}C_1 \tag{2.30}$$

$$C_{m+1}=\frac{(r-m(r-\mu-\lambda)-\frac{1}{2}m(m-1)\sigma^2)}{(m+1)\lambda\bar{V}}C_m \tag{2.31}$$

for all integer $m>2$.

Thus, the solution to equation (2.24) is given by

$$F(V)=\sum_{n=0}^{\infty}C_nV^n \tag{2.32}$$

where C_0 is selected so that the boundary conditions are satisfied.

The series representation is not very useful for our purpose, since it diverges, and does not provide a way to generate numerical solutions. To study the characteristics of this model, the numerical solution of the reserve value, and the optimal investment rules have been computed numerically by using the 4th order Runge-Kutta method. (Appendix C)

2.3.3 Second Order Mean-Reverting Process

When the value of developed reserves follow a second order mean-reverting process, we mean that

$$dV=\eta(\bar{V}-V)Vdt+\sigma Vdz \tag{2.33}$$

To obtain the optimal investment rule, we employ the contingent claims analysis as before. First, calculate the expected percentage rate of change in V which is given by,

$$\frac{1}{dt}E(\frac{dV}{V}) = \eta(\bar{V} - V) \qquad (2.34)$$

Like before, δ is given by the difference between μ, the risk-adjusted discount rate, and the rate of change in V,

$$\delta = \mu - \eta(\bar{V} - V) \qquad (2.35)$$

Next, substitute this equation into the differential equation (2.12). $F(V)$ must satisfy

$$\frac{1}{2}\sigma^2 V^2 F_{VV} + [r - \mu + \eta(\bar{V} - V)]VF_V - rF = 0, \qquad (2.36)$$

and also the boundary conditions (2.13 - 2.15).

To solve for equation (2.36), we begin by assuming a solution of the form

$$F(V) = AV^{\theta}h(V), \qquad (2.37)$$

where A and θ are constants and $h(V)$ is a function in V to be determined. We substitute this expression into equation (2.36) and arrive at

$$[\tfrac{1}{2}\sigma^2\theta(\theta - 1) + (r - \mu + \eta\bar{V})\theta - r]V^{\theta}h(V) +$$

$$[\frac{1}{2}\sigma^2 V h_{VV}(V) + (\sigma^2\theta + r - mu + \eta\bar{V} - \eta V)h_v(V) - \eta\theta h(V)]V^{\theta+1} = 0 \qquad (2.38)$$

This equation must hold true for all values of V, thus both the bracketed expression must equal zero. The first expression lead us to a value for θ which has to be positive in order to satisfy the boundary condition, $F(0) = 0$.

$$\frac{1}{2}\sigma^2\theta(\theta - 1) + (r - \mu + \eta\bar{V})\theta - r = 0 \qquad (2.39)$$

$$\theta = \frac{1}{2} + \frac{(\mu - r - \eta\bar{V})}{\sigma^2} + \sqrt{[\frac{(r - \mu + \eta\bar{V})}{\sigma^2} - \frac{1}{2}]^2 + \frac{2r}{\sigma^2}} \tag{2.40}$$

The second expression of equation (2.38) allow us to compute $h(V)$,

$$\frac{1}{2}\sigma^2 V h_V V(V) + (\sigma^2\theta + r - \mu + \eta\bar{V} - \eta V)h_V(V) - \eta\theta h(V) = 0 \tag{2.41}$$

By letting $S = \frac{2\lambda}{\sigma^2}V$, we can rewrite $h(V)$ as $g(S)$ and equation (2.41) becomes

$$Sg_{SS}(S) + (b - S)g_S(S) - \theta g(S) = 0 \tag{2.42}$$

where $b = 2\theta + 2\frac{(r-\mu+\eta\bar{V})}{\sigma^2}$.

Equation (2.42) is known as the Kummer's Equation, and it has as its solution the confluent hypergeometric function, given by:

$$H(S;\theta;b) = 1 + \frac{\theta}{b}S + \frac{\theta(\theta+1)}{b(b+1)}\frac{S^2}{2!} + \frac{\theta(\theta+1)(\theta+2)}{b(b+1)(b+2)}\frac{S^3}{3!} + \ldots\ldots \tag{2.43}$$

Thus the solution to equation (2.36), which gives us the value of the discovered but undeveloped oil reserve, is

$$F(V) = AV^{\theta}H(\frac{2\eta}{\sigma^2}V;\theta;b) \tag{2.44}$$

where A is a constant to be determined. Both A and V^*, can be found using the remaining two boundary conditions (2.14) and (2.15). Since the confluent hypergeometric function converges, numerical solutions can be computed directly from the analytical expression (2.44).

Chapter 3

Explanation of Solution

This chapter describes the behavior of the optimal investment rules, and the value of the undeveloped oil reserves, with respect to the various parameters. By varying the parameters, I will establish a link between our model and the real world situation. At the same time, I will elaborate on the inadequacy of the Net Present Value method, and thus the need to introduce a more effective tool to aid in the firm's decision making process.

3.1 Geometric Brownian Process

To begin the study, I will set $I = 1$, $r = 0.04$, $\sigma = 0.2$, $\delta = 0.04$ (where $\mu - \alpha = \delta$), unless otherwise stated.

The Net Present Value rule states that a firm should invest when the net present value of the developed reserves, V, exceeds the invested amount, I. So, as long as $V \geq I = 1$, the firm should invest.

In contrast, we look at figure 3.1 which gives the solution to the geometric Brownian model. In this case,

$$F(V) = \tfrac{1}{4}V^2 \text{ for } V \leq 2$$
$$F(V) = V - I \text{ for } V > 2$$

where $\beta = 2$, $V^* = 2 = 2I$, and $a = \frac{1}{2}$. Since $(V - I)$ is the net payoff a firm

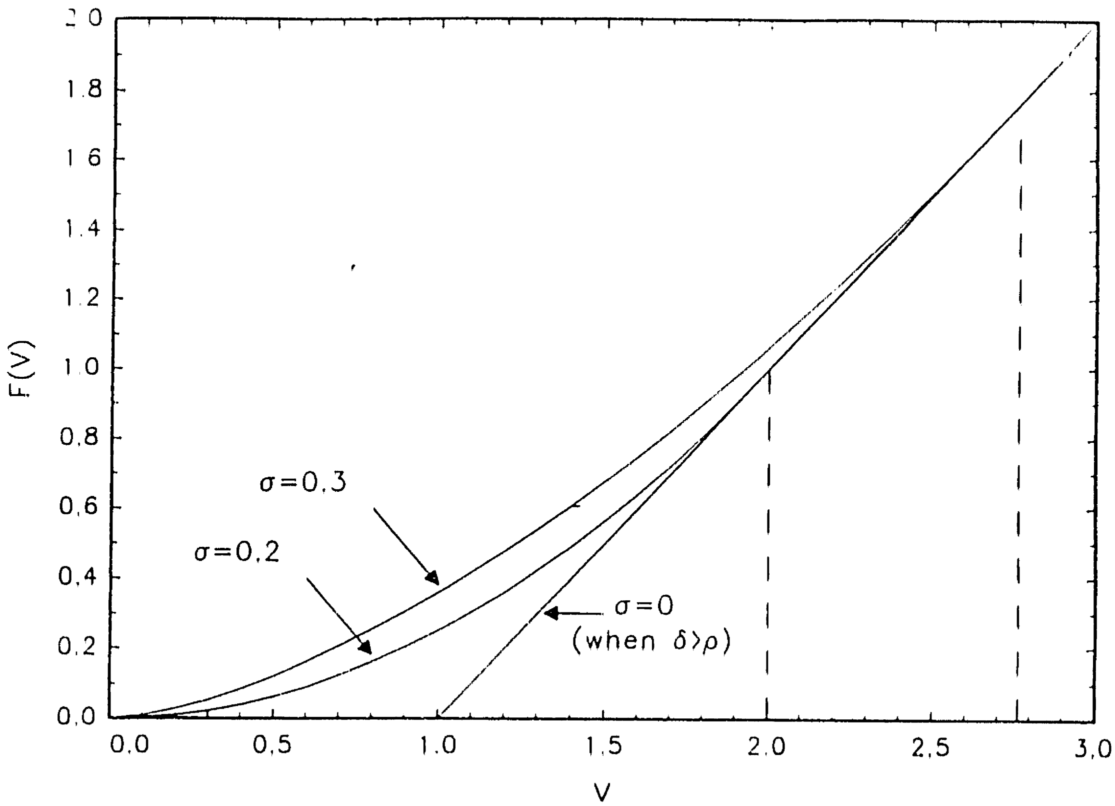

Figure 3-1: Plot of $F(V)$ vs V for the geometric Brownian Model

receives when it invests at V, the optimal investment rule states that the firm should invest only when $V \geq V^*$, so to minimize any foregone value, $F(V) - (V - I)$. This foregone value is the opportunity cost in deciding to invest now rather than waiting.

In the same figure 3.1, we observe the plot of $F(V)$ vs V for $\sigma = 0$ and $\sigma = 0.3$. The critical value, V^*, which occurs at the point of tangency between $F(V)$ and $V - I$, increases as σ increases. This occurs because the option to wait becomes more valuable when there is greater uncertainty. In the real world situation, this relation imply that uncertainty over future oil prices tend to increase the value of a firm's investment opportunities, $F(V)$, and decreasing the actual amount of investment made by the firm. Thus an increase in volatility associated with economic conditions should lead to an increase in the firm's market value, while reducing reserve development and production. To check for the validity of this relationship, one can perform a study on the correlation between the stock prices of various oil companies, the volatility associated with that period, and the number of development projects undertaken.

We conclude that an increase in σ will always lead to an increase in V^* even if the

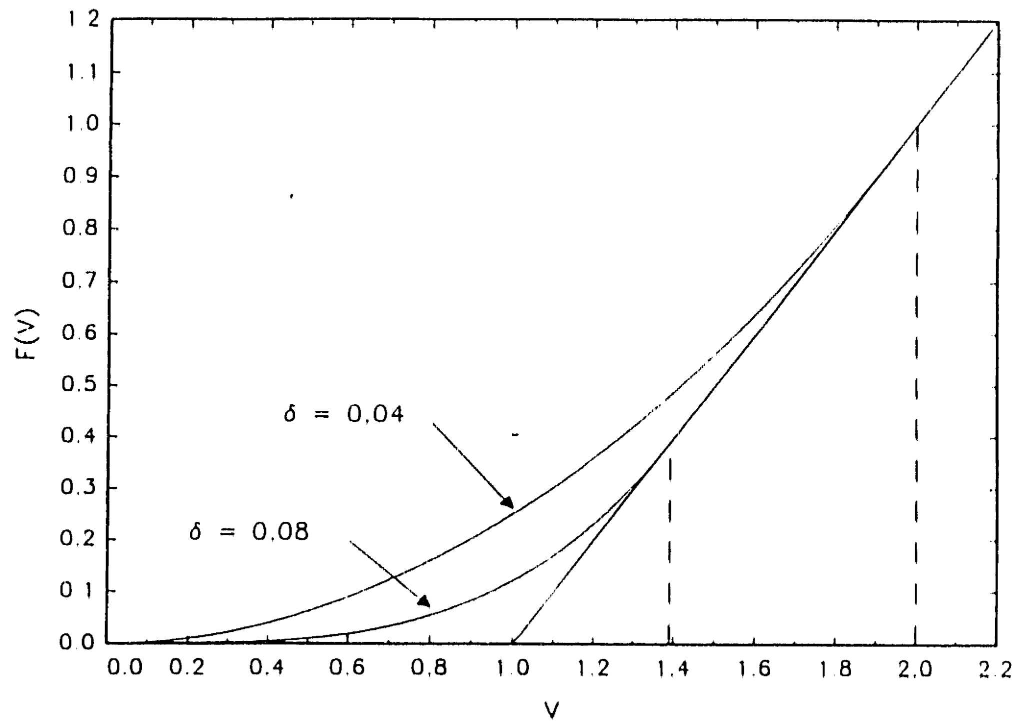

Figure 3-2: Plot of $F(V)$ vs V for different δ

firm is risk neutral, and even if the stochastic changes in V is completely diversifiable.

Figures 3.2 and 3.3 lead us to a discussion on the effect of changing the payout rate, δ. Note that an increase in δ leads to a decrease in V^*, as mentioned in section 2.1.3. The reason is that as δ increases, it becomes more expensive to wait, since the firm has to pay δV to hold onto the option to develop.

Figure 3.4 shows that an increase in the risk free rate, r, increases $F(V)$ and V^*. Since the investment expenditure made at a later time T has a present value of, Ie^{-rT}, an increase in r leads to a lower cost of investment, giving the firm a greater incentive to wait longer. At the same time, we note that the present value of the developed reserve remains at $Ve^{-\delta T}$, and there is no reduction in the expected payoff as long as δ is fixed. Thus the Geometric Brownian model predicts an increase in the value of undeveloped oil reserves, and a decrease in actual development when the risk free rate, r, increases.

To conclude this section, note that the different parameters discussed are usually dependent on each other. Thus, it is important to be careful when interpreting

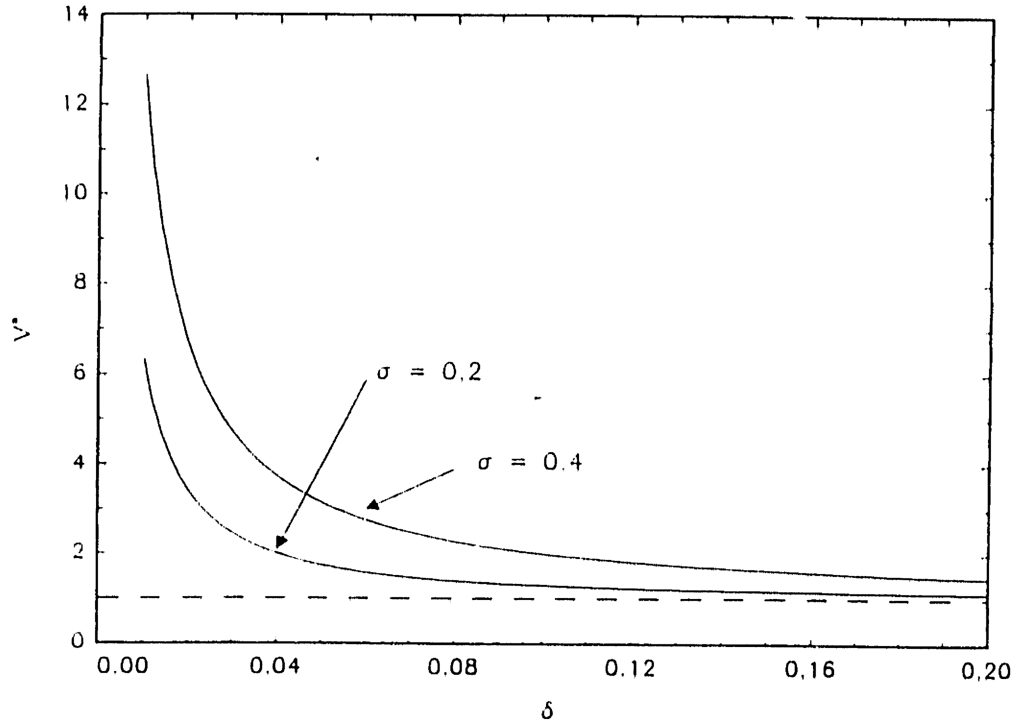

Figure 3-3: Plot of V^* vs δ

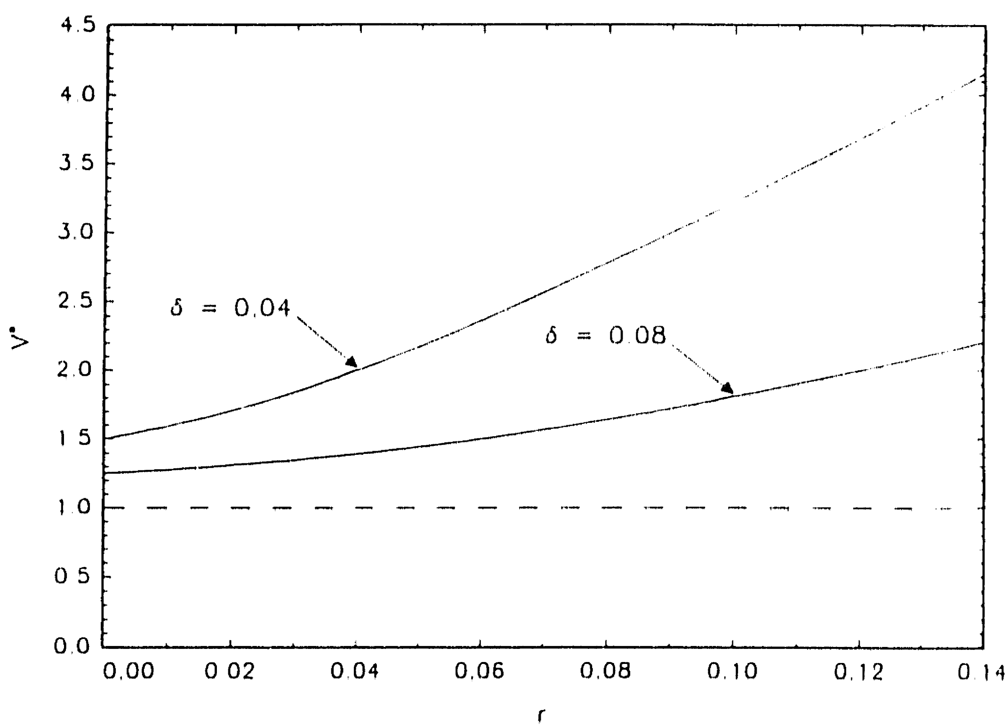

Figure 3-4: Plot of V^* vs r

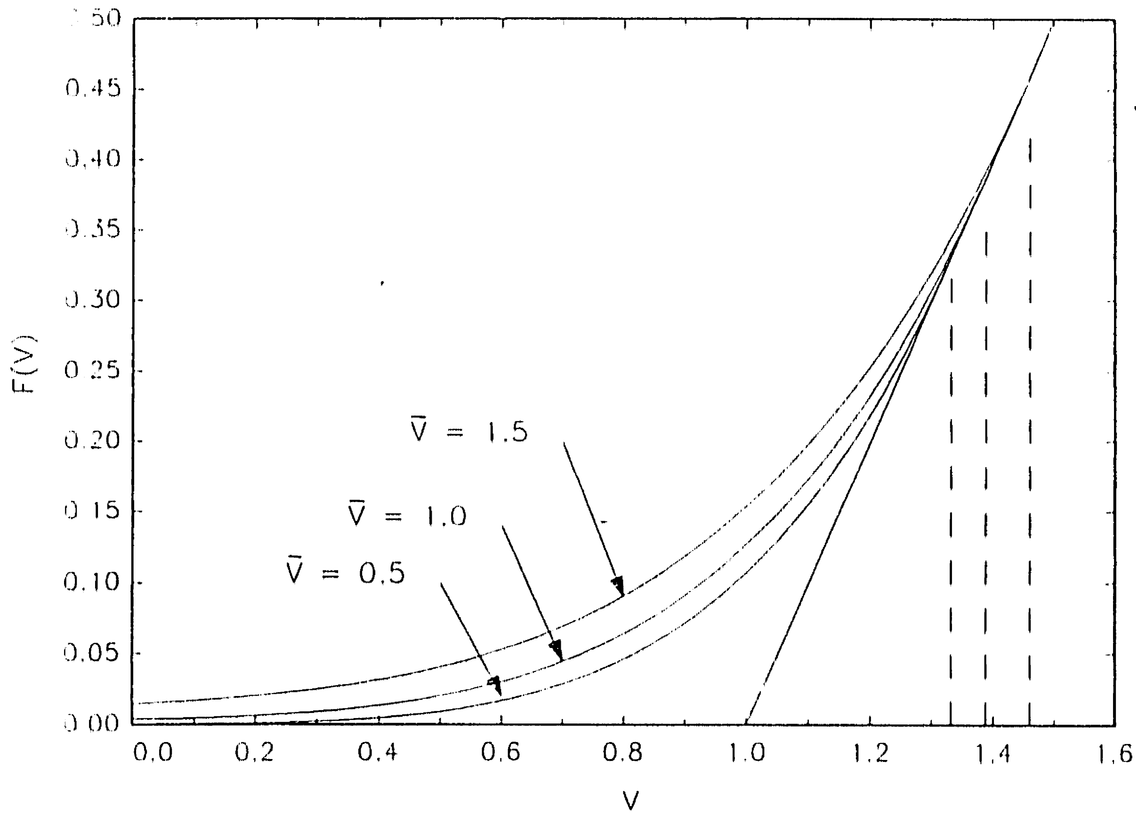

Figure 3-5: Plot of $F(V)$ vs V for $\lambda = 0.02$

comparative static results. An increase in the risk-free rate, r, could very well lead to an increase in the risk-adjusted interest rate, μ, which in turn could lead to an increase in the payout rate, δ. We have to keep in mind such interdependencies when analyzing how a change in some market-driven parameter will affect the value of the undeveloped oil reserves and the optimal investment rules.

3.2 First Order Mean-Reverting Process

To begin our study of the numerical solution of the first order mean reverting model, assume $I = 1$, $r = 0.04$, $\mu = 0.08$, and $\sigma = 0.2$, unless otherwise stated. Observe how the optimal investment rules behave when we vary the rate of mean-reversion, λ, and the value it reverts to, $\bar{V}$.

The first two graph, figure 3.5 and 3.6, shows a plot of $F(V)$ vs V for three different $\bar{V} = 0.5$, 1, and 1.5. An increase in $\bar{V}$ leads to an increase in V^*, and at the same time raises the value of the option to invest, $F(V)$. The logic behind such

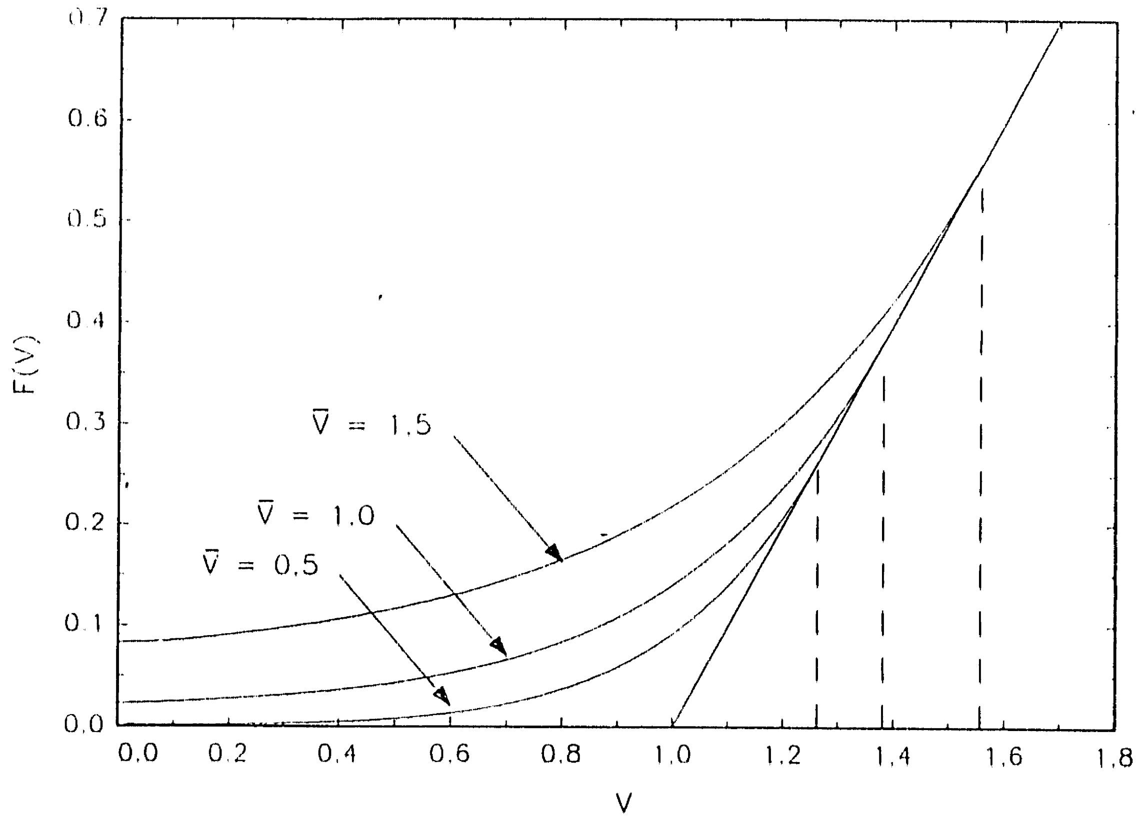

Figure 3-6: Plot of $F(V)$ vs V for $\lambda = 0.05$

a result is clear. When firms expect the price of oil to revert to a higher value, they would naturally find it beneficial to wait for a higher price before investing.

In figure 3.7, we observe how V^* varies with respect to the rate of mean reversion, λ, for various $\bar{V}$. For $\bar{V} > I$, a greater λ implies a greater pull towards a higher value. This would naturally increase the value of the option to wait, since the value of the project will increase faster and sooner. However, V^* levels off as λ increases and as V^* exceeds $\bar{V}$. For a $\bar{V} < I$, the effect is opposite. An increase in λ will tend to pull the project value down, leading to a decrease in V^*. In this situation when a firm expects the price to drop to a mean value below the cost of investment, the option to wait becomes less valuable.

The next figure 3.8, illustrates the effects of λ and μ, on V^*. For a lower λ, implying weaker mean-reversion, V^* starts at a higher value and decreases to a lower value as μ increases. This behavior can be explained as follows. A lower μ, and thus a lower δ, means that it costs the firm less to wait. A smaller rate of mean-reversion implies that the value of the project, V, is affected less by mean-reversion,

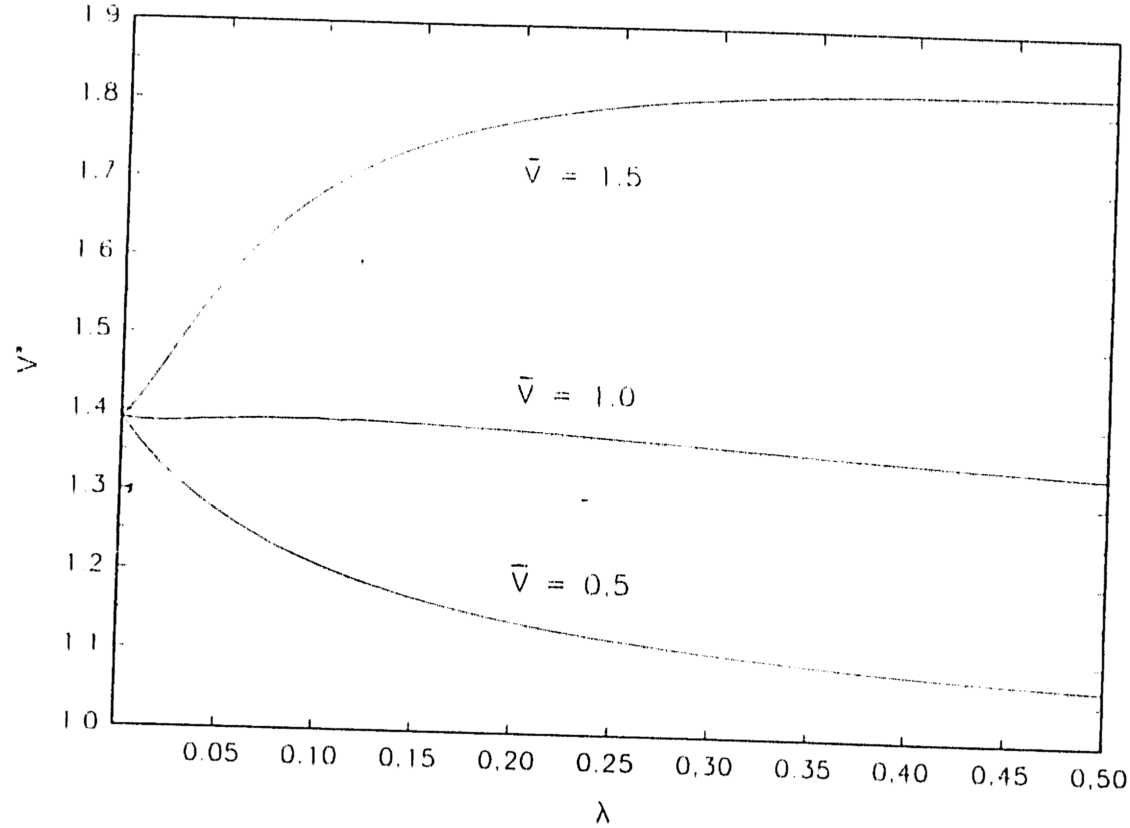

Figure 3-7: Plot of V^* vs λ for $\bar{V} = 0.5, 1.0, 1.5$

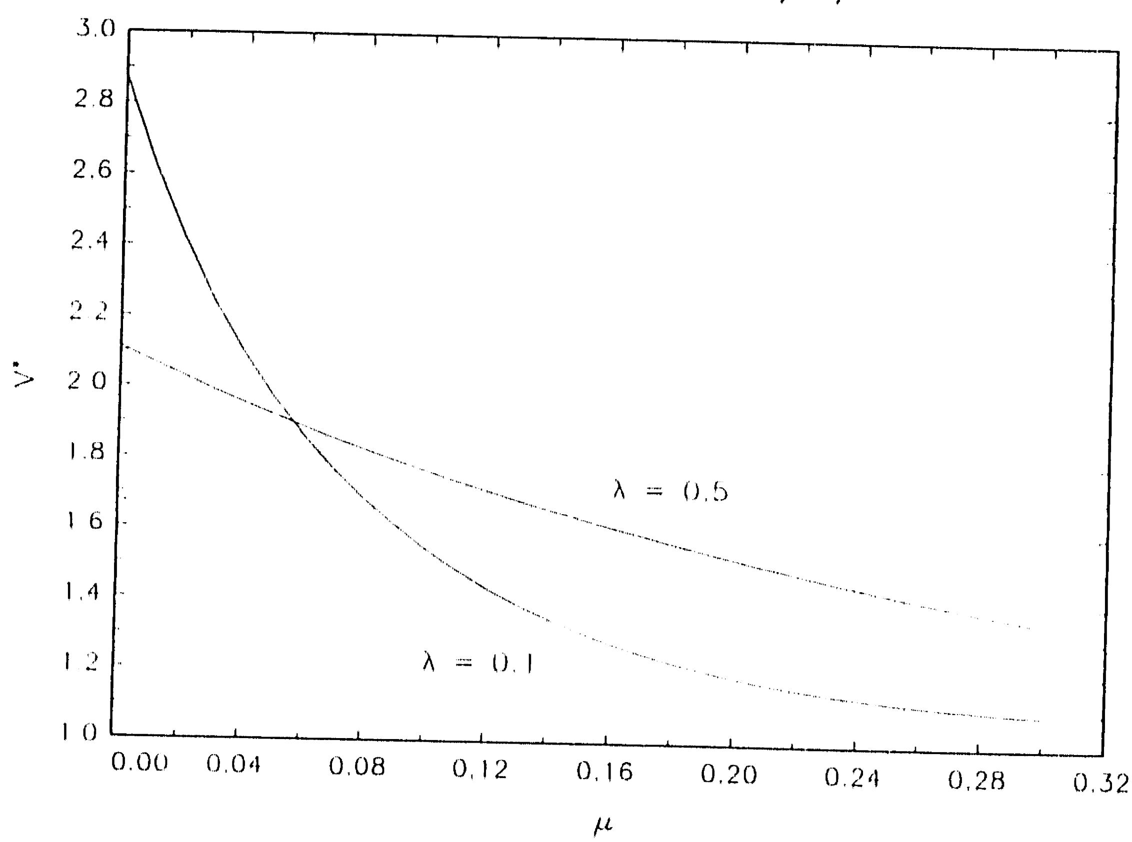

Figure 3-8: Plot of V^* vs μ for $\lambda = 0.1$ and 0.5

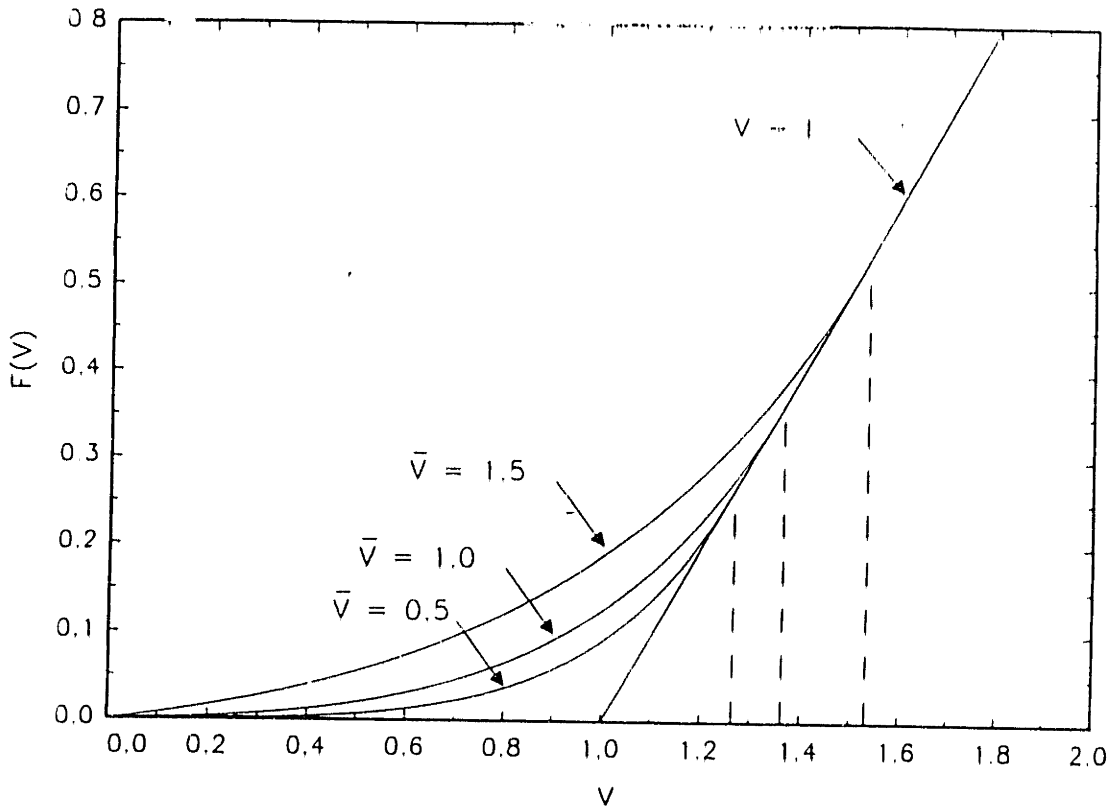

Figure 3-9: Plot of $F(V)$ vs V for $\eta = 0.05$

and more by any sudden increase. In the presence of greater uncertainty, low δ, and low λ, the firm would find waiting a more attractive option, and thus V^* is higher. As μ increases, and so does the cost of waiting, such an option to wait becomes less valuable.

3.3 Second Order Mean-Reverting Process

Assume that all parameters have similar values as before. The effects of mean-reversion on the optimal investment rules should essentially be the same as in the first order case, except that a second order mean-reverting process asserts a pull proportional to $\bar{P} - P$ as well as to P. This means that the pull towards a high mean, when P is low, decreases as P drops. The main difference lies in the boundary condition requiring $F(0) = 0$. This difference is noticed when we compare figure 3.6 with figure 3.9.

For the second order case, as P approaches zero, the value of the option to wait is

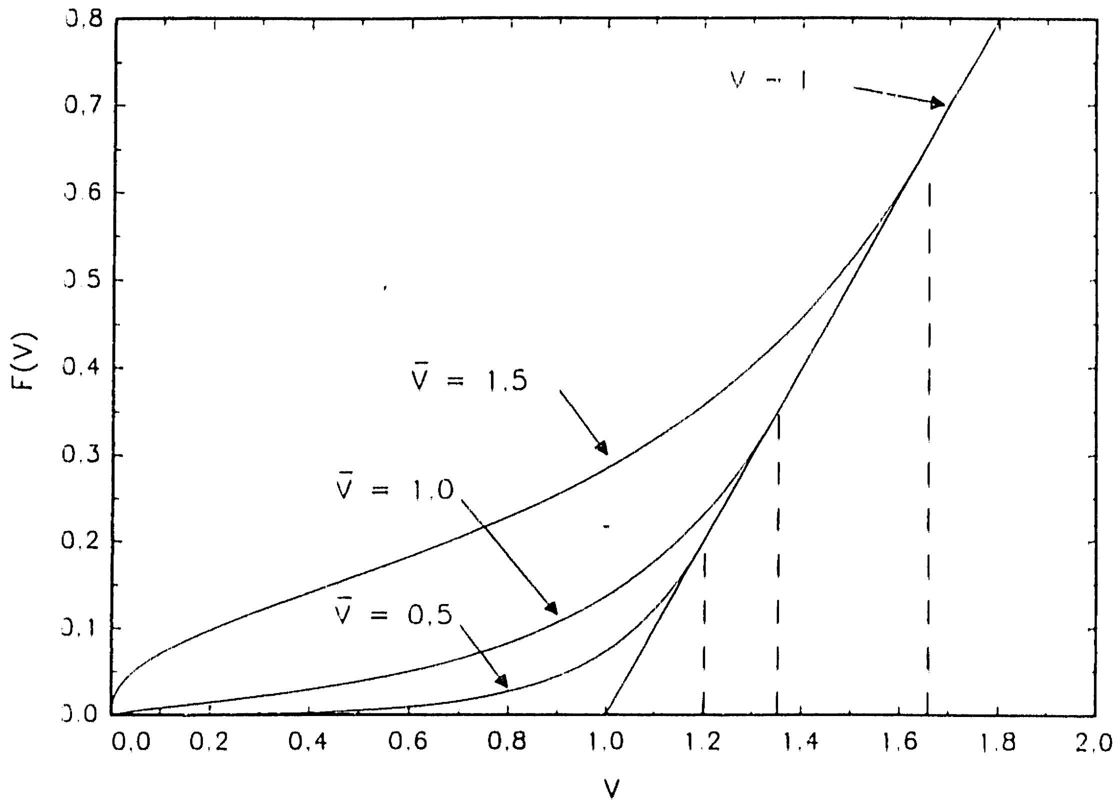

Figure 3-10: Plot of $F(V)$ vs V for $\eta = 0.1$

driven down to zero. This is because of the nature of the second order mean-reverting equation, which states that the rate of change towards the the mean value decreases as P drops, and when P finally hits zero, it will remain at zero. The value curve, $F(V)$, is concave in shape when η is small. However, as we increase η, even a small P can lead to a large pull towards the mean. To explain for this behavior, the shape of $F(V)$ must now be convex, as shown in figure 3.10. In the first order case, the option to wait can have a value greater than zero even if P were to fall to zero. This added flexibility may seem more realistic, since it would be unthinkable if oil prices drops to zero one day and remains that way forever. However, one must not immediately dismiss the second order model, since it may still give a good approximation of the real world situation when prices of oil remain in a reasonable range, close to its mean value.

Figure 3.11, which shows how the critical value, V^*, varies with η, can essentially be explain as in the first order case. However, when μ is decreased to a value of 0.04, we arrive at figure 3.12, which shows a change in behavior from our original graph.

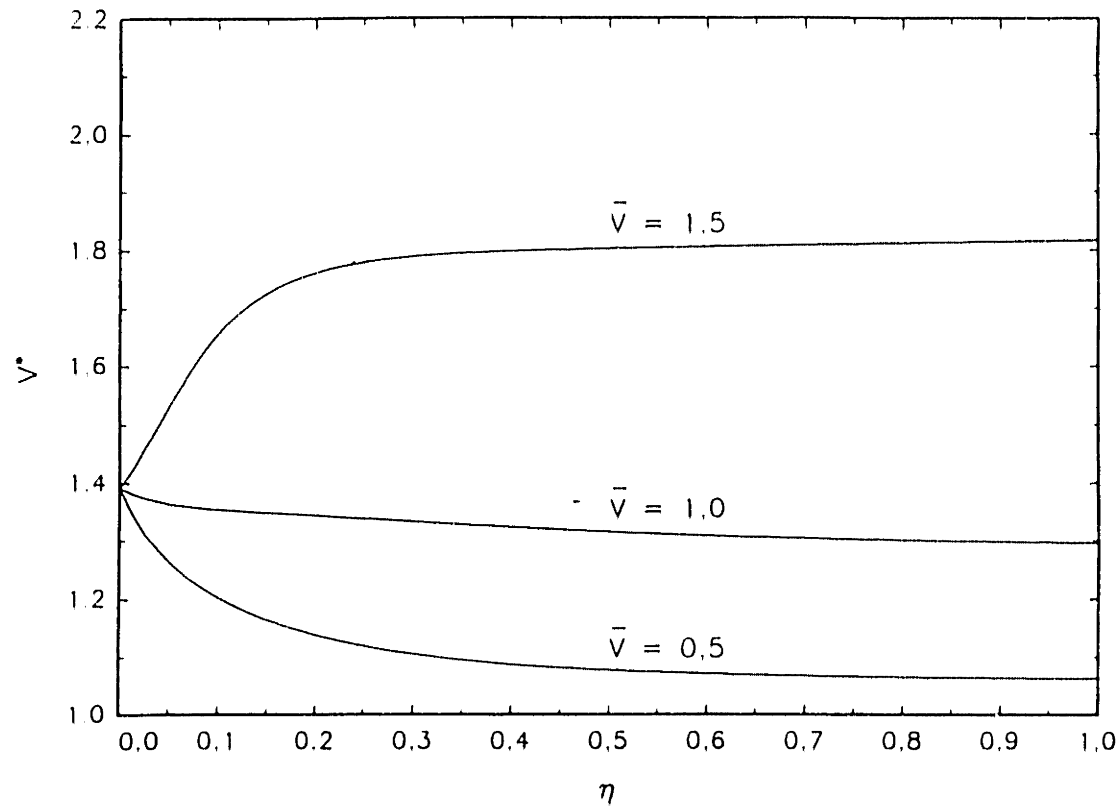

Figure 3-11: Plot of V^* vs η for $\mu = 0.08$

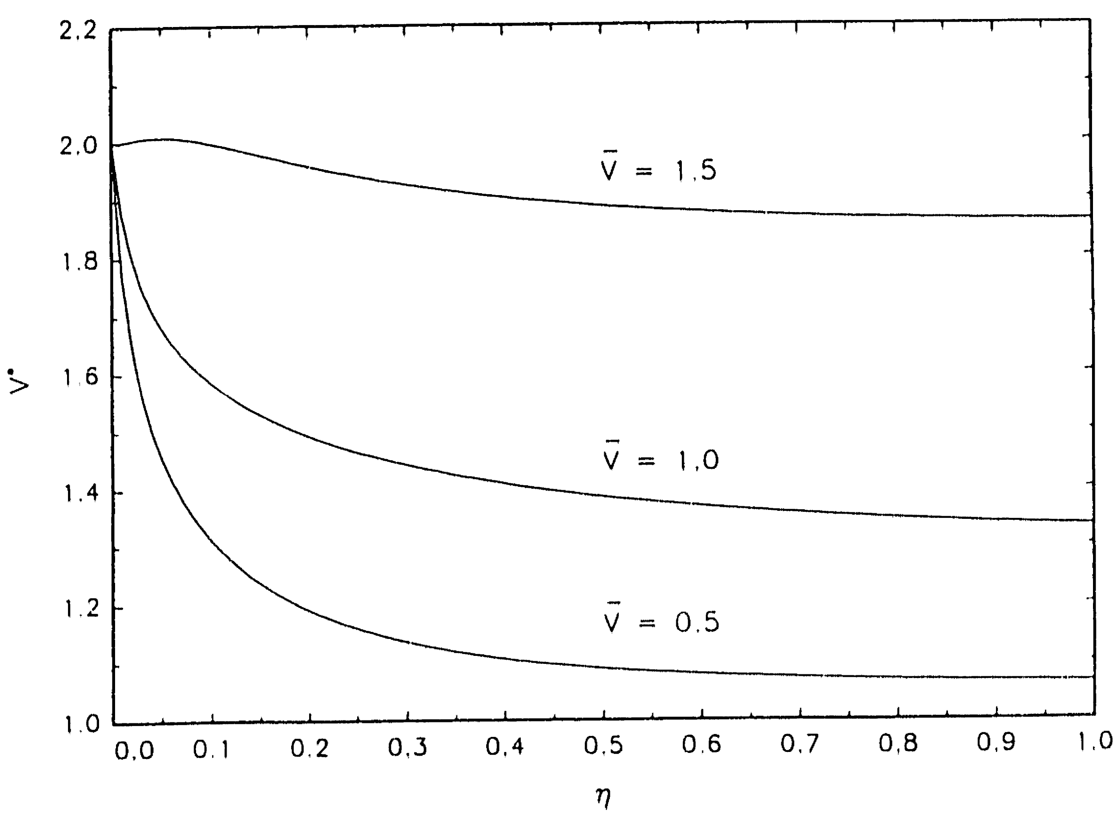

Figure 3-12: Plot of V^* vs η for $\mu = 0.04$

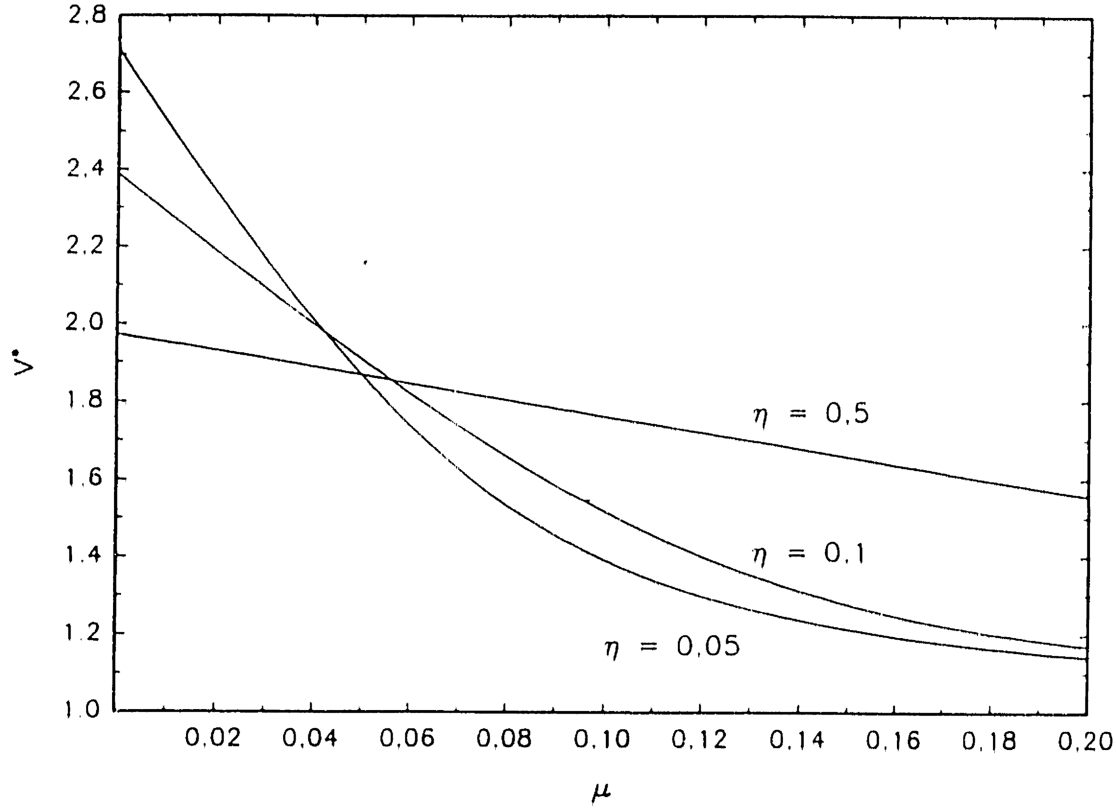

Figure 3-13: Plot of V^* vs μ

A smaller μ leads to a smaller payoff, δ, whose effect have been explained in the previous section. All things equal, an increase in μ will lead to a higher $F(V)$ and thus a higher V^* when $\lambda = 0$. Since this critical investment value, V^* when $\lambda = 0$, is now higher than before, an increase in λ would lead to a decrease in V^*, unless $\bar{V}$ is large enough so that it exceeds V^* when $\lambda = 0$.

The last figure 3.13, shows a similar behavior as in the first order case. However, we note that there is a greater pull towards $\bar{V} = 1.5$, which lends itself to a higher V^*.

Chapter 4

Mean-Reversion in Oil Investment Analysis

In this chapter, I discuss the significance and problems related to including mean-reversion in the investment analysis. In the first section, I discuss which of the two mean-reversion processes best models the market behavior of oil prices, and I attempt to estimate η and λ. In the second section, I compare the results of the random walk simulation to the results of the mean-reversion case, to see if the valuation process and the optimal investment rules are significantly affected by mean-reversion.

4.1 Modeling Oil Prices

To see if the first and the second order mean-reversion process sufficiently models the real world movement of oil prices, we start by estimating a value for η and λ. Industry experts generally agree that it takes about 4-5 years for any huge changes in oil prices to revert back to its mean. Using that estimate, we search for trends of mean-reversion using the 120 years of market data. We can calculate λ and η from the following equations,

$$\lambda = -\frac{1}{t} ln[\frac{P(t) - \bar{P}}{P(0) - \bar{P}}] \tag{4.1}$$

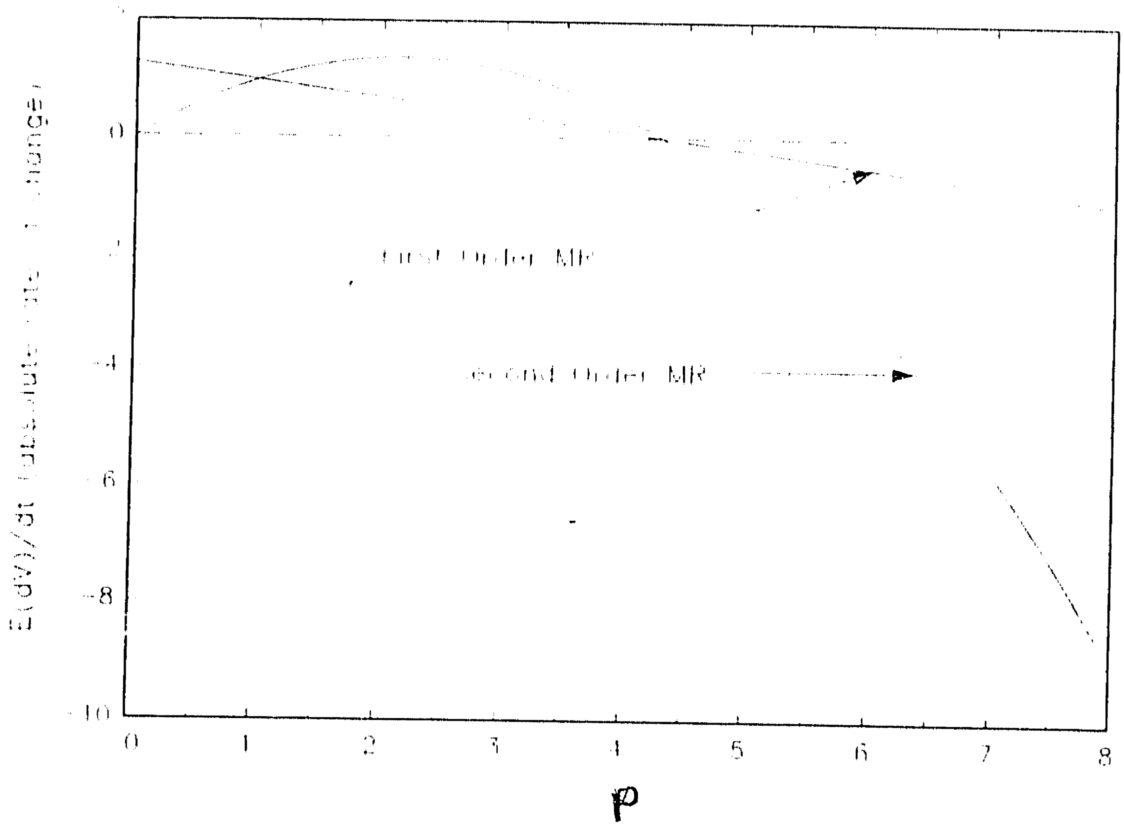

Figure 4-1: Absolute rate of change, $\lambda = \eta = 0.3$ and $\bar{P} = 4.29$

and

$$\eta = -\frac{1}{t} ln[\frac{P(0)(P(t) - \bar{P})}{P(t)(P(0) - \bar{P})}] \tag{4.2}$$

The 1980-1985 period is a good example, when prices fell from 13.94 to 5.46 in 5 years. Plugging into equation 4.1 and 4.2, we obtain $\lambda \approx 0.42$ and $\eta \approx 0.23$. We also noticed that on many occasions when prices falls below the mean to about 2 dollars, and takes about 3 years to recover. That approximation would give us an estimate for $\lambda \approx 0.25$ and $\eta \approx 0.3$. I will therefore assume that $\lambda = \eta \approx 0.3$ without proof.

To more fully understand the behavior of the two mean-reverting processes, we plot the absolute rate of change as a function of P (Figure 4.1). Notice that the first order mean-reverting process exerts a linear pull proportional to the difference between P and $\bar{P}$. The second order process, on the other hand exerts a pull which is a quadratic function in P.

In deciding which model best describes the behavior of oil prices, one can study the consistency of the value of η and λ. The simple calculation above seems to suggest that the second mean reverting process is a better model. This method of evaluation can be extended by looking at oil price data from the past 120 years. Notice that any decrease in oil prices was always followed by a gradual recovery path. Any sharp increase in oil prices, on the other hand, was always followed by a steep decline that could not be accounted for by the first order model. However, due to insufficient data on oil prices, we cannot conclusively test the validity of either models. What could be said though, is that given the usual range of oil prices (1.5 – 14 dollars), the second model is a better approximation of the real world situation than the first.

4.2 The Significance of Mean-Reversion

For the purpose of comparing the results of the geometric Brownian model and the second order mean-reversion model, we begin this section by setting values for the variables:

$$r = 0.04$$

$$\mu = 0.10$$

$$\sigma = 0.2$$

$$I = \frac{V}{1.6} = 3.75 \text{ (assuming mean oil price of \$20 per barrel)}$$

For the geometric Brownian model, we select

$$\alpha = 0$$

For the second order mean-reversion model, we select

$$\bar{P} = 20$$

$$\bar{V} = 0.3\bar{P} = 6$$

$$\eta = 0.3$$

We observe from figure 4.2 that generally, including mean-reversion in the investment analysis of oil reserves brings about two changes. First, in the evaluation of undeveloped oil reserves, mean-reversion leads to an increase in the value of reserves, provided $\bar{V}$ is high enough. Second, in the optimal investment rule, the inclusion of mean-reversion means a higher optimal investment value, V^*.

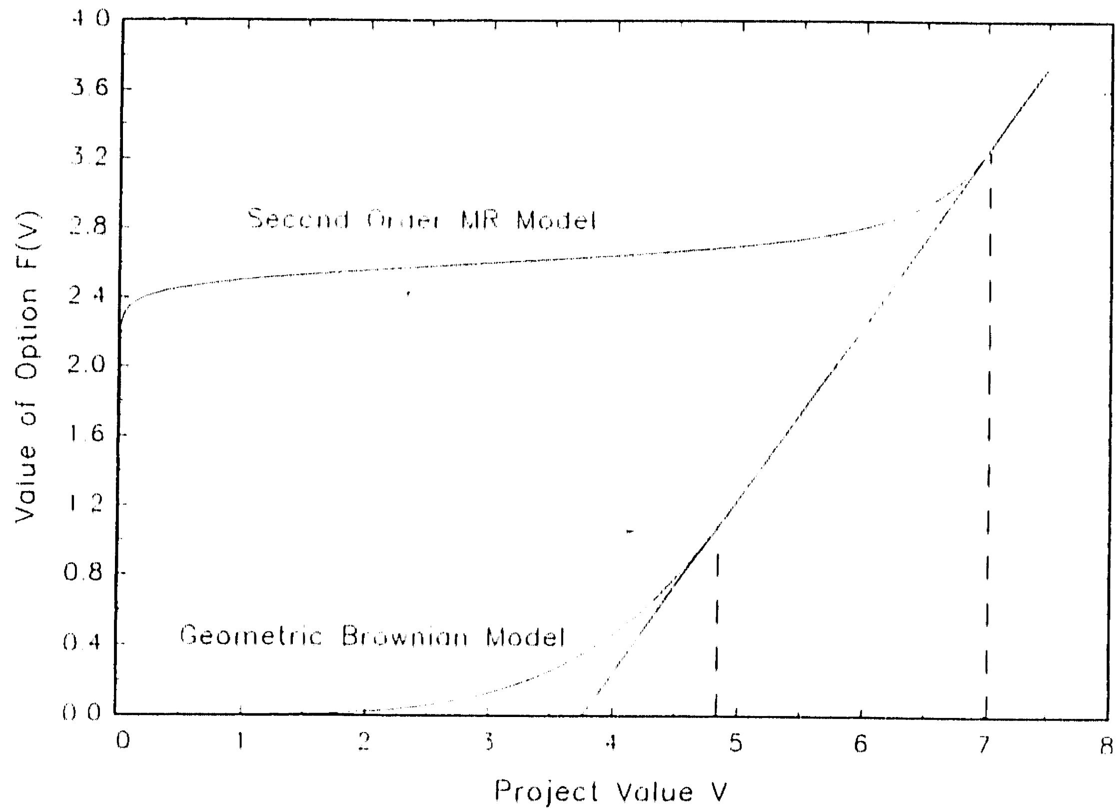

Figure 4-2: The geometric Brownian model and the second order mean-reversion model

However, we note that the results shown by figure 4.2 is only true under the assumed values for the different variables. We need to more fully understand the extent of influence of mean-reversion on the value of the reserves and on the optimal investment rules even as the variables change. Tables 4.1a and 4.2a show percentage change in the optimal investment value of the second order mean-reversion model relative to the geometric Brownian model(where the subscript rw stands for random walk and the subscript mr stands for mean-reversion):

$$Change = \frac{V^*_{mr} - V^*_{rw}}{V^*_{rw}} \times 100\%$$

Tables 4.1b and 4.2b show the percentage change in the value of the undeveloped oil reserves at $V = V^*_{rw}$ for the mean-reversion case with respect to the random walk model. We omit cases when V^*_{rw} is greater than V^*_{mr}.

$$Change = \frac{F_{mr}(V^*_{rw}) - F_{rw}(V^*_{rw})}{F_{rw}(V^*_{rw})} \times 100\%$$

	$\eta = \mathbf{0.05}$	$\eta = \mathbf{0.3}$	$\eta = \mathbf{0.9}$
$\mu = \mathbf{0.08}$	33.07%	36.01%	37.94%
$\mu = \mathbf{0.10}$	35.50%	45.10%	48.15%
$\mu = \mathbf{0.12}$	34.24%	50.75%	54.97%

Table 4.1a: Percentage change in Optimal Investment Value

	$\eta = \mathbf{0.05}$	$\eta = \mathbf{0.3}$	$\eta = \mathbf{0.9}$
$\mu = \mathbf{0.08}$	47.88%	89.86%	105.26%
$\mu = \mathbf{0.10}$	66.40%	147.36%	173.32%
$\mu = \mathbf{0.12}$	76.58%	204.44%	243.08%

Table 4.1b: Percentage change in Value of Oil Reserve

Tables 4.1a and 4.1b compares the results of the two models as we vary η, the rate of mean-reversion, and μ. Since $\bar{V} = 6$, we note that as the rate of mean-reversion increases, the pull towards $V = 6$ grows, thus pushing the values of $F_{mr}(V^*_{rw})$ and V^*_{mr} higher. This explains the increase in percentage change when η increases.

	$\bar{V} = \mathbf{4}$	$\bar{V} = \mathbf{6}$	$\bar{V} = \mathbf{8}$
$\mu = \mathbf{0.08}$	−4.06%	36.01%	78.57%
$\mu = \mathbf{0.10}$	2.09%	45.10%	90.94%
$\mu = \mathbf{0.12}$	5.81%	50.75%	98.82%

Table 4.2a: Percentage change in Optimal Investment Value

	$\bar{V} = \mathbf{4}$	$\bar{V} = \mathbf{6}$	$\bar{V} = \mathbf{8}$
$\mu = \mathbf{0.08}$	na	89.86%	224.74%
$\mu = \mathbf{0.10}$	1.25%	147.36%	329.57%
$\mu = \mathbf{0.12}$	5.81%	204.44%	435.45%

Table 4.2b: Percentage change in Value of Oil Reserve

Tables 4.2a and 4.2b compares the results of the two models by varying $\bar{V}$. As expected, a higher mean-reverting value leads to an increase in value of oil reserves as well as in the optimal invesment value.

In figure 4.2, the percentage change in the optimal investment value is 45.01%, while the percentage change in value of oil reserves is 147.36%. When mean-reversion is included in the analysis, and as we vary η, μ, and $\bar{V}$, we see that the results differ significantly from the random walk model.

This simple comparison we have performed suggests that mean-reversion is an important aspect of the investment model that should not be ignored. Just as the Net Present Value model fails to account for the value of the option to wait, the random walk model fails to account for the value added due to the mean-reverting properties of oil prices.

Chapter 5

Conclusion & Suggestion for Future Research

5.1 Conclusion

Oil companys' valuation of an undeveloped oil reserve, and their decision of when to develop the reserve, depends heavily on the behavior of oil prices. High volatility in oil prices increases the value of a firm's option to wait before developing. The random walk model was suggested by Paddock, Siegel, and Smith, as a replacement for the conventional net present value approach. Although the random walk model takes the volatility of oil prices into consideration, it fails to take the mean-reversion of oil prices into account.

In this thesis, we studied the implications of including mean-reversion into the investment analysis process, and we found that mean-reversion does significantly change the results obtained by the random walk model. This difference can mean a whole lot to the oil companies, especially in the evaluation of undeveloped oil reserves. A 150% difference in value as suggested by the second order mean-reversion model implies a miscalculation of value in the order of thousand of dollars.

There is of course a trade-off when we include mean-reversion into the investment analysis process. Mean-reversion leads to a more complicated evaluation process. Solutions obtained usually cannot be expressed analytically, and must be generated numerically. Such a trade-off in including mean-reversion may afterall be worthwhile and necessary, because mean-reversion leads to a significant change in the results. With the increasing power and availability of desk-top computers, such a trade-off is made even more attractive.

However, future research is necessary to determine if the mean-reversion property of oil prices should be included into the investment analysis process, and if so, what is the best way it should be done.

5.2 Suggestions for Future Research

First, the models presented in this thesis is by no means complete representation of the real world situation. In fact, parameters are not static in reality. This problem can be overcomed by representing the parameters as functions in terms of time, t. Thus r and σ can be replaced by $r(t)$ and $\sigma(t)$ in our model. Of course this will lead to a more complex problem and the value of investment opportunity will then be expressed as a function of V and t. Another problem one will face is in finding the correct function for each of the parameters.

Second, although we were able to verify that oil prices follow a mean-reversion process, limited market data makes the task of an accurate description difficult. Future research should focus on this problem of identifying a simple and accurate representation of mean-reversion, typical to oil prices, that could be easily applied to valuate petroleum properties and the associated optimal investment rules.

Finally, values for η and λ quoted in this thesis were rough estimates. There is a need to more accurately measure the rate of mean-reversion of oil prices.

Appendix A

Volatility of Oil Prices

The measure of volatility , σ, of oil prices is the measure of standard deviation of the annual rate of change in oil prices. Thus at year t, the rate of change in oil prices, R_t, is given by

$$R_t = \frac{P_t - P_{t-1}}{P_{t-1}} \times 100\% \tag{A.1}$$

Figure A-1 shows a plot of R for the period 1870-1990. The standard deviation of R for that period is found to be 21.2%.

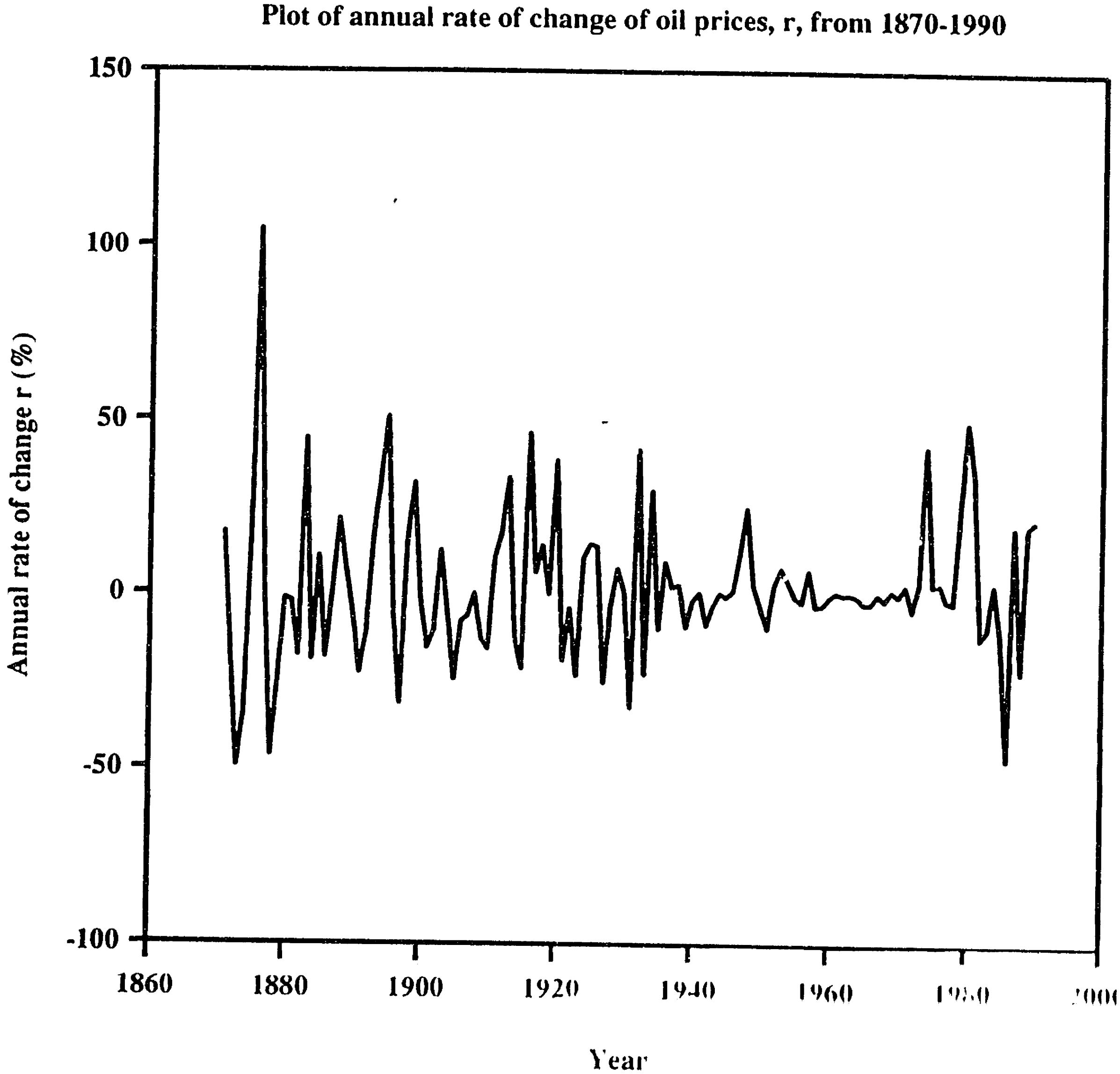

Figure A-1: Plot of annual rate of change, R, from 1870-1990

Appendix B

Unit Root Test on Oil Prices

Let's assume the oil prices is a random walk process, and can be described by the following equation:

$$P_t = a + bt + cP_{t-1} + d\Delta P_{t-1} + \varepsilon_t \tag{B.1}$$

where $\Delta P_{t-1} = P_{t-1} - P_{t-2}$, which accounts for the possible lag effect. A simple F test is performed to test the random walk hypothesis (if c is significantly different from 1). F is calculated as follows:

$$F = \frac{(N-k)(ESS_R - ESS_{UR})}{q(ESS_{UR})} \tag{B.2}$$

where N is the number of observations, k is the number of estimated parameters in the unrestricted regression, q is the number of estimated parameters in the restricted regression, while ESS_R and ESS_{UR} are the sum of squared residuals in the restricted and unrestricted regression.

Using 120 years of data, we test the hypothesis that the price of oil is a random walk process, i.e $b = 0$, $c = 1$. Using the Ordinary Least Squares method, we run an unrestricted regression

$$P_t - P_{t-1} = a + bt + (c-1)P_{t-1} + d\Delta P_{t-1} \tag{B.3}$$

which gives an Error Sum of Squares (ESS_{UR}) of 114.133:

Variable	Parameter Estimate	Standard Error	T for H0: Parameter=0
intercept	0.4086	0.2284	1.789
t	0.0071	0.0028	2.538
P_{t-1}	-0.1981	0.0463	-4.269
ΔP_{t-1}	0.2266	0.087621	2.586

The restricted regression

$$P_t - P_{t-1} = a + d\Delta P_{t-1} \tag{B.4}$$

gives an Error Sum of Squares (ESS_R) of 133.200

Variable	Parameter Estimate	Standard Error	T for H0: Parameter=0
intercept	-0.0054	0.0970	-0.055
ΔP_{t-1}	0.1588	0.0915	1.734

$F = 9.689$. Thus, according to the Table below, we can reject the hypothesis that oil prices follow a random walk process (i.e. $b = 0$ and $c = 1$) at a 1% level.

Sample Size	Prob = 0.90	Prob = 0.95	Prob = 0.99
25	5.91	7.24	10.61
50	5.61	6.73	9.31
100	5.47	6.49	8.73
250	5.39	6.34	8.43

(Table obtained from Dickey and Fuller, pg 1063 1981)

Reapeating the test for the last 40 years, we obtain for the unrestricted regression an $ESS_UR = 51.05501$:

Variable	Parameter Estimate	Standard Error	T for H0: Parameter=0
intercept	-2.2809	1.7699	-1.289
t	0.0368	0.0206	1.788
P_{t-1}	0.2929	0.1599	1.832
ΔP_{t-1}	-0.2404	0.0950	-2.529

For the restricted case, we obtain $ESS_R = 59.91360$:

Variable	Parameter Estimate	Standard Error	T for H0: Parameter=0
intercept	0.0745	0.1937	0.385
ΔP_{t-1}	0.1661	0.1602	1.037

This gives an $F = 3.123$, and we cannot reject the hypothesis that oil prices follow a random walk process, even at a 10% level.

Appendix C

Runge-Kutta Method of Fouth Order

The following is the algorithm that computes the solution of the initial value problem $y' = f(x, y)$, and $y(x_0) = y_0$ at equidistant points, ie.

$x_n = x_0 + nh$ where $0 \leq n \leq N$.

Note that f is such that the problem has a unique solution on the interval $[x_0, x_N]$. We begin by stating the initial values x_0, y_0, stepsize h, and the number of steps N. Then, we approximate y_{n+1} to the solution $y(x_{n+1})$ at $x_{n+1} = x_0 + (n+1)h$, where $n = 0, 1, ..., N-1$.

For $n = 0, 1, ..., N-1$, we perform the following:

$k_1 = hf(x_n, y_n)$

$k_2 = hf(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_1)$

$k_3 = hf(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_2)$

$k_4 = hf(x_n + \frac{1}{2}h, y_n + k_3)$

$x_{n+1} = x_n + h$

$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

The step size h should not be greater than a certain value which depends on the desired accuracy. h should be small such that $\kappa = h\mathcal{K}$ lies between 0.01 and 0.05. $\mathcal{K}$ is the close upper bound for $|\frac{\delta f}{\delta y}|$. For the fourth order method,

$$\kappa \approx 2|\frac{k_3 - k_2}{k_2 - k_1}|. \tag{C.1}$$

To apply the Runge-Kutta Method to the second order equation (2.24), we have to write the equation as a combination of two first order equations. By introducing the variable Z, and representing $f_n = F(V_n)$we have

$$f_n' = Z_n \tag{C.2}$$

and

$$f_n'' = \frac{2}{\sigma^2 V_n^2}[rf_n - (r - \mu - \lambda)V_n Z_n - \lambda \bar{V} Z_n] \tag{C.3}$$

We perform the Runge-Kutta for both equations with the starting values $V_0 = V^*$, $f_0 = V^* - I$, $Z_0 = 1$, and stepsize h, where $h < 0$. At $x_n = 0$, we check that f_n and Z_n satisfies the boundary conditions (2.25) and (2.26). In other words, a shooting algorithm should be employed to select the correct starting value V^* that would lead us to a solution satisfying all boundary conditions.

References

1. W. Press, W. Vetterling, S. Teukolsky, and B. Flannery, *Numerical Recipes in C : The Art of Scientific Computing*, Cambridge Press 1992

2. R. Pindyck and D. Rubinfeld, *Econometric Models and Economic Forecasts*, McGraw Hill, 1991

3. M. Adelman, H. DeSilva, and M. Koehn, *The Valuation of Oil Reserves* 1946 – 1986, MIT-EL 87-014WP

4. D. Ott, and D. Norman, *An Empirical Analysis of The Determinants of Petroleum Drilling*, American Petroleum Institute research study no.32

5. Paddock, Siegel, and Simth, *Option Valuation of Claims on Real Assets : The Case of Offshore Petroleum Leases*, MIT Energy Lab WP83-005

6. R. Pindyck, *Investments of Uncertain Cost*, CEPR WP92-005

7. M. Adelman, *Finding and Development Costs in the US* 1945 – 1985, MIT Energy Lab Working Papaer no. MIT-EL 86-008WP