Treatment of Constraints in Discrete Dynamics

by

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Abstract

In most undergraduate courses in discrete linear dynamics, a formal treatment of constraints is not typically introduced. A constraint is any factor external or internal to a mechanical system of concern that restricts the motion of the system. Although in most dynamical systems there are ways to avoid the need of a formal approach to incorporating the constraints into the equations of motion, there are some cases in which a direct approach is the only way that the state equations of motion may be derived. The goal of this thesis is to translate a formal approach of the treatment of constraints from some intermediate level texts into a level that may be understood at an undergraduate level. Although most of the concepts are not original, the examples, solutions to problems, and explanations were not taken from any published texts.

This thesis begins with some terminology and the classification of the various constraints. The discussion continues with the effects of constraints on the degrees of freedom and the generalized coordinates. Following this discussion is an introduction to the approach of treating constraint equations as forces. Finally, a systematic approach to the derivation of the state equations of a system with constraints is suggested and verified. Nonholonomic examples, systems that are not normally treated in introductory courses in dynamics, are used to demonstrate the concepts of this thesis.

Thesis Supervisor: James H. Williams, Jr.
Title: SEPTE Professor of Engineering

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VARIABLES

L	=	Lagrangian, all energy from conservative elements
T^*	=	Kinetic coenergy of a system
V	=	energy from the conservative forces
Ξ	=	energy from the nonconservative forces
u_s	=	nondescript generalized coordinate
du_s	=	nondescript infinitesimal displacement
δu_s	=	nondescript virtual displacement
λ_s	=	Lagrange multiplier
X, Y, Z	=	cartesian coordinates in meters
Θ, Φ, Ψ	=	angular coordinates in radians

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Chapter 1

Classification and Definition of Constraints

In order to study the response of a system in the presence of constraints, one must first develop a system of classifying the various constraints. The term 'constraints' is a broad term that can mean any restriction on the motion of a system of particles. However, it is difficult to master the treatment of constraints without limiting one's focus to the specific category concerned.

For example, one might consider the problem of two molecules moving in unbounded space. Although the two particles are free to move without any apparent restrictions on their motion, there is, in fact, one constraint that may be recognized. There is a theorem of physics that states the following:

> In a system of two or more particles, unconstrained motion does not exist.[1]

In other words, a constraint is created because the two particles may not exist in the same space at the same time. This particular constraint will have no effect on the state equations of the system unless the two molecules collide. In order to solve the equations of motion for this system, the problem must be broken down into two cases: the occurrence of collision and the unobstructed motion of the two particles.

On the other hand, consider the situation in which the two molecules are bound by a bar of a specific length. Again, this is a two particle system, but the constraint is of a considerably different nature. The distance between the coordinates of the two molecules is constant. Solving the equations of motion for this problem would involve an entirely different approach, which will be discussed later in the thesis.

Before introducing the categorization of constraints, this chapter will define and discuss the terms 'finite variables' and 'infinitesimal displacements.' These are the mathematical variables that will help categorize the constraints as well as eventually enable the incorporation of the constraints into the state equations.

Secondly, the concepts of the rheonomic constraints and scleronomic constraints will be defined. This discussion will include some examples that will enable one to delineate the two categories of constraints.

Finally, this chapter will include the comparison of holonomic versus nonholonomic constraints. Although it is not always obvious whether a particular restriction given on the behavior of a system is a holonomic constraint, there is a simple test that will make this distinction.

1.1 Finite variables and infinitesimal displacements

The term finite variables is used to define the set of coordinates which specify the location of an object or system of objects at a given time. The time variable t is also included in the set of finite variables. There are always several ways to identify the locations of a system, or the configuration space, by changing the coordinate system. However, the minimum number of finite variable necessary to fully define a configuration is independent of the particular coordinate system used. Any such minimum set of finite variables is called the set of generalized coordinates.

Consider the example following, which is constrained to two dimensions. There are two masses connected to a wall by springs as shown in Figure 1 on the following page. As a consequence of the presence of ledges connected to the wall, the masses, which are linked together by a spring, are constrained to move perpendicular to the wall.

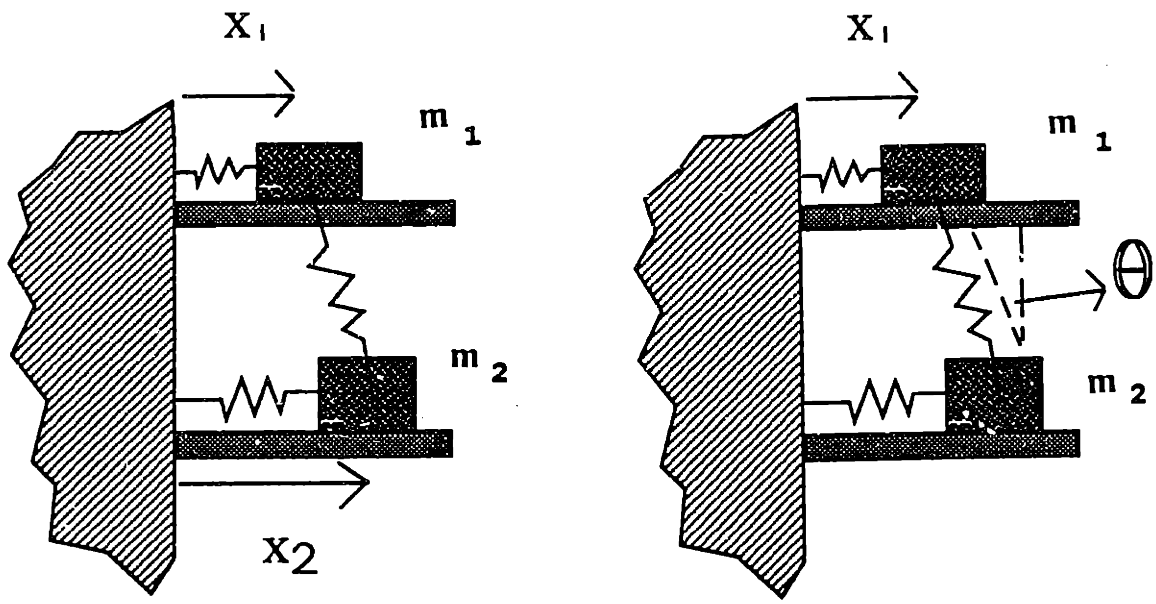

Figure 1: A two particle mass-spring system.

On the left side of the figure is a set of generalized coordinates selected entirely in cartesian coordinates. The configuration space may be fully identified by X_1, X_2, and t, where the generalized coordinates X_1 and X_2 are the respective distances of mass m_1 and mass m_2 from the wall.

On the right half of the figure, the generalized coordinates X_1 and θ locate the positions of both masses. As was true with the cartesian coordinates, in this system three finite variables (X_1,θ, and t) are required to describe the configuration space. When referring to a set of finite variables, it would be redundant to use X_1, X_2, θ, and t since only two generalized coordinates (and t) are necessary.

In general, when manipulating a set of finite variables in any arbitrary coordinate system in order derive an equation or theorem, the following notation of u's is often used to represent the set of variables:

$$(u_1, u_2, u_3, ..., u_N, t)$$

When working with specific problems, however, an appropriate coordinate system must be chosen, and the finite variables would be labeled accordingly.

Infinitesimal displacements are simply infinitely small motions of a particle in the direction indicated. For any finite coordinate u_k, the corresponding infinitesimal displacement is notated by du_k. For example, the infinitely small displacement of a pendulum might be represented by $d\theta$.

Both finite variables and infinitesimal displacements are vital instruments in describing constraints in equation form. Constraints may either contain limitations on finite variables, or they may include restrictions involving infinitesimal displacements. This concept can be seen through examples.

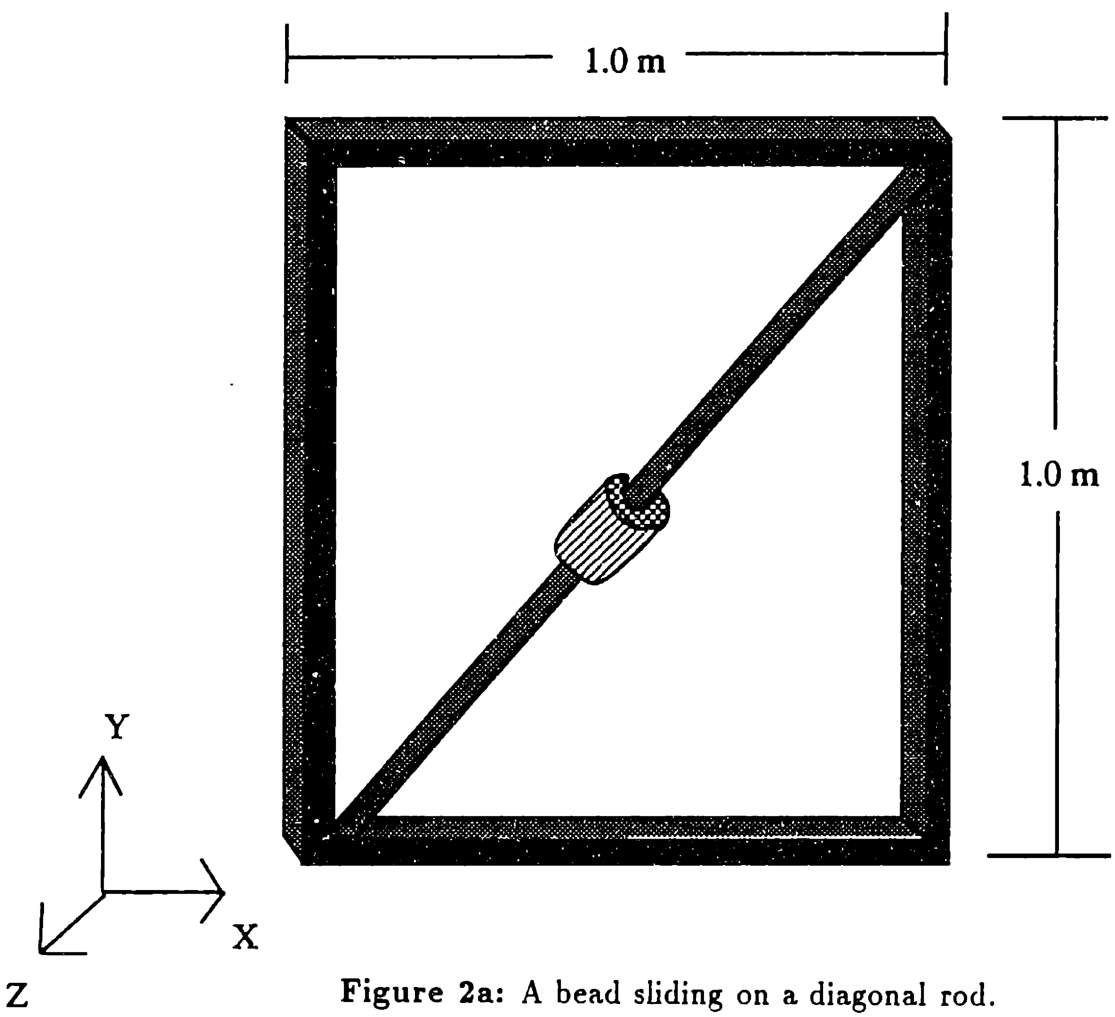

Figure 2a: A bead sliding on a diagonal rod.

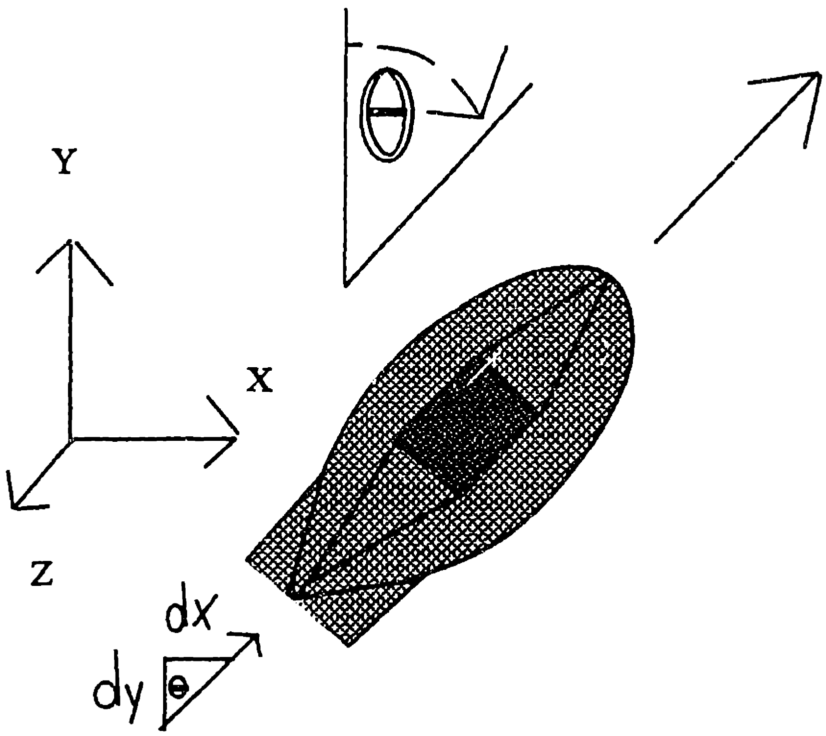

Figure 2b: A boat with forward velocity.

In the first example (Fig. 2a), there is a bead sliding along a diagonal rod across a square frame. One set of generalized coordinates to describe the location of the bead might be X, Y, and Z. In three dimensions, there are two constraints on this system. The first is that the bead is restricted to the plane of the fame. This constraint is represented by the equation:

$$Z = 0$$

The second constraint is that the distance of the bead from the left side of the frame is identical to the distance of the bead from the bottom side of the frame. Once again, this constraint may described in an equation using only the finite variables.

$$X = Y$$

The second example, in Figure 2b, is somewhat more complicated. There is a boat moving in a body of water in this example. The boat is free to turn and move forward, but it is not able to move along any direction other than in the angle of its orientation. This angle θ and the variables x and y can completely describe the location and orientation of the boat. However, there is one constraint in this problem. As shown in the figure, if the boat has an orientation of θ and it moves forward an infinitesimal distance, the infinitesimal displacements will be related by the following geometric equation:

$$\frac{dx}{dy} = tan\theta$$

Unlike the first example, this one involves a limitation on the infinitesimal displacements, not necessarily a restriction of the configuration space. The equation above can be reorganized into the more commonly expressed form below:

$$dx - tan\theta \times dy = 0$$

This particular notation is called the Pfaffian form.

The Pfaffian form is the set of constraint equations which relates infinitesimal displacements and the finite variables in the following format:

$$\sum_{s=1}^{N} A_{rs} du_s + A_r dt = 0 \quad (r = 1, 2, ..., L) \tag{1.1}$$

In this equation, it is assumed that there exist L constraints involving infinitesimal displacements which can be arranged in the form suggested by Equation 1.1, and there are N coordinates of concern. The terms A_{rs} and A_r are functions of the generalized coordinates. For convenience, it is normally desired that all constraints on a system be converted to the Pfaffian form. This is true not only for constraints involving infinitesimal motions but also for those placing restrictions on finite variables. However, it is possible through algebra alone to convert an equation involving only finite variables directly into the Pfaffian form. This statement may be cleared though reexamining the example in Figure 2a.

Consider the bead moving along the diagonal rod. The most obvious constraint on the system was that $X = Y$ at all positions of the bead. However, this constraint also implies that if the bead is displaced by some minute distance, the slope of the motion is equal to one. The displacement in X is equal to the displacement in Y. In mathematical terms:

$$\frac{dy}{dx} = 1 \longrightarrow dx + (-1)dy = 0$$

Therefore, this constraint of finite variables implies a constraint on the displacements, which could be converted to Pfaffian form.

It so happens that all constraints of finite variables similarly imply a constraint of the displacements. Although the conversion of finite restrictions to the Pfaffian form may not always be intuitive, there is a simple way making the conversion mathematically. In multivariable calculus, it is taught that to approximate a change in any function in three dimensional space, the following equation can be used:

$$\Delta F(x,y,z) = \frac{\partial F(x,y,z)}{\partial x} \times \Delta x + \frac{\partial F(x,y,z)}{\partial y} \times \Delta y + \frac{\partial F(x,y,z)}{\partial z} \times \Delta z + g(x,y,z)$$

where $g(x,y,z)$ is a function of higher order differential terms. This equation represents the Linear Approximation Theorem. When calculating with infinitesimal displacements, the changes in the variables are so small that the higher order terms of $g(x,y,z)$ vanish relative to the first derivatives. Applying the Linear Approximation Theorem to a function with some arbitrary number of variables, one would derive the following equation:

$$df(u_1,u_2,...,u_n) \approx \frac{\partial f(u_1,u_2,...,u_N)}{\partial u_1} \times du_1 + \frac{\partial f(u_1,u_2,...,u_N)}{\partial u_2} \times du_2 + \frac{\partial f(u_1,u_2,...,u_N)}{\partial u_N} \times du_N$$

Taking this one step further, if our function of finite variables includes the time variable, there is no mathematical reason that it should be treated differently than the generalized coordinates in differentiating our function. Our general formula for converting a finite constraint to infinitesimals becomes:

$$df(u_1,u_2,...,u_N,t) = \sum_{s=1}^{N} \frac{\partial f}{\partial u_s} \times du_s + \frac{\partial f}{\partial t} \times dt \qquad (1.2)$$

Therefore, by manipulating a constraint of the finite variables to the form $F(u_1, u_2, ..., u_N, t) = 0$, which is not difficult, is is possible to derive a constraint relating infinitesimals. Simply through differentiation by parts, it is possible to discover the Pfaffian form.

The example of the bead on the diagonal rod may be used to demonstrate this technique. From the constraint $X = Y$, we can derive $\partial X - \partial Y = 0$:

$$X = Y \longrightarrow f(X, Y) = X - Y = 0$$

$$d[f(X, Y)] = d[X - Y] = d[0] = 0$$

$$d(X - Y) = \frac{\partial(X - Y)}{\partial X} \times dX + \frac{\partial(X - Y)}{\partial Y} \times dY = dX + (-1)dY = 0$$

The same displacement constraint in Pfaffian form was discovered through differentiation as the one that was concluded through physical intuition. This verifies that it is possible, by use of Equation 1.2, to derive the Pfaffian form quite easily given any constraint.

1.2 Rheonomic and scleronomic constraints

This section will briefly define the terms scleronomic and rheonomic constraints. They refer only to constraints that involve equivalences relating the finite variables. Constraints that include restrictions on infinitessimals displacements that cannot be removed from the constraint equation are neither rheonomic nor scleronomic. For example, the constraint equation:

$$dx - tan\theta \times dy = 0$$

derived from the example in Figure 2b is neither a scleronomic nor a rheonomic constraint. There are infinitesimal displacements that cannot be removed from the equation. It is impossible to convert this particular constraint to a form containing a relation between only the finite variables.

A scleronomic constraint is any constraint establishing some equation between the generalized coordinates. It may be recalled that the time variable t is not a generalized coordinate. Therefore, a scleronomic constraint makes some restriction on the coordinate space which holds true for all time and independently of time. A rheonomic constraint is a constraint which implies some equation between the finite variables including a time dependence. There exists some restriction on the coordinate space which varies with time.

The difference between the two terms may be demonstrated in the following examples:

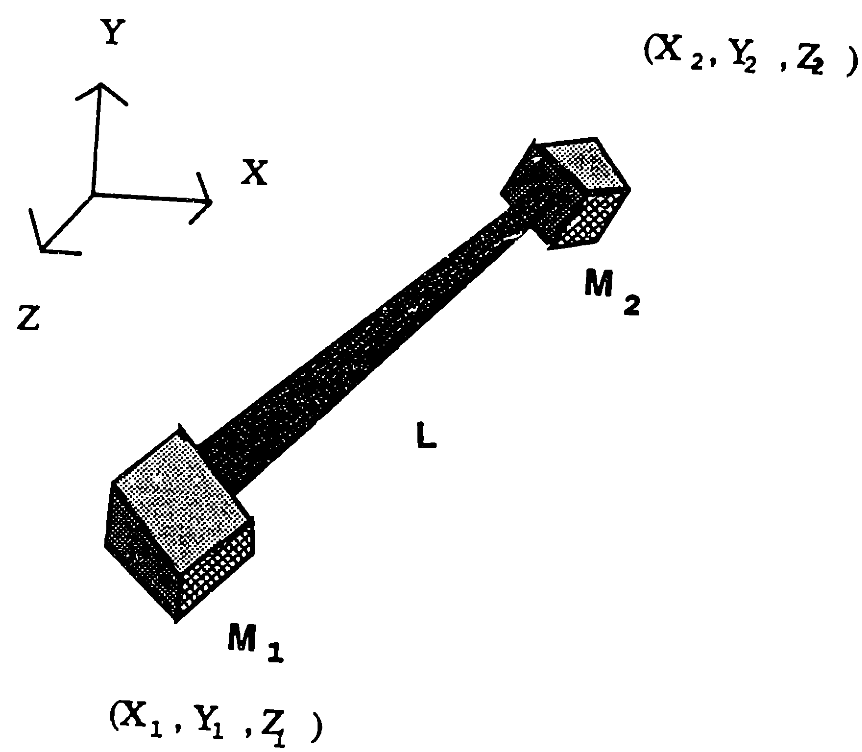

Figure 3a: Two masses connected by a rod of length L.

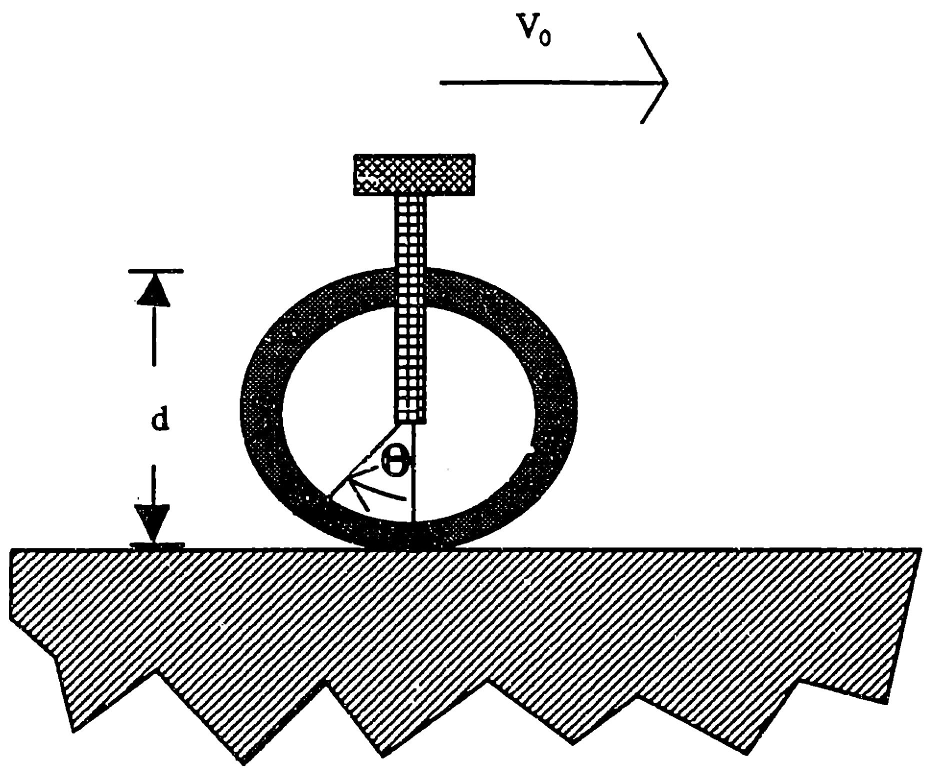

Figure 3b: A unicycle on a flat plane at constant velocity.

In the example in Figure 3a, there are two masses in three dimensional space connected by a rod of length L. The position of the two masses may be identified by the cartesian coordinate (X_1, Y_1, Z_1) and (X_2, Y_2, Z_2) indicating the location of the masses M_1 and M_2 respectively from some reference. The constraint in this problem is that the distance between the two masses is constant at L. In other words,

$$(X_1 - X_2)^2 + (Y_1 - Y_2)^2 + (Z_1 - Z_2)^2 = L^2.$$

This is apparently a scleronomic constraint. There is an equation for the constraint relating the finite variables without including the time variable.

On the other, the example of the unicycle in Figure 3b is different in this respect. The unicycle moves forward at some constant known velocity V_o, while some hole in the tire, initially on the pavement at $\theta = 0$ rotates around the hub of the tire. If the location of the hole is identified by θ, and the diameter of the wheel is d, the following constraint can be inferred:

$$\theta = \frac{V_o t}{d/2} = \frac{2V_o t}{d}.$$

This is, therefore, a rheonomic constraint. There is no way to remove the time variable from the constraint on θ.

1.3 Holonomic versus nonholonomic constraints

One of the most important steps in attempting to solve a dynamical problem with constraints is to determine whether the constraints are holonomic or nonholonomic. In later chapters, it is shown how this identification provides helpful information pertaining to counting the number of degrees of freedom as well as provides clues necessary for deriving the equations of motion. However, for now it will suffice to define and distinguish the two.

A holonomic constraint is any constraint that implies a relation among the finite variables which can be placed into the form as follows:

$$F(u_1, u_2, ..., u_N, t) = 0. \tag{1.3}$$

There may be time dependence in a holonomic constraint, but it is not a requirement. Any constraint on the finite variables alone (as opposed to the infinitesimal displacements), or any constraint which can, through integration or other means, be converted into an equation involving only finite variables, may be organized into the form of Equation 1.3 is holonomic. Consequently, any constraint that is rheonomic or scleronomic is a holonomic constraint, and conversely, any constraint that is neither rheonomic nor scleronomic is not holonomic.

By definition, any constraint that is not holonomic and cannot be arranged into the form of Equation 1.3 is a nonholonomic constraint. The three examples following help to illustrate the difference between holonomic and nonholonomic constraints.

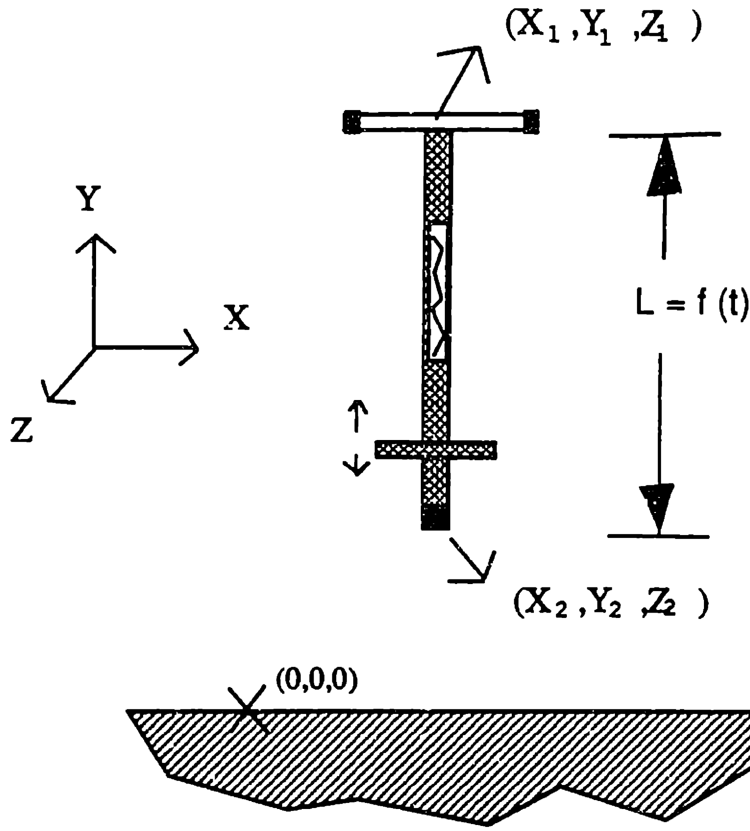

Figure 4a: A pogo stick.

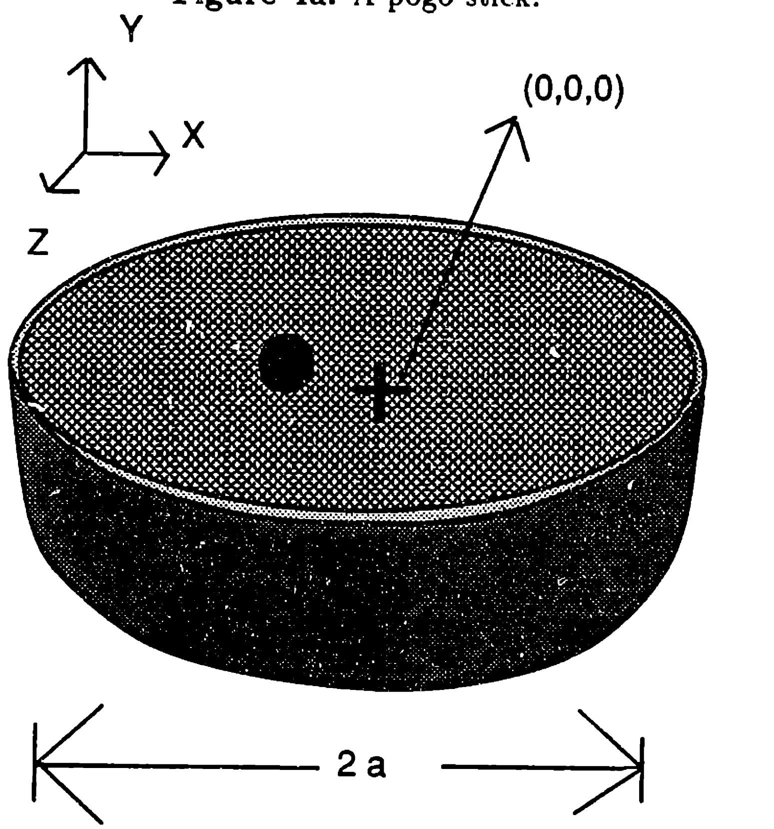

Figure 4b: A marble bounded by a hemispherical bowl.

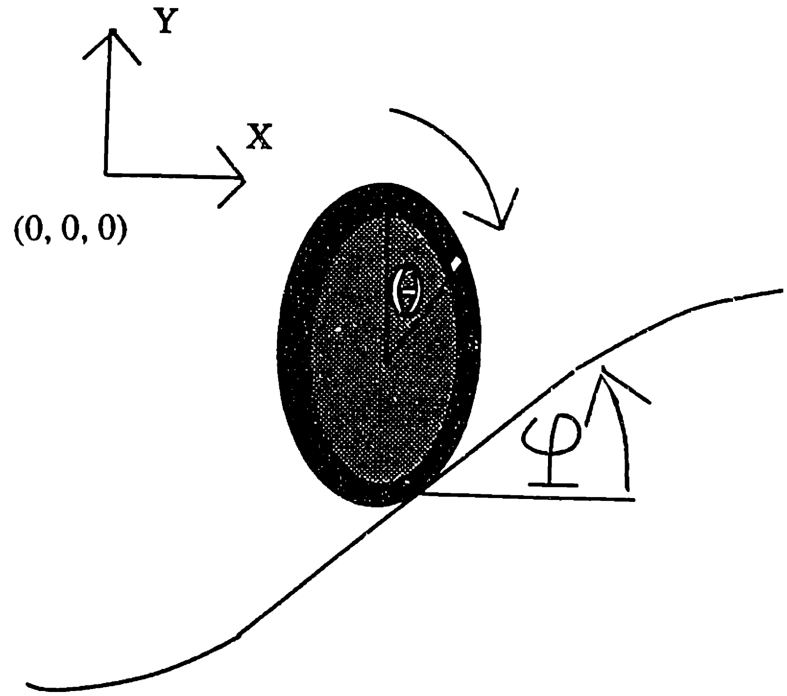

Figure 4c: A coin rolling on a flat surface.

The first example in Figure 4a is a pogo stick. The pogo stick may be considered as a two particle system with the handlebars as being the first particle and the bottom of the stick as particle two. The cartesian coordinates (X_1, Y_1, Z_1) and (X_2, Y_2, Z_2) locate the particles one and two from some reference point. The length of the pogo stick is some known function of time, which will be notated as $f(t)$. The constraint on the locations of particles one and two, respectively, is represented by the equation:

$$f^2(t) = (X_1 - X_2)^2 + (Y_1 - Y_2)^2 + (Z_1 - Z_2)^2.$$

This is clearly a rheonomic constraint. Therefore, it must be holonomic. By moving the terms to one side, the form of the constraint matches that of Equation 1.3.

$$F(X_1, Y_1, Z_1, X_2, Y_2, Z_2, t) = (X_1 - X_2)^2 + (Y_1 - Y_2)^2 + (Z_1 - Z_2)^2 - f(t)^2 = 0.$$

On the other hand, Figure 4b shows a marble constrained to the interior of a bowl. It may be assumed that the bowl is capped, preventing the marble from leaving the interior of the bowl. For simplicity, the location, but not the orientation of the marble is relevant. Therefore, this is a one particle system. If the origin of reference is at the center of the circular lid covering top of the hemisphere as shown, one constraint is that the marble cannot leave the top of the bowl. Another is that it cannot pass though the walls of the hemisphere. Therefore, the constraints for this problem become:

$$Y < 0.$$

$$X^2 + Y^2 + Z^2 \leq a^2.$$

There is no way to eliminate the inequality in these constraints. Consequently, it is not possible to convert either to the form of Equation 1.3. These are nonholonomic constraints. However, these are a special type of nonholomic constraints called piecewise holonomic. When the marble is rolling on the side of the hemisphere, a holonomic constraint is imposed of the form $X^2 + Y^2 + Z^2 - a^2 = 0$. Otherwise, when the particle is floating in the interior of the bowl, no constraint is present. Therefore, if the nonholonomic constraint is broken into cases, the constraint no longer becomes nonholonomic.

However, this is not the case for all nonholonomic systems. Consider the example in Figure 4c of a coin rolling on a flat surface. In this example, (X, Y) defines the location of the center of mass of the coin with respect to some specified point on the plane of the flat surface, as illustrated in Figure 4c. The symbol φ defines the angle of the heading of the coin with respect to the X-axis. θ is the clockwise turn of the coin from vertical. Using these four coordinates, it is possible to fully identify the location and orientation of the coin. Assuming that the coin can turn in φ and rotate in θ but cannot slip, there two constraints pertaining to the infinitesimal displacements of the coin. Headed at a given angle φ, an infinitesimal rotation of the coin will cause the coin to move forward in the X and Y directions by the amounts:

$$dx = a \cos \varphi d\theta.$$

$$dy = a \sin \varphi d\theta.$$

It is impossible, from the two constraints above, to derive a constraint involving the finite variables alone. These two constraints are, therefore, nonholonomic constraints. In future references to constraints in this paper, it will be these types of constraints to which the discussion is referring when the term nonholonomic is used, as opposed to the piecewise holonomic constraints.

A question that often arises is how one can recognize if a nonholonomic constraint is truly nonholonomic. Earlier it was shown that any constraint of finite variables can be converted to an equation involving infinitesimal displacements. However, there is also a way to test if an equation involving infinitesimals can be returned to the form of Equation 1.3. This method is called the integrability test.

In order to test for integrability, one must reorganize the constraint equation of concern into the Pfaffian form, which was defined to be:

$$\sum_{s=1}^{N} A_s du_s + A dt = 0$$

Using this form, it can be proven that if the equation may be integrated so as to eliminate the infinitesimals, the following equation must be satisfied:

$$A_i(\frac{\partial A_j}{\partial u_k} - \frac{\partial A_k}{\partial u_j}) + A_j(\frac{\partial A_i}{\partial u_j} - \frac{\partial A_j}{\partial u_i}) + A_k(\frac{\partial A_i}{\partial u_j} - \frac{\partial A_j}{\partial u_i}) = 0(i, j, k = 1, 2, ..., N). \quad (1.4)$$

In other words, if a constraint is holonomic, then any combonation of three infinitesimal displacements du_i, du_j, and du_k from the Pfaffian equation and their corresponding expressions of finite variables by which they are preceded, A_i, A_j and A_k, must satisfy Equation 1.4 above. In this test, the time variable must be treated as any other.

For example, suppose a constraint was given in the Pfaffian form as follows:

$$X_1 dX_1 + Y_1 dY_1 + Z_1 dZ_1 - f(t)\frac{\partial f(t)}{\partial t}dt = 0.$$

There are four independent variables including time in this example, indicating four tests to see if the constraint is holonomic. (There are four way to select any three variables.) If all four combinations satisfy Equation 1.4, the constraint is holonomic.

$$X_1(\frac{\partial Y_1}{\partial Z_1} - \frac{\partial Z_1}{\partial Y_1}) + Y_1(\frac{\partial Z_1}{\partial X_1} - \frac{\partial X_1}{\partial Z_1}) + Z_1(\frac{\partial X_1}{\partial Y_1} - \frac{\partial Y_1}{\partial X_1}) = 0.$$

$$X_1(\frac{\partial Y_1}{\partial t} - \frac{\partial[f(t)(\frac{\partial f(t)}{\partial t})]}{\partial Y_1}) + Y_1(\frac{\partial[f(t)(\frac{\partial f(t)}{\partial t})]}{\partial X_1} - \frac{\partial X_1}{\partial t}) + [f(t)(\frac{\partial f(t)}{\partial t})](\frac{\partial X_1}{\partial Y_1} - \frac{\partial Y_1}{\partial X_1}) = 0.$$

$$X_1(\frac{\partial Z_1}{\partial t} - \frac{\partial[f(t)(\frac{\partial f(t)}{\partial t})]}{\partial Z_1}) + Z_1(\frac{\partial[f(t)(\frac{\partial f(t)}{\partial t})]}{\partial X_1} - \frac{\partial X_1}{\partial t}) + [f(t)(\frac{\partial f(t)}{\partial t})](\frac{\partial X_1}{\partial Z_1} - \frac{\partial Z_1}{\partial X_1}) = 0.$$

$$Y_1(\frac{\partial Z_1}{\partial t} - \frac{\partial[f(t)(\frac{\partial f(t)}{\partial t})]}{\partial Z_1}) + Z_1(\frac{\partial[f(t)(\frac{\partial f(t)}{\partial t})]}{\partial Y_1} - \frac{\partial Y_1}{\partial t}) + [f(t)(\frac{\partial f(t)}{\partial t})](\frac{\partial Y_1}{\partial Z_1} - \frac{\partial Z_1}{\partial Y_1}) = 0.$$

All four equations are satisfied, and the constraint is verified to be holonomic.

This particular constraint was very similar to the rheonomic example in Figure 4a. Instead of two free points, though, the second point (the lower end of the pogo stick) is fixed at the origin. This gives the constraint equation:

$$f^2(t) - X_1^2 + Y_1^2 + Z_1^2 = 0.$$

Through differentiation, it is easy to show that the pogo stick with a fixed end was indeed identical to the Pfaffian constraint just tested.

However, consider one of the constraint equations from the nonholonomic constraint of example 4c. For example, the constraint $dx - a\cos\varphi d\theta = 0$ might be tested by Equation 1.4. Since there are three independent finite variables involved in this constraint equation, all three must have infinitesimal displacements, dx, $d\theta$, and $d\varphi$, which correspond to the terms du_j, d_k, and du_l, respectively, from Equation 1.4. The constraint equation may be written in term of these three infinitesimal displacements in the form $(1)dx + (-a\cos\varphi)d\theta + (0)d\varphi = 0$. This implies that the term A_j is (1), A_k is $(-a\cos\varphi)$, and A_l is (0). Substituting these terms back into Equation 1.4 gives the result:

$$1(\frac{\partial(-a\cos\varphi)}{\partial\varphi} - \frac{\partial 0}{\partial\theta}) - a\cos\varphi(\frac{\partial 0}{\partial X} - \frac{\partial 1}{\partial\varphi}) + 0(\frac{\partial 1}{\partial\theta} - \frac{\partial(-1\cos\varphi}{\partial X}) = -a\sin\varphi \neq 0.$$

Therefore, the integration test verifies that the constraint in example 4c is nonholonomic.

Chapter 2

Degrees of Freedom

Finding whether a constraint is holonomic or not plays a significant role in determining different features of a dynamical system, such as the number of degrees of freedom and the number of possible infinitesimal displacements that a particle in a system can move. It also helps in determining if a system is holonomic or nonholonomic.

A holonomic system is a system that contains either only holonomic constraints or no constraints at all. Therefore, any problem that contains one or more constraints which are not holonomic is called a nonholonomic system. The nonholonomic systems cover a wide range of problems. For example, the simple spring-mass system in Figure 5 is one example.

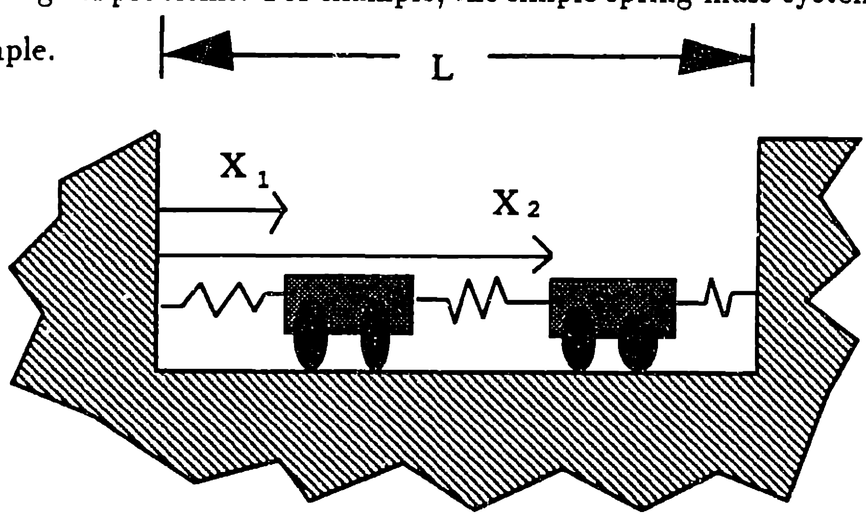

Figure 5: A linear two mass-spring system.

The location of the two masses are indicated by the distances X_1 and X_2 from the left wall. There are three nonholonomic constraints as follows:

$$X_1 > 0, \quad X_1 < X_2, \quad and X_2 < L$$

Therefore, technically this is a nonholonomic system. Nevertheless, in this chapter any further reference to a nonholonomic system refers to a system with a constraint that implies a nonintegrable equality. The example of the rolling coin was one such nonholonomic system.

Although this line of discussion may seem useless in any practical sense, in this chapter it will be shown that by determining whether the constraints are holonomic or not, one can calculate the number of degrees of freedom. It will be proven how holonomic constraints restrict the accessibility of the configuration space, whereas the nonholonomic constraints may only restrict the motion of the system at any given time.

Next, this chapter will demonstrate how complex treatment of constraints in the case of a holonomic system may be avoided through certain shortcuts. The reduction by one of the number of finite variables is often equivalent to the effects of a holonomic constraint. In other words, holonomic constraint equations, in some cases, may be eliminated by means of some coordinate conversion or substitution, resulting in the elimination of a generalized coordinate. Although this trick is not valid for nonholonomic constraints, it becomes very convenient in solving holonomic systems.

2.1 Effects on the degrees of freedom

With the new vocabulary from chapter one, it is possible to closely examine the effects of constraints upon the number of degrees of freedom. This exercise may be useful in accounting for the number of equations and independent variables for which one must solve in order to give a complete solution of the dynamics of a system. When a constraint limits the dimensions of the motion of a system, often fewer equations of motion are required to identify the exact location and motion of a system. This section will show that the reduction of the state equations depends, however, if the constraint is holonomic or not.

In the physical world of three dimensions, any system of N particles, free from constraints, must have $3N$ degrees of freedom. Admittedly, earlier in the paper a theorem was mentioned stating that any system of two particles or more has some type of constraint. Nevertheless, if the system is piecewise-unconstrained, it will take $3N$ finite variables to fully locate a system. The term piecewise-unconstrained means that a system, in its current position, faces no restrictions of motion.

In the case of a constrained system, the following theorem relates the number of holonomic constraints to the number of degrees of freedom:

> In a holonomic system with k constraints, there need be only $3N - k$ independent variables to completely describe the configuration space of the system.[2]

This theorem can be verified both through algebra and by example. In fact, more generally, it can be proven that any holonomic constraint subtracts one degree of freedom from the system. After proving theorem 2, the effects of nonholonomic constraints on the number of free variables will be examined.

To begin, one may recall that any holonomic constraint can be converted to the form in Equation 1.4. For the sake of discussion, the following formula will represent a specific constraint a certain dynamic system:

$$f(x_1, x_2, ..., x_N, t) = 0.$$

What this means is that a certain constraint equation gives some relationship of equality between some number of the generalized coordinates. Not all, but at least some of the coordinates must be involved in the equation. One can, therefore, rewrite the constraint equation into the following:

$$f(x'_1, x'_2, ..., x'_Z, t) = 0$$

where only the variables actually affected by the constraint equation are included, each given a corresponding primed variable. Therefore, since variables which are independent of the constraint were not given a primed variable, $Z \leq N$. Now it can be observed that the value of any particular variable x'_m, where $1 \leq m \leq Z$, can be equated to some function of the remaining $Z - 1$ primed variables. The original constraint equation may now be reorganized to the form:

$$x'_m = g(x'_1, x'_2, ..., x'_{m-1}, x'_{m+1}, ...x'_Z, t) = 0. \tag{2.1}$$

Similarly, if there existed a second constraint equation which could be written as a function of Y finite variables, one could similarly revise it to the form:

$$x''_m = h(x''_1, x''_2, ..., x''_{m-1}, x''_{m+1}, ...x''_Y, t) = 0. \tag{2.2}$$

The double prime marks indicate that the variables in this constraint are different from those related in the first constraint. If one substitutes the Equation 2.1 from the first constraint into the state equations of the response of the system everywhere that an x'_m arises, the result will be the elimination of the generalized coordinate x''_m. Furthermore, if the constraints in Equation 2.2 is replaced into the state equations everywhere the variable x''_m occurs, assuming $x''_m \neq x'_m$, yet another variable will have been eliminated. If by coincidence $x''_m = x'_m$, then Equation 2.2 may simply be rearranged equating x''_{m-1} to a function of the remaining variables of the second constraint. By this process, one can always eliminate k variables with k holonomic constraints, assuming that there exists fewer constraints then the largest number of variables related by those constraints. If, however, there are more constraints than related coordinates, there must either be a contradiction among the equations, or else there is a redundant constraint equation. This fact comes from the algebraic rule by Gibbs that k equations suffice to solve for k unknowns. Therefore, since each constraint equation eliminates one finite variable, it is verified that a system with N particles and k holonomic constraints must have $3N - k$ dimensions in the configuration space

An example can be used to demonstrate how, by substituting a holonomic constraint into a state equation, it is possible to reduce the number of free variables involved. A state equation is a second order differential equation giving the motion of the system in terms of the generalized coordinates. Consider the problem of a spring pendulum sliding on a horizontal bar as shown here:

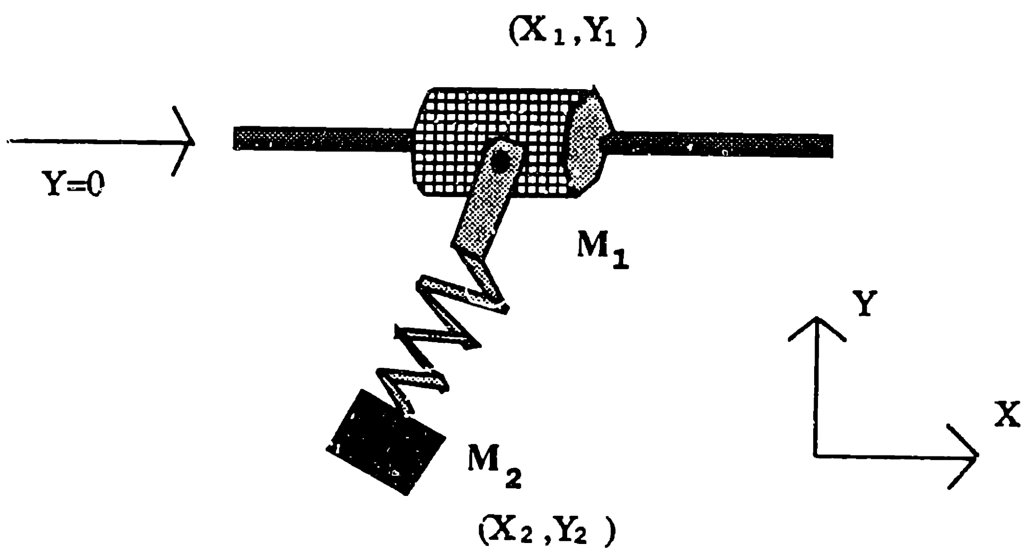

Figure 3: A sliding spring pendulum.

One mass is constrained to slide along a beam while the other, connected to the first by a linear spring, is free to oscillate in the X-Y plane. The location of the sliding and free mass are specified by X_1, Y_1 and X_2, Y_2 respectively. For simplicity, this problem will be treated as a two-dimensional problem with one constraint. That constraint is simply $Y_1 = 0$. In fact, for accounting purposes, there are two additional holonomic constraints coming from the fact the system is to be treated in two dimensions. They are $Z_1 = 0$ and $Z_2 = 0$, where Z is the perpendicular distance of a particle from the plane of the pendulum. Consequently, one would expect:

$$3N - k = 3 \times 2 - 3 = 3 \ \textit{degrees of freedom}.$$

To derive the state equations which will be consequently simplified, one can first ignore the single constraint in our two dimensional system. Using the Lagrangean techniques taught in introductory dynamics classes, it should be possible to derive four differential equations of motion. The four expressions necessary to use this technique are as follows:

$$InertialForces:$$

$$\mathbf{T}^{*} = \frac{1}{2}M_1(\dot{X}_1^{\,2} + \dot{Y}_1^{\,2}) + \frac{1}{2}M_2(\dot{X}_2^{\,2} + \dot{Y}_2^{\,2})$$

$$Conservativeforces:$$

$$\mathbf{V} = M_1 g Y_1 + M_2 g Y_2 + \frac{1}{2}k((X_1 - X_2)^2 + (Y_1 - Y_2)^2)$$

$$Nonconservativeforces:$$

$$\Xi_{\mathbf{X_1}} = \Xi_{\mathbf{X_2}} = \Xi_{\mathbf{Y_1}} = \Xi_{\mathbf{Y_2}} = 0$$

$$Lagrangian:$$

$$\mathbf{L} = \mathbf{T}^{*} - \mathbf{V} = \frac{1}{2}M_1(\dot{X}_1^{\,2} + \dot{Y}_1^{\,2}) + \frac{1}{2}M_2(\dot{X}_2^{\,2} + \dot{Y}_2^{\,2} - M_1 g Y_1 - M_2 g Y_2 - \frac{1}{2}k((X_1 - X_2)^2 + (Y_1 - Y_2)^2)$$

Therefore, after substituting the Lagrangian into Lagrange's differential equations, (which are $\frac{d}{dt}(\frac{\partial L}{\partial \dot{u}_s}) - \frac{\partial L}{\partial u_s} - \Xi_{u_s} = 0$), the following state equations of motion are derived:

$$M_1\ddot{X}_1 + k(X_1 - X_2) = 0. \tag{2.3}$$

$$M_2\ddot{X}_2 + k(X_2 - X_1) = 0. \tag{2.4}$$

$$M_1\ddot{Y}_2 + k(Y_2 - Y_1) + Mg = 0. \tag{2.5}$$

$$M_2\ddot{Y}_1 + k(Y_1 - Y_2) + Mg = 0. \tag{2.6}$$

Now, it may be recalled that by arranging the constraint equation $Y_1 = 0$ to the form of Equation 2.1, which happens to be $Y_1 = 0$, one can place the constraint into the state equations. In doing so, both the number of differential equations as well as the number of independent variables has dropped by one. The new set of state equations includes the equations 2.3 and 2.4 as is. Equation 2.5 is modified through substitution of the constraint equation, resulting in the derivation of an equation no longer containing the variable Y_1:

$$M_1\ddot{Y}_2 + kY_2 + Mg = 0.$$

Finally, the fourth equation of the set of differential state equations (Equation 2.6) may be omitted since it is unnecessary as well as incorrect. There is no way to have an equation including the second derivative of Y_1 when Y_1 is fixed. The constraint on Y_1 invalidates the use of the Lagrange's equation on that variable. In later chapters, it will be discovered that the reason for this discrepancy is that the constraint implies some nonconservative force holding the coordinate fixed. In conclusion, substitution of a holonomic constraint into the state equation reduced the number of equations of motion by one.

2.2 Reduction of finite variables

Although the technique of substitution of a holonomic constraint in the form of Equation 2.1 has been a proven method of eliminating one of the variables involved in the state equations, it is by no means the easiest. In the case of many constraints, the reduction of the degrees of freedom can be accomplished by simply removing a variable and replacing it with some constant or function of time. This type of simplification may be done preceding any other. For example, in the previous example, the system was restricted to two dimensions. Inadvertently, the Z coordinates were removed from the equationns and replaced with zeros.

Consider the example in the following sketch (Figure 7). This system is single particle sliding on a vibrating table:

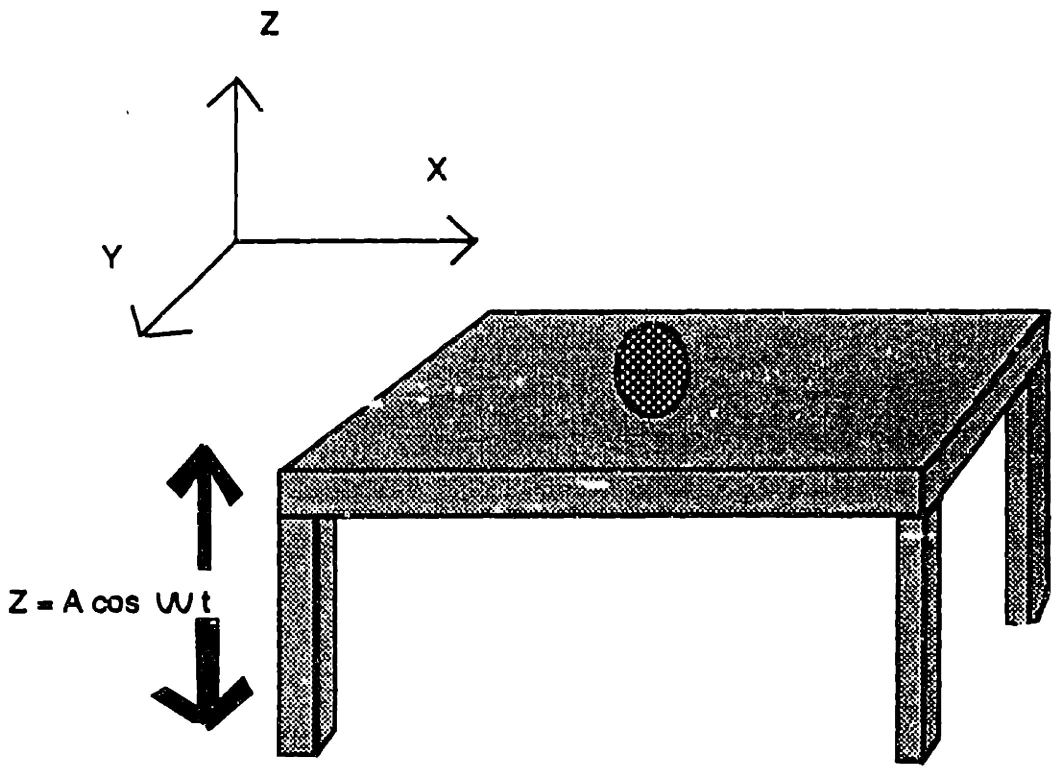

Figure 7: A particle sliding on a table.

The table is oscillating at a frequency of ω at an amplitude of A. Instantly, one can recognize the constraint $Z = A\cos\omega t$. Rather than include the vertical locator Z in the derivation of the state equations and then eliminate it, one can replace all occurrences of the variable by the time function from the start. The variable $\dot{Z}$ would be replaced by $-At\omega\sin\omega t$. The derivation of the two state equations would, therefore, be as follows:

$$L = \frac{1}{2}M(\dot{X}^2 + \dot{Y}^2 + \dot{Z}^2 - MgZ.$$

$$L = \frac{1}{2}M(\dot{X}^2 + \dot{Y}^2 + (-At\omega\sin\omega t)^2) - Mg(A\cos\omega t).$$

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{X}}) - \frac{\partial L}{\partial X} = \Xi_X \Rightarrow \ddot{X} = 0.$$

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{Y}}) - \frac{\partial L}{\partial Y} = \Xi_Y \Rightarrow \ddot{Y} = 0.$$

In this example, the constraint on Z resulted in transforming the system into a two variable problem. Since Z was some known function of time regardless of the coordinates, it could be replaced by its known response. This is quite often the case in holonomic constraints.

However, in many cases, a coordinate conversion is necessary to remove a variable from a Lagrangian. Changing the set of coordinates can sometimes reform a constraint in the form of Equation 2.1 into a constraint in which one variable is constant or known time function. In such a case, that variable can be more easily replaced by its known value and therefore be eliminated from the list of independent coordinates. The following example of the bead and ring is a good example of how a convenient conversion of coordinates can facilitate reducing the number free variables of a system.

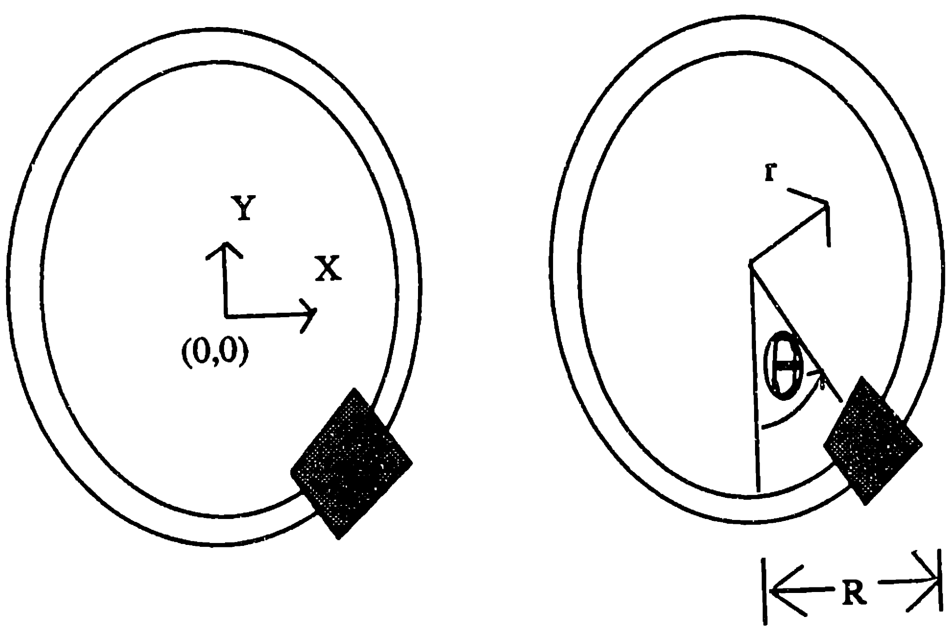

Figure 8: A bead sliding on a vertical ring: two coordinates.

Suppose the cartesian coordinate system indicated on the left half of Figure 8 was originally selected to locate the position of the bead. The Lagrangian could easily be derived in this coordinate system to be the following:

$$\mathbf{L} = \mathbf{T}^* - \mathbf{V} = \frac{1}{2}\mathbf{M}(\dot{\mathbf{X}}^2 + \dot{\mathbf{Y}}^2) + \mathbf{MgY}.$$

However, there exists nonconservative forces that create a hindrance with this coordinate system. The constraint created by the rigid hoop is that the distance of the bead from the center of the hoop is constant. In other words $X^2 + Y^2 - R^2 = 0$. The nonconservative forces coinciding with this constraint are that the hoop causes a normal force at the contact point of the bead and the hoop. This complicated force may be notated as $F(X, Y, \dot{X}, \dot{Y})$. This force must be broken into vertical and horizontal components, F_X and F_Y. Consequently, the two following state equations are derived.

$$M\ddot{X} = F_X(X, Y, \dot{X}, \dot{Y}).$$

$$M\ddot{Y} + Mg = F_Y(X, Y, \dot{X}, \dot{Y}).$$

Now, there are two state equations, two free variables, and two unknown nonconservative forces involved. The problem has gotten unnecessarily confusing. In order to convert the two equations containing two variables into one equation with one variable, one could first establish a geometric relationship with F_X and F_Y, which will be shown in later chapters. After that, the two state equations can be combined. Next, the constraint equation, in the form $X = \sqrt{L^2 - Y^2}$, can reduce the problem to a single variable (Y), single state equation problem.

However, rather than do this complex manipulation, the cylindrical coordinates, on the right sketch of Figure 8 may facilitate the derivation of a single state equation. The constraint is now simply $r = R$. This type of constraint problem may be solved by replacing the constant R wherever the variable r would occur. This type of process of variable reduction was just explained just earlier. There are also no nonconservative forces of concern (the normal forces from the hoop are in the r direction, which is known already). Therefore, in cylindrical coordinates, the state equations are easily solved:

$$L = \frac{1}{2}M((r\dot{\theta})^2 + \dot{r}^2) + Mgr(1 - \cos\theta) = \frac{1}{2}M(R\dot{\theta})^2 + MgR(1 - \cos\theta).$$

$$Mr\dot{\theta}^2 + Mg\sin\theta = 0.$$

Through a coordinate conversion, the simplification of the system to a single degree of freedom was much easier that it might otherwise have been.

2.3 Nonholonomic system

Whereas this chapter has primarily served in demonstrating how holonomic constraints affect the degrees of freedom of a system, the discussion may be extended to nonholonomic systems. The holonomic constraints both restricted the degrees of freedom and dimensions of the configuration space. However, there is a distinction between the degrees of freedom and the dimensions of configuration space. The number of the dimensions of configuration space is the minimum number of independent coordinates required to locate a system. On the other hand, the number of degrees of freedom pertains to the possible directions of motion of a system. It is equal to the minimum number of infinitesimal displacements needed to identify any infinitely small motion in the system. All constraints, by definition of a constraint, must either restriction the motion or the location of a system (or both).

Now, one can relate the discussion of holonomic constraints with nonholonomic systems. If there exists a nonholonomic constraint, it cannot (see definition of nonholonomic) be converted to the form:

$$f(u_1, u_2, ...u_N, t) = 0.$$

Now suppose it would be possible to define one variable in terms of the others from the nonholonomic constraint. In other words, suppose there exists a nonholonomic constraint that could be manipulated into the following configuration:

$$X_M = g(X_1, X_2, ..., X_{M-1}, X_{M+1}, ..., X_N, t)$$

This was the form that allowed the elimination of the number of dimensions of the configuration by a holonomic constraint by substitution. However, subtracting the X_M term from both sides of the equation implies that:

$$g(X_1, ..., X_{M-1}, X_{M+1}, ..., X_N, t) - X_M = f(u_1, u_2, ...u_N, t) = 0.$$

This means that the constraint must have been holonomic if it could be substituted into the state equations to eliminate the number of free variables. Therefore, there is no mathematical way to eliminate the dimensions of the configuration space with nonholonomic constraints. Nevertheless, as stated before, a constraint restrains either the motion or the configuration space of a system. For a nonholonomic system, then, the restriction must be on the degrees of freedom of motion. It is not easy to prove this fact. However, one can get an intuitive sense that any constraint (except for inequalities or other discontinuities) will restrict all motion normal to the constraint. For example, the constraint $dx = dy$ prevents any infinitesimal motion perpendicular to the $x = y$ direction. This must reduce the degrees of freedom by one. This effect can be seen, once again, in the case of the rolling disk.

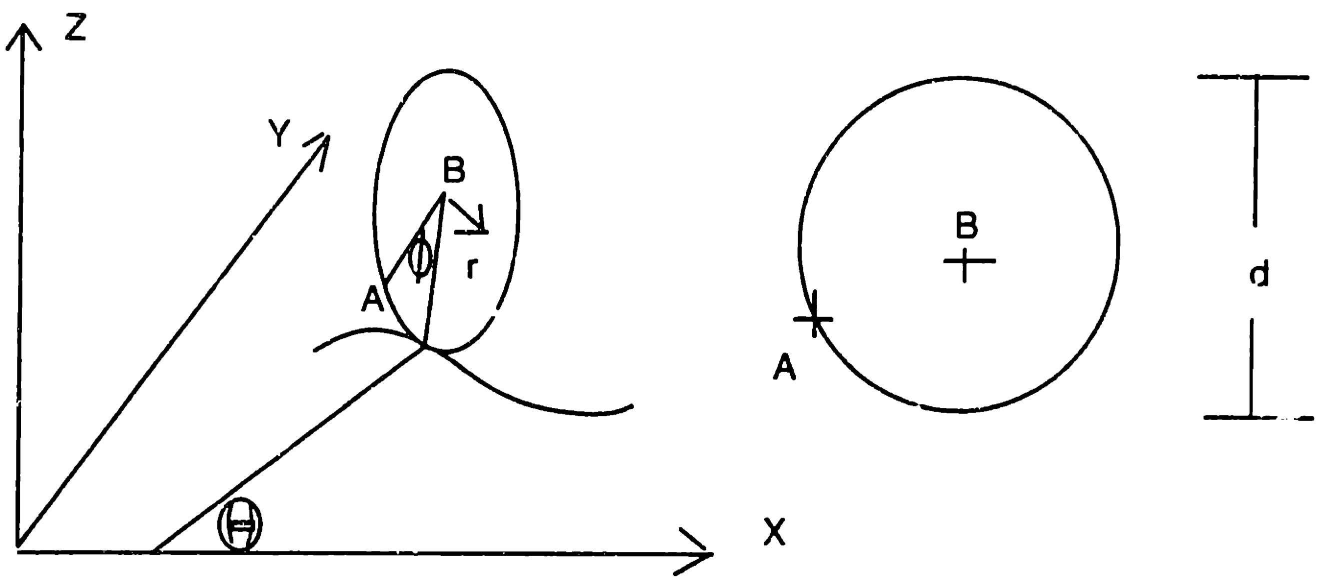

Figure 9: The rolling disk problem.

This was a nonholonomic system. The rolling disk with no slip is actually a two particle system. This can be seen from the sketch to the right. One can fully specify the coordinates of the system by locating points A and B in three dimensions. Ignoring constraints, this specification can be done with $3N = 3 \times 2 = 6$ variables. The coordinates of B may be specified in cartesian coordinates relative to some fixed point of the planar surface. On can use spherical coordinates relative to B to locate the position of A, some fixed point on the edge of the disk. This will be the equivalent of the coordinate system indicated in the left sketch of Figure 9. There are two holonomic constraints on the configuration space of particles A and B.

$$Z = \frac{d}{2}; \; r = \frac{d}{2}$$

This leaves $3(N) - k = 3 \times 2 - 2 = 4$ independent variables. As from the technique in the last section, the variables Z and r may be replaced by their constant variables, leaving X, Y, θ, and ϕ to describe the configuration space. There are also two nonholonomic constraints in this system:

$$\dot{X} = r\dot{\phi}\sin\theta = \frac{d}{2}\dot{\phi}\sin\theta.$$

$$\dot{Y} = -r\dot{\phi}\cos\theta = -\frac{d}{2}\dot{\phi}\cos\theta.$$

Now, there are four independent variables and two nonholonomic constraints. Despite the two constraints of infinitesimal motion, the constraints do not limit the number of variables necessary to locate the particles A and B. Over some period of time any set of X, Y, θ and ϕ may coincide with the location of the system. However, the degrees of freedom has been reduced to $4 - 2 = 2$ degrees of freedom by the two nonholonomic constraints. This is consistent with one's intuition. The disk is free to roll and to pivot, but nothing else. It is the angles of the rolling ϕ and pivoting θ which cause the center of the disk X, Y to move to a specific corresponding location. In conclusion, all constraints of which imply that a certain equality holds reduces the degrees of freedom by one, whereas only the holonomic constraints also reduce the number of independent variables describing the configuration space.

Chapter 3

Constraint forces

For the purpose of studying the degrees of freedom of a system and its configuration space, it is sufficient to treat constraints as an equality between some function of the finite and infinitesimal variables. On the other hand, when attempting to actually derive a set of differential equations defining the motion of a system, one must take another perspective. In the previous chapter, there were several examples in which, through some substitution of variables or conversion of coordinate systems, it was possible to derive the set of state equations directly from the constraint equations. However, the examples in which this was possible were somewhat contrived. In more complex system and in nonholonomic systems, difficulties arise in this approach.

However, there is a systematic way of incorporating constraints into the differential equations of motion. To explain this process is, in fact, the primary goal of this thesis. In order to understand this method, one must first understand the perspective of constraints being treated as indeterminate forces acting to hold the system to some restricting equation. As a result of this type of reasoning, the constraint suddenly becomes the equivalent of a nonconservative force which may be included into Lagrange's equations. This reduces the problem of solving any system containing constraints into a routine mathematical procedure.

In this chapter, first some definitions pertaining to displacement vectors will be given. The concepts of actual, possible and virtual displacement vectors will be introduced. These definitions will be useful in the discussion of constraint forces.

Afterwards the intuition behind treating a constraint as a nonconservative force will be given. Some examples should help to both justify the conversion of constraint equations to forces as well as demonstrate the process of doing so. The definition of the Lagrange multiplier will conclude this chapter. The Lagrange multiplier is the tool that will enable the inclusion of the constraint forces into the equations of motion. Actual demonstration of the solution of a system using the Lagrange multiplier will be reserved for the following chapter.

3.1 Displacements

In order to understand the concept of Lagrange multipliers, a knowledge of the three categories of displacements is a prerequisite. These terms often appear in discussions of restricted motion of a system.

An actual displacement is the set of functions which satisfy both the equations of motion and the constraints. The vector of the following form which specifies these functions for all coordinates is called the actual displacement vector:

$$U(t) = (u_1(t), u_2(t), ..., u_N(t)) \tag{3.1}$$

The components of the actual displacement vector are functions of time since the vector locates the position for all time. Therefore, all information and physical properties of the system, including initial and boundary conditions, must be known in order to derive the actual displacement vector. A traveling mass spring example may be used to demonstrate this definition.

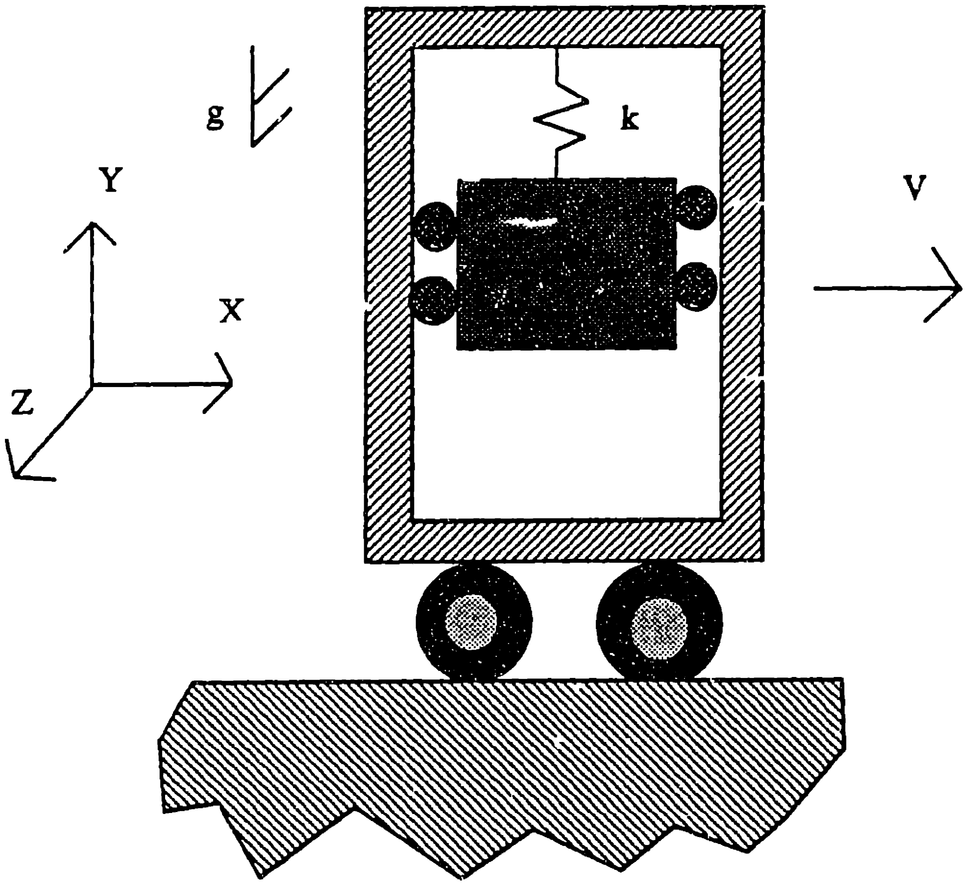

Figure 10: A horizontally moving guided mass-spring system.

In the above example, there is a mass guided by frictionless rollers hanging vertically by a spring. The entire system is moving in the positive X direction at the velocity of $V = 1$. Finding the X and Z components of the actual displacement is trivial since the constraint imposed by the rollers is $Z = 0$ and $X = Vt = t$. On the other hand, in order to find the Y component, one must solve the differential equation:

$$M\ddot{Y} + kY + Mg = 0$$

If the mass moves at $\dot{Y_0}$ from the displacement Y_0 at time zero, the displacement vector is found to be:

$$\vec{U}(t) = t\hat{i} + [Y_0 \cos\sqrt{\frac{k}{M}}t + \frac{\dot{Y_0}}{\sqrt{\frac{k}{M}}}\sin\sqrt{\frac{k}{M}}t - \frac{Mg}{k}]\hat{j} + 0\hat{k}$$

This is a very convenient form of displacement vector, since it already gives the precise location of the mass at all time. In most problems of dynamics, one is not given the actual displacements, however, since they are usually what one is trying to derive in the first place.

Instead one is often given specifications in a problem which enables the derivation of the possible displacements. The possible displacements is any set of infinitesimal displacements that satisfy all of the constraints in the Pfaffian form, which has been defined to be:

$$\sum_{s=1}^{N} A_{rs} du_s + A_r dt = 0 (r = 1, 2, ..., L).$$

Any particular set of possible displacements may be written in the following vector notation:

$$d\vec{U} = (du_1, du_2, ..., du_N). \tag{3.2}$$

Suppose, in a given dynamical system, it can be determined that the system is restricted to follow the following constraint:

$$f(u_1, u_2, ..., t) = 0.$$

This constraint must be converted into Pfaffian form through differentiation, as demonstrated in Chapter 1. One can simply input the function $f(u_1, u_2, ..., t)$ into equation 1.2. A possible displacement vector is any set of functions of finite variables which, when substituted into its corresponding infinitessimal displacement in the Pfaffian form, obeys that Pfaffian equation. Consider the example in Figure 10 of the moving mass spring system. When converted to Pfaffian, the following restrictions are placed on the possible displacement vector:

$$f(u_1, u_2, ..., t) = X - Vt = X - t = 0$$

$$\sum_{s=1}^{N} \frac{\partial f}{\partial u_s} \times du_s + \frac{\partial f}{\partial t} \times dt = \frac{\partial (X - t)}{\partial X} dX + \frac{\partial (X - t)}{\partial t} dt = dX - dt = 0.$$

$$g(u_1, u_2, ..., t) = Z = 0.$$

$$\sum_{s=1}^{N} \frac{\partial g}{\partial u_s} \times du_s + \frac{\partial g}{\partial t} \times dt = \frac{\partial Z}{\partial Z} dZ = dZ = 0.$$

The possible displacement vector, therefore, covers a large variety of vectors. The actual displacement vector is just one of many possible displacement vector. In the case of the moving oscillator, one possible displacement vector could be the actual displacement vector:

$$d\vec{U} = (dX, dY, dZ) = (t, [Y_0 \cos\sqrt{\frac{k}{M}}t + \frac{\dot{Y}_0}{\sqrt{\frac{k}{M}}} \sin\sqrt{\frac{k}{M}}t - \frac{Mg}{k}], 0)$$

One might notice that the actual displacement vector does satisfy the equations $dX - dt = 0$ and $dZ = 0$. However, the possible displacement vector can be anything that obeys the two constraints. Although every actual constraint is a possible constraint, the reverse is not true. The possible displacements may be independent of initial conditions, as well as any specifics about the components of the system, such as the spring constant, the damping, and the mass values. For example, the Pfaffian constraints of guided mass spring system would also be satisfied by the possible displacement vector $(dX, dY, dZ) = (17, t^2, 0)$. Since dt may be set at any value in an attempt to overcome the restrictions on the possible displacement vectors, in this example $dt = 17$. Therefore, the Pfaffian equation $dX - dt = 17 - 17 = 0$ is satisfied. So long as the components of a vector satisfy $dX - dt = 0$ and $dZ = 0$, the conditions for a possible displacement are satisfied.

Finally, the term virtual displacements must be defined in order to study Lagrange multipliers. The virtual displacements are defined to be the set of infinitesimal displacements which satisfy a set of equations of constraint, transformed from the original Pfaffian form into the set of time-independent equations:

$$\sum_{s=1}^{N} A_{rs} du_s = 0 \quad (r = 1, 2, ..., L).$$

The virtual displacements give the set of displacements consistent with the geomtric constraints at any fixed time. The modified Pfaffian form differs from the complete form of Equation 1.1 in that the time increment dt is set to zero. As may be recognized, the modified set of Pfaffian equations may not even be consistent with the actual constraint equations since the time increment dt was simply erased from the Pfaffian form. Nevertheless, any set of virtual displacements must satisfy the equations $\sum_{s=1}^{N} A_{rs} du_s = 0$. A complete set of virtual displacements may be used to compose the virtual displacement vector. The virtual displacement vector is typically notated in the following way:

$$\delta\vec{U} = (\delta u_1, \delta u_2, ..., \delta u_N). \tag{3.3}$$

Once again, this displacement vector also is a set of infinitesimal displacements, this time labeled by δu's. However, the constraints to which the displacements must be limited are different concerning the virtual and possible displacements. The two types of displacement vectors, in general, do not coincide. Only if each of the constraints on a system is time independent will the virtual and possible displacement vectors be identical. For example, the example from Figure 10, which has the rheonomic constraint $dX - dt = 0$, will have different virtual and possible displacement vectors. Dropping the time factor $A_t dt$ from the constraint, which is $-dt$, would leave the constraint equation:

$$dX = 0.$$

Now, the vector $(\delta X, \delta Y, \delta Z) = (17, t^2, 0)$ does not obey the constraint $dX = 0$ and is therefore not a virtual displacement vector. Instead, a virtual displacement vector might look something like $(0, t^2, 0)$. As is the case with this particular example, the actual displacement vector might not be a virtual displacement vector. The difference between the two types of constraints reflects the effect of time dependence on the displacement of the mass in Figure 10. The possible displacement vectors give the possible directions that the mass may move in time. Since the system moves horizontally as the masss is free to move in the vertical direction, any vector in the X-Y plane is a possible displacement vector. On the other hand, the virtual displacement vectors give the possible motion of the system in fixed time. In fixed time, the system does not move horizontally, but the mass is not restricted in the vertical direction. Therefore, $\delta X = \delta Z = 0$ in virtual displacement vectors.

3.2 Constraint forces and the Lagrange multiplier

Now that the necessary terminology for understanding the various displacement vectors has been defined, it is possible to introduce the concept of constraint forces. By treating constraints as forces on a system, one can comprehend the Lagrange Multiplier rule, a fail-proof rule which enables the solution of any problem involving constraints.

In previous chapters, constraints have been defined as restrictions on the degrees of freedom or on the dimensions of the configuration space. In this perspective, there appears to be some abstract boundaries or impenetrable planes which restricts the motion of the system. In reality, however, there are no imaginary boundaries which cause the constraints. Instead, constraints might be the results of rigid bodies external to the system or the physical properties of the surroundings of the system limiting a particle's motion. These surroundings must impose forces in order to affect the motion of a system. Although, these external forces may vary in time with the motion of the system, if the forces were ignored completely, the system would experience no constraints.

An example of a constraint which may be perceived as the result of forces may be seen in the following example in Figure 11:

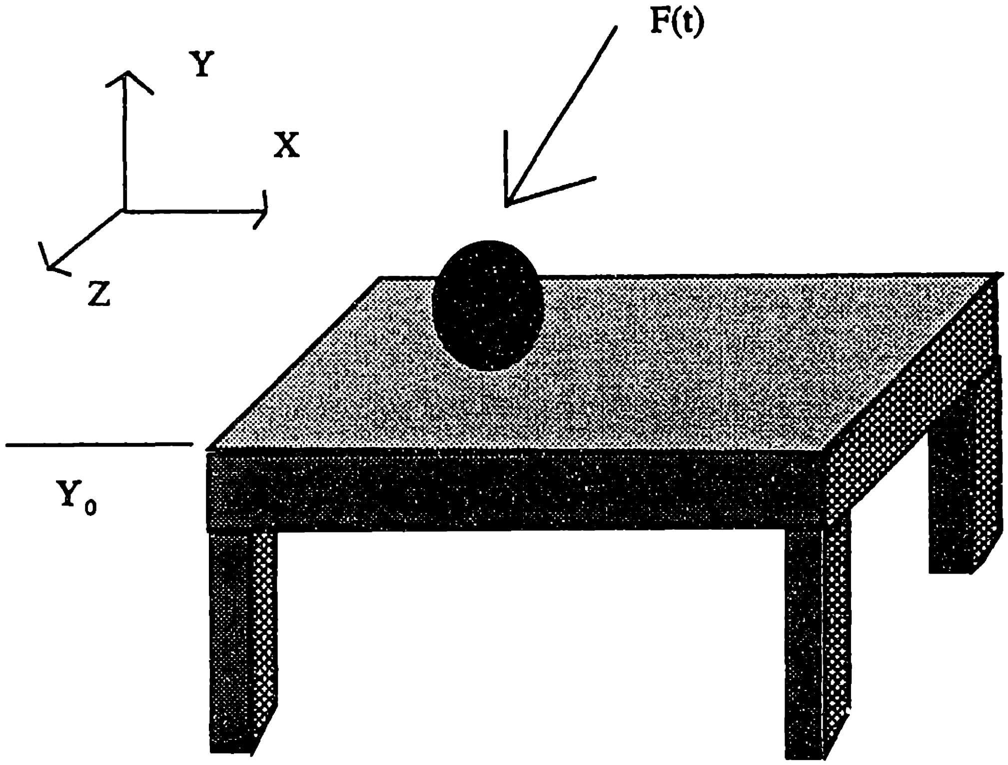

Figure 11: An iron sphere on a magnetic table.

One may visualize a sphere rolling on a magnetic table as in Figure 11. There is a an externally imposed force $F(t)$ which is acting on the sphere. This force may be broken into the components:

$$F(t) = F_x(t)\hat{i} + F_y\hat{j} + F_z\hat{k}.$$

Assuming that the sphere does not roll off the edge of the table, the holonomic constraint $Y = Y_0$ is imposed on the particle, where Y_0 is the height of the table. That constraint may be considered the result of the upward normal forces from the table and the downward magnetic forces. Ignoring the constraints, one could derive a set of equations based on the dynamics of a particle acted on by gravity and some force $F(t)$. However, with the constraining forces, the response in the Y direction must be altered as the result of the constraining forces of the table's surface and the magnetism. As far as the accounting of forces is concerned, the constraint equation

$Y - Y_0 = 0$ is identical to saying that the table is imposing a force in the Y direction exactly sufficient to cancel the effects of all other vertical forces acting on the sphere. With this perspective, it is possible to understand the definition of constraint forces. A constraint force is defined to mean any force which does no work through any virtual displacement. These forces are, in the sense of work, lost. In the example of the table in Figure 11, all vertical forces are constraint forces, since the sphere is incapable of being displaced from the surface of the table. That includes the normal forces of the table, the magnetic forces, the gravitational force, as well as the vertical component of the imposed force, $F(t)$. One may recognize that the forces of the table are nonconstant, adjusting to counteract the imposed force.

Now one may consider a slightly more complicated problem, also involving a holonomic constraint of the form:

$$f(u_1, u_2, \ldots, u_N) = 0.$$

The following sketch shows a marble confined to roll along the inside of a cylindrical duct:

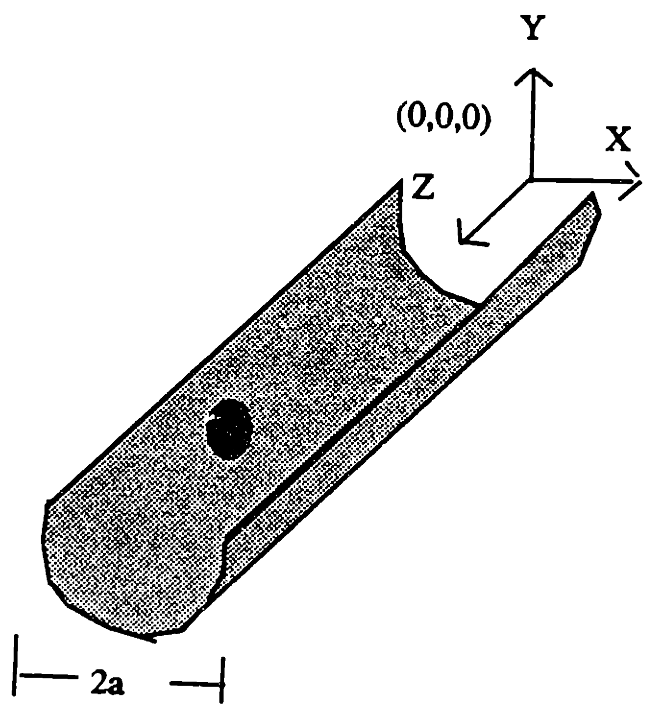

Figure 12: A marble rolling inside a cylindrical duct.

Just as it is assumed that the marble cannot leave the surface of the duct, it is also assumed that the marble cannot rise out of the duct. Therefore $y \leq 0$ using the cartesian coordinate system implied in the sketch. Ignoring the inequality constraint, there is a holonomic constraint created by the forces of gravity and the surface of the duct holding the marble to the inside surface:

$$X^2 + Y^2 - a^2 = 0.$$

The nature of the constraint forces is some unknown combination of gravity, inertial forces, and normal forces from the walls of the duct. However, it is known the the constraint forces must be perpendicular to the surface of the duct. The surface of the duct happens to be specified by the constraint equation above. Therefore, the constraint forces must be normal to the equation of constraint. This was true in the case of the table in Figure 10. In fact, the definition of a constraint force may be modified to mean any force that is normal to all virtual displacement vectors.

Suppose this claim that any constraint force is perpendicular to all virtual displacements were incorrect. If this were not true, then there must exist some constraint force not perpendicular to a particular virtual displacement vector. This would imply that the constraint force could be broken into a component perpendicular to the displacement vector and a component parallel to the displacement vector. The Figure 13 below may help to illustrate this:

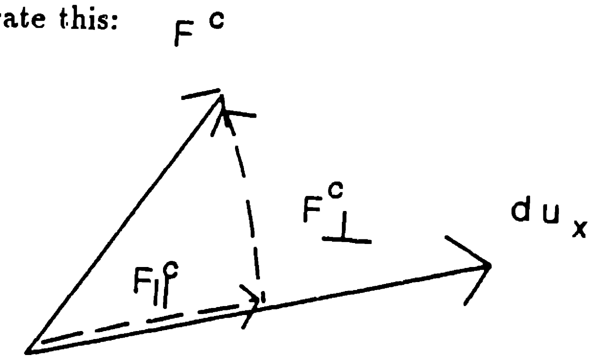

Figure 13: Displacement vector and constraint force.

Now if a particle were to be displaced along the virtual displacement vector shown, which represents the actual motion of a particle, virtual work would have been done by the parallel component of the constraint force. This contradicts the original definition of constraint forces. Therefore, it is true that constraints are simply the confinements of a system created from forces normal to the restrictions.

3.3 Lagrange multipliers

This concept of normality leads to a rule called d'Alembert's Principle, which states that the individual constraint forces may be disregarded in the dynamics problems of systems. Since the constraint forces do not affect the virtual work of a system, they should be eliminated from the differential equations of motion. For example, in the case of the sphere on the table, the gravitational force alone had no effect on the dynamics of the sphere, nor did the vertical component of the external force, although the constraint $Y = Y_0$ must be incorporated into the solution of the system. It is for this reason that one may make use of the concept of Lagrange multipliers. This will be explained shortly.

Consider a case, such as the one of Figure 12, that contains a constraint of the form $f(u_1, ..., u_N) = 0$. As shown in the last section, the constraint forces are perpendicular to the plane of constraint. Therefore, the constraint forces in this case are some magnitude times the gradient of f. In multivariable calculus, the gradient of a function is the vector normal to the function. If the constraint is already in Pfaffian form, the function containing the infinitesimal displacements becomes the vector perpendicular to the set of possible displacements specified by the Pfaffian equation. Therefore, the left side of the Pfaffian equation, Equation 1.1, becomes the direction of the constraint forces. One might consider the example of the spherical pendulum below.

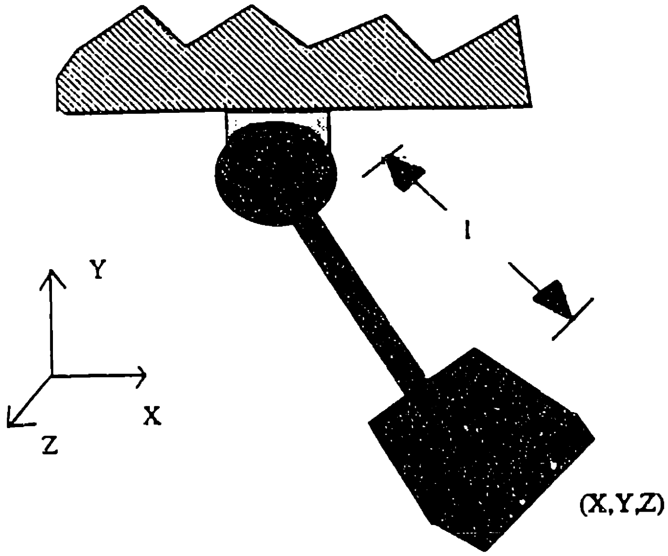

Figure 14: Spherical Pendulum.

The variables used to located the pendulum's mass are cartesian coordinates. The constraint forces are the forces along the rod of the pendulum since the distance of the mass to the pivot is fixed. One might be able to see intuitively that the constraint forces of the rod are in the direction of the vector $\vec{F} = X\hat{X} + Y\hat{Y} + Z\hat{Z}$, where $\hat{X}$, $\hat{Y}$, and $\hat{Z}$ are the unit vectors in the directions of the inite variables X,Y, and Z respectively. This may be derived mathematcally by either deriving the Pfaffian or taking the gradient of the constraint $f(u_1, ..., t) = X^2 + Y^2 + Z^2 - l^2 = 0$. Since the method of finding the Pfaffian form is more general, applying to any constraint, this method is shown below:

$$From Equation 1.2 \qquad \sum_{s=1}^{N} \frac{\partial f(u_1, ..., t}{\partial u_s} \times du_s + \frac{\partial f(u_1, ..., t)}{\partial t} \times dt = 0$$

$$\frac{\partial(X^2 + Y^2 + Z^2 - l^2)}{\partial X} dX + \frac{\partial(X^2 + Y^2 + Z^2 - l^2)}{\partial Y} dY + \frac{\partial(X^2 + Y^2 + Z^2 - l^2)}{\partial Z} dZ = 0$$

$$2XdX + 2YdY + 2ZdZ = 0 \Rightarrow XdX + YdY + ZdZ = 0$$

Since the infinitesimal displacements dX,dY, and dZ, specify the set of possible displacements, one can equate them to some multiple of the corresponding unit vectors, $\hat{X}$, $\hat{Y}$, and $\hat{Z}$. Therefore, converting the left side of the Pfaffian form to find the perpendicular constraint forces yields the following:

$$\Sigma Constraint forces \propto X\hat{X} + Y\hat{Y} + Z\hat{Z}$$

Now, the concept of the Lagrange multiplier will be introduced. The Lagrange multiplier is an unknown coefficient, usually represented by the symbol λ, that represents the magnitude of the unknown constraint forces. Although by the definition of the constraint forces, which was the set of all forces which do no work, the sum must equal zero in order for there to be no motion perpendicular to the constraint plane. However, the unknown constraint forces may not cancel out, and the magnitude of these forces may be specified by λ. A constraint of the form in Equation 1.1, the sum of the unknown constraint forces, having a magnitude λ and a direction perpendicular to the possible displacement vectors, may be expressed by the Equation 3.4:

$$\sum_{s=1}^{N} A_{rs}\hat{u}_s + +A_r\hat{t} = 0 \quad (r = 1, 2, ..., L). \tag{3.4}$$

Consider, the example of the spherical pendulum. It is often inconvenient to identify the inertial and gravitational forces parallel the the rod of the pendulum in order to cancel them according to the constraint. However, it is possible to determine the dynamics of a mass under gravitational and inertial forces without constraint, and then to incorporate the unknown remaining constraints forces from Equation 3.4, which are parallel to the vector taken from Pfaffian form. It was mentioned earlier, in d'Alembert's principle, that the constraint forces do no work and should be canceled from the state equations of motion. It is for this reason that the Lagrange multiplier is constructed. Although it represents some nonconstant force of the unknown constraint forces, which in the pendulum example is the tension in the rod, the force magnitude is placed in a single variable which may be eventually removed algebraicly.

3.4 Lagrange Multiplier Rule

The Lagrange multiplier rule gives a way to incorporate the unknown constraint forces into the equations of motion. This rule in itself does not include the convenience of using the Lagrange equations in its methods, requiring solving problems in the original style of Newtownian mechanics. However, the final chapter of this thesis will demonstrate how the Lagrange multiplier rule may be applied to the Lagrange equations.

From simple Newtonian mechanics, it is taught that the sum of the external forces on an object in a particular coordinate direction is equal and opposite to the kinetic forces of the object along the same coordinate. In mathematical terms, this means that $m_s \ddot{u}_s = \Sigma F$, where ΣF includes both the known external forces F_s as well as the unknown constraint forces F_s^c. The balanced force equation for any coordinate is:

$$m_s \ddot{u}_s = F_s + F_s^c (s = 1, 2, 3, ..., N) \qquad (3.5)$$

By multiplying each of the forces in the above equation by the infinitesimal displacement corresponding the the coordinate direction of the forces, the force equation becomes a work equation. Consequently, the following equality is derived:

$$(m_s \ddot{u}_s - F_s - F_s^c) du_s = 0 (s = 1, 2, 3, ..., N). \qquad (3.6)$$

This formula represents the concept called the Lagrange multiplier rule. The Lagrange multiplier rule is useful because it shows how to include unknown constraint forces into the equations of motion.

In order to derive a state equation in the form of Equation 3.6, first one can solve a mechanical system ignoring the unknown constraint forces. Determining the sum of known forces in a given coordinate direction will lead to an equation of the form $m_s \ddot{U}_s - F_s = 0$. From this point, the forces may be converted to work given an infinitesimal displacement by multiplying the equation by the variable du_s. Now, for the work equation to really balance, one must add the component of the Pfaffian in the du_s direction, which is multiplied by some arbitrary force magnitude λ. The component of the Pfaffian times the corresponding infinitesimal displacement represents the work along that coordinate. When a constraint in the Pfaffian contains several infinitesimals, each component must be multiplied by the same Lagrange multiplier, since the direction of the constraint force is predetermined.

Once a set of equations in the form of Equation 3.6 is derived, one can simply remove the infinitesimal displacement that multiplies the whole equation, leaving Equation 3.5. Since the displacements are arbitrary and unrelated to the finite variables, the equation holds when each of the displacements equals one and may be erased. Finally, now that there is a system of differential equations containing Lagrange multipliers, the Lagrange multiplier may be removed from the problem by combining the equations of motion.

This is a lot of information to understand at once. However, this example might clarify the Lagrange multiplier rule.

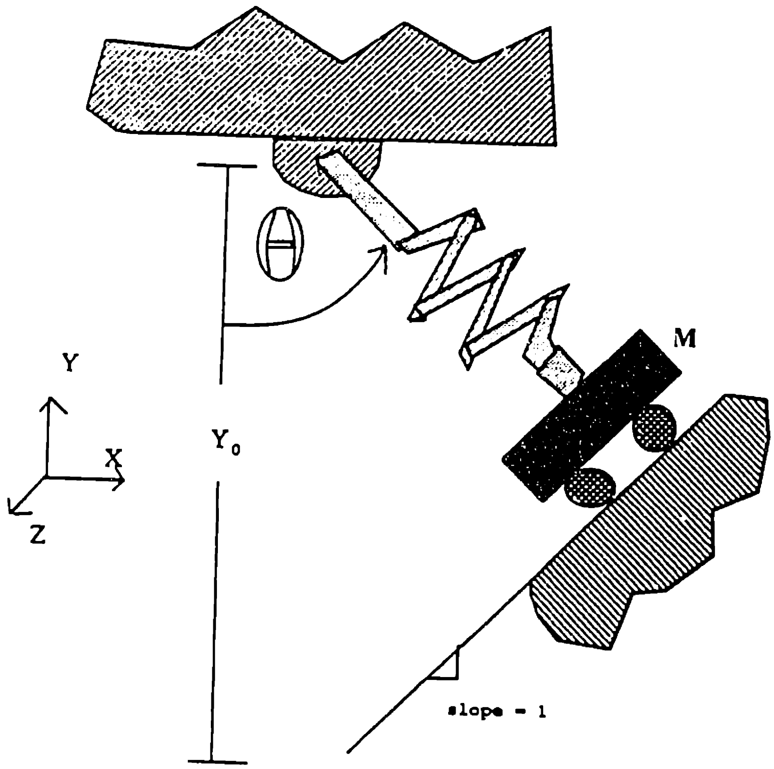

Figure 15: A sloped spring pendulum.

Consider the sloped spring pendulum system illustrated above in two dimensions. Cartesian coordinates may be used to locate the position of the rolling mass. The angle θ as label in Figure 15 is just for reference. The vertical distance between the slope and pivot is y_o. The origin is at the pivot. The first step is to solve the problem ignoring the unknown constraint forces. The unknown constraint force is the normal force of the slope on the bottom of the mass. The known forces are the spring, the gravity, and the inertial forces. The spring force, which is the spring constant times the distance of the mass to the pivot, is $k\sqrt{x^2+y^2}$. This force must be broken into the vertical and horizontal forces:

$$Horizontal : k\sqrt{x^2+y^2} \times \sin\theta = k\sqrt{x^2+y^2}\frac{x}{\sqrt{x^2+y^2}} = kx.$$

$$Vertical : k\sqrt{x^2+y^2} \times \cos\theta = k\sqrt{x^2+y^2}\frac{y}{\sqrt{x^2+y^2}} = ky.$$

The kinetic forces are simply $m\ddot{x}$ and $m\ddot{y}$ as is stated in Newton's second law. Finally, there is a gravitational force in the vertical direction of mg. Now, the force balance equations of the known forces may be multiplied by their corresponding infinitesimal displacement to derive the desired work equations:

$$Horizontal : m\ddot{x} + kx = 0 \Rightarrow (m\ddot{x} + kx)dx = 0.$$

$$Vertical : m\ddot{y} + ky + mg = 0 \Rightarrow (m\ddot{y} + ky + mg)dy = 0.$$

Of course these equations of work are not really valid without adding the effects of the constraint forces from the slope.

Now the constraint equation may be introduced into the state equations. The sliding mass is restricted by the slope forces to follow the linear curve:

$$y = x - y_0 \Rightarrow x - y + y_0 = 0.$$

Now, through differentiating this constraint in Equation 1.2 to get the Pfaffian form, one discovers the equation:

$$dx - dy = 0.$$

Since this equation of possible displacements implies a constraint force perpendicular to this plane of motions, the work from the unknown constraint force of magnitude λ must be denoted as:

$$\sum_{s=1}^{N} F_s^c du_s = \lambda(dx - dy) = \lambda dx + (-\lambda)dy = 0.$$

Since the work from the unknown constraint force of the slope is broken into the horizontal and vertical components, these components may be added onto the ends of the corresponding state equations ignoring details of the unknown constraints. The following equations, which are completely valid, completely sum the total work in each coordinate direction:

$$Horizontal : (m\ddot{x} + kx + \lambda)dx = 0 \Rightarrow m\ddot{x} + kx + \lambda = 0.$$

$$Vertical : (m\ddot{y} + ky + mg + (-\lambda))dy = 0 \Rightarrow m\ddot{y} + ky + mg + (-\lambda) = 0$$

Recalling that the displacements are independent of the finite variable, one can convert the work equations back into force balanced equations as shown above.

Finally, the two equations of motion may be combined to eliminate the unknown constraint force λ. In this particular example, the two force equations may be added together to eliminate the multiplier. This results in the two variable equation:

$$m(\ddot{x} + \ddot{y}) + k(x + y) + mg = 0.$$

One might recall that holonomic constraints such as the one in this example reduce the number of independent variables. This fact suggests that by substituting the constraint equation $y = x - y_0$ into the two variable state equation, a single-variable equation may be found. The resulting equation is indeed:

$$m\ddot{x} + kx = \frac{1}{2}(ky_0 - mg).$$

This is the differential equation which describes the motion of the system! In order to find the solution for the vertical motion, one can just subtract y_0 from the horizontal solution.

One last point should be made on the use of Lagrange multipliers. If there are multiple constraints on a system, there are the same number of groups of unknown constraint forces. In such cases, there will be several independent Lagrange multipliers. There may be several different multipliers in the same force balanced equation. Nevertheless, these multipliers should be eliminated by merging the equations of force, no differently than is done with one unknown constraint force.

In conclusion, by using the principles of Newtonian mechanics along with following the steps implied by the Lagrange multiplier rule, there is a fail-proof way of deriving the state equations in a system containing constraints.

Chapter 4

Lagrange's equations with constraints

Until this chapter, the various definitions of constraints were defined. Next, the distinction between holonomic and nonholonomic constraints was made, and the affect of constraints upon the number of dimensions in the configuration space and number of degrees of freedom was examined. After that, the concept of constraints being interpreted as undetermined forces which do no work was introduced. Finally, this concept was extended to actually deriving the state equations from a system of known components and constraints though Newtonian mechanics and the Lagrange multiplier rule.

However, as is often suggested in introductory dynamics classes, the derivation of state equations from Newtonian mechanics can get quite complicated. Trying to identify all of the forces on an object and breaking the force vectors into an orthogonal set of coordinates is often tricky and is subject to human errors. The Lagrange's equations provides an easier way of deriving the equations of motion. The sources of kinetic and potential energy along with nonconservative forces and components that cause dissipation of energy are much easier to identify in a complex system than are the orthogonal components of each of the forces. Through some differentiation technique specified by Lagrange's equations, one can indirectly solve for the state equations with little difficulty.

In this final chapter, the technique for incorporating constraint forces into the technique of Lagrange's equations for finding the state equations of motion will be introduced. The justification for this technique, which will be given, comes from the Lagrange multiplier rule. An alternative explanation of Lagrange's equations with unknown constraints will also be suggested. The demonstration of this process will involve the solution of a nonholonomic system.

A second demonstration of this technique will involve the case of a simple two dimensional pendulum. This is a holonomic example which is easily solved with a coordinate conversion, as was done in a previous chapter. However, solving the system in cartesian coordinates, in which case the constraint does not vanish, will prove that any problem can be solved with a systematic application of Lagrange's equations. This solution should coincide with the shortcut method of changing variables.

4.1 Lagrange's equations and a nonholonomic example

In order to derive the Lagrange's equations including constraint forc:s, one must return to the Lagrange multiplier rule. This rule was simple a balance of forces. Consider the set of k constraint equations in the Pfaffian form as is shown below:

$$A_{1r}du_1 + A_{2r}du_2 + ... + A_{Nr}du_N + A_{tr}dt = 0.(r = 1, 2, ..., k).$$

Multiplying each of the N Pfaffian equations by some force magnitude (Lagrange multiplier) λ_r converts the constraints into sums of components to be placed into the work equations. The Lagrange multiplier rule was previously defined to be the following equation:

$$(m_s\ddot{u}_s - F_s - F_s^c)du_s = 0.(s = 1, 2, ..., N).$$

Now it is possible to replace the value of each $F_s^c du_s$ by the sum of the effects that each of the constraints have on the work in the u_s direction. This will modify the Lagrange multiplier rule to the following form:

$$(m_s \ddot{u}_s - F_s - \sum_{r=1}^{k} \lambda_r A_{sr}) du_s = 0 (s = 1, 2, ..., N).$$

Notice that the $A_{tr}dt$ has no way of being accounted into the state equations since the direction of the constraint forces depends on the geometry of the constraints. As an aside, this suggests the idea that the constraint forces are perpendicular to the virtual displacements. Returning to the derivation of Lagrange's equations with constraints, these new equations of work may now be converted back to a summation of forces by dropping the infinitesimal displacement terms du_s. The force balance equation may be slightly reorganized to the form:

$$m_s \ddot{u}_s - F_s = \sum_{r=1}^{k} \lambda_r A_{sr} (s = 1, 2, ..., N). \tag{4.1}$$

However, if one reexamines this equation, one might notice that the left hand side of the equation, $m_s \ddot{u}_s - F_s$ is simply the sum of the known forces. These forces include the spring forces, the known nonconservative forces, and the kinetic forces. In other words, the sum of the known forces, or $m_s \ddot{u}_s + F^s$, is equal to the forces derived ignoring the unknown constraint forces. The right hand side of the equation above, the components of the unknown constraints, are simply taken from the Pfaffian form of the constraints. The left hand side may be a more difficult term to find without using Lagrangian mechanics. However, since the left side is the sum of the forces ignoring unknown constraint forces, one can equate it to the results of placing the Lagrangian L into the Lagrange's equations, ignoring the unknown forces. In mathematical terms:

$$m_s \ddot{u}_s - F_s = \frac{d}{dt}(\frac{\partial L}{\partial \dot{u}_s}) - \frac{\partial L}{\partial u_s} - \Xi_{u_s}.(s = 1, 2, ..., N).$$

Substituting this equality into the modified Lagrange multiplier rule (Equation 4.1) results in a complete force balance equation that allows the use of the Lagrange's equations on the components of the dynamical system other that the unknown constraint forces.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{u}_s}\right) - \frac{\partial L}{\partial u_s} - \Xi_{u_s} - \sum_{r=1}^{k} \lambda_r A_{sr} = 0. \quad (s = 1, 2, ..., N). \tag{4.2}$$

There is an intuitive, yet less mathematically rigorous way of thinking about why the Lagrange's equations with constraints (Equation 4.2) is correct. Rather than try to derive the relationship from the Lagrange multiplier rule to Lagrange's equations, one might attempt to visualize the unknown constraint forces simply as nonconservative forces. Each of these forces $\lambda_r A_{sr}$ is like an externally imposed force. Consider the example of the spherical pendulum. One would derive the same response of the system though the Lagrange multiplier rule as one would by modeling the problem as a free mass being forced by some external nonconservative force vector which just happens to point to the location of the pivot at all times. With this reasoning, one can just add the constraint forces to the other known nonconservative forces (such as damping and known external forces).

$$\Xi_{u_s}(old) + \sum_{r=1}^{k} \lambda_r A_{sr} \Rightarrow \Xi_{u_s}(modified).$$

Now, by simply rewriting the classical Lagrange's equations with the new definition of nonconservative forces, one discovers the same equation for finding the state equations as Equation 4.2.

Unfortunately, in the form of Equation 4.2, there might be several equations each with several Lagrange multipliers λ_r. Although the derivation of these equations is quite systematic for a linear system, one might think that a more convenient approach of solving the equations of motion in which some of the multiplier are already eliminated during the process of deriving the differential state equations. While considering the simplicity achieved in the two dimensional pendulum problem through coordinate conversion, one might attempt to derive a shortcut. Unfortunately, the results in Equation 4.2 are the best one can do to find a systematic fail-proof way of solving dynamical systems requiring the use of the λ-multipliers. It is important not to attempt to simplify the problem by substituting the constraint into the Lagrangian $L = T^{*} - V$, a common mistake. This process would, in fact, simplify the problem for holonomic system, but it would lead to incorrect solutions for systems with complications such as nonholonomic constraints. Although this thesis will not attempt to show when shortcuts may be used, it does show that the elimination of the λ-multipliers may be accomplished though a routine procedure, however tedious (Appendix B).

Now to demonstrate the effectiveness of the newly derived Equation 4.2, a nonholonomic system with a time dependent constraint will be presented. This illustration Figure 16 helps describe one such example.

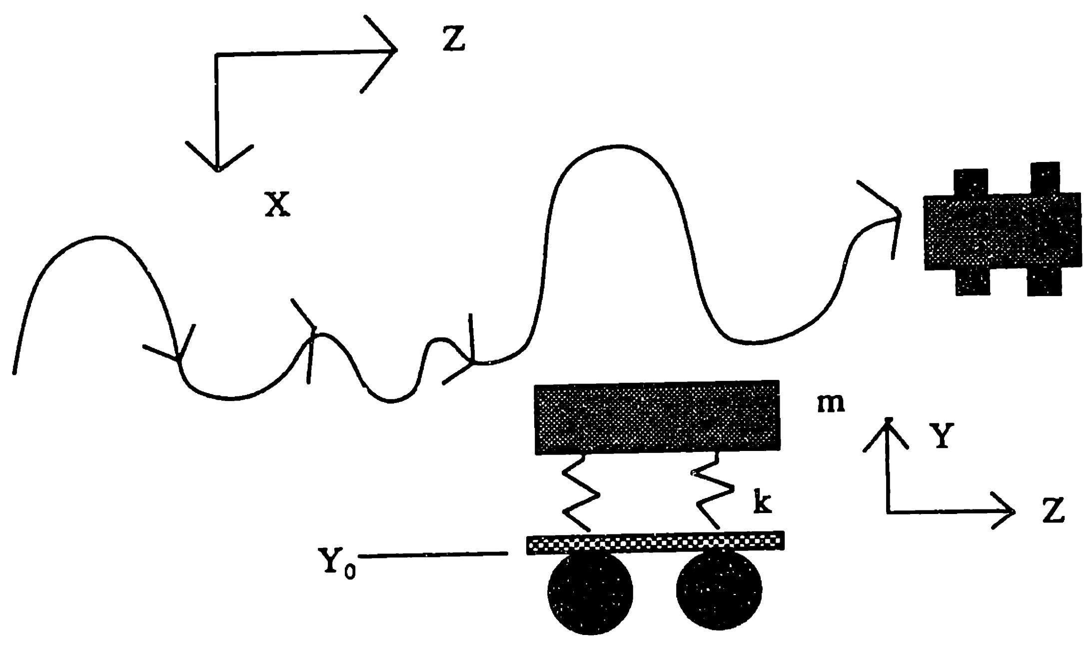

Figure 16: A sinusoidally guided automobile.

The illustration shows an automobile moving on the x-z plane. Although the velocity as a function of time of the car is unknown and externally controlled, the steering has been programmed to follow a sinusoidal path. To further complicate the problem, the cabin car is suspended vertically (Y-direction) on some spring of total spring stiffness k. However, to simplify things, the cabin will be treated as a single particle, so that the kinetic energy from the angular rotation of the car is negligible; the orientation of the cabin is not a necessary variable to locate the cabin as long as the infinitesimal displacements are related by the sinusoidal function of time. This will allow the use of the cartesian x, y, and z alone to specify the generalized coordinates. Therefore, there appears to be two constraints on this system, both nonholonomic. The first, which states that the cabin must lie above some constant height of the wheels, which may be called Y_0, is $y > Y_0$. This constraint may be ignored for reasonably small displacements. However, the second constraint, which states that the angular direction, or slope of the displacements, of the motion of the cart must be equal to $\sin \omega t$. In equation form:

$$\frac{dx}{dz} = \sin \omega t.$$

This equation must be placed in the Pfaffian to be of use in the Lagrange's equations. The rearranged form of the equation is:

$$(1)dx + (0)dt + (-\sin \omega t)dz = 0.$$

There must be some force, in this case a frictional force from the pavement, acting in the direction $(1)\lambda\hat{x} + (-\sin \omega t)\lambda\hat{z}$ (see Equation 3.4). The magnitude of the force, which depends on the unknown velocity and orientation of the cabin, is incorporated into to the Lagrange multiplier λ. One can see that in terms of Equation 4.2, one can equate the terms $\lambda_1 A_{x1}$ with λ, $\lambda_1 A_{z1}$ with $-\lambda \sin \omega t$, and 0 with any other $\lambda_r A_{rs}$.

Now, this constraint may be tested to show that it is clearly a nonholonomic constraint. One can plug the components of the Pfaffian form into the integrability test of Equation 1.4 using x, t, and z as the three variables of concern. The expressions A, B_t, and C represent the functions (1), (0), and $(-\sin \omega t)$ from the Pfaffian equation, giving the following result:

$$A(\frac{\partial B_t}{\partial z} - \frac{\partial C}{\partial t}) + B_t(\frac{\partial C}{\partial x} - \frac{\partial A}{\partial z}) + C(\frac{\partial A}{\partial t} - \frac{\partial B_t}{\partial x}) = \omega \cos \omega t \neq 0.$$

Therefore, since the integrability condition is not satisfied, the constraint must be nonholonomic. The Lagrangian is simple to derive in this example:

$$\mathbf{L} = \mathbf{T}^{*} - \mathbf{V} = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - \frac{1}{2}ky^2.$$

The nonconservative known forces are all zero ($\Xi_s = 0$). Substituting the Lagrangian and the into the Lagrange's equations with unknown constraints (Equation 4.2), one would derive the following set of equations:

In terms of the variable y,

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{y}}) - \frac{\partial L}{\partial y} - \Xi_y - \sum_{r=1}^{k} \lambda_r A_{yr} = 0.$$

$$m\ddot{y} + ky = 0. \tag{4.3}$$

There were no Lagrange multipliers resulting from Lagrange's in the y direction since $A_{y1} = 0$. There was no component of the unknown constraint forces acting in the y direction. Consequently, the Equation 4.3 is derived as a final solution of the motion of the system in the y direction.

Now, Equation 4.2 may be applied in term of the finite variables x and z:

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{x}}) - \frac{\partial L}{\partial x} - \Xi_x - \sum_{r=1}^{k} \lambda_r A_{xr} = 0.$$

$$\sum_{r=1}^{k} \lambda_r A_{xr} = \lambda_1 A_{x1} = \lambda(1).$$

$$m\ddot{x} - \lambda(1) = 0.$$

Finally, for the coordinate z:

$$\frac{d}{dt}(\frac{\partial L}{\partial \dot{z}}) - \frac{\partial L}{\partial z} - \Xi_z - \sum_{r=1}^{k} \lambda_r A_{zr} = 0.$$

$$\sum_{r=1}^{k} \lambda_r A_{zr} = \lambda_1 A_{z1} = \lambda(-\sin \omega t).$$

$$m\ddot{z} - \lambda(-\sin\omega t) = m\ddot{z} + \lambda\sin\omega t = 0.$$

Now, in order to make these equations of motion more meaningful, one must attempt to eliminate the multiplier from the set of state equations. This may be done by combining the second two equations above:

$$m\ddot{x}\sin\omega t + m\ddot{z} = 0. \tag{4.4}$$

At this point, the problem is essentially solved. Another differential state equation of motion has been derived from a system with constraints. However, it is usually more desired to have the state equations in the form of single variables. The Equation 4.3, the Equation 4.4 for vertical motion, and a third equation will allow a separation of variables into three independent state equations. Whereas the holonomic constraints were converted to equalities between finite variables to be substituted into the mixed variable state equations, nonholonomic constraints often imply some relation between velocities, which may be of use. In this example, if the displacement of the cabin is forced to follow a sinusoidal path such that $dx = dz\sin\omega t$, the direction of velocities must also be pointed in that direction. This implies that:

$$\dot{x} = \dot{z}\sin\omega t. \tag{4.5}$$

This is the third equation needed to separate the variables in the state equations. One can rearrange these three equations to get single variable differential equations.

The rearranging and combining of Equations 4.3, 4.4, and 4.5 to get single variable equations is not very complicated at this stage. One can take the time derivative of Equation 4.5 as shown:

$$\ddot{x} = \ddot{z}\sin\omega t + \omega\dot{z}\cos\omega t = 0.$$

It is also possible to rearrage Equation 4.4 into the equation $\ddot{x} = -\frac{\ddot{z}}{\sin\omega t}$. Now one can substitute the rearanged Equation 4.4 into the time derivative of equation 4.5 to get a state equation as a function of z alone:

$$(\frac{1}{\sin\omega t} + \sin\omega t)\ddot{z} + \omega\dot{z}\cos\omega t = 0 \tag{4.6}$$

Equation 4.6 is the second state equation of motion in terms of one variable, the first being Equation 4.3.

After completing this task, one can rearrange Equations 4.3, 4.4, and 4.5 to get an equation in x. This can be done by first dividing the equation 4.5 by $\sin\omega t$:

$$\dot{z} = \frac{\dot{x}}{\sin\omega t}.$$

Now, once again, the modified form of Equation 4.5 may be differentiated with respect to time:

$$\ddot{z} = \frac{\ddot{x}}{\sin\omega t} - \dot{x}\frac{\omega\cos\omega t}{sin^2\omega t}.$$

It is straightfoward to modify equation 4.4 into the equation $\ddot{z} = -\ddot{x}\sin\omega t$. Once again, the modified Equation 4.4 may be substituted into the new differentiated form of Equation 4.5:

$$(\sin\omega t + \frac{1}{\sin\omega t})\ddot{x} - (\frac{\omega\cos\omega t}{\sin^2\omega t})\dot{x} = 0. \tag{4.7}$$

The result of this algebraic manipulation is a differential equation in term of x (Equation 4.7), another in terms of y (Equation 4.3), and a third in terms of z (Equation 4.6). Although the final state equations to this problem do not appear simple, the Lagrange's modified equations led to the solution almost systematically.

4.2 The two dimensional pendulum

In holonomic systems, one is often unaccustomed to using the techniques involving the Lagrangian multiplier, particularly at an introductory level in the study of dynamics. The reason for this is not that constraints which have an impact upon the equations of motion of a system are uncommon, but rather that it is possible to avoid them through various tricks such as coordinate transformations. However, it can be demonstrated that even these holonomic problems, with slightly more arithmetic difficulty, can be solved systematically with the modified Lagrange's equations. One might attempt to solve the solve the two dimensional pendulum with cartesian coordinates this way.

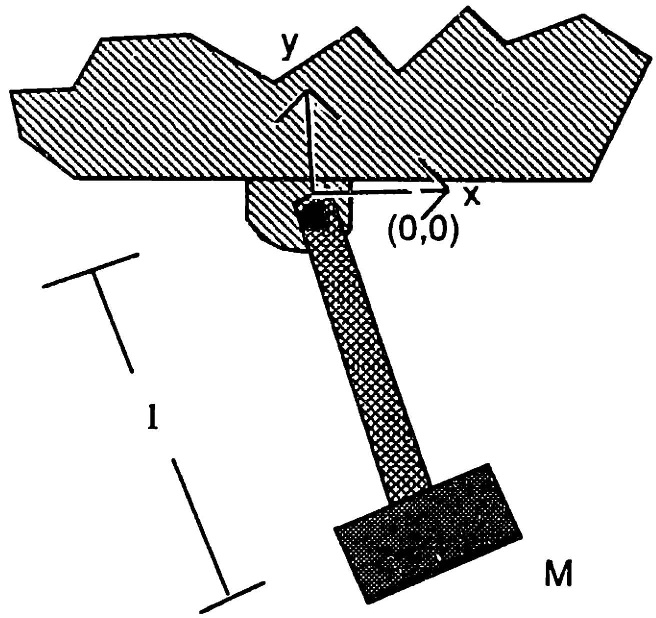

Figure 17: A simple two dimensional pendulum.

The simple pendulum is a mass connected to a rigid rod confined to the x-y plane. This problem is identical to the bead on a hoop analyzed in Chapter 2 (Figure 8). In the second chapter, the problem was solved by converting to a cylindrical coordinate system. The resulting solution was the equation:

$$ml^2\ddot{\theta} + mgl\sin\theta = 0. \tag{4.8}$$

This type of shortcut involved some intuition and luck. It was through recognizing that the constraint force of the beam was always perpendicular to the angular motion of the pendulum and never had any impact upon the response of the variable θ that allowed this simplification to be made.

However, a systematic approach may be taken using the cartesian coordinates. This system is a one particle system with one holonomic constraint. The origin of reference of the mass is at the pivot of the pendulum. Therefore, since the distance of the mass from the pivot is constant, the constraint equation is:

$$x^2 + y^2 - l^2 = 0.$$

The kinetic energy, energy from conservative forces, and energy from nonconservative forces of a mass in two dimensions without constraints are:

$$T^* = \frac{1}{2}m[(\dot{x}^2 + \dot{y}^2)]$$

$$V = mg(y - l)$$

and

$$\Xi_x = \Xi_y = 0.$$

According to the technique for Lagrange's equations with constraints, the constraint must be converted to Pfaffian form in order to determine the direction of the constraint force. This can be done simply by differentiating the holonomic constraint.

$$(2x)dx + (2y)dy = (x)dx + (y)dy = 0.$$

Now, that the direction of the constraint force is known, the magnitude may be specified by λ. The constraint force, (Equation 3.4) becomes $F^c = x\hat{x} + y\hat{y}$. Therefore, in the notation from Equation 4.2, the terms $\lambda_1 A_{x1} = \lambda(x)$, $\lambda_1 A_{y1} = \lambda(y)$, and $A_{sr} = 0$ for any other r and s. One can use the Lagrange's differential equations to get the following set of state equations:

$$L = T^* - V$$

$$\frac{\partial}{\partial t}(\frac{\partial L}{\partial \dot{x}}) - \frac{\partial L}{\partial x} + \lambda x = 0$$

$$m\ddot{x} + \lambda x = 0. \tag{4.9}$$

In the Y-direction,

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} + \lambda y = 0$$

$$m\ddot{y} - mg + \lambda y = 0. \tag{4.10}$$

Now, the combining Equations 4.9 and 4.10 in order to eliminate the Lagrange multiplier may be accomplished by adding y times Equation 4.9 to $-x$ times the Equation 4.10. The result of this arithmetic is the following two variable equation:

$$ym\ddot{x} - xm\ddot{y} + mgx = 0. \tag{4.11}$$

This final Equation 4.11, when added to the constraint equation $x^2 + y^2 - l^2$, gives two equations of motion in the finite variables x and y. The separation of these variables may be performed by combining the two equations. This was somewhat more involved algebraicly than was the coordinate transformation from Chapter 2. However, it can be shown that the result in Equation 4.11 is identical to that in Equation 4.8. In order to convert from the cartesian equation to the polar equation, one must establish a link between the two systems such as equating $x = l\sin\theta$ and $y = l\cos\theta$. Differentiating the equations twice gives the relations:

$$\ddot{x} = l(-\sin\theta\dot{\theta}^2 + \cos\theta\dot{\theta}^2).$$

$$\ddot{y} = l(-\sin\theta\dot{\theta}^2 - \cos\theta\dot{\theta}^2).$$

Substituting these equalities into the Equation 4.9 in cartesian variables, one finds that:

$$ym\ddot{x} - xm\ddot{y} + mgx = 0.$$

$$m(l\cos\theta)(-l\sin\theta\dot{\theta}^2 + l\cos\theta\dot{\theta}^2) - m(l\sin\theta)(-\sin\theta\dot{\theta}^2 - \cos\theta\dot{\theta}^2) + mgl\sin\theta = 0.$$

$$ml^2(\cos^2 + \sin^2)\ddot{\theta} + mgl\sin\theta = 0.$$

$$ml^2\ddot{\theta} + mgl\sin\theta = 0.$$

This is identical to Equation 4.6.

In conclusion, although holonomic systems may have some shortcut to eliminate the necessity of the Lagrange multipliers, the method involving the modified Lagrange's equations is a failproof way of finding the state equations of a system. Nonholonomic systems must always be solved using the Lagrange multipliers, and often more easily with Lagrange's modified equations.

Appendix A

Appendix A: The Rolling Disk

In the body of this thesis, many terms and characteristics relating to constraints were introduced. Although some examples were worked completely, from the physical description of a system to the derivation of the state equations, constant explanation of the theory was included. In this example, a simple summary outline of procedures is given on how to solve a problem involving constraints with no theoretical verification of the method to interfere.

In this example, the classical nonholonomic example of a disk or ring rolling on a flat surface will be solved. This problem is used very often in textbooks in discrete dynamics to demonstrate a nonholonomic constraint. The illustration of the problem is given below:

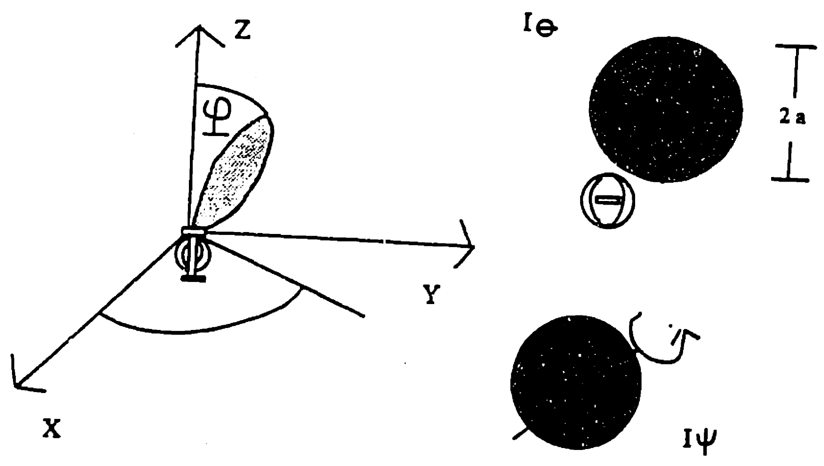

Figure 18: A rolling disk or ring.

As shown in the sketch, the disk may tilt at an angle Ψ and rotate in the direction of Θ around the center. The disk is also headed in the direction specified by Φ. The center of the coin is located at (X, Y) by cartesian coordinates. These generalized coordinates are sufficient to describe the configuration space. The goal of this problem is to derive the state equations of this system. There are two nonholonomic constraints on the displacements of the disk. Since there are five finite variables, there must be three degrees of freedom of motion. This is consistent with intuition, since a coin can roll, tilt, or turn, but nothing else. With this knowledge, one can now outline how to solve a problem with constraints.

Step One: Define the Lagrangian.

One must first derive the Lagrangian of the system and nonconservative forces ignoring the unknown constraints. In this example, this is not very easy. First, one must find the kinetic energy T^*. The kinetic energy is created by motion of the center of mass, rotation around the center point of the coin, and spinning or tilting of the disk. The kinetic energy of the center of mass may be written in cartesian coordinates as $T_m = \frac{1}{2}M(\dot{X}^2 + \dot{Y}^2 + \dot{Z}^2)$. Of course Z was not one of the generalized coordinates. However, it may be replaced by the equivalence $Z = \cos\Psi$. Now, the result is the following:

$$T_m = \frac{1}{2}M(\dot{X}^2 + \dot{Y}^2 + \sin^2\Psi\dot{\Psi}^2).$$

The kinetic energy created by the rotation of the coin does not only include motion in the Θ direction, but also a component of the changing of the heading angle Φ. In the extreme case, a coin may lie nearly flat on the ground fixed at one point of the edge to the ground. Although the coin may not roll in this position, it may pivot about that point resulting in the rotation of the mass about the center of the coin of the amount $\dot{\Phi}$. Adding the two angular velocity vectors, one reasons that:

$$T_\theta = \frac{1}{2}I_\theta(\dot{\Phi} + \dot{\Phi}\sin\Psi)^2.$$

I_θ is the moment of inertia of the coin around its center. In the total kinetic energy from rotations around the diameter of the disk, energy from turning in the Φ direction is combined with the kinetic energy of the tilting around some axis parallel to the ground. Since both are separate perpendicular motions, the energies, not the velocies are summed.

$$T_\psi = \frac{1}{2} I_\psi (\dot{\Psi}^2 + \dot{\Phi}^2 \cos^2 \Psi).$$

For a radially symetrical disk, the moment of inertia around the diameter I_ψ is half of I_θ. The conservative forces are much easier to find, since the only factor involved is gravity. Therefore, $V = Mga(1 - \cos\Phi)$. Adding up the components of the Lagrangian, one concludes that:

$$L = \frac{1}{2} M(\dot{X}^2 + \dot{Y}^2 + \sin^2 \Psi \dot{\Psi}^2) + \frac{1}{2} I_\theta (\dot{\Phi} + \dot{\Phi} \sin\Psi)^2 + \frac{1}{2} I_\psi (\dot{\Psi}^2 + \dot{Phi}^2 \cos^2 \Psi) - Mga(1 - \cos\Phi).$$

Aside from the constraints, there are no nonconservative forces.

Step Two: Define the Constraints in Pfaffian Form.

The second step in solving a system is to define the constraints. Each constraint must be designated a Lagrangian multiplier. This step often requires an ability to visualize the motion of an object in three dimensions. There are two nonholonomic constraints in the case of the rolling coin. The first involves the tilting of the coin, and the second involves the rolling and turning of the disk. In the first constraint of the disk, there is some relation between the location and motion of the center of mass of the disk and its motion in the Ψ direction. Since this only involves the way vertical motion and location affects the movement of the center, the infinitesimals of the angles Θ and Φ will not be present in this constraint equation. After some visualization of the tilting motion of the system, particularly at the extreme cases, one might come up with the constraint:

$$\cos\Phi dx + \sin\Phi dy - a\cos\Psi d\Psi = 0.$$

The Lagrange multiplier for the constraint will be λ. A second constraint equation may be derived concerning the rolling and turning of the disk. These displacements specify the infinitesimal motion of the center of the disk perpendicular to the displacements caused by the tilting of the disk. In order to reason what this constraint equation might be, one must determine the relationship of the displacements of the center in the dX and dY directions to separate displacements in the Θ and Φ directions. After considering the response of the system as the result of infinitesimal displacements in the rolling angle and orientation, one should conclude that:

$$-sin\Phi dX + \cos\Phi dY + ad\Theta + a\sin\Phi d\Phi = 0.$$

Rather than use the notion λ_2 for a second multiplier, the Lagrange multiplier for this constraint will be labeled μ. Intuitively deriving the constraint equations may not be an easy task, but the number of equations was known from the beginning of the problem since the number of degrees of freedom of motion was easy to visualize. If there had been any holonomic constraints in finite variables, they would be converted to Pfaffian form at this step.

Step Three: Substitute the Lagrangian and constraints into Lagrange's equations.

Although this step may be tedious and detailed, the mathematics behind this step is simply differentiation. Careful substitution into the Lagrange's equations for each finite variable results in the following five differential equations:

In the X direction:

$$M\ddot{X} - \lambda\cos\Phi + \mu\sin\Phi = 0. \tag{A.1}$$

In the Y direction:

$$M\ddot{Y} - \lambda\sin\Phi + \mu\cos\Phi = 0. \tag{A.2}$$

In the Ψ direction:

$$Ma^2(\sin 2\Psi\dot{\Psi}^2+\sin^2\Psi\ddot{\Psi})+I_\psi(\ddot{\Psi}+\frac{\dot{\Phi}^2}{2}\sin 2\Psi)+I_\theta(\dot{\Theta}\cos\Psi+\frac{\dot{\Phi}}{2}\sin 2\Psi)+Mgasin\Psi-\lambda a\cos\Psi = 0. \tag{A.3}$$

In the Φ direction:

$$I_\psi(\ddot{\Phi}\sin^2\Psi+\sin\Psi\cos\Psi\dot{\Psi}\dot{\Phi})+I_\theta(\ddot{\Theta}\sin\Psi+\dot{\Theta}\cos\Psi\dot{\Psi}+\ddot{\Phi}\sin^2\Psi+2\dot{\Psi}sin\Psi cos\Psi\dot{\Phi})-\mu a\sin\Psi = 0. \tag{A.4}$$

In the Θ direction:

$$I_\theta(\ddot{\Theta} + \ddot{\Theta} \sin \Psi + \dot{\Phi} \cos \Psi \dot{\Psi}) - \lambda a = 0. \tag{A.5}$$

Now at the end of this step, there are five equations including the multipliers, two unknowns, and five variables.

Step Four: Establish equations from the constraints.

The constraints are currently in Pfaffian form. In order to use them in the process of elimination of the Lagrange multipliers, one must convert them into equations involving the finite variables and their derivatives. In holonomic constraints, the constraint equations usually originate in a form relating the finite variables. In a nonholonomic constraint relating the infinitesimals, one must use a trick. If an equation holds relating the infinitesimal displacements, the velocities in the respective coordinates also follow the same relationship. This makes intuitive sense because the all components of a Pfaffian equation may be simply divided by an infinitesimal unit of time. In the case of the constraints from Step 2, the following equations are created:

$$\cos \Phi \dot{x} + \sin \Phi \dot{y} - a \cos \Psi \dot{\Psi} = 0. \tag{A.6}$$

$$-sin\Phi\dot{X} + \cos \Phi \dot{Y} + a\dot{\Theta} + a \sin \Phi \dot{\Phi} = 0. \tag{A.7}$$

Now there are seven equations, two unknown multipliers, and five variables.

Step Five: Eliminate the Lagrange multipliers.

The Lagrange multipliers must be removed for the differential equations to be of any significance. Usually this is not difficult to do, but matrix inversion may be necessary if there are several multipliers (See Appendix B). In this particular example, Equations A.1 and A.2 may be combined to find a set of equations defining the two multipliers:

$$\lambda = M(\ddot{X} \cos \Phi + \ddot{Y} \sin \Phi). \tag{A.8}$$

$$\mu = M(-\ddot{X} \sin \Phi + \ddot{Y} \cos \Phi). \tag{A.9}$$

It would be redundant to replace these equations for the multipliers back into the Equations A.1 and A.2. However, it may be placed into Equations A.3 through A.5, thereby eliminating the multiplier. After the newly defined multipliers are replaced into the state equations, there are now five equations with five variables containing no unknowns. Three equations are generated by substitution of Equations A.6 and A.7 into the state equations, and two were created from step four.

Step Six: Separate the variables.

There is an algebraic rule that states that a system of k equations and k variables can be solved. This rule implies that it is possible to separate the variables so that no variable is dependent any of the others (except for the time variable). If this were not true, then there would be no one solution of the equations. Therefore, in this problem of five equations and five generalized coordinates, it is possible to separate the variables into their own equations. This final step shall be omitted for the example of the rolling disk, since the algebra behind such a system of equations is complicated and not useful to the understanding the process of solving a discrete dynamical system.

Step Seven: Integrate each of the single variable equations.

Usually the final desired solution to the response of dynamical system is the actual position and orientation of the system as a function of time. For example, in a simple spring-mass system, the integration of the state equation $M\ddot{x} + kx = 0$ is the function of x with respect to time, in particular $x = A\cos(kt/M + \phi)$. In order to get the actual solution of the differential equations for a system, one must know the initial conditions of the system. For example, in the example of the rolling coin, if one knows the initial rolling speed, tilting veiocity, turning velocity, position, and precise orientation, theoretically one can actually find a function of time for each of the variables. Again the mathematical complexity at accomplishing this step for the example of the coin is to difficult to perform in this thesis.

Appendix B

Appendix B: Elimination of Several Multipliers

In this brief appendix, it can be shown that for any system of N state equations and k multipliers, where the number of equations exceeds the number of multipliers, the multipliers may be eliminated at the expense of the k equations. Therefore, if there are x generalized coordinates, there must be $x - N + k$ constraint equations to supplement the N state equations in order to solve the system. If there are any more, there must be redundancy or a contradiction. If there are any fewer, there is a piece of information about the system that is missing.

Consider the set of equations as follows:

$$F_s(u_1, u_2, ..., \dot{u}_1, \dot{u}_2, ..., \ddot{u}_1, ..., \ddot{u}_x) = \sum_{r=1}^{k} \lambda_r A_{sr} (s = 1, 2, ..., N). \tag{B.1}$$

These are the set of N state equations derived from Lagrange's modified equation. Each equation relates the generalized coordinates and their first and second derivatives to the multipliers. This set of equations may be translated to vector notation. In order to accomplish this, one must first take the first k equations and leave the last $N - k$ equations for later use. The first k functions of the finite variables from the left side of Equation B.1 would correspond to a column vector of length k as shown below:

$$\{F\} = \begin{vmatrix} F_1(u_1, ..., \ddot{u_x}) \\ F_2(u_1, ..., \ddot{u_x}) \\ ... \\ F_{k-1}(u_1, ..., \ddot{u_x}) \\ F_k(u_1, ..., \ddot{u_x}) \end{vmatrix}$$

The right hand side of the Equation B.1 contains the Lagrange multipliers. Since for every equation there are k multipliers, the right side of the Equation B.1 corresponds to a $k \times k$ matrix. This matrix is shown below:

$$[L] = \begin{vmatrix} A_{11}\lambda_1 & A_{12}\lambda_2 & ... & ... & A_{1k}\lambda_k \\ A_{21}\lambda_2 & A_{22}\lambda_2 & ... & ... & A_{2k}\lambda_k \\ ... & ... & ... & ... & ... \\ A_{k-1,1}\lambda_1 & A_{k-1,2}\lambda_2 & ... & ... & A_{k-1,k}\lambda_k \\ A_{k1}\lambda_1 & A_{k2}\lambda_2 & ... & ... & A_{kk}\lambda_k \end{vmatrix}$$

However, it is quite simple to separate the matrix $[L]$ into a matrix and a column vector. This may be accomplished as follows:

$$[M]\{\lambda\} = \begin{vmatrix} A_{11} & A_{12} & ... & ... & A_{1k} \\ A_{21} & A_{22} & ... & ... & A_{2k} \\ ... & ... & ... & ... & ... \\ A_{k-1,1} & A_{k-1,2} & ... & ... & A_{k-1,k} \\ A_{k1} & A_{k2} & ... & ... & A_{kk} \end{vmatrix} \begin{vmatrix} \lambda_1 \\ \lambda_2 \\ ... \\ \lambda_{k-1} \\ \lambda_k \end{vmatrix}$$

Now, the matrix $[L]$ has been broken into the matrix $[M]$ and the vector $\{\lambda\}$. The equality between the matrix $[L]$ and the vector $\{F\}$ must hold just as Equation B.1 is identically valid for all values of s. At this stage, a simple matrix inversion will solve all k multipliers simultaneously as indicated in Equation B.2:

$$\{\lambda\} = [M]^{-1} \cdot \{F\}. \tag{B.2}$$

However, by solving these equations for the multipliers, they are no longer of any use as state equations. The multipliers were found assuming the first k state equations were correct. If one were to replace the multipliers into any of the first k equations by their newly discovered values, the resulting equation would give a useless identity equation, such as $0 = 0$ or $a - a = 0$.

Nevertheless, one can replace the multiplier in the the remaining $N - k$ equations with the values recently discovered. The result will be $N - k$ state equations containing no undetermined multipliers. If there are x free variables, the total of constraint equations and state equation with no multipliers must equal to x. Therefore, as claimed earlier, there must be $x - N + k$ constraint equations to solve the system.

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