Elliptic Fibrations among Toric Hypersurface Calabi-Yau Manifolds and Mirror Symmetry of Fibrations

by

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B.S., National Taiwan University (2011)
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Submitted to the Department of Physics in partial fulfillment of the requirements for the degree of

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Abstract

In this thesis, we investigate the prevalence of elliptic and genus one fibrations among toric hypersurface Calabi-Yau threefolds by (1) constructing explicitly elliptically fibered Calabi-Yau threefolds with large Hodge numbers using Weierstrass model techniques motivated by F-theory, and comparing the Tate-tuned Weierstrass model set with the set of Calabi-Yau threefolds constructed using toric hypersurface methods, and (2) systematically analyzing directly the fibration structure of 4D reflexive polytopes by classifying all the 2D subpolytopes of the 4D polytopes in the Kreuzer and Skarke database of toric Calabi-Yau hypersurfaces.

With the classification of the 2D fibers, we then study the mirror symmetry structure of elliptic toric hypersurface Calabi-Yau threefolds. We show that the mirror symmetry of Calabi-Yau manifolds factorizes between the toric fiber and the base: if there exist 2D mirror fibers of a pair of mirror reflexive polytopes, the base and fibration structure of one hypersurface Calabi-Yau determine the base of the other.

Thesis Supervisor: Washington Taylor IV
Title: Professor of Physics
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means a lot to me the gradual change in his attitude from when I just started working with him he would tend to persuade me out of the string theory area because it is just too competitive if I would really want an academic career, to that he thought I could do some great stuff and should try to apply. I realized the significance and great happiness to me was no longer whether I have a lifelong academic career, but that there has been a five-year period in my life where I enjoyed doing physics and was lucky enough to work with a great physicist I respect, who came to understand me and see my effort, which made me further motivated and look forward to every meeting when I got interesting results to show, or to have fun discussing and solving problems together.

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Chapter 1

Introduction and Overview

Calabi-Yau manifolds play a central role in string theory; these geometric spaces can describe extra dimensions of spacetime in supersymmetric compactifications of the theory [1]. While Calabi-Yau manifolds have been a major subject of study in mathematics and physics over the last three decades, the set of these geometries is still relatively poorly understood; it has not even yet been proved whether the number of distinct topological types of Calabi-Yau threefolds is finite or infinite. There are many interesting questions that connect physics and mathematics to explore about Calabi-Yau manifolds; one of the most intriguing aspects is probably the dualities between two string theories compactified on respectively a pair of Calabi-Yau manifolds that have the roles of certain topological quantities reversed; the equivalence is known as mirror symmetry. In this thesis, we study some aspects of Calabi-Yau manifold finiteness through the following two approaches to Calabi-Yau threefold constructions:

Following the approach of Batyrev [2], in 2000 Kreuzer and Skarke (KS) carried out a complete analysis of all reflexive polytopes in four dimensions, giving a systematic classification of those Calabi-Yau (CY) threefolds that can be realized as hypersurfaces in toric varieties [3, 4]. For many years the resulting database [4] has represented the bulk of the known set of Calabi-Yau threefolds, particularly at large Hodge numbers. Moreover, this largest known set of Calabi-Yau threefolds constructed from the class of over 400 million reflexive 4D polytopes exhibit the mirror symmetry structure [5, 6] that the Calabi-Yau threefolds generated by hypersurfaces in toric varieties
from the reflexive polytopes have the property that for every Calabi-Yau threefold $X$ with Hodge numbers $h^{1,1}(X), h^{2,1}(X)$ there is a corresponding mirror threefold $\tilde{X}$ with Hodge numbers $h^{1,1}(\tilde{X}) = h^{2,1}(X), h^{2,1}(\tilde{X}) = h^{1,1}(X)$.

On the other hand, a specific class of Calabi-Yau manifolds that admit a genus one or elliptic fibration is of particular interest to the approach to string theory known as F-theory. More recently, the study of F-theory [7, 8] has motivated an alternative method for the systematic construction of Calabi-Yau threefolds that have the structure of an elliptic fibration. By systematically classifying all bases that support an elliptically fibered CY [10, 11, 12, 13] and then systematically considering all possible Weierstrass tunings [14, 15] over each such base, it is possible in principle to construct all elliptically fibered Calabi-Yau threefolds. Unlike the general class of Calabi-Yau threefolds, it is known that the number of distinct topological types of elliptic and genus one Calabi-Yau threefolds is finite [16] (See also [17] for earlier work that laid the foundation for this proof, and [18] for a more constructive and explicit argument for finiteness). In recent years, an increasing body of circumstantial evidence has suggested that in fact a large fraction of the known Calabi-Yau manifolds admit an elliptic or genus one fibration. The analysis of complete intersection Calabi-Yau manifolds and generalizations thereof has shown that these classes of Calabi-Yau threefolds and fourfolds are also overwhelmingly dominated by elliptic and genus one fibrations [20, 22, 21, 24, 25, 26].

In this thesis, we first review briefly in Chapter 2 some background knowledge to set the context for the rest of the thesis; in the bulk of our work, we investigate the prevalence of elliptic fibrations among toric hypersurface Calabi-Yau threefolds, and explore the mirror symmetry structure of toric hypersurface elliptic Calabi-Yau threefolds. The work consists of a series of three parts:

In Chapter 3, which is based on the work in [36], we systematically construct from the F-theory point of view the Tate models of elliptically fibered Calabi-Yau with one or both Hodge numbers at least 240, and identify their corresponding toric hypersurface Calabi-Yau threefolds in the KS database that admit elliptic fibrations. Through the comparison of the two constructions, we found tuned Weierstrass models
of (1) large $h^{1,1}$ that had not been much studied before, which involve big toric bases or tunings of huge gauge groups, (2) a novel Tate tuning of $su(6)$ giving rise to exotic 3-index antisymmetric matter, (3) non-abelian symmetries over non-toric curves and higher genus curves in the bases, and (4) non-flat elliptic fibrations over toric bases that resolve into flat elliptic fibrations over non-toric bases.

In Chapter 4, which is based on the work in [37], we carry out a direct analysis of the toric hypersurface Calabi-Yau manifolds in the Kreuzer-Skarke database. There are 16 reflexive 2D polytopes that can act as fibers of a 4D polytope describing a Calabi-Yau threefold; the presence of any of these fibers in the 4D polytope indicates that the corresponding Calabi-Yau threefold hypersurface is genus one or elliptically fibered. We systematically consider all polytopes in the Kreuzer-Skarke database that are associated with Calabi-Yau threefolds with one or both Hodge numbers at least 140. We show that with only four exceptions these Calabi-Yau threefolds all admit an explicit elliptic or more general genus one fibration that can be seen from the toric structure of the polytope. We furthermore find that for toric hypersurface Calabi-Yau threefolds with small $h^{1,1}$, the fraction that lack a genus one or elliptic fibration decreases roughly exponentially with $h^{1,1}$. Together these results strongly support the notion that genus one and elliptic fibrations are quite generic among Calabi-Yau threefolds, particularly at large Hodge numbers.

In Chapter 5, which is based on the work in [38], we study the notion of mirror symmetry at the level of fibrations. With our results of the classification of all fibers of a given polytope, we can easily identify first the mirror fibers of a given pair. We further explore the relationship between the bases of the mirror fibration pair, which depends more subtly on the specific way in which the rays of the base lie over the fiber. We find that mirror symmetry “factorizes” between the fiber and the base for many fibrations: given a mirror pair of polytopes $\nabla$ and $\Delta$ that both have 2D fiber slices, there exists a mirror pair of slices $\nabla_2$ of $\nabla$ and $\Delta_2$ of $\Delta$; moreover, the base $\bar{B}$ of the elliptic Calabi-Yau threefold $\bar{X}$ associated with $\Delta$ with respect to $\Delta_2$ is encoded in the line bundles of the base $B$ of the elliptic Calabi-Yau threefold $X$ associated with $\nabla$ with respect to $\nabla_2$, and vice versa.
Chapter 2

Physics and Mathematics

Background

This chapter reviews some background knowledge on the basics of string compactifications, string dualities, M-theory, F-theory, and mathematical tools for the rest of this work. Most discussion is based on materials in textbooks [27, 28, 29] and lecture notes [30, 31, 32, 33], to which we direct readers interested in further details.

2.1 Compactifications and Dualities

This section reviews briefly some basic ideas in string compactifications and dualities between string theories that are relevant to the context of the thesis, and introduces F-theory and its relations to geometry.

2.1.1 Type II Superstring Theories

In the Neveu-Ramond-Schwarz formalism, superstring theories are described by sigma models from the 2D string worldsheet to the spacetime target where the worldsheet theory is an $\mathcal{N} = (1,1)$ superconformal field theory (SCFT). Quantum consistencies of the 2D SCFT worldsheet theory requires the spacetime dimension to be $D = 10$; furthermore, the modular invariance of the partition function requires removal of
Table 2.1: Ramond-Ramond fields and D-branes in type II string theories. As the Hodge dual is between the field strength in $D = 10$ dimensional spacetime, the subscripts of a pair of dual forms are summed to eight. The values of $p$, for which there are $D_p$ branes, are such that there is a $(p + 1)$-form $C_{p+1}$ or $(p + 1)$-dual form $C_{p+1}$ for each.

<table>
<thead>
<tr>
<th></th>
<th>$C_n$ and the dual forms $C_{D-n-2}$</th>
<th>$D_p$ branes</th>
</tr>
</thead>
<tbody>
<tr>
<td>IIA</td>
<td>$C_1, C_3, C_5, C_7$</td>
<td>D0, D2, D4, D6</td>
</tr>
<tr>
<td>IIB</td>
<td>$C_0, C_2, C_4, C_6, C_8$</td>
<td>D(-1), D1, D3, D5, D7</td>
</tr>
</tbody>
</table>

half of the sigma model spectrum, which is known as GSO projection. Two of the three possible GSO projections result in 10D $\mathcal{N} = (1, 1)$ and $\mathcal{N} = (2, 0)$ spacetime supersymmetric theories, which are known as type IIA and type IIB superstring theories respectively, as the corresponding low energy effective theories are IIA and IIB supergravity theories respectively. ¹

The resulting string spectrum after GSO projection organizes into spacetime bosonic and the corresponding fermionic fields in four sectors. The bosonic fields are in Neveu-Schwarz-Neveu-Schwarz (NSNS) and Ramond-Ramond (RR) sectors, and the fermionic fields are in the NSR and RNS sectors. The low energy field content in the NSNS sector is common to both type II theories, which contains a graviton field $G_{\mu\nu}$, a dilaton field $\phi$ (which is a scalar field that controls the perturbative expansion string parameter $g_s = e^{\phi}$), and a 2-form field $B$. The RR sector contains higher $n$-form field potentials $C_n$: in IIA theory, it contains 1-form and 3-form field potential, and in IIB theory it contains 0-form (scalar), 2-form, self-dual 4-form field potential (the resulting 5-form field strength satisfies $dC_4 = *(dC_4)$, where $*$ is the Hodge star operator on the spacetime.)

Another feature of superstring theories is the presence of solitonic BPS solutions to the supergravity theories (which preserve half of the supersymmetries). These are extended objects in spacetime called $D_p$-branes for values of $p$ for which there are corresponding $(p + 1)$-form or dual form fields, and $p$ specifies the spatial dimension of the extended object (see Table 2.1). $D_p$-branes are electrically charged under the

¹In any dimensions, $\mathcal{N}$ denotes the ratio of the number of supercharges to the smallest spinor representation; however, the allowed minimal representations, which could be Weyl, Majorana, or Majorana-Weyl spinors, depend on dimensions. When the (symplectic) Majorana condition is imposed (in $D = 2, 6 \mod 8$), each of the left and right chiral sectors of the Weyl representations are self-conjugate and independent, and therefore $\mathcal{N}$ is labeled by two separated numbers.
corresponding \((p + 1)\)-form, and the electric charge is measured by integrating the field strength over a sphere in the transverse spatial dimension enclosing the brane object

\[
q_e = \int_{S^{D-(p+1)-1}} *(dC_{p+1}) \in \mathbb{Z}. \tag{2.1}
\]

Dp-branes are also magnetically charged under the dual \((D - (p + 1) - 2)\)-forms, and the magnetic charge is similarly measured by integrating the dual field strength over the sphere

\[
q_m = \int_{S^{D-(p+1)-1}} ** (dC_{D-(p+1)-2}) \in \mathbb{Z}. \tag{2.2}
\]

For example, a D7-brane in the IIB theory is magnetically charged under the RR scalar \(C_0\).

From the perturbative perspective, IIA and IIB are theories of only closed strings, but with the branes included, they become theories of both open and closed strings. The open strings have boundaries on branes which inherit the non-abelian gauge symmetries of a stack of D-branes at their end points, and therefore D-branes and open string degrees of freedom are the sources of spacetime non-abelian gauge symmetry.

There are also NS5 branes coupled to the dual of the \(B\) field. When the spacetime has a compact component \(X\), \(B\) is defined only up to shifts by integral units: \(B \rightarrow B + \delta B\), where \(\delta B \in H^2(X, \mathbb{Z})\).

### 2.1.2 Circle Compactification and T-duality

From the worldsheet perspective, the basic quantum numbers for a quantized string state are left and right oscillator numbers \(N, \tilde{N}\). In the Kaluza-Klein (KK) compactification on a circle \(S^1\) of radius \(R\) (impose \(R\)-periodic boundary condition on one of the spatial direction), there is an additional quantum number \(n\) from the quantization of momentum in the circle direction, and there is also a topological quantity \(w\) that accounts for the number of times a closed string wraps around the circle. The spacetime mass formula of a state specified by these numbers is given by

\[
m^2 = \frac{n^2}{R^2} + \frac{w^2 R^2}{\alpha'^2} + \frac{2}{\alpha'}(N + \tilde{N} - 2), \tag{2.3}
\]
where $\alpha'$ is the only free parameter in the string theory that sets the string scale (the string tension is $T = \frac{1}{2\pi \alpha'}$); $\alpha'$ is occasionally set to one. Note that the formula (2.3) is unchanged under simultaneous exchange of $R \leftrightarrow \frac{\alpha'}{R}$ and $n \leftrightarrow w$. This led to the original idea of T-duality that string theory was invariant under this operation.

From the spacetime perspective, the KK compactification of the type II theories results in 9D low energy effective theories (higher KK modes are integrated out):

<table>
<thead>
<tr>
<th>10D fields</th>
<th>$S^1$ reduction</th>
<th>9D fields</th>
</tr>
</thead>
<tbody>
<tr>
<td>The graviton</td>
<td>$\rightarrow$</td>
<td>a graviton, a 1-form, a scalar</td>
</tr>
<tr>
<td>The dilaton</td>
<td>$\rightarrow$</td>
<td>a scalar</td>
</tr>
<tr>
<td>The anti-symmetric $B$ field</td>
<td>$\rightarrow$</td>
<td>a 1-form, a 2-form</td>
</tr>
<tr>
<td>RR fields $C_n$</td>
<td>$\rightarrow$</td>
<td>an $n$-form, an $(n - 1)$-form</td>
</tr>
</tbody>
</table>

The degrees of freedom of the overall 9D spectrum is the same in both IIA and IIB cases (note that as $C_4$ in IIB is self-dual, the reduced 4-form and 3-form are dual to each other in the 9D theory, and only one of them is counted for independent degrees of freedom.) The 9D theory is a $U(1) \times U(1)$ gauge theory, where the two KK gauge fields come from the reduction of the graviton and the $B$ field. At the $R = \sqrt{\alpha'}$ self-dual point, the spacetime gauge symmetry is enhanced to $SU(2) \times SU(2)$.

T-duality reverses the two string theories: type IIA compactified on a circle of radius $R$ is equivalent to type IIB on a circle of radius $\alpha'/R$. Note while the KK compactifications of IIA and IIB are completely identical (in particular, the chirality of the IIB theory is lost in the compactification) from the worldsheet perspective (perturbative closed strings only), with the open strings and D-branes included, where different boundary conditions are specified in IIA and IIB, the RR sector degrees of freedom are distinguished; in particular, D$p$-branes are turned into D$(p + 1)$ branes.

### 2.1.3 S-duality and SL(2, Z) Symmetry in IIB String Theory

Type IIB supergravity action is invariant up to field redefinition under the exchange of the 2-forms $B$ and $C_2$ if the string coupling is inversed $g_s \rightarrow 1/g_s$. This suggests a duality between the strong and the weak coupling limits in the IIB theory, which
is known as S-duality. Furthermore, as $C_0 \rightarrow C_0 + 1$ is an invariant transformation of the action, it is natural to introduce a complex quantity, which is known as the axiodilaton field, in the upper half complex plane

$$\tau = C_0 + \frac{i}{g_s}. \quad (2.4)$$

Then both $\tau \rightarrow \tau + 1$ and $\tau \rightarrow -1/\tau$ are symmetries that leave IIB theory invariant. These two transformations generate the entire $SL(2, \mathbb{Z})$ group. This is the famous $SL(2, \mathbb{Z})$ symmetry of the type IIB string theory. \footnote{In fact, the IIB supergravity action is invariant under larger continuous $SL(2, \mathbb{R})$ transformations. However, as we discussed in §2.1.1 that fields are coupled electrically or magnetically to various branes in the IIB string theory, the fields then must be represented by integer cohomology in order for the action to be invariant. Therefore, the allowed transformations in the string theory are restricted to integral shifts.}

The BPS states in IIB string theory then organize into $SL(2, \mathbb{Z})$ doublets as they are charged under the corresponding fields, which are collectively denoted as $(p, q)$ -branes, where $p, q$ are relatively prime. The doublets are $(p$ fundamental strings, $q$ D1-branes) strings, $(p$ NS5-branes, $q$ D5-branes) 5-branes, and D7-branes are promoted to $(p, q)$ 7-branes. \footnote{The self-dual RR $C_4$ was left unchanged in the transformation, so do D3 branes, which are the objects charged under the field strength.}

### 2.1.4 M-theory, F-theory, and M/F-Theory Duality

M-theory is a quantum theory associated with 11D supergravity, which is the largest dimension that allows a supergravity theory. It can be viewed as a strong coupling limit of the type IIA string theory. The bosonic fields are simply a graviton and a 3-form gauge potential $C_3^M$. If we perform a KK reduction of M-theory on a circle, we obtain a 10D graviton, a 1-form, and a scalar from the 11D graviton; and a 10D 2-form and a 3-form from the 11D 3-form. These 10D fields match exactly with the fields in the 10D IIA supergravity discussed in §2.1.1.

Consider compactifying M-theory on a two torus $T^2$ ($S^1 \times S^1$): The reduction of the M-theory graviton results in a 9D graviton, two 1-forms, and the scalar fields are described by the parameters of the flat metric of the compact space $T^2$, which are
the area of the torus and the ratio of the periods \( \tau = \frac{R_A}{R_B} \mod \text{SL}(2, \mathbb{Z})^d \). The reduction of the 3-forms \( C^M_3 \) on \( T^2 \) results in one 9D massless 3-form (two indices of the 3-form are on the torus), two 2-forms (one of the indices is on the torus where there are two choices), one 1-form (none of the indices is on the torus). This 9D spectrum maps to the spectrum of the IIB theory on a circle discussed in §2.1.2. Moreover, the geometrical \( \text{SL}(2, \mathbb{Z}) \) invariance of two-tori \( \tau \) in the M-theory compactification maps over to the IIB \( \text{SL}(2, \mathbb{Z}) \) invariance discussed in §2.1.3 of the axiodilaton field \( \tau \) in equation (2.4).

Consider IIB theory on a spacetime of the form \( B^d \times \mathbb{R}^{1,9-d} \) with 7-branes \( \Sigma_{d-2} \times \mathbb{R}^{1,9-d} \) that completely fill up the \( \mathbb{R}^{1,9-d} \) space, and \( \Sigma_{d-2} \) is a complex codimension one locus in \( B^d \). The axiodilaton field is shifted by \( \tau \to \tau + 1 \) going around \((1,0)\) 7-branes (D7-branes), or more generally, going around \((p,q)\) 7-branes

\[
\tau \to \frac{a\tau + b}{c\tau + d}
\]

where \( a, b, c, d \) are the monodromy matrix

\[
M_{[p,q]} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} p & r \\ q & s \end{pmatrix} M_{[1,0]} \begin{pmatrix} p & r \\ q & s \end{pmatrix}^{-1}
\]

for \( \begin{pmatrix} p & r \\ q & s \end{pmatrix} \in \text{SL}(2, \mathbb{Z}) \), and \( M_{[1,0]} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \) is the monodromy matrix of D7-branes. Further compactifying one of the flat direction \( (B^d \times \mathbb{R}^{1,9-d} \to B^d \times S^1 \times \mathbb{R}^{1,8-d}) \) would correspond to an M-theory compactification picture where the fixed two-torus would have to be promoted to a family of two-tori moving around over \( B^d \), and going around a circle in the parameter space of the two-tori would come back to the same torus but with a different basis having been chosen for the homology of the torus (this is known as monodromy); i.e.,

IIB theory on \( B^d \times S^1 \times \mathbb{R}^{1,8-d} \) is dual to M-theory on \( X^{d+2} \times \mathbb{R}^{1,8-d} \), where \( X^{d+2} \) is a \( T^2 \) fibration over \( B^d \); and by T-duality, the IIB theory in the circle decompactified.

\(^4\)The \( \text{SL}(2, \mathbb{Z}) \) symmetry is about choosing a different basis for the cycles on the two-torus.
limit $B^d \times \mathbb{R}^{1,9-d}$ corresponds to taking the area parameter of the M-theory torus to the zero limit.

To explore the SL(2,\mathbb{Z}) symmetry (the axiodilaton profile $\tau$) over the geometry of the compact space $B^d$ gives rise to the study of F-theory.

### 2.1.5 Supersymmetric Compactifications Geometry

Gauge theories can be described in the mathematical framework of principal bundles (fiber bundles with the structure group to be the gauge group). Gauge fields and matter fields are encoded the principal bundles and their associated representation bundles, respectively.

The requirement of unbroken supersymmetry theory in the lower dimensional spacetime in compactifications imposes certain conditions on bundles over the compact part of the 10D spacetime. Vanishing supersymmetric variations requires a non-trivial covariantly constant spinor (a section of the spinor bundle on 10D spacetime), which is independent of the uncompactified part. This is guaranteed if the compact space on which M-theory or type II theories are compactified is Ricci flat. For examples, the $S^1 = T^1, T^2$ compactifications that we saw in §2.1.2, §2.1.4 are Ricci flat. If we require unbroken supersymmetry in the M-theory compactified on $X^{d+2} \times \mathbb{R}^{1,8-d}$, the $T^2$ fibration $X^{d+2}$ has to be Ricci flat.

From the F-theory perspective, the supersymmetry condition requires

$$R_{MN} = \nabla_M \nabla_N \phi$$ \hspace{1cm} (2.7)

in the presence of 7-branes, where $R_{MN}$ is the Ricci tensor of the base $B^{d-2}$ and $\phi$ is the dilaton field in IIB theory. By the M/F-theory duality and the mapping of the varying $\tau$ discussed at the end of §2.1.5, the condition in equation (2.7) must be equivalent to the Ricci flat condition in the M-theory compactification, which is indeed the case as will be discussed in §2.2.
2.2 Elliptically Fibered Calabi-Yau Manifolds

This section briefly reviews the basics of Calabi-Yau Manifolds, which are the mathematical objects being studied in supersymmetric string compactifications; and introduces Weierstrass models, which provides the algebraic geometry description of elliptic fibration being used in F-theory.

2.2.1 Calabi-Yau Manifold Basics

We discussed in §2.1.5 that in order to preserve supersymmetry in lower dimensions, the compact spacetime must admit a Ricci-flat metric. Such kind of manifolds are known as Calabi-Yau (CY) manifolds. An equivalent condition for a $D = n$ Calabi-Yau manifold (CY$_n$) is that it is a compact $n$-dimensional Kahler manifold with SU($n$) holonomy (there exists a unique nowhere vanishing holomorphic $n$-form). Another equivalent condition proved by Yau shows that for any compact Kahler manifold with vanishing first Chern class $c_1(X_n) = 0$, it admits a unique Ricci-flat metric.

The topology of a Calabi-Yau $n$-fold $X_n$ is characterized by Hodge numbers $h^{p,q}(X_n)$, which are the dimensions of the Dolbeault cohomology groups $H^{p,q}(X_n)$ of a complex manifold; i.e., $h^{p,q}$ is the dimension of the $(p,q)$-forms on $X_n$, and $0 \leq p, q \leq n$. For any complex manifold, $h^{0,0} = 1$, corresponding to the constant function; by complex conjugation, $h^{p,q} = h^{q,p}$; for Kahler manifolds, $h^{p,q} = h^{n-p,n-q}$ (Poincare duality). This thesis will be focused on Calabi-Yau threefolds, the independent Hodge numbers of which are $h^{3,0} = 1, h^{2,0} = h^{1,0} = 0$ by the full SU(3) holonomy condition, $h^{0,0} = 1, h^{1,1} = $ the number of deformations of Kahler structure, and $h^{2,1} = $ the number of deformations of complex structure.

2.2.2 Elliptically Fibered Calabi-Yau Manifolds in F-theory

The elliptic fibration in F-theory is a holomorphic map from a complex $n$-dimensional Calabi-Yau manifold $X_n$ to a complex $(n - 1)$-dimensional compact spacetime manifold $B_{n-1}$ (the base).\(^5\) The general fiber is a two-torus away from the locus of 7-branes;\(^5\)The notation in §2.1 denotes real dimensions in superscripts.
along the $7$-brane loci, the fiber degenerates. The axiodilaton field $\tau$ is a holomorphic section of the fibration map, and variation of the axiodilaton field defines a holomorphic line bundle $\mathcal{L}$ over $B_{n-1}$. In terms of the bundle data, the supersymmetry condition in equation (2.7) is equivalent to

$$c_1(B_{n-1}) = c_1(\mathcal{L}).$$

Therefore $c_1(X_n) = c_1(B_{n-1}) - c_1(\mathcal{L}) = 0$; i.e., $X_n$ has to be a Calabi-Yau manifold. The study of supersymmetric compactifications in F-theory has thus turned into the study of elliptically fibered Calabi-Yau manifolds and mappings between geometry and physics.

### 2.2.3 Weierstrass Models of Elliptic Fibration

F-theory models can then be systematically studied by first choosing a base $B_n$ and then specifying an elliptic fibration $X_{n+1}$. A Weierstrass model is an algebraic equation which describes an elliptic fibration. It provides an efficient way for us to describe the data needed for a varying family of $\tau \mod \text{SL}(2, \mathbb{Z})$ subject to supersymmetry constraints in F-theory physics.

The Weierstrass model of an elliptic fibration with a section over a base $B_n$ is given by

$$y^2 = x^3 + fxz^4 + gz^6,$$

where $f \in \pi^*\mathcal{L}^4$, $g \in \pi^*\mathcal{L}^6$ are sections of line bundles. More abstractly, we take the weighted projective bundle

$$\pi : P = \mathbb{P}_{2,3,1}[\mathcal{L}^2 \oplus \mathcal{L}^3 \oplus \mathcal{O}_{B_n}] \to B_n,$$

The Calabi-Yau condition (2.8) on the total space $X_{n+1}$ requires that $\mathcal{L} = \mathcal{O}(-K_{B_n})$, where $K_{B_n}$ is the canonical class of the base. Therefore, $f \in \mathcal{O}(-4K_{B_n})$, $g \in \mathcal{O}(-6K_{B_n})$. Over a fixed point in the base, the elliptic fiber is cut out by the equation (2.9) now an equation in terms of weighted projective coordinates $[x : y : z]$ of the
fiber ambient space $\mathbb{P}^{2,3,1}$; as in the weighted projective bundle, $x \in \mathcal{O}_P(2) \otimes \pi^* \mathcal{L}^2, y \in \mathcal{O}_P(3) \otimes \pi^* \mathcal{L}^3, z \in \mathcal{O}_P(1)$ are sections of line bundles.

Consider $n = 2$ (an elliptic Calabi-Yau threefold $X_3$ over a complex two-dimensional base $B_2$), so the divisors in the base are curves. The elliptic fiber becomes singular over the codimension-one loci in the base where the discriminant

$$\Delta = 4f^3 + 27g^2$$

vanishes. Kodaira classified the types of singular fiber at a generic point along an irreducible component $\{\sigma = 0\}$ of the discriminant locus $\{\Delta = 0\}$, which is determined by the orders of vanishing of $f$, $g$, and $\Delta$ in an expansion in $\sigma$. These are the singular fibers that can be resolved in a way such that the resolved manifold $\tilde{X}_3$ is still a Calabi-Yau. The resolved fibers consist of irreducible $\mathbb{P}^1$ components that intersect to form the Dynkin diagram of certain non-abelian symmetries. The singularity types and the corresponding symmetry algebras are listed in Table 2.2.

The gauge algebras that are further determined by monodromy conditions [46, 61] are those of types $I_n, I^*_n, IV, IV^*$, where some factorizability conditions are imposed on the terms of $f, g, \Delta$ of lowest degrees of vanishing order along $\{\sigma = 0\}$. These conditions are summarized in Table 2.3, in terms of the first non-vanishing sections $f_i(\zeta), g_j(\zeta), \Delta_k(\zeta)$ in the local expansions

$$f(\sigma, \zeta) = f_0(\zeta) + f_1(\zeta)\sigma + \cdots,$$  \hspace{1cm} (2.12)
$$g(\sigma, \zeta) = g_0(\zeta) + g_1(\zeta)\sigma + \cdots,$$  \hspace{1cm} (2.13)
$$\Delta(\sigma, \zeta) = \Delta_0(\zeta) + \Delta_1(\zeta)\sigma + \cdots,$$  \hspace{1cm} (2.14)

where $\{\zeta = 0\}$ defines a divisor that intersects $\{\sigma = 0\}$ transversely so that $\sigma, \zeta$ together serve as local coordinates on an open patch of base.

The short form Weierstrass model in equation (2.9) is the most general form for an elliptic Calabi-Yau threefold with a section. Most cases discussed in this thesis are elliptically fibered Calabi-Yau threefolds that always have a section and
<table>
<thead>
<tr>
<th>Type</th>
<th>$\text{ord}(f)$</th>
<th>$\text{ord}(g)$</th>
<th>$\text{ord}(\Delta)$</th>
<th>singularity</th>
<th>nonabelian symmetry algebra</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I_0$</td>
<td>$\geq 0$</td>
<td>$\geq 0$</td>
<td>$0$</td>
<td>none</td>
<td>none</td>
</tr>
<tr>
<td>$I_n$</td>
<td>$0$</td>
<td>$0$</td>
<td>$n \geq 2$</td>
<td>$A_{n-1}$</td>
<td>$\text{su}(n)$ or $\text{sp}([n/2])$</td>
</tr>
<tr>
<td>$II$</td>
<td>$\geq 1$</td>
<td>$1$</td>
<td>$2$</td>
<td>none</td>
<td>none</td>
</tr>
<tr>
<td>$III$</td>
<td>$1$</td>
<td>$\geq 2$</td>
<td>$3$</td>
<td>$A_1$</td>
<td>$\text{su}(2)$</td>
</tr>
<tr>
<td>$IV$</td>
<td>$\geq 2$</td>
<td>$2$</td>
<td>$4$</td>
<td>$A_2$</td>
<td>$\text{su}(3)$ or $\text{su}(2)$</td>
</tr>
<tr>
<td>$I_0^*$</td>
<td>$\geq 2$</td>
<td>$\geq 3$</td>
<td>$6$</td>
<td>$D_4$</td>
<td>$\text{so}(8)$ or $\text{so}(7)$ or $\text{g}_2$</td>
</tr>
<tr>
<td>$I_n^*$</td>
<td>$2$</td>
<td>$3$</td>
<td>$n \geq 7$</td>
<td>$D_{n-2}$</td>
<td>$\text{so}(2n-4)$ or $\text{so}(2n-5)$</td>
</tr>
<tr>
<td>$IV^*$</td>
<td>$\geq 3$</td>
<td>$4$</td>
<td>$8$</td>
<td>$E_6$</td>
<td>$\epsilon_6$ or $\epsilon_4$</td>
</tr>
<tr>
<td>$I_0^*$</td>
<td>$3$</td>
<td>$\geq 5$</td>
<td>$9$</td>
<td>$E_7$</td>
<td>$\epsilon_7$</td>
</tr>
<tr>
<td>$II^*$</td>
<td>$\geq 4$</td>
<td>$5$</td>
<td>$10$</td>
<td>$E_8$</td>
<td>$\epsilon_8$</td>
</tr>
<tr>
<td>non-min</td>
<td>$\geq 4$</td>
<td>$\geq 6$</td>
<td>$\geq 12$</td>
<td>none</td>
<td>does not occur in F-theory</td>
</tr>
</tbody>
</table>

Table 2.2: Kodaira classification of singularities in the elliptic fiber along codimension one loci in the base in terms of orders of vanishing of the parameters $f, g$ in the Weierstrass model (2.9) and the discriminant locus $\Delta$.

<table>
<thead>
<tr>
<th>ord($f$)</th>
<th>ord($g$)</th>
<th>ord($\Delta$)</th>
<th>algebra</th>
<th>monodromy condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I_n$</td>
<td>$0$</td>
<td>$0$</td>
<td>$n$</td>
<td>$\text{su}(n)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>since $\Delta_0 = 0$, locally $f_0(\zeta) = -\frac{1}{3}u_0^2$ and $g_0(\zeta) = \frac{2}{37}u_0^3$ for some $u_0(\zeta)$, which is a perfect square $\text{sp}([n/2])$ otherwise</td>
</tr>
<tr>
<td>$IV$</td>
<td>$\geq 2$</td>
<td>$2$</td>
<td>$4$</td>
<td>$\text{su}(3)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$g_2(\zeta)$ is a perfect square $\text{su}(2)$ otherwise</td>
</tr>
<tr>
<td>$I_0^*$</td>
<td>$\geq 2$</td>
<td>$\geq 3$</td>
<td>$6$</td>
<td>$\text{so}(8)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$x^3 + f_2(\zeta)x + g_3(\zeta)$ $= (x - a)(x - b)(x + a + b)$ for some $a(\zeta), b(\zeta)$ $\text{so}(7)$ otherwise</td>
</tr>
<tr>
<td>$I_n^*$</td>
<td>$2$</td>
<td>$3$</td>
<td>$n \geq 7$</td>
<td>$\text{so}(2n-4)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>since $\Delta_6 = 0$, locally $f_2(\zeta) = -\frac{1}{3}u_1^2$ and $g_3(\zeta) = \frac{2}{37}u_1^3$ for some $u_1(\zeta)$; $\Delta_n(\zeta)$ is a perfect square for odd $n$ $\Delta_n(\zeta)$ is a perfect square for even $n$ $\text{so}(2n-5)$ otherwise</td>
</tr>
<tr>
<td>$IV^*$</td>
<td>$\geq 3$</td>
<td>$4$</td>
<td>$8$</td>
<td>$\epsilon_6$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$g_4(\zeta)$ is a perfect square $\epsilon_4$ otherwise</td>
</tr>
</tbody>
</table>

Table 2.3: Monodromy conditions for certain algebras to satisfy in additional to the desired orders of vanishing of $f, g, \Delta$: $f_i(\zeta), g_j(\zeta), \Delta_k(\zeta)$ are coefficients of the expansions in equations (2.12)-(2.14).
therefore in principle admit a short form Weierstrass form realization. There can also be genus one fibered Calabi-Yau threefolds (lacking a global section), which can be related to Weierstrass models of elliptic Calabi-Yau threefolds through the Jacobian construction (described from the physics perspective in [48, 51]). The physics of these threefolds is more subtle, involving discrete gauge groups [49, 50, 52, 53, 54].

2.3 6D F-theory from Geometry of Elliptically Fibered Calabi-Yau Threefolds

This section briefly reviews the 6D supergravity theory in F-theory compactification, and the relations between the geometry of the elliptically fibered Calabi-Yau threefolds and the 6D F-theory physics. Most this section is based on the materials in [55], to which we direct readers interested in further details.

2.3.1 6D $\mathcal{N} = (1, 0)$ F-theory/5D $\mathcal{N} = 1$ M-theory Duality

As discussed in §2.1.4, IIB theory on $B_2 \times \mathbb{R}^{1,5}$ is dual to M-theory on $X_3 \times \mathbb{R}^{1,4}$, where $X_3$ is a $T^2$ fibration over $B_2$ in the circle decompactified limit where the volume of the M-theory torus is shrunk to zero size. We are interested in theories with eight supercharges; i.e., $\mathcal{N} = (1, 0)$ in 6D and $\mathcal{N} = 1$ in 5D, in which two kinds of supermultiplets are allowed: vector multiplets and hypermultiplets; in addition, 6D $\mathcal{N} = 1$ theory has also tensor supermultiplets, which are neutral with respect to gauge symmetries.

In the 6D theory, the vector multiplets are in the adjoint representations of non-abelian gauge group factors $G_i$, and the hypermultiplets are in any other representations of the groups. The bosonic content of various multiplets are:

- gravity multiplet: the graviton, a self-dual 2-form
- vector multiplet: a vector
- hypermultiplet: two complex scalars
• tensor multiplet: an anti-self-dual 2-form, one real scalar

The numbers of vector multiplets, hypermultiplets, tensor multiplets are denoted by $V, H, T$, respectively.

On the other hand, in the 5D theory there are

• gravity multiplet: the graviton, a vector

• vector multiplet: a vector, a scalar

• hypermultiplet: two complex scalars

The massless spectrum of a six-dimensional effective theory from F-theory compactification is related to the geometric data of the internal manifold, which is a singular elliptically fibered Calabi-Yau threefold ($\text{CY3}$) $X_3$ over a two-dimensional base $B_2$ (complex dimensions). The physics data can be extracted by studying the singular fibers by means of the Weierstrass models discussed in §2.2.3. The non-abelian gauge symmetries of the 6D effective theory arise from the seven-branes located at the discriminant locus (equation 2.11), and correspond exactly to the non-abelian symmetries of the singular fiber (see Table 2.2). At the intersections of seven-branes there are localized matter fields that are hypermultiplets in the 6D theory; the representations of the matter fields can be determined from the detailed form of the singularities over the codimension-two points in the base (see e.g. [45, 46, 47]).

On the other hand, the dual M-theory is compactified on a smooth Calabi-Yau threefold $\tilde{X}_3$, which is the resolution of $X_3$ by resolving the singular fibers, and the non-abelian gauge symmetry in F-theory becomes abelian gauge symmetry in the dual M-theory. The Hodge numbers, $h^{1,1}$ and $h^{2,1}$, of $\tilde{X}$ can be related to the (massless) matter content of the 6D theory:

$$h^{1,1}(\tilde{X}) = r + T + 2,$$

(2.15)

---

6 The non-abelian theory moves to the “Coulomb branch” in the M-theory picture where M2-branes in the M-theory wrap the $\mathbb{P}^1$ irreducible components of the resolved fibers are massive vector fields in the Coulomb branch. If the $\mathbb{P}^1$'s were shrunk to zero size, the vector bosons became massless and the non-abelian symmetry was restored (F-theory limit).
where the number of tensor multiplets $T$ is determined already by the choice of base $B_2$,
\[ T = h^{1,1}(B_2) - 1, \tag{2.16} \]
and $r = r_{\text{abelian}} + \sum_i r_i$ is the total rank of the gauge group,
\[ G = U(1)^{r_{\text{abelian}}} \times \prod_{\text{non-abelian factors } i} G_i, \tag{2.17} \]
of the 6D effective theory. We also have
\[ h^{2,1}(\hat{X}) = H_{\text{neutral}} - 1, \tag{2.18} \]
where $H_{\text{neutral}}$ is the number of hypermultiplets that are neutral under the Cartan subalgebra\(^7\) of the gauge group $G$ of the 6D F-theory.

There can also be abelian gauge symmetries, which arise from additional rational sections of the elliptic fibration [8]. The study of $u(1)$ symmetries is more subtle in that it relates to the global structure of the fibration, as opposed to non-abelian symmetries where we can just study singular fibers locally.

### 2.3.2 Anomaly Constraints on Matter Content

Six-dimensional $\mathcal{N} = (1, 0)$ supergravity theories potentially suffer from gravitational, gauge, and mixed gauge-gravitational anomalies. We focus in this thesis primarily on non-abelian gauge anomalies, though similar considerations hold for abelian gauge factors. In principle, the matter content of a 6D theory can be determined by a careful analysis of the codimension two singularities in the geometry. In many situations, however, the generic matter content of a low-energy theory is uniquely determined by the gauge group content and anomaly cancellation.

\(^7\)In other words, this counts fields that are neutral matter fields in the 5D M-theory sense but may transform under the unhiggsed non-abelian factors of the 6D F-theory. Often, matter charged under the non-abelian factors is still charged under the Cartan subalgebra, but for certain representations of some non-abelian groups there can be charged matter that is neutral under the Cartan subalgebra.
On the one hand, the anomaly information can be encoded in an 8-form $I_8$, which is built from the 2-forms characterizing the non-abelian field strength $F$ and the Riemann tensor $R$, and which has coefficients that can be computed in terms of $T, V, H$, and the explicit numbers of chiral matter fields in different representations. On the other hand, the anomalies can be cancelled through a generalized Green-Schwarz term if $I_8$ factorizes for some constant coefficients $a_i^\alpha, b_i^\beta$ in the vector space $\mathbb{R}^{1,T}$ associated with self-dual and anti self-dual two-forms in the gravity and tensor multiplets,

$$I_8 = \frac{1}{2} \Omega_{\alpha\beta} X^\alpha_4 X^\beta_4,$$  \hspace{1cm} (2.19)

where

$$X^\alpha_4 = \frac{1}{2} a^\alpha \text{tr} R^2 + \sum_i b_i^\alpha \frac{2}{\lambda_i} \text{tr} F_i^2.$$  \hspace{1cm} (2.20)

Here $\Omega_{\alpha\beta}$ is a signature $(1, T)$ inner product on the vector space, and $\lambda_i$ are normalization constants for the non-abelian gauge group factors $G_i$. Then, using the notation and conventions of [63], the conditions for anomaly cancellation are obtained by equating the coefficients of each term from the two polynomials

$$R^4 : \quad H - V = 273 - 29T,$$  \hspace{1cm} (2.21)

$$F^4 : \quad 0 = B_{Adj}^i - \sum_R x^i_R B^i_R,$$  \hspace{1cm} (2.22)

$$(R^2)^2 : \quad a \cdot a = 9 - T,$$  \hspace{1cm} (2.23)

$$F^2 R^2 : \quad a \cdot b_i = \frac{1}{6} \lambda_i \left( A^i_{Adj} - \sum_R x^i_R A^i_R \right),$$  \hspace{1cm} (2.24)

$$(F^2)^2 : \quad b_i \cdot b_i = \frac{1}{3} \lambda_i^2 \left( \sum_R x^i_R C_R^i - C_{Adj}^i \right),$$  \hspace{1cm} (2.25)

$$F_i^2 F_j^2 : \quad b_i \cdot b_j = 2 \sum_{R,S} x^i_{RS} A^i_RA^j_S, \quad i \neq j,$$  \hspace{1cm} (2.26)
where $A_R, B_R, C_R$ are group theory coefficients defined by
\[
\begin{align*}
\text{tr}_R F^2 &= A_R \text{tr}_{\text{fund.}} F^2, \\
\text{tr}_R F^4 &= B_R \text{tr}_{\text{fund.}} F^4 + C_R (\text{tr}_{\text{fund.}} F^2)^2,
\end{align*}
\tag{2.27}
\tag{2.28}
\]

$x_R^i$ is the number of matter fields in the representation $R$ of the non-abelian factor $G_i$, and $x_{RS}^{ij}$ is the number of matter fields in the $(R, S)$-representation of $G_i \otimes G_j$.

For 6D theories coming from an F-theory compactification, the vectors $a, b^i$ are related to homology classes in the base $B_2$ through the relations
\[
\begin{align*}
a & \leftrightarrow K_{B_2}, \\
b_i & \leftrightarrow C_i,
\end{align*}
\tag{2.29}
\tag{2.30}
\]

where, again, $K_{B_2}$ is the canonical class of $B_2$, and $C_i \in H_2(B_2, \mathbb{Z})$ are irreducible curves in the base supporting the singular fibers associated with the non-abelian gauge group factors $G_i$. With this identification, the Dirac inner products between vectors in $\mathbb{R}^{1,7}$ are related to intersection products between divisors in the base. The generic matter content can then simply be determined from the values of the vectors $a, b^i$.  

The gravitational anomaly cancellation condition (2.21) gives $H = V - 273 - 29T$, where $V$ is the number of vector multiplets; i.e., the dimension of the gauge group $G$, and $H = H_{\text{charged}} + H_{\text{neutral}}$ is the total number of hypermultiplets (separated into neutral and charged matter under the Cartan of the gauge group $G$), and therefore there is another useful expression for $h^{2,1}$ besides equation (2.18)
\[
\begin{align*}
h^{2,1}(\hat{X}) &= 272 + V - 29T - H_{\text{charged}}.
\end{align*}
\tag{2.31}
\]

\begin{itemize}
\item[A summary of $A_R, B_R, C_R$ in different representations and $\lambda_i$ for different non-abelian gauge groups can be found in appendix B in [15].
\item[9]For each representation the matter content contains one complex scalar field and a corresponding field in the conjugate representation. For special representations like the 2 of $SU(2)$, the representation is pseudoreal, so that the conjugate need not be included; such a field is generally referred to as a “half-hypermultiplet”.
\item[10]For most of the theories we consider here the matter content follows uniquely in this way from the values of $a, b^i$. In some situations, however, more exotic matter representations can arise; we encounter some cases of this later in chapter 3, such as the three-index antisymmetric representation of $SU(6)$.
\end{itemize}
2.4 Calabi-Yau Hypersurfaces in Toric Varieties

Following [29], we first review some basic features of toric geometry, and we introduce Batyrev's Calabi-Yau hypersurface construction in toric variety and the Kreuzer-Skarke database, which has classified all the Calabi-Yau threefolds of the construction.

2.4.1 Brief Review of Toric Varieties

An $n$-dimensional toric variety $X_\Sigma$ can be constructed by defining the fan of the toric variety. A fan $\Sigma$ is a collection of strongly convex cones in $N_\mathbb{R} = N \otimes \mathbb{R}$, which are generated by a finite number of vectors in $N$ and which contain no line through the origin, each with the apex at the origin, and where $N$ is a rank $n$ lattice, satisfying the conditions that

- Each face of a cone in $\Sigma$ is also a cone in $\Sigma$.
- The intersection of two cones in $\Sigma$ is a face of each.

Then $X_\Sigma$ can be described by the homogeneous coordinates $z_i$ corresponding to the one-dimensional cones $v_i$ (also called rays) of $\Sigma$; $X_\Sigma$ may be constructed as the quotient of an open subset in $\mathbb{C}^k$ ($k$ is the number of rays), by a group $G$, 

$$X_\Sigma = \frac{\mathbb{C}^k - Z(\Sigma)}{G}, \quad \text{ (2.32)}$$

where

- $Z(\Sigma) \subset \mathbb{C}^k$ is the union of the zero sets of the polynomial sets $S = \{z_i\}$ associated with the sets of rays $\{v_i\}$ that do not span a cone of $\Sigma$.
- $G \subset (\mathbb{C}^*)^k$ is the kernel of the map

$$\phi : (\mathbb{C}^*)^k \rightarrow (\mathbb{C}^*)^n, (z_1, \ldots, z_k) \mapsto \left(\prod_{j=1}^{k} z_j^{v_j,1}, \ldots, \prod_{j=1}^{k} z_j^{v_j,n}\right),$$

where $v_{j,l}$ specifies the $l$th component of the ray $v_j$ in the coordinate representation in $\mathbb{C}^n$. 

39
Toric divisors $D_i$ are given by the sets $D_i \equiv \{ z_i = 0 \}$ associated to all the rays $v_i$. The anti-canonical class $-K$ of a toric variety is given by the sum of toric divisors

$$-K = \sum_i D_i.$$  \hspace{1cm} (2.33)

Smooth two-dimensional toric varieties are particularly simple. The irreducible effective toric divisors are rational curves with one intersecting another forming a closed linear chain. This is easily seen from the 2D toric fan description, where each ray of the 2D fan corresponds to an irreducible effective toric divisor. The intersection products are also easy to read off from the fan diagram, where (including divisors cyclically by setting $D_{k+1} \equiv D_1$, etc.)

$$D_i \cdot D_{i+1} = 1,$$  \hspace{1cm} (2.34)

and the self-intersection of each curve is

$$D_i \cdot D_i = m_i,$$  \hspace{1cm} (2.35)

where $m_i$ is such that

$$-m_i v_i = v_{i-1} + v_{i+1},$$  \hspace{1cm} (2.36)

and zero otherwise. We will generally denote the data defining a smooth 2D toric base by the sequence of self-intersection numbers. (The 2D fan can be recovered given the intersections, up to lattice automorphisms.)

In the context of this thesis, toric varieties play two distinct but related roles. On the one hand, we can use toric geometry to describe many of the bases that support elliptically fibered Calabi-Yau threefolds. On the other hand, toric geometry can be used to describe ambient fourfolds in which CY threefolds can be embedded as hypersurfaces, as we describe in the next section.
2.4.2 Batyrev’s Construction of Calabi-Yau Manifolds

Given a lattice polytope, which is the convex hull of a finite set of lattice points (in particular, the vertices are lattice points), we may define a face fan by taking rays to be the vertices of the lattice polytope, and the top-dimensional \((n\)-dimensional) cones to correspond to the facets of the polytope. By including more lattice points in addition to vertices of the polytope as rays, and thus subdividing (“triangulating”) the facets of the polytope into multiple smaller top-dimensional cones, we can refine the fan to impose further properties such as simpliciality or smoothness.\(^{11}\) In this way, a lattice polytope can be associated with a toric variety. In general, a given lattice polytope can lead to many different varieties, depending upon the refinement of the face fan. Even for a given set of additional rays added, there can be many different triangulations of the fan.

We will be interested in particular in the fans from reflexive polytopes, which are defined as follows. Let \(N\) be a rank \(n\) lattice, \(N_\mathbb{R} \equiv N \otimes \mathbb{R}\). A lattice polytope \(\nabla \subset N\) containing the origin is reflexive if its dual polytope \(\nabla^*\) is also a lattice polytope. The dual of a polytope \(\nabla\) in \(N\) is defined to be

\[
\nabla^* = \{ u \in M_\mathbb{R} = M \otimes \mathbb{R} : \langle u, v \rangle \geq -1, \forall v \in \nabla \},
\]

where \(M = N^* = \text{Hom}(N, \mathbb{Z})\) is the dual lattice. If \(\nabla\) is reflexive, its dual polytope \(\Delta = \nabla^*\) is also reflexive as \((\nabla^*)^* = \nabla\). We call the pair of reflexive polytopes a mirror pair. Both of them contain the origin as the only interior lattice point. Calabi-Yau manifolds in Batyrev’s construction [6] are built out of reflexive polytopes. Given a mirror pair \(\nabla \subset N\) and \(\Delta \subset M\), the (possibly refined) face fan of \(\nabla\) describes a toric ambient variety, in which a Calabi-Yau hypersurface is embedded using the anti-canonical class of the ambient toric variety, so that the hypersurface itself has

\[^{11}\text{A fan is simplicial if all its cones are simplicial. A cone is simplicial if its generators are linearly independent over } \mathbb{R}. \text{ A fan is smooth if the fan is simplicial and for each top-dimensional cone its generators generate the lattice } N.\]
trivial canonical class. Explicitly, a section of the anti-canonical bundle is given by

$$p = \sum_{i}^{\# \text{lattice points in } \Delta} c_i m_i, \quad (2.38)$$

where $c_i$ are generic coefficients taking values in $\mathbb{C}$ and each monomial $m_i$ is given by an associated lattice point $w_i$ in $\Delta$

$$m_i = \prod_{j}^{\# \text{rays}} z_j^{(w_i, v_j) + 1}, \quad (2.39)$$

where $z_j$ is the homogeneous coordinate associated with the ray $v_j$ in the fan associated to $\nabla$. The well-definedness of each $m_i$ in terms of the homogeneous coordinates $z_j$ is guaranteed by the linear equivalence relations among the divisors associated to $v_j$'s, and holomorphicity in the $z_j$s by the reflexivity of $\nabla$. We can check that Equation (2.38) indeed defines a section of the anti-canonical class, so that a CY hypersurface is cut out by $p = 0$. We can determine the class by looking at any one of the monomials; we pick the origin since we know it is always an interior point. Its associated monomial by equation (2.39) is simply the product of all homogeneous coordinates associated to all toric divisors $\prod_{j=1}^{\# \text{rays}} z_j$, which immediately we see by equation (2.33) is a section in the anti-canonical class. For the smoothness of the Calabi-Yau, as mentioned previously, there exists a refinement\textsuperscript{12} of the face fan of $\nabla$ such that the fan is simplicial so the ambient toric variety will have at most orbifold singularities. In the case of $n \leq 4$, with the anti-canonical embedding, a hypersurface will generically avoid these singularities and therefore is generically smooth.

\textsuperscript{12}Appropriate subdivisions of the face fan of the toric ambient variety by additional lattice points in the facets of the polytope give the resolved description of the embedded Calabi-Yau, where extra coordinates in equation (2.39) define the exceptional divisors in the resolution of the ambient space. The added lattice points that do not lie in the interior of the facets also correspond to exceptional divisors in the resolution of the Calabi-Yau. (Generic hypersurface CYs do not meet the divisors that correspond to interior points of facets.)
2.4.3 Mirror Symmetry in Toric Hypersurface CY

Mirror symmetry relates the type IIA string theory compactified on one Calabi-Yau manifold $X$ of odd dimension to the type IIB string theory compactified on another Calabi-Yau manifold $\tilde{X}$, and vice versa; then $X$ and $\tilde{X}$ is said to be a mirror pair [34, 35]. From the string worldsheet point of view, mirror symmetry relates the SCFT moduli spaces of the two Calabi-Yau manifolds, which builds local isomorphisms between the Kahler moduli space of one manifold and the complex-structure moduli space of its mirror partners. As the roles of Kahler and complex structure moduli are reversed, we expect the exchange of the Hodge numbers for a mirror pair of Calabi-Yau threefolds

$$h^{1,1}(X) = h^{2,1}(\tilde{X}), \quad \text{and} \quad h^{2,1}(X) = h^{1,1}(\tilde{X}).$$

(2.40)

This equivalence can be considered as an analogue of T-duality discussed in §2.1.2, which was about a duality between the two type II theory compactifications where the only geometric parameter is the radius of the compact circle which is identified through $R \leftrightarrow \frac{1}{R}$ in the duality. Mirror symmetry extends to comparing of IIA and IIB theories on various compact spaces. One thing important in string compactification is keeping track of all of the geometric parameters that go into the compactification. The identifications between the geometric parameter spaces of the two theories to realize the duality are more complicated in mirror symmetry.

In the Batyrev’s construction of hypersurface CYs §2.4.2, for a pair of reflexive polytopes $\nabla$ and $\Delta$, $\nabla$ was taken as the polytope defining the toric fan. Conversely, we can start from $\Delta$ and associate it with the polytope that defines the fan of the ambient space, and calculate monomials associated with lattice points in $\nabla$. Then the hypersurface CY is mirror to the previous one. The Hodge numbers of mirror pairs are related by $h^{p,q}(CY_{\nabla}) = h^{d-p,q}(CY_{\Delta})$, where $d = n - 1$ is the complex dimension of the CY; in particular, we will look at 4 dimensional reflexive polytopes for CY

\[13\] while this thesis focuses on Calabi-Yau threefolds, there is also a mirror symmetry statement on even dimension Calabi-Yau manifolds is that IIA theories on $X$ and on $\tilde{X}$ are related, as are the IIB theories on $X$ and on $\tilde{X}$ [34].
threefolds, where the only non-trivial Hodge numbers are $h^{1,1}$ and $h^{2,1}$, and mirror CY hypersurfaces have exchanged values for $h^{1,1}$ and $h^{2,1}$. As $\nabla$ and $\Delta$ are a pair of 4D reflexive polytopes, there is a one-to-one correspondence between $l$-dimensional faces $\theta$ of $\Delta$ and $(4-l)$-dimensional faces $\tilde{\theta}$ of $\nabla$ related by the dual operation

$$\theta^* = \{y \in \nabla, \langle y, pt \rangle = -1 \mid \text{for all pt that are vertices of } \theta\}. \quad (2.41)$$

For the CY associated with $\nabla$, the Hodge numbers are given by

$$h^{2,1} = \text{pts}(\Delta) - \sum_{\theta \in F^\Delta_2} \text{int}(\theta) + \sum_{\theta \in F^\Delta_1} \text{int}(\theta)\text{int}(\theta^*) - 5, \quad (2.42)$$

$$h^{1,1} = \text{pts}(\nabla) - \sum_{\tilde{\theta} \in F^\nabla_3} \text{int}(\tilde{\theta}) + \sum_{\tilde{\theta} \in F^\nabla_2} \text{int}(\tilde{\theta})\text{int}(\tilde{\theta}^*) - 5, \quad (2.43)$$

where $\theta$ are faces of $\Delta$, $\tilde{\theta}$ are faces of $\nabla$, $F^\nabla/\Delta$ denotes the set of $l$-dimensional faces of $\nabla$ or $\Delta$ ($l < n$), and pts$(\nabla/\Delta) := \text{number of lattice points of } \nabla \text{ or } \Delta$, int$(\theta/\tilde{\theta}) := \text{number of lattice points interior to } \theta \text{ or } \tilde{\theta}$. The correspondence (2.41) makes the duality between the Hodge number formulae manifest. Note that the Hodge numbers depend only on the polytope and not on the detailed refinement of the fan.

### 2.4.4 Kreuzer-Skarke Database of 4D Reflexive Polytopes

M. Kreuzer and H. Skarke have classified all 473,800,776 four-dimensional reflexive polytopes for the Batyrev Calabi-Yau construction [4, 67]. A pair of reflexive polytopes in the KS database are described in the format: $M:$# lattice points, # vertices (of $\Delta$) $N:$# lattice points, # vertices (of $\nabla$) $H:$ $h^{1,1}$, $h^{2,1}$. We will refer to $\nabla$ as the (fa)$N$ polytope and $\Delta$ as the M(onomial) polytope to remind ourselves that $\nabla$ gives the fan of the ambient toric variety for the CY anti-canonical embedding and $\Delta$ determines the monomials of the anti-canonical hypersurface. In many cases, it is sufficient to specify polytopes with the information given in the format above, but sometimes there can be distinct polytopes with identical information of this type, in which case we will either give further the vertices of the $N$ polytope to specify the polytope more
precisely, or indicate its numerical order as it appears among those with identical
data in the KS database website (http://hep.itp.tuwien.ac.at/~kreuzer/CY/) with a
superscript, e.g., M:165 11 N:18 9 H:9,129\textsuperscript{[1]} or M:165 11 N:18 9 H:9,129\textsuperscript{[2]}.
Chapter 3

Tate-tuned Models in the Kreuzer-Skarke Database

In this chapter, which is based on our work in [36], we study how the set of Calabi-Yau threefolds produced by toric hypersurface constructions through reflexive polytopes in the Kreuzer-Skarke database can be related to the general construction of elliptic Calabi-Yau threefolds through tuned Weierstrass models. The approach we take is to identify a specific subclass of tuned models that match with toric hypersurface constructions. In particular, we begin with the set of toric bases identified in [11] and consider Tate tunings over these bases. We focus on a region of Hodge numbers where we expect a single toric fiber type to dominate.

Our primary approach in this chapter is to systematically construct Tate tunings that should have counterparts as reflexive polytopes in the Kreuzer-Skarke database. Thus, we start from the F-theory construction and develop an algorithm to systematically classify Tate-tuned models that give polytopes and elliptic Calabi-Yau threefolds with large Hodge numbers; these models are all expected to have a corresponding standard $\mathbb{P}^{2,3,1}$-fibered polytope, and we compare the two constructions and match the results with the known data in the KS database. This gives us something like a “sieve” that leaves behind a set of special cases of KS data not produced by our algorithm. After implementing this sieve, we have then considered separately those few examples in the KS database in the range of interest that were not found
by our F-theory construction. We have found that there are a few polytopes in the
KS database that can be described in terms of the standard $\mathbb{P}^{2,3,1}$-fibered type; i.e.,
have Tate forms, but were nonetheless not found with the initial sieve. This turns
out to be because they involve such extensive tunings that the starting bases needed
are outside the range we considered. There are also data in the KS database that we
did not identify in the original sieve because they are accompanied by more sophisti-
cated constructions involving $u(1)$ tunings, novel $\mathfrak{su}(6)$ tunings associated with exotic
matter representations, or tunings of generic models over non-toric bases, which we
had not considered. Moreover, we encounter a type of novel models that did not arise
from our systematic construction because they are involved with tunings on non-
toric curves in the base; they turn out nonetheless to also be described by reflexive
polytopes with toric fibers associated with elliptic fibrations.

3.1 Weierstrass/Tate Models of 6D F-theory

In this section, we review and introduce some more algebraic descriptions of elliptic
fibration and the bases that support elliptically fibered CYs, and their relations to
the physics of 6D F-theory.

3.1.1 Generic, Tuned Weierstrass Models and Non-Higgsable
Clusters

F-theory compactified on a (possibly singular) elliptically fibered Calabi-Yau three-
fold $X_3$ gives a 6D effective supergravity theory. Such a compactification of F-theory
is equivalent to M-theory on the resolved Calabi-Yau $\check{X}_3$ in the decompactification
limit of M-theory, where in the F-theory picture the resolved components of the elliptic
fiber are shrunk to zero size. F-theory can also be thought of as a nonperturbative
formulation of type IIB string theory. In this picture the type IIB theory is compact-
ified on the base $B_2$. In this F-theory description, spacetime filling 7-branes sit at the
codimension-one loci in the base where the fibration degenerates. The non-abelian
gauge symmetries of the 6D effective theory arise from the seven-branes and can be inferred from the singularity types of the elliptic fibers along the codimension-one loci in the base, according to the Kodaira classification (Table 2.2). At the intersections of seven-branes there are localized matter fields that are hypermultiplets in the 6D theory; the representations of the matter fields can be determined from the detailed form of the singularities over the codimension-two points in the base (see e.g. [45, 46, 47]). Therefore the physics data can be extracted by studying the singular fibers by means of the Weierstrass models (short form) discussed in §2.2.3 or the Tate models (long form) of X that we review in §3.1.3.

A generic Weierstrass model refers to one with coefficients of the monomial terms in the sections \( f \) and \( g \) in equation (2.9) to be at a generic point in the moduli space. This corresponds physically to a maximally Higgsed phase. In the maximally Higgsed phase over many bases the gauge group and matter content are still nontrivial. In 6D F-theory (compactified on elliptically fibered Calabi-Yau threefolds), the minimal gauge algebras and matter configuration associated with a given base \( B_2 \) are carried by *non-Higgsable clusters* (NHCs) [10], which are isolated rational curves in the base of self-intersection \( m \), \(-12 \leq m \leq -3\), and clusters of multiple rational curves of self-intersection \( \leq -2 \): \( \{-2, -3\} \), \( \{-2, -2, -3\} \), and \( \{-2, -3, -2\} \). The sections \( f, g, \Delta \) automatically vanish to higher orders along these curves in any Weierstrass model over the given base. This can be understood geometrically as an effect in which the curvature over the negative self-intersection curves must be cancelled by 7-branes to maintain the Calabi-Yau structure of the elliptic fibration. The orders of vanishing and the corresponding minimal gauge groups on these NHCs are listed in the last two columns of Table 3.1.

Starting from the generic model over a given base \( B \), we can systematically tune the Weierstrass model coefficients \( f \) and \( g \) to increase the order of vanishing over various curves beyond what is imposed by the NHCs, producing additional or enhanced gauge groups on some curves in the base. Many aspects of such tunings are described in a systematic fashion in [15]. While over some bases there is a great deal of freedom to tune many different gauge group factors on various curves in the
Table 3.1: Tops over NHCs and the corresponding Tate vanishing orders. This table provides a dictionary of NHCs between Tate/Weierstrass models, which encode singular fibers in terms of degree of vanishing orders of sections, and elliptic toric hypersurface constructions (discussed in §3.2 and §3.3), which encode singular fibers in terms of “tops” (see §3.2.5 and §3.2.7). In each case the first example is the top and associated minimal Tate tuning associated with the “dual of the dual” construction described in §3.3.1.
Weierstrass model, there are also limitations imposed by the constraint that there be no codimension one loci over which \( f, g \) vanish to orders \( (4, 6) \). In this thesis we also avoid cases with codimension two \( (4, 6) \) loci by blowing up such points on the base as part of the resolution process. Such singularities can be related to 6D superconformal field theories; in the geometric picture such singularities are associated with non-flat fibers\(^1\) and a resolution of the singularity can generally be found by first blowing up the \( (4,6) \) point in the base, which modifies the geometry of the base \( B_2 \), increasing \( h^{1,1}(B_2) \) by one. While in many cases the extent to which enhanced gauge groups can be tuned in the Weierstrass model over any given base can be determined by considerations such as the low-energy anomaly consistency conditions, the precise set of possible tunings is most clearly described in terms of an explicit description of the Weierstrass coefficients. In the case of toric bases, the complete set of monomials in \( f, g \) has a simple description (see e.g. \([11, 15]\)) and we have very strong control over the parameters of the Weierstrass model.

### 3.1.2 Base Surfaces for 6D F-theory Models

There is a finite set of complex base surfaces that support elliptic Calabi-Yau threefolds. It was shown by Grassi \([17]\) that all such bases can be realized by blowing up a finite set of points on the minimal bases \( \mathbb{P}^2, F_m \) with \( 0 \leq m \leq 12 \), and the Enriques surface. This leads to a systematic constructive approach to classifying the set of allowed F-theory bases. The structure of non-Higgsable clusters limits the configurations of negative self-intersection curves that can arise on any given base, so we can in principle construct all allowed bases by blowing up points in all possible ways and truncating the set of possibilities when a disallowed configuration such as a curve of self-intersection \(-13\) or below arises. This was used in \([11]\) to classify the full set of toric bases \( B_2 \) that can support elliptic Calabi-Yau threefolds (toric geometry is described in more detail in the following section). While further progress has been

\(^1\)Resolution of non-flat fibers in related cases of tuned Weierstrass models has recently been considered for example in \([57, 58]\); the explicit connection between resolutions giving non-flat fibrations and flat fibrations over a resolved base through sequences of flops are described explicitly in the papers \([59, 60]\) that appeared after the initial appearance of this preprint.
made [12, 13] in classifying non-toric bases, we focus here primarily on toric base surfaces, as these are the primary bases that arise in the toric hypersurface construction of Calabi-Yau threefolds. Note, however, that as we discuss later in the chapter, particularly in e.g. §3.4, §3.6.1, there are cases in the Kreuzer-Skarke database where a toric polytope corresponds to an elliptic fibration over a non-toric base. The primary context in which this distinction is relevant involves curves of self-intersection $-9, -10, -11$. As discussed in [10], the Weierstrass model over such curves automatically has 1, 2, or 3 points on the curve where $f, g$ vanish to degrees $(4, 6)$. Such points on the base can be blown up for a smooth Calabi-Yau resolution, so that the actual base supporting the elliptic fibration is generally a non-toric complex surface.\footnote{More precisely, as described in [48] and §3.1.1, and discussed in more detail in §3.4, the original base supports an elliptic fibration that is “non-flat,” meaning that the fiber becomes two dimensional at some points, while the elliptic fibration over the blown up base is a flat elliptic fibration.}

In the simplest cases, such as $F_{11}$ and $F_{10}$, the blown up base still has a toric description; in other simple cases, such as $F_9$, the resulting surface is a “semi-toric” surface admitting only a single $\mathbb{C}^*$ action [12], but on surfaces with, for example, multiple curves of self-intersection $-9, -10, -11$, the blow-up of all $(4, 6)$ points in the base gives generally a non-toric base that is neither toric nor admits a single $\mathbb{C}^*$ action. Despite this complication, this blow-up and resolution process is automatically handled in a natural way in the framework of the toric hypersurface construction, so that (non-flat) elliptic fibrations over bases with these types of curves arise naturally in the Kreuzer-Skarke database, as we will discuss in §3.2.7. Thus, we include toric bases with curves of self-intersection $-9, -10, -11$ in the set of bases we consider for Tate/Weierstrass constructions. The complete set of such bases was enumerated in [11], where it was shown that there are 61,539 toric bases that support elliptic CY3’s. Generic elliptic Calabi-Yau threefolds over these bases give rise to a range of Hodge number pairs that fill out the range of known Calabi-Yau Hodge numbers, in a “shield” shape with peaks at $(11, 491)$, $(251, 251)$, and $(491, 11)$ [56]. As we see in §3.6, in some cases the base needed for a tuned Weierstrass model to match a toric hypersurface construction is even more exotic than those arising from blowing up points on curves of self-intersection $-9, -10, -11$. In these more complicated cases
as well, however, the general story is the same. The polytope construction gives rise to a non-flat elliptic fibration with codimension two (4, 6) points on the toric base. Blowing these curves up gives rise to another, generically non-toric, base with multiple additional curves. After these blow-ups, there is a tuned Weierstrass model giving a (flat) elliptic fibration over the new base. While the toric base is what arises most clearly from the polytope construction, the structure of the blown up base admitting the flat elliptic fibration is relevant when considering F-theory models and anomaly cancellation.

We consider Tate tunings over the toric bases, which we now describe in more detail, and compare to the toric hypersurface construction of elliptic Calabi-Yau three-folds in §3.2.

### 3.1.3 Tate Form and Tate Algorithm

The Tate algorithm is a systematic procedure for determining the Kodaira singularity type of an elliptic fibration, and provides a convenient way to study Kodaira singularities in the context of F-theory [46, 61]. The associated “Tate forms” for the different singularities match up neatly with the most generic toric hypersurface construction of elliptically fibered CY3 that we focus on in this chapter. We start with an equation for an elliptic curve in the general form

\[ y^2 + a_1xyz + a_3yz^3 = x^3 + a_2x^2z^2 + a_4xz^4 + a_6z^6, \]  

(3.1)

where for an elliptic fibration \( a_n \) are sections of line bundles \( \mathcal{O}(-nK_B) \). The general form (3.1) can be related to the standard Weierstrass form (2.9) by completing the

---

3 The most generic construction corresponds to models where the toric fiber ambient space is \( \mathbb{P}^{2,3,1} \), and the projective coordinate \( z \) corresponding to the ray \( v_z \) in \( 2v_x + 3v_y + v_z = 0 \) belongs to the trivial bundle in the base. See discussion in §3.2.2. We will encounter models of other fiber types and different “twists” of fiber bundle over the base \( B_2 \) in Chapter 4.
square in \( y \) and shifting \( x \), which gives the relations

\[
  b_2 = a_1^2 + 4a_2, \quad (3.2)
\]
\[
  b_4 = a_1a_3 + 2a_4, \quad (3.3)
\]
\[
  b_6 = a_3^2 + 4a_6, \quad (3.4)
\]
\[
  b_8 = a_2a_6 - a_1a_3a_4 + a_2a_3^2 - a_4^2, \quad (3.5)
\]
\[
  f = -\frac{1}{48}(b_2^2 - 24b_4), \quad (3.6)
\]
\[
  g = -\frac{1}{864}(-b_2^3 + 36b_2b_4 - 216b_6), \quad (3.7)
\]
\[
  \Delta = -b_2^2b_8 - 8b_4^2 - 27b_6^2 + 9b_2b_4b_6. \quad (3.8)
\]

An advantage of the general form (3.1) is that by requiring specific vanishing orders of the \( a_n \)’s according to Table 3.2, specific desired vanishing orders of \((f, g, \Delta)\) can be arranged to implement any of the possible gauge algebras. Moreover, the monodromy conditions in Table 2.3 imposed by some gauge algebras on \( f, g, \) or \( \Delta \) are also satisfied automatically by these “Tate form” models. For example, for tunings of fiber types \( I_m \) or \( I_m^* \), where \( \Delta \) is required to vanish to a certain order while \( \text{ord}(f) \) and \( \text{ord}(g) \) are kept fixed, the vanishing order of \( a_n \)’s prescribed by the Tate algorithm immediately give the desired \( \text{ord}(\Delta) \). This makes the Tate form much more convenient for constructing these singular fibers by only requiring the order of vanishing of the \( a_n \)’s to be specified, in contrast to the Weierstrass form (2.9) where it is necessary to carefully tune the coefficients of \( f \) and \( g \) to arrange for a vanishing of \( \Delta \) to higher order. The Tate forms described in Table 3.2 are also connected very directly to the geometry of reflexive polytopes. As we discuss in the subsequent sections, tuning a Tate form can be described by simply removing certain monomials from the general form (3.1), which corresponds geometrically to removing certain points from a lattice in the toric construction. We refer to tunings of this type as “Tate tunings” in contrast to tunings of the coefficients of \( f \) and \( g \); when applied to the polytope toric construction, we refer to Tate tunings as “polytope tunings”.

Note that Table 3.2 has incorporated some results of the present study into the
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<th>$a_3$</th>
<th>$a_4$</th>
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<td>$2n+1$</td>
<td>(2n)*</td>
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<tr>
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<td>1</td>
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<td>$2n+2$</td>
<td>(2n+1)*</td>
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Table 3.2: Tate forms: Extends earlier versions of table by including alternative $SU(2n)$ and $SO(2k)$ tunings that can be realized purely by orders of vanishing without additional monodromy constraints. In particular, alternate tuning $^\circ$ of $SU(6)$ gives alternate exotic matter content; see text for further details. Groups and tunings marked with $^*$ require additional monodromy conditions.
Tate table originally described in the F-theory context in [46] and later modified in [61]. The most significant new feature is an alternate Tate form for the algebras \( \text{su}(2n) \), with \( a_2 \) vanishing to order 2. For \( n = 3 \), in particular, this Tate form gives a tuning with exotic 3-index antisymmetric \( \text{SU}(6) \) matter. An example of a polytope that realizes this tuning is described in §3.6.1. For higher \( n \), in cases where \( a_1 \) is a constant — i.e. on curves of self-intersection \(-2\) — this simply gives an alternate Tate tuning of \( \text{SU}(2n) \). On any other kind of curve, at the codimension two loci where \( a_1 = 0 \) there is a codimension two \((4, 6)\) singularity when \( n > 3 \). This can immediately be seen from the fact that at the locus \( a_1 = 0 \), each \( a_k \) vanishes to order \( k \) so that (3.2–3.8) give a vanishing of \((f, g, \Delta)\) to orders \((4, 6, 12)\). Resolving this singularity generally involves blowing up a point on the base, so that the resulting elliptic fibration is naturally thought of as living on a base with larger \( h^{1,1} \), but this kind of Tate model for \( \text{SU}(8) \) and higher would be relevant in a complete analysis of all reflexive polytopes.

We have also identified Tate tunings of \( \text{so}(4n + 4) \), like those of \( \text{so}(4n + 2) \) that do not require an extra monodromy condition and only require the vanishing order of \( a_i \)'s; this arises naturally in the context of the geometric constructions of polytopes. We discuss briefly how these two types of Tate tunings are relevant in the constructions. For \( \text{so}(4n + 4) \), if \( a_6 \) is of order \( 2n + 1 \), then the necessary monodromy condition is that \([61, 62] (a_4^2 - 4a_2a_6)/z^{2n+2}|_{z=0} \) is a perfect square. This condition is clearly automatically satisfied if \( a_6 \) is actually of order \( 2n + 2 \), so can be guaranteed simply by setting certain monomials in the Tate coefficients to vanish (in a local coordinate system, which can become global in the toric context used in the later sections of the chapter). On the other hand, if the leading terms in \( a_2, a_4, a_6 \) are each constrained to be powers of a single monomial \( m, m^{n+1}, m^{2n+1} \), then the monodromy condition will be automatically satisfied with \( a_6 \) of order \( 2n + 1 \) without specifying any particular coefficients for these monomials. We encounter both kinds of situation in this chapter. For \( \text{so}(8) \), the monodromy condition when \( a_6 \) is of order 4 is that \((a_4^2 - 4a_4)/z^2|_{z=0} \) is a perfect square [61]. This can be satisfied if \( a_2, a_4 \) contain only a single monomial.

\[^4\text{To relate this to the condition stated in Table 2.3, note that the leading term in the discriminant}\]
each \( m, m^2 \) at leading order, but cannot be imposed by simply setting the orders of vanishing of each \( a_i \). The situation is similar when \( a_6 \) is of order 3, though the monodromy condition is more complicated when \( a_2, a_4, a_6 \) are not single monomials \( m, m^2, m^3 \). This is the only gauge algebra with no monodromy-independent Tate tuning except through this kind of single monomial condition. Finally, for \( \mathfrak{so}(4n+2) \), with \( a_6 \) of order \( 2n \), the monodromy condition is that \( (a_3^2 + 4a_6)/z^{2n}|_{z=0} \) is a perfect square, satisfied in particular if \( a_6 \) is actually of order \( 2n + 1 \) or if the leading terms in \( a_3, a_6 \) are each a single monomial proportional to \( m, m^2 \). We explore further, for example, in §3.4.3 for \( \mathfrak{so}(12) \) the subtleties in using the Tate tuning \( \{1, 1, 3, 3, 5\} \) described in [46], which requires an additional monodromy condition, vs. our alternative tuning \( \{1, 1, 3, 3, 6\} \); In fact, analogous situations occur in tuning all gauge algebras with monodromies.

3.1.4 The Zariski Decomposition

A central feature of the geometry of an F-theory base surface is the structure of the intersection form on curves (divisors) in \( B_2 \). The intersection form on \( H_2(B, \mathbb{Z}) \) has signature \((1, T)\). Curves of negative self-intersection \( C \cdot C < 0 \) are rigid. A simple but useful algebraic geometry identity, which follows from the Riemann-Roch theorem, is that

\[
C \cdot (C + K_B) = 2g - 2, \tag{3.9}
\]

for any curve \( C \) of genus \( g \). We are primarily interested in rational (genus 0) curves, for which therefore \( C \cdot C = -K_B \cdot C - 2 \). All toric curves on a toric base \( B_2 \) are rational, and the intersection product of toric curves has a simple structure that we review in the following section.

To study the orders of vanishing of \( f, g \) and \( \Delta \) along some irreducible divisors in
the base, aside from looking explicitly at the sets of monomials of \( f, g \) and \( \Delta \), it is convenient to consider the more abstract “Zariski decomposition”, in which an effective divisor \( A \) is decomposed into (minimal) multiples of irreducible effective divisors \( C_i \) of negative self-intersection and a residual part \( Y \)

\[
A = \sum_i q_i C_i + Y, \quad q_i \in \mathbb{Q};
\]

where \( Y \) is effective and satisfies

\[
Y \cdot C_i = 0, \quad \forall i.
\]

Then the order of vanishing along the curve \( C_i \) of a section of the line bundle corresponding to the divisor \( A \) must be at least \( c_i = \lceil q_i \rceil \). Mathematically, the Zariski decomposition is normally considered over the rationals, so \( q_i \in \mathbb{Q} \). Here, however, we are simply interested in the smallest integer coefficient of \( C_i \) compatible with the decomposition over the ring of integers. For example, consider the decomposition

\[
- nK_B = \sum_i c_i C_i + Y
\]

The goal is to find the minimal set of integer values \( c_i \) such that the conditions \( Y \cdot C_i \geq 0 \) are satisfied. Taking the intersection product on both sides with \( C_j \), the conditions can be rewritten as the set of inequalities

\[
v_{j,n} - \sum_i M_{ji} c_i \geq 0, \quad \forall j,
\]

where \( M_{ji} \equiv C_j \cdot C_i \) are pairwise intersection numbers (non-negative for \( i \neq j \)) and self-intersection numbers \( M_{jj} = C_j \cdot C_j \equiv m_j \), and \( v_{j,n} \equiv -nK_B \cdot C_j \).

The Zariski decomposition of \(-4K_B\) and \(-6K_B\) was used in [10] to analyze the non-Higgsable clusters that can arise in 6D theories. More generally, we can use the same approach to analyze models where we tune a given gauge factor on a specific divisor beyond the minimal content specified by the non-Higgsable cluster structure.
In such a situation, we would choose by hand to take some values of $c_i$ in (3.12) to be larger than the minimal possible values; this may in turn force other coefficients $c_j$ to increase. As a simple example, consider a pair of $-2$ curves (i.e. curves of self-intersection $-2$) $C, D$ that intersect at a point $(C \cdot D = 1)$. The Zariski decomposition of the discriminant locus gives simply $-12K_B = Y$, since $K_B \cdot C = K_B \cdot D = 0$ from (3.9), so the discriminant need not vanish on $C$ or $D$. If, however, we tune for example an $su(4)$ gauge algebra on $D$ so that $\Delta$ vanishes to order 4 on $D$ then we have the Zariski decomposition $-12K_B - 4D = 2C + Y'$, since $-4D \cdot C = -4$, implying that $\Delta$ must also vanish to order 2 on $C$, so that $C$ must therefore also carry at least an $su(2)$ gauge algebra.

### 3.1.5 Zariski Decomposition of a Tate Tuning

A particular application of the Zariski decomposition that we use here extensively is in the context of a Tate tuning. In particular, assume that we have an elliptic fibration in the Tate form (3.1) over a complex surface base $B$, and we have a set of curves $C_j$ in the base that includes all curves of negative self-intersection. The parameter space of the elliptic fibration is given by the five sections $a_n \in \mathcal{O}(-nK), n = 1, 2, 3, 4, 6$. We denote by $c_{j,n}$ the order of vanishing of $a_n$ on $C_j$. The minimal necessary order of vanishing of each $a_n$ on each curve $C_j$ can be determined by applying the Zariski decomposition for $-nK$. This gives rise to a set of vanishing orders $c_{j,n}$ associated with each non-Higgsable cluster, which we list in Table 3.3. These are the minimal values $c_{j,n} = c_{j,n}^{\text{NHC}}$ that satisfy the inequalities (3.13) for each value of $n \in \{1, 2, 3, 4, 6\}$. In doing a Tate tuning, we impose the additional condition that over certain curves $C_j$, the vanishing order is at least some specified value that is higher than the minimum imposed by the NHCs, $c_{j,n} \geq c_{j,n}^{\text{tuned}} \geq c_{j,n}^{\text{NHC}}$. We can then use the Zariski decomposition to determine the minimum values of the $c_{j,n}$ compatible with this lower bound that also satisfy the inequalities (3.13).

More concretely, to determine the unique minimum set of values $c_{j,n}$ that satisfy the inequalities (3.13), we proceed iteratively, following an algorithm described in appendix A of [10]. For each $n$, we begin with an initial assignment of vanishing
orders

\[ c_{j,n}^{(0)} = c_{j,n}^{\text{tuned}} \]  

(3.14)

when we are imposing a given tuning. When we are computing the minimal values from NHC’s without tuning we simply use the minimal order of vanishing from the Zariski decomposition on each isolated curve of self-intersection \( m_j = M_{jj} \),

\[
c_{j,n}^{(0)} = \begin{cases} \left\lceil \frac{n(2+m_j)}{m_j} \right\rceil , & m_j \leq -3, \\ 0, & m_j > -3. \end{cases} \]

(3.15)

We can then use the inequalities (3.13) to determine the minimal correction that is needed to each vanishing order (label \( n \) dropped for clarity of the notation),

\[
\Delta c_j^{(1)} = \text{Max} \left( 0, \left\lfloor \frac{v_j - \sum_i M_{ji} (c_i^{(0)})}{m_j} \right\rfloor \right). \]

(3.16)

The second corrections are obtained similarly, replacing \( c^{(0)} \) on the RHS with \( c^{(0)} + \Delta c_j^{(1)} \). We continue to repeat this procedure until the corrections in the \( f \)-th step all become zero, \( \Delta c_j^{(f)} = 0 \) for all \( j \). The final solutions \( \{c_j\} \) are obtained iteratively this way by adding the non-negative correction values \( \{\Delta c_j^{(k)}\} \):

\[
c_j = c_j^{(0)} + \Delta c_j^{(1)} + \Delta c_j^{(2)} + \cdots + 0,
\]

where \( \Delta c_j^{(t+1)} = \text{Max} \left( 0, \left\lfloor \frac{v_j - \sum_i M_{ji} (c_i^{(0)} + \sum_{k=1}^{t} \Delta c_i^{(k)})}{m_j} \right\rfloor \right) \).  

(3.17)

At each step this algorithm clearly increases the orders of vanishing in a minimal way, so when the algorithm terminates the solution is clearly a minimal solution of the inequalities (3.13). Note that in some cases, the algorithm leads to a runaway behavior when there is no acceptable solution without \( (4, 6) \) loci. When this occurs, or when one of the factors of the gauge algebra exceeds that desired by the tuning, we terminate the algorithm and do not consider this tuning as a viable possibility.

As an example, consider the set of curves \( \{C_j\} \) to be the NHC \( \{-3, -2\} \), so
Table 3.3: The minimal vanishing orders of sections \( a_{1,2,3,4,6} \) over NHCs

\[
\begin{array}{|c|c|}
\hline
\text{NHC} & \{c_{j,n}^{\text{NHC}}\} \\
\hline
\{-3\} & \{(1, 1, 1, 2, 2)\} \\
\{-4\} & \{(1, 1, 2, 3)\} \\
\{-5\} & \{(1, 2, 2, 3, 4)\} \\
\{-6\} & \{(1, 2, 2, 3, 4)\} \\
\{-7\} & \{(1, 2, 3, 3, 5)\} \\
\{-8\} & \{(1, 2, 3, 3, 5)\} \\
\{-12\} & \{(1, 2, 3, 4, 5)\} \\
\{-3, -2\} & \{(1, 1, 2, 2, 3), \{1, 1, 1, 1, 2\}\} \\
\{-3, -2, -2\} & \{(1, 1, 2, 2, 3), \{1, 1, 2, 2, 2\}, \{1, 1, 1, 1, 1\}\} \\
\{-2, -3, -2\} & \{(1, 1, 1, 1, 2), \{1, 2, 2, 2, 4\}, \{1, 1, 1, 1, 2\}\} \\
\hline
\end{array}
\]

Then the vanishing orders calculated from (3.17) are \( \{c_{1,n}\} = \{1, 1, 2, 3\} \) and \( \{c_{2,n}\} = \{1, 1, 1, 1, 2\} \), as shown in Table 3.3.

Note that a tuning beyond that shown in Table 3.3 does not necessarily increase the gauge group on any of the curves. In particular, for some gauge groups there are multiple possible Tate tunings. Both for the generic gauge group associated with the generic elliptic fibration over a given base and for constructions with gauge groups that are enhanced through a Tate tuning, this means that there may be distinct Tate tunings with the same physical properties. As we will see later, these distinct Tate tunings can correspond through distinct polytopes to different Calabi-Yau threefold constructions. Note also that for the toric bases we are studying here, an essentially equivalent analysis could be carried out by explicitly working with the various monomials in the sections \( a_n \), which in the toric context are simply points in a dual lattice, as we discuss in the next section. We use the Zariski procedure because it is more efficient and more general; the results of this analysis should, however, match an explicit toric computation in each case.
3.1.6 Global Symmetry Constraints in F-theory on Tuning

In many cases, the anomaly cancellation conditions impose constraints not only on the matter content of the theory but also on what gauge groups may be combined on intersecting curves, corresponding to vectors $b'$ with non-vanishing inner products in the low-energy theory (see the mapping in equation (2.30)). For example, two gauge factors of $g_2$ or larger in the Kodaira classification cannot be associated with vectors $b, b'$ having $b \cdot b' > 0$; in the low-energy supergravity theory this is ruled out by the anomaly conditions while in the F-theory picture this would correspond to a configuration with a codimension two $(4, 6)$ point at the intersection between the corresponding curves. In addition to these types of constraints, another set of constraints on what combination of gauge groups can be tuned on specific negative self-intersection curves in a base $B_2$ can be derived from the low-energy theory by considering the maximum global symmetry of an SCFT that arises by shrinking a curve $C$ of self-intersection $n < 0$ that supports a given gauge factor $G_i$ [64]. While in most cases these global symmetry conditions simply match with the expectation from anomaly cancellation, in some circumstances the global symmetry condition imposes stronger constraints. For example the "$E_8$ rule" [65] states that the maximal global symmetry on a $-1$ curve that does not carry a nontrivial gauge algebra is $e_8$; i.e., the direct sum of the gauge algebras carried by the curves intersecting the $-1$ curve should be a subalgebra of $e_8$. While the global symmetry constraints are completely consistent with F-theory geometry, they may not be a complete and sufficient set of constraints; for example a similar constraint appears to hold in F-theory for the algebras on a set of curves intersecting a 0 curve [15], though the low-energy explanation for this is not understood in terms of global constraints from SCFT's.

The maximal global symmetry groups realized in 6D F-theory for each possible algebra on a curve of self-intersection $m \leq -1$ are worked out in [64]. We use their results in our algorithm to constrain possible gauge algebra tunings. More explicitly, given a gauge algebra on a curve, the maximal global symmetry on the curve is determined, so the direct sum of the algebras on the curves intersecting it should be
a subalgebra of the maximal global symmetry algebra. For instance, consider a linear
chain of three curves \( \{C_1, C_2, C_3\} \) carrying gauge algebras \( \{g_1, g_2, g_3\} \). These can be
either minimal or enhanced algebras, but they have to satisfy \( g_3 \oplus g_1 \subset g_2^{(\text{glob})} \), where
\( g_2^{(\text{glob})} \) is the maximal global symmetry algebra given \( g_2 \) on the curve \( C_2 \), as enumerated
in the tables in [64]. This will be useful for us to constrain the possible tunings on
a curve when the gauge symmetries on its neighboring curves are known, making
our search over possible tunings more efficient. This is also convenient sometimes
for us to determine the gauge algebras that have monodromy conditions without
having to figure out the monodromy directly, which we use in for example §3.4.3. We
also include the "E8 rule" in our algorithm in §3.5, corresponding to the case where
\( m = -1 \) and \( g_2 \) is trivial.

3.1.7 Hodge Numbers of Tuned Weierstrass Models

In general, the anomaly constraints on 6D theories provide a powerful set of con-
sistency conditions that we use in many places in this thesis to analyze and check
various models that arise through tunings; in particular, using the anomaly condi-
tions to determine the matter spectrum gives a direct and simple way without having
to analyze geometry explicitly in many cases to compute the Hodge numbers of the
associated elliptic Calabi-Yau manifold in equations (2.15) and (2.18/2.31) that can
be matched to the Hodge numbers of a toric hypersurface construction in equations
(2.43) and (2.42). Equation (2.31) is more useful for some of our purposes than equa-
tion (2.18). In particular, as we are interested in studying various specializations
(tunings) of a generic elliptically fibered CY3 over a given base \( B_2 \). The number of
tensors \( T \) is fixed for a given base. Thus, if we start with known Hodge numbers \( h^{1,1} \)
and \( h^{2,1} \) for the generic elliptic fibration over a given (e.g. toric [11, 56]) base, and
specialize/tune to a model with a larger gauge group and increased matter content,
then the Hodge numbers of the tuned model can be simply calculated by adding to
Table 3.4: Rank preserving tunings: tunings of these four classes of gauge algebras do not change $h^{1,1}$ or $h^{2,1}$.

<table>
<thead>
<tr>
<th>Rank $r$</th>
<th>Algebras</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$su(3), g_2$</td>
</tr>
<tr>
<td>3</td>
<td>$su(4), so(7)$</td>
</tr>
<tr>
<td>4</td>
<td>$so(8), so(9), f_4$</td>
</tr>
<tr>
<td>$r \geq 5$</td>
<td>$so(r), so(r + 1)$</td>
</tr>
</tbody>
</table>

Those of the generic models respectively the shifts

$$\Delta h^{1,1} = \Delta r, \quad (3.18)$$

$$\Delta h^{2,1} = \Delta V - \Delta H_{\text{charged}}, \quad (3.19)$$

Such a specialization/tuning amounts physically to undoing a Higgsing transition, and the second of these relations simply expresses the physical expectation that the number of matter degrees of freedom that are lost ("eaten") in a Higgsing transition is equal to the number of gauge bosons lost to symmetry breaking. Note that the data on the right hand sides are associated in general with tuned non-abelian gauge symmetries but also in some special cases involve abelian factors. Note also that the right-hand sides of (3.18) and (3.19) are always non-negative and non-positive respectively for any tuning. In most cases, the gauge group increases in rank and some of the $h^{2,1}$ moduli are used to implement the tuning. In rank-preserving tunings, however, the Hodge numbers do not change (see Table 3.4) — $h^{1,1}$ of course does not change in a rank-preserving enhancement; $h^{2,1}$ does not change either in these tunings, as one can check by considering carefully the matter charged under the Cartan subalgebra (cf. footnote 7 in Chapter 2.)

### 3.2 Elliptic Fibration Toric Hypersurfaces

In this section, we study reflexive polytopes that are associated with elliptically fibered CY3s in the Batyrev’s toric hypersurface CY constructions discussed in §2.4.2. Such a polytope contains a 2D subpolytope which serves as an ambient space of the elliptic
fiber, and the projection along the fiber direction gives a 2D toric base. We discuss in this section the 16 toric fiber types of generic fiber embedding, tops of singular fibers, and toric bases from projection. In particular, we study fibered polytopes of a specific structure: the standard $\mathbb{P}^{2,3,1}$ fibered polytopes defined in §3.2.2, which are associated naturally to Tate models. The gauge symmetries associated with the Tate tunings can be read off from the tops $[39, 70, 72, 73]$ of the polytopes.

The other 15 fiber types, however, implicitly constrain the Weierstrass model associated with an elliptic fibration. We explain in §3.2.4 some constraints on the other 15 fiber types, which are related to the structure of the base. Based on these constraints, we expect that when we confine the range of Hodge numbers to relatively large values, as we do in section 3.5, the simplest $\mathbb{P}^{2,3,1}$ fiber type will dominate the set of polytopes. 

### 3.2.1 Fibered Polytopes in the KS Database

For the purpose of studying (often singular) elliptically fibered Calabi-Yau threefolds that arise in the KS database, we will be interested in 4D reflexive fibered polytopes $[68, 69, 70, 48]$. A fibered polytope $\nabla$ is a polytope in the $N$ lattice that contains a lower-dimensional subpolytope, $\nabla_2 \subset N_2 = \mathbb{Z}^2$, which passes through the origin. We are interested in the case where $\nabla_2$ is itself a reflexive 2D polytope, containing the origin as an interior point. Such a fibered polytope $\nabla$ admits a projection map $\pi : \nabla \to N_B$ such that $\pi^{-1}(0) = \nabla_2$, and $N_B = \mathbb{Z}^2$ is the quotient of the original lattice $N$ by the projection. We can construct a set of rays $v_i^{(B)}$ in $N_B$ that are the primitive rays with the property that an integer multiple of $v_i^{(B)}$ arises as the image $\pi(v_i)$ of a ray in $\nabla$. (A primitive ray $v \in N$ is one that cannot be described as an integer multiple $v = nw$ of another ray $w \in N$, with $n > 1$.) We define the base $B_2$ to be the 2D toric variety given by the 2D fan $\Sigma_B$ with $v_i^{(B)}$ taken to be the 1D cones; the 2D cones are uniquely defined for a 2D variety. Note that in higher dimensions, the base of the fibration is not uniquely defined as a toric variety since the cone structure

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5 That this expectation is correctly borne out is also verified explicitly with a systematic analysis of the KS database in Chapter 4.
of the base may not be unique.

In the toric geometry language, a fan morphism is a projection \( \pi : \Sigma \rightarrow \Sigma_B \) with the property that for any cone in \( \Sigma \) the image is contained in a cone of \( \Sigma_B \). Such a fan morphism can be translated to a map between toric varieties \( \pi : X_\Sigma \rightarrow B_2 \). Such a map is a toric morphism, which is an equivariant map with respect to the torus action on the toric varieties that maps the maximal torus in \( X_\Sigma \) to the maximal torus in \( B_2 \). As far as the authors are aware, it is not known whether in general every fibered polytope admits a triangulation leading to a compatible fan morphism and toric morphism. Note, however, that the elliptic fiber structure of the polytope does not depend upon the existence of a triangulation with respect to which there is a fan morphism \( \pi : \Sigma_\Sigma \rightarrow \Sigma_B \). Thus, to recognize an elliptic Calabi-Yau threefold in the KS database, it is only necessary to find a reflexive subpolytope \( \nabla_2 \subset \nabla \). The Calabi-Yau manifold defined by an anti-canonical hypersurface in \( X_\Sigma \) through the Batyrev construction with reflexive polytopes will then be an elliptically fibered Calabi-Yau threefold over the base \( B_2 \)[69]. A primary goal of this chapter is to relate reflexive polytopes in the Kreuzer-Skarke database that have this form to elliptic fibrations of tuned Weierstrass models.

There are in total 16 2D reflexive polytopes, which give slightly different realizations of an elliptic curve when an anti-canonical hypersurface is taken [48, 41, 71]. The hypersurface equations \( p = 0 \), with \( p \) given in (2.38), of all 16 types of fibered polytopes can be brought into the Weierstrass form (2.9) by the methods described in Appendix A in [48]; this gives an equivalent description of the same Calabi-Yau as long as the fibration has a global section. The Kreuzer-Skarke database of reflexive polytopes and associated Calabi-Yau hypersurface constructions contains a wide range of polytopes that include fibered polytopes with many different examples of the 16 fiber types.

For a given base \( B_2 \) and a given fiber type, there can be a variety of different polytopes corresponding to configurations with different “twists” of the fibration, associated with different embeddings of the rays \( v_i \) defining the base \( B_2 \) with respect to the fiber subpolytope \( \nabla_2 \). For example, the Hirzebruch surfaces \( \mathbb{F}_m \) are each as-
associated with fibered polytopes with fiber and base \( \mathbb{P}^1 \), distinguished by the different twists of the fibration. For a fibered polytope \( \nabla \) with a reflexive subpolytope \( \nabla_2 \), the dual \( \Delta \) admits a projection to \( \Delta_2 = (\nabla_2)^* \).

### 3.2.2 Standard \( \mathbb{P}^{2,3,1} \)-fibered Polytopes and Corresponding Tate Models

One of the findings of this work is that the bulk of KS models with large Hodge numbers appear to have a description in the form of a standard \( \mathbb{P}^{2,3,1} \)-fibered type, with a specific form for the twist of the fiber over the base surface \( B_2 \); these models can be connected directly to the Tate form for elliptic fibrations, and in fact can be constructed from that point of view directly. On the one hand, we describe the structure of this type of standard polytope in this section, with the result that the anti-canonical hypersurface equations from (suitably refined) of standard \( \mathbb{P}^{2,3,1} \)-fibered polytopes are in the form of equation (3.1). On the other hand, we describe the direct construction of polytopes by carrying out Tate tunings on the effective curves in the toric bases in §3.3. For a given base \( B_2 \) there are generally many distinct polytopes that have the standard \( \mathbb{P}^{2,3,1} \)-fibered structure; as we describe in the following section, these correspond to different Tate tunings over the same base.

The fiber polytope \( \nabla_2 \) that provides a natural correspondence with the Tate form models (3.1) is associated with the toric fan giving the weighted projective space \( \mathbb{P}^{2,3,1} \); this is a toric variety given by the rays \( v_x = (-1,0), v_y = (0,-1), v_z = (2,3) \) (see Figure 3-1a). Given a \( \mathbb{P}^{2,3,1} \)-fibered polytope \( \nabla \) over a toric base \( B_2 \), where the fiber is defined by three rays satisfying \( 2v_x + 3v_y + v_z = 0 \), we can always perform a \( SL(2,\mathbb{Z}) \) transformation to put the rays in the fiber into the coordinates

\[
v_x = (0,0,-1,0), v_y = (0,0,0,-1), v_z = (0,0,2,3) .
\]

We can define a standard\(^6\) \( \mathbb{P}^{2,3,1} \)-fibered polytope over the base \( B_2 \) as one where there

\(^6\)Because the rays of the base are “stacked” in (3.21) over the vertex \( (2,3) \) of the fiber, we sometimes refer to constructions of this form as “stacking” fibrations. The “standard stacking” we
(a) The toric fan for $\mathbb{P}^{2,3,1}$. The convex hull of $\mathbb{P}^{2,3,1}$ plays the role of the reflexive subpolytope $\nabla_2$ for standard $\mathbb{P}^{2,3,1}$-fibered polytopes $\nabla$ in the $N$ lattice. The rays $v_x, v_y, v_z$ are associated with the homogeneous coordinates $x, y, z$, respectively, in the hypersurface equation.

(b) The dual polytope $\Delta_2$ to $\nabla_2$ in the $M_2$ lattice. Projection onto the $M_2$ plane projects the lattice points in $\Delta$ into seven lattice points in $\Delta_2$. These lattice points correspond to the five sections $a_1, a_2, a_3, a_4, a_6$ in the Tate form of the Weierstrass model, indicated in the figure by $xyz, x^2z^2, yz^3, xz^4, z^6$, respectively, and to the coefficients of the remaining two terms $x^3, y^2$ in the hypersurface equation.

Figure 3-1: The reflexive polytope pair for the $\mathbb{P}^{2,3,1}$ ambient toric fiber.
is a coordinate system after an $SL(4, \mathbb{Z})$ transformation such that the vectors

$$v_i^{(a)} = (v_i^{(B)}, v_i^{(B)}), 2, 3) \quad (3.21)$$

are contained within $\nabla$ for every ray $v_i^{(B)} = (v_i^{(B)}, v_i^{(B)})$ in $\Sigma_B$. Note that in fact, these lattice points are all on the boundary of $\nabla$ since the only interior point of a reflexive polytope is the origin. This particular choice of fiber and twist geometry represents a very specific class of fibered polytopes that produce elliptically fibered Calabi-Yau threefolds as hypersurfaces. These standard $\mathbb{P}^{2,3,1}$-fibered polytopes play a central role in the analysis of this chapter, and are a generalization of the well-studied 3D reflexive polytope for a K3 surface that is an elliptic fibration over a $\mathbb{P}^1$ base [70]. As mentioned above, these polytopes appear to be highly prevalent in the Kreuzer Skarke database at large Hodge numbers. This seems to occur for several reasons. The $\mathbb{P}^{2,3,1}$ fiber is the only one of the 16 reflexive 2D polytopes that is possible in the presence of curves of very negative self intersection in the base (see discussion in §3.2.4). And the natural correspondence between tuned Tate models and the particular twist structure defined by (3.21) makes this twist structure particularly compatible with the reflexive polytope Calabi-Yau construction. We do, however, encounter some specific examples of other twists in later sections.

For a standard $\mathbb{P}^{2,3,1}$-fibered polytope, the lattice points of $\Delta \subset M$ in this coordinate system organize into the following sets of points:

$$\{(0, 0, 1, -1), (0, 0, -2, 1), (-, -, 0, 0), (-, -, -1, 1), (-, -, 1, 0), (-, -, 0, 1), (-, -, 1, 1)\}. \quad (3.22)$$

The elliptically fibered CY hypersurface equation $p = 0$ with $p$ from (2.38) then takes precisely the Tate form (3.1). The sets of points in (3.22) are associated with the monomials $y^2, x^3, xy, x^2, y, x, 1$ respectively; $y^2$ and $x^3$ have a single overall coefficient,
and the monomials in the base associated with the other five sets of points correspond precisely to monomials in the five sections \(\{a_1, a_2, a_3, a_4, a_6\}\) (see figure 3-1.) In particular, the condition that \(\Delta\) is the dual polytope of \(\nabla\) precisely imposes the condition that \(a_n \in \mathcal{O}(-nK_B)\). For example, for \(a_6\) we have the condition on the monomial associated with the point \((m_1, m_2, 1, 1)\) that \(v_1^{(B)} m_1 + v_2^{(B)} m_2 + 2 + 3 \geq -1\) for each ray \(v^{(B)} = \pi((v_1^{(B)}, v_2^{(B)}, 2, 3))\) in the fan of the base \(B_2\), so \((m_1, m_2)\) represents a section of \(-6K_{B_2}\), in much the same way that the monomials in (2.39) represent sections of \(-K\) of the ambient toric variety. A similar computation for each \(a_n\) confirms that the corresponding monomials satisfy \(v_1^{(B)} m_1 + v_2^{(B)} m_2 \geq -n\), and the degree \(d\) in the variable \(z^{(B)}\) associated with the ray \(v^{(B)}\) of a monomial \((m_1, m_2)\) is given by \(v_1^{(B)} m_1 + v_2^{(B)} m_2 = -n + d\). An analogous computation shows that for the points associated with \(y^2\) and \(x^3\) the condition is \(v_1^{(B)} m_1 + v_2^{(B)} m_2 \geq 0\); for any compact base this implies that \(m_1 = m_2 = 0\), so the first two points in (3.22) are the only points of the form \((m_1, m_2, 1, -1)\) and \((m_1, m_2, -2, 1)\) and are associated with constant functions on the base. This matches with the fact that these are sections of the trivial bundle \(\mathcal{O}\) over the base, and the fact that the only global holomorphic functions on any compact base are constants. This proves that for any standard \(\mathbb{P}^{2,3,1}\)-fibered polytope, the lattice points in \(\Delta\) are associated precisely with the Tate form of a Weierstrass model over the base, as stated above.

In the simplest cases, all the lattice points of the polytope \(\nabla\) are simply given by the vectors (3.20) and the vectors of the form (3.21). This corresponds to the generic elliptic fibration over a toric base \(B_2\) without non-Higgsable clusters. In other cases, however, there are lattice points in \(\nabla\) other than those given by (3.20) and (3.21). This corresponds to situations with NHCs or gauge groups tuned over curves in \(B_2\) by removing Tate monomials. The set of monomials in \(\Delta\) completely span the set of sections of the appropriate line bundles \(\mathcal{O}(-nK_B)\) for the generic elliptic fibration over a given base. In the case of NHCs, in particular, the monomials in \(\Delta\) span the appropriate set of sections, while in the case of gauge group tunings, some of these monomials are set to zero. From the point of view of the Calabi-Yau geometry, the lattice points in \(\nabla\) other than those given by (3.20) and (3.21)
reflect the singular nature of the resulting Calabi-Yau hypersurface. Up to some monodromy subtleties that we discuss further in §3.3, the set of new lattice points introduced together with $v_i^{(a)}$ in $\pi^{-1}(v_i^{(B)})$ is known as a top [39, 72, 73], which forms the extended Dynkin diagram of the gauge algebra of the singular fiber over the associated divisor $D_i^{(B)}$, with $v_i^{(a)}$ the affine root (this is the only inverse image when the fiber is smooth). In section 3.3 we describe in more detail the dictionary between Tate tunings and toric/polytope geometry for specific gauge groups on particular local curve configurations in the base geometry.

### 3.2.3 Fibered Polytopes Analysis: Fiber Types and 2D Toric Bases

We provide here an method to explicitly analyze given reflexive polytope whether it is fibered or not, and what are the toric fiber and the base in the case of fibered polytopes. This is useful in the later section to study the polytopes left out by the sieve of our systematic construction. We can learn from this analysis whether one of the 16 reflexive fiber types is a fiber of the polytope in question; we then define the 2D toric base from the fibered polytope. As we describe later in the chapter, we can thereby determine the singularities of the elliptic fiber over the curves in the base, and then we check that the Hodge numbers of the inferred tuned model are consistent with those of the polytope model. Here we briefly summarize the first piece of this analysis: the algorithm to determine if a given reflexive 2D polytope is a fiber of a 4D polytope. There are also software programs like Sage [74] with built in routines to identify the reflexive subpolytopes of a given polytope.

1. We assume that we are interested in a fiber described by the 2D reflexive polytope $\nabla_2$. To increase the efficiency of the algorithm in the case that the number of lattice points in $\nabla$ is large (which is true in the case of large $h^{1,1}$ that we are focusing on), we begin by focusing on only a subset of these lattice points that can possibly play a role as the points in a fiber $\nabla_2$. As mentioned in §3.2.1, the presence of a fiber subpolytope $\nabla_2 \subset \nabla$ implies that there is a projection from
\[ \Delta \rightarrow \Delta_2. \] Let us call the maximum value of the inner product for any pair of vectors in the fiber and its dual

\[ M_{\text{max}} = \max v \cdot w, \ v \in \nabla_2, w \in \Delta_2. \quad (3.23) \]

For example, for \( \mathbb{P}^{2,3,1} \), \( M_{\text{max}} = 5 \), and for \( \mathbb{P}^{1,1,2} \), \( M_{\text{max}} = 3 \). We can then check for each lattice point \( v \in \nabla \) whether there exists a vertex \( w \) in \( \Delta \) with \( v \cdot w > M_{\text{max}} \). If there is, then \( v \) cannot be a ray in a fiber \( \nabla_2 \). We collect the subset of rays in \( \nabla \) that are not ruled out by this condition:

\[ S = \{ v \in \nabla : v \cdot w \leq M_{\text{max}} \ \forall w \ \text{vertex of} \ \Delta \}. \quad (3.24) \]

2. We then look for a subset of rays of \( S \) that satisfy the necessary linear relations to be elements of the fiber \( \nabla_2 \). For example, for \( \mathbb{P}^{2,3,1} \), we want to find rays \( \{v_x, v_y, v_z\} \) that satisfy

\[ 2v_x + 3v_y + v_z = 0. \quad (3.25) \]

In this case we can look at all pairs of rays \( v, v' \) in \( S \), and check to see if \( 2v + 3v' \) is also an element of \( S \). If so, we can then check that the intersection of \( \nabla \) with the plane spanned by \( v, v' \) precisely contains the 7 points in the polytope \( \nabla_2 \) shown in Figure 3-1a. If this is the case then \( \nabla \) has a fiber \( \nabla_2 \). The other fiber types can be checked in a similar fashion.

By equations (2.39), (3.23) and the projection \( \Delta \rightarrow \Delta_2 \), the maximum exponent of all monomials in the variables associated with the rays in the fiber should be \( M_{\text{max}} + 1 \), and the monomials can be grouped according to the powers of the fiber coordinates into sets that are in one-to-one correspondence with the lattice points in \( \Delta_2 \). For example, for \( \mathbb{P}^{2,3,1} \)-fibered polytopes (see Figure 3-1b), we have the maximum exponent in \( z \) among all fiber coordinates; \( M_{\text{max}} + 1 = 6 \), and the lattice points in \( \Delta_2 \) are in one-to-one correspondence with the sections

\[ \{y^2, xyz, yz^3, x^3, x^2z^2, xz^4, z^6\}. \quad (3.26) \]
Note that, following the definition of a standard $\mathbb{P}^{2,3,1}$-fibered polytope from §3.2.2, the lattice points in $\Delta_2$ are in one-to-one correspondence with the sections of the line bundles $\mathcal{O}(-nK_B)$, and the monomials $x^3$ and $y^2$ are the only two independent of the base coordinates.

Similarly, the sections of the $\mathbb{P}^{1,1,2}$-fibered polytope (see Figure 3-6) are

$$\{y^2, yz^2, xyz, x^2y, z^4, xz^3, x^2z^2, x^3z, x^4\} \quad (3.27)$$

when the associated rays are such that

$$v_x + 2v_y + v_z = 0, \quad (3.28)$$

and $M_{\text{max}} + 1 = 4$. The first step in the algorithm above is only used to speed up the algorithm, but particularly when the number of lattice points in $\nabla$ is large, this speedup is significant. For example, for the polytope associated with the Calabi-Yau with Hodge numbers $H:491, 11$, the number of lattice points in $\nabla$ is 680, while the number in $S$ is only 9. Since the second step of the algorithm is quadratic in the number of lattice points considered, this represents a speedup by a factor of hundreds or thousands of times in many cases. While in this work we are only considering a few examples, such a speedup is useful when considering larger datasets. In the next chapter we will describe the systematic application of this algorithm to all elements of the KS database with large Hodge numbers.

Once we have determined the fiber, we can then compute the base $B_2$ of the fibration. We define the set of rays of the fan describing $B_2$ to be

$$\{v_i^{(B)}/\text{GCD}(v_i^{(B)}, v_j^{(B)}), \forall v_i \in \nabla\}, \quad (3.29)$$

where $v_i^{(B)} \equiv \pi(v_i) = (v_{i,1}^{(B)}, v_{i,2}^{(B)})$ and $\pi$ is the projection along the fiber subpolytope ($\pi(\nabla_2) = 0$). The division by GCD($v_{i,1}^{(B)}, v_{i,2}^{(B)}$) is done to restrict to primitive rays in the image, as discussed in §3.2.1. Given the rays $v_i^{(B)}$, we associate a 2D cone with each pair of adjacent rays, giving a unique toric structure to the base geometry $B_2$. 

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Note that the base defined this way gives a flat toric fibration, but not necessarily a flat elliptic fibration [48]. We discuss this point in more detail in later sections.

In the regions of the Hodge numbers that we study in this chapter, we also encounter polytopes that have no standard $\mathbb{P}^{2,3,1}$ fiber. These polytopes can be described using two different types of models. One of these other types of model that we encounter is very similar to the standard $\mathbb{P}^{2,3,1}$-fibered polytopes, but has a fiber that is a single blowup of $\mathbb{P}^{2,3,1}$. This $\text{Bl}_{[0,0,1]}\mathbb{P}^{2,3,1}$ fiber, which is one of the other 16 reflexive 2D fiber types, is shown in Figure 3-5. The corresponding fiber subpolytope $\Delta_2$ is identical to that for the $\mathbb{P}^{2,3,1}$ fiber except that it has an additional vertex at $(-1, -1)$, so that the number of lattice points in the plane of the fiber subpolytope is 8 rather than 7. From the Tate point of view, such a fiber occurs when all the monomials in the coefficient $a_6$ are taken to vanish. This vanishing of $a_6$ forces a global $u(1)$ symmetry that we mentioned earlier [41, 53]. We describe an explicit example of this type of model at the end of §3.6.1. Models with this fiber can be treated in essentially the same fashion as standard $\mathbb{P}^{2,3,1}$-fibered polytopes.

The other unusual kind of fibration that we encounter in a few models is a fibered polytope with fiber $\Delta_2$ given by the usual $\mathbb{P}^{2,3,1}$ polytope, but with a different “twist” to the $\mathbb{P}^{2,3,1}$ bundle over the base. In other words, while there is a projection of $\Delta$ to the dual polytope $\Delta_2$ of the $\mathbb{P}^{2,3,1}$ fiber, the base rays in $\nabla$ do not all lie in a plane that contains the vector $v_2$; i.e., the base of the polytope defined in (3.29) can not be constituted by a set of rays all in the form (3.21). The consequence of this is that the hypersurface equations (2.38) for these Calabi-Yau threefolds do not take on the Tate form (3.1). In particular, there is generically more than one lattice point projected to the points in $\Delta_2$ associated with $y^2$ and/or $x^3$. To determine the Weierstrass form (2.9) for the models of this type that we found and analyze their structure, we found that it was useful to view them as essentially “$\mathbb{P}^{1,1,2}$-fibered polytopes” (or more precisely, $\mathbb{P}^{1,1,2}$ with two more blowups) rather than the standard $\mathbb{P}^{2,3,1}$-fibered polytopes (see figure 3-6 for comparison). This allows us to follow the method for

7 Note that in higher dimensions, the cone structure of the fan is not uniquely determined by the rays.
analyzing $\mathbb{P}^{1,1,2}$-fibered models described in Appendix A of [48] to bring them into Weierstrass form. This type of novel model gives rise to an enhancement over non-toric curves as we mentioned earlier. We refer to this type of models as *non-standard* $\mathbb{P}^{2,3,1}$-fibered polytopes, and describe their analysis in more detail in section 3.6.2. The treatment of non-standard $\mathbb{P}^{2,3,1}$ models in terms of models with a blow-up of $\mathbb{P}^{1,1,2}$ as a fiber is closely analogous to the analysis of models with a $\text{Bl}_{[0,0,1]}\mathbb{P}^{2,3,1}$ fiber as special cases of $\mathbb{P}^{2,3,1}$ Weierstrass/Tate models.

### 3.2.4 Bases with Large Hodge Numbers

In this work we have confined our study to the simplest $\mathbb{P}^{2,3,1}$ fiber type polytopes. In part this is because the standard fiber type matches with the Tate structure of the Weierstrass model as discussed previously. Also, however, this fiber type dominates the structure at large Hodge numbers. In particular, we can explicitly identify constraints on the bases that can be used for the other 15 fiber types. These constraints are such that the other fiber types all lead to problematic codimension one $(4, 6)$ singularities on some divisor in the base when the base contains curves of sufficiently negative self-intersection. In particular, none of the other 15 fibers can be supported over any base that contains a curve of self-intersection less than $-8$. This immediately constrains the set of constructions at large Hodge number, since the generic elliptic fibrations with the largest Hodge numbers almost always involve $-12$ curves in the base (though there are notable exceptions to this general principle, including the other one of the two fibrations of the H:491:11 polytope).

We leave a more detailed analysis of the constraints on different fiber types for future work, but briefly outline the issue that arises for other fiber types besides the $\mathbb{P}^{2,3,1}$ fiber. Consider for example the $\mathbb{P}^{2}$ fiber type. Carrying out the analogue of the standard stacking procedure for a $\mathbb{P}^{2}$ fiber, we find that there are 10 dual monomials analogous to the coefficients $a_1, \ldots, a_6$. These 10 monomials are sections of line bundles $\mathcal{O}(-K), \mathcal{O}(-2K)$ and $\mathcal{O}(-3K)$. Any section of a line bundle $-nK$ must vanish over a $-12$ curve to at least degree $n$ when $n < 5$ by the Zariski decomposition. This immediately leads to the presence of a codimension one $(4, 6)$ singularity over
any $-12$ curve in the base. Similar issues arise for the other fiber types.

Considering the toric bases, we can simply consider the complete enumeration carried out in [11] and identify the bases with largest Hodge numbers that have curves of self-intersection no smaller than $-8$. The base of this type with the largest $h^{2,1}_2$ for the generic elliptic fibration is $F_8$, over which the generic elliptic fibration has Hodge numbers $(10, 376)$. Even over $F_8$, the largest $h^{2,1}_2$ value that can be achieved for a tuning with any fiber other than $\mathbb{P}^{2,3,1}_2$ is quite restricted; over this base, for example, there are 5 other fiber types including $\mathbb{P}^{1,1,2}_1$ that are possible; the generic fibration with each of these fiber types gives an elliptic Calabi-Yau threefold with Hodge numbers $(11, 227)$. Any other fibration with these or any other fibers other than $\mathbb{P}^{2,3,1}_2$ over any base would seem to give a Calabi-Yau threefold with an even smaller value of $h^{2,1}_2$. Thus, by restricting to Hodge numbers above $h^{2,1}_2 \geq 240$, we can expect that the threefolds in the KS database that admit elliptic fibrations should be all or almost all described by the $\mathbb{P}^{2,3,1}_2$ fiber type.

Similarly, the largest value of $h^{1,1}_1$ that can arise for a base with no curves of self-intersection below $-8$ is 224. The corresponding base has a set of toric curves of self-intersection $(0, -8/-7/-8/-8/-8/-8/-8/-8/-8/-8/-8/-8/-7/-8)$, where $/-$ denotes the sequence $-1, -2, -3, -2, -1$ associated with $E_7$ chains (see e.g. [11]), and a generic elliptic fibration with Hodge numbers $(224, 18)$. There is nothing that can be tuned over this base without producing a curve of self-intersection below $-8$ so it seems that confining attention to threefolds with $h^{1,1}_1 \geq 240$ should again restrict us to primarily $\mathbb{P}^{2,3,1}_2$ fiber types. As we see in §3.6, however, there are a few unusual cases in which bases that have generic elliptic fibrations with rather small values of $h^{1,1}_1$ admit extreme tunings that dramatically increase the value of $h^{1,1}_1$ without producing curves of highly negative self-intersection. In a companion paper [37], we study the fibration structure of the hypersurface models in the KS database more directly, and confirm both the prevalence of $\mathbb{P}^{2,3,1}_2$ fibers at large Hodge numbers and the existence of exceptions involving extreme tunings.
3.2.5 Tate Tuning and Polytope Tops

We saw in §3.2.2 that for a standard \( \mathbb{P}^{2,3,1} \)-fibered polytope, the lattice points of \( \Delta \) that project to each of the different lattice points of \( \Delta_2 \) (figure 3-1) correspond precisely to the sets of monomials in the coefficients of the Tate form (3.1). The lattice points of \( \Delta \) are thus divided into 5 groups corresponding to the 5 sections \( a_n \in \mathcal{O}(-nK_B) \) and another 2 points corresponding to the constant coefficients of \( y^2 \) and \( x^3 \). In the previous subsection we described generic elliptic fibrations over weak Fano bases, where the “standard stacking” procedure immediately gives a reflexive 4D polytope, and no additional rays are needed in \( \nabla \), corresponding to a physics model with no nonabelian gauge group. We now wish to consider how this story changes when there is a nontrivial nonabelian gauge group due either to an NHC in the base or a Tate tuning of the monomials in the Tate form.

The presence of an NHC in the base or an explicit Tate tuning can force some of the coefficients in the \( a_n \)s to vanish to some specified order along a particular base divisor \( D_i^{(B)} \). This absence of monomials in \( \Delta \) gives rise to a corresponding enlargement of \( \nabla \) from the standard stacking. The additional lattice points in the fan polytope \( \nabla \) correspond to the exceptional divisors that resolve the singularities of the associated fibered geometry. These additional lattice points form the “top” [39, 70, 72, 73] of the enhanced gauge symmetries over \( D_i^{(B)} \). In coordinate representation, a lattice point in the top of \( D_i^{(B)} \) is of the form

\[
( (l v_i^{(B)})_{1,2}, (pt_{1,2,3,4,5,6,7})_{3,4}), \tag{3.30}
\]

where

\[
pt_{1,2,3,4,5,6,7} = (2, 3), (1, 2), (1, 1), (0, 1), (0, 0), (-1, 0), (0, -1) \tag{3.31}
\]

are the 7 lattice points in the 2D reflexive fiber subpolytope \( \mathbb{P}^{2,3,1} \), \( v_i^{(B)} \) is the associated 2D ray, and \( l \in \mathbb{N} \) specifies the “level” of the point away from the fiber plane (see figure 3-2). We adopt the shorthand notation \( pt_j^{(l)} \) or \( pt_j^{l,...} \), where the number of primes specifies the level parameter \( l \). When we denote a top, the points
Table 3.5: The tops of $\mathfrak{su}(n)$ algebras. The coordinates of the points $pt_{1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17}$ are given in equations (3.31) and (3.32). All lattice points in these tops are of level one, and correspond to affine Dynkin nodes. The rank of each algebra is the number of the nodes minus one.

<table>
<thead>
<tr>
<th>$n$</th>
<th>Tate form</th>
<th>Top/Affine Dynkin nodes</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>${0, 1, 3, 4, 7}$</td>
<td>${pt'_1, pt'_2, pt'_3, pt'_4, pt'_5, pt'_6, pt'_7}$</td>
</tr>
<tr>
<td>8</td>
<td>${0, 1, 4, 8}$</td>
<td>${pt'_1, pt'_2, pt'_3, pt'_4, pt'_5, pt'_6, pt'_7, pt'_8}$</td>
</tr>
<tr>
<td>9</td>
<td>${0, 1, 4, 5, 9}$</td>
<td>${pt'_1, pt'_2, pt'_3, pt'_4, pt'_5, pt'_6, pt'_7, pt'_8, pt'<em>9, pt'</em>{11}}$</td>
</tr>
<tr>
<td>10</td>
<td>${0, 1, 5, 10}$</td>
<td>${pt'_1, pt'_2, pt'_3, pt'_4, pt'_5, pt'_6, pt'_7, pt'<em>8, pt'<em>9, pt'</em>{10}, pt'</em>{11}}$</td>
</tr>
<tr>
<td>11</td>
<td>${0, 1, 5, 6, 11}$</td>
<td>${pt'_1, pt'_2, pt'_3, pt'_4, pt'_5, pt'<em>6, pt'<em>7, pt'<em>8, pt'</em>{10}, pt'</em>{11}, pt'</em>{14}}$</td>
</tr>
<tr>
<td>12</td>
<td>${0, 1, 6, 12}$</td>
<td>${pt'_1, pt'_2, pt'_3, pt'_4, pt'<em>5, pt'<em>6, pt'<em>7, pt'<em>8, pt'</em>{10}, pt'</em>{11}, pt'</em>{12}, pt'</em>{14}}$</td>
</tr>
<tr>
<td>13</td>
<td>${0, 1, 6, 7, 13}$</td>
<td>${pt'_1, pt'_2, pt'_3, pt'<em>4, pt'<em>5, pt'<em>6, pt'<em>7, pt'<em>8, pt'</em>{10}, pt'</em>{11}, pt'</em>{12}, pt'</em>{14}, pt'</em>{17}}$</td>
</tr>
</tbody>
</table>

with fewer than the maximal number of primes over each point are omitted and implied by the point of most primes with the same index; e.g. $\{pt''_1, pt''_2, pt'_3, pt'_4\} = \{pt'_1, pt''_2, pt'_3, pt'_4, pt'_5, pt'_6\}$. The tops of the various gauge algebras have been worked out in the previous literature. Tops for gauge algebras of rank no greater than eight that arise in reflexive polytopes can be looked up for example in Table 3.2 in [39]. We have explicitly calculated a few more cases, including the tops of $\mathfrak{so}(n)$ and $\mathfrak{su}(n)$ gauge algebras to rank 12 in both cases and list the results in tables Table 3.5 and Table 3.6, respectively. In [40], Vincent Bouchard and Harald Skarke generalized the notion of tops (including those which may not have a completion to reflexive polytopes) to include all fiber types, and they classified all such “tops in the dual space” (i.e., the $M$ lattice space), including higher rank $\mathfrak{so}(n)$ and $\mathfrak{su}(n)$ tops. The tops in Table 3.5 and Table 3.6 were explicitly obtained from reflexive polytope constructed from successive Tate tunings, and we have cross-checked the $\mathfrak{so}(n)$ cases with the results of [40] in the dual space, which agree up to a $GL(2, \mathbb{Z})$ transformation. Note that for higher rank $\mathfrak{so}(n)$ and $\mathfrak{su}(n)$ algebras, the $\nabla$ polytope grows in the fiber subpolytope direction (as opposed to the level direction), and more $pts$ projecting to the fiber plane are involved. We list the ones we need in Table 3.5 and Table 3.6:

$$pt_{8,9,10,11,12,13,14,15,16,17} = (-1, -1), (-2, -1), (-3, -2), (-2, -2), (-4, -3), (-5, -4), (-3, -3),$$
$$(-6, -5), (-7, -6), (-4, -4).$$

(3.32)
Table 3.6: The tops of $\mathfrak{so}(n)$ algebras. The coordinates of the points $p_{t.2.3.4.5.6.7.8.9.10.12.13.14.15.16}$ are given in equations (3.31) and (3.32). (Only the highest level point for each $p_t$ is listed in each top, and the lattice points of the lower levels are implied.) $\mathfrak{so}(4n - 1)$ and $\mathfrak{so}(4n)$ in the table have the same top but different (numbers of) affine Dynkin nodes as the ranks (which differ from the number of the nodes by one) are different. These tops match those found in [40] after an appropriate coordinate transformation.
Figure 3-2: A 3D visualization of the lattice points that appear in a top over $v_i^{(B)}$; in standard $\mathbb{P}^{2,3,1}$ models, a top over a ray in the base $v_i^{(B)}$ (in the direction $H$) is a set of lattice points stacked over the 7 lattice points of the fiber subpolytope $\mathbb{P}^{2,3,1}$ (in the X-Y plane). The level (the multiple of $v_i^{(B)}$) where points are located is indicated by the number of primes. When the gauge algebra is trivial over the associated divisor $D_i^{(B)}$, $pt'$ (equation (3.21)) is the only point in the top; while otherwise there are additional points (cf. Table 3.1) forming the extended Dynkin diagram of the gauge algebra with $pt'$ the affine node.

There is a simple and precise correspondence between tunings of the Tate form and tops. This correspondence holds independent of whether the Tate form corresponds to an NHC or an explicit tuning. Consider for example a situation where the standard $\mathbb{P}^{2,3,1}$-fibered polytope $\nabla$ contains the lattice point $pt' = (v_{1,2}, 1, 2)$. Recall that the lattice point $pt' = (v_{1,2}, 2, 3)$ imposes the conditions on the dual lattice points $(m_1, 2, 1, 1)$ associated with monomials in $a_6$ that $v^{(B)} \cdot m + 5 \geq -1 \Rightarrow v^{(B)} \cdot m \geq -6$ as expected for a section of $\mathcal{O}(-6K)$. The point $pt'_1$ imposes the stronger condition $v^{(B)} \cdot m + 3 \geq -1 \Rightarrow v^{(B)} \cdot m \geq -6 + 2$, corresponding to the condition that $a_6$ vanish to order 2 over the corresponding $D^{(B)}$. A similar calculation shows that $(a_1, a_2, a_3, a_4, a_6)$ vanish to orders at least $(0, 0, 1, 1, 2)$ respectively when the point $pt'_2$ is present in $\nabla$. Indeed, this goes both ways: only when the $a_n$s vanish at least to orders $(0, 0, 1, 1, 2)$, associated with the absence of a certain set of lattice points in $\Delta$, can the point $pt'_2$ appear in $\nabla$, and indeed if all the $a_n$s vanish to these orders then the point $pt'_2$ must appear in the polytope $\nabla$ dual to $\Delta$. Thus, there is a precise local correspondence between Tate tunings of the $a_n$ coefficients over a certain ray in the base, associated with lattice points absent from $\Delta$, and the toric top in $\nabla$. 

80
Table 3.7: Some examples of the correspondence between additional lattice points in $\nabla$ associated with a ray $v^{(B)}$ in the base and the associated tuning of the Tate coefficients $(a_1, a_2, a_3, a_4, a_6)$ over the associated divisor.

<table>
<thead>
<tr>
<th>point</th>
<th>ord($a_1, a_2, a_3, a_4, a_6$)</th>
<th>group</th>
<th>type</th>
</tr>
</thead>
<tbody>
<tr>
<td>$pt_2'$</td>
<td>(0, 0, 1, 1, 2)</td>
<td>SU(2)</td>
<td>$I_2$</td>
</tr>
<tr>
<td>$pt_3'$</td>
<td>(0, 1, 1, 2, 3)</td>
<td>SU(3)</td>
<td>$I_3^s$</td>
</tr>
<tr>
<td>$pt_1''$</td>
<td>(1, 1, 2, 2, 3)</td>
<td>$G_2$</td>
<td>$I_0^{ns}$</td>
</tr>
<tr>
<td>$pt_4'$</td>
<td>(0, 0, 2, 2, 4)</td>
<td>Sp(2)</td>
<td>$I_4^{ns}$</td>
</tr>
</tbody>
</table>

Note that just as multiple Tate tunings can correspond to the same gauge algebra, the corresponding multiple tops also correspond to the same gauge algebra. The multiplicity of constructions for a given gauge algebra was studied from the point of view of tops in [41]. One particular situation in which multiple tops are possible for a fixed gauge algebra corresponds to monodromy-dependent Tate tuning configurations, which we discuss further in §3.4.1.

This correspondence leads to a natural association of reflexive polytopes with elliptic fibrations over toric bases that have Tate forms. Over a given base, various gauge groups can arise from a combination of non-Higgsable clusters and Tate tunings. The interplay between extra vertices in $\nabla$ over nearby divisors and the absence of monomials in $\Delta$ leads to local interactions between the sets of lattice points in the polytope that are affected by adjacent rays in the base. We consider more explicitly in the following section how this leads to consistent reflexive polytopes in both the NHC and Tate tuning cases.

### 3.2.6 Singular Fiber Resolution and Lattice Points in Tops

We will not deal systematically with the explicit triangulation of $\nabla$, corresponding to the resolution of the Calabi-Yau threefold, but make some comments here on the relationship between extra rays in $\nabla$ and the resolution of the singular fiber associated with a tuned or non-Higgsable gauge group. Many of the details of this correspondence were worked out in [72, 73]. When the gauge algebra is non-trivial
over a divisor $D^{(B)}$, there are lattice points in the top over $v^{(B)}$ in addition to just $pt'$. Specifically, in the cases where there are no lattice points in the top lying in the interior of the 2-dimensional faces of $\nabla$, the lattice points in the top that do not lie in the interior of the 3-dimensional faces of $\nabla$ form the Dynkin diagram of the gauge algebra. These correspond to the exceptional divisors that arise in the resolution of the corresponding singularities. However, when there are lattice points lying in the interior of the 3-dimensional and the 2-dimensional faces of $V$, they contribute to the second and third terms, respectively, in Batyrev’s $h^{1,1}$ formula (2.43). The second term corresponds to components that miss the hypersurface, and contributions to the third term arise when the singularity is not resolved by a toric divisor but rather by a non-toric deformation, so the Dynkin diagram is not fully visible from the top. This happens exactly in those gauge algebras with an additional monodromy condition that is automatically satisfied.

In summary, $\nabla$ models are divided into two types according to whether there is a nonzero third term in the $h^{1,1}$ formula (2.43): (1) Trivial third term: There is no lattice point lying in the interior of any two-dimensional face. Gauge algebras can be read off directly from tops (the nodes of the Dynkin diagram are given by the lattice points in the top that do not lie in interior of facets), which are those in the literature. The Tate forms are those with no additional monodromy condition, which again match those in the literature. The nodes also correspond to exceptional divisors resolving the singular fiber. (2) Non-vanishing third term: There are lattice points lying in the interior of two-dimensional faces. These cases give rise to the additional Tate forms we have described. For example, in the gauge algebras involved with monodromy conditions, there are Tate forms of lower degrees, which achieve the gauge algebras by satisfying the additional monodromy conditions automatically. The singular fiber is (partially) resolved by deformation. Therefore, there are fewer exceptional divisors in the top, in which the “Dynkin diagram” would seem to be the lower rank gauge algebra counterpart.
3.2.7 Tops over $-9, -10, -11$-curves and 2D Fiber Resolution

While standard $\mathbb{P}^{2,3,1}$-fibered polytopes associated with singular fibers of Kodaira types only the resolved singular elliptic fiber is one dimensional embedded in a 2D toric variety described by the top and the resolved smooth elliptic fibration is flat as discussed in the preceding section; the standard stacking construction of a polytope for a standard $\mathbb{P}^{2,3,1}$-fibered model over a base surface containing $-9, -10, -11$-curves produces a flat toric fibration that leads to a hypersurface that is a non-flat elliptic fibration. There are $(4,6)$-points in the $-9, -10, -11$-curves where the fiber becomes two-dimensional; the singular fiber is resolved into an irreducible component of the non-generic toric fiber, which is two-dimensional, as the hypersurface CY equation restricting to the component is trivially satisfied over these points. For a flat elliptic fibration, the $(4,6)$-points in the base must be blown up, which in general leads to a non-toric base. Note that in the Calabi-Yau hypersurface picture, some flops may be necessary before the blow-ups can be done in the toric picture [48]. Nonetheless, this provides a clear correspondence between the non-flat elliptic fibrations associated with polytopes leading to $(4,6)$ points in the base and flat elliptic fibrations over blown up bases, which provide Calabi-Yau threefolds with the same Hodge numbers.

We go through the details of these constructions for the Hirzebruch surface bases $\mathbb{F}_m, m = 9, 10, 11$.

The flat toric fibration of $\mathbf{M:560 6 N:26 6 H:14,404}$ gives a non-flat elliptic fibration model over the toric base $\mathbb{F}_{m=9}$. The vertices of the $\nabla$ polytope are

$$\{(0,0,-1,0),(0,-6,2,3),(-1,-m,2,3),(1,0,2,3),(0,1,2,3),(0,0,0,-1)\}.$$

(3.33)

We associate the base coordinates $\{b_1, b_2, b_3, b_4\}$ to the toric curves $\{0,-m,0,m\}$ whose corresponding rays in the base are $\{(1,0),(0,-1),(-1,-m),(0,1)\}$. The set of lattice points in the top over the $-m$-curve is given by the set of lattice points in
\n
∇ of the form \((0, a, x, y)\),

\[
\{(0, -6, 2, 3), (0, -5, 2, 3), (0, -4, 1, 2), (0, -4, 2, 3), (0, -3, 1, 1), (0, -3, 1, 2),
(0, -3, 2, 3), (0, -2, 0, 1), (0, -2, 1, 2), (0, -2, 2, 3), (0, -1, 0, 0),
(0, -1, 0, 1), (0, -1, 1, 1), (0, -1, 1, 2), (0, -1, 2, 3)\}\].

(3.34)

Each of these points represents an irreducible component of the 2-dimensional non-generic toric fiber over the \(-m\)-curve \(\{b_2 = 0\}\) and projects to the corresponding base ray \((0, -1)\). Over a generic point on the \(-m\) curve, the hypersurface \(CY, p\) given by equation (2.38), intersects with only the irreducible components on the boundary of the top giving a \(\mathbb{P}^1\) for each, which combine to form the \(E_8\) affine Dynkin diagram. These nine components are

\[
\{((0, -6, 2, 3), (0, -5, 2, 3), (0, -4, 2, 3), (0, -3, 2, 3), (0, -2, 2, 3), (0, -1, 2, 3),
(0, -1, 1, 2), (0, -1, 0, 1), (0, -1, 0, 0)\}\].

(3.35)

where the set of components in first line forms the longest leg of the diagram, and the sets \{\((0, -6, 2, 3), (0, -4, 1, 2), (0, -2, 0, 1)\)\} and \{\((0, -6, 2, 3), (0, -3, 1, 1)\)\} form the other two legs. \((0, -6, 2, 3)\) is the node where three legs connect, and \((0, -1, 2, 3)\) is the affine node.

However, \(p\) also intersects the full irreducible component \((0, -1, 0, 0)\) over three points in the \(-m\)-curve, but does not meet the component over the other points: \(p\) restricted to the divisor \(I = (0, -1, 0, 0)\) is

\[
p|_I = b_2^2 (c_4 b_1^3 + c_{284} b_1^2 b_3 + c_{285} b_1 b_3^2 + c_3 b_3^3) b_4^7,
\]

(3.36)

where \(c_3, c_4, c_{284}, \text{ and } c_{285}\) are some complex structure moduli. This vanishes identically over the three points \(\{c_4 b_1^3 + c_{284} b_1^2 b_3 + c_{285} b_1 b_3^2 + c_3 b_3^3 = 0\}\) in the \(-m\)-curve (\(I\) projects to the ray of the \(-m\)-curve), and is otherwise a constant. It is these three points in the toric base that must be blown up to give the \(-12\)-curve and the
semi-toric base over which the elliptic fibration model becomes flat and gives a good model for F-theory compactification.

Similarly, the flat toric fibrations of $M:600 \times 6 \times N:26 \times 6 \times H:13,433$ and $M:640 \times 6 \times N:26 \times 6 \times H:12,462$ give non-flat elliptic fibration models over the toric bases $F_{m=10}$ and $F_{m=11}$, respectively. Both vertex sets are given by equation (3.33), and the tops over the $-m$-curves are the same as that over the $-9$ curve in equation (3.34). We know that a $-10$-curve (resp. a $-11$-curve) would need two blowups (resp. one blowup) to become a $-12$-curve, so we expect there are two $(4,6)$ points (resp. one point) in the $-m$-curve over which the resolved fiber is two-dimensional. Indeed, we calculate the CY hypersurface in equation (2.38), and restrict it on each component in (3.34), and we find

$$p|_I = b_2^5(c_4 b_1^2 + c_3 b_3 b_3 + c_3 b_3^2) b_3 b_4^7$$

in the case of $m = 10$, and

$$p|_I = b_2^5(c_4 b_1 + c_3 b_3) b_4^7$$

in the case of $m = 11$. Over a generic point in the $-m$-curve, $p|_I$ is non-vanishing, and $p$ intersects with the nine components in (3.35), each giving a $\mathbb{P}^1$ that corresponds to a node in the extended $E_8$ Dynkin diagram.

The correspondence between the non-flat and the flat models may be thought of as encoding the relationship between the irreducible component of the 2-dimensional fiber over a $(4,6)$ point and divisors that resolve the $-m$-curve to a $-12$-curve in the base.

### 3.3 Reflexive Polytope Construction from Elliptic Fibrations

We defined in §3.2.2 a standard $\mathbb{P}^{2,3,1}$-fibered polytope, and showed that there is always a corresponding Tate model. Now we are trying to do the converse — given a toric base and a corresponding Tate model, we wish to construct a corresponding reflexive polytope.
A generic Tate-form elliptic fibration over a given toric base can always be constructed starting from the “standard stacking” procedure; this procedure uses the \( \mathbb{P}^{2,3,1} \) fiber type with a specific twisting. Tuning the resulting generic Tate model by removing monomials in the dual polytope then leads to a set of possible tunings corresponding to further reflexive polytopes that can appear in the database: we describe in §3.3.1 the construction of reflexive polytopes from elliptic fibrations corresponding to F-theory models from generic models with or without NHCs to constructions with tunings with an example in §3.3.2. In 3.3.3, we consider polytopes associated with global models of combining Tate tunings that are each possible locally into an allowed Tate-tuned global model over a given base.

### 3.3.1 Reflexive Polytopes from Elliptic Fibrations

The recipe for the construction of a 4D standard \( \mathbb{P}^{2,3,1} \)-fibered polytope for an elliptically fibered threefold is the natural generalization of the 3D reflexive polytope for a K3 surface that is an elliptic fibration over a \( \mathbb{P}^1 \) base as described in e.g. [70].

To construct a 4D \( \mathbb{P}^{2,3,1} \)-fibered polytope, we start with the 2D \( \mathbb{P}^{2,3,1} \) fiber and a 2D base, and we construct the polytope in a straightforward way to have the desired fibration structure over the base. We denote the toric fan associated with the base \( B \) by \( \Sigma_B \), with the set of rays being \( \{v_i^{(B)}\} \). Taking the fan of \( \mathbb{P}^{2,3,1} \) to be the ambient space of the elliptic fiber, we can embed this in the 4D coordinates such that the three rays are \( \{v_x, v_y, v_z\} = \{(0, 0, -1, 0), (0, 0, 0, -1), (0, 0, 2, 3)\} \). Since in the Weierstrass or Tate model framework of equation (2.10) the fiber coordinate \( z \) is associated with the trivial bundle over the base, the lattice point associated with the ray \( v_z = (0, 0, 2, 3) \) should be in the plane of the base. Thus, we define a polytope \( \mathcal{V} \) to be the convex hull of the set

\[
\{(v_{i,1}^{(B)}, v_{i,2}^{(B)}, 2, 3)|v_i^{(B)} \text{ rays in } \Sigma_B\} \cup \{(0, 0, -1, 0), (0, 0, 0, -1)\},
\]

where \( v_{i,1}^{(B)}, v_{i,2}^{(B)} \) are the first and the second components of the 1D ray \( v_i^{(B)} \) in the smooth 2D toric base \( B \). From the definition in the previous section, this is a stan-
dard \( \mathbb{P}^{2,3,1} \)-fibered polytope; we refer sometimes to this construction as the "standard stacking" approach to construction of a polytope. Note that the 4D rays \( v_i = (v_{i,1}^{(B)}, v_{i,2}^{(B)}, 2, 3) \) can be vertices of \( \tilde{\nabla} \) only if \( v_i^{(B)} \) are associated with curves of self-intersection \( D_i \cdot D_i > -2 \) (see Table 3.8). We now wish to check that \( \tilde{\nabla} \) is reflexive, so it can be used as the reflexive polytope \( \nabla \) in Batyrev's construction of a Calabi-Yau threefold. In some cases \( \tilde{\nabla} \) is immediately reflexive, and in other more complicated cases it must be modified to make it reflexive.

### Fibrations without singular fibers

We start with the simplest case, in which we have a generic elliptically fibered Calabi-Yau over a toric base \( B \) that contains no non-Higgsable clusters (i.e., no curves with self-intersection less than \(-2\)). In this case, the Weierstrass/Tate model of the Calabi-Yau is smooth and there is no gauge group in the 6D supergravity theory. In this context, lattice points associated to curves of self-intersection \(-2\) lie on the 1D faces of \( \tilde{\nabla} \) that are boundaries of the 2D face \( \theta_B \), which is the 2D face associated with the base; and there are no interior points in \( \theta_B \) other than \((0, 0, 2, 3)\). We can now check directly that in these simple cases \( \tilde{\nabla} \) is reflexive without further modification. The vertices of the polytope dual to the convex hull of the set of vertices \((3.39)\), in any

<table>
<thead>
<tr>
<th>NHC</th>
<th>{-3}</th>
<th>{-2, -3}</th>
<th>{-2, -2, -3}</th>
<th>{-2, -3, -2}</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fan</td>
<td><img src="image1" alt="Fan1" /></td>
<td><img src="image2" alt="Fan2" /></td>
<td><img src="image3" alt="Fan3" /></td>
<td><img src="image4" alt="Fan4" /></td>
</tr>
</tbody>
</table>

Table 3.8: Non-convexity of NHCs: The rays corresponding to an NHC cannot be vertices; hence, the vertex contribution from the base can only come from curves of self-intersection \( \geq -1 \) (isolated \(-2\) curves will be on a 1D face, and also cannot be vertices).
where \((i, j)\) are taken to run over all pairs of labels of base rays that correspond to adjacent vertices of \(\theta_B\). The vertices in (3.40) will lie on the \(M\) lattice only when the denominators \(\text{Det}[v_i^{(B)}, v_j^{(B)}]\) are cleared so that all entries are integers. For a smooth 2D base fan, \(\text{Det}[v_i^{(B)}, v_{i+1}^{(B)}] = 1\), so we have a lattice point whenever \(j = i + 1\) (including the boundary case \(j = 1, i = n\)); i.e., we get lattice points as long as there are no non-convex base rays, which would be skipped. We also get a lattice point as long as \(v_i^{(B)}\) and \(v_j^{(B)}\) are separated only by some number \(k\) of -2 curves. In this case \(v_i^{(B)} - v_j^{(B)} = kw\), where \(w\) is a primitive vector, and \(\text{Det}[v_i^{(B)}, v_j^{(B)}] = k\), so we again have a cancellation and the vertex of the dual polyhedra is an integral lattice point.

Thus, as long as the base \(B\) contains no non-Higgsable clusters, the set of vertices (3.39) immediately provides a reflexive polytope.\(^8\)

Simple examples of polytopes realized in this way are the elliptically fibered Calabi-Yau threefolds over the toric bases \(\mathbb{P}^2\), \(\mathbb{F}_{n=0,1,2}\), whose vertex sets of the \(M\) polytopes \(\Delta\) are (3.40), with the first set of vertices respectively being \((-6, -6, 1, 1), (12, -6, 1, 1), (-6, 12,\)
\((-6, -6, 1, 1), (6(1 + n), -6, 1, 1), (6(-n + 1), 6, 1, 1), (-6, 6, 1, 1)\}, given the respective base rays \{(1, 0), (0, 1), (-1, -1)\}, \{(1, 0), (0, 1), (-1, -n), (0, -1)\}. The \(\mathbb{P}^2\) model gives the only polytope (up to lattice automorphism) with Hodge numbers \(H:2,272\) in the KS database and the \(\mathbb{F}_{n=0,1,2}\) models give exactly the three data points with Hodge numbers \(H:3,243\).

The bases described by toric varieties with no curves of self-intersection less than -2 are weak Fano varieties, and correspond to reflexive 2D polytopes, as we have just verified explicitly. We now want to describe the generalization of this construction to situations where there is a gauge group arising either from a non-Higgsable cluster in the base or a Tate tuning. The realization of reflexive 4D polytopes in these

\(^8\)As we will discuss in \(\S3.3.1\), the set (3.39) still gives a reflexive polytope in certain cases when the base contain NHCs, but those lattice points corresponding to the curves in the base that carry the NHCs are not vertices.
cases arises from a general relationship between Tate tunings and “tops” in the toric language.

NHCs with immediately reflexive polytopes

Now consider models where the base has a non-Higgsable cluster. We begin with the simplest cases, where the NHC contains a single curve of negative self-intersection \(-m\), and \(m|12\). In these cases, the same standard stacking construction described in equations (3.39) and (3.40) for without NHC cases still lead directly to a reflexive polytope. This can be understood from several points of view. Due to the factor 6 in the numerators of the first two coordinates in (3.40), those cases where a ray is skipped and \(\text{Det}[v_i^{(B)}, v_j^{(B)}] = 3\), or 6 also give lattice points; i.e., when the skipped rays are NHCs \(-3\) and \(-6\); furthermore, the NHCs \(-4\) and \(-12\) are fine as well because of extra factors of 2 that arise from the difference terms in the numerators. Therefore the set (3.40) should also be sufficient to give the \(\Delta\) polytopes of the models with the NHCs \(-3, -4, -6, -12\), so that the standard stacking polytope \(\nabla\) defined through (3.39) is reflexive. The values of \(m\) compatible with the standard stacking can also be understood from the bounds on the set of monomials in \(a_6\) controlled by the \(-m\) curve. Other than the vertices from \(x^3, y^2\), all vertices of \(\Delta\) come from lattice points associated with monomials in \(a_6\). Choosing local toric coordinates for a set of adjacent rays \(v_1^{(B)}, v_2^{(B)}, v_3^{(B)}\) in the base \(B\) so that the ray \(v_2\) corresponds to the \(-m\) curve,

\[
v_1^{(B)} = (1, 0), v_2^{(B)} = (0, -1), v_3^{(B)} = (-1, -m),
\]

the monomials \((m_1, m_2)\) in \(a_6 \in \mathcal{O}(-6K_B)\) are then bounded by \(m_1 \geq -6, m_2 \leq 6\), and \(6 - mm_2 \geq m_1\). The first and the third constraints intersect at an integral point precisely when \(m|12\). This intersection point is a vertex of \(\Delta\), so \(\Delta\) can only be a lattice polytope when \(m|12\). Note that \(6 - 12/m\) is the order of vanishing of \(a_6\) over the divisor associated with \(v_2^{(B)}\) since there are no points in the dual lattice with \(m_2 > 12/m\).

As an example, the reflexive polytope model for the generic elliptically fibered CY 89
over the base $\mathbb{F}_{12}$ has \( \{v_i^{(B)}\} = \{(1, 0), (0, -1), (-1, -12), (0, 1)\} \) (the self-intersection numbers of the toric divisors are \( \{0, -12, 0, 12\} \)); the vertices of the 2D convex polygon are \( i = 1, 3, 4 \), and the dual vertices arise from the pairs \( \{(i, j)\} = \{(1, 3), (3, 4), (4, 1)\} \); so with these pairs, (3.40) gives the vertices of the dual polytope $\Delta$, which is a lattice polytope. Indeed, this polytope has vertices

\[
\{(-6, 1, 1, 1), (78, -6, 1, 1), (-6, -6, 1, 1), (0, 0, -2, 1), (0, 0, 1, -1)\},
\]

and is the only reflexive polytope in the $M$ lattice (up to lattice automorphism) associated with the Hodge pair H:11,491 in the KS database.

We can understand the reflexive polytopes formed in this way in terms of the dual Tate tunings and tops described in the previous subsection. For example, consider the case of the $-3$ curve NHC. Using again the local toric coordinates (3.41) with $m = 3$, the polytope $\nabla$ has vertices from (3.39), $(1, 0, 2, 3)$ and $(-1, -3, 2, 3)$. Considering a 3D slice of $\nabla$ that contains the fiber polytope $\nabla_2$ and the ray $v_2^{(B)} = (0, -1)$, we have a picture like Figure 3-2, where $v_i^{(B)}$ is identified with $v_2^{(B)}$. The boundary of the polytope $\nabla$ intersects the vertical line \( \{H = 0\} \) plane, at $(X, Y, H) = (2, 3, 3/2)$; this corresponds in the polytope to the midpoint $(0, -3/2, 2, 3)$ of the line between the two vertices $(1, 0, 2, 3)$ and $(-1, -3, 2, 3)$. The boundary of the polytope in the 3D slice is therefore the 2-plane passing through the points $(2, 3, 3/2), (0, -1, 0), (-1, 0, 0)$. This plane passes through the point $pt_2' ((X, Y, H) = (1, 2, 1)$ in the Figure), so the reflexive polytope associated with a standard stacking from a base with a $-3$ curve automatically has the point $pt_2' = (0, -1, 1, 2)$ in the top in $\nabla$. Using the same methodology as in the $n = 6$ example above, we see that the orders of vanishing of the $a_n$s in the dual polytope are $(1, 1, 1, 2, 2)$. From Table 3.2, we see that this is a type $IV$ singularity; in this case the monodromy condition for the gauge algebra $su(3)$ is automatically satisfied, so this actually corresponds to an $su(3)$ top, as indicated in the first line of Table 3.1.
Other NHCs: reflexive polytopes from the dual of the dual

The rest of the NHCs have the issue that there are fractions in the vertices of the dual polytope described by (3.40). Let us denote the convex hull of the set of vertices defined by (3.39) by $\tilde{\Delta}$, and its dual by $\tilde{\Delta}$. If $\tilde{\Delta}$ is not a lattice polytope then $\tilde{\Delta}$ is not a reflexive polytope. We have to supply $\tilde{\Delta}$ with additional lattice points to make it into a reflexive polytope $\Delta$ so that $\Delta = \Delta^*$ is a lattice polytope.

We can turn $\tilde{\Delta}$ into a reflexive polytope in a minimal fashion by taking the “dual of the dual”. We begin by defining the lattice polytope $\Delta^0 = \text{convex hull}(\tilde{\Delta}^* \cap M)$ to be the polytope defined by the convex hull of the set of integral points of $\tilde{\Delta}$; the polytope $\Delta^0$ then has itself a dual $\nabla = (\Delta^0)^*$. This gives us the minimal reflexive polytope $\nabla \supset \tilde{\Delta}$ in the $\mathbb{N}$ lattice that we are looking for; for any base with NHCs, as we have confirmed by explicit computation in each case, the resulting $\nabla$ indeed has a dual $\Delta = \nabla^*$ that is a lattice polytope.

This “dual of the dual” procedure adds points in the $\mathbb{N}$ lattice that are needed to complete the tops associated with the gauge symmetries coming from the NHCs in all cases other than those of a single curve with self-intersection $n|12$. For example, take the generic model over $\mathbb{F}_5$ described by the set of rays $\{(1, 0), (0, 1), (-1, -5), (0, -1)\}$; if we took just (3.39) as the set of vertices, we would have $\{(i, j) = (1, 2), (2, 3), (3, 1)\}$ in (3.40) and there would be a non-lattice point vertex $(-6, 12/5, 1, 1)$ from $(i, j) = (3, 1)$. This problem can be seen as arising from the absence of a sufficient set of lattice points in $\tilde{\Delta}$ over the NHC $-5$-curve $\nu_4^{(B)}$ to form a complete $f_4$ top. While the top in $\tilde{\Delta}$ (the convex hull of the standard stacking polytope) over $\nu_4^{(B)}$ is $\{pt'_1, pt''_1, pt'_2, pt'_3\}$, it is $\{pt'_1, pt''_1, pt''_2, pt''_3, pt'_2, pt'_3, pt'_4\}$ in $\nabla$; the latter is exactly the $f_4$ top as described in earlier literature, which is obtained explicitly via the $\Delta^0$ construction we just described above.

For each of the NHC’s, Table 3.1 describes the tops that arise over the divisors supporting the NHC, the corresponding Tate forms, and the vanishing orders of $f, g, \Delta$ along with the resulting gauge algebra. The minimal top associated with the $\Delta^0$ construction of $\nabla$ as the dual of the dual is in each case the first top listed. In a number
<table>
<thead>
<tr>
<th>Hodge pair</th>
<th>Mult. in KS</th>
<th>$\mathbb{F}_n$ base</th>
<th>Gauge symmetry</th>
<th>Top over the $-n$-curve</th>
</tr>
</thead>
<tbody>
<tr>
<td>(3,243)</td>
<td>3</td>
<td>$\mathbb{F}_2$</td>
<td>trivial</td>
<td>${p_{t_1}'}$ (affine node)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\mathbb{F}_1$</td>
<td>trivial</td>
<td>${p_{t_1}'}$ (affine node)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\mathbb{F}_0$</td>
<td>trivial</td>
<td>${p_{t_1}'}$ (affine node)</td>
</tr>
<tr>
<td>(5,251)</td>
<td>3</td>
<td>$\mathbb{F}_3$</td>
<td>$su(3)$</td>
<td>${p_{t_1}', p_{t_2}'}$ (&quot;$su(2)$&quot;)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\mathbb{F}_3$</td>
<td>$g_2$ enhanced on -3</td>
<td>${p_{t_1}'', p_{t_2}'', p_{t_3}'}$</td>
</tr>
<tr>
<td>(7,271)</td>
<td>4</td>
<td>$\mathbb{F}_4$</td>
<td>$so(8)$</td>
<td>${p_{t_1}'', p_{t_2}'', p_{t_3}'}$ (&quot;$so(7)$&quot;)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\mathbb{F}_4$</td>
<td>$f_4$ enhanced on -4</td>
<td>${p_{t_1}''', p_{t_2}''', p_{t_3}''', p_{t_4}'}$</td>
</tr>
<tr>
<td>(7,295)</td>
<td>1</td>
<td>$\mathbb{F}_5$</td>
<td>$f_4$</td>
<td>${p_{t_1}''', p_{t_2}''', p_{t_3}''', p_{t_4}'}$</td>
</tr>
<tr>
<td>(9,321)</td>
<td>3</td>
<td>$\mathbb{F}_6$</td>
<td>$\mathfrak{e}_6$</td>
<td>${p_{t_1}''', p_{t_2}''', p_{t_3}''', p_{t_4}'}$ (&quot;$f_4$&quot;&quot;)</td>
</tr>
<tr>
<td>(10,348)</td>
<td>1</td>
<td>$\mathbb{F}_7$</td>
<td>$\mathfrak{e}_7$ (w/ matter $\frac{1}{2}56$)</td>
<td>${p_{t_1}''', p_{t_2}''', p_{t_3}''', p_{t_4}'}$</td>
</tr>
<tr>
<td>(10,376)</td>
<td>2</td>
<td>$\mathbb{F}_8$</td>
<td>$\mathfrak{e}_7$ w/o matter</td>
<td>${p_{t_1}''', p_{t_2}''', p_{t_3}''', p_{t_4}'}$</td>
</tr>
<tr>
<td>(11,491)</td>
<td>1</td>
<td>$\mathbb{F}_{12}$</td>
<td>NHC -12 curve: $\mathfrak{e}_8$</td>
<td>${p_{t_1}''', p_{t_2}''', p_{t_3}''', p_{t_4}'}$</td>
</tr>
</tbody>
</table>

Table 3.9: Polytope models associated with generic elliptic fibrations over the Hirzebruch surfaces $\mathbb{F}_{0,1,...,8,12}$, as well as all other models with the same Hodge numbers. Alternate constructions include multiple tops, some due to monodromy conditions in Tate tunings, as well as rank-preserving tunings (§3.4.1).

There are other higher Tate tunings that give different tops but the same gauge algebra, as discussed further in §3.4.1. The global models describing generic elliptic fibrations over the Hirzebruch surfaces that incorporate each of the single-curve NHC’s are also described explicitly in Table 3.9, showing how this construction works in the context of the global polytopes. While in this work we focus on the systematic construction of polytopes through tuning of Tate forms (corresponding to the structure of $\Delta$), one could also construct general polytopes by considering the different tops over each base and thus classifying polytopes $\nabla$; in Table 3.1 we also list the possible new vertices that may arise in the polytope $\nabla$ for each top.

### Reflexive polytopes from Tate tunings

We can understand Tate tunings in the polytope in a similar fashion. Consider starting with the reflexive polytope $\nabla$ associated with the generic elliptic fibration...
over a given toric base $B$, constructed as above using the standard stacking procedure and the dual of the dual if needed for NHC's. We take $\Delta$ to be the dual polytope of $\nabla$, which is also a lattice polytope. We can produce an additional gauge group beyond the minimum imposed from the NHC's by performing a tuning in the Tate description of the model, which corresponds to removing certain vertices from the polytope $\Delta$. Using a Tate tuning from Table 3.2 gives us the set of lattice points that should be removed from $\Delta$ associated with certain coefficients in the $a_n$'s over the divisor(s) in $B$. Calling the new $M$ polytope that results from removing these lattice points $\hat{\Delta}$, we get an enlarged $N$ polytope $\hat{\nabla} = (\hat{\Delta})^*$, which has extra lattice points. In general, each Tate tuning in $\Delta$ gives a corresponding top in $\nabla$, giving a new reflexive polytope $\hat{\nabla}$. This gives a large class of constructions for Tate tunings that should have reflexive polytope analogues in the KS database.

As a simple example (and a more detailed example in the next section), consider the polytope $\nabla$ associated with the generic elliptic fibration over $\mathbb{F}_2$. As discussed in §3.3.1 this polytope follows from the standard stacking procedure and has vertices given by

$$\nabla = \text{Conv}\{(1, 0, 2, 3), (-1, -m, 2, 3), (0, 1, 2, 3), (0, 0, -1, 0), (0, 0, 0, -1)\} \quad (3.43)$$

with $m = 2$. This is a reflexive polytope, identified in the Kreuzer-Skarke database as M:335 5 N:11 5 H:3,243. The dual polytope $\Delta$ has vertices

$$\{(6, -6, 1, 1), (0, 0, -2, 1), (18, -6, 1, 1), (0, 0, 1, -1), (-6, 6, 1, 1)\}. \quad (3.44)$$

Now consider a Tate tuning of the algebra $\mathfrak{su}(2)$ over the $-2$ curve $C$ in the base, which corresponds to the 2D toric vector $(0, -1)$. This is achieved by setting $a_1, a_2, a_3, a_4, a_6$ to vanish to orders $\{0, 0, 1, 1, 2\}$ in the coordinate associated with $C$, which is the second coordinate in $\Delta$. The set of the lattice points that have to be removed from $\Delta$ to achieve the required vanishing orders is $\{(-6, 5, 1, 1), (-6, 6, 1, 1), (-5, 5, 1, 1), (-4, 4, 0, 1), \ldots\}$.
The resulting new $M$ polytope after the reduction is

$$\hat{\Delta} = \text{Conv}\{(-6, -6, 1, 1), (-6, 4, 1, 1), (-2, 2, -1, 1), (-2, 4, 1, 1), (0, 0, -2, 1), (0, 0, 1, -i), (18, -6, 1, 1)\}.$$ (3.45)

This gives the reflexive polytope $\hat{\nabla}$ given by augmenting $\nabla$ from (3.43) with the additional lattice point $(0, -1, 1, 2)$, which gives the $\mathfrak{su}(2)$ top over $C$, as described in Table 3.7. The resulting polytope corresponds to the example $M:3297 N:126 H:4238$ in the KS database. The Hodge numbers from the polytope data are consistent with those from the anomaly cancellation calculation in equations (3.18) and (3.19) with a tuning of $\mathfrak{su}(2)$ on the isolated $-2$ curve: $\Delta h^{1,1} = \text{rank}(\mathfrak{su}(2)) = 1, \Delta h^{2,1} = \dim(\mathfrak{su}(2)) - 4 \times 2 = 3 - 8 = -5$.

In general, we find that the correspondence described in the last few subsections between Tate tunings and tops immediately provides reflexive polytopes for most Tate tuning constructions. There are several subtleties in this construction, which we elaborate in §3.4.

### 3.3.2 An Example of Polytope Tunings

We go through in details how standard $\mathbb{P}^{2,3,1}$-fibered polytope tuning corresponds to Tate tuning of Tate models with an example of polytopes for elliptic fibrations with tuned $\mathfrak{su}_3, \mathfrak{g}_2$ over the curve of self-intersection $-2$ in the Hirzebruch surface base $\mathbb{F}_2$.

Standard $\mathbb{P}^{2,3,1}$-fibered polytopes naturally correspond to Tate (tuned) models. In principle, as long as the Tate tunings on adjacent curves do not lead to $(4, 6)$ singularities\footnote{Although in some cases such $(4, 6)$ singularities still lead to reflexive polytopes that can be associated with flat elliptic fibrations over blown-up bases, as encountered in the examples of §3.6.1.}, and are not merely further specialization of existing tunings that do not change the gauge algebra\footnote{None of the lattice points corresponding to such further specialization are vertices of $\Delta$, so removing those points does not affect the polytope.}, removing the lattice points corresponding to a given tuning gives a different reflexive polytope, associated with the resolved CY of the tuned singular model. The Hodge numbers of the new resolved polytope model can
be computed either directly from the polytopes or through F-theory by anomaly cancellation.

As an example, consider tuning a type $I_3^* \text{su}(3)$ gauge algebra on the $-2$ curve in the base $\mathbb{F}_2$. The polytope model for the generic CY is $M:335 \text{ N:}11 \text{ H:}3,243$, of which the set of vertices of $\nabla$ is

$$\{(1,0,2,3),(0,1,2,3),(-1,-2,2,3),(0,0,-1,0),(0,0,0,-1)\}, \quad (3.46)$$

and the set of vertices of $\Delta$ is

$$\{(-6,-6,1,1),(0,0,-2,1),(18,-6,1,1),(0,0,1,-1),(-6,6,1,1)\}. \quad (3.47)$$

The projection along the fiber gives the rays in the base $\{v_i^{(B)}\} = \{(1,0),(0,1),(-1,-2),(0,-1)\}$ corresponding to curves of self-intersection $\{0,2,0,-2\}$. We calculate the hypersurface equation (2.38) and take the set of homogeneous coordinates $\{z_j\} = \{x,y,z,b_4\}$ associated respectively to rays $v_x,v_y,v_z$ in the fiber plane and $(v_{B_4}^{(1)},v_{B_4}^{(2)},2,3)$ in the base plane to get

$$y^2 + a_1 xyz + a_3 yz^3 = x^3 + a_2 x^2 z^2 + a_4 x z^4 + a_6 z^6, \quad (3.48)$$

where the 5 sections $a_i$ in the coordinates $b_4$ and some second coordinate $\zeta$ in the base have the forms

$$a_1(b_4,\zeta) = a_{1,0}(\zeta) + a_{1,1}(\zeta)b_4 + a_{1,2}(\zeta)b_4^2, \quad (3.49)$$
$$a_2(b_4,\zeta) = a_{2,0}(\zeta) + a_{2,1}(\zeta)b_4 + a_{2,2}(\zeta)b_4^2 + a_{2,3}(\zeta)b_4^3 + a_{2,4}(\zeta)b_4^4, \quad (3.50)$$
$$a_3(b_4,\zeta) = a_{3,0}(\zeta) + a_{3,1}(\zeta)b_4 + \cdots + a_{3,5}(\zeta)b_4^5 + a_{3,6}(\zeta)b_4^6, \quad (3.51)$$
$$a_4(b_4,\zeta) = a_{4,0}(\zeta) + a_{4,1}(\zeta)b_4 + \cdots + a_{4,7}(\zeta)b_4^7 + a_{4,8}(\zeta)b_4^8, \quad (3.52)$$
$$a_6(b_4,\zeta) = a_{6,0}(\zeta) + a_{6,1}(\zeta)b_4 + \cdots + a_{6,11}(\zeta)b_4^{11} + a_{6,12}(\zeta)b_4^{12}. \quad (3.53)$$

The numbers of monomials (lattice points) in the sections $a_i$ are $\{9, 25, 49, 81, 169\}$; together with the two points associated with $x^3$ and $y^2$ these compose the
total set of 335 lattice points in the M polytope $\Delta$. The number of monomials in each section can be further divided according to the power of the monomials in the $b_4$ expansion. According to Tate Table 3.2, the vanishing orders have to reach $\{0, 1, 2, 3\}$ in $b_4$ to tune an $I_3^s \mathfrak{su}(3)$ over $D_{B4}$, so all lattice points contributing to $a_{2,0}, a_{3,0}, a_{4,0}, a_{4,1}, a_{6,0}, a_{6,1}, a_{6,2}$ should be removed. As one can check those are

\[
a_{2,0} \leftrightarrow \{(-2,2,-1,1)\}, \tag{3.54}
\]
\[
a_{3,0} \leftrightarrow \{(-3,3,1,0)\}, \tag{3.55}
\]
\[
a_{4,0} \leftrightarrow \{(-4,4,0,1)\}, \tag{3.56}
\]
\[
a_{4,1} \leftrightarrow \{(-4,3,0,1),(-3,3,0,1),(-2,3,0,1)\}, \tag{3.57}
\]
\[
a_{6,0} \leftrightarrow \{(-6,6,1,1)\}, \tag{3.58}
\]
\[
a_{6,1} \leftrightarrow \{(-6,5,1,1),(-5,5,1,1),(-4,5,1,1)\}, \tag{3.59}
\]
\[
a_{6,2} \leftrightarrow \{(-6,4,1,1),(-5,4,1,1),(-4,4,1,1),(-3,4,1,1),(-2,4,1,1)\} \tag{3.60}
\]

After reduction, the vertex set of the new dual polytope $\Delta'$ for the tuned model becomes

\[
\{(-6,-6,1,1), (0,0,-2,1), (18,-6,1,1), (0,0,1,-1), (-6,3,1,1), \tag{3.61}
\]
\[
(-3,2,1,0), (-1,1,0,0), (-1,2,1,0), (0,3,1,1)\}, \tag{3.62}
\]

This new polytope is again reflexive, and corresponds to the example M:320 9 N:13 7 H:5,233 in the KS database. Comparing the two sets of data (for the generic and tuned models), the difference in the number of lattice points of the monomial polytopes $\Delta$ and $\Delta'$, $335 - 320 = 15$, is the number of the lattice points being removed. On the other hand, the fan polytope is enlarged $\nabla \rightarrow \nabla'$, and the increased number $N, 13 - 11 = 2$, comes from lattice points $\{(0,-1,1,1), (0,-1,1,2)\}$, which together with the affine node $(0,-1,2,3)$ form exactly the $\mathfrak{su}(3)$ top. The Hodge shifts $\{5,233\} - \{3,243\} = \{2,-10\}$ match exactly with the calculation from anomalies for tuning the algebra $\mathfrak{su}(3)$ on an isolated $-2$-curve.

There are two polytopes in the KS database with Hodge numbers $\{5,233\}$. The
other polytope M:316 6 N:14 6 H:5,233 is the polytope arising from an enhancement to a \( \mathfrak{g}_2 \) gauge algebra by further removing from the \( \mathfrak{su}(3) \) model

\[
\begin{align*}
    a_{1,0} & \leftrightarrow \{(-1,1,0,0)\}, \\
    a_{3,1} & \leftrightarrow \{(-3,2,1,0), (-1,2,1,0), (-2,2,1,0)\};
\end{align*}
\]

so that the vanishing orders along the \(-2\)-curve becomes \(\{1,1,2,2,3\}\), and the number of lattice points in \(\Delta (M)\) decreases by 4. Comparing the fan polytope of the \(\mathfrak{g}_2\) tuning model to that of the generic model, there are three more lattice points \(\{(0,-2,2,3), (0,-1,1,1), (0,-1,1,2)\}\), which together with \( (0,-1,2,3)\) form the \(\mathfrak{g}_2\) top. The Hodge numbers are the same as those of the \(\mathfrak{su}(3)\) model, since \(\mathfrak{su}(3) \to \mathfrak{g}_2\) is a rank-preserving tuning (see Table 3.4).

### 3.3.3 Combining Tunings and Tate-Zariski Test

A final important issue that we must consider in attempting to systematically construct global models associated with polytopes is whether given a generic model over a given base, all combinations of Tate tunings that are each possible locally can be combined into an allowed global model. This depends on the global structure of the base and can be tested by the Tate-Zariski decomposition discussed in §3.1.5. As discussed there, we can perform a Zariski decomposition, with the initial values of \(\{c_{j,n}/n\}\) over each curve set to be the initial values we want in Table 3.1. We then carry out the Tate-Zariski iteration procedure and if the Zariski decomposition with the desired vanishing values and corresponding gauge groups does not exist, there will not be a corresponding polytope model. In general, if the Zariski decomposition works out, there is a corresponding polytope. We do not have a proof of this in general but as we see later, at least the Hodge numbers of every elliptic Calabi-Yau threefold constructed in this way arise from a polytope in the KS database. This analysis of combined tunings through Tate-Zariski is the essential analysis we carry out in our systematic enumeration of Tate tunings that should have corresponding polytopes. To illustrate the issues that can arise we give a couple of simple examples.
Figure 3-3: The toric fan of the base of a generic model with small \( h^{1,1} \): \( \{23, 107, \{3, -2, -2, -1, -6, -1, -2, -3, -1, -1, -1\}\}. Each \(-1\) curve in the base corresponds to a vertex of \( \nabla \).

here, where one but not all of the possible Tate tunings over a given curve in the base are consistent with a global model.

Let us consider first as a concrete example the generic model over the base with toric curves of self-intersection numbers \( \{-3, -2, -2, -1, -6, -1, -2, -3, -1, -1, -1\} \), for which the toric rays take coordinates \( \{v_i(B)\} = \{(1, 1), (3, 2), (5, 3), (7, 4), (2, 1), (5, 2), (3, 1), (1, 0), (0, -1), (-1, -1), (-1, 0), (0, 1)\} \) (figure 3-3). We consider Tate tunings that keep the gauge group the same as in the generic model, determined by the NHCs. From Table 3.1 and the discussion in the preceding subsection, we see that there are three different possible Tate tunings over the \(-6\) curve: \( \{1, 2, 2, 3, 4\} \), \( \{1, 2, 3, 5\} \), \( \{1, 2, 2, 4, 6\} \). We wish to know which of these three tunings leads to a consistent Tate-Zariski decomposition, and which corresponding polytopes exist.

For the polytope \( \nabla \) in each of these three cases, we have the vertices from the fiber
\[
\{(0, 0, -1, 0), (0, 0, 0, -1)\},
\]
the vertices from the base, which come from the \(-1\)'s:
\[
\{(7, 4, 2, 3), (5, 2, 2, 3), (0, -1, 2, 3), (-1, -1, 2, 3), (-1, 0, 2, 3), (0, 1, 2, 3)\}.
\]
and vertices from the tops of the NHCs

-3, 2, 2: \{(2, 2, 2, 3)\},
-2, 3: \{(3, 1, 1, 2), (2, 0, 2, 3)\},
-6 with three choices of different possible top vertices.

We now consider each of the tunings in turn over the -6 curve:

1. Minimal tuning \{1, 2, 2, 3, 4\}, corresponding to no additional top vertex from Table 3.1. This construction leads to a consistent Zariski decomposition, which gives rise to the generic polytope model M:148 11 N:33 11 H:23,107^2: we start with the initial configuration

\[\{(1, 1, 2, 2, 3), (1, 1, 1, 1), (0, 0, 0, 0, 0), (1, 2, 2, 3, 4), (0, 0, 0, 0, 0),\]
\[\{1, 1, 1, 2, (1, 1, 2, 3), (0, 0, 0, 0, 0), (0, 0, 0, 0, 0), (0, 0, 0, 0, 0)\}\].

(3.67)

After the iteration procedure, the configuration becomes

\[\{(1, 1, 2, 2, 3), (1, 1, 2, 2, 2), (1, 1, 1, 1, 1), (1, 1, 0, 0, 0), (1, 2, 2, 3, 4), (1, 1, 0, 0, 0),\]
\[\{1, 1, 1, 2, (1, 1, 2, 3), (0, 0, 0, 0, 0), (0, 0, 0, 0, 0), (0, 0, 0, 0, 0)\}\],

where each curve still has their suitable Tate vanishing orders, which persist as \{1, 2, 2, 3, 4\} on -6.

2. Tate tuning \{1, 2, 2, 3, 5\}, corresponding to the additional top vertices \{(pt_3^c, pt_3^l) = (2, 1, 0, 0), (4, 2, 1, 1)\} over the -6 curve. This works as well and gives the generic polytope model M:147 12 N:35 13 H:23,107^[1]: we start with the initial configuration in (3.67) but with the vanishing orders along -6 replaced by \{1,2,2,3,5\}. The configuration after iteration becomes

\[\{(1, 1, 2, 2, 3), (1, 1, 2, 2, 2), (1, 1, 1, 1, 1), (1, 1, 0, 0, 0), (1, 2, 2, 3, 5), (1, 1, 0, 0, 1),\]
\[\{1, 1, 1, 2, (1, 1, 2, 3), (0, 0, 0, 0, 0), (0, 0, 0, 0, 0), (0, 0, 0, 0, 0)\}\],

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3. Tate tuning \( \{1, 2, 2, 4, 6\} \), which would correspond to the additional top vertex \( \{pt'_{1}\} = \{(2, 1, 0, -1)\} \). This does not give a consistent polytope. The iteration of the initial configuration

\[
\{\{1, 1, 2, 2, 3\}, \{1, 1, 2, 2\}, \{1, 1, 1, 1, 1\}, \{0, 0, 0, 0, 0\}, \{1, 2, 2, 4, 6\}, \{0, 0, 0, 0, 0\}, \\
\{1, 1, 1, 1, 2\}, \{1, 1, 2, 2, 3\}, \{0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0\}\}
\]

becomes

\[
\{\{1, 1, 2, 2, 3\}, \{1, 1, 2, 2, 3\}, \{1, 1, 1, 1, 2\}, \{1, 1, 0, 2, 3\}, \{1, 2, 2, 4, 6\}, \{1, 1, 0, 2, 3\}, \\
\{1, 1, 1, 2, 3\}, \{1, 1, 2, 2, 3\}, \{0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0\}\},
\]

where the vanishing orders over the NHC \(-2, -2, -3\) are disturbed. Hence, unlike the case of the \( F_6 \) base where there is a generic polytope model of vanishing order \( \{1, 2, 2, 4, 6\} \) on \(-6\), the third Tate tuning and corresponding top realization does not exist for this base.

As another illustrative example, consider the polytopes associated with Hodge numbers \( H:416,14 \), which match those of the generic elliptic Calabi-Yau threefold over the base \( \{416, 14, \{-12//-11//-12//-12//-12//-12//-12//-12//-12//-12//-12\}\} \).
12//–12//–12, –1, –2, –2, –3, –1, –5, –1, –3, –2, –1, –8, –1, –2, –3, –2, –1, –8, 0} \}
(see figure 3-4, by // we denote the sequence of curves –1, –2, –2, –3, –1, –5, –1, –3, –2, –2, –1; there are in total 163 toric curves in the base, with curves 153 and 162 being the –8 curves). There are only two polytope models in the KS database with H:416,14, and both give polytope models of the CY with generic gauge group over the given base, with different detailed Tate tuning/top structure. Naively one might expect four models, since there are two different \( e_7 \) tunings possible over each –8 curve. Analyzing the structure of the polytopes, however, we find:

1. M:26 6 N:576 6 H:416,14

- A vertex from the 0-curve in the base. In particular, note that all –1 curves in // do not contribute to vertices.
- Vertices from NHC tops
  (a) \( D_{B13}([-11]): pt_1^{(6)} \)
  (b) \( D_{B153}(-3 \text{ in } [-3,-2]): pt_1'' \)
  (c) \( D_{B162}([-8]): pt'_4 \)
- and vertices \( v_x, v_y \).


- Vertex contributions from the base and the fiber are the same as the first case.
- Vertices from NHC tops
  (a) \( D_{B13}([-11]): pt_1^{(6)} \)
  (b) \( D_{B153}(-3 \text{ in } [-3,-2]): pt_1'' \)
  (c) \( D_{B156}([-8]): pt'_4 \)
  (d) \( D_{B162}([-8]): pt'_6 \)

In the first model, the top over the first –8-curve \( (D_{B156}) \) is \{\( pt_1^{m'}, pt_2^{m'}, pt_3^{m'}, pt_4^{m'}, pt_5^{m'} \}\) while over the second \( (D_{B162}) \) is \{\( pt_1^{m''}, pt_2^{m''}, pt_3^{m''}, pt_4^{m''}, pt_5^{m''}, pt_6^{m''} \}\); in the second model,
it is \( \{pt''_1', pt''_2', pt''_3', pt''_4', pt''_5'\} \) over both \(-8\)-curves; however there is no model of top \( \{pt''''_1, pt''''_2, pt''''_3, pt''''_4, pt''''_5, pt''''_6\} \) over \( DB_{156} \). This matches with the observation that there is no corresponding Zariski decomposition — the vanishing orders can not be \( \{1, 2, 3, 3, 6\} \) along \( DB_{156} \).

Note that these models also illustrate another point: a vertex of the base can only come from curves with self-intersection number \( m \) greater than \(-2\), but all curves with \( m > -2 \) will not necessarily be vertices. Though this generally is the case for small \( h^{1,1} \), exceptions increase as \( h^{1,1} \) increases, since additional rays can expand the convex hull of the base polytope. Also, a vertex associated with a top can only come from those possibilities listed in the third column of Table 3.1, but the entries in that column are not always vertices, though they are always lattice points in the \( N \) polytope \( \nabla \). This fact can be seen in the first model in the second example: \( pt''''_4 \) over \( DB_{156} \) \([-8\]) is not a vertex point.

### 3.4 Multiplicity in the KS Database

Given a pair of Hodge numbers \( h^{1,1}, h^{2,1} \), there are in general many distinct polytopes in the KS database. There are many ways in which such a multiplicity may arise. Of course, generic or tuned elliptic fibrations over distinct bases may coincidentally give the same Hodge numbers. However, there are also many closely related constructions that give identical Hodge numbers. Different realizations of the same gauge algebra through different Tate tunings may contribute, often related to monodromy tunings. There are also rank-preserving tunings that change the gauge algebra but not the Hodge numbers. And in some cases there are non-toric deformations that can give additional multiplicity. A complete analysis of the KS database that accounts for these multiplicities exactly would require a complete and systematic tracking of all distinct possible Tate tunings for each gauge algebra combination and a clear and systematic analysis of the non-toric deformation possibilities. We have not attempted such a systematic analysis here. Rather, we focus on constructing distinct possible gauge groups through Tate tunings and identifying the distinct Hodge numbers that
can arise for reflexive polytopes in this way. In this section we discuss some aspects of the multiplicity question.

### 3.4.1 Multiple Tops

One thing that we have found in considering the variety of Tate tunings and the corresponding models in the KS database is that for many gauge algebras there are multiple distinct tops that can arise in the \(N\)-polytope \(\nabla\). This multiplicity of tops was also discussed in [41]. These different tops correspond to distinct Tate tunings of the same gauge algebra. In many cases these arise in situations where the gauge algebra in the Weierstrass model depends upon some monodromy condition, which may be satisfied automatically in certain cases by the Tate tuning.

**Monodromy tunings**

For some gauge algebras such as \(su_3\), \(e_6\), and \(so(n)\) with \(n\) even, there are multiple tops associated with distinct Tate tunings, where one tuning involves an additional monodromy condition. As can be seen from Table 3.2, for each of these gauge algebras there are two distinct Tate tunings that realize the algebra, with one (both in the case of \(so(8)\)) involving a monodromy condition. (Note that these forms in the table expand on earlier versions of the table appearing in the literature, which did not include all these possibilities.) As discussed in §3.3, the monodromy condition for the weaker Tate tuning can be realized automatically when the leading terms in certain \(a_i\)s are powers of a single monomial, corresponding in the polytope language to a condition that the associated set of lattice points contain only a single element with appropriate multiplicity properties.

As an example of this phenomenon, consider the generic model over the \(F_{m=3}\) base,

\[
\nabla = \text{Conv}\{(1, 0, 2, 3), (-1, -3, 2, 3), (0, 1, 2, 3), (0, 0, -1, 0), (0, 0, 0, -1)\}. \quad (3.68)
\]

This is already a reflexive polytope, \(M:348 \ 5 \ N:12 \ 5 \ H:5,251\), with the top over the \(-3-\)
curve \( \{pt', pt''\} \) that we found at the end of §3.3.1. Naively from Table 3.7, this might appear to be an “\(su(2)\)” top; however looking explicitly at the Tate form associated to the polytope \(\Delta\), the vanishing orders along the \(-3\)-curve are \(\{1,1,1,2,2\}\) in terms of the five sections \(a_n\), and \(\{2,2,4\}\) in terms of \(\{f,g,\Delta\}\), and the \(su(3)\) monodromy condition is satisfied - hence the gauge algebra is indeed \(su(3)\) (indeed, we know from the presence of the \(-3\) NHC that \(su(2)\) is not possible in this geometry.) In §3.1.3 (see in particular Table 3.2), we described two distinct Tate tunings for \(su(3)\). In this case, the geometry matches the alternate Tate form for \(IV^*\) associated with vanishing of \(a_6\) to order 2 and an additional monodromy condition, and the “top” is a non-standard \(su(3)\) top. There also exists a polytope model with the “usual” \(su(3)\) top: adding \(pt'_3\) \(\{(0,-1,1,1)\}\) to the top gives another polytope model \(M:347 7 N:13 6 H:5,251\), which has the standard \(su(3)\) top; on the Tate side this model can be obtained by the reduction of the \(M\) polytope such that the vanishing orders along the \(-3\)-curve become \(\{1,1,1,2,3\}\) - the standard Tate form for \(IV^*\). Analogous situations arise for the NHCs \(-4\) and \(-6\) as well: in these cases, as discussed above, \(\nabla\) in equation (3.68) is already a reflexive polytope model of the generic CY over \(\mathbb{F}_{m=4,6}\). The tops over the \(-m\) curves in these cases look like those appearing in the literature for gauge algebras \(\mathfrak{g}_2, \mathfrak{f}_4\) respectively, and the vanishing orders along the \(-m\)-curves are \(\{1,1,2,2,3\}\), \(\{1,2,2,3,4\}\) for \(m = 4,6\), which would naively be tunings for \(\mathfrak{g}_2, \mathfrak{f}_4\). In these cases, however, the gauge algebras are actually \(so(8), \mathfrak{e}_6\) with monodromy conditions satisfied. Just like the case for \(\mathbb{F}_3\), there are also generic polytope models over \(\mathbb{F}_{4,6}\) that have the usual \(so(8), \mathfrak{e}_6\) tops and Tate vanishing orders of \(so(8), \mathfrak{e}_6\). The extra lattice points in the tops of these \(\nabla\) polytopes precisely correspond to the reduction in Tate monomials of the \(M\) polytope \(\Delta\).

**Further specializations in polytope construction**

In addition to multiple tops associated with monodromy conditions in Tate tunings, there are also other Tate tuning/top combinations that can arise for certain gauge groups. We have not attempted a systematic analysis of all possibilities, but we have encountered a range of possibilities simply in analyzing the polytopes of the KS.
database with fixed Hodge numbers and associated Tate tunings for the dual polytopes. To give a sense of the possibilities that arise, we list the structures of the polytopes in the KS database that have the Hodge numbers of generic elliptically fibered CYs over $\mathbb{F}_m$ bases for $0 \leq m \leq 12$ in Table 3.9. The details of the corresponding Tate forms for the $-m$ NHCs are given in Table 3.1. Note in particular, that in addition to those models mentioned above, there is a third polytope model associated with the Hodge numbers of the generic elliptic fibration over $\mathbb{F}_6$ in addition to the monodromy construction and the standard construction discussed above.

This third possibility involves a further specialization of the vanishing orders of the standard construction along the $-6$-curve, giving a further reduced $M$ polytope $\Delta$. Another interesting case of multiple tops that arises in these tables is the possibility of a second type of Tate tuning/top for $\mathfrak{e}_7$ on a $-8$ curve. In this case there is no monodromy issue\(^{11}\), but a second Tate tuning where the degree of vanishing of $a_6$ is enhanced, associated with a second $\mathfrak{e}_7$ top and corresponding reflexive polytopes.

Note however that further Tate tunings of a given algebra may not give rise to a new reflexive polytope, even if the higher vanishing orders still have a valid Zariski decomposition. We describe briefly several examples here: There is only one polytope in the KS database with $H:4,226$, which corresponds to the type $I_2$ $\mathfrak{su}(2)$ tuning $\{0, 0, 1, 1, 2\}$ on the $-2$-curve of the $\mathbb{F}_2$ base, but there is no polytope that corresponds to the type $III$ $\mathfrak{su}(2) \{1, 1, 1, 1, 2\}$. It is even more interesting to compare the H:5,233 models discussed in §3.3.2 and H:5,251 in table 3.9: there is no $IV$ $\mathfrak{su}(3)$ for the former since it is just a specialization of the type $I_3$ $\mathfrak{su}(3)$ tuning, while there are two different $IV$ $\mathfrak{su}(3)$ realizations for the latter; and both of these sets have the rank-preserving tuning $g_2$ model. Similarly, type $J_{2n}^{as}$ and type $I_{2n+1}^{as} \mathfrak{sp}(n)$ tunings do not give rise to different polytopes. Also for three different types $I_0, I_1, II$ of the trivial algebra, only the one with the lowest vanishing orders that has a Tate-Zariski decomposition has a reflexive polytope construction. An amusing exercise is illustrated in Table 3.10, where we can see the changes in three different types of trivial algebra under various

\(^{11}\)However, note that the same Tate vanishing orders $\{1, 2, 3, 3, 5\}$ may also give the $\mathfrak{e}_7$ algebra over $-7$ curves where there is also charged matter.
Table 3.10: Some rank-preserving tunings over the $F_1$ base. Notice that the Tate vanishing orders of the trivial algebra on the $-1$-curve change in the Tate-Zariski decomposition as the vanishing orders of the two 0-curves get higher. The last row gives an example of a general observation that when the gauge algebra tuning is only a further specialization of an existing gauge algebra tuning (but not the case of gauge algebras realized by different monodromy tunings listed in Table 3.2 with $\ast$, which involves with the requirements of additional conditions), there would not be a corresponding polytope in the KS database even if the Tate-Zariski configuration is stable. The example illustrates that since there is the $su(3)$ model in the second row realized by $I_3$, there is no model realized by $IV^\ast$.

3.4.2 Rank-preserving Tunings

Another source of multiplicity comes from tunings of rank-preserving type as described in the end of §3.1.7. Recall from Table 3.4 that these are tunings of certain gauge algebras that leave the Hodge numbers of an elliptic Calabi-Yau unchanged. These are also associated with further Tate tunings on $\Delta$ and additional tops in $\nabla$ that do not change the Hodge numbers from the generic elliptic fibration over a given base. We have seen several examples in global models: $H:7,271$ of rank 4 $so(8)$, $so(9)$, and $f_4$ tunings in table 3.9, $H:9,129$ of different combinations of rank preserving tunings in table 3.10, and we will see $H:338,22$ of rank 5 $so(10)$ and $so(11)$ tunings in table 3.11 in the context of $so(n)$ tunings.

Notice that it is not always true that tuning gauge algebras with the same rank will lead to the same $h^{2,1}$ shift. For example, $su(7)$ and $e_6$ are not subalgebras of each other, and the tunings give different $h^{2,1}$s.
3.4.3 Combinatorial Multiplicity

To systematically analyze multiplicities of different Tate tunings of the same algebra, we would need to consider all combinations of tuning multiplicity in the preceding sections: monodromy and non-monodromy tunings of algebras like $\text{su}_3, \text{e}_6, \text{so}(n)$ etc. Over bases with many curves allowing such tunings this could give a large combinatorial multiplicity. However, it is not the case that the naive enumeration of all combinations would give rise to new reflexive polytopes. In addition to the issue of whether local tunings can be combined into an allowed global tunings discussed in 3.3.3, there are on the other hand cases that pass Tate-Zariski test but do not correspond to additional reflexive polytopes. We do not have general rules for combinatorial multiplicity of polytopes, but only report some empirical results through examples; particularly, we discuss some aspects of $\text{so}(n)$ tunings and the associated reflexive polytopes, which have some unique features. 12

Multiplicity Realizations of Combining Tunings

Consider the two polytope models in the first block of Table 3.11. We start with the minimal $\{1, 1, 2, 2, 3\}$ Tate vanishing orders for all three $-4$ curves, which together do have a corresponding Tate-Zariski decomposition, so there is a corresponding polytope construction. Then we tune the vanishing orders on the middle $-4$-curve alone to be $\{1, 1, 2, 2, 4\}$. After iteration, the other two $-4$-curves are forced to also have $\{1, 1, 2, 2, 4\}$ vanishing, giving the second generic model with all $-4$ curves reaching the second realization. This exhausts the possibilities. So from what might appear to in principle be 8 possible combinations of tunings, only two are actually consistent. It can also happen that only the lower-order realization exists, while the higher-order realization does not have an acceptable Zariski decomposition and there is no corresponding polytope, as we have seen for example in the failure to realize the third

12 In the analysis in the remainder of this chapter we focus on classifying the possible elliptic fibrations constructions through the set of Tate tunings. One could also, however, imagine classifying different reflexive polytopes by considering all ways of augmenting the set of vertices associated with all possible tops. Proceeding in this fashion would require a systematic way of identifying the complete set of tops for each possible tuned gauge group and a set of rules to combine them.
Table 3.11: An example contrasting the absence and the existence of multiple realizations: successive Tate tunings of generic CYs over the toric base \{-12//-11//-12//-12//-12//-12//-12//-12/-12/-1,-2,-2,-3,-1,-4,-1,-4,-1,-4,0\}. The Tate vanishing orders on the last seven curves \{-1,-4,-1,-4,-1,-4,0\} are indicated. All polytope models in the KS database with the Hodge pairs \{225, 23\}, \{338, 22\}, \{339, 21\} are listed in each of the three blocks. In models with Hodge pair \{338, 22\}, both the weaker and the stronger versions of the tuning of so(10) on the middle -4-curve exist - the weaker version can not correspond to so(9) by the global symmetry constraint on the -4-curve. On the other hand, there is only one model with Hodge pair \{339, 21\}, the weaker version of the tuning of so(10) on the last -4-curve does not exist in the KS database - the same Tate tuning gives so(9) on the last -4-curve in the model M:38 8 N:466 8 H:338,22.

In general, the realization of any given combination must be checked by performing a global Tate-Zariski decomposition, as local information may not be completely adequate to rule in or out a possible tuning. An example is given by the models in Table 3.12, where there is no global Zariski decomposition of the \{1,1,3,3,6\} realization of so(12), and the reflexive polytope model does not exist over the given global base, though it would seem to be fine if we were to analyze the tuning pattern with the focus on the local sequence \{-1,-3,-1,-4,-1,0\} only.

model of H:23,107 with the generic gauge group over a -6 curve in section 3.3.3.
Table 3.12: An example of the non-existence of the stronger version of the Tate form: a tuning of a generic model over the base \{-12/-11/-12/-12/-12/-12/-12/-12/-12/-1,-2,-3,-1,-5,-1, -3,-1,-4,-1,0\} on the last -4 curve with a \(\mathfrak{so}(12)\) gauge algebra (which forces gauge algebras on nearby curves). The Tate vanishing orders on the last six curves \{-13,-1,-4,-1,0\} are indicated. While the weaker version of the Tate form \{1,1,3,3,6\} exists in the KS database, the stronger version \{1,1,3,3,6\} does not give rise to a Tate-Zariski decomposition with the desired gauge algebras.

**Multiplicity Realizations of \(\mathfrak{so}(n)\) Gauge Algebras**

As for \(\mathfrak{su}_3, \mathfrak{e}_6\), we find that both kinds of Tate tunings of the \(\mathfrak{so}(2n)\) gauge algebras can arise in corresponding polytopes in the KS database, corresponding to the usual condition that a global Tate-Zariski decomposition is possible. We also note, however, that when the algebra \(\mathfrak{so}(2n-1)\) can be realized on one polytope over a given curve, then the monodromy realization of \(\mathfrak{so}(2n)\) is generally not possible, though the higher Tate tuning generally is. This basically corresponds geometrically to the question of whether the minimally tuned Tate model with the weaker vanishing condition has the appropriate single monomials in the \(a_i\)s, or not. By the same token, the gauge algebra \(\mathfrak{so}(8)\), which has only monodromy realizations, can only be realized when neither \(\mathfrak{g}_2\) or \(\mathfrak{so}(7)\) is possible over a given curve, which essentially reduces the appearance of this algebra to the NHC structure of -4 curves.

To illustrate these points we give a few examples:

For a first example, consider a chain of curves \{-1,-4,-1,-4,-1,-4,0\}; by requiring Tate vanishing orders \{0,0,1,1,2\} (\(\mathfrak{sp}(1)\) gauge algebra) on \(D_{B3}\) and \(D_{B5}\), the Tate vanishing orders on each of the curves become \{0,0,0,0,0\}, \{1,1,2,3,4\}, \{1,0,1,2,2\}, \{1,1,2,3,4\}, \{0,0,0,0,0\}\}. Without taking into account the monodromy conditions, it would appear in this case that the enhanced algebras were \{\(\mathfrak{so}(9)\) \(\oplus \mathfrak{sp}(1)\) \(\oplus \mathfrak{so}(9)\) \(\oplus \mathfrak{sp}(1)\) \(\oplus \mathfrak{so}(9)\) \}\}; explicitly analysis of the monomials, however, shows that while \(D_{B2}\) and \(D_{B5}\) are indeed \(\mathfrak{so}(9)\) algebras, there is
really a so(10) algebra on DB4, since the so(10) monodromy condition is automatically satisfied. This can also be understood from the perspective of global symmetry
constraints [64]; when the gauge algebra is so(9) on a -4-curve, the maximal global
symmetry algebra is sp(1), so it is not possible for so(9) to appear on DB4 next to
two sp(1)'s. Thus, DB4 indeed must carry the gauge algebra so(1O), for which the
maximal global symmetry algebra is sp(2) D sp(1)

e

sp(1).

For a similar example, for tunings ofso(4k+3) andso(4k+4) consider the sequence
of curves {-1, -3, -1, -4, -1, 0}; by requiring vanishing orders of {1, 1, 3, 3, 5} on
DB4

and {O,0, 3,3,6} on DB5, the other vanishing orders are forced to {{0, 0, 0,0 0},

{11,1, 2,2,3}, {1, 0, 2,1,2},{1,1,3,3, 5}, {0, 0, 3, 3,6}, {0, 0, ,0, 2}}, which gives the
gauge algebras {- e92 sP (1) E so(12)Dsp(3) e -};the algebra so(11) is not possible
on DB4 by global symmetry constraints. Examples of these tunings in the context of
global constructions are given in Tables 3.11 and 3.12.
In the examples just given, on certain curves the so(2n - 1) gauge algebra cannot arise, and the lower Tate tuning with the monodromy condition is realized. As
mentioned above, when the so(2n -1)

tuning is allowed, there is not generally a poly-

tope in the KS database with the same Tate tuning and the monodromy condition
automatically satisfied, and one has to use the higher Tate tuning to guarantee the
condition. These facts can be seen in contrasting the polytope models, for example, of
so(9) and the two realizations of so(10) in table 3.11. There is only one model in the
KS database with the Hodge pair{339,21}, M:36 9 N:467 9 H:339,21, which corre{-12// - 11// - 12// - 12// - 12//

-

sponds to tuning of the generic model {335, 23,

12// - 12// - 12// - 12// - 12// - 12, -1,-2, -2, -3, -1, --4, -1, -4, -1,
on { -1-41,-41
so(10)

e

-4,

0}

-4, 0}}

to gauge algebras {. e so(9) esp(1) e so(10) esp(1)

e

is {1, 1, 2, 3, 5}.

There is

-}. The Tate tuning along the last -4-curve

not a second polytope with the same Hodge numbers corresponding to the weaker
Tate realization {1, 1, 2, 3, 4} of the gauge algebra so(10) along the last -4-curve.
This matches with the observation that the absence of multiple data in the KS
database for a given tuning is due to the existence of the same Tate tuning appearing in the lower rank gauge algebras: There is already the case M:38 8 N:466
110


8 H:338,22, corresponding to tuning of the same generic model to gauge algebras \(\mathfrak{so}(9) \oplus \mathfrak{sp}(1) \oplus \mathfrak{so}(10) \oplus \mathfrak{sp}(1) \oplus \mathfrak{so}(9) \oplus \cdot\), and the Tate tuning along the last \(-4\)-curve is \(\{1, 1, 2, 3, 4\}\) giving an \(\mathfrak{so}(9)\) there. On the other hand, there are two models with H:338,22, M:39 7 N:465 7 and M:38 8 N:466 8, corresponding to the tuning \(\{\cdot \otimes \mathfrak{so}(9) \oplus \mathfrak{sp}(1) \oplus \mathfrak{so}(10) \oplus \mathfrak{sp}(1) \oplus \mathfrak{so}(9) \oplus \cdot\}\) giving the two different Tate realizations of the \(\mathfrak{so}(10)\). In this case, the weaker tuning satisfies the monodromy condition automatically, which is expected as \(\{\cdot \otimes \mathfrak{so}(9) \oplus \mathfrak{sp}(1) \oplus \mathfrak{so}(9) \oplus \mathfrak{sp}(1) \oplus \mathfrak{so}(9) \oplus \cdot\}\) is not allowed as mentioned.

There is a similar story between \(\mathfrak{so}(11)\) and \(\mathfrak{so}(12)\). For example, we can tune an \(\mathfrak{so}(11)\) on the \(-3\)-curve of the generic model over \(\mathbb{F}_3\) by requiring Tate vanishing orders of \(\{1, 1, 3, 3, 5\}\), which gives rise to M:328 8 N:18 7 H:8,242 in KS database. Then to get a polytope corresponding to a tuning of \(\mathfrak{so}(12)\), we need to use \(\{1, 1, 3, 3, 6\}\), which has a good Zariski decomposition, and therefore a corresponding reflexive polytope exists, M:318 10 N:19 8 H:9,233. The Hodge numbers of all these examples are consistent with calculations from anomalies.

As we have mentioned, there is a special situation for the \(\mathfrak{so}(8)\) algebra and related polytopes in the KS database: all realizations of \(\mathfrak{so}(8)\) involve monodromy constraints. Thus, there are no polytopes where there is a Tate tuning of the algebra \(\mathfrak{so}(8)\), and this algebra only arises over the NHC \(-4\). In the case of the NHC \(-4\), \(\mathfrak{so}(8)\) is the minimal gauge algebra, so either vanishing orders \(\{1, 1, 2, 2, 3\}\) or \(\{1, 1, 2, 2, 4\}\) will automatically satisfy the \(\mathfrak{so}(8)\) monodromy condition in any Tate tuning over a base with a \(-4\) curve. This unique aspect of \(\mathfrak{so}(8)\) matches with the observation that a tuned \(\mathfrak{so}(7)\) cannot be ruled out through the global symmetry group since the global symmetry group on a tuned \(\mathfrak{so}(7)\) curve contains that on a tuned \(\mathfrak{so}(8)\) curve. Thus, any Tate tuning of \(\{1, 1, 2, 2, 3\}\) or \(\{1, 1, 2, 2, 4\}\) over a curve with self-intersection greater than \(-4\) will lead to a model with, if not \(\mathfrak{g}_2\), \(\mathfrak{so}(7)\) enhancement.

### 3.4.4 Multiple Base Resolution/Non-flat Fibrations

Lastly, multiplicity can come from situations where the elliptic fibration over a toric base has \((4, 6)\) points that must be blown up. As discussed in §3.1.2, over toric bases
containing curves with self-intersection number $-9, -10, -11$ the generic elliptic fibration is non-flat and the base must be blown up at the $(4, 6)$ points to give $-12$-curves, over which there is a flat elliptic fibration. In general the base resulting from these blow-ups will be non-toric, and the blowups give extra tensor multiplets contributing to anomaly cancellation [11, 12]. In some cases, however, the base is still toric after blowing up one or more of the $(4, 6)$ points; in such cases there will be multiple entries in the KS database associated with these distinct bases. In general we expect that these all represent smooth Calabi-Yau threefolds that can be viewed as non-flat elliptic fibrations over toric bases or flat elliptic fibrations over the non-toric bases resolved at the non-toric $(4, 6)$ points, though we have not checked explicitly that this is true in all cases. Examples of some non-flat elliptic fibrations of this type are analyzed in [48, 57, 58]. To illustrate this structure, in §3.2.7 we analyze the non-flat elliptic fibration structure of the toric hypersurfaces associated with (flat) toric fibrations of the reflexive fibered polytopes over the Hirzebruch surfaces $F_9, F_{10}, F_{11}$. In these cases, we see explicitly that the fiber over the $(4, 6)$ points in the $-9, -10, -11$-curves contains extra irreducible components that may naturally be associated with divisors in the blown up space. The multiplicity with which the Hodge pairs for the generic elliptic fibration models over the suitably blown up Hirzebruch surfaces $F_{9/10/11}$ are listed in Table 3.13, the tops over the $-9/-10/-11$-curves are listed in the second block in Table 3.1. This illustrates the way in which the same smooth Calabi-Yau threefold can be realized as a non-flat elliptic fibration over one or more toric bases as well as sometimes a flat elliptic fibration over another toric base, with each fibration structure realized in a different polytope in the KS database. For example, as illustrated in the table there are 6 distinct polytopes at Hodge numbers $H:14,404$, which correspond to toric realizations of elliptic fibrations over different "semi-toric" bases that admit only a single $\mathbb{C}^*$ action (including various limits in which $-2$ curves arise).

<table>
<thead>
<tr>
<th>Hodge pair</th>
<th>Mult. in KS</th>
<th>Bases</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

112
(14,404)  6

\[
\begin{align*}
\{0, -9, 0, 9\} & \quad \{0, 10, 0, 10\} \\
\{-1, -1, -11, -1, 9\} & \quad \{-1, -1, -11, 0, 10\} \\
\{-1, -2, -1, -12, -1, 9\} & \quad \{-1, -2, -1, -12, 0, 10\}
\end{align*}
\]
Table 3.13: A variety of polytope models arise for the Hodge pairs associated with the generic elliptic fibrations over the Hirzebruch surfaces $F_{9/10/11}$. The possibilities are enumerated in this table. The first graph for each Hodge pair is the generic model, where the $(4, 6)$ singularities on the $-9, -10, \text{or } -11$ curve are at non-toric points and the elliptic fibration is non-flat. In these cases the blow-ups are handled automatically by the resolution of the toric geometry, giving a resolved model corresponding to a flat elliptic fibration over a “semi-toric” base. There are also toric bases that arise by blowing up one or more of the $(4, 6)$ points at toric points, giving polytopes with toric fibrations over blow-ups of the Hirzebruch surfaces. When multiple $(4, 6)$ points coincide this corresponds to a limit with a $-2$ curve in the base. For each polytope the base of the fibration is a toric surface given by the curves on the outside of the diagram, with self-intersections as labeled.

### 3.5 Systematic Construction of Tate-tuned Models in the KS Database

Kreuzer and Skarke have classified all $473,800,776$ 4D reflexive polytope models, which give $30,108$ distinct Hodge pairs. It was found in [56] that the set of Hodge pairs $\{h^{1,1}, h^{2,1}\}$ of all generic elliptically fibered CYs over toric bases is a subset of all the Hodge pairs in the KS database. We gave in §3.3.1 a general construction of reflexive polytope models of these generic elliptic fibrations over toric bases with NHCs, and we expect that all generic elliptic fibration models over toric bases have these corresponding reflexive polytope models in the KS database. We wish to carry out a more comprehensive comparison by matching tuned Weierstrass models of CYs over 2D toric bases with 4D reflexive polytope models of Calabi-Yau hypersurfaces at large Hodge numbers.

Of the $30,108$ distinct Hodge pairs in the KS database, $1,827$ have either $h^{1,1} \geq 240$ or $h^{2,1} \geq 240$ (only the Hodge pair $\{251, 251\}$ satisfies both inequalities). To compare the two constructions at large Hodge numbers, the next step would be to construct
roughly this number of distinct Weierstrass models of tuned CYs in these regions. Not all Weierstrass models correspond to reflexive polytope constructions, however. Nonetheless, as discussed in §3.2, there is a close relation between Tate-tuned models and $\mathbb{P}^{2,3,1}$-fibered polytopes, which dominate at large Hodge numbers as argued in §3.2.4. Therefore, our approach is to construct systematically all Tate-tuned models via tunings of generic Tate models over 2D toric bases. As a preliminary to this analysis, however, we begin with a simpler systematic analysis of which gauge group tunings may be possible based on more general Weierstrass tunings, and then we refine the analysis to Tate tunings. We describe the logic of this analysis in more detail in §3.5.1.

All Hodge pairs of the Tate-tuned models from this algorithm fall within those in the KS database. However, there are certain Hodge numbers in the KS database in the regions of interest at which our initial analysis identified no matching Tate-tuned model. We therefore have analyzed directly, via the method described in §3.2.3, the polytope models with the Hodge numbers that were not found in our systematic tuning construction; the analysis of these cases is described in §3.6. It turns out that all these remaining polytope models can be described as somewhat more exotic Weierstrass or Tate tunings over bases that are either toric bases or blow-ups thereof. This completes the comparison of the two constructions at the level of Hodge numbers. At a basic level the result of this analysis is that in the regions of interest all the Hodge pairs in the KS database are realized through generic or tuned elliptic fibrations. This matches with the results through a direct analysis of the fibration structure in the next chapter [37]. The more detailed analysis we carry out here, however, gives much more insight into the structure of these fibrations and the complex variety of Weierstrass tunings and geometries that are realized through simple toric hypersurface constructions.

We also discuss briefly the limits of Tate tuning in section 3.5.4, where we collect some results on tunings that are compatible with the global symmetry constraints but can't not be realized by Tate tunings. These tunings may be realized by Weierstrass models and in such cases give new Hodge pairs outside the KS database.
3.5.1 Algorithm: Global Symmetries and Zariski Decomposition for Weierstrass Models

We give an algorithm in this section to systematically construct all tunings of enhanced gauge groups over a given 2D toric base, starting with the generic model. Our goal is to construct all Tate-tuned models over toric bases that give elliptic Calabi-Yau threefolds with Hodge numbers in the regions $h^{1,1} \geq 240$ or $h^{2,1} \geq 240$. As we saw in §3.1.6, global symmetry constraints on each curve put upper bounds on the gauge algebras that can be tuned on intersecting curves. On the other hand, as discussed in §3.3.3, there is an issue of whether local tunings on subsets of curves can be combined into a global model over some toric base $B_2$. This can be tested by the Zariski decomposition. More specifically, our goal is to carry out explicitly arbitrary combinations of the Tate tunings from §3.1.3 on the curves in the base, applying the variant of the Zariski decomposition described in §3.1.5 to determine which combinations are globally compatible. While in principle we could simply iterate over all possible gauge algebra combinations, using the global symmetry constraints on what gauge algebras can arise on the curves intersecting a curve of negative self intersection helps prune the tree and make the algorithm more efficient. Global symmetries are also helpful in limiting the set of possible monodromy-dependent gauge groups that can arise on sequences of intersecting curves in ways that are not apparent at the level of the Zariski decomposition.

Although ultimately we wish to analyze Tate tunings, we perform an initial analysis of Weierstrass tunings using global symmetry constraints and the Zariski decomposition. This gives us a set of possible tunings that we expect may be possible at the level of the gauge algebras. Not all these constructions, however, are compatible with Tate tunings and with polytopes. We begin the discussion by focusing on the Weierstrass tunings and then in §3.5.4 we use the results of the Weierstrass tunings to check which Tate tunings are possible.

Given a 2D toric base $B$, which is represented by a set of $K$ irreducible toric curves $\{C_j, j = 1, 2, \ldots, K\}$ intersecting each other in a linear chain, we first obtain for the
generic model over the base $B$ the orders of vanishing $\{c_{j,4}, c_{j,6}, c_{j,12}\}$ of $f$, $g$, and $\Delta$ along each curve. The sets of values $\{c_{j,4}\}$, $\{c_{j,6}\}$, and $\{c_{j,12}\}$ can be determined by the Zariski decomposition via the procedure described in equations (3.14)-(3.17) with $n = 4$, $n = 6$, and $n = 12$, respectively, or can be directly read off from the non-Higgsable cluster structure of the curves $\{C_j\}$.

Now let us consider all possible (Weierstrass) tunings of the generic model. We describe a procedure to determine an allowed pattern $\{g_j\}$ of tuned algebras $g_j$ on each curve $C^j$ in the base. Note that in this algorithm we assume that there are no toric $(4,6)$ points in the base, even after the tuning; such a point would be blown up to form a different toric base, and the tunings over the blown up base would be found directly by tunings over that base. We do allow non-toric $(4,6)$ points in the case where the base contains $-9$, $-10$ or $-11$ curves; in these cases we essentially treat the curve as a $-12$ curve supporting an $\epsilon_8$, understanding that the polytope hypersurface construction will automatically resolve these singularities and effectively blow up the non-toric points in the base, in accord with the discussions in §3.1.2 and §3.4.4.

3.5.2 Main Structure of the Algorithm: Bases with a Non-Higgsable $\epsilon_7$ or $\epsilon_8$

We consider first the simplest cases, where there is at least one curve in the toric base of self-intersection $m \leq -9$; such a curve necessarily carries a non-Higgsable $\epsilon_8$ gauge algebra. We start the procedure by choosing a specific curve with a non-Higgsable $\epsilon_8$ and first considering the possible tunings on one of the adjacent curves. Let us label the curve with the $\epsilon_8$ using the index $j = 1$, the curve we attempt the first tuning on by $j = 2$, the subsequent curve by $j = 3$, etc. This choice of the initial configuration is convenient to serve as the starting point of a branching algorithm because an $\epsilon_8$ algebra cannot be further enhanced; moreover, nothing can be tuned next to an $\epsilon_8$, without producing a $(4,6)$ singularity at a toric point, which we are assuming does not happen as discussed above. Therefore, the gauge algebras on $C_1$ and $C_2$ are fixed: $g_1 = \epsilon_8$ and $g_2$ has to be a trivial algebra.
We then pass to tunings $g_3$ on $C_3$. The possible tunings on $C_3$ are constrained by the global symmetry group $g_2^{(\text{glob})}$ on $C_2$, which is determined by the self-intersection number of $C_2$ and the gauge algebra $g_2$ on $C_2$. Let the set $\{g_{3,\alpha}\}$ be the set of algebras that satisfy the constraint $g_1 \oplus g_{3,\alpha} = e_8 \oplus g_{3,\alpha} \subset g_2^{(\text{glob})}$. For the global symmetries, we used the results in Table 6.1 and Table 6.2 in [64] for the maximal global symmetry group $g_j^{(\text{glob})}$ on a curve $C_j$ of negative self-intersection $m$ carrying a gauge algebra $g_j$.

Additionally, the curves of negative self-intersection that do not support an NHC can carry trivial gauge algebras, of types $I_0, I_1, II$; therefore in such cases when $g_j = -$, we use $g_j^{(\text{glob})} = e_8$ and $g_j^{(\text{glob})} = su(2)$ for $m = -1$ and $m = -2$, respectively. We used the results tabulated in [75] for the subgroups of a global symmetry group $g_j^{(\text{glob})}$ to obtain the restricted set of algebras $\{g_{j,\alpha}\}$ satisfying the constraint $g_{j-2} \oplus g_{j,\alpha} \subset g_{j-1}^{(\text{glob})}$.

We attempt tunings one-by-one for each $g_{3,\alpha}$. For each possible algebra we replace the original orders of vanishing $\{c_{3,4}, c_{3,6}, c_{3,12}\}$ with the desired orders of vanishing corresponding to $g_{3,\alpha}$ using Table 2.2. We then perform the Zariski iteration procedure on all curves with the new $\{c_{3,4}, c_{3,6}, c_{3,12}\}$ for $n = 4, 6$ and 12, respectively. If all the gauge algebras on the curves prior to and including $C_3$ stay unchanged after the iteration, tuning $g_{3,\alpha}$ is not ruled out. If any of the gauge algebras on the curves prior to $C_3$ have changed, or the vanishing orders $\{c_{3,4}, c_{3,6}, c_{3,12}\}$ do not produce the desired gauge algebra $g_{3,\alpha}$ in the new configuration after the iteration, tuning $g_{3,\alpha}$ is not allowed on $C_3$; in such cases we terminate the procedure with this $g_{3,\alpha}$ branch, and attempt the next tuning $g_{3,\alpha+1}$ on $C_3$. Note, however, that the fact that the gauge algebras stay unchanged does not mean that the set of values $\{c_{j \leq 3,4}\}, \{c_{j \leq 3,6}\}, \{c_{j \leq 3,12}\}$ stay unchanged under the iterations. Indeed, often it is the case that the orders of vanishing on curves near $C_3$ are increased, but without modifying the gauge algebra on $C_2$. In other words, in this case $g_2$ should be the trivial algebra, but it may be type $I_0, I_1$, or II (cf. also examples in Table 3.10.)

Note that the vanishing orders $\{c_{j > 3,4}, c_{j > 3,6}, c_{j > 3,12}\}$ can obtain new values after the initial set of iterations just described. If these increase beyond those determined by the initial NHC configuration, we use the larger vanishing orders as the starting points in further iterations of the tuning. We denote by $g_{j[i]}$ the gauge algebra on
curve $j$ after the iteration procedure associated with curve $C_i$. For $i = j$, $g_{j[j]}$ denotes a choice of gauge algebra in a branch, $g_{j[j]} \in \{g_{j,\alpha}\}$, and $g_{j[j]} \supseteq g_{j[j-1]}$. Note that we must have $g_{j[k]} = g_{j[j]}$ for all $k > j$ as we require in the branch that the gauge algebra on $C_j$ stays unchanged in tuning gauge algebras on $C_{k>j}$, but the orders of vanishing realizing the gauge algebra may be different. We can proceed with the new configuration to the next step of tuning algebras on $C_4$, as long as $g_{4[3]} \oplus g_{2[3]} \subset g_{3[3]}^{(\text{glob})}$ is satisfied, where now $g_{4[3]}$ is the gauge algebra on $C_4$ in the new configuration and $g_{3[3]}^{(\text{glob})}$ is determined by the self-intersection of $C_3$ and the gauge algebra $g_{3[3]} \in \{g_{3,\alpha}\}$. For example, let us start with $g_{3[3]} = g_{3,1}$. We terminate the procedure with the $g_{3,1}$ branch and attempt the next branch of tuning $g_{3,2}$ on $C_3$ if $g_{4[3]} \oplus g_{2[3]} \subset g_{3,1}^{(\text{glob})}$ is violated in the new configuration.

Assume $g_{3,1}$ passes the tests above. We then continue the procedure similarly to tune the curve $C_4$ in the $g_{3,1}$ branch with the new configuration: The set of possible tunings $\{g_{4,\beta}\}$ we attempt on $C_4$ is constrained by $g_{4,\beta} \oplus g_{2[3]} \subset g_{3[3]}^{(\text{glob})}$. The branch $g_{4,1}$ can be continued only if $g_{4,1}$ passes the two tests (1) the set of gauge algebras $\{g_{j \leq 4}\} = \{g_{3,4}, g_{4,1}\}$ stays unchanged after performing Zariski iterations on $\{c_{j,4}\}$, $\{c_{j,6}\}$, and $\{c_{j,12}\}$ with the desired degrees of vanishing $\{c_{4,4}, c_{4,6}, c_{4,12}\}$ of the tuned gauge algebra plugged into the configuration, and (2) $g_{5[4]} \oplus g_{3[4]} \subset g_{4,1}^{(\text{glob})}$ is satisfied, where $g_{5[4]}$ is the gauge algebra on $C_5$ after the iterations in the newest updated configuration, and $g_{4,1}^{(\text{glob})}$ is again determined by the self-intersection of $C_4$ and $g_{4,1}$.

The procedure continues similarly until the second to the last curve $C_{K-1}$ is met. As the last curve $C_K$ is connected back to the first curve $C_1$, we need to consider also the global symmetry constraint on $C_K$ to close the tuning pattern. The set of possible tunings $\{g_{K-1,\gamma}\}$ on $C_{K-1}$ is constrained by $g_{K-1,\gamma} \oplus g_{K-3[K-2]} \subset g_{K-2[K-2]}^{(\text{glob})}$. First, the usual two conditions have to be satisfied for $g_{K-1,\gamma}$ to be allowed: in the new configuration after the iterations associated with the tuning of $g_{K-1,\gamma}$ (1) the prior gauge algebras are held fixed, and (2) the global symmetry constraint on $C_{K-1}$ is satisfied. Moreover, there is the additional third condition: (3) the global symmetry constraint on the curve $C_K$ has to be satisfied; i.e., $g_{K-1,\gamma} \oplus g_1 \subset g_{K[K-1]}^{(\text{glob})}$, where $g_1$
is held fixed and is $c_8$ in the simplest cases, and $g_{K[K-1]}^{(\text{glob})}$ is determined by the self-intersection of the curve $C_K$ and the gauge algebra $g_K$ after the Zariski iteration for the tuning $g_{K-1,\gamma}$. In fact, $g_K$ is only allowed to be a trivial algebra in the simplest cases as $C_1$ carries an $c_8$ algebra, so no tuning is allowed on $C_K$. Hence, if the global symmetry constraint on $C_K$ is satisfied, we are basically done to this point in the procedure searching for a tuning pattern. In this case, we obtain a tuning pattern

$$\{c_8, \cdots, g_{3[K-1]}, g_{4[K-1]}, \cdots, g_{K-3[K-1]}, g_{K-2[K-1]}, g_{K-1[K-1]}, \cdots\}.$$  

We check all $g_{K-1,\gamma}$'s in order similarly to complete the scan through all possible tuning patterns compatible with the initial viable possibility for $g_{3[K-1]}, \cdots, g_{K-2[K-1]}$. After all $g_{K-1,\gamma}$'s are processed, we proceed iteratively with a nested loop, continuing with the next possible value of $g_{K-2}$, etc. so that all possible combinations of gauge group tunings are considered.

All tunings increase $h^{1,1}$ and decrease $h^{2,1}$ with respect to the generic model over a given base. Thus, to classify all tuned models of $h^{2,1} \geq 240$, we need only consider toric bases for which the generic elliptic fibration has $h^{2,1} \geq 240$. In our initial scan, we also restricted to bases that have generic models with $h^{1,1} \geq 220$. As we describe in more detail in the following section, this misses a few cases where there is a large amount of tuning that significantly changes $h^{1,1}$. On the other hand, as bases associated with generic models having $h^{1,1} > 224$ always contain at least one curve carrying an $c_8$ algebra, the algorithm as described above is quite effective in dealing with tunings in the large $h^{1,1}$ region as we always have a simple starting point for the iteration. In fact, the algorithm actually can work in the same way for tunings of generic models with a curve carrying $c_7$ in the base; i.e., generic models with a curve of self-intersection $m \leq -7$ in the base. This is because $c_7$ algebras also cannot be further enhanced without modifying the base — an enhancement to an $c_8$ algebra would give additional $(4,6)$ points that must be blown up. And no non-trivial algebra can be tuned next to an $c_7$ algebra. Thus, in this case we similarly can make the convenient choice that the initial configuration is fixed to be $g_1 = c_7, g_2 = \cdot$.

We make some final comments on two technical issues in the tuning procedure. As mentioned above, in tuning the curve $C_j$, not only do the orders of vanishing
on $C_{j+1}$ (and in some cases on further curves $C_{j+2}, \ldots$) also change in general, but
the new vanishing orders $\{c_{j+1,4}, c_{j+1,6}, c_{j+1,12}\}$ can in some cases correspond to a
different gauge algebra. However, because the three Zariski iterations were performed
independently, sometimes these vanishing orders do not correspond to any algebras
in the Kodaira classification. We encountered a few cases of this type, for example
where $\{c_{j+1,4}, c_{j+1,6}, c_{j+1,12}\} = \{1, 2, 4\}$; this can happen for example if a previous
$\mathfrak{su}(n)$ tuning ($\{0, 0, n\}$) pushes up the order of vanishing of $\Delta$ more significantly than
$f, g$ (where some required orders of $f, g$ already imposed on the curve); however, note
that, this can never happen in a real $\Delta$ as calculated in a complete model from $f$ and
g in equation (2.11). In such situations, we modify the orders of vanishing on $C_{j+1}$ to
fit with those that correspond to the gauge algebra that arises by increasing the values
c_{j+1,4}, c_{j+1,6}$ minimally. Then we perform the iteration again after the modification,
and use the resulting configuration to test the conditions (1) and (2).

Another detail to take care is the tuning of algebras only distinguished
by monodromy conditions. For those cases where there are distinct algebras associated with
different monodromy conditions, we retain all the possibilities allowed by global sym-
metries; in the list of possible tunings we attach an additional label to the orders
of vanishing using a fourth entry $\{c_{j,4}, c_{j,6}, c_{j,12}, \text{algebra}\}$ to ensure that all possible
tunings are considered.

3.5.3 Special Cases: Bases Lacking Curves of Self-intersection

$m \leq -7$ and/or Having Curves of Non-negative Self-intersection

The algorithm described in the preceding subsection relies on the presence of a curve
of self-intersection $m \leq -7$ in the base, where we can begin the iteration process
in a simple fashion as the gauge algebra on the initial curve is fixed. In the regions
we are considering, there are very few bases that lack such curves; we describe here
briefly how these cases are handled. Of course, one could simply use a brute force
algorithm of choosing an arbitrary starting point and looping over all tunings on the
initial curve $C_1$. In principle, however, for efficiency we would like to choose the curves $C_1, C_2$ such that there are fewer allowed combinations $\{g_1, g_2\}$. For example, for the generic model $\{11, 263, \{-1, -1, -6, -1, -1, 4\}\}$, we may choose to rotate the sequence of the curves to be $\{-6, -1, -1, 4, -1, -1\}$, so that there are only two initial configurations on the $-6$ curve $C_1$, which are the generic gauge algebra $\{e_6, \}$ and an enhancement on $C_1 \{e_7, \}$. Note that in this case there cannot be any enhancement on $C_2$ as the global symmetry algebra is always the trivial algebra on $-6$-curves without an further enhancement to $e_7$, so no tunings are allowed on any intersecting curves. In fact, in the Hodge number regions we are considering, there are very few cases that lack non-Higgsable $e_7$ or $e_8$ gauge algebras. Every base with a generic elliptic fibration having $h^{1,1} \geq 220$ has a curve of self-intersection $-7$ or below. In the region $h^{2,1} \geq 240$, there are 14 generic models that contain no curve in the base carrying an $e_7$ or $e_8$ algebra; the generic models over bases $F_{0 \leq m \leq 6}$ and $P^2$ compose 9 of these 14 models, and are discussed further below. In the remaining cases, there is no choice of the initial configuration that uniquely determines the initial configuration, and we have to enumerate and specify different initial configurations $\{g_1, g_2\}$ over a curve of minimal self-intersection to perform the algorithm.

A further issue arises for bases that have curves of non-negative self intersection. On such curves, there is no global constraint on the adjacent algebras from the SCFT point of view. While there are some analogous constraints in the case of curves of non-negative self intersection [15], the constraints are weaker and less completely understood. So we do not impose global constraints in these cases. In principle this can be handled by simply iterating over all gauge groups, however in practice the number of cases where this issue is relevant is rather limited and can be handled efficiently using more specific methods.

From the minimal model point of view we can fairly easily classify the types and configurations of non-negative self intersection curves that can arise. The minimal model bases $P^2$ and $F_m$ have three consecutive curves of non-negative self intersection. Any blow-up of one of these bases has either only one such curve or two adjacent such curves, since blow-ups reduce the self-intersection of curves containing the blow-up
point and do not introduce new curves of non-negative self intersection. Blow-ups of the resulting bases again have at most two curves of non-negative self intersection and when there are two they are always adjacent. So the possibilities are actually quite limited.

In general, the way we deal with the cases having one or two non-negative curves for bases with large $h^{2,1}$ is by performing the algorithm separately in both opposite directions from a good starting point (curve of maximally negative self intersection) to get two “half-patterns” of tunings, and connect them appropriately. In other words, we start from a chosen curve $C_1$, run the algorithm in both directions, and stop the tuning procedures when the first non-negative curve is met in both directions. We do not impose any global conditions for the curves of non-negative self intersection. The combination of the two sets of the half-patterns connected in this way gives all tuning patterns of a generic model with one or two non-negative curves in the base. For bases with large $h^{1,1}$, there is generally at most one non-negative self intersection curve and the nearby gauge group is generally constrained by global symmetries and nearby large negative self-interactions; in some of these cases we have used simpler heuristics to complete the analysis in the presence of non-negative self-intersection curves.

For the cases $\mathbb{P}^2$ and $\mathbb{F}_{0 \leq m \leq 12}$ that have three non-negative curves, most tunings in fact decrease the Hodge number $h^{2,1}$ below the value 240 of interest. For example [15], tuning an $\text{su}(2)$ on a $+1$ curve of $\mathbb{P}^2$ changes the Hodge numbers from $(2, 272)$ to $(3, 231)$. There are some exceptions: for example tuning an $\text{su}(2)$ on the $+12$ curve of $\mathbb{F}_{12}$ gives a Calabi-Yau with Hodge numbers $(12, 318)$. But it turns out (as we see explicitly from the analysis of the following section) that all these cases with $h^{2,1} \geq 240$ are also realized in other ways by generic or tuned models over other toric bases. So we do not need to explicitly include these in our analysis since we are not trying to reproduce the precise multiplicity of models at each Hodge number pair.

Although we have only focused on tuning models in the large Hodge number regions, one can in principle classify all allowed tuning patterns of non-abelian gauge algebras on any toric base with the algorithm described here; though slightly different
methods are needed for tunings over the bases $\mathbb{P}^2$ and $\mathbb{F}_{0 \leq m \leq 12}$, an exhaustive search is straightforward in these cases as there are only a few curves in these bases (three curves in $\mathbb{P}^2$ and four curves in $\mathbb{F}_m$).

### 3.5.4 Tate-tuned Models

The analysis described so far in terms of Weierstrass models gives a large collection of possible gauge algebra tunings over each toric base. Not all of these gauge algebra combinations correspond to hypersurfaces in reflexive polytopes. There are several reasons for this. First, not every Weierstrass tuning can be realized through a Tate form, so some of these tunings on toric curves will not have standard $\mathbb{P}^{2,3,1}$-fibered polytope constructions. Further, some of the combinations of gauge groups that are allowed by the Zariski analysis and global constraints still cannot be realized in practice in a global model — we alluded for example at the end of §3.5.2 to the fact that monodromy conditions are not really taken care of properly in the Zariski decompositions of $n = 4, 6, 12$. Indeed, an explicit check shows that not all the Hodge pairs calculated via equations (3.18) and (3.19) from the Weierstrass tuning patterns we got from §3.5.2 and §3.5.3 lie in the KS database.

We are interested in constraining to a subset of tuning constructions for which we expect direct polytope constructions. Hence, for each gauge algebra tuning combination that satisfies the Weierstrass Zariski analysis and global conditions, we attempt to construct an explicit Tate-type model by specifying Tate orders of vanishing according to Table 3.2 for each tuning in a tuning pattern. We then perform the Zariski decomposition of the Tate tunings described in §3.1.5. A tuning pattern gives a genuine Tate-tuned model if it has the Zariski decompositions of Tate tunings. In performing this analysis, we used in our systematic analysis only the stronger version of the Tate forms for the algebras with multiple realizations and/or monodromy conditions. In particular, we did not use any of the tunings marked with $\circ$ or $\star$ in Table 3.2. The second version of the $I_{2n}^8$ Tate tuning (marked with $\circ$) was in fact previously not known and was identified through the analysis of the next section. For the $so$ algebras, some of the alternate monodromy tunings were not previously known.
(for example, the non-* version of $so(4n + 4)$ algebras); also, we wished to restrict attention to cases where the algebra is guaranteed simply by the order of vanishing of the Tate coefficients. In general, as we have noted in the examples in §3.3.3 and §3.4.3, the polytope constructions do not satisfy the monodromy conditions for the higher rank gauge algebras in these cases.

These principles give us a set of gauge group and Tate tunings over each toric base that we believe should have direct correspondents in the KS database through standard $\mathbb{P}^{2,3,1}$-fibered polytopes, given the correspondence that we established in §3.3. We have carried out an explicit comparison of these two sets, and indeed the Hodge numbers of this more limited class of Tate-tuned gauge groups all correspond to values that appear in the KS database. Furthermore, the Hodge pairs from the original Weierstrass analysis that are not in the KS database are exactly those of the tuning patterns that can not be realized by Tate tuning. In fact, given this restricted set of tunings we reproduce almost all of the 1,827 distinct Hodge pairs in the range $h^{1,1} \geq 240$ or $h^{2,1} \geq 240$. Only 18 of the Hodge pairs in this range were not found by a “sieve” using the Tate constructions described above. In the next section we consider the analysis of these 18 outlying polytope constructions.

A question that we do not explore further here, but which is relevant to the more general problem of understanding the full set of Calabi-Yau threefolds and the classification of 6D F-theory models, is the extent to which tunings are possible that look like they should be allowed from the Weierstrass Zariski analysis and anomaly cancellation conditions, but do not correspond to Tate constructions. Various aspects of this “Tate tuning swampland” were analyzed in [15]. In the context of this project, we did a local analysis of the Weierstrass tuning patterns that are not Tate tuning patterns. We reproduced some parts of the known Tate tuning swampland and also found new obstructions. Some examples of the problematic tunings in the Tate construction are listed in Table 3.14. An interesting question for further research is which of these can be realized through good global Weierstrass models when the indicated sequence of curves arises as part of a toric (or non-toric) base.
| su(3) ⊕ sp(3), su(3) ⊕ sp(4),  
g_2 ⊕ so(10),  
sO(9) ⊕ su(4), so(9) ⊕ sp(2), so(10) ⊕ su(4), so(10) ⊕ sp(2),  
sO(11) ⊕ su(3), so(11) ⊕ sp(2),  
sO(13) ⊕ sp(1), so(13) ⊕ su(2) |
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<td>Miscellaneous Tate swamp (some examples)</td>
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<td>Gauge groups</td>
<td>Local geometry</td>
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<tr>
<td>so(7) ⊕ su(2) ⊕</td>
<td>-3, -2, -2</td>
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<tr>
<td>· ⊕ su(2) ⊕ sp(2) (or su(4))</td>
<td>-2, -2, -1/0</td>
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<tr>
<td>· ⊕ su(2) ⊕ g_2 ⊕ sp(3)</td>
<td>-2, -2, -2, -1/0</td>
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Table 3.14: Tate tuning swamp: We list all subalgebras allowed by the "E_8 rule" that however can not be realized by Tate tunings. We also give some examples of the tuning patterns we found that do not violate global symmetry constraints but that can not be realized by Tate tunings (i.e. violate Tate-Zariski decomposition).

### 3.6 Direct Polytope Analysis for Exceptional Cases

As discussed above, there are only 18 Hodge pairs in the regions h^{1,1} ≥ 240 or h^{2,1} ≥ 240 in the KS database that are not produced by our Tate tuning algorithm. One of these missing 18 Hodge pairs is in the large h^{2,1} region, \{45, 261\}, and the other 17 (see Table 3.15) are in the large h^{1,1} region. In this section we analyze the polytopes in the Kreuzer-Skarke database associated with these 18 Hodge number pairs.

By studying these 18 classes of Calabi-Yau manifolds, we have identified new tuning constructions that we had not known previously; the KS database provides us with global models utilizing these constructions that we did not expect a priori in our original analysis. We study the fibration structure of the 18 outstanding classes by analyzing the polytopes in the way described in section 3.2.3. All the polytopes associated with these 18 Hodge pairs have a \mathbb{P}^{2,3,1} fibered polytope structure (though in some cases it is really the more specialized Bl[0,0,1]\mathbb{P}^{2,3,1} fiber that occurs), but not all of them are the standard \mathbb{P}^{2,3,1}-fibered polytopes that we have defined in §3.2.2. In particular, the CY hypersurface of a standard \mathbb{P}^{2,3,1}-fibered polytope (or Bl[0,0,1]\mathbb{P}^{2,3,1}-fibered polytope) has a Tate form, while this is not the case for other fibration structures that use the same fiber but a different "twist". We analyze the
two different types of polytopes arising from the 18 Hodge pairs separately. In §3.6.1 we analyze the standard $\mathbb{P}^{2,3,1}$-fibered polytopes in the KS database that we have not obtained in our systematic construction of Tate-tuned models. In §3.6.2 we analyze the polytopes that do not have the standard $\mathbb{P}^{2,3,1}$-fibered structure. We also include in §3.6.2 some further examples in the KS database that are outside the range of focus of this work but that illustrate some further interesting exotic structure associated with gauge groups on non-toric curves in the base.

### 3.6.1 Fibered Polytope Models with Tate Forms

Of the 18 missing Hodge pairs, there are $1 + 9$ Hodge pairs in the large $h^{2,1}, h^{1,1}$ regions respectively in which there is a standard $\mathbb{P}^{2,3,1}$-fibered polytope (or a standard $\text{Bl}_{[0,0,1]} \mathbb{P}^{2,3,1}$-fibered polytope), which has a Tate form. Therefore, we analyze the Tate models explicitly from these polytopes to learn about the Tate tunings that we missed in our initial construction.

The Hodge pair in the large $h^{2,1}$ region, $\{45, 261\}$, has only one polytope. This polytope reveals a second tuning of the type $I_{2m}^*$ singular fiber that is not just a specialization of the known Tate tuning. We also find that applying this novel Tate tuning $\mathfrak{su}(6)$ on a $m \geq -1$-curve gives models with the three-index antisymmetric representation as opposed to the generic fundamental and two-index antisymmetry representations. We describe this analysis in detail in §3.6.1. The polytopes of the nine missing Hodge pairs at large $h^{1,1}$ with the standard fibration structure are either

![Table 3.15: The Hodge number pairs in the KS database at large $h^{1,1}$ that we did not obtain from straightforward Tate-tuned models. However, all these can be reproduced by some flat elliptic fibrations that we discuss in this section: The standard $\mathbb{P}^{2,3,1}$ models, which have a Tate form, are studied in §3.6.1, and the non-standard $\mathbb{P}^{2,3,1}$ models, which involve tunings on non-toric curves in the base, are studied in §3.6.2.]

<table>
<thead>
<tr>
<th>Fibered Polytopes</th>
<th>Hodge Pairs</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Standard $\mathbb{P}^{2,3,1}$-fibered polytopes</td>
<td>huge tuning ${240, 48}, {244, 10}, {250, 10}, {261, 9}$</td>
<td>non-toric base ${258, 60}$ (&quot;$c_5$-tuning&quot;)</td>
</tr>
<tr>
<td>$\text{Bl}_{[0,0,1]} \mathbb{P}^{2,3,1}$-fibered polytopes</td>
<td>global $u(1)$ tuning</td>
<td>non-toric base ${263, 32}, {251, 35}, {247, 35}, {240, 37}$ (&quot;$\mathfrak{so}(n \geq 13)$-tuning&quot; on a $-3$-curve)</td>
</tr>
<tr>
<td>Non-standard $\mathbb{P}^{2,3,1}$-fibered polytopes</td>
<td>tuning on non-toric curve</td>
<td>${261, 51}, {261, 45}, {260, 62}, {260, 54}, {259, 55}, {258, 84}, {254, 56}, {245, 57}$</td>
</tr>
</tbody>
</table>
extremely tuned models, with bases having generic elliptic fibrations with $h^{1,1} < 220$ (described in §3.6.1), or are non-flat elliptic fibration models over a toric base (described in §3.6.1). In the non-flat elliptic fibration cases, as we have discussed in §3.4.4, the CY resolution of (4,6) singularities in terms of the polytope model produces irreducible components of the ambient toric fiber (as the hypersurface equation restricted to the components is trivially satisfied over the (4,6) points). Therefore, at these points the dimension of the fiber jumps to two giving the non-flat elliptic fibration structure. Associating the additional divisors with blow-ups in the base allows us to describe the resulting Calabi-Yau threefolds alternatively as flat elliptic fibrations over the blown up base. The resulting models in the cases found here give rise to $\mathfrak{e}_8$ tunings or $\mathfrak{so}(n \geq 13)$ tunings on $-3$-curves, and are also involved with tuned Mordell-Weil sections, which are associated with $U(1)$ factors and $U(1)$-charged hypermultiplets.

**Type $I_{2n}^5$ Tate tunings and exotic matter**

The polytope model $M:357$ $N:65$ $H:45,261$ is a standard $\mathbb{P}^{2,3,1}$-fibered polytope, and is a Tate-tuned model of the generic model

$$\{38, 290, \{-2, -2, -1, -6, -1, -3, -1, -5, -1, -3, -2, -1, -12, 0, 6\}\}.$$

The data $\{a_1, a_2, a_3, a_4, a_6\}$ of the Tate form show the orders of vanishing along each curve

$$\{\{0, 2, 2, 4, 6\}, \{0, 2, 1, 4, 5\}, \{0, 2, 0, 4, 4\}, \{1, 2, 2, 4, 5\}, \{1, 2, 0, 4, 2\},$$
$$\{1, 2, 1, 4, 3\}, \{1, 2, 0, 4, 4\}, \{1, 2, 2, 4, 4\}, \{1, 2, 1, 4, 1\}, \{1, 2, 2, 4, 3\},$$
$$\{1, 2, 2, 4, 2\}, \{1, 2, 2, 4, 1\}, \{1, 2, 2, 4, 0\}, \{1, 2, 3, 4, 5\}, \{0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0\}\}.$$

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In terms of \( \{f, g, \Delta\} \) (equations (3.2)-(3.8)), the orders of vanishing are

\[
\{\{0, 0, 6\}, \{0, 0, 3\}, \{0, 0, 0\}, \{3, 4, 8\}, \{1, 0, 0\}, \{2, 2, 4\}, \{1, 0, 0\}, \{3, 4, 8\}, \\
\{2, 1, 2\}, \{3, 3, 6\}, \{3, 2, 4\}, \{3, 1, 2\}, \{3, 0, 0\}, \{4, 5, 10\}, \{0, 0, 0\}, \{0, 0, 0\}\},
\]

which shows that there is an \( \text{su}(6) \) enhanced on the first \(-2\)-curve, \( D_1 \equiv \{b_1 = 0\} \), and an \( \text{su}(3) \) on the second \(-2\) curve. However, the corresponding Tate tuning is not just a specialization of the \( \text{su}(6) \) Tate tuning \( \{0, 1, 3, 3, 6\} \) in the literature. Via this example, we found the second version of the \( \text{su}(2n) \) tuning, which we have included in the Tate tunings listed in Table 3.2, indicated by \( \text{su}(2n)^\circ \).

As this is the only polytope associated with the Hodge pair \( \{45, 261\} \), it seems that the traditional \( \text{su}(2n) \) tuning is somehow not allowed in this configuration. We checked explicitly by performing a tuning where we substitute in the vanishing order \( \{0, 1, 3, 3, 6\} \) over \( D_1 \), and perform the Tate-Zariski decomposition. The vanishing orders after iteration become

\[
\{\{0, 1, 3, 3, 6\}, \{0, 1, 3, 2, 5\}, \{0, 1, 3, 1, 4\}, \{1, 2, 3, 3, 5\}, \{1, 2, 3, 1, 2\}, \\
\{1, 2, 3, 2, 3\}, \{1, 2, 3, 1, 1\}, \{1, 2, 3, 3, 4\}, \{1, 2, 3, 2, 1\}, \{1, 2, 3, 3, 3\}, \\
\{1, 2, 3, 3, 2\}, \{1, 2, 3, 3, 1\}, \{1, 2, 3, 3, 0\}, \{1, 2, 3, 4, 5\}, \{0, 0, 0, 0, 0\}, \{0, 0, 0, 0\}\};
\]

or in terms of \( \{f, g, \Delta\} \),

\[
\{\{0, 0, 6\}, \{0, 0, 4\}, \{0, 0, 2\}, \{3, 5, 9\}, \{1, 2, 3\}, \{2, 3, 6\}, \{1, 1, 2\}, \{3, 4, 8\}, \\
\{2, 1, 2\}, \{3, 3, 6\}, \{3, 2, 4\}, \{3, 1, 2\}, \{3, 0, 0\}, \{4, 5, 10\}, \{0, 0, 0\}, \{0, 0, 0\}\},
\]

which is problematic as the global symmetry constraint on the \(-6\)-curve \( D_4 \) is violated. This confirms again that there has to be a Tate-tuned pattern that is consistent under the Tate-Zariski decomposition for a corresponding polytope to exist. And we cannot obtain a polytope of these Hodge numbers using the standard tuning methods because the \( \text{su}(2n)^\circ \) tunings, \( \{0, 2, n - 1, n + 1, 2n\} \), are not specializations of the
standard $\text{su}(2n)$ tunings, $\{0, 1, n, n, 2n\}$.

In the case of a $-2$ curve, as in the example encountered at large $h^{2,1}$, the matter content associated with the physics of the exotic $\text{su}(6)^o$ tuning is equivalent to that of a standard $\text{su}(6)$ tuning over a $-2$ curve. After incorporating these alternative $\text{su}(2n)^o$-tunings into our algorithm, however, we discovered that this second Tate realization of $\{f, g, \Delta\} = \{0, 0, 2n\}$ gives rise to the non-generic three-index antisymmetric (20) representation of $\text{su}(6)$ when the tuning is performed on curves of self-intersection $m \geq -1$. We describe an example of this explicitly, in the context of a global model that lies outside the regions of primary interest $h^{1,1}, h^{2,1} \geq 240$.

The polytope model M:280 11 N:28 9 H:18,206 is a Tate-tuned model of

$$\{11, 263, \{-1, -2, -1, -6, 0, 4\}\}. \quad (3.70)$$

There is an $\text{su}(6)^o$ tuned on the $-1$-curve $D_1$ and an $\text{su}(3)$ tuned on the $-2$-curve $D_2$. Interestingly, by explicit analysis, we find that the $f, g$ from the polytope data automatically satisfy the conditions for the codimension-two singularity on $D_1$ to support the three-index antisymmetric representation of $\text{su}(6)$, as described in [47]. To see this, we fix the complex structure moduli of $f, g$ to some general enough $\mathbb{Z}$ values to avoid accidental cancellations, expand $f$ and $g$ in terms of $\sigma \equiv b_1$ where the coefficients are in terms of a second local coordinate that we choose to be $b_2$

$$f(\sigma, b_2) = f_0(b_2) + f_1(b_2)\sigma + f_2(b_2)\sigma^2 + \cdots, \quad (3.71)$$
$$g(\sigma, b_2) = g_0(b_2) + g_1(b_2)\sigma + g_2(b_2)\sigma^2 + \cdots; \quad (3.72)$$

then we find (following the notation in [47])

- $\Delta_0 = 0 : f_0 \sim -\frac{1}{48}\phi_0^4$ and $g_0 \sim \frac{1}{864}\phi_0^6$; we choose to set $\phi_0 = 57 + 46b_2$.
- $\Delta_1 = 0 : g_1 = -\frac{1}{12}\phi_0^2 f_1$.
- $\Delta_2 = 0 : f_1 \sim \frac{1}{2}\phi_0 \psi_1 \Rightarrow \psi_1 = -(1/6)b_2(37 + 62b_2)\phi_0^2$ and $g_2 = \frac{1}{4}\psi_1^2 - \frac{1}{12}\phi_0^2 f_2$. 

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\( \Delta_3 = 0 : \psi_1 \sim -\frac{1}{3} \phi_0 \phi_1 \Rightarrow \phi_1 = (1/2)b_2(37 + 62b_2)\phi_0 \) and \( g_3 = -\frac{1}{12} \phi_0^2 f_3 - \frac{1}{3} \phi_1 f_2 - \frac{1}{2} \phi_1^3. \)

\( \Delta_4 = 0 : f_2 + \frac{1}{3} \phi_1^2 = \frac{1}{2} \phi_0 \phi_2 \Rightarrow \psi_2 = -(1/12)b_2(-972 + 321867b_2 + 818194b_2^2 + 770316b_2^3 + 257716b_2^4) \) and \( g_4 = \frac{1}{4} \phi_2^2 - \frac{1}{12} \phi_0^2 f_4 - \frac{1}{3} \phi_1 f_3. \)

\( \Delta_5 = 0 : \alpha = \text{GCD} [\phi_0, \phi_2] = 1 \Rightarrow \beta = \phi_0, \phi_2 = -3\psi_2, \nu = (1/2)b_2(37 + 62b_2). \)
\( f_3 = -\frac{1}{3} \nu \phi_2 - 3\lambda \beta \Rightarrow \lambda = (1/72)b_2^2(358621 + 1496554b_2 + 1733688b_2^2 + 656328b_2^3) \) and \( g_5 = -\frac{1}{12} \phi_0^2 f_5 - \frac{1}{3} \phi_1 f_4 + \phi_2 \lambda. \)

Hence, \( \alpha \neq 0 \) and \( \beta = 0 \) over the codimension-two point \( \sigma = \phi_0 = 0 \), which gives rise to a 3-index antisymmetric matter field. Indeed, we have to use the representations \( 15 \times 6 + 1/2 \times 20 \), as opposed to the ordinary \( 14 \times 6 + 1 \times 15 \) of \( \text{su}(6) \) on \( -1 \)-curves, to obtain the correct shifts of the Hodge number \( h^{1,1} \) from anomaly cancellation: \( \Delta h^{1,1} = 2 + 5 = 7 \), and \( \Delta h^{2,1} = (8 + 35) + (6 \times 3 + 15 \times 6 + 1/2 \times 20 - 3 \times 6 \text{ (shared)}) = -57. \)

The conclusion that the \( \text{su}(6)^\circ \)-tuning on the \(-1\)-curve leads to this exotic matter representations is not particular to this specific global model. Following the same steps, we performed a local analysis on an isolated \(-1\) curve; when we tune the Tate form \( \{0, 2, 2, 4, 6\} \) on the curve, we see that \( \alpha \neq 0 \) but \( \beta = 0 \) over a point on the curve, while the Tate form \( \{0, 1, 3, 3, 6\} \) leads to \( \alpha = 0 \) over a point but \( \beta \neq 0 \). Although there is no corresponding polytope model with ordinary \( \text{su}(6) \) matter in case of the global model studied above (there is no polytope in the KS database that gives a Calabi-Yau with Hodge numbers \{18, 207\}, and the tuning \( \{0, 1, 3, 3, 6\} \) over the base (3.70) does not lead to a good global Zariski decomposition), we can contrast the two tunings of \( \text{su}(6) \) on a \(-1\)-curve in polytopes that describe tunings of the generic model over the \( F_1 \) base M:335 6 N:11 6 H:3,243. Both models exist in the KS database: the \( \text{su}(6) \)-tuning gives the model M:242 12 N:16 9 H:8,179 and the \( \text{su}(6)^\circ \)-tuning gives the model M:236 10 N:16 8 H:8,178.

The two different Tate forms of \( \text{su}(6) \) automatically give different representations on all curves with self-intersection \( m \geq -1 \) (there is only matter in the fundamental representation on \(-2\)-curves). For example, consider tuning the generic model over \( F_1 \) now with \( \text{su}(6) \) and \( \text{su}(6)^\circ \) respectively on a 0-curve. The \( \text{su}(6) \)-tuning gives the
<table>
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<tr>
<th>Tate form</th>
<th>Ordinary matter</th>
<th>Exotic matter</th>
</tr>
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<tbody>
<tr>
<td>Representations</td>
<td>${0, 1, 3, 3, 6}$</td>
<td>${0, 2, 2, 4, 6}$</td>
</tr>
<tr>
<td></td>
<td>$(16 + 2m)6 + (m + 2)15$</td>
<td>$(16 + 3m + 2)6 + (m + 2)\frac{1}{2}20$</td>
</tr>
</tbody>
</table>

Table 3.16: Representations of $su(6)$ and $su(6)^0$-tuning on curves of self-intersection $m \geq -2$.

model with ordinary matter $M:204\;11\;N:16\;9\;H:8,152$ while the $su(6)$-tuning gives the model with exotic matter (two half-hypermultiplets in the $20$ representation) $M:197\;9\;N:16\;8\;H:8,150$. The Hodge numbers from the polytope data are consistent with the calculation from anomaly cancellation with the respective matter representations (see table 3.16).

**Large Hodge number shifts**

Four of the “extra” Hodge number pairs in the region $h^{1,1} \geq 240$ turn out to come from standard Tate tunings of generic models that have $h^{1,1} < 220$, outside the region we considered for starting points. These are listed as “huge tunings” in Table 3.15. These models each contain a chain of $\{-1, -4\}$s, which allows $so(n)$ with $n$ very large to be enhanced on the $-4$-curves, producing huge shifts of the Hodge numbers. While there are only four specific models of this type among the 18 Hodge pairs in the region of interest not found by Tate tunings, it seems that this large tuning structure on chains of $-1, -4$ curves is a common feature and there are many other examples of this in the database, both increasing multiplicities at large Hodge numbers in cases that also have Tate tuned realizations, and also occurring at Hodge numbers outside the range of interest here.

We work out one example here in detail; the others have similar structure. The example with the largest $h^{1,1}$ (from the four “extra” models of this type) is the polytope $M:20\;6\;N:352\;7\;H:261,9$, which is a Tate-tuned model of the generic polytope model

\[
\{135, 15, \{-12, -1, -2, -2, -3, -1, -4, -1, -4, -1, -4, -1, -1, -4, -1, -4, -1, -1, -4, -1, -4, -1, -3, -2, -2, -1, -12, 0\}\},
\]

(3.73)
as can be determined by explicitly computing the base polytope of the toric fibration. Therefore, the enhanced tunings should give \( \{ \Delta h^{1,1}, \Delta h^{2,1} \} = \{ 126, -6 \} \). Explicit analysis of the polytope gives the data \( \{ m, \{ a_1, a_2, a_3, a_4, a_6 \}, \{ f, g, \Delta \} \} \) of each \( m \)-curve.

\[
\{ \{ -12, \{ 1, 2, 3, 4, 5 \}, \{ 4, 5, 10 \} \}, \{-1, \{ 1, 1, 5, 5, 0 \}, \{ 2, 0, 0 \} \}, \{-2, \{ 1, 1, 5, 5, 1 \}, \{ 2, 1, 2 \} \}, \\
\{-2, \{ 1, 1, 5, 5, 2 \}, \{ 2, 2, 4 \} \}, \{-3, \{ 1, 1, 5, 5, 3 \}, \{ 2, 3, 6 \} \}, \{-1, \{ 1, 0, 7, 6, 1 \}, \{ 0, 0, 1 \} \}, \\
\{-4, \{ 1, 1, 5, 5, 4 \}, \{ 2, 3, 7 \} \}, \{-1, \{ 1, 0, 7, 6, 3 \}, \{ 0, 0, 3 \} \}, \{-4, \{ 1, 1, 5, 5, 5 \}, \{ 2, 3, 8 \} \}, \\
\{-1, \{ 1, 0, 7, 6, 5 \}, \{ 0, 0, 5 \} \}, \{-4, \{ 1, 1, 5, 5, 6 \}, \{ 2, 3, 9 \} \}, \{-1, \{ 1, 0, 7, 6, 7 \}, \{ 0, 0, 7 \} \}, \\
\{-4, \{ 1, 1, 5, 5, 7 \}, \{ 2, 3, 10 \} \}, \{-1, \{ 1, 0, 7, 6, 9 \}, \{ 0, 0, 9 \} \}, \{-4, \{ 1, 1, 5, 5, 8 \}, \{ 2, 3, 11 \} \}, \\
\{-1, \{ 1, 0, 7, 6, 11 \}, \{ 0, 0, 11 \} \}, \{-4, \{ 1, 1, 5, 5, 9 \}, \{ 2, 3, 12 \} \}, \{-1, \{ 1, 0, 7, 6, 12 \}, \{ 0, 0, 12 \} \}, \{-4, \{ 1, 1, 5, 5, 9 \}, \{ 2, 3, 12 \} \}, \\
\{-1, \{ 1, 0, 7, 6, 12 \}, \{ 0, 0, 12 \} \}, \{-4, \{ 1, 1, 5, 5, 9 \}, \{ 2, 3, 12 \} \}, \{-1, \{ 1, 0, 7, 6, 12 \}, \{ 0, 0, 12 \} \}, \{-4, \{ 1, 1, 5, 5, 9 \}, \{ 2, 3, 12 \} \}, \\
\{-1, \{ 1, 0, 7, 6, 12 \}, \{ 0, 0, 12 \} \}, \{-4, \{ 1, 1, 5, 5, 9 \}, \{ 2, 3, 12 \} \}, \{-1, \{ 1, 0, 7, 6, 11 \}, \{ 0, 0, 11 \} \}, \{-4, \{ 1, 1, 5, 5, 8 \}, \{ 2, 3, 11 \} \}, \\
\{-1, \{ 1, 0, 7, 6, 9 \}, \{ 0, 0, 9 \} \}, \{-4, \{ 1, 1, 5, 5, 7 \}, \{ 2, 3, 10 \} \}, \{-1, \{ 1, 0, 7, 6, 7 \}, \{ 0, 0, 7 \} \}, \{-4, \{ 1, 1, 5, 5, 6 \}, \{ 2, 3, 9 \} \}, \{-1, \{ 1, 0, 7, 6, 5 \}, \{ 0, 0, 5 \} \}, \\
\{-4, \{ 1, 1, 5, 5, 5 \}, \{ 2, 3, 8 \} \}, \{-1, \{ 1, 0, 7, 6, 3 \}, \{ 0, 0, 3 \} \}, \{-4, \{ 1, 1, 5, 5, 4 \}, \{ 2, 3, 7 \} \}, \\
\{-1, \{ 1, 0, 7, 6, 1 \}, \{ 0, 0, 1 \} \}, \{-3, \{ 1, 1, 5, 5, 3 \}, \{ 2, 3, 6 \} \}, \{-2, \{ 1, 1, 5, 5, 2 \}, \{ 2, 2, 4 \} \}, \\
\{-2, \{ 1, 1, 5, 5, 1 \}, \{ 2, 1, 2 \} \}, \{-1, \{ 1, 1, 5, 5, 0 \}, \{ 2, 0, 0 \} \}, \{-12, \{ 1, 2, 3, 4, 5 \}, \{ 4, 5, 10 \} \}, \\
\{0, \{ 0, 0, 0, 0, 0 \}, \{ 0, 0, 0 \} \} \}.
\]

The gauge algebras on -4 and -1 curves are only determined from this analysis up to monodromies. We can, however, determine the algebras without explicitly analyzing monomials. First, from the anomaly constraint analyzed in [15], \( \text{su} \) cannot be adjacent to \( \text{so} \), so the algebras on -1 curves have to be \( \text{sp} \). The choice \( \text{so}(2n-5) \) or \( \text{so}(2n-4) \) on -4 is determined from global symmetry constraints. For example, it has to be \( \text{so}(20) \) rather than \( \text{so}(19) \) between two \( \text{sp}(6) \) algebras for the global symmetry on the
-4 curve to be satisfied; while the lower rank $so(17), so(19)$ has to be chosen for two
-4's connecting to $sp(5)$ for the global symmetry constraint on the -1 curve to be satisfied. Hence, the corresponding gauge algebras are

$$\{\{-12, e_8\}, \{-1, \cdot\}, \{-2, \cdot\}, \{-2, su(2)\}, \{-3, g_2\}, \{-1, \cdot\}, \{-4, so(9)\}, \{-1, sp(1)\},$$

$$\{-4, so(11)\}, \{-1, sp(2)\}, \{-4, so(13)\}, \{-1, sp(3)\}, \{-4, so(15)\}, \{-1, sp(4)\}, \{-4, so(17)\},$$

$$\{-1, sp(5)\}, \{-4, so(19)\}, \{-1, sp(6)\}, \{-4, so(20)\}, \{-1, sp(6)\}, \{-4, so(20)\}, \{-1, sp(6)\},$$

$$\{-4, so(20)\}, \{-1, sp(6)\}, \{-4, so(20)\}, \{-1, sp(6)\}, \{-4, so(20)\}, \{-1, sp(6)\},$$

$$\{-4, so(19)\}, \{-1, sp(5)\}, \{-4, so(17)\}, \{-1, sp(4)\}, \{-4, so(15)\}, \{-1, sp(3)\}, \{-4, so(13)\},$$

$$\{-1, sp(2)\}, \{-4, so(11)\}, \{-1, sp(1)\}, \{-4, so(9)\}, \{-1, \cdot\}, \{-3, g_2\}, \{-2, su(2)\}, \{-2, \cdot\},$$

$$\{-1, \cdot\}, \{-12, e_8\}, \{0, \cdot\}\},$$

which give the correct Hodge number shifts (in particular, one can quickly check that according to the rank of the gauge algebras $\Delta h^{1,1} = 126$ as expected above).

**Tate-tuned models corresponding to non-toric bases**

We have not considered tuning an $e_8$ algebra on any curve of self-intersection $m \geq -8$, as it leads to a violation of the anomaly conditions that corresponds to the appearance of a $(4, 6)$ singularity. Similarly, tunings of $so(n \geq 13)$ on $-3$-curves are also ruled out by anomaly cancellation. Nonetheless, there are polytope models in the KS database that appear to contain these tunings, which give rise to Hodge pairs that we have not obtained in Tate tunings of Kodaira type. This set of tunings can be understood as more complicated generalizations of the non-flat structure we have already described for fibrations over $-9, -10$ and $-11$ curves. As we discussed already in that context, over $(4, 6)$ points the resolved fiber in the polytope model is two-dimensional, but we can understand the Calabi-Yau geometry by resolving the base at these points to obtain a corresponding flat elliptic fibration model over a blown up base that is generally non-toric. In this section we describe models that involve $e_8$ algebras tuned on $-8$ curves and models involving tunings of $so(n \geq 13)$ on $-3$-curves. In the latter
case, the “extra” models in the KS database in our region of interest involve a further complication in which a nontrivial Mordell-Weil group is generated associated with an abelian $U(1)$ factor in the F-theory gauge group; a detailed example with that additional structure is discussed in the next subsection of §3.6.1.

We begin with an example of a tuned $c_8$ on a $-8$-curve. This occurs in the model $M:88 \ N:356 \ H:258,60$. The $\nabla$ polytope has vertices $\{(0,0,0,-1),(7,6,2,3),(-1,-1,2,3),(-1,-1,1,2),(0,6,2,3),(0,0,-1,0),(-42,-36,2,3),(-15,-13,2,3)\}$. It describes a non-flat Tate-tuning of the generic elliptic fibration

$$\{252,78,\{-12/,-11/,-12/,-12/,-12/,-12/,-12//-8,-1,-2,-1,0\}\}$$

where $//\,$ stands for $\{-1,-2,-2,-3,-1,-5,-1,-3,-2,-2,-1\}$, and there are in total 101 curves $D_i$ in the base. There is an $c_8$ tuned on $D_{97}$ and an $su(2)$ tuned on $D_{100}$, where the orders of vanishing are enhanced to $\{1,2,3,4,5\}$ and $\{0,0,1,1,2\}$, respectively. As it needs four blowups for a $-8$-curve to become a $-12$-curve, which carries the $c_8$ gauge algebra without $(4,6)$ points, we expect that there are four $(4,6)$ points on the $D_{97}$ over which the resolved fiber become two-dimensional.

The $(4,6)$ points and the 2D fiber can be understood by an explicit analysis of the hypersurface $p$ in equation (2.38) restricting to each irreducible component, which corresponds to a lattice point in the $c_8$ top in equation (3.34) of the non-generic toric fiber over $D_{97}$. Analogous to the models over Hirzebruch surfaces $\mathbb{F}_9/\mathbb{F}_{10}/\mathbb{F}_{11}$ in §3.2.7, we find in this case that over a generic point on the $-8$-curve, $p$ intersects the 9 components in equation (3.35) that are the boundary of the 3-dimensional face in a locus comprising nine $\mathbb{P}^1$’s, which form the $c_8$ extended Dynkin diagram, but over four distinct $(4,6)$ points on $D_{97}$, $p$ intersects also the whole irreducible component corresponding to $((v_0^{(B)})_{1,2},0,0)$ ($p_t'$) in the top; i.e., $p|_{D_{97}} = 0$ is trivially satisfied over these four points, and the elliptic fiber over the toric base contains this irreducible component, which is two-dimensional, at these four points.

The corresponding flat elliptic fibration model has a non-toric base where the four points on $D_{97}$ are blown up and the proper transform $-12$-curve intersects with the
Table 3.17: Polytope tunings of M:348 5 N:12 5 H:5,251 (generic model over $\mathbb{F}_3$): $\mathfrak{so}(n)$-tunings on the $-3$-curve with $n < 13$. These are flat elliptic fibration models, where the Hodge numbers can be directly calculated from the anomaly cancellation conditions.

<table>
<thead>
<tr>
<th>$n$</th>
<th>Tate form</th>
<th>polytope model</th>
<th>top over the $-3$-curve</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>${1,1,2,2,4}$</td>
<td>M:342 8 N:15 7 H:6,248</td>
<td>${p_{l''}, p_{l'}, p_{l''}, p_{l'}}$</td>
</tr>
<tr>
<td>9</td>
<td>${1,1,2,3,4}$</td>
<td>M:339 8 N:16 7 H:7,247</td>
<td>${p_{l''}, p_{l'}, p_{l''}, p_{l'}}$</td>
</tr>
<tr>
<td>10</td>
<td>${1,1,2,3,5}$</td>
<td>M:332 10 N:17 8 H:8,242</td>
<td>${p_{l''}, p_{l'}, p_{l''}, p_{l'}}$</td>
</tr>
<tr>
<td>11</td>
<td>${1,1,3,3,5}$</td>
<td>M:328 8 N:18 7 H:8,242</td>
<td>${p_{l''}, p_{l'}, p_{l''}, p_{l'}}$</td>
</tr>
<tr>
<td>12</td>
<td>${1,1,3,3,6}$</td>
<td>M:318 10 N:19 8 H:9,233</td>
<td>${p_{l''}, p_{l'}, p_{l''}, p_{l'}}$</td>
</tr>
</tbody>
</table>

four exceptional divisor $-1$-curves. Now we can calculate the Hodge number shifts of the flat elliptic fibration model via anomaly cancellation: $\Delta T = 4$ (each blowup contributes one additional tensor multiplet), $\Delta r = (8 - 7) + 1$, $\Delta V = (248 - 133) + 3$, and $\Delta h_1 = 10 \times 2$; therefore, by equations (3.18) and (3.19), $\Delta h_{1,1} = 6$ and $\Delta h_{2,1} = -18$, which gives $\{252, 78\} + \{6, -18\} = \{258, 60\}$, as needed.

The remaining four Hodge pairs corresponding to standard $\mathbb{P}^{2,3,1}$-fibered polytopes at large $h_{1,1}$ that were missed in our Tate tuning set have a combination of two novel features: they have apparent $\mathfrak{so}(n \geq 13)$ tunings on $-3$ curves, and also have extra sections associated with a nontrivial Mordell-Weil rank and corresponding $U(1)$ factors in the F-theory physics. For clarity, we delegate a complete example of one of the “extra” models of this type to the next subsection, and focus in the rest of this subsection on the issue of $\mathfrak{so}(n \geq 13)$-tunings on $-3$-curves in the context of simpler models with relatively small $h_{1,1}$ that do not also involve the $U(1)$ issue.

As mentioned above, $\mathfrak{so}(n)$-tunings on $-3$-curves give rise to $(4,6)$ singularities and two-dimensional resolved fibers when $n \geq 13$. While the anomaly conditions impose an upper bound of $n = 12$ for $\mathfrak{so}(n)$-tunings over $-3$-curves, there is no bound on $-4$-curves from anomaly conditions [15]. Therefore, in these cases the corresponding flat elliptic fibration models can be obtained by resolving the $-3$-curves to $-4$-curves that support $\mathfrak{so}(n \geq 13)$ without suffering from $(4,6)$ points.

We start with a generic polytope model over the Hirzebruch surface $\mathbb{F}_3$, M:348 5 N:12 5 H:5,251, and perform successive tunings of $\mathfrak{so}(n)$ on the $-3$-curve. For
<table>
<thead>
<tr>
<th>$n$</th>
<th>polytope model</th>
<th>{2D component, (4, 6) point}</th>
</tr>
</thead>
<tbody>
<tr>
<td>13</td>
<td>M:312 N:207 H:10232 ${p_5', c_3 b_1 + c_4 b_3}$</td>
<td></td>
</tr>
<tr>
<td>14</td>
<td>M:299 N:218 H:11221 ${p_5', c_4 b_1 + c_5 b_3}$</td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>M:292 N:227 H:11221 ${p_5', c_3 b_1 + c_4 b_3}$</td>
<td></td>
</tr>
<tr>
<td>16</td>
<td>M:276 N:238 H:12206 ${p_5', c_4 b_1 + c_5 b_3}$</td>
<td></td>
</tr>
<tr>
<td>17</td>
<td>M:267 N:247 H:13205 ${p_5', c_3 b_1 + c_4 b_3, p_5', c_3 b_1 + c_4 b_3}$</td>
<td></td>
</tr>
<tr>
<td>18</td>
<td>M:248 N:258 H:14188 ${p_5', c_4 b_1 + c_5 b_3, p_5', c_4 b_1 + c_5 b_3}$</td>
<td></td>
</tr>
<tr>
<td>19</td>
<td>M:238 N:267 H:14188 ${p_5', c_3 b_1 + c_4 b_3, p_5', c_3 b_1 + c_4 b_3}$</td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>M:216 N:278 H:15167 ${p_5', c_4 b_1 + c_5 b_3, p_5', c_4 b_1 + c_3 b_3}$</td>
<td></td>
</tr>
<tr>
<td>21</td>
<td>M:204 N:287 H:16166 ${p_5', c_3 b_1 + c_4 b_3, p_5', c_3 b_1 + c_4 b_3, p_{11}, c_3 b_1 + c_4 b_3}$</td>
<td></td>
</tr>
<tr>
<td>22</td>
<td>M:179 N:298 H:17143 ${p_5', c_4 b_1 + c_5 b_3, p_5', c_4 b_1 + c_5 b_3, p_{11}, c_4 b_1 + c_5 b_3}$</td>
<td></td>
</tr>
<tr>
<td>23</td>
<td>M:166 N:307 H:17143 ${p_5', c_3 b_1 + c_4 b_3, p_5', c_3 b_1 + c_4 b_3, p_{11}, c_3 b_1 + c_4 b_3}$</td>
<td></td>
</tr>
<tr>
<td>24</td>
<td>M:138 N:317 H:18116 ${p_5', c_3 b_1 + c_4 b_3, p_5', c_3 b_1 + c_4 b_3, p_{11}, c_3 b_1 + c_4 b_3}$</td>
<td></td>
</tr>
<tr>
<td>25</td>
<td>M:123 N:326 H:19115 ${p_5', c_2 b_1 + c_3 b_3, p_5', c_2 b_1 + c_3 b_3, p_{11}, c_2 b_1 + c_3 b_3}$</td>
<td></td>
</tr>
</tbody>
</table>

Table 3.18: Polytope tunings of M:3485 N:125 H:5251 (generic model over $F_3$): $so(n)$-tunings on the $-3$-curve with $13 \leq n < 26$. These are non-flat elliptic fibration models. The last column gives the (4, 6) points and the corresponding 2D toric fiber components contained in the hypersurface CY (see Table 3.6 for pt in tops). The Hodge numbers can be calculated from the associated flat elliptic fibration model over the non-toric base where the (4, 6)-point are blown up.
7 ≤ n ≤ 12, all these polytope tunings, except \(\mathfrak{so}(8)\)\(^{13}\), give a model in the KS database as expected, and the Hodge numbers of these polytope models agree with the Hodge numbers calculated from anomalies. We list these polytope models in Table 3.17. Note that the tuning from \(\mathfrak{so}(10)\) to \(\mathfrak{so}(11)\) is a rank-preserving tuning (see Table 3.4), so the Hodge numbers for these cases are identical.

Consider now the \(\mathfrak{so}(13)\) polytope tuning on the \(-3\)-curve. This also gives a reflexive polytope, M:312 8 N:20 7 H:10,232, which is still of the standard \(\mathbb{P}^{2,3,1}\)-fibered form over the \(\mathbb{F}_3\) base. But this is a non-flat elliptic fibration. In fact, we know immediately from the Hodge numbers that there is some additional subtlety in this tuning. Naively, \(\mathfrak{so}(12)\) to \(\mathfrak{so}(13)\) would be a rank-preserving tuning, and, as for the \(\mathfrak{so}(10)\) to \(\mathfrak{so}(11)\) tuning, in the absence of other issues these should have the same Hodge numbers, but they clearly do not. An explicit analysis shows that over a generic point on the \(-3\)-curve, the hypersurface equation intersects with seven components associated with the seven lattice points \(\{pt'_1, pt''_1, pt'_2, pt'_3, pt''_3, pt'_6, pt''_6\}\) in the \(\mathfrak{so}(13)\) top in a locus containing \(\mathbb{P}^1\)'s which form the \(\mathfrak{so}(13)\) extended Dynkin diagram, and there is a \((4,6)\) point on the \(-3\)-curve, over which the fiber contains the whole irreducible component associated with the lattice point \(pt'_5\) in the top.

Again, we can calculate the Hodge numbers by considering the corresponding flat elliptic fibration model over the base where the \((4,6)\) point on the \(-3\)-curve is blown up. This blow-up produces an exceptional \(-1\)-curve that intersects the proper transform \(-4\)-curve, and which can support any \(\mathfrak{so}(n)\) tunings without producing \((4,6)\) points. Therefore, \(\Delta h^{1,1} = \Delta T + \Delta r = 1 + (6-2) = 5\) and \(\Delta h^{2,1} = \Delta V - 29 \Delta T - \Delta H_c = (78 - 8) - 29 - 5 \times (13 - 1) = -19,\(^{14}\) which agrees with \(\{10, 232\} - \{5, 251\}\).

In the flat elliptic fibration model over the resolved base, as we keep increasing the \(\mathfrak{so}(n)\) tuning, an additional gauge factor \(\mathfrak{sp}(m)\) is forced to arise on the exceptional \(-1\)-curve starting at \(n = 17\): a simple local analysis shows that tuning \(\mathfrak{so}(n)\) on a

\(^{13}\)We do not expect tuned \(\mathfrak{so}(8)\) in reflexive polytope models; see §3.4.3 for discussion.

\(^{14}\)Note that although the representations of \(\mathfrak{so}(13)\) tuning on an \(-4\)-curve are \(5 \times 13\), the components that are charged under the Cartan are \(5 \times (13 - 1)\) (the Cartan subgroup of \(SO(2N+1)\) is the same as \(SO(2N)\)). As \(\mathfrak{so}(13)\) is a rank-preserving tuning of \(\mathfrak{so}(12)\), we can also do the calculation as if it were a \(\mathfrak{so}(12)\) tuning, in which case \(\Delta h^{1,2} = \Delta V - 29 \Delta T - \Delta H_c = (66 - 8) - 29 - 4 \times 12 = -19\). The two Hodge number shifts are the same.
-4-curve forces $\mathfrak{sp}([n/4] - 4)$ on an intersecting -1-curve. The forced $\mathfrak{sp}(m)$ is not apparent in the $(f, g, \Delta)$ of the polytope model, which is the non-flat model over the original $\mathbb{F}_3$ base where the -1-curve does not exist. But we have to carefully consider this forced gauge algebra on the exceptional -1-curve in computing the Hodge numbers from the anomaly equations (2.15) and (2.31). For example, tuning $\mathfrak{so}(22)$ on the -3-curve gives rise to the model $M:179$ 10 $N:29$ 8 $H:17,143$. The corresponding flat fibration model has a -4-curve intersecting an exceptional -1-curve replacing the -3-curve, and the $\mathfrak{so}(22)$ is tuned on the -4-curve, which forces an $\mathfrak{sp}(2)$ on the -1-curve. Therefore, the shifts of the Hodge numbers are $\Delta h^{1,1} = \Delta T + \Delta r = 1 + ((11 - 2) + 2) = 12$ and $\Delta h^{2,1} = \Delta V - 29\Delta T - \Delta H_c = ((231 - 8) + 10) - 29 - (14 \times 22 + 12 \times 4 - 1/2 \times 22 \times 4 \text{ (shared)}) = -108$, which agree with the Hodge numbers from the polytope. The Hodge numbers of the polytope models from the successive tunings can be calculated this way up to $\mathfrak{so}(26)$, at which point all monomials in $a_6$ are tuned off, and a $U(1)$ global factor comes into play. See below for an explicit analysis of one of the models associated with the missing Hodge pairs where such a $U(1)$ becomes relevant. We list the non-flat polytope models of tuning $\mathfrak{so}(n)$, $13 \leq n < 26$, over the -3 curve of $\mathbb{F}_3$ in Table 3.18.

An example with a nonabelian tuning that forces a $U(1)$ factor

In this subsection, we work through the details of an example of the missing Hodge pairs in the last part of Table 3.15: $M:47$ 11 $N:362$ 11 $H:263,32$. This example involves huge tunings, a blow-up from an $\mathfrak{so}(n)$ tuning on a -3 curve, and the further feature of a forced nontrivial Mordell-Weil group giving a $U(1)$ factor. After describing the geometry, we do a detailed calculation of the Hodge numbers through the associated flat elliptic fibration model.

The rational sections of an elliptic fibration form the Mordell-Weil group, which is a finitely generated group of the form $\mathbb{Z}^{\text{rank}} \times \text{(torsion subgroup)}$. If an elliptically fibered Calabi-Yau has a non-trivial Mordell-Weil rank, the F-theory compactification on it has an abelian sector $U(1)^{\text{rank}}$ [9]. The Weierstrass model of an elliptically fibered Calabi-Yau automatically comes with a zero section $z = 0$. Addi-
(a) $\nabla_2$: the additional ray blown up from $\mathbb{P}^{2,3,1}$ resolves $u(1)$-tuned models.

(b) $\Delta_2$: all monomials in the section $a_6$ are removed in the tuning, which leads to a global $u(1)$ factor in $\text{Bl}_{[0,0,1]}[\mathbb{P}^{2,3,1}]$-fibered polytopes.

Figure 3-5: The reflexive polytope pair for the $\text{Bl}_{[0,0,1]}[\mathbb{P}^{2,3,1}]$ ambient toric fiber.

tional sections can be produced through constraints in the toric geometry [41]. For instance, an abelian global $u(1)$ symmetry is forced when we set all the monomials in the section $a_6$ to vanish (the condition $a_6 = 0$ in [53].) While this can be simply imposed as a constraint to tune a $U(1)$ factor, this condition can also be imposed when we tune a large enough set of nonabelian gauge algebras on the toric curves. The lack of the monomials in $a_6$ occurs in this way in the four missing Hodge pairs $\{263, 32\}, \{251, 35\}, \{247, 35\}, \{240, 37\}$ in Table 3.15, which are therefore $\text{Bl}_{[0,0,1]}[\mathbb{P}^{2,3,1}]$-fibered polytope models (see Figure 3-5).

The $\nabla$ polytope of $M:47$ 11 $N:362$ 11 $H:263,32$ is $\text{Bl}_{[0,0,1]}[\mathbb{P}^{2,3,1}]$-fibered over the base

$$\{-4, -1, -3, -1, -4, -1, -4, -1, -4, 0, 2\}.$$  \hspace{1cm} \text{(3.74)}

The generic model over this base has Hodge numbers $\{28, 160\}$. The polytope of interest can be obtained by Tate-tuning a polytope model, for example $M:225$ 6 $N:31$ 6 $H:28,160$, associated with the generic model over the base eq 3.74. Indeed, $\nabla$ is a

140
standard $\mathbb{P}^{2,3,1}$-fibered polytope, where the tunings are

$$\{-4, \text{so}(38)\}, \{-1, \text{sp}(29)\}, \{-3, \text{so}(92)\}, \{-1, \text{sp}(36)\}, \{-4, \text{so}(68)\}, \{-1, \text{sp}(24)\}, \{-4, \text{so}(44)\}, \{-1, \text{sp}(12)\}, \{-4, \text{so}(20)\}, \{0,\}, \{2,\}.$$ (3.75)

The non-flat fiber results from the $\text{so}(92)$ on the $-3$-curve, as it exceeds the upper bound $\text{so}(12)$ associated with anomaly conditions. As the non-abelian tuning uses all of the monomials in $a_6$, the dual fiber subpolytope $V_2$ becomes a blowup of $\mathbb{P}^{2,3,1}$, $\text{Bl}_{[0,0,1]}\mathbb{P}^{2,3,1}$ (see Fig. 3-5).

Now we compute the Hodge numbers from the associated flat elliptic fibration model over the resolved base

$$-1$$

$-4, -1, -4, -1, -4, -1, -4, 0, 2,$

with tuned gauge symmetries

$$\text{sp}(19)$$

$$\text{so}(38), \text{sp}(29), \text{so}(92), \text{sp}(36), \text{so}(68), \text{sp}(24), \text{so}(44), \text{sp}(12), \text{so}(20), \cdots.$$ (3.77)

The $\text{sp}(19)$ on the exceptional $-1$-curve is forced by the $\text{so}(92)$ on the intersecting $-4$-curve. Again, the tuned non-abelian symmetries force a global $U(1)$. The dimensions of the non-abelian gauge group factors in equation (3.78) are

$$741$$

$$703, 1711, 4186, 2628, 2278, 1176, 946, 300, 190, 0, 0,$$

which differ from the total dimension of the gauge groups in the NHCs in (3.74) by

$$\Delta V_{\text{non-abelian}} = 14859 - (4 \times 28 + 8) = 14739.$$ The representations of the gauge group
factors on the individual curves are [15]

\[ 46 \times 38 \]  
\[ 30 \times 38, 66 \times 58, \quad 84 \times 92, \quad 80 \times 72, 60 \times 68, 56 \times 48, 36 \times 44, 32 \times 24, 12 \times 20, \ldots \]  

But some representations are shared between each pair of intersecting curves. The representations that are charged under both of the two corresponding group factors, \( \mathfrak{so}(n) \oplus \mathfrak{sp}(m) \), are:

\[ \frac{1}{2} \cdot 92 \cdot 38 \]  
\[ \frac{1}{2} \cdot 38 \cdot 58, \quad \frac{1}{2} \cdot 58 \cdot 92, \quad \frac{1}{2} \cdot 92 \cdot 72, \quad \frac{1}{2} \cdot 72 \cdot 68, \quad \frac{1}{2} \cdot 68 \cdot 48, \quad \frac{1}{2} \cdot 48 \cdot 44, \quad \frac{1}{2} \cdot 44 \cdot 24, \quad \frac{1}{2} \cdot 24 \cdot 20, \ldots \]

where the \( \frac{1}{2} \) factors come from the group theoretic normalization constant of \( \mathfrak{so}(n) \).

Hence, \( \Delta H_{\text{non-abelian charged}} = (\text{sum of all terms in } (3.78) - \text{sum of all terms in } (3.79)) = 14830 \). Note that all representations of a forced non-abelian gauge group are shared: \( 1/2(92) = 46 \) on the exceptional \(-1\)-curve are shared. All representations on the blown up \(-4\)-curve are also shared: \( 1/2(38 + 58 + 72) = 84 \), so the gauge symmetries can not be enhanced further on the three intersecting \(-1\)-curves.

The final piece needed is the \( U(1) \) charged matter. These fields are not charged under the non-abelian group, and therefore have not yet been taken into account in our computations. These matter fields are localized at codimension two on the \( I_1 \) component (away from the non-abelian components) of the discriminant locus (equation (2.11)), and the number of the \( U(1) \) charged matter fields corresponds to the number of the nodes, over which the fiber is Kodaira \( I_2 \), on the \( I_1 \) component [77].

Concretely, as described for example in [49], we calculate the discriminant locus of the \( I_1 \) with respect to one of the two local coordinates, which we choose to be \( b_1 \) associated with the 2-curve and \( b_2 \) associated with the 0-curve; then the \( I_1 \) discriminant locus factors into

\[ \Delta_{I_1}(b_2) = p_1(b_2)(p_2(b_2))^2(p_3(b_2))^3, \]

where \( p_1 \) is a polynomial of degree 76 in \( b_2 \), \( p_2 \) is a polynomial of degree 9 in \( b_2 \), and \( p_3 \) is
a polynomial of degree 63 in $b_2$. The degrees of the polynomials $p_2$, and $p_3$ correspond to the number of nodes and cusps on the $I_1$, respectively. The hypermultiplets charged only under the $U(1)$ are localized at the nodes, and therefore $\Delta H_{\text{abelian charged}} = 9$ in this example.

Summing up all the pieces, we obtain $\Delta h^{1,1} = \Delta T + \Delta r_{\text{non-abelian}} + \Delta r_{\text{abelian}} = 1 + (251-18)+1 = 235$ and $\Delta h^{2,1} = (\Delta V_{\text{non-abelian}} + \Delta V_{\text{abelian}}) - 29\Delta T - (\Delta H_{\text{non-abelian charged}} + \Delta V_{\text{abelian charged}}) = (14739 + 1) - 29 - (14830 + 9) = -128$, which agrees with the differences in Hodge numbers from the polytopes: $\{263, 32\} - \{28, 160\} = \{235, -128\}$.

### 3.6.2 Weierstrass Models from Non-standard $\mathbb{P}^{2,3,1}$-fibered Polytopes

For the remaining eight Hodge pairs with large $h^{1,1}$ in the KS database that were missed by our Tate construction (see table 3.15), the CY hypersurface equations (2.38) with suitable homogeneous coordinates cannot be in Tate form, although the $\nabla$ polytopes are still $\mathbb{P}^{2,3,1}$ fibered. The failure to be in the Tate form arises from the feature that there are lattice points in $\Delta$ that give rise to non-trivial base dependence in the coefficients of the monomials $x^3$ or $y^2$; i.e., these should be sections of non-trivial line bundles over the base.

These $\nabla$ polytopes do not the form of standard $\mathbb{P}^{2,3,1}$-fibered polytopes that we have defined in §3.2.2, although they still have $\mathbb{P}^{2,3,1}$ fibers. We refer to such polytopes as non-standard $\mathbb{P}^{2,3,1}$-fibered polytopes. In fact, the feature of having base-dependent terms in $x^3$ or $y^2$ is equivalent to being a non-standard $\mathbb{P}^{2,3,1}$-fibered polytope. Geometrically this feature corresponds to the condition that there is only a single lattice point in $\Delta$ that projects to each of the vertices associated with these monomials. We prove this equivalence as follows: Without loss of generality, we choose a coordinate system such that the three vertices of the $\mathbb{P}^{2,3,1}$ subpolytope $\nabla_2$ are as given in equation (3.20), and such that the projection matrix to the base is $\pi = \{(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 0, 0), (0, 0, 0, 0)\}$. Therefore, the set of the vertices of the dual subpolytope $\Delta_2$ is $\{(-2, 1), (1, -1), (1, 1)\}$, and the lattice points in $\Delta$ are
all in one of the forms in the set \{(_-, -, 1, -1), (_-, -, -2, 1), (_-, -, 0, 0), (_-, -, -1, 1),
(_-, -, 1, 0), (_-, -, 0, 1), (_-, -, 1, 1)\}. Let us first show the forward direction: We already showed in §3.2.2 that the standard \(\mathbb{P}^{2,3,1}\)-fibered polytope construction in the coordinates given in (3.21) gives a dual polytope \(\Delta\) that contains at most the single points corresponding to \(\mathcal{O}(0)\) at the vertices \((-2, 1), (1, -1)\) associated with the \(y^2, x^3\) terms (assuming that the base is compact), and both of these points must be present for the polytope \(\Delta\) to contain the origin as an interior point. Thus, any standard \(\mathbb{P}^{2,3,1}\)-fibered polytope can be put in a coordinate system where it has only the points \((0, 0, -2, 1)\) and \((0, 0, 1, -1)\) that project to \((-2, 1)\) and \((1, -1)\) in \(\Delta_2\). We can prove the backward direction as follows: Assume there is only a single lattice point in \(\Delta\) taking each of the forms \((-, -, -2, 1)\) and \((-, -, 1, -1)\). There is always a linear transformation that leaves the last two coordinates fixed that moves these to the points \((0, 0, -2, 1)\) and \((0, 0, 1, -1)\); this linear transformation will also leave the form of the fiber fixed as (3.20). The presence of these two points in \(\Delta\) shows that every lattice point \((v_{i,1}^{(B)}, v_{i,2}^{(B)}, \xi, \eta)\) has coordinates \(\xi, \eta\) that satisfy \(\eta \leq \xi + 1, \eta \geq 2\xi - 1\). For each ray in the base, however, the presence of any such lattice point imposes conditions on the points in \(\Delta\) over each of the points other than \((-2, 1)\) and \((1, -1)\) that are at least as strong as those imposed by the ray \((v_{i,1}^{(B)}, v_{i,2}^{(B)}, 2, 3)\); the conditions over these two points can be weaker, but as long as there is only the one point \((0, 0, -2, 1), (0, 0, 1, -1)\) over these two points in the dual fiber, the ray \((v_{i,1}^{(B)}, v_{i,2}^{(B)}, 2, 3)\) will be included in the polytope. Thus, for each ray in the base \((v_{i,1}^{(B)}, v_{i,2}^{(B)}, 2, 3) \in \nabla\) in this coordinate system. This proves that the presence of a single lattice point of each of the forms \((-, -, -2, 1)\) and \((-, -, 1, -1)\) implies that the polytope \(\nabla\) has the form of a standard \(\mathbb{P}^{2,3,1}\)-fibered polytope.

We would like to have a Weierstrass description of the non-standard \(\mathbb{P}^{2,3,1}\)-fibered polytopes so that we can use the methodology of F-theory to understand and analyze the geometry. To this end, we treat the \(\mathbb{P}^{2,3,1}\) fiber as a twice blown up \(\mathbb{P}^{1,1,2}\) fiber, as depicted in figure 3-6a; following the procedure in appendix A of [48] to obtain the Weierstrass model of the associated Jacobian fibration model of a \(\mathbb{P}^{1,1,2}\)-fibered polytope, we can obtain similarly that of the blownup \(\mathbb{P}^{1,1,2}\)-fibered polytope. Note
that because even non-standard $\mathbb{P}^{2,3,1}$-fibered polytopes give elliptic Calabi-Yau hypersurfaces that have a global section, the Jacobian fibration should have the same geometry as the original Calabi-Yau hypersurface; this would not be true for example if the original elliptic fibration had no section [50]. Explicitly in coordinates, instead of treating the elliptic fiber as being embedded in the $\mathbb{P}^{2,3,1}$ ambient fiber with $v_x = (0, 0, -1, 0), v_y = (0, 0, 0, -1), v_z = (0, 0, 2, 3)$, we treat the $\mathbb{P}^{2,3,1}$ fiber as a blowup $\mathbb{P}^{1,1,2}$, and embed the elliptic fiber in this blowup $\mathbb{P}^{1,1,2}$ ambient fiber with $v_x = (0, 0, -1, 0), v_y = (0, 0, 0, -1), v_z = (0, 0, 1, 2)$. The blowup rays of $\mathbb{P}^{1,1,2}$ reflect the fact that two of the nine sections of a $\mathbb{P}^{1,1,2}$-fibered polytope model are completely tuned off (see figure 3-6b) — the hypersurface equation of a non-standard $\mathbb{P}^{2,3,1}$-fibered polytope is a specialization of that of a generic $\mathbb{P}^{1,1,2}$-fibered polytope, and the blowups of the $\mathbb{P}^{1,1,2}$ fiber resolve the singularities of the tunings.

The Weierstrass models obtained in this way from non-standard $\mathbb{P}^{2,3,1}$-fibered
polytopes have the novel feature that they can have gauge groups tuned over non-toric curves in the base. Moreover, unlike the toric curves, which are always genus zero curves (isomorphic to \( \mathbb{P}^1 \)), non-toric curves can be of higher genus, and this class of global Weierstrass models gives examples of tunings of gauge groups over higher genus curves in the bases. As a check on this picture, we can verify that the Hodge numbers of these Weierstrass models calculated from anomaly cancellation match with those of the polytope data.

We give some examples of Weierstrass models from non-standard \( \mathbb{P}^{2,3,1} \)-fibered polytopes in the following subsections. In §3.6.2, we give a simple example that illustrates the non-toric curve enhancement feature. In §3.6.2, we analyze the eight remaining polytope data with large \( h^{1,1} \) from the KS database that were missing in the Tate-tuned construction. We also give some further examples of interesting geometries from the KS database at smaller Hodge numbers to illustrate the unusual nature of the non-standard \( \mathbb{P}^{2,3,1} \)-fibered polytope construction. In §3.6.2, we give a model with an \( \text{su}(2) \) tuning on a non-toric curve of genus one in the base.

**A warmup example**

As an illustration of the two different types of \( \mathbb{P}^{2,3,1} \)-fibered polytopes, we contrast the two polytopes in the KS database associated with Calabi-Yau threefolds having Hodge numbers \( \{8, 250\} \): M:346 8 N:16 7 H:8,250 and M:345 8 N:17 7 H:8,250.

The second \( \nabla_{2nd} \) polytope is a standard \( \mathbb{P}^{2,3,1} \)-fibered polytope, with vertices (in the standard coordinates) \( \{(0,0,-1,0),(-1,-4,2,3),(0,-2,1,2),(0,1,2,3),(1,0,1,2),
\)
\( (1,0,2,3),(0,0,0,-1)\} \). This is a Tate-tuned model over the base \( \mathbb{F}_4 \), with \( \text{so}(9) \oplus \text{sp}(1) \) enhanced on the \(-4\)-curve and the 0-curve \( \{b_2 = 0\} \). The base rays are
\( \{(0,1),(1,0),(0,-1),(-1,-4)\} \), and in particular, \( \{(0,1,2,3),(1,0,2,3),(0,-1,2,3),(-1,-4,2,3)\} \) are lattice points. This polytope can be obtained by tuning the \( \Delta_{\text{so}(8)} \) polytope of either one of the \( \text{so}(8) \) KS models (a generic elliptic fibration over \( \mathbb{F}_4 \)) by requiring the vanishing orders with respect to the coordinate \( b_2 \sim (1,0,2,3) \) to be \( \{0,0,1,1,2\} \) and those for the coordinate \( b_3 \sim (0,-1,2,3) \) to be \( \{1,1,2,3,4\} \). The \( \nabla_{2nd} \) polytope, which is then the dual of the reduction of the \( \Delta_{\text{so}(8)} \) polytope, has an \( \text{so}(9) \) top.
over the base ray \((0, -1)\) and an \(\mathfrak{sp}(1)\) top over \((1, 0)\).

The first \(\nabla_{1st}\) polytope is, on the other hand, of a non-standard \(\mathbb{P}^{2,3,1}\)-fibered form. The data of this polytope can be obtained by removing the vertex \((1, 0, 2, 3)\) from \(\nabla_{2nd}\) or equivalently by adding the lattice point \((-1, 0, -2, 1)\) to \(\Delta_{2nd}\), which becomes a vertex of \(\Delta_{1st}\). The one lattice point reduction of \(\nabla_{2nd}\) corresponds to the one lattice point enhancement of \(\Delta_{2nd}\). Let us now show explicitly that \(\nabla_{1st}\) is a non-standard \(\mathbb{P}^{2,3,1}\)-fibered polytope and check that it satisfies each of the two equivalent conditions (i.e., the absence of an appropriate preimage of the base in \(\nabla\) and the condition that \(\Delta\) has lattice points associated with monomials in \(x^3\) or \(y^2\) that have base dependence): The base rays of \(\nabla_{1st}\) are the same as those of \(\nabla_{2nd}\), but the ray \((1, 0)\) lacks the preimage \((1, 0, 2, 3)\) that we have removed; instead, the base ray \((1, 0)\) comes from the projection of the 4D ray \((1, 0, 1, 2)\); nonetheless \(\nabla_{1st}\) still has \(\mathbb{P}^{2,3,1}\) as a subpolytope, and therefore \(\nabla_{1st}\) is a non-standard \(\mathbb{P}^{2,3,1}\)-fibered polytope. The equivalent condition for a non-standard \(\mathbb{P}^{2,3,1}\)-fibered polytope is also satisfied from the \(\Delta\) point of view: let us associate base coordinates \(\{b_1, b_2, b_3, b_4\}\) to the set of 4D rays \(\{(0, 1, 2, 3), (1, 0, 1, 2), (0, -1, 2, 3), (-1, -4, 2, 3)\}\), and calculate the set of monomials. The two lattice points \((-1, 0, -2, 1), (0, 0, -2, 1)\) give monomials of the form \(x^3\) with base-dependent coefficients, \(b_4x^3\) and \(b_2x^3\) respectively.

Although we do not have a Weierstrass model from a Tate form for this polytope, we instead have a Weierstrass form for the hypersurface in the \(\text{Bl}_2\mathbb{P}^{1,2}\)-fibered polytope (where we have substituted some generic \(Z\) values in the complex structure
moduli):

\[
\begin{align*}
  f &= 1/48b_3^2(-1009274573279509056 + 34622237106205930350000b_3 \\
  &\quad - 274589065851262777907525390625b_3^2 - 528582381600b_3^3b_4^{22} \\
  &\quad - 22258660320b_3^6b_4^{23} + 388841808b_3^6b_4^{24}), \\
  g &= -(1/864)b_3^3(34420563383589981388800 \\
  &\quad + 19265477065422776388836404147200b_3 \cdots + 6291082311776640b_3^9b_4^{35} \\
  &\quad + 27125536688271b_3^9b_4^{36}), \\
  \Delta &= 19683/2b_3^7(35b_2 + 24b_4)^2(109370724968448b_1^7b_2^2 \\
  &\quad + 588208065199776b_1^{16}b_2^2b_3 + 1344055426083360b_1^{15}b_2^{10}b_3^2 \\
  &\quad + \cdots + 681083735457852b_2b_3^{17}b_4^{69} + 217077176379771b_3^{17}b_4^{70}).
\end{align*}
\]

According to this analysis, there is an \(\mathfrak{so}(9)\) enhancement on the \(-4\)-curve \((b_3 = 0)\) and an \(\mathfrak{sp}(1)\) enhancement on the non-toric 0-curve \(\{35b_2 + 24b_4 = 0\}\). Note that this non-toric curve is a (rational) 0-curve because it is in the same class as the two toric 0-curves. The curve supporting the \(\mathfrak{sp}(1)\) algebra intersects both the \(-4\)- and the 4-curve at one point. This is essentially the same configuration as the second model, so the Hodge numbers from an anomaly calculation also give the same result, \(\{8, 250\}\), in both cases. While in this case, the non-toric curve supporting the \(\mathfrak{sp}(1)\) can be trivially transformed into a toric curve by a simple linear change of variables, this is not the case in the more complicated examples that we consider in the later subsections.

**The eight remaining missing cases at large \(h^{1,1}\)**

Now let us come back to the polytopes of the eight Hodge pairs in the large \(h^{1,1}\) region that we did not obtain through Tate tunings and that have non-standard \(\mathbb{P}^{2,3,1}\) fibration structure. We go through one example in detail; the others have similar structure.

As a specific example, we consider the polytope M:65 8 N:357 8 H:261,45. The
vertex set of $\Delta$ is

\[
\{(-3, -3, 1, 1), (0, 0, -2, 1), (1, -7, 1, 1), (-3, 1, 1, 1), (-1, 1, -1, 1), (0, 1, 1, 1), \\
(-1, 1, 1, -1), (0, 0, 1, -1)\},
\]

where both the lattice points in the second line contribute to a $y^2$ term but with base dependence. Performing the projection we find that $\nabla$ is a non-standard $\mathbb{P}^{2,3,1}$-fibered polytope over the base

\[
\{-12/-11/-12/-12/-12/-12/-12/-12/-9,-1,-2,-2,-1,0\}. \tag{3.82}
\]

There are in total 102 base rays, and all rays but $v_{i=98,99,100,101}$ have a preimage of the form $(\_,\_,2,3)$.

The generic Weierstrass model over this base has the Hodge numbers \{257, 77\}, so the tunings must be such that the shifts are \{4, -32\}. We analyze the Weierstrass model of the non-standard $\mathbb{P}^{2,3,1}$ polytope; as in the preceding example we treat $\nabla$ as a Bl2 $\mathbb{P}^{1,1,2}$-fibered polytope (in particular, the fiber coordinates are associated to \{u, v, v\} = \{(0, 0, -1, 0), (0, 0, -1, 0), (0, 1, 2)\}), and find the associated tuned Weierstrass model. The resulting computation of \{f, g, \Delta\} shows that

- Over the toric curve $D_{100} \equiv \{b_{100} = 0\}$ the vanishing order is enhanced to \{0, 0, 2\}, which corresponds to an $\text{su}(2)$ gauge symmetry on the $-2$-curve.

- Over the non-toric curve $D_{\text{non-toric}} \equiv \{b_{\text{non-toric}} = 0\}$, where

\[
b_{\text{non-toric}} = c_7b_{100}b_{101}b_{98}b_{99} \\
+ c_6b_{10}b_{12}b_{13}b_{14}b_{15}b_{16}b_{17}b_{18}b_{19}b_{20}b_{21}b_{22}b_{23}b_{24}b_{25}b_{26}b_{27}b_{28}b_{29}b_{30}b_{31} \\
b_{27}b_{28}b_{29}b_{30}b_{31}b_{32}b_{33}b_{34}b_{35}b_{36}b_{37}b_{38}b_{39}b_{40}b_{41}b_{42}b_{43}b_{44}b_{45}b_{46}b_{47}b_{48}b_{49}b_{50}b_{51}b_{52}b_{53}b_{54} \\
b_{55}b_{56}b_{57}b_{58}b_{59}b_{60}b_{61}b_{62}b_{63}b_{64}b_{65}b_{66}b_{67}b_{68}b_{69}b_{70}b_{71}b_{72}b_{73}b_{74}b_{75}b_{76}b_{77} \\
b_{78}b_{79}b_{80}b_{81}b_{82}b_{83}b_{84}b_{85}b_{86}b_{87}b_{88}b_{89}b_{90}b_{91}b_{92}b_{93}b_{94}b_{95}b_{96}, \tag{3.83}
\]

the vanishing order is enhanced to \{0, 0, 3\}, which corresponds to an $\text{su}(3)$ gauge.
symmetry on the non-toric curve. In this expression, $c_i$ are constant coefficients, while $b_k$ are the variables associated with toric divisors $D_k$. Note that the non-toric curve intersects the two toric curves \{\(b_{102} = 0\)\} and \{\(b_{97} = 0\)\} \((b_{102} \text{ and } b_{97} \text{ are the only coordinates that do not appear in equation (3.83), and there are no intersections between the divisors associated with the variables in the first and second terms})\). As in the preceding example, this non-toric curve is a 0-curve, and is linearly equivalent to the combination of curves $D_{98} + D_{99} + D_{100} + D_{101}$, as can be seen from the first term in (3.83). The complicated combination of powers in the second term in (3.83) arise from the structure of the toric rays and the sequence of blow-ups needed to build those rays from a fiber of a minimal model Hirzebruch base.

- Over the curve $D_{97} \equiv \{b_{97} = 0\}$ (a $-9$-curve), there is a two-dimensional resolved fiber; due, however, to the enhancement over the non-toric curve intersecting the $-9$-curve, there are some differences between the fiber structure over this $-9$-curve and the one in an isolated $-9$-curve such as we discussed in §3.4 (see also §3.2.7): The top is the same as that in equation (3.34), so the 9 components that are the boundary of the 3-dimensional face intersect with the CY over a generic point in the $-9$-curve in a locus of $\mathbb{P}^1$s that compose the extended $E_8$ Dynkin diagram, just as in equation (3.35). However, as opposed to having three distinct $(4,6)$-points, as occur in the isolated $-9$-curve, there is only one $(4,6)$ point. Over this point the CY intersects the four irreducible components interior to the 3-face (while in the previous case, there is only one irreducible component that intersects the CY)

\[
S = \{(-3, -3, 1, 2), (-2, -2, 1, 2), (-1, -1, 1, 2), (-1, -1, 0, 1)\}. \quad (3.84)
\]
Explicitly,

\[ p|_I = c_7 b_{100} b_{101} b_{98} b_{99} + c_6 b_1^{22} b_2^{29} b_3^{36} b_7^{14} b_9^{34} b_{10}^{27} b_{11}^{20} b_{12}^{33} b_{13}^{13} b_{14}^{22} b_{15}^{19} b_{16}^{25} b_{17}^{10} b_{18}^{31} b_{19}^{14} b_{20}^{25} b_{21}^{24} b_{22}^{25} b_{23}^{24} b_{24}^{25} b_{25}^{26} b_{26}^{27} b_{27}^{29} b_{28}^{33} b_{29}^{23} b_{30}^{29} b_{31}^{28} b_{32}^{15} b_{33}^{15} b_{34}^{21} b_{35}^{26} b_{36}^{31} b_{37}^{15} b_{38}^{19} b_{39}^{14} b_{40}^{14} b_{41}^{42} b_{42}^{43} b_{43}^{44} b_{44}^{45} b_{45}^{46} b_{46}^{47} b_{47}^{48} b_{48}^{49} b_{49}^{50} b_{50}^{51} b_{51}^{52} b_{52}^{53} b_{53}^{54} b_{54}^{55} b_{55}^{56} b_{56}^{57} b_{57}^{58} b_{58}^{59} b_{59}^{60} b_{60}^{61} b_{61}^{62} b_{62}^{63} b_{63}^{64} b_{64}^{65} b_{65}^{66} b_{66}^{67} b_{67}^{68} b_{68}^{69} b_{69}^{70} b_{70}^{71} b_{71}^{72} b_{72}^{73} b_{73}^{74} b_{74}^{75} b_{75}^{76} b_{76}^{77} b_{77}^{78} b_{78}^{79} b_{79}^{80} b_{80}^{81} b_{81}^{82} b_{82}^{83} b_{83}^{84} b_{84}^{85} b_{85}^{86} b_{86}^{87} b_{87}^{88} b_{88}^{89} b_{89}^{90} b_{90}^{91} b_{91}^{92} b_{92}^{93} b_{93}^{94} b_{94}^{95} b_{95}^{96} b_{96}^{97} b_{97}^{98} b_{98}^{99} b_{99}^{100}, \forall I \in \mathbb{S} 3.85 \)

Moreover, by comparing equations (3.83) and (3.85), we know that the (4,6) point is exactly at the intersection of the divisors \( \{b_{97} = 0\} \) and \( \{b_{\text{non-toric}} = 0\} \).

We now find the associated flat elliptic fibration model, so that we can use F-theory techniques to compute the Hodge number shifts. We first identify the resolved base, which is semi-toric, and then determine the tunings. Since there is the only one (4,6) point, we blow up successively three times at this point to turn the \(-9\)-curve into a \(-12\)-curve, and the non-toric \(0\)-curve is replaced by a chain of curves of self-intersection numbers \(-1, -2, -2, -1\) (similar to the last graph of the Hodge pair \(\{14, 404\}\) in Table 3.13). The divisor classes of the curves after the blow-up process can be determined in the usual fashion: The \(-12\)-curve \(\tilde{D}_{97}\) is the proper transform
of the $-9$-curve after the three blowups

$$\tilde{D}_{97} = D_{97} - E_1 - E_2 - E_3,$$

(3.86)

where $E_1, E_2, E_3$ are the exceptional divisors associated with the three blowups. The proper transform of the non-toric curve is

$$\tilde{D}_{\text{non-toric}} = D_{\text{non-toric}} - E_1,$$

(3.87)

which is a $-1$-curve. The three curves, $-2, -2, -1$, connecting $\tilde{D}_{97}$ and $\tilde{D}_{\text{non-toric}}$ are respectively

$$\tilde{E}_1 = E_1 - E_2, \quad \tilde{E}_2 = E_2 - E_3, \quad \text{and} \quad E_3.$$

(3.88)

Now we figure out the gauge symmetries on these divisors. There was an $\text{su}(3)$ on the 0-curve, $D_{\text{non-toric}}$, which is now on the $-1$-curve, $\tilde{D}_{\text{non-toric}}$. This forces an $\text{su}(2)$ on the $-2$-curve, $\tilde{E}_1$, connecting to $\tilde{D}_{\text{non-toric}}$. The configuration of the intersecting curves and the symmetry enhancements are drawn in Fig. 3-7.

Remarkably, the described configuration gives the correct counting of the shifts in Hodge numbers through the anomaly calculation. The contributions to $h^{1,1}$ and $h^{2,1}$ from the tunings through equations (3.18) and (3.19) are

- $\text{su}(2)$ on $D_{100}$: $\Delta h^{1,1} = \Delta r = +1$ and $\Delta h^{2,1} = \Delta V - \Delta H_{\text{charge}} = 3 - 4 \times 2 = -5$.

- $\text{su}(2) \oplus \text{su}(3)$ on $\{\tilde{E}_1, \tilde{D}_{\text{non-toric}}\}$: $\Delta h^{1,1} = +1 + 2$ and $\Delta h^{2,1} = (3 + 8) - (12 \times 3 + 4 \times 2 - 2 \times 3$ (shared)) $= -27$.

The total gives the desired Hodge number shifts $\{\Delta h^{1,1}, \Delta h^{2,1}\} = \{4, -32\}$. Note that the final Hodge numbers correspond to the flat elliptic fibration over the non-toric base, and correctly reflect the associated contribution of three tensor multiplets from the three blow-ups of the base.\footnote{The Hodge numbers denoted in a generic model containing a $-9/-10/-11$-curve are understood to be those of the flat elliptic fibration over the base that has been resolved at the (4,6) points.} The correspondence between the non-flat and the flat models may be considered as that the four irreducible components of the
2-dimensional fiber over the (4, 6) point transform to the three divisors resolving the 
−9-curve in the base and one divisor in the fiber to resolve the forced su(2) on \(\tilde{E}_1\).

We now consider the remaining non-standard \(\mathbb{P}^{2,3,1}\) fibered "extra" cases at large 
h^{1,1}. The model associated with Hodge numbers \(\{245, 57\}\) has a gauge symmetry that
is enhanced on the non-toric curve, but there are no (4, 6) singularities involved, which
is similar to the example in 3.6.2; we must, however, be careful to properly include
the shared matter contribution to the matter multiplets, as a curve intersecting the
non-toric curve also carries a gauge symmetry. The models with Hodge numbers
\(\{261, 51\}, \{260, 62\}, \{260, 54\}, \{259, 55\}, \{258, 84\}, \) and \(\{254, 56\}\) are all similar to
that of \(\{261, 45\}\) that we have treated in detail here.

We conclude the discussion of these cases by briefly summarizing the details of
the model M:82 10 N:351 10 H:254,56, where we need to include one extra tensor
multiplet in the Hodge number counting.

- **generic model (total 100 toric curves in the base)**

\[
\{251, 79, \{-12/ / -11/ / -12/ / -12/ / -12/ / -12/ / -12/ / -12, (3.89)
\]
\[-1, -2, -2, -3, -1, -5, -1, -3, -2, -1, -7, -1, -2, -1, 0\}\}

- **one (4, 6) point on the \(-7\)-curve \(D_{g6}\).**

- **gauge symmetry enhancements**

1. su(2) on the \(-2\)-curve \(D_{g8}\)

2. su(2) on the non-toric 0-curve \(D_{\text{non-toric}}\) intersecting the 0-curve \(D_{100}\) and
   \(-7\)-curve \(D_{g6}\) at the (4, 6) point.

The corresponding flat elliptic fibration model has

- **base structure:**

The (4, 6) is blown up. \(D_{g6}\) is resolved into a \(-8\)-curve \(\tilde{D}_{g6} = D_{g6} - E_1\) and
\(D_{\text{non-toric}}\) into \(\tilde{D}_{\text{non-toric}} = D_{\text{non-toric}} - E_1\), where \(E_1\) is the exceptional divisor of
the blowup.
• enhanced gauge symmetries:

1. $\text{su}(2)$ on the $-2$-curve $D_{08}$, which shifts $h^{1,1}$ by $r(\text{su}(2)) = 1$ and $h^{2,1}$ by $V(\text{su}(2)) - H_{\text{charged}}(\text{su}(2) \text{ on } -2\text{-curve}) = 3 - 4 \times 2 = -5$.

2. $\text{su}(2)$ on the $-1$-curve $\tilde{D}_{\text{non-toric}}$, which shifts $h^{1,1}$ by $r(\text{su}(2)) = 1$ and $h^{2,1}$ by $V(\text{su}(2)) - H_{\text{charged}}(\text{su}(2) \text{ on } -1\text{-curve}) = 3 - 10 \times 2 = -17$.

• Hodge numbers

1. contributions from the enhanced gauge symmetries $\Delta h^{1,1} = 1 + 1 = 2$, $\Delta h^{2,1} = -5 - 17 = -22$

2. contribution from the tensor multiplet associated with the one extra blowup in the base $\Delta h^{1,1} = 1$, $\Delta h^{2,1} = -29$

3. compensation of $\Delta h^{2,1} = \frac{1}{2}56 = 28$ due to the fact that there are half-hyper multiplets $\frac{1}{2}56$ on the NHC $-7$-curve, but there are no localized matter fields on the NHC $-8$-curve.

In total we have $h^{1,1} = 251 + 2 + 1 = 254$ and $h^{2,1} = 79 - 22 - 29 + 28 = 56$, which agree with the Hodge numbers of the polytope model.

Example: a model with a tuned genus one curve in the base

In the final part this section we consider an additional non-standard $\mathbb{P}^{2,3,1}$-fibered models that has the further interesting feature that a gauge group is tuned on a non-toric curve that has nonzero genus. While this phenomenon does not occur in the “extra” models at large Hodge numbers that we have focused on here, the fact that this non-toric tuning structure can arise even in the context of toric hypersurface Calabi-Yau constructions seems sufficiently interesting and novel that we provide some details for understanding the structure of models of this type.

We study in particular a model with an $\text{su}(2)$ tuning on a non-toric curve of genus
one in the base: M:2237 N:106 H:3,165. The vertex set of Δ is

\[ \{((-1, -1, -2, 1), (2, -1, -2, 1), (-1, 2, -2, 1), (-4, -4, 1, 1), (8, -4, 1, 1), (-4, 8, 1, 1), (0, 0, 1, -1))\} \] (3.90)

where the first three lattice points contribute to a \( x^3 \) term with base dependence. Then \( V \) is a non-standard \( \mathbb{P}^{2,3,1} \)-fibered polytope over \( \mathbb{P}^2 \). The base rays are

\[ \{(1, 0), (0, 1), (-1, -1)\} \rightarrow \{b_1, b_2, b_3\}, \] (3.91)

which come from the projection of the 4D rays \( \{(1, 0, 1, 2), (0, 1, 1, 2), (-1, -1, 1, 2)\} \) (in fact, these are the only three lattice points in \( V \) that do not project to \( (0, 0) \), so none of the preimages are in the form \( ((v(B))_{1,2}, 2, 3) \)).

We analyze the Weierstrass model of the non-standard \( \mathbb{P}^{2,3,1} \) polytope: Treating \( V \) again as the \( B_1 \mathbb{P}^{1,1,2} \)-fibered polytope (in particular, the fiber coordinates are associated to \( \{v_x, v_y, v_z\} = \{(0, 0, -1, 0), (0, 0, 0, -1), (0, 0, 1, 2)\} \)) we find the associated tuned Weierstrass model. The orders of vanishing of \( \{f, g, \Delta\} \) are enhanced to \( \{0, 0, 2\} \) on the curve in the base \( D_{\text{non-toric}} \equiv \{I_{\text{su}(2)} = 0\} \), where

\[
I_{\text{su}(2)} = c_1b_1^3 + c_{179}b_2b_3^2 + c_{180}b_2^2b_3 + c_{181}b_1b_3^2 + c_{182}b_1b_2b_3 + c_{183}b_1b_2^2 + c_{184}b_1^2b_3 + c_{185}b_1^2b_2 + c_2b_2^3 + c_3b_3^2.
\] (3.92)

In particular, the result for the discriminant \( \Delta \) is

\[ \Delta = I_{\text{su}(2)}^2 I_1, \] (3.93)

where the \( I_1 \) component of \( \Delta \) is a degree 30 polynomial in the homogeneous coordinates. Note that \( D_{\text{non-toric}} \) is a smooth curve of genus one, which can be calculated
by the formula (3.9)

\[ 3[b_1] \cdot (3[b_1] - ([b_1] + [b_2] + [b_3])) = 0 = 2g - 2 \Rightarrow g = 1. \]  (3.94)

We calculate Hodge numbers from the anomaly conditions: the matter representations of $\text{su}(2)$ on a $g = 1$ curve of self-intersection $D_{\text{non-toric}}^2 = 9$ is $54 \times 2 + 3$ [15], but only two components of the adjoint representation $3$ are charged under the Cartan (see the footnote in §3.1.7). Therefore, $H_{\text{charged}} = 108 + 2 = 110$. Then $h^{1,1} = \Delta r = 1$ and $h^{2,1} = \Delta V - H_{\text{charged}} = 3 - 110 = -107$, which agree with \{3, 165\} - \{2, 272\} = \{+1, -107\}.

3.7 Conclusions

3.7.1 Summary of Results

In this chapter we have carried out a systematic comparison of elliptic Calabi-Yau threefolds with large Hodge numbers that are realized by tuning Tate-form Weierstrass models over toric bases and those that are realized as hypersurfaces in toric varieties through the Batyrev construction. Specifically, we have considered a class of Tate-tuned models over toric bases that have nonabelian gauge groups tuned over toric divisors. These tunings give a specific class of "standard" $\mathbb{P}^{2,3,1}$-fibered reflexive polytopes, all of which give Calabi-Yau threefolds with Hodge numbers that appear in the Kreuzer-Skarke database.

- Almost all Hodge number pairs of known CY3's in the regime studied come from elliptically fibered Calabi-Yau threefolds associated with polytopes constructed in this fashion that are associated with an explicit Tate/Weierstrass construction of the restricted class that we considered in our initial analysis.

- We have explicitly analyzed the structure of the Calabi-Yau threefolds in the Kreuzer-Skarke database for the 18 Hodge number pairs not found in our initial analysis from Tate constructions. All of these admit elliptic fibrations of slightly
more complicated forms.

- Thus, we have found explicit realizations of elliptic Calabi-Yau threefolds that produce all Hodge number pairs with \( h^{1,1} \geq 240 \) or \( h^{2,1} \geq 240 \) that are known to be possible for Calabi-Yau threefolds. This matches with the results of the next chapter [37] showing that all polytopes in the KS database giving Calabi-Yau threefolds with \( h^{1,1} \geq 150 \) or \( h^{2,1} \geq 150 \) have a genus one fibration, and have an elliptic fibration whenever \( h^{1,1} \geq 195 \) or \( h^{2,1} \geq 228 \). These results provide additional evidence that virtually all known Calabi-Yau threefolds with large Hodge numbers are elliptically fibered, building on a variety of other recent work that has led to similar observations [14, 15, 19, 22, 23, 24, 25, 26]. Since the number of elliptic Calabi-Yau threefolds is finite, this in turn suggests that the number of distinct topological classes of Calabi-Yau threefolds is in fact finite.

- In the course of this analysis we have encountered some novel structures in the Tate/Weierstrass tunings needed to reproduce certain CY3’s associated with polytopes in the KS database. This has led to new insights into the Tate algorithm as well as in the structure of fibrations that may occur through polytopes.
  
  — A novel Tate tuning of SU(6) gives rise to exotic 3-index antisymmetric matter, of a form recently studied in [47, 76].
  
  — Some polytopes in the KS database correspond to tunings of very large gauge algebras with components like \( \mathfrak{so}(20) \).
  
  — Polytopes in the KS database include non-flat elliptic fibrations over toric bases that resolve into flat elliptic fibrations over more complicated non-toric bases including not only blow-ups of \(-9, -10, -11\) curves, but also more exotic structure such as an \( \epsilon_8 \) over a \(-8\) curve that must be blown up four times, or tunings of \( \mathfrak{so}(n), n \geq -13 \) on \(-3\) curves that must be blown up to \(-4\) curves to satisfy anomaly conditions. In some of the \( \mathfrak{so}(n) \) cases the resolved geometry also gives rise to a nontrivial Mordell-Weil group associated with a \( \text{U}(1) \) factor in the gauge group.
  
  — Some polytopes in the KS database have elliptic fibrations over toric bases in which nonabelian gauge algebras are tuned over non-toric curves in the base.
We worked out the tops associated with the gauge algebras $\mathfrak{so}(n)$, $13 \leq n \leq 25$, as well as the tops associated with gauge algebras $\mathfrak{su}(n)$, $7 \leq n \leq 13$. For $\mathfrak{so}(n)$, these match the tops found in [40] after an appropriate linear transformation; our construction gives explicit realizations of these tops in reflexive polytopes for the range of algebras listed, which is not guaranteed from the construction of [40]. The tops associated with $I_n$ and $I_n^*$ types have the feature that they develop along the fiber direction, and the projection to the fiber plane falls outside the $\mathbb{P}^{2,3,1}$ fiber sub-polytope. Another interesting feature of the $\mathfrak{so}(n)$ tops is that there can be multiple distinct tops for certain gauge algebras, corresponding to monodromy conditions on the associated Tate tunings.

3.7.2 Possible Extensions

In [37] discussed in the next chapter, we carry out a complementary analysis to what we have done here. Here we have started from the Tate tuning picture and matched to data in the Kreuzer-Skarke database. One can instead start with the polytopes in the database and try to derive the elliptic fibration structure. This is essentially the approach taken by Braun in [48], in which the database was scanned for elliptic fibrations over the base $\mathbb{P}^2$. In [37], we take that point of view and analyze the fibration structure of the polytopes in the KS database directly. The approach taken here, however, shows that at large Hodge numbers most Calabi-Yau threefolds have a standard elliptic fibration structure; the "sieve" approach taken here enables us to identify some of the most interesting cases that present novel features.

There are several closely related analyses that could be carried out that we have not done here or in [37]; each of these represents an opportunity for further work that would give increased understanding of the set of Calabi-Yau threefolds, the role of elliptic fibrations, and the landscape of 6D F-theory models.

First, we have started from the point of view of tuning Tate models and used the output of that analysis to match Hodge numbers in the KS database. In principle, we could have tried to reproduce all the polytopes in the database, i.e. included multiplicity information. For reasons discussed in §3.4, this would be a more complicated
analysis. In many cases there are multiple local Tate tunings that give equivalent
gauge groups, and we have in each case systematically taken only the lowest possible
choice for NHCs and the lowest order choice with no further monodromy condition
required for a given gauge group tuning. For bases with many toric divisors, the
number of combinatorial possibilities of local tunings can become quite large. There
are also many equivalent models that correspond to carrying out explicitly different
subsets of toric blow-ups to partially resolve $(4, 6)$ singularities. We have checked in
some cases that the multiplicity of Hodge numbers in the KS database is reproduced
by distinct Tate/Weierstrass tunings of elliptic fibrations, but we have not approached
this systematically. This would be a natural next step for this kind of analysis, and
might reveal additional novel structures for the elliptic fibrations found in the KS
database.

Second, we have restricted to large Hodge numbers in part because we have only
focused on Tate models associated with the most generic $\mathbb{P}^{2,3,1}$ fiber structure for
the polytope. There are 16 distinct possible toric fibers, analyzed in detail in the
F-theory context in [48, 41, 71], each leading to a distinct class of Weierstrass tuning
types with characteristic nonabelian and abelian gauge structure, and in principle we
could systematically analyze all tunings that correspond to each of the different fiber
types. This would be necessary to extend the analysis of this work systematically
to smaller Hodge numbers, where the other fiber types become prevalent [37]. We
leave such an endeavor for future work. It would also be interesting to see whether
the more general class of fibers considered in [78] may give further insights into other
Weierstrass tuning types that may be possible with complete intersection fibers.
Chapter 4

Classification of 2D Fiber Subpolytopes

In this chapter, which is based on our work in [37], we carry out a systematic scan through the Kreuzer-Skarke database to determine which reflexive polytopes associated with Calabi-Yau threefolds that have large Hodge numbers or small $h^{1,1}$ have toric reflexive 2D fibers that indicate the existence of an elliptic or genus one fibration for the associated Calabi-Yau threefold. There are 16 reflexive 2D polytopes that can act as fibers of a 4D polytope describing a Calabi-Yau threefold; the presence of any of these fibers in the 4D polytope indicates that the corresponding Calabi-Yau threefold hypersurface is genus one or elliptically fibered. We systematically consider all polytopes in the Kreuzer-Skarke database that are associated with Calabi-Yau threefolds with one or both Hodge numbers at least 140. We show that with only four exceptions these Calabi-Yau threefolds all admit an explicit elliptic or more general genus one fibration that can be seen from the toric structure of the polytope. We furthermore find that for toric hypersurface Calabi-Yau threefolds with small $h^{1,1}$, the fraction that lack a genus one or elliptic fibration decreases roughly exponentially with $h^{1,1}$. Together these results strongly support the notion that genus one and elliptic fibrations are quite generic among Calabi-Yau threefolds.  

\footnote{Note that our analysis here only identifies elliptic and genus one fibrations that are manifest in the polytope structure.}
The outline of this chapter is as follows: In Section 4.1 we describe the 16 types of toric fibers of the polytope that can lead to a genus one or elliptic fibration of the hypersurface Calabi-Yau and our methodology for analyzing the fibration structure of the polytopes. In Section 4.2, we give our results on those Calabi-Yau threefolds with the largest Hodge numbers that do not admit an explicit elliptic or genus one fibration in the polytope description, as well as some results on the distribution of fiber types and multiple fibrations. In Section 4.3 we discuss some simple aspects of the likelihood of the existence of fibrations and compare to the observed frequency of fibrations in the KS database at small $h^{1,1}$. Section 4.4 contains some concluding remarks. We are making the results of the fiber analysis of polytopes in the Kreuzer-Skarke database associated with Calabi-Yau threefolds having Hodge numbers $h^{1,1} \geq 140$ or $h^{2,1} \geq 140$ available in Mathematica form [83].

4.1 Identifying Toric Fibers

A introductory review of the toric hypersurface construction and how elliptic fibrations are described in this context is given in §2.4.2 and §3.2.1, respectively. For the convenience of the reader, here we give a brief summary of the essential points for this chapter.

4.1.1 Toric Hypersurfaces and the 16 Reflexive 2D Fibers

The basic framework for understanding Calabi-Yau manifolds through hypersurfaces in toric varieties was developed by Batyrev [2]. A lattice polytope $\nabla$ is defined to be the set of lattice points in $N = \mathbb{Z}^n$ that are contained within the convex hull of a finite set of vertices $v_i \in N$. The dual of a polytope $\nabla$ is defined to be

$$\nabla^* = \{ u \in M_\mathbb{R} = M \otimes \mathbb{R} : \langle u, v \rangle \geq -1, \forall v \in \nabla \}, \quad (4.1)$$

where $M = N^* = \text{Hom}(N, \mathbb{Z})$ is the dual lattice. A lattice polytope $\nabla \subset N$ containing the origin is reflexive if its dual polytope is also a lattice polytope. When $\nabla$ is reflexive,
we denote the dual polytope by $\Delta = \nabla^*$. The elements of the dual polytope $\Delta$ can be associated with monomials in a section of the anti-canonical bundle of a toric variety associated to $\nabla$. A section of this bundle defines a hypersurface in $\nabla$ that is a Calabi-Yau manifold of dimension $n - 1$.

When the polytope $\nabla$ has a 2D subpolytope $\nabla_2$ that is also reflexive, the associated Calabi-Yau manifold has a genus one fibration. There are 16 distinct reflexive 2D polytopes, listed in Appendix A.1. These fibers are analyzed in the language of polytope "tops" [39] in [40]. The structure of the genus one and elliptic fibrations associated with each of these 16 fibers is studied in some detail in the F-theory context in [48, 41, 71].

Of the 16 reflexive 2D polytopes listed in Appendix A.1, 13 are always associated with elliptic fibrations. This can be seen, following [41], by observing that the anti-canonical class $-K_2$ of the toric 2D variety associated with a given $\nabla_2$ is $\sum C_i$ where $C_i$ are the toric curves associated with rays in a toric fan for $\nabla_2$. The intersection of the curve $C_i$ with the genus one fiber associated with the vanishing locus of a section of $-K_2$ is thus $C_i \cdot (-K_2) = 2 + C_i \cdot C_i$, so $C_i$ defines a section associated with a single point on a generic fiber only for a curve of self-intersection $C_i \cdot C_i = -1$. The three fibers $F_1, F_2, F_4$ are associated with the weak Fano surfaces $\mathbb{P}^2, F_0 = \mathbb{P}^1 \times \mathbb{P}^1$, and $F_2 = \mathbb{P}^2[1,1,2]$, which have no $-1$ curves, while the other 13 fibers $F_i$ all have $-1$ curves. Thus, polytopes $\nabla$ with any fiber $\nabla_2$ that is $F_n, n \notin \{1,2,4\}$ give CY3s with elliptic fibrations, while those $\nabla$ with only fibers of types $F_1, F_2, F_4$ are genus one fibered but may not be elliptically fibered.

### 4.1.2 Algorithm for Checking a Polytope for Fibrations

We generalize the algorithm that we used in Chapter 3 §3.2.3 to check for reflexive 2D fibers of a 4D reflexive polytope. Except for a small tweak to optimize efficiency, this is essentially the approach outlined in [41]. The basic idea is to check a given polytope for each of the possible 16 reflexive subpolytopes. For a given polytope $\nabla$ and potential fiber polytope $\nabla_2$, we proceed in the following two steps:
1. To increase the efficiency of the analysis we start by determining the subset $S$ of the lattice points in $\nabla$ that could possibly be contained in a fiber of the form $\nabla_2$, using a simple criterion. For each fixed fiber type $\nabla_2$, there is a maximum possible value $I_{\text{max}}$ of the inner product $v^{(F)} \cdot m$ for any $v^{(F)} \in \nabla_2, m \in \Delta_2$. For example, for the 2D $\mathbb{P}^{2,3,1}$ polytope $(F_{10})$, $I_{\text{max}} = 5$. The values of $I_{\text{max}}$ for each of the reflexive 2D polytopes $\nabla_2$ are listed in Appendix A.1. When $\nabla_2$ is a fiber of $\nabla$, which implies that there is a projection from $\Delta$ to $\Delta_2$, $I_{\text{max}}$ is also the maximum possible value of the inner product $v^{(F)} \cdot m$ for any $m \in \Delta$. Thus, we define the set $S$ to be the set of lattice points $v \in \nabla$ such that $v \cdot m \leq I_{\text{max}}$ for all vertices $m$ of $\Delta$. Particularly for polytopes $\nabla$ that contain many lattice points, generally associated with Calabi-Yau threefolds with large $h^{1,1}$, this step significantly decreases the time needed for the algorithm.

2. We then consider each pair of vectors $v, w$ in $S$ and check if the intersection of $\nabla$ with the plane spanned by $v, w$ consists of precisely a set of lattice points that define the 2D polytope $\nabla_2$. If such a pair of vectors exists then $\nabla$ has a fiber $\nabla_2$ and the associated Calabi-Yau threefold has an elliptic fibration structure defined by this fiber type.

In practice, we implement these steps directly only to check for the presence of the minimal 2D subpolytopes $F_1, F_2, F_4$ within a 2D plane; all the other 2D reflexive polytopes contain the points of $F_1$ as a subset (in some basis). These three cases use the values $I_{\text{max}} = 2, 1, 3$ respectively as shown in Appendix A.1. The three minimal 2D polytopes do not contain any other 2D reflexive polytopes, and it requires a minimal number of linear equivalence relations among the toric divisors to check if these minimal polytopes are present as a subset of the points in $\nabla$ that are in a plane defined by a non-collinear pair $v, w \in S$:

- $F_1$: $-(v + w) \in S$
- $F_2$: $-v, -w \in S$
- $F_4$: $-(v + w)/2 \in S$
We could in principle use this kind of direct check to determine the presence of the larger subpolytopes as well, though this becomes more complicated for the other fibers and we proceed slightly more indirectly. After identifying all the 2D planes that are spanned by non-colinear pairs $v, w$ and contain one of the three minimal 2D subpolytopes, we calculate the intersection of the 4D polytope with the 2D plane to obtain the full subpolytope that contains the minimal 2D subpolytope. This intersection can be determined by identifying all lattice points $x \in \nabla$ that give rise to a $4 \times 4$ matrix of rank two with another three non-colinear vectors in the 2D plane. Note that this intersection must give a 2D reflexive polytope, since there can only be one interior point in the 2D fiber polytope as any other interior point besides the origin would also be an interior point of the full 4D polytope, which is not possible if the 4D polytope is reflexive.

Let us call the sets of subpolytopes containing $F_1, F_2,$ and $F_4$ respectively $S_1, S_2,$ and $S_4$. We can then efficiently determine which fiber type arises in each case by some simple checks. Observing that all the 2D polytopes other than the three minimal ones contain the $F_1$ polytope, we immediately have

- $\{\nabla_2^{F_2}\} = S_2 \setminus S_1,$
- $\{\nabla_2^{F_4}\} = S_4 \setminus S_1.$

Then we group the fibers associated with the rest of the 2D polytopes, which are all in $S_1$, by the number of lattice points:

- 5 points: $F_3$
- 6 points: $F_5, F_6$
- 7 points: $F_7, F_8, F_9, F_{10}$
- 8 points: $F_{11}, F_{12}$
- 9 points: $F_{13}, F_{14}, F_{15}$
- 10 points: $F_{16}$

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This immediately fixes the fibers $F_3$ and $F_{16}$. To distinguish the specific fiber types for the remaining four groups a number of approaches could be used. We have simply used a projection to compute the self-intersections of each curve in a given fiber and the sequence of these self-intersections. (Note that in a toric surface, the self intersection of the curve associated with the vector $v_i$ is $m$, where $v_{i-1} + v_{i+1} = -mv_i$.) By simply counting the numbers of $-2$ curves we can identify $F_{5-13}$. Finally, $F_{14}, F_{15}$ have the same numbers of curves of each self-intersection, so we use the order of the self-intersections of the curves in the projection to distinguish these two subpolytopes.

4.1.3 Stacked Fibrations and Negative Self-intersection Curves in the Base

In Chapter 3, we have found that at large Hodge numbers many of the polytopes in the KS database belong to a particular “standard stacking” class of $\mathbb{P}^{2,3,1}$ fiber type $(F_{10})$ fibrations over toric base surfaces, which are $F_{10}$ fibrations where all rays in the base are stacked over a specific vertex $v_s$ of $F_{10}$. This simple class of fibrations corresponds naturally to Tate-form Weierstrass models over the given base, which take the form $y^2 + a_1yx + a_3y = x^3 + a_2x^2 + a_4x + a_6$. In this Chapter we systematically consider the distribution of different fiber types, and also analyze which of the $\mathbb{P}^{2,3,1}$ fibrations are of the “standard stacking” type. As background for these analyses, we describe in this section the more general “stacked” form of polytope fibrations and perform some further analysis of which stacked fibration types can occur over bases with curves of given-intersection; since certain fibers cannot arise in fibrations over bases with extremely negative self-intersection curves (at least in simple stacking fibrations), this helps to explain the dominance of $\mathbb{P}^{2,3,1}$ fibers at large $h^{1,1}$.

Stacked fibrations

As discussed in more detail in Chapter 3, the presence of a reflexive fiber $F = \nabla_2 \subset \nabla$ gives rise to a projection map $\pi : \nabla \to \mathbb{Z}^2$, where $\pi(F) = 0$, associated with a genus one or elliptic fibration of the Calabi-Yau hypersurface $X$ over a toric complex surface.
The "stacked" form of a fibration refers to a polytope in which the rays of the base all have pre-images under $\pi$ that lie in a plane in $\nabla$ passing through one of the points in the fiber polytope $\nabla_2$. Specifically, a polytope $\nabla$ that is in the stacked form can always be put into coordinates so that the lattice points in $\nabla$ contain a subset

$$\{(v_i^{(B)})_{1,2}; (v_s^{(F)})_{1,2})|v_i^{(B)} \in \{\text{vertex rays in } \Sigma_B\}\} \cup \{(0,0,(v_i^{(F)})_{1,2})|v_i^{(F)} \in \{\text{vertices of } \nabla_2\}\},$$

(4.2)

where $\Sigma_B$ is the toric fan of the base $B$ and $v_s^{(F)}$ is a specified point of the fiber subpolytope $\nabla_2$. We refer to such polytopes as $v_s^{(F)}$ stacked $F$-fibered polytopes.

In some contexts it may be useful to focus attention on the stacked fibrations where the point $v_s^{(F)}$ is a vertex of $\nabla_2$, as these represent the extreme cases of stacked fibrations, and have some particularly simple properties\(^2\). We can refer to these as "vertex stacked" fibrations. The standard $\mathbb{P}^{2,3,1}$ fibrations discussed in Chapter 3 (sometimes there called "standard stacking" fibrations) refer to the cases of stacked fibrations where the fiber is $F_{10}$ and the specified stacking point is the vertex $v_s^{(F)} = (-3, -2)$.\(^3\) These are based on a standard type of construction in the toric hypersurface literature (see e.g. [70]). In the case of a standard stacking, the monomials in $\Delta$ match naturally with the set of monomials in the Tate-form Weierstrass model. Generalizing this analysis gives bounds on what kinds of curves can be present in the base supporting a stacked fibration with different fiber types.

**Negative curve bounds**

For any stacked fibration with a given fiber type $F$ and specified point $v_s^{(F)}$ for the stacking, the monomials in the dual polytope $\Delta$ are sections of various line bundles $\mathcal{O}(-nK_B)$. By systematically analyzing the possibilities we see that many fibers cannot be realized in stacked fibrations over bases with curves of very negative self-

\(^2\) In particular, the analysis in §3.6.2 can be easily generalized to show that a fibration has a vertex stacking on $v_s^{(F)} \in \nabla_2$ iff there is a single monomial over every point in the dual face of $\Delta_2$ and these monomials all lie in a linear subspace of $\Delta$.

\(^3\) Note that in Chapter 3, we have a different convention for $\mathbb{P}^{2,3,1}$ which uses slightly different coordinates from those one we use here, so that the vertex in the notation in Chapter 3 is $v_s^{(F)} = (2, 3)$. 

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intersection without giving rise to singularities in the fibration over these curves that go outside the Kodaira classification and have no Calabi-Yau resolution.

We analyze this explicitly as follows. To begin with, the lattice points of the dual polytope \( \Delta \) of an \( F \)-fibered polytope \( \nabla \) are of the form

\[
\{(m^{(2)}_{1,2}, m^{(F)}_2 | m^{(F)}_j \in \Delta^{(F)}_2; (m^{(2)}_1, m^{(2)}_2) \in \mathbb{Z}\},
\]

where \( \Delta^{(F)}_2 \) is one of the 16 dual subpolytopes that are given in detail in Appendix A.2. For a given base \( B \), we have the condition

\[
m^{(2)} \cdot v^{(B)}_i \geq -n, \forall i \Leftrightarrow m^{(2)} \text{ gives a section in } \mathcal{O}(-nK_B).
\]

Given that \( ((v^{(B)}_i^{(1,2)}, v^{(F)}_j^{(1,2)}) \in \nabla \) for all \( i \) in a fibration that has the “stacked” form (4.2), the reflexive condition \( m \cdot v \geq -1, m \in \Delta, v \in \nabla \) implies that a lattice point \( m = ((m^{(2)}_1, (m^{(F)}_j)_1, (m^{(F)}_2) \in \Delta \) gives a section in \( \mathcal{O}(-(v^{(F)}_i \cdot m^{(F)}_j + 1)K_B)\). (See Figure 4-1 for examples with the \( F_{10} \) fiber type, using the three different vertices \( v^{(F)}_j \) of \( V_2 \) as the specified points for three different stackings, including the “standard stacking” in which the monomials over the different lattice points in \( \Delta_2 \) correspond to sections \( a_n \) of different line bundles in the Tate-form Weierstrass model.) Note that the lattice points in \( \Delta \) that project to the same lattice point in \( \Delta_2 \) always give sections that belong to the same line bundle, since the line bundle depends only on \( m^{(F)}_j \).

This shows that the allowed monomials in any polytope dual to a stacked fibration construction over a base \( B \) take values as sections of various line bundles \( \mathcal{O}(-nK_B) \). For each vertex \( v^{(F)}_i \) of the 2D polytope \( \nabla_2 \), and for each fiber type \( F \), the number of lattice points in \( \Delta_2 \) corresponding to the resulting line bundle \( \mathcal{O}(-nK) \) is listed in the third column in Table 4.2. For points \( v^{(F)}_i \) in \( \nabla_2 \) that are not vertices, the numbers of such points will interpolate between the vertex values; the largest values of \( n \) are found from vertex stackings.

The line bundles in which the monomials take sections place constraints on the structure of the base. The order of vanishing of a section \( \sigma_n \in \mathcal{O}(-nK_B) \) over a
Figure 4-1: Different choices of the point $v_{s}^{(F)}$ used to specify a stacking construction are associated with different “twists” of the $F$-fiber bundle over the base $B$. The different choices of $v_{s}^{(F)}$ for a given fiber type give rise to monomials in the dual polytope that are sections of different line bundles over the base, illustrated here for three different choices of $v_{s}^{(F)}$ as vertices of the fiber $F_{10} = \mathbb{P}^{2,3,1}$. In the stacking construction, each lattice point in $\Delta_{2}$ is associated with a line bundle $\mathcal{O}(-v_{s}^{(F)} \cdot m_{j}^{(F)} + 1)K_{B})$, $m_{j}^{(F)} \in \Delta_{2}$. The dashed lines are normal to the corresponding $v_{s}^{(F)}$. The lattice points in $\Delta_{2}$ on the same dashed line are associated with sections of the same line bundle over the base. (cf. the $F_{10}$ data in Table 4.3 and Table 4.2.)
| $F$ | $v_s^{(F)}$ | $\{\#\text{pts}_{\Delta_2}(O(-nK_B))| n = 0, 1, 2, 3, 4, (5), 6\}$ | fibered-polytope | $B$ |
|-----|------------|--------------------------------------------------|-----------------|-----|
| $F_1$ | $(1,0), (0,1), (-1, -1)$ | $\{3, 3, 2, 1, 0, (0), 0\}$ | M:171 5 N:11 5 H:11,131 | $F_6$ |
| $F_2$ | $(1,0), (0,1), (0, -1), (-1,0)$ | $\{2, 3, 4, 0, 0, (0), 0\}$ | M:119 8 N:11 6 H:9,93 | $F_4$ |
| $F_3$ | $(1,0), (0,1)$ | $\{3, 3, 2, 1, 0, (0), 0\}$ | M:170 9 N:14 7 H:11,131 | $F_6$ |
| $F_4$ | $(1,0), (0,1), (-1, -1)$ | $\{3, 3, 2, 1, 0, (0), 0\}$ | M:90 8 N:11 6 H:10,76 | $F_4$ |
| $F_5$ | $(1,0), (0,1)$ | $\{2, 3, 3, 0, 0, (0), 0\}$ | M:169 13 N:17 9 H:11,131 | $F_6$ |
| $F_6$ | $(1,0), (0,1), (-1, -1)$ | $\{2, 3, 2, 1, 0, (0), 0\}$ | M:166 9 N:14 7 H:11,131 | $F_6$ |
| $F_7$ | $(1,1), (1,0), (0,1), (0, -1), (-1,0)$ | $\{2, 3, 2, 0, 0, (0), 0\}$ | M:88 16 N:15 10 H:10,76 | $F_4$ |
| $F_8$ | $(1,0), (0,1)$ | $\{2, 2, 3, 0, 0, (0), 0\}$ | M:106 8 N:14 6 H:9,93 | $F_8$ |
| $F_9$ | $(0,1), (0,1), (0,-1), (1,-1)$ | $\{2, 2, 2, 1, 0, (0), 0\}$ | M:144 14 N:19 10 H:12,120 | $F_4$ |
| $F_{10}$ | $(1,0), (0,1)$ | $\{2, 2, 2, 1, 0, (0), 0\}$ | M:125 5 N:17 5 H:11,131 | $F_6$ |
| $F_{11}$ | $(0,1), (0,-1)$ | $\{2, 2, 2, 1, 0, (0), 0\}$ | M:680 5 N:26 5 H:11,491 | $F_{12}$ |
| $F_{12}$ | $(0,1), (0,-1)$ | $\{2, 2, 2, 1, 0, (0), 0\}$ | M:51 9 N:16 7 H:11,59 | $F_4$ |
| $F_{13}$ | $(0,1), (0,-1)$ | $\{2, 2, 2, 1, 0, (0), 0\}$ | M:257 9 N:24 7 H:11,277 | $F_8$ |
| $F_{14}$ | $(1,0), (0,1)$ | $\{2, 2, 2, 1, 0, (0), 0\}$ | M:60 13 N:18 9 H:12,58 | $F_4$ |
| $F_{15}$ | $(1,0), (0,1)$ | $\{2, 2, 2, 1, 0, (0), 0\}$ | M:78 12 N:16 8 H:10,76 | $F_4$ |
| $F_{16}$ | $(1,0), (0,1)$ | $\{2, 2, 2, 1, 0, (0), 0\}$ | M:68 8 N:17 6 H:10,76 | $F_4$ |

Table 4.2: Line bundles in the $v_s$ stacking $F$-fibered construction, with examples over Hirzebruch surfaces $F_m$, where $-m$ saturates the negative curve bound in each case.
Table 4.3: Curves \( C \) with self-intersection \( C \cdot C \) that are allowed in the base of a stacked \( F \)-fibered polytope for the 16 fiber types \( F \). The numbers below the labels of the 16 fiber types count the numbers of the vertices of \( F \) that give vertex stacked-form fibrations where the corresponding curve can appear in the base. (Note that \(-3\) and \(-4\) curves are allowed in all cases, so the first and second rows give the total number of the vertices of a given fiber, and the most negative curve that can occur for a given fiber corresponds to the position of the last non-empty entry in the column.) The second column gives the orders of vanishing of \( \sigma_n \in \mathcal{O}(-nK) \) along \( C \), \( n = 1, 2, 3, 4, (5), 6 \) (none of the fibered polytopes has \( \mathcal{O}(-nK) \) for either \( n \geq 7 \) or \( n = 5 \)). A \((4,6)\) singularity arises along the whole curve unless there exists a section \( \sigma_n \in \mathcal{O}(-nK) \) such that \( \text{ord}_C(\sigma_n) < n \). The existence of such a section depends on the fiber type and the specified vertex of the base used for the stacking. Curves with \(-13 \leq C \cdot C \leq -3\) are considered (while curves \( C^2 \geq -2\) are always allowed since \( \{\text{ord}_C(\sigma_n)|n = 1, 2, 3, 4, (5), 6\} = \{0, 0, 0, 0, (0), 0\} \), there is always a \((4,6)\) singularity along the whole curve when \( C^2 \leq -13\) since \( \text{ord}_C(\sigma_n) = n \) for all \( n = 1, 2, 3, 4, 5, 6 \).

The orders of vanishing \( \{\text{ord}_C(\sigma_n)|n = 1, 2, 3, 4, (5), 6\} \) for each \( m \), \(-3 \geq m \geq -13\), are listed in the second column in Table 4.3. Note that none of the 16 fiber types gives a section of \( \mathcal{O}(-5K_B) \) (see the third column in Table 4.2).

For a Weierstrass model, where the coefficients \( f, g \) are sections of the line bundles \( \mathcal{O}(-4K_B) \) and \( \mathcal{O}(-6K_B) \), the Kodaira condition that a singularity have a Calabi-Yau resolution is that \( f, g \) cannot vanish to orders 4 and 6. For the more general class of fibrations we are considering here, the necessary condition is that at least one section \( \sigma_{n=1,2,3,4,(5),6} \) exists with \( \text{ord}_C(\sigma_n) < n \). This condition is necessary so that

\[
\text{ord}_C(\sigma_n) = \left\lceil \frac{n(m+2)}{m} \right\rceil.
\]

The calculation can be simply done by using the Zariski decomposition, along the lines of [10].
Table 4.4: For each \( m \), the minimal value of \( n \) such that a section \( \sigma_n \in \mathcal{O}(-nK_B) \) exists preventing \((4,6)\) points over a curve of self-intersection \( m \). Note that since there are no \( \sigma_n \)s in any cases (see the third column in Table 4.2), \( \min(n) \) jumps from 4 to 6 between \( m = -8 \) and \( m = -9 \).

When the sections are combined to make a Weierstrass form, the resulting \( f, g \) give either a section in \( \mathcal{O}(-4K_B) \) or a section in \( \mathcal{O}(-6K_B) \), respectively, whose order of vanishing does not exceed 4 or 6. Note that as the absolute value \( |m| \) of the self-intersection of the curve \( C \) increases, the minimal \( n \) that satisfies \( \text{ord}_C(\sigma_n) < n \) is non-decreasing. The minimum value \( \min(n) \) so that this condition is satisfied is listed for each \( m \) in Table 4.4. Therefore, given a fiber type \( F \) with a specified point \( v_F \), the allowed negative curves in the base that are allowed for a stacking construction using the point \( v_s^{(F)} \) that gives a resolvable Calabi-Yau construction are such that the following two conditions are satisfied: the existence of a section \( \sigma_{n=1,2,3,4,6} \) such that (1) \( \sigma_n \in \mathcal{O}(-(v_s^{(F)} \cdot m_j^{(F)} + 1)K_B) \) and (2) \( \text{ord}_C(\sigma_n) < n \). For each fiber type \( \nabla_2 \), we have considered the stacking constructions over each vertex. The most negative self-intersection curve that is allowed in the base for each fiber type is tabulated in the last non-empty entry in the corresponding column in Table 4.3, and a \( v_s^{(F)} \) that gives rise to stacked fibrations in which the most negative curve is allowed, and the corresponding line bundles associated with lattice points in \( \Delta_2 \) are given in Appendix A.2. Note that since for any lattice point in \( \Delta_2 \), the largest value of \( n \) such that for any choice of stacking point \( v_s^{(F)} \) the corresponding points in \( \Delta \) are sections of \( \mathcal{O}(-nK_B) \) arises from a vertex, it is sufficient to consider the maximum \( n \) across the possible choices of vertices \( v_s^{(F)} \).

This analysis shows that any polytope that has the stacked form with a given fiber type \( F \) gives a genus one fibration over a base \( B \) in which the self-intersection of the curves has a lower bound given by the last nonempty entry in the corresponding column of Table 4.3. For the fiber \( F_{10} \), this bound is more general. It is not possible to find any elliptic fibration with a smooth Calabi-Yau resolution over a base that contains curves of self-intersection \( C \cdot C < -12 \). While we have not proven it for
polytopes that do not have the stacking form described here, it seems plausible to conjecture that the bounds on curves in the base for each fiber type given in Table 4.3 will also hold for arbitrary fibrations (i.e. for general "twists" of the fibration that do not have the stacking type). We have not encountered any cases in our analysis that would violate this conjecture. And it is straightforward to see using the analysis done here already that these curve bounds will still hold when there is a coordinate system where each ray of the base has a pre-image living over some ray \( v_F \in \nabla_2 \), even when these rays are not all the same \( v_s^{(F)} \) as in the stacking case, since the bound applying for each curve will match that of some choice of \( v_s^{(F)} \). If the more general conjecture is correct, then, for example, it would follow in general that any reflexive polytope with a fiber \( F_4 \) can only have curves in the base of self-intersection \( \geq -8 \), those with a fiber \( F_1 \) can only have curves in the base of self-intersection \( \geq -6 \), etc. We leave, however, a general proof of this assertion to further work.

4.1.4 Explicit Construction of Reflexive Polytopes from Stackings

In Chapter 3, we showed that the standard stacking construction with the fiber \( \mathbb{P}^{2,3,1} \), combined with a large class of Tate-form Weierstrass tunings, can be used to explicitly construct a large fraction of the reflexive polytopes in the Kreuzer-Skarke database at large Hodge numbers. The stacking construction with other fibers can be used similarly to construct other reflexive polytopes in the KS database.

Explicitly, given the negative curve bounds on the base determined above, we can construct a stacked \( F \)-fibered polytope over \( B \) as follows, following a parallel procedure to that described in Chapter 3 for the \( \mathbb{P}^{2,3,1} \)-fibered standard stackings: Given a fiber \( F \) with a specified ray \( v_s^{(F)} \), and a smooth 2D toric base \( B \) in which the self-intersections of all curves are not lower than the negative curve bound associated with \( v_s^{(F)} \), we start with the minimal fibered polytope \( \tilde{\mathcal{V}} \subset N \) (which may not be reflexive) that is the convex hull of the set in equation (4.2). If \( \tilde{\mathcal{V}} \) is reflexive, then we are done; otherwise we adopt the "dual of the dual" procedure used in §3.3.1 to
resolve $\hat{\nabla}$: define $\Delta^\circ = \text{convex hull}(\nabla)^* \cap M$. As long as the negative curve bound is satisfied (no (4,6) curves), $\Delta^\circ$ is a reflexive polytope, and the resolved polytope in $N$ is $\nabla \equiv (\Delta^\circ)^*$.

Explicit examples of $F$-fibered polytopes over Hirzebruch surfaces $\mathbb{F}_m$ are given in Table 4.2, for each fiber type $F$. The base $\mathbb{F}_m$ is in each case chosen such that $-m$ saturates the negative curve bound associated with the specific vertex $v_s^{(F)}$ for a given fiber type (see Appendix A.2 for the possible choices of $v_s^{(F)}$ for each fiber type that allow the most negative self-intersection curves in the base). For example, the standard stacked $\mathbb{P}^{2,3,1}$-fibered polytopes considered in Chapter 3 have bases stacked over the vertex $(-3, -2)$ of the fiber $F_{10}$ in Appendix A.1, and there exist sections in $\mathcal{O}(-nK_B)$ for $n = 1, 2, 3, 4, 6$ (see Figure (c) in Table 4-1), so models in this class correspond naturally to the Tate-form Weierstrass models where $a_n = \sigma_n$, and the negative curve bound is $-12$. The model listed in Table 4.2 is the generic elliptically fibered CY over $\mathbb{F}_{12}$.

The construction just described above gives the minimal reflexive $F$-fibered polytope over $B$ that contains the set in equation (4.2). While the $F_{10}$ fiber type with $v_s^{(F)} = (-3, -2)$ gives the most generic elliptic Calabi-Yau over any given toric base $B$ through this construction, using the other fiber types or the other specified points of $F_{10}$ for stacked stacking polytopes give models with enhanced symmetries (these can include discrete, abelian, and non-abelian symmetries). Further tunings of the polytope analogous to Tate-tunings for the standard $\mathbb{P}^{2,3,1}$ polytope can reduce $\Delta$ and enlarge $\nabla$, giving a much larger class of reflexive polytopes for Calabi-Yau threefolds. The explicit construction of the polytopes corresponding to Tate tuned models via polytope tunings of the standard $F_{10}$-fibered polytope with $v_s^{(F)} = (-3, -2)$ were discussed in §3.3. We have not attempted systematic polytope tunings for the other fiber types, but in principle one can work out tuning tables analogous to the Tate table for the other fiber types.
4.1.5 Lattice Automorphism and Fibrations

Lattice automorphisms are elements of $GL(n, \mathbb{Z})$ group that leave the polytope invariant; each such automorphism acts as a permutation on the set of vertices. In some cases the number of fibrations is enhanced by the existence of automorphism symmetries of the polytope. While a generic polytope has no symmetries, some polytopes with large numbers of fibrations also have many symmetries. In such cases the number of inequivalent fibrations can be smaller than the total number of fibrations. This issue is also addressed in [48, 26].

We illustrate the symmetry and fibration structure of polytopes with an example of the polytope data $M:7$ $N:201$ $H:149,1$ $[[296]]$ associated with the Calabi-Yau having Hodge numbers $(149, 1)$. This is the polytope with the largest number of fibrations (including multiplicities in orbits of automorphism symmetries).

The polytope $\nabla$ in question has five vertices:

\begin{align*}
A &= (1, -1, -1, -1) \quad (4.6) \\
B &= (-1, -1, -1, -1) \quad (4.7) \\
C &= (-1, -1, -1, 7) \quad (4.8) \\
D &= (-1, -1, 7, -1) \quad (4.9) \\
E &= (-1, 7, -1, -1). \quad (4.10)
\end{align*}

These vertices satisfy the linear condition

$$4A + B + C + D + E = 0. \quad (4.11)$$

The possible symmetries allowed by this equation include all permutations on the vertices $B, C, D, E$. The polytope is clearly symmetric under all permutations on $C, D, E$, as these can be realized by permutations on the axes 2, 3 and 4. One can also check that the polytope is symmetric under the linear transformation that swaps
B and C while leaving D and E fixed,

\[ T = \begin{pmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \]  

(4.12)

This matrix in \( SL(2, \mathbb{Z}) \) satisfies (acting on the right on row vectors)

\[ B \cdot T = C, C \cdot T = B, A \cdot T = A, D \cdot T = D, E \cdot T = E, \]  

(4.13)

and is thus a symmetry of the polytope. This shows that all 24 permutations on \( B, C, D, E \) are symmetries.

Explicitly, let the column vectors \( a, b, c, d, e \) be defined as

\[ a = (1, 0, 0, 0)^T \]  

(4.14)

\[ b = (0, 1, 0, 0)^T \]  

(4.15)

\[ c = (0, 0, 1, 0)^T \]  

(4.16)

\[ d = (0, 0, 0, 1)^T \]  

(4.17)

\[ e = (-4, -1, -1, -1)^T, \]  

(4.18)

which are the five vertices of the \( \Delta \) polytope. The 24 linear transformation matrices that leave the \( \nabla \) polytope invariant are

\[ \{(a, b, c, d), (a, b, c, e), (a, b, e, d), (a, b, d, e), (a, b, e, c), (a, b, d, c), (a, c, b, d), (a, c, b, e), (a, e, b, d), (a, d, b, e), (a, e, b, c), (a, d, b, c), (a, c, e, d), (a, c, d, e), (a, e, c, d), (a, d, c, e), (a, e, d, c), (a, d, e, c), (a, c, e, b), (a, c, d, b), (a, e, c, b), (a, d, c, b), (a, e, d, b), (a, d, e, b)\}. \]

The different fibrations go into orbits of this 24-element symmetry group. For example, there are 12 \( F_3 \) fibers; one of them is \( \{(0, -1, -1, 0), (0, -1, 1, -1), (0, 0, 2, -1), (0, 1, -1, 1)\}, \)
and all the 12 fibers are generated by

\{(a, b, c, d), (a, b, c, e), (a, c, b, d), (a, c, b, e), (a, d, b, c), (a, e, b, c), (a, b, d, c), (a, b, e, c),
(a, b, e, d), (a, b, d, e), (a, d, b, e), (a, e, b, d)\}.

4.2 Results at Large Hodge Numbers

We have systematically run the algorithm described in Section 4.1.2 to check for a manifest elliptic or genus one fibration realized through a reflexive 2D fiber for each polytope in the Kreuzer-Skarke database that gives a Calabi-Yau threefold \(X\) with \(h^{1,1}(X)\) or \(h^{2,1}(X)\) greater or equal to 140. The number of polytopes that give rise to Calabi-Yau threefolds with \(h^{1,1} \geq 140\) is 248305. Since the set of reflexive polytopes is mirror symmetric (Hodge numbers \(h^{1,1}, h^{2,1}\) are exchanged in going from \(\nabla \leftrightarrow \Delta\)), this is also the number of polytopes with \(h^{2,1} \geq 140\). (Note, however, that the mirror of an elliptic Calabi-Yau threefold is not necessarily elliptic.) There are 495515 polytopes with at least one of the Hodge numbers at least 140, and from these numbers it follows that the number of polytopes with both Hodge numbers at least 140 is 1095. While as described in Section 4.1.2, we have made the algorithm reasonably efficient for larger values of \(h^{1,1}\), our implementation in this initial investigation was in Mathematica, so a complete analysis of the database using this code was impractical. We anticipate that in the future a complete analysis of the rest of the database can be carried out with a more efficient code, but our focus here is on identifying the largest values of \(h^{1,1}, h^{2,1}\) that are associated with polytopes that give Calabi-Yau threefolds with no manifest elliptic fiber. In §4.3 we analyze the distribution of fibrations at small \(h^{1,1}\).

4.2.1 Calabi-Yau Threefolds without Manifest Genus One Fibers

Of the 495515 polytopes analyzed at large Hodge numbers, we found that only four lacked a 2D reflexive polytope fiber, and thus the other 495511 polytopes all lead to Calabi-Yau threefolds with a manifest genus one fiber. The Hodge numbers of the
four Calabi-Yau threefolds without a manifest genus one fiber are

$$(h^{1,1}, h^{2,1}) = (1, 149), (1, 145), (7, 143), (140, 62).$$  (4.19)

(See Figure 4-2.) It is of course natural that any Calabi-Yau threefold with $h^{1,1} = 1$ cannot be elliptically fibered; by the Shioda-Tate-Wazir formula [106], any elliptically fibered Calabi-Yau threefold must have at least $h^{1,1} = 2$, with one contribution from the fiber and at least one more from $h^{1,1}$ of the base, which must satisfy $h^{1,1}(B) \geq 1$.

We also expect that any genus one fibered CY3 will have at least a multi-section [49, 50], so $h^{1,1} \geq 2$ in these cases for similar reasons.

The examples $(1, 145)$ and $(1, 149)$ are the only Hodge numbers from polytopes in the Kreuzer-Skarke database with $h^{1,1} = 1, h^{2,1} \geq 140$. Note that the quintic, with Hodge numbers $(1, 101)$, is another example of a Calabi-Yau threefold with $h^{1,1} = 1$ that has no elliptic or genus one fibration.

We list here the polytope structure of the two examples from (4.19) that have $h^{1,1} > 1$, in the form given in the Kreuzer-Skarke database. $M$ refers to the numbers of lattice points and vertices of the dual polytope $\Delta$, while $N$ refers to the numbers of lattice points and vertices of the polytope $\nabla$, and $H$ refers to the Hodge numbers $h^{1,1}$ and $h^{2,1}$. The vectors listed are the vertices of the polytope in the $N$ lattice. The
numbers in parentheses for each polytope refer to the position in the list of polytopes in the Kreuzer-Skarke database that give CY3s with those specific Hodge numbers.

- M:196 5 N:10 5 H:7,143 (1\textsuperscript{st}/54)
  Vertices of $\nabla$: \{(-1, 4, -1, -2), (-1, -1, 1, 1), (1, -1, 0, 0), (-1, -1, 0, 1), (-1, -1, 0, 3)\}

- M:88 8 N:193 9 H:140,62 (6\textsuperscript{th}/255)
  Vertices of $\nabla$: \{(-1, 2, -1, 4), (-1, 0, 4, -1), (1, -1, -1, -1), (-1, -1, -1, 19),
  (-1, -1, 5, 1), (-1, 1, 0, -1), (-1, 1, -1, -1), (-1, -1, -1, -1), (-1, -1, 5, -1)\}

Note that we have not proven that these Calabi-Yau threefolds do not have elliptic or genus one fibers, we have just found that such fibers do not appear in a manifest form from the structure of the polytope. We leave for further work the question of analyzing non-toric elliptic or genus one fibration structure of these examples, or others with smaller Hodge numbers that also lack a manifest genus one fiber; such an analysis might be carried out using methods similar to those of [26].

4.2.2 Calabi-Yau Threefolds without Manifest Elliptic Fibers

Of the 495515 polytopes analyzed, only 384 had fibers of only types $F_1, F_2, F_4$. These cases are associated with genus one fibered Calabi-Yau threefolds that have no manifest toric section, and therefore are not necessarily elliptically fibered. Note that we have not proven that these Calabi-Yau threefolds do not have elliptic fibers; in fact, many toric hypersurface Calabi-Yau threefolds have been found to have non-toric fibrations [41]. It would be interesting to study these examples further for the presence of non-toric sections.

The largest values of $h^{2,1}$ and $h^{1,1}$ for these genus one fibered Calabi-Yau threefolds without a manifest toric section are realized by the examples:

- M:311 5 N:15 5 H:11, 227 (1\textsuperscript{st}/19)
  Vertices of $\nabla$: \{(-1, 0, 4, -3), (-1, 2, -1, 0), (1, -1, -1, 1), (-1, 0, -1, 1), (-1, 0, -1, 3)\}
The fiber type $F_4$ is the only fiber that arises in these five polytopes. In the first case, with Hodge numbers $(11, 227)$, the base of the elliptic fibration is the Hirzebruch surface $F_8$. Analysis of the F-theory physics of the genus one fibration associated with this polytope suggests that there should in fact be an elliptic fiber with a non-toric global section. For further work, it would be nice to prove this and find the non-toric section explicitly. Further analysis of the F-theory physics of the other cases may also be interesting, as well as the question of whether these threefolds admit elliptic fibrations that are not manifest in the toric description.

---

5In the F-theory analysis, we consider the Jacobian fibration associated with the $F_4$ fibration. This is an elliptic fibration with a section, for which a detailed analysis shows that there are no further enhanced non-abelian gauge symmetries. There are, however, 150 nodes in the $I_1$ component of the discriminant locus in the base. Since the generic elliptic fibration model over $F_8$ has Hodge numbers $(10, 376)$, this analysis suggests that there should be an additional section in this case, which should correspond to a non-toric section in the original polytope and in the Jacobian model would give rise to a $U(1)$ abelian factor where the 150 nodes correspond to matter fields charged under the $U(1)$; the anomaly cancellation condition is satisfied for the resulting Jacobian model, matching with the shift in Hodge numbers $(10, 376) + (1, 1 - 150) = (11, 227)$. 

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- Vertices of $\nabla$:

- $7^{th} \{(-1, 0, 4, -1), (-1, 2, -1, -1), (1, -1, -1, -1), (-1, -1, 1, -1), (-1, -1, 6, -1), (-1, 1, 0, 6), (-1, -1, -1, 28), (-1, 1, -1, 10), (-1, -1, 6, 0)\}$

- $8^{th} \{(-1, 0, 4, -1), (-1, 2, -1, -1), (1, -1, -1, -1), (-1, -1, -1, -1), (-1, -1, 6, -1), (-1, 1, 0, 6), (-1, -1, -1, 28), (-1, 0, -1, 19), (-1, -1, 6, 0)\}$

- $9^{th} \{(-1, 0, 4, -1), (-1, 2, -1, -1), (1, -1, -1, -1), (-1, -1, -1, -1), (-1, -1, 5, -1), (-1, 1, 0, 6), (-1, -1, -1, 28), (-1, 1, -1, 10), (-1, -1, 5, 4)\}$

- $10^{th} \{(-1, 0, 4, -1), (-1, 2, -1, -1), (1, -1, -1, -1), (-1, -1, -1, -1), (-1, -1, 5, -1), (-1, 1, 0, 6), (-1, -1, -1, 28), (-1, 0, -1, 19), (-1, -1, 5, 4)\}$
4.2.3 Fiber Types

The numbers of distinct polytopes in the regions $h^{1,1}, h^{2,1} \geq 140$ that have each fiber type (not counting multiplicities) are

<table>
<thead>
<tr>
<th>Fiber Type</th>
<th>$F_1$</th>
<th>$F_2$</th>
<th>$F_3$</th>
<th>$F_4$</th>
<th>$F_5$</th>
<th>$F_6$</th>
<th>$F_7$</th>
<th>$F_8$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>612</td>
<td>1</td>
<td>1279</td>
<td>40218</td>
<td>32</td>
<td>19907</td>
<td>20</td>
<td>8579</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Fiber Type</th>
<th>$F_9$</th>
<th>$F_{10}$</th>
<th>$F_{11}$</th>
<th>$F_{12}$</th>
<th>$F_{13}$</th>
<th>$F_{14}$</th>
<th>$F_{15}$</th>
<th>$F_{16}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>2067</td>
<td>487387</td>
<td>24811</td>
<td>850</td>
<td>27631</td>
<td>2438</td>
<td>273</td>
<td>58</td>
</tr>
</tbody>
</table>

In Appendix A.3, we have included a set of figures that show the distribution of polytopes containing each fiber type, according to the Hodge numbers of the associated Calabi-Yau threefolds. We have shaded the data points of Hodge pairs varying from light to dark with increasing multiplicities; two factors contribute to the multiplicity in these figures: the multiplicity of the polytopes associated with the same Hodge pair and the multiplicity of fibers of the same type for a given polytope (note that the latter multiplicity is not included in the numbers in the table above). We discuss multiple fibrations in the next subsection.

We can see some interesting patterns in the distribution of polytopes with different fiber types. As discussed in §4.1.3, at least for polytopes with the stacked fibration form, the only fiber type that can arise over a base with a curve of self-intersection less than $-8$ is the $\mathbb{P}^{2,3,1} (F_{10})$ fiber (see Table 4.3). From the graphs in Appendix A.3, it is clear that this fiber dominates at large Hodge numbers. The other fiber types that can arise over a base with a curve of self-intersection less than $-6$ are $F_4, F_{13}$ (with two possible specified vertices) and $F_6, F_8, F_{11}$ (with only one specified vertex). The Hodge numbers of Calabi-Yau threefolds coming from polytopes with fiber types $F_4, F_6, F_8$ extend to $h^{1,1} = 263$, and $F_{11}$ extends to $h^{1,1} = 377$; in fact, the right most data point of the fiber types $F_4, F_6, F_8, F_9, F_{12}, F_{15}$ is the same: $\{263, 23\}$, and the right most data point of the fiber types $F_{11}$ and $F_{14}$ is the same: $\{377, 11\}$. The fiber $F_{13}$ also continues out to the largest values of $h^{1,1}$ as $F_{10}$ does. Since the largest value of $h^{1,1}$ for a generic elliptic fibration over a toric base $B$ containing no
curves of self-intersection < -8 is 224 [11, 56, 36], these large values of $h^{1,1}$ for fibers other than $F_{10}$ must involve tuning of relatively large gauge groups.

For $h^{1,1} > 377$ the only fibers that arise are $F_{10}$ and $F_{13}$. In fact, the Calabi-Yau threefold with the largest $h^{1,1}$, which has Hodge numbers $(491, 11)$, has two distinct fibrations: one has the standard $\mathbb{P}^{2,3,1}$ fiber over the 2D toric base $\{-12// -11// -12// -12// -12// -12// -12// -12// -12// -12// -12// -12// -11// -12// -11// -12, 0\}$, represented by the self-intersection numbers of the toric curves, where $//$ stands for the sequence $-1, -2, -2, -3, -1, -5, -1, -3, -2, -2, -1$; the other fibration has the fiber $F_{13}$ over the base $\{-4, -1, -3, -1, -4, -1, -4, -1, -4, 0, 2\}$. We leave a more detailed analysis of the alternative fibration of this Calabi-Yau threefold for future work.

On the other hand, the fiber $F_2$, which is most restricted, arises from only one $\nabla$ polytope, with multiplicity one: $M: 40 6 N: 186 6 H: 149, 29$, which also has two different $F_{10}$ subpolytopes.

These observations tell us that, as we might expect, $h^{1,1}$ extends further for the fiber subpolytopes that admit more negative curves in the base. Almost half of the fiber types do not arise for any polytopes at all in the region $h^{2,1} \geq 140$: $F_2, F_5, F_7, F_{12}, F_{14}, F_{15}$, and $F_{16}$. None of these is allowed over any base with a curve of self-intersection less than $-6$ (at least in the stacking construction of §4.1.3).

### 4.2.4 Multiple Fibrations

Another interesting question is the prevalence of multiple fibrations. This question was investigated for complete intersection Calabi-Yau threefolds in [25, 26], where it was shown that many CICY threefolds have a large number of fibrations. In the toric hypersurface context we consider here, a polytope can have both multiple fibrations by different fiber types and by the same fiber type. In this analysis, as in the rest of this chapter, we consider only fibrations that are manifest in the toric description. We have found that the total number of (manifest) fibrations in a polytope in the two large Hodge number regions ranges from zero to 58. The total numbers of fibrations and the number of polytopes that have each number of total fibrations are listed in
# Table 4.5: Table of the number of polytopes in the large Hodge number regions $h'^1, h'^2 > 140$ that have a given number of distinct (manifest) fibrations. Numbers in parentheses are after modding out by automorphism symmetries (see §4.1.5).

Table 4.5.

There are 16 polytopes in the region $h'^1 \geq 140$ or $h'^2 \geq 140$ with a non-trivial action of the automorphism symmetry on the fibers. We list these 16 polytopes in Appendix A.4. For example, the polytope giving a Calabi-Yau with Hodge numbers $(149, 1)$ has an automorphism symmetry of order 24, associated with an arbitrary permutation on 4 of the 5 vertices of the polytope; the number of distinct classes of fibrations modulo automorphisms in this case is reduced to only 8 instead of 58 (see §4.1.5).

The polytopes that we have found with a large total number of (manifest) fibrations are generally in the large $h'^1$ region; in fact, polytopes in the large $h'^2$ region have at most three fibrations:

<table>
<thead>
<tr>
<th># total fibrations</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td># polytopes with large $h'^2$</td>
<td>3</td>
<td>240501</td>
<td>7775</td>
<td>26</td>
</tr>
</tbody>
</table>

The four polytopes with the two largest numbers of total fibrations (58, 37 without
modding out by automorphisms) are respectively

\[
\{\{7,5,201,5,149,1,296\}, \{0,0,12,12,0,0,0,12,0,0,15,0,3,4\}\}
\]

and

\[
\{\{7,5,196,5,145,1,288\}, \{0,0,6,0,6,0,0,12,0,0,9,3,0,1\}\}
\]

\[
\{\{8,6,195,7,144,2,284\}, \{0,0,6,0,6,0,0,12,0,0,9,3,0,1\}\},
\]

\[
\{\{9,7,192,10,144,4,280\}, \{0,0,0,0,9,0,0,0,15,3,0,6,3,0,1\}\},
\]

where the numbers are in the format

\[
\{\# \text{ lattice points of } A, \# \text{ vertices of } A, \# \text{ lattice points of } V, \# \text{ vertices of } V, h^{1,1}, h^{2,1}, \text{Euler Number}\}, \{\#F_1, \#F_2, \ldots, \#F_{16}\}\}.
\]

Note that the first two polytopes are, respectively, the mirrors of the first two polytopes (with \(h^{1,1} = 1\)) without any fibrations in equation (4.19).

We also note that in general, the polytopes with larger numbers of total manifest fibrations fall within a specific range of values of \(h^{1,1}\) and \(h^{2,1}\) (at least in the ranges we have studied here). The ranges of \(h^{1,1}\) and \(h^{2,1}\) of the polytopes that have 8 or more fibrations (without considering automorphisms) are listed in Table 4.6. It may be interesting to note that in a somewhat different context, it was found in [84] that a large multiplicity of elliptically fibered fourfolds arises at a similar locus in the space of Hodge numbers, at intermediate values of \(h^{1,1}\) and small values of \(h^{3,1}\) (which counts the number of complex structure moduli, as does \(h^{2,1}\) for Calabi-Yau threefolds). It would be interesting to understand whether these observations stem from a common origin.

It is also interesting to note that while every Calabi-Yau threefold with \(h^{1,1} > 335\) or \(h^{2,1} > 256\) has more than one fibration, the polytopes associated with the largest values of \(h^{1,1}\) have precisely two manifest fibrations, and the average number of fibrations at large \(h^{1,1}\) is close to 2. In Figure 4-3, we show the average number of fibrations for the polytopes associated with Calabi-Yau threefolds of Hodge numbers
Table 4.6: Ranges of Hodge numbers in which the polytopes with the largest numbers of fibrations (not including automorphisms) are localized.

\[ h^{1,1} \geq 140. \]

The maximal number of fibrations for each specific fiber type in a polytope is

\[
\begin{array}{cccccccccccc}
F_1 & F_2 & F_3 & F_4 & F_5 & F_6 & F_7 & F_8 & F_9 & F_{10} & F_{11} & F_{12} & F_{13} & F_{14} & F_{15} & F_{16} \\
4 & 1 & 12 & 12 & 2 & 9 & 1 & 4 & 4 & 15 & 4 & 2 & 15 & 6 & 3 & 4 \\
(4) & (1) & (6) & (8) & (2) & (9) & (1) & (4) & (4) & (15) & (4) & (2) & (9) & (6) & (3) & (1)
\end{array}
\]

Numbers in parentheses are after modding out by automorphism symmetries; for example, the maximal number of \(F_{16}\) fibers, which comes from the polytope associated with the Hodge pair (149,1), reduces from four to one (see the last row of the table in Appendix A.4).

If we count the distinct fiber types in a polytope, we find that the maximum number of fiber types that a polytope in the large Hodge number regions can have is eight. The eight polytopes that have the maximum number of eight distinct fiber
Figure 4-3: Average number of fibrations for polytopes associated with Calabi-Yau threefolds with $h^{1,1} \geq 140$.

In Table 4.7, we show the distribution of all polytopes, polytopes with large $h^{1,1}$, and polytopes with large $h^{2,1}$ according to the number of distinct fiber types. There are at most three distinct fiber types in the polytopes in $h^{2,1} \geq 140$. While all fiber types occur in the large $h^{1,1}$ region, the only fiber types that occur in the large $h^{2,1}$ region are $F_1, F_3, F_4, F_6, F_8, F_{10}, F_{11}$, and $F_{13}$.

Finally, it is interesting to note that only the plot of $F_{10}$ in Appendix A.3 seems to exhibit mirror symmetry to any noticeable extent. We do not expect elliptic
fibrations to respect mirror symmetry, so this may simply arise from a combination of the observation that the total set of hypersurface Calabi-Yau Hodge numbers in the Kreuzer-Skarke database is mirror symmetric and the observation that in the large Hodge number regions that we have considered most of the Calabi-Yau threefolds admit elliptic fibrations described by a $F_{10}$ fibration of the associated polytope.

### 4.2.5 Standard vs. Non-standard $\mathbb{P}^{2,3,1}$-fibered Polytopes

In Chapter 3, we compared elliptic and toric hypersurface Calabi-Yau threefolds with Hodge numbers $h^{1,1} \geq 240$ or $h^{2,1} \geq 240$. We found that in the large $h^{1,1}$ region, there were eight Hodge pairs in the KS database that were not realized by a simple Tate-tuned model, and do not correspond to a “standard stacking” $\mathbb{P}^{2,3,1}$-fibered polytope. We found, however, that these eight outlying polytopes have a description in terms of a $\mathbb{P}^{2,3,1}$ fiber structure that is not of the standard $(v_s^{(F)} = (-3, -2))$ stacking form, and furthermore it can be seen do not respect the stacking framework of §4.1.3. The Weierstrass models of these Calabi-Yau threefolds all have the novel feature that they can have gauge groups tuned over non-toric curves, which can be of higher genus, in the base. As discussed in Chapter 3, the definition of a standard $\mathbb{P}^{2,3,1}$-fibered polytope $\nabla$ (where the base is stacked over the vertex $(-3, -2)$ of the $F_{10}$ fiber) turns out to be equivalent to the condition that the corresponding $\Delta$ has a single lattice point for

<table>
<thead>
<tr>
<th># distinct fiber types</th>
<th># polytopes</th>
<th># polytopes with $h^{1,1} \geq 140$</th>
<th># polytopes with $h^{2,1} \geq 140$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>4</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>393788</td>
<td>153601</td>
<td>229443</td>
</tr>
<tr>
<td>2</td>
<td>86008</td>
<td>78995</td>
<td>6460</td>
</tr>
<tr>
<td>3</td>
<td>13354</td>
<td>13347</td>
<td>7</td>
</tr>
<tr>
<td>4</td>
<td>1755</td>
<td>1755</td>
<td>-</td>
</tr>
<tr>
<td>5</td>
<td>469</td>
<td>469</td>
<td>-</td>
</tr>
<tr>
<td>6</td>
<td>112</td>
<td>112</td>
<td>-</td>
</tr>
<tr>
<td>7</td>
<td>17</td>
<td>17</td>
<td>-</td>
</tr>
<tr>
<td>8</td>
<td>8</td>
<td>8</td>
<td>-</td>
</tr>
</tbody>
</table>

Table 4.7: Distribution of polytopes by number of distinct fiber types
Figure 4-4: Hodge pairs with only non-standard $F_{10}$-fibered polytopes. The grey dots correspond to all Hodge pairs with $F_{10}$ fibers. The black dots correspond to Hodge pairs with only non-standard $F_{10}$-fibered polytopes. The vertical and horizontal dashed line correspond to $h^{1,1} = 240$ and $h^{2,1} = 240$, respectively.

<table>
<thead>
<tr>
<th></th>
<th>total # fibrations</th>
<th># fibrations in polytopes with $h^{1,1} \geq 140$</th>
<th># fibrations in polytopes with $h^{2,1} \geq 140$</th>
</tr>
</thead>
<tbody>
<tr>
<td>standard</td>
<td>433827</td>
<td>242652</td>
<td>192218</td>
</tr>
<tr>
<td>non-standard</td>
<td>183818</td>
<td>130255</td>
<td>53705</td>
</tr>
<tr>
<td>non-standard fraction</td>
<td>0.297611</td>
<td>0.349381</td>
<td>0.218381</td>
</tr>
</tbody>
</table>

Table 4.8: Fractions of fibrations by the fiber $F_{10}$ that take the "standard stacking" form versus other fibrations.

Of the choices $m_2^{(F)} = (1, -1)$ and $m_3^{(F)} = (-1, 2)$ in equation (4.3) (where we have numbered the vertex with the largest multiple of $-K_B$ as $m_1^{(F)} = (-1, -1)$), and there is furthermore a coordinate system in which this lattice point has coordinates $m^{(2)} = (0, 0)$ in both cases. We have scanned through the $F_{10}$-fibered polytopes and used this feature to compute the fraction of $F_{10}$-fibered polytopes that have the standard versus non-standard form; the results of this analysis are shown in Table 4.8.

Of the $488119$ $F_{10}$-fibered polytopes, $98758$ have more than one $F_{10}$ fiber. Most of these polytopes have both standard and non-standard types of fibrations. There are $103$ Hodge pairs that have only the non-standard fibered polytopes. These may give rise to more interesting Weierstrass models, like those we have studied with $h^{1,1} \geq 240$.
in §3.6.2. As a crosscheck to the "sieving" results there, we have confirmed that none of these 103 Hodge pairs are in the region $h^{2,1} \geq 240$, and the 12 Hodge pairs of these 103 pairs that have $h^{1,1} \geq 240$ are exactly the Hodge pairs associated with non-standard $\mathbb{P}^{2,3,1}$-fibered polytopes in Table 3.15, together with the four Hodge pairs of $\text{Bl}_{[0,0,1]}\mathbb{P}^{2,3,1}$-fibered polytopes; the latter four polytopes, in other words, happen to also be $F_{11}$-fibered, and can be analyzed as blowups of standard $\mathbb{P}^{2,3,1}$ model $(U(1)$ models). We list the remaining 91 Hodge pairs that only have non-standard $\mathbb{P}^{2,3,1}$ fiber types below (see also Figure 4-4):

- $140 \leq h^{1,1} < 240$
  $\{\{149, 1\}, \{154, 7\}, \{179, 8\}, \{177, 16\}, \{179, 22\}, \{207, 22\}, \{235, 22\}, \{184, 23\}, \{228, 24\}$, \{178, 27\}, \{206, 27\}, \{177, 28\}, \{205, 28\}, \{211, 38\}, \{232, 38\}, \{233, 38\}, \{182, 39\}, \{217, 39\}$, \{223, 40\}, \{194, 41\}, \{221, 41\}, \{210, 43\}, \{203, 44\}, \{174, 45\}, \{207, 45\}, \{145, 46\}, \{193, 46\}$, \{205, 46\}, \{159, 48\}, \{180, 49\}, \{187, 53\}, \{239, 53\}, \{150, 55\}, \{231, 55\}, \{225, 57\}, \{231, 57\}$, \{204, 63\}, \{231, 63\}, \{175, 64\}, \{237, 65\}, \{141, 66\}, \{208, 66\}, \{228, 66\}, \{199, 67\}, \{211, 67\}$, \{193, 69\}, \{201, 69\}, \{190, 70\}, \{200, 70\}, \{161, 71\}, \{160, 73\}, \{190, 76\}, \{214, 82\}, \{185, 83\}$, \{198, 84\}, \{181, 85\}, \{193, 85\}, \{229, 85\}, \{164, 86\}, \{200, 86\}, \{160, 88\}, \{185, 93\}, \{177, 101\}, \{197, 101\}, \{148, 102\}, \{171, 105\}, \{147, 119\}, \{141, 123\}, \{140, 126\}$

- $140 \leq h^{2,1} < 240$
  $\{\{3, 141\}, \{3, 165\}, \{3, 195\}, \{4, 142\}, \{4, 148\}, \{4, 154\}, \{4, 162\}, \{4, 166\}, \{4, 178\}, \{5, 141\}$, \{5, 143\}, \{5, 149\}, \{5, 153\}, \{11, 176\}, \{22, 217\}, \{23, 182\}, \{23, 200\}, \{24, 183\}, \{31, 170\}$, \{95, 155\}, \{110, 144\}, \{111, 141\}$.

### 4.3 Fibration Prevalence as a Function of $h^{1,1}(X)$

In this section we consider the fraction of Calabi-Yau threefolds at a given value of the Picard number $h^{1,1}(X)$ that admit a genus one or elliptic fibration. We begin in §4.3.1 with a summary of some analytic arguments for why we expect that an increasingly small fraction of Calabi-Yau threefolds will fail to have such a fibration as $h^{1,1}$ increases; we then present some preliminary numerical results in §4.3.2.
4.3.1 Cubic Intersection Forms and Genus One Fibrations

For some years, mathematicians have speculated that the structure of the triple intersection form on a Calabi-Yau threefold may make the existence of a genus one or elliptic fibration increasingly likely as the Picard number $\rho(X) = h^{1,1}(X)$ increases. The rationale for this argument basically boils down to the fact that a cubic in $k$ variables is increasingly likely to have a rational solution as $k$ increases. In this section we give some simple arguments that explain why in the absence of unexpected conspiracies this conclusion is true. If this result could be made rigorous it would be a significant step forwards towards proving the finiteness of the number of distinct topological types of Calabi-Yau threefolds.

As summarized in [25], the following conjecture is due to Kollár [85]:

**Conjecture 1.** Given a Calabi-Yau $n$-fold $X$, $X$ is genus one (or elliptically) fibered iff there exists a divisor $D \in H^2(X, \mathbb{Q})$ that satisfies $D^n = 0$, $D^{n-1} \neq 0$, and $D \cdot C \geq 0$ for all algebraic curves $C \subset X$.

Basically the idea is that $D$ corresponds to the lift $D = \pi^{-1}(D(B))$ of a divisor $D(B)$ on the base of the fibration, where the $(n-1)$-fold self-intersection of $D$ gives a positive multiple of the fiber $F = \pi^{-1}(p)$, with $p$ a point on the base. This conjecture was proven already for $n = 3$ by Oguiso and Wilson [86, 87] under the additional assumption that either $D$ is effective or $D \cdot c_2(X) \neq 0$. In the remainder of this section, as elsewhere in this chapter, we often simply refer to a Calabi-Yau as genus one fibered as a condition that includes both elliptically fibered Calabi-Yau threefolds and more general genus one fibered threefolds.

In the case $n = 3$, to show that a Calabi-Yau threefold is genus one fibered, we thus wish to identify an effective divisor $D$ whose triple intersection with itself vanishes. The triple intersection form can be written in a particular basis $D_i$ for $H^2(X, \mathbb{Z})$ as

$$\langle A, B, C \rangle = \sum_{i,j,k} \kappa_{ijk} a_i b_j c_k.$$  \hspace{1cm} (4.20)$$

where $A = \sum_i a_i D_i$, etc., and $D_i \cap D_j \cap D_k = \kappa_{ijk}$ The condition that there is a divisor
\( D = \sum_i d_i D_i \) satisfying \( D^3 = 0 \) is then the condition that the cubic intersection form on \( D \) vanishes

\[
D^3 = \langle D, D, D \rangle = \sum_{i,j,k} \kappa_{ijk} d_i d_j d_k = 0. \tag{4.21}
\]

We are thus interested in finding a solution over the rational numbers of a cubic equation in \( k = \rho(X) \) variables. The curve condition provides a further constraint that \( D \) lies in the positive cone defined by \( D \cdot C \geq 0 \) for all algebraic curves \( C \subset X \). Note that identifying a rational solution \( D \) to (4.21) immediately leads to a solution over the integers \( \tilde{d}_i \in \mathbb{Z} \forall i \), simply by multiplying by the LCM of all the denominators of the rational solution \( d_i \).

There are basically two distinct ways in which the conditions for the existence of a divisor in the positive cone satisfying \( D^3 = 0 \) can fail. We consider each in turn. Note that even when the condition \( D^3 = 0 \) is satisfied, the condition for an elliptic fibration can fail if \( D^2 = 0 \), in which case \( D \) itself corresponds to a K3 fiber; this class of fibrations is also interesting to consider but seems statistically likely to become rarer as \( \rho \) increases.

**Number theoretic obstructions**

There can be a number theoretic obstruction to the existence of a solution to a degree \( n \) homogeneous equation over the rationals such as (4.21).\(^6\) For example, there cannot be an integer solution in the variables \( x, y, z, w \) of the equation

\[
x^3 + x^2 y + y^3 + 2z^3 + 4w^3 = 0. \tag{4.22}
\]

This can be seen as follows: if all the variables \( x, y, z, w \) are even, we divide by the largest possible power of 2 that leaves them all as integers. Then there must be a solution with at least one variable odd. The variable \( x \) cannot be odd, since if \( y \) is odd or even the LHS is odd. Similarly, \( y \) cannot be odd. So \( x, y \) must be even in the minimal solution. But \( z \) cannot be odd or the LHS would be congruent to 2 mod 4. And \( w \) cannot be odd if the others are even since then the LHS would be congruent

\(^6\)Thanks to Noam Elkies for explaining to us various aspects of the mathematics in this section.
to 4 mod 8.

Such number-theoretic obstructions can only arise for small numbers of variables $k$. It was conjectured long ago that for a homogeneous degree $n$ polynomial the maximum number of variables for which such a number-theoretic obstruction can arise is $n^2$ [88]. While there is a counterexample known for $n = 4$, where there is an obstruction for a quartic with 17 variables, it was proven in [89] that every non-singular cubic form in 10 variables with rational coefficients has a non-trivial rational zero. And the existence of a rational solution has been proven for general (singular or non-singular) cubics in 16 or more variables [90]. Thus, no number-theoretic obstruction to the existence of a solution to $D^3 = 0$ can arise when $\rho(X) = h^{1,1}(X) > 15$, and there are also quite likely no obstructions for $\rho(X) > 9$ though this stronger bound is not proven as far as the authors are aware.

Cone obstructions

If the coefficients in the cubic conspire in an appropriate way, the cubic can fail to have any solutions in the Kähler cone. We now consider this type of obstruction to the existence of a solution. For example, the cubic

$$\sum_i d_i^3 + \sum_{i,j} d_i^2 d_j + \sum_{i,j,k} d_i d_j d_k = 0 \quad (4.23)$$

has no nontrivial solutions in the cone $d_i \geq 0$ since all coefficients are positive. The absence of solutions in a given cone becomes increasingly unlikely, however, as the number of variables increases (again, in the absence of highly structured cubic coefficients). A somewhat rough and naive approach to understanding this is to consider adding the variables one at a time, assuming that the coefficients are random and independently distributed numbers. In this analysis we do not worry about the existence of rational solutions; in any given region, the existence of a rational solution should depend upon the kind of argument described in the previous subsection. We assume for simplicity that the cone condition states simply that $d_i \geq 0 \forall i$; a more careful analysis would consider cones of different sizes and angles. For two variables
Now assume that \( x \) is some fixed value \( x \geq 0 \). This cubic always has at least one real solution \((x, y)\). If the coefficients in the cubic are randomly distributed, we expect roughly a 1/2 chance that \( y \geq 0 \) for this real solution. Now add a third variable. If the above procedure gives a solution \((x, y, z = d_3 = 0)\) in the positive cone, we are done. If not, we plug in some fixed positive values \( x, y \geq 0 \) and the condition becomes a cubic in \( z \). Again, there is statistically roughly a 1/2 chance that a given real solution for \( z \) is positive. So for 3 variables we expect at most a probability of roughly 1/4 that there is no solution in the desired cone. Similarly, for \( k \) variables, this simple argument suggests that most a fraction of \( 1/2^{k-1} \) of random cubics will lack a solution in the desired cone.

This is an extremely rough argument, and should not be taken particularly seriously, but hopefully it illustrates the general sense of how it becomes increasingly difficult to construct a cubic that has no solutions in \( k \) variables within a desired cone. Interestingly, the rate of decrease found by this simple analysis matches quite closely with what we find in a numerical analysis of the Kreuzer-Skarke data at small \( k = \rho(X) = h^{1,1}(X) \).

### 4.3.2 Numerical Results for Calabi-Yau Threefolds at Small \( h^{1,1}(X) \)

We have done some preliminary analysis of the distribution of polytopes without a manifest reflexive 2D fiber for cases giving Calabi-Yau threefolds with small \( h^{1,1} \). The results of this are shown in Table 4.9.

It is interesting to note that the fraction of polytopes without a genus one (or elliptic) fiber that is manifest in the toric geometry decreases roughly exponentially, approximately as \( p(\text{no fiber}) \sim 0.1 \times 2^{5-h^{1,1}} \) in the range \( h^{1,1} \sim 4—7 \). Comparing to the total numbers of polytopes in the KS database that lack a manifested genus one
<table>
<thead>
<tr>
<th>$h^{1,1}$</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total # polytopes</td>
<td>36</td>
<td>244</td>
<td>1197</td>
<td>4990</td>
<td>17101</td>
<td>50376</td>
</tr>
<tr>
<td># without reflexive fiber $\nabla_2$</td>
<td>23</td>
<td>91</td>
<td>256</td>
<td>562</td>
<td>872</td>
<td>1202</td>
</tr>
<tr>
<td>% without reflexive fiber</td>
<td>0.639</td>
<td>0.373</td>
<td>0.214</td>
<td>0.113</td>
<td>0.051</td>
<td>0.024</td>
</tr>
</tbody>
</table>

Table 4.9: The numbers of polytopes without a 2D reflexive fiber, corresponding to Calabi-Yau threefolds without a manifest genus one fibration, for small values of $h^{1,1}$.

fiber, if this fraction continues to exhibit this pattern, the total number of polytopes out of the 400 million in the full KS database would be something like 14,000. (Note, however, that the polytope identified in the database that has no manifest fibration and corresponds to a Calabi-Yau with $h^{1,1} = 140$ would be extremely unlikely if this exponential rate of decrease in manifest fibrations continues; this suggests that the tail of the distribution of polytopes lacking a manifest fibration does not decrease quite so quickly at large values of $h^{1,1}$. Because the analytic argument of the previous section involves all fibrations, not just manifest ones, it may be that this asymptotic is still a good estimate of actual fibrations if most of the polytopes at large $h^{1,1}$ that lack manifest fibrations actually have other fibrations that cannot be seen from toric fibers.)

The naive distribution of the estimated number of polytopes from the simple exponentially decreasing estimate is shown in the black dots in Figure 4-5. Even with some uncertainty about the exact structure of the tail of this distribution, this seems to give good circumstantial evidence that at least among this family of Calabi-Yau threefolds, the vast majority are genus one or elliptically fibered, and that the Calabi-Yau threefolds like the quintic that lack genus one fibration structure are exceptional rare cases, rather than the general rule.

4.4 Conclusions

The results reported in this work indicate that most Calabi-Yau threefolds that are realized as hypersurfaces in toric varieties have the form of a genus one fibration. At large Hodge numbers almost all Calabi-Yau threefolds in the Kreuzer Skarke database satisfy the stronger condition that they are elliptically fibered. This contributes to
the growing body of evidence that most Calabi-Yau threefolds lie in the finite class of elliptic fibrations. We have shown that all known Calabi-Yau threefolds where at least one of the Hodge numbers is greater than 150 must have a genus one fibration, and all CY3's with $h^{1,1} \geq 195$ or $h^{2,1} \geq 228$ have an elliptic fibration. We have also shown that the fraction of toric hypersurface Calabi-Yau threefolds that are not manifestly genus one fibered decreases exponentially roughly as $0.1 \times 2^{5-h^{1,1}}$ for small values of $h^{1,1}$. These results correspond well with the recent investigations in [25, 26, 91], which showed that over 99% of all complete intersection Calabi-Yau (CICY) threefolds have a genus one fibration (and generally many distinct fibrations), including all CICY threefolds with $h^{1,1} > 4$, and that similar results hold for the only substantial known class of non-simply connected Calabi-Yau threefolds.

Taken together, these empirical results, along with the analytic arguments described in §4.3.1, suggest that it becomes increasingly difficult to form a Calabi-Yau geometry that is not genus one or elliptically fibered as the Hodge number $h^{1,1}$ increases. Proving that any Calabi-Yau with Hodge numbers beyond a certain value must admit an elliptic fibration is a significant challenge for mathematicians; progress in this direction might help begin to place some explicit bounds that would help in proving the finiteness of the complete set of Calabi-Yau threefolds.
There are a number of ways in which the analysis of this work could be extended. Clearly, it would be desirable to analyze the fibration structure of the full set of polytopes in the Kreuzer-Skarke database, which could be done by implementing the algorithm used in this work using faster and more powerful computational tools. It is also important to note that while the simple criteria we used here showed already that most known Calabi-Yau threefolds at large Hodge numbers are elliptic or more generally genus one fibered, the cases that are not recognized as fibered by these simple criteria may still have genus one or elliptic fibers. In particular, while we have identified a couple of Calabi-Yau threefolds with $h^{1,1} > 1$ and either $h^{1,1}$ or $h^{2,1}$ greater than 140 that do not admit an explicit toric genus one fibration that can be identified by a 2D reflexive fiber in the 4D polytope, it seems quite likely that the Calabi-Yau threefolds associated with these polytopes may have a non-toric genus one or elliptic fibration structure. Such fibrations could be identified by a more extensive analysis along the lines of [26].

For Calabi-Yau threefolds that do not admit any genus one or elliptic fibration, it would be interesting to understand whether there is some underlying structure to the triple intersection numbers that is related to those of elliptically fibered Calabi-Yau manifolds, and whether there are simple general classes of transitions that connect the non-elliptically fibered threefolds to the elliptically fibered CY3’s, which themselves all form a connected set through transitions associated with blow-ups of the base and Higgsing/unHiggsing processes in the corresponding F-theory models. We leave further investigation of these questions for future work.

Finally, it of course would be interesting to extend this kind of analysis to Calabi-Yau fourfolds. An early analysis of the fibration structure of some known toric hypersurface Calabi-Yau fourfolds was carried out in [92]. The analysis of fibration structures of complete intersection Calabi-Yau fourfolds in [22] suggests that again most known constructions should lead predominantly to Calabi-Yau fourfolds that are genus one or elliptically fibered. The classification of hypersurfaces in reflexive 5D polytopes has not been completed, although the complete set of $3.2 \times 10^{11}$ associated weight systems has recently been constructed [93]. In fact, recent work on
classifying toric threefold bases that can support elliptic Calabi-Yau fourfolds suggests that the number of such distinct bases already reaches enormous cardinality on the order of $10^{3000}$ [94, 84]. Thus, at this point the known set of elliptic Calabi-Yau fourfolds is much larger than any known class of Calabi-Yau fourfolds from any other construction.
Chapter 5

Mirror Symmetry Factorization in Calabi-Yau Fibrations

An increasing body of evidence [56, 19, 22, 14, 24, 25, 26, 91, 36, 37] suggests that a large fraction of known Calabi-Yau threefolds have the property that they can be described as genus one or elliptic fibrations over a complex two-dimensional base surface. We showed in Chapter 4 that this is true of all but at most 4 Calabi-Yau threefolds in the Kreuzer-Skarke database having one or the other Hodge number $h^{2,1}, h^{1,1}$ at least 140, and that at small $h^{1,1}$ the fraction of polytopes in the Kreuzer-Skarke database that lack an obvious elliptic or genus one fibration decreases roughly as $0.1 \times 2^{5-h^{1,1}}$. In this chapter, which is based on our work in [38], we show that the structure of these fibrations gives a natural way of “factorizing” mirror symmetry for many elliptic and genus one fibered toric hypersurface Calabi-Yau threefolds, so that the fiber of $X$ determines the fiber of the mirror threefold $\tilde{X}$, and the base and fibration structure of $X$ determine the base of $\tilde{X}$. A key aspect of this factorization involves the observation that a simple additional condition on the fibration structure of an elliptic toric hypersurface Calabi-Yau threefold $X$ implies that the mirror $\tilde{X}$ is also elliptic and has a mirror elliptic fiber characterized by a 2D polytope dual to the one that contains the genus one or elliptic fiber of $X$. Such mirror fibers were also studied in the related context of K3 surfaces and heterotic/F-theory duality in [68, 95, 96, 19, 97], and in the context of elliptic fibers for F-theory in [71].
The outline of this chapter is as follows: In §5.1, we review some basic aspects of toric hypersurface Calabi-Yau manifolds and elliptic and genus one fibrations; we then describe in general the way in which mirror symmetry can factorize for toric hypersurface Calabi-Yau manifolds. In §5.2, we consider the simplest examples of this factorization: when \( X \) is the generic CY elliptic fibration over any toric base surface \( B \) that supports an elliptic Calabi-Yau threefold, the mirror \( \tilde{X} \) has a simple description as an elliptic fibration over a dual base \( \tilde{B} \) that has a simple description in toric geometry in terms of \( B \). In §5.3, we consider some further examples, including Weierstrass/Tate tunings of generic models over a toric base, “stacked” fibrations with non-generic fiber types, and analogous factorization for elliptic Calabi-Yau fourfolds. We conclude in §5.4 with a summary and some open questions.

## 5.1 Toric Hypersurface Calabi-Yau Manifolds, Fibrations and Mirror Symmetry

In this section we review some basic aspects of toric hypersurface Calabi-Yau threefolds and fibrations for the convenience of the reader; much of the material reviewed in the first part of this section is covered in more detail in Chapter 3 §3.2 and Chapter 4 §4.1; and we describe the basic framework of mirror symmetry factorization that applies for many elliptic and genus one fibered toric hypersurface Calabi-Yau manifolds.

### 5.1.1 Toric Hypersurfaces and Fibrations

A broad class of Calabi-Yau manifolds can be described as hypersurfaces in toric varieties following the approach of Batyrev [2]. A lattice polytope \( \nabla \) is defined to be the set of lattice points in \( N = \mathbb{Z}^n \) that are contained within the convex hull of a finite set of vertices \( v_i \in N \). The dual of a polytope \( \nabla \) is defined to be

\[
\nabla^* = \{ u \in M_{\mathbb{R}} = M \otimes \mathbb{R} : \langle u, v \rangle \geq -1, \forall v \in \nabla \},
\]

(5.1)
where \( M = N^* = \text{Hom}(N, \mathbb{Z}) \) is the dual lattice. A lattice polytope \( \nabla \subset N \) containing the origin is reflexive if its dual polytope is also a lattice polytope. For any reflexive polytope, the origin is the unique interior point.

When \( \nabla \) is reflexive, we denote the dual polytope by \( \Delta = \nabla^* \). The elements of the dual polytope \( \Delta \) can be associated with monomials in a section of the anticanonical bundle of a toric variety associated to \( \nabla \). A section of this bundle defines a hypersurface in the toric variety associated to \( \nabla \); this hypersurface is a Calabi-Yau manifold of dimension \( n - 1 \). The polytopes \( \nabla \) and \( \Delta \) describe toric hypersurface Calabi-Yau manifolds that are related by mirror symmetry [6]. As \( \nabla \) and \( \Delta \) are a pair of 4D reflexive polytopes, there is a one-to-one correspondence between \( l \)-dimensional faces \( \theta \) of \( \Delta \) and \((4 - l)\)-dimensional faces \( \tilde{\theta} \) of \( \nabla \) related by the dual operation

\[
\theta^* = \{ y \in \nabla, \langle y, pt \rangle = -1 \} \text{ for all } pt \text{ that are vertices of } \theta \}.
\tag{5.2}
\]

For the CY associated with \( \nabla \), the Hodge numbers are given by

\[
h^{2,1} = \text{pts}(\Delta) - \sum_{\theta \in F_3^\nabla} \text{int}(\theta) + \sum_{\theta \in F_2^\nabla} \text{int}(\theta) \text{int}(\theta^*) - 5, \tag{5.3}
\]

\[
h^{1,1} = \text{pts}(\nabla) - \sum_{\tilde{\theta} \in F_3^\nabla} \text{int}(\tilde{\theta}) + \sum_{\tilde{\theta} \in F_2^\nabla} \text{int}(\tilde{\theta}) \text{int}(\tilde{\theta}^*) - 5, \tag{5.4}
\]

where \( \theta \) are faces of \( \Delta \), \( \tilde{\theta} \) are faces of \( \nabla \), \( F_i^{\nabla/\Delta} \) denotes the set of \( l \)-dimensional faces of \( \nabla \) or \( \Delta \) \( (l < n) \), and \( \text{pts}(\nabla/\Delta) := \text{number of lattice points of } \nabla \) or \( \Delta \), \( \text{int}(\theta/\tilde{\theta}) := \text{number of lattice points interior to } \theta \) or \( \tilde{\theta} \). The correspondence (5.2) makes the duality between the Hodge number formulae manifest.

When the polytope \( \nabla \) has a 2D subpolytope \( \nabla_2 \) that is also reflexive, the associated Calabi-Yau manifold has a genus one fibration [69].\(^1\) This fibration is characterized

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\(^1\)In [98] it is argued that in some cases the condition for a fibration is more subtle; in particular, the clearest way of constructing a fibration uses a toric morphism that must be compatible with a triangulation of \( \nabla \), which may not be possible in some cases, particularly in higher dimensions. We do not analyze the detailed structure of triangulations in this work; for the simple cases considered here there does not seem to be any obstruction to the existence of a triangulation giving a toric morphism compatible with the fibration, but this in principle should be checked in detail, particularly for higher-dimensional varieties where the triangulation of the base is not unique.
by a projection \( \pi \) on \( N = \mathbb{Z}^n \) that maps the fiber \( \nabla_2 \) to 0. For a 4D polytope \( \nabla \), the base \( B \) of the fibration is described by the 2D toric variety associated with the set of primitive rays in the image of \( \nabla \) under the projection \( \pi : \mathbb{Z}^4 \to \mathbb{Z}^2 \), where a primitive ray is one that is not an integer multiple of another element of the lattice. Since the base in this case is two-dimensional, the triangulation in the resulting toric variety is uniquely determined by the ordering of the rays; in higher dimensions, there may be many distinct triangulations possible.

There are 16 distinct reflexive 2D polytopes, listed in Appendix A.1. The structure of the genus one and elliptic fibrations associated with each of these 16 fibers is studied in some detail in [40] and in the F-theory context in [48, 41, 71, 37]. As discussed in these papers, all fibers other than \( F_1, F_2, F_4 \) contain at least one curve of self-intersection \(-1\); such a curve gives rise to a global section so that the fibration is elliptic and not just genus one. Associated with each of the 16 reflexive 2D polytopes \( F = \nabla_2 = F_i \) is a dual fiber \( \tilde{F} \), given by \( \tilde{F}_i = \Delta_2 = F_{17-i} \) for all \( i \) except \( i = 7, 8, 9, 10 \) in which cases \( \tilde{F}_i \cong F_i \) under a linear change of coordinates. We will refer to \( \tilde{F} \) as the mirror fiber of \( F \). The anti-canonical hypersurfaces in \( F, \tilde{F} \) represent a mirror pair of 1D Calabi-Yau varieties (genus one curves). Aspects of these dual fibers and associated mirror curves have previously been encountered and studied in the contexts of K3 fibrations, heterotic/F-theory duality and F-theory fibers in [68, 95, 96, 19, 71, 97].

When a polytope \( \nabla \) admits a toric fibration of this kind, the lattice points in the dual polytope \( \Delta \) can be associated with monomials in various line bundles over the base \( B \). We can choose a coordinate system on \( N \) so that the vertices of the fiber \( F = \nabla_2 \) lie in the plane \( (0, 0; \cdot, \cdot) \). Each lattice point \( v \in \nabla \) can then be represented in the form

\[
v = (v_1, v_2; v_3, v_4) = (v^{(I)}; v^{(II)}),
\]

where the first two coordinates \( v^{(I)} = (v_1, v_2) \) correspond to the base direction, and the last two coordinates \( v^{(II)} \equiv (v_3, v_4) \) correspond to the toric fiber direction. For
each point \(v^F \in \nabla_2\), the point \(v = (0; v^F)\) lies in \(\nabla\).\(^2\) The primitive rays defining the base as a toric variety are those that are not integer multiples of another ray,

\[
\{v^B\} = \{v^{(I)} = (v_1, v_2)/(\text{GCD}(v_1, v_2)), \exists v^{(II)} : v = (v^{(I)}, v^{(II)}) \in \nabla\}. \quad (5.6)
\]

The existence of the projection \(\pi : v = (v_1, v_2; v_3, v_4) \rightarrow (v_1, v_2; 0, 0)\) taking \(\nabla_2 \rightarrow 0\) is equivalent to the condition that there is a projection on the dual lattice \(\rho : m = (m_1, m_2; m_3, m_4) \rightarrow (0, 0; m_3, m_4)\) that maps \(\Delta\) to the mirror fiber \(\bar{F} = \Delta_2\).\(^3\) For each \(m^{(II)} \in \Delta_2\), the set of lattice points in \(\Delta\) that map under \(\rho\) to \(m^{(II)}\) can be thought of as monomials that are sections of a specific line bundle over \(B\). Each ray of the form (5.5) satisfies the condition \(v \cdot m \geq -1\), which implies \((v_1, v_2) \cdot (m_1, m_2) \geq -1 - v^{(II)} \cdot m^{(II)}\). When \(v^B = (v_1, v_2)\) is a primitive ray and corresponds to a toric curve \(C\) in the base \(B\), this means that \(m^{(I)} = (m_1, m_2)\) is a section of a line bundle that can vanish to order \(v^{(II)} \cdot m^{(II)} + 1\) on \(C\).

The simplest examples of the utility of these conditions can be seen in polytopes that have the “stacked” form described in §4.1.3, where there is a fixed lattice point \(v_s \in \nabla_2\) so that for every ray \(v^B\) in the base there exists a ray of the form \((v^B; v_s) \in \nabla\) for that particular lattice point \(v_s\). In these cases, the monomials over \(m^{(II)}\) represent sections of the line bundle \(\mathcal{O}(-nK_B)\), where \(n = 1 + v^{(II)} \cdot m^{(II)}\) and \(-K_B\) is the anti-canonical class of the base. In particular, when the fiber is \(F_{10} = \mathbb{P}^{2,3,1}\), and \(v_s = (-3, -2)\) the resulting “standard stacking” form gives monomials over the points in \(\Delta_2\) that naturally describe the general (Tate) form of the Weierstrass model for an elliptic fibration, and can be described as sections of \(\mathcal{O}(-nK_B)\), with \(n = 1, 2, 3, 4, 6\).

\(^2\)Note that \(v^{(II)}\) in the fiber direction may lie outside the fiber \(\nabla_2\) for a general point \((v^{(I)}; v^{(II)})\) in \(\nabla\). In fact, as we discuss further later in this chapter, this has to be the case for some lattice points in \(\nabla\) when the dual polytope \(\Delta\) is not a fibered polytope.

\(^3\)This can be easily proven as follows: Given the projection \(\pi\), we know that the fiber \(\nabla_2\) lies in the plane \((0, 0; \ldots)\). This implies that any \(m = (m_1, m_2; m_3, m_4) = (m^{(I)}; m^{(II)})\) satisfies \(m^{(II)} \cdot v^F \geq -1, \forall v^F \in \nabla_2\), which implies \(m^{(II)} \in \Delta_2\), i.e., the existence of the projection \(\rho\) onto \(\Delta_2\). To see the existence of the fiber given the projection \(\rho\), every point in the form \(v = (0, 0; v^F)\), where \(v^F \in \nabla_2\), satisfies \(v \cdot m \geq -1, \forall m \in \Delta\) since \(v^F \cdot m^{(II)} \geq -1, \forall m^{(II)} \in \Delta_2\); we therefore have \(\nabla_2 \subset \nabla\), and the projection taking \(\nabla_2 \rightarrow 0\) takes the form \(\pi\).
5.1.2 Faces of the Base Polytope and Chains of Non-Higgsable Clusters

Certain chains of self-intersections of curves in the base, associated with characteristic combinations of non-Higgsable clusters, have been observed both in 6D supergravity theories [10, 11] and 6D superconformal field theories constructed from F-theory [107, 101, 108]. In the context of the toric bases we consider here, these can be seen as arising simply from the sequences of primitive rays in a toric base associated with a face of the bounding polytope at different distances from the origin. We encounter the $E_8$ sequence connecting $-12$ curves in the example in §5.2.2, and the simple $SO(8)$ sequence connecting $-4$ curves in the example in §5.3.4. We briefly discuss here all the constructions of this type to illustrate how they arise in a unifying context in this framework.

The simplest case is that of the self-intersection sequence $\{-1, -4, -1, -4, -1, \ldots\}$. This can be seen as arising from the set of primitive rays associated with a face of the toric base polytope at distance 2 from the origin (Figure 5-1). The primitive rays in this case are of the forms $(n, 1) \forall n$ and $(2k + 1, 2) \forall k$. For example, starting from $(0, 1)$, the sequence of rays in a toric diagram associated with a polytope having a face along the line $(x, 2)$ is

$$\begin{align*}
(0, 1), & (1, 2), (1, 1), (3, 2), (2, 1), \ldots
\end{align*}$$

(5.7)

Since from toric geometry we know that a toric curve $v_i$ has self-intersection $-n$ when
\[ n v_i = v_{i-1} + v_{i+1}, \]
we can read off the self-intersection sequence \([-4, -1, -4, -1, -4, \ldots] \) from the ray sequence (5.7). This corresponds to a sequence of non-Higgsable gauge groups \( SO(8), \cdot, SO(8), \cdot, SO(8) \ldots \) in the F-theory picture.

Performing a similar analysis at other distances we see that the following sequences arise:

\[
\begin{align*}
    d = 2 & \rightarrow \ [[-4, -1, -4, \ldots]] (SO(8)) \\
    d = 3 & \rightarrow \ [[-6, -1, -3, -1, -6, \ldots]] (E_6 \times SU(3)) \\
    d = 4 & \rightarrow \ [[-8, -1, -2, -3, -2, -1, -8, \ldots]] (E_7 \times (SU(2) \times SO(7) \times SU(2)))^2 \\
    d = 6 & \rightarrow \ [[-12, -1, -2, -2, -3, -1, -5, -1, -3, -2, -2, -1, -12]] \\
    & (E_8 \times F_4 \times (G_2 \times SU(2))^2)
\end{align*}
\]

where in each case the sequence repeats and we have indicated the non-Higgsable gauge group for a single cycle of the sequence. These are precisely the maximal connected sequences identified in [10] and associated with e.g. "\(E_8\) matter" in [101].

### 5.1.3 Factorization of Mirror Symmetry

From the preceding characterization of polytope fibrations, it clearly follows that under certain circumstances when the polytope \(\nabla\) has a subpolytope \(\nabla_2 = F_i\) that gives a genus one fibration of the associated Calabi-Yau hypersurface \(X\) the polytope \(\Delta\) will also have a subpolytope \(\Delta_2 = \tilde{F}_i\) that gives a genus one fibration of the mirror Calabi-Yau \(\tilde{X}\). This will occur whenever there is a coordinate system such that the point \((0, 0; m^{(II)})\) is a point in \(\Delta\) for all \(m^{(II)} \in \Delta_2\). A necessary and sufficient condition for this to occur is that there exist a coordinate system so that every lattice point in \(\nabla\) can simultaneously be put in the form (5.5), with \(v^{(II)} = v^F \in \nabla_2\) (the values of \(v^F\) need not be the same for different lattice points in \(\nabla\) but they must all lie in \(\nabla_2\), i.e. that there is a projection in the space \(N\) onto the fiber polytope \(\nabla_2\).

\(\text{This condition was encountered in the context of K3 surfaces and heterotic F-theory duality in several earlier papers; mirror K3 fibrations were described in terms of slices and projections in [68, 19], and described in terms of symplectic cuts in [96], motivated by some examples found by Candelas. This type of construction has been further used in studying mirror symmetry of } G_2 \)
In any situation where these conditions hold, we have a mirror pair of Calabi-Yau manifolds $X, \tilde{X}$, each of which is elliptically or genus one fibered. Furthermore, in the toric presentation, the 2D toric fibers associated with the elliptic or genus one fibers of $X, \tilde{X}$ themselves have mirror hypersurface curves, with $F = F, \tilde{F} = \tilde{F}$. We refer to this situation as a "factorization" of mirror symmetry for elliptic Calabi-Yau manifolds.

This kind of factorization is really in some sense a semi-factorization. In particular, the relationship between the base $B$ of the elliptic fibration of $X$ and the base $\tilde{B}$ of the mirror fibration depends upon the "twist" of the fibration of $X$ encoded in the specific way in which the rays of the base lie over the fiber in (5.5). In general, this relationship can be rather complex, though it can always be determined from the condition described in §5.1.1 that implies that each primitive base ray in the mirror is associated with a section $m^{(I)}$ of a line bundle whose degree of vanishing on the associated curve is constrained by the set of inner products $v^{(I)} \cdot m^{(I)}$.

In many cases, the structure is particularly simple and the base $\tilde{B}$ of the mirror elliptic fibration can be associated with a line bundle $O(-nK_B)$ for a fixed $n$, so that $\tilde{B}$ can be read off in a simple way from $B$. In particular, this occurs when all the rays of the base are stacked over a particular point $v_s \in \nabla_2$ in the fiber, and there are no rays in $\nabla$ (associated with tops) representing rays in the base over other fiber points that impose extra constraints on the points in the mirror polytope. In this case, the monomials in the dual polytope $\tilde{B}$ can be associated with sections of line bundles $O(-nK_B)$, with $n = 1 + v_s \cdot m^{(II)}$, and the base $\tilde{B}$ can be associated with a polytope manifolds [99]. The strong prevalence of mirror symmetric pairs of K3-fibered CY3s observed in [19] is closely related to the prevalence of mirror symmetric pairs of elliptically-fibered CY3s studied here; indeed, many of the examples we consider here are also K3 fibered.

The polytopes in the KS database [4] are associated with the monomial polytopes $\Delta$. Given a mirror pair of fibered polytopes $\nabla, \Delta$, where $\nabla$ is the "fan polytope" associated with the fibration $X$ with the Hodge numbers $(h^{1,1}(X), h^{2,1}(X))$, it is the $\Delta$ polytope associated with the data $M$: # lattice points, # vertices (of $\Delta$) $N$: # lattice points, # vertices (of $\nabla$) $H$: $h^{1,1}(X), h^{2,1}(X)$ that is listed on the website. Then $\Delta$ is the fan polytope associated with the fibration $\tilde{X}$ with the Hodge numbers $(h^{1,1}(\tilde{X}) = h^{2,1}(X), h^{2,1}(\tilde{X}) = h^{1,1}(X))$. While the lattice points in the fan polytope $\nabla$ associated with the fibration $X$ are denoted by $v$, when $\Delta$ is viewed as the fan polytope associated with the fibration $\tilde{X}$, we sometimes denote the lattice points in $\Delta$ by $w$, while we use the same symbol $m$ to denote the lattice points in either $\Delta$ or $\nabla$ in the cases when they are used as the monomial polytope.
built from the primitive rays in the set of points in the 2D polytope associated with $-nK_B$ with the largest value of $n$ realized from the points $m(I) \in \Delta_2$:

$$\{w_i^B\} = \{w = (w_1, w_2) | \ GCD(w_1, w_2) = 1, w \cdot v^B \geq -n_{vs} \forall v^B \in \Sigma_B\},$$

where $\Sigma_B$ is the toric fan for $B$ and $n_{vs} = \max\{(1 + v_s \cdot m(I)) | m(I) \in \Delta_2\}$. In Appendix A.1 we illustrate for each $v_s \in \nabla_2$ the maximum value of $1 + v_s \cdot m(I)$. Note that unlike in (5.6), where we are dealing with projections of 4D rays, when an integer multiple $kw$ of a primitive 2D vector $w$ satisfies $kw \cdot v^B \geq -n$ then the vector $w$ also satisfies this condition, so we can construct all primitive vectors by simply taking those with unit GCD on the coordinates. The simplest cases in which (5.12) applies is for the “standard stacking” $F_{10}$ fiber constructions associated with the generic Tate form for an elliptic fibration over a toric base, as discussed at the end of §5.1.1, in which case $n_{vs} = 6$. We describe a number of examples of this type in §5.2. When there is a further tuning of the monomials in $\Delta$, associated with a nontrivial “top” in $\nabla$, the construction of the mirror base $\tilde{B}$ is similar but depends on the tuning, as we discuss in more detail in §5.3.

All of the analysis just outlined is equally relevant taking mirror symmetry the other way. Starting with $\Delta$, the base $B$ can similarly in the complementary cases, such as when $\Delta$ is a standard stacking associated with a generic elliptic fibration over $\tilde{B}$, be calculated using (5.12) from $O(-n_{w_s}K_{\tilde{B}})$, where $w_s \in \Delta_2$ is the stacking point of the stacked polytope $\Delta$, and $n_{w_s} = \max\{(1 + w_s \cdot m(I)) | m(I) \in \nabla_2\}$, and the same kind of generalization is used when there is a tuning of the monomials in $\nabla$. In the remainder of this chapter we consider primarily examples of this type, where there is a clear factorization of the mirror symmetry that allows a ready identification of both the fiber and the base of the mirror fibered Calabi-Yau variety. We leave a further analysis of more general cases for future work.
5.2 Generic Calabi-Yau Elliptic Fibrations over Toric Base Surfaces

In this section we consider the simplest and perhaps clearest class of examples of the factorization of mirror symmetry described above: Calabi-Yau threefolds $X$ that are generic CY elliptic fibrations over a toric base surface $B$. The closely related class of threefolds resulting from tuned Tate-form Weierstrass models over a toric base provide a larger class of examples where the mirror symmetry factorization can be understood easily; examples of this broader class are given in §5.2.4, §5.3.1, and §5.3.2.

5.2.1 General Case

It was shown in [11] that there are 61,539 toric bases that support an elliptic Calabi-Yau threefold with a smooth resolution. The Hodge numbers of the generic elliptic fibrations over all these bases were analyzed in [56]. It was shown in §3.3.1 (or see §4.1.4) that for each toric base $B$, a reflexive 4D polytope in the Kreuzer-Skarke database can be constructed by starting with a “standard stacking” polytope defined by the convex hull of the set of points of the form (5.5), where the first two coordinates are taken across all toric rays $v^B$ in the fan of $B$, and the last two coordinates $v_s = (-3, -2)$ correspond to a vertex of the fiber $F = F_{10} = \mathbb{P}^{2,3,1}$, and then taking the “dual of the dual” of the resulting polytope. In the simplest cases, where $B$ only has curves of self-intersection $-n$ where $n|12$, the initial polytope is already reflexive. In all these cases, there is a corresponding reflexive 4D polytope in the Kreuzer-Skarke database that has an explicit $F_{10}$ fiber.

For each of these generic elliptic fibrations over a toric base, the dual polytope $\Delta$ associated with the mirror Calabi-Yau threefold $\tilde{X}$ contains lattice points that

---

6 Note that these bases include those with curves of self-intersection $-9, -10, -11$. Such bases can be blown up at non-toric points to achieve a smooth flat fibration Calabi-Yau resolution; Calabi-Yau threefolds can also be realized as non-flat fibrations over these bases. This technicality is handled automatically through the resolution process for the toric hypersurface Calabi-Yau threefolds in the Kreuzer-Skarke database (see §3.2.7 and §3.4.4, also examples in Table 5.2 and an example in §5.3.3.)
can be interpreted as sections of line bundles $O(-nK_B)$ with $n = 1, 2, 3, 4, 6$. These correspond to the coefficient polynomials in the “Tate form” of a Weierstrass model\textsuperscript{7}

$$y^2 + a_1 y x + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6,$$

where $a_n$ is a section of $O(-nK_B)$. Since the origin is contained in each of the 2D sets of points over each $m^{(II)} \in \Delta_2$, for all these models the dual polytope $\Delta$ has a subpolytope $\tilde{\Delta} = \Delta_2$ lying on the plane $m_1 = m_2 = 0$, so the resulting mirror $\tilde{X}$ also has an elliptic fiber given by an anti-canonical curve in the toric fiber $\tilde{F} \cong F = F_{10}$. These correspond to one of the simplest classes of elliptic toric hypersurface Calabi-Yau threefolds with a simple and manifest factorization of mirror symmetry. In these cases, the dual polytope has a base that is described as a toric variety by the set of primitive rays associated with the monomials in $O(-6K_B)$, which lie over the point $m^{(II)} = (-1, -1) \in \Delta_2$. The Hodge numbers of the generic elliptic fibrations over toric bases are plotted in Figure 5-2 [56]; these include many of the largest Hodge number pairs in the KS database.\textsuperscript{8}

In the remainder of this section we describe some examples of these factorized mirror pairs explicitly. From this construction, it is clear that similarly, an arbitrary Tate tuning of the generic elliptic fibration over a toric base $B$, which is realized by a reduction in the set of monomials in $\Delta$ and an increase in the set of rays in $\nabla$ (often described in the language of tops), as described in more detail in Chapter 3 §3.3.2, will also lead to a mirror pair of Calabi-Yau threefolds that are both elliptically fibered with the self-dual toric 2D fiber type $F_{10}$. In general, tuning the fibration $X$ will reduce the size of the polytope associated with the mirror base $\tilde{B}$, which will then be described by a toric fan that contains as rays only a subset of the primitive rays in $-6K_B$.\textsuperscript{9} We describe examples of such tunings in §5.3.1-5.3.3.

\textsuperscript{7}The term “Tate form” is used often by physicists for this general form of the Weierstrass model because of its use in the context of F-theory in the Tate algorithm for constructing models with a particular desired gauge group.

\textsuperscript{8}The simplest subset of these cases, where both sides of the mirror symmetry are generic elliptic fibrations without tuning, and some of the patterns appearing in these cases, were noted in the context of an earlier project with Braun and Wang [100].

\textsuperscript{9}Note that in some cases when all monomials over some points $m^{(II)} \in \Delta_2$ are set to vanish, this
5.2.2 Example: Generic Elliptic Fibration over $\mathbb{P}^2$ (Hodge Numbers (2, 272))

As a simple first case, we consider the case of the base $B = \mathbb{P}^2$, with a generic elliptic fiber given by an anti-canonical curve in the toric fiber space $\mathbb{P}^{2,3,1}$ (i.e. the toric fiber $F_{10}$). In this case the polytope $\nabla$ has vertices

$$\{v_i\} = \{(0, 0; 1, 0), (0, 0; 0, 1), (1, 0; -3, -2), (0, 1; -3, -2), (-1, -1; -3, -2)\}.$$

Here the fiber is given by the slice with vertices $(0, 0; 1, 0), (0, 0; 0, 1), (0, 0; -3, -2)$, and the projection onto the base projects the last two coordinates to 0. Since $B$ contains no curves of self-intersection $-n$ with $n$ not a divisor of 12, $\nabla$ is immediately

Furthermore will correspond to reducing the size of $\Delta_2$, with a corresponding increase in the size of $\nabla_2$; for example, setting $a_6 = 0$ in the Tate model changes the dual fiber to $\tilde{F}_{13}$, so that the fiber of $\nabla$ acquires two additional points and becomes $F_{13}$. 

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The base $B = \mathbb{P}^2$ and the base $\tilde{B}$ of the mirror of the generic elliptic fibration over $B$, shown as toric varieties. Rays that are red, blue, purple, green correspond to curves in the toric base $\tilde{B}$ that carry non-Higgsable $E_8, F_4, G_2,$ and $SU(2)$ gauge groups respectively.

reflexive. The dual polytope $\Delta$ is easily seen to have vertices

$$\{m_i\} = \{(0, 0; -1, 2), (0, 0; 1, -1), (-6, 12; -1, -1), (12, -6; -1, -1), (-6, -6; -1, -1)\}.$$  

(5.15)

Under a coordinate transformation on the last two coordinates this takes the form

$$\{w_i\} = \{(0, 0; 1, 0), (0, 0; 0, 1), (-6, 12; -3, -2), (12, -6; -3, -2), (-6, -6; -3, -2)\}.$$  

(5.16)

This dual polytope $\Delta$ is again fibered by an $F_{10}$ fiber in the $(0, 0; \cdot, \cdot)$ plane, and the projection onto the base gives a base $\tilde{B}$ that has a toric description using the primitive rays in the 2D polytope with vertices $(-6, 12), (12, -6), (-6, -6)$ (See Figure 5-3). The dual base polytope consists of the points in $\Delta$ that can be associated with sections of the line bundle $-6K_B$. The mirror polytopes $\nabla, \Delta$ both appear in the Kreuzer-Skarke database, and give rise to elliptic toric hypersurface Calabi-Yau threefolds with hodge numbers $(2, 272)$ and $(272, 2)$ respectively (see footnote 5).

We can characterize the mirror base $\tilde{B}$ as a toric variety by the sequence of self-
intersections of the toric rays calculated by equation (5.12)

$$\tilde{B} \rightarrow [\ldots]$$

where the notation // denotes the sequence of self-intersections

$$// = -1, -2, -2, -3, -1, -5, -1, -3, -2, -2, -1.$$ (5.18)

This sequence of self-intersections is a familiar sequence that connects $-12$ curves that support $E_8$ (Kodaira type $II^*$) singularities in the elliptic fibration (see e.g. [10, 101]). This sequence of self-intersections characterizes a face of the base at distance 6 from the origin; similar structure for faces at different distances from the origin are described in §5.1.2.

Using methodology motivated by F-theory we can compute the Hodge numbers of the generic elliptic fibrations over $B, \tilde{B}$ directly from the geometry of the bases [8, 56]. For $B$, all curves have self-intersection above $-3$, so there is no non-Higgsable gauge group. From the Shioda-Tate-Wazir formula we have

$$h^{1,1}(X) = h^{1,1}(B) + 1 = 2.$$ (5.19)

From the gravitational anomaly condition, we have

$$h^{2,1}(X) = 273 - 29(h^{1,1}(B) - 1) - 1 = 272.$$ (5.20)

For the mirror the computation is a bit more complicated. On each $-11$ curve there is a single $(4, 6)$ point that must be blown up so that the total space has a smooth Calabi-Yau resolution [10]. Before these blowups the number of toric curves is 108, so $h^{1,1}(\tilde{B}) = 108 - 2 + 3 = 109$. The non-Higgsable gauge group from the curves of negative self-intersection below $-1$ is $G = E_8^9 \times F_4^9 \times (G_2 \times SU(2))^{18}$, with rank 162.
Figure 5-4: The base $B = \tilde{B}$ over which the generic elliptic fibration is the self-mirror Calabi-Yau threefold with Hodge numbers $(251, 251)$. Note that the inner product between each pair of vertices is $-6$, so that $B = B \sim -6K_B$. Rays that are red, blue, purple, green correspond to curves in the toric base $\tilde{B}$ that carry non-Higgsable $E_8, F_4, G_2,$ and $SU(2)$ gauge groups respectively.

We then have

$$h^{1,1}(\tilde{X}) = h^{1,1}(\tilde{B}) + \text{rank } G + 1 = 109 + 162 + 1 = 272. \quad (5.21)$$

On the other hand, each $G_2 \times SU(2)$ non-Higgsable factor is associated with 8 charged matter hypermultiplets, so we have

$$h^{2,1}(\tilde{X}) = 273 - 29(h^{1,1}(\tilde{B}) - 1) + \text{dim } G - m_{NH} - 1 = 273 - 29(108) + 9(248 + 52 + 34) - 144 - 1 = 2. \quad (5.22)$$

### 5.2.3 Example: Self-mirror Calabi-Yau Threefold with Hodge Numbers $(251, 251)$

We now consider the self-mirror Calabi-Yau threefold with Hodge numbers $(251, 251)$ at the central peak of the “Hodge shield”. This polytope can be put into a coordinate system where the vertices are

$$\{v_i\} = \{(0, 0; 1, 0), (0, 0; 0, 1), (-1, 6; -3, -2), (0, -1; -3, -2), (42, 6; -3, -2)\}.$$

$$ \quad (5.23)$$

This polytope is self-dual, $\nabla = \Delta$, up to a coordinate transformation, and is clearly a $\mathbb{P}^{2,3,1}$ fibration over the base $B$ with vertices $(-1, 6), (0, -1), (42, 6)$. Since the inner product between each pair of these vertices is $-6$, the procedure of constructing the mirror base $\tilde{B}$ from monomials in $\mathcal{O}(-6K_B)$ gives $B = \tilde{B}$. The primitive rays in this
base (Figure 5-4) give a toric surface with a sequence of self-intersections

\[ [[0, 6, -12], [-11], [-12], [-12], [-12], [-12], [-12], [-12]] . \] (5.24)

Again using the relationship between geometry and F-theory physics we can compute the Hodge numbers directly from the geometry of the base. There are 99 toric curves in the base, with one $-11$ curve that must be blown up, so $h^{1,1}(B) = 99 - 2 + 1 = 98$. The non-Higgsable gauge group is $G = E_8^9 \times F_4^8 \times (G_2 \times SU(2))^{16}$, with rank 152, so

\[ h^{1,1}(X) = h^{1,1}(B) + \text{rk} \, G + 1 = 251 . \] (5.25)

Similarly,

\[ h^{2,1}(\tilde{X}) = 273 - 29(h^{1,1}(B) - 1) + \text{dim} \, G - m_{NH} - 1 = 251 . \] (5.26)

### 5.2.4 Example: Generic Elliptic Fibration over $\mathbb{F}_n$ (Hodge Numbers $\elltr{3, 243}{\ldots, (11, 491)}$)

As further examples we consider the generic elliptic fibrations over the Hirzebruch surfaces $B = \mathbb{F}_n$. In each case the mirror Calabi-Yau is elliptic over a base constructed from $-6K_B$, though in some cases the mirror is not a generic elliptic fibration but has some tuning. We describe several specific cases in detail and summarize the complete set for all $n = 0, \ldots, 12$ in Table 5.1 and Table 5.2. The Hirzebruch surface $\mathbb{F}_n$ can be described by the toric rays $(0, 1), (1, 0), (0, -1), (-1, -n)$. For $n = 0, 1$ these are the vertices of the associated 2D polytope, while for $n > 1 (0, -1)$ is not a vertex and the other three are. The dual base $\tilde{B} \sim -6K_B$ is thus characterized by the toric variety with a fan given by all primitive rays in the polytope defined by the vertices

\[ \{(-6, -6), (-6, 12/n), (6(n + 1), -6)\} , \] (5.27)

when $n \geq 2$, $\{(-6, -6), (-6, 12), (6, 0), (6, -6)\}$ for $n = 1$, and $\{(-6, -6), (-6, 6), (6, 6), (6, -6)\}$ for $n = 0$. 

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When $n|12, n > 1$, the vertices of the polytope (5.27) containing the dual base are integral; from this it follows that the polytope $\nabla$ is reflexive, since $\Delta$ has the form of (5.15), where the last set of vertices are given by $(m(I); -1, -1)$, with $m(I)$ vertices of the polytope (5.27). The polytope $\nabla$ has vertices

$$\{v_i\} = \{(0, 0; 1, 0), (0, 0; 0, 1), (1, 0; -3, -2), (0, 1; -3, -2), (-1, -n; -3, -2)\}.$$  

(5.28)

As in the preceding examples we can read off the sequence of self-intersections of the toric rays associated with primitive rays in (5.27). We give a few explicit examples.

**F$_{12}$:**

The polytope $\nabla$ is reflexive, and the Hodge numbers of the associated Calabi-Yau threefold are $(11, 491)$; this is the largest possible value of $h^{2,1}(X)$ for any elliptic Calabi-Yau threefold [56]. In this case the vertex $(-6, 1)$ of $\tilde{B}$ is primitive and corresponds to a curve of self-intersection 0, and the vertices $(-6, -6)$ and $(78, -6)$ each go to primitive rays $(-1, -1)$ and $(13, -1)$ associated with curves of self-intersection 11. The sequence of self-intersections of the toric rays for $\tilde{B}$ is then

$$[[-12// -11//(-12//){13} - 11// - 12, 0]]$$  

(5.29)

Blowing up the base at two points on the $-11$ curves we can confirm that the Hodge numbers of the generic elliptic fibration, corresponding to the polytope $\Delta$, are $(491, 11)$ as expected.

**F$_6$:**

In this case again $\nabla$ defined through (5.28) is immediately reflexive. The Hodge numbers for the associated Calabi-Yau threefold are $(9, 321)$, which can be immediately determined from the non-Higgsable $E_6$ gauge group over the $-6$ curve in $B$. In this case, however, the vertex $(-6, 2)$ is not primitive. The top part of the toric diagram for $\tilde{B}$ is shown in Figure 5-5, and contains the sequence of curves of self-intersection $-1, -2, -2, -2, -2, -2, -1$. Unlike in the case of $F_{12}$ the mirror polytope $\Delta$ is not a generic elliptic fibration over $\tilde{B}$. And indeed, the generic elliptic fibration over $\tilde{B}$
has Hodge numbers \((317, 17)\) rather than the values of \((321, 9)\) expected from mirror symmetry. This can be understood from the fact that the vertex \((-6, 2)\) is not present in \(\hat{B}\), so the monomials in \(-6K_B\) are not simply the points in \(B = \mathbb{F}_6\) but also include the lattice points \((0, -k), k \in \{4, 5, 6\}\). In the polytope \(\nabla\) these monomials are all set to vanish. Computing the resulting gauge group structure on \(\hat{B}\) we find that there is a gauge group \(SU(2) \times G_2 \times SU(2)\) tuned on the middle sequence of three \(-2\) curves. This gives an additional rank contribution of 4 to \(h^{1,1}(\hat{X})\) from the gauge group and there are contributions to \(h^{2,1}(\hat{X})\) of +20 from the dimension of the tuned gauge group, and \(-28\) from the charged matter fields,\(^{10}\) so that the correct Hodge numbers are found for the tuned Calabi-Yau

\[
(321, 9) = (317, 17) + (4, -8).
\] (5.30)

When \(n\) does not divide 12, there are additional vertices of \(\nabla\) that must be included to attain a reflexive polytope from the simple stacking of \(\mathbb{F}_n\).

\(\mathbb{F}_5\):

For example, when \(n = 5\), \((0, -3, -3, -2)\) is an additional vertex that must be included.

\(^{10}\)Note that the correct counting of \(h^{2,1}\) considers only the matter fields charged under the Cartan subalgebra. The counting in this case is equivalent to that of the rank preserving tuning of \(SU(2) \times SU(3) \times SU(2)\), where the \(SU(3)\) has 6 hypermultiplets charged in the fundamental \((3)\) representation, 4 of which are in bifundamentals with the \(SU(2)\) factors. In this case the \(SU(3)\) fundamentals combine in pairs into \(G_2\) fundamentals, and the \(G_2\) has one additional fundamental \((7)\), which contributes another 6 hypermultiplets charged under the Cartan, canceling the difference in dimension between \(SU(2)\) and \(G_2\). This rank-preserving tuning between \(SU(2)\) and \(G_2\) connects two phases of the same Calabi-Yau geometry.
cluded with the set in (5.28) for \( \nabla \) to be reflexive\(^{11} \), so the projection of the whole \( \nabla \) polytope to the base plane is instead the polytope defined by the vertices

\[
\{(1, 0), (0, 1), (-1, -5), (0, -3)\},
\]  

(5.31)

and the convex hull of the toric fan of \( B \) lies within this polytope. On the dual side, \( \tilde{X} \) is a generic elliptic fibration with Hodge numbers \((295, 7)\). The polytope defined by all monomials in \( \mathcal{O}(-6K_B) \) (defined using the rays in \( B \) from equation (5.31)) has the set of vertices

\[
\{(-6, 2), (-4, 2), (-6, -6), (36, -6)\}\).  

(5.32)

This is the projected polytope of the whole \( \Delta \) to the base. The primitive rays in this projected polytope define the base \( \tilde{B} \), which is characterized by the self-intersection sequence

\[
[[-12// - 11//(-12//)6 - 11// - 12, -1, -2, -2, -3, -1, -3, -2, -2, -1]], \quad (5.33)
\]

and the vertices of the convex hull of the toric fan of \( \tilde{B} \) are

\[
\{( -5, 2), (-6, 1), (1, 1), (-6, -5), (-5, -6), (31, -5), (35, -6)\}\}.  

(5.34)

which is contained in the projected polytope. In this case the mirror \( \Delta \) is again a generic (non-tuned) elliptic fibration over the mirror base \( \tilde{B} \), and the Hodge numbers can be computed directly from the non-Higgsable clusters on the base with intersections (5.33).

We list the results of the remaining \( \mathbb{F}_n \) cases in Table 5.1 for \( 0 \leq n \leq 8, n = 12 \). For \( n = 9, 10, 11 \), additional blowups at points in the base that may be toric or non-toric are required to support a flat elliptic fibration, and there are various ways to resolve these bases (see Table 3.13). We find that the corresponding mirror fibrations are generic models over different dual bases \( \tilde{B} \) for different resolutions. The results are

\(^{11}\) Additional vertices from tops for all generic fibrations over Hirzebruch surfaces can be looked up in Table 3.1.
listed in Table 5.2.
<table>
<thead>
<tr>
<th>$B$</th>
<th>$(h^{1,1}, h^{2,1})$</th>
<th>Mirror base $\tilde{B}$</th>
<th>tuning</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbb{F}_0$</td>
<td>(3,243)</td>
<td>$(-12/-11/)^3 - 12/-11/-1,-2,-2,-3,-1,-5,-1,-3,-2,-2,-1$</td>
<td></td>
</tr>
<tr>
<td>$\mathbb{F}_1$</td>
<td>(3,243)</td>
<td>$-12/-11/(-12/)^2 - 11/-12/-11/-11,-1,-2,-2,-3,-1,-5,-1,-3,-2,-2,-1$</td>
<td></td>
</tr>
<tr>
<td>$\mathbb{F}_2$</td>
<td>(3,243)</td>
<td>$-12/-11/(-12/)^3 - 11/-12/-10,-1,-2,-2,-3,-1,-5,-1,-3,-2,-2,-1$</td>
<td></td>
</tr>
<tr>
<td>$\mathbb{F}_3$</td>
<td>(5,251)</td>
<td>$-12/-11/(-12/)^4 - 11/-12,-1,-2,-2,-3,-1,-5,-1,-3,-1,-5,-1,-3,-2,-2,-1$</td>
<td>$\mathfrak{g}_2$</td>
</tr>
<tr>
<td>$\mathbb{F}_4$</td>
<td>(7,271)</td>
<td>$-12/-11/(-12/)^5 - 11/-12,-1,-2,-2,-3,-1,-4,-1,-3,-2,-2,-1$</td>
<td>$\mathfrak{f}_4$</td>
</tr>
<tr>
<td>$\mathbb{F}_5$</td>
<td>(7,295)</td>
<td>$-12/-11/(-12/)^6 - 11/-12,-1,-2,-2,-3,-1,-3,-2,-2,-1$</td>
<td></td>
</tr>
<tr>
<td>$\mathbb{F}_6$</td>
<td>(9,321)</td>
<td>$-12/-11/(-12/)^7 - 11/-12,-1,-2,-2,-2,-2,-2,-1$</td>
<td>$\mathfrak{su}(2) \oplus \mathfrak{g}_2 \oplus \mathfrak{su}(2)$</td>
</tr>
<tr>
<td>$\mathbb{F}_7$</td>
<td>(10,348)</td>
<td>$-12/-11/(-12/)^8 - 11/-12,-1,-2,-2,-2,-2,-1$</td>
<td>$\mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$</td>
</tr>
<tr>
<td>$\mathbb{F}_8$</td>
<td>(10,376)</td>
<td>$-12/-11/(-12/)^9 - 11/-12,-1,-2,-2,-2,-2,-1$</td>
<td>$\mathfrak{sp}(1)$</td>
</tr>
<tr>
<td>$\mathbb{F}_{12}$</td>
<td>(11,491)</td>
<td>$-12/-11/(-12/)^{13} - 11/-12,0$</td>
<td></td>
</tr>
</tbody>
</table>

Table 5.1: For each Hirzebruch base $B = \mathbb{F}_n$, $0 \leq n \leq 8$, $n = 12$ the Hodge numbers of the generic elliptic fibration $X$ over $B$, the toric structure of the base $\tilde{B}$ of the mirror Calabi-Yau threefold $\tilde{X}$, and any gauge algebra tuned (on the boldfaced curves) in $\tilde{B}$. Note that the tunings on the mirror bases for $\mathbb{F}_3, \mathbb{F}_4$ are rank-preserving tunings that do not change the Hodge numbers (corresponding to a different phase of the same Calabi-Yau); the generic elliptic CY fibrations over the mirror bases in these cases are mirror to corresponding rank-preserving tunings on the $-3, -4$ curves of the original $\mathbb{F}_3, \mathbb{F}_4$ respectively.
5.3 Some Further Examples

In this section we consider some further examples that go beyond generic CY elliptic fibrations over toric bases. We first consider some cases of Tate tunings of the generic elliptic fibrations. A Tate tuning over a toric base generally gives a reflexive 4D polytope with a standard stacking form and the $F_{10}$ fiber, where the Tate tuning corresponds to reducing the monomials in $\Delta$. There is a natural correspondence between these Tate tunings and tops involving additional lattice points in $\nabla$. In general, for a Tate tuning where $a_1, \ldots, a_6$ vanish to orders $[a_n]_i$ on the toric divisor associated with the ray $v_i^B$, the vanishing of the monomials in $a_n$ corresponds to a removal of the points in $\Delta$ that lie over the point in $\Delta_2$ associated with sections of the line bundle $\mathcal{O}(-nK_B)$. For many Tate tunings, as for the generic elliptic fibration, the remaining points in the toric representation of $\mathcal{O}(-6K_B)$ are a superset of the remaining points in the other $\mathcal{O}(-nK_B)$'s. This is equivalent to the condition that the mirror polytope $\Delta$ has a standard stacking form so that all rays in the base have a preimage under the fiber projection of the form $(\cdot, \cdot; -1, -1)$. In such situations, the set of rays of the dual base becomes

$$\{w_i^B\} = V^B(a_6),$$

(5.35)

where\(^1\)

$$V^B(a_n) = \{w = (w_1, w_2) | \text{GCD}(w_1, w_2) = 1, w \cdot v_i^B \geq -n + [a_n]_i \ \forall v_i^B \in \Sigma_B\}.$$  

(5.36)

We describe explicit examples of this in §5.3.1 and §5.3.2. More generally, there are some Tate tunings (such as $SU(6)$) that place such stronger constraints on $a_6$ than on other coefficients $a_n$. This can occur when $6 - [a_6]_i < n - [a_n]_i$ for some $i$ and $n \leq 4$. In such cases, the mirror $\Delta$ is no longer a standard stacking but we can still

\(^1\)Note that $V^B(a_n)$ can also be described simply as a set of primitive rays in 2D toric coordinates associated with the monomials in $a_n$, with the product $\Pi_i z_i$ of the toric variables $z_i$ associated with the rays of $\Sigma_B$ taken as the origin.
<table>
<thead>
<tr>
<th>$B$</th>
<th>$(h^{1,1}, h^{2,1})$</th>
<th>Resolved $B$</th>
<th>Mirror base $\hat{B}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F_9$</td>
<td>$(14, 404)$</td>
<td></td>
<td>$-12!/!-11!/!(-12!/!)^{10} - 11!/!-9, 0$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(*)</td>
<td>$-12!/!-11!/!(-12!/!)^{10} - 11!/!-12, -1, -2, -2, -1$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$-11!/!-11!/!(-12!/!)^{10} - 11!/!-10, 0$</td>
</tr>
<tr>
<td></td>
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<td></td>
<td>$-12!/!-11!/!(-12!/!)^{10} - 11!/!-10, -1, -1$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$-12!/!-11!/!(-12!/!)^{10} - 11!/!-11, -1, -2, -1$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$-11!/!-11!/!(-12!/!)^{10} - 11!/!-11, -1, -1$</td>
</tr>
<tr>
<td>$F_{10}$</td>
<td>$(13, 433)$</td>
<td></td>
<td>$-12!/!-11!/!(-12!/!)^{11} - 11!/!-10, 0$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(*)</td>
<td>$-12!/!-11!/!(-12!/!)^{11} - 11!/!-12, -1, -2, -1$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$-11!/!-11!/!(-12!/!)^{11} - 11!/!-11, 0$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>$-12!/!-11!/!(-12!/!)^{11} - 11!/!-11, -1, -1$</td>
</tr>
<tr>
<td>$F_{11}$</td>
<td>$(12, 462)$</td>
<td></td>
<td>$-12!/!-11!/!(-12!/!)^{12} - 11!/!-11, 0$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(*)</td>
<td>$-12!/!-11!/!(-12!/!)^{12} - 11!/!-12, -1, -1$</td>
</tr>
</tbody>
</table>

Table 5.2: For each Hirzebruch base $B = F_n$, $n = 9, 10, 11$ the Hodge numbers of the generic elliptic fibration $X$ over resolved $B$, the toric structure of the resolved base and the base $\hat{B}$ of the mirror Calabi-Yau threefold $\hat{X}$. The bases marked with (*) are resolved at generic non-toric points, and correspond to the naive stacking models with the given Hirzebruch base, while the other bases are resolved at combinations of toric and non-toric points.
give an explicit description of the base $\tilde{B}$ in terms of $B$ and the Tate tuning,

$$\{w_i^{\tilde{B}}\} = \cup_{n \in \{1, 2, 3, 4, 6\}} V^B(a_n).$$  (5.37)

This is essentially a rewriting of (5.6) for the mirror polytope $\Delta$ when $\nabla$ is a standard stacking type polytope with the fiber $F_{10}$. We describe an explicit example of this in §5.3.3. In terms of the monomial polytope $\Delta$, the set of points determined by equation (5.36) are given by

$$V^B(a_n) = \{(m_1, m_2)/\text{GCD}(m_1, m_2)\mid m = (m_1, m_2, m^{\tilde{F}}(n)) \in \Delta\},$$  (5.38)

where $m^{\tilde{F}}(n)$ represents the two coordinates of the particular lattice point $m^{(II)} \in \Delta_2$ that is associated with monomials in $a_n$. This formulation was used in some of the following explicit computations. In the coordinate system we adopt in this chapter

$$m^{\tilde{F}}(n = 1, 2, 3, 4, 6) = \{(0, 0), (-1, 1), (0, -1), (-1, 0), (-1, -1)\},$$  (5.39)

$$m^F(n = 1, 2, 3, 4, 6) = \{(0, 0), (-1, 0), (-1, -1), (-2, -1), (-3, -2)\}.  (5.40)$$

For example, $V^B(a_6) = \{(m_1, m_2)/\text{GCD}(m_1, m_2)\mid m \in \Delta$ of the form $(m_1, m_2, -1, -1)\}$.

In principle, we can use this same kind of analysis to describe other kinds of fibrations with general fiber types $F_i$, though details of the structure of tuned models will be different and depend on tops for the different fibers [40]. We leave a more complete analysis of the general situation to future work.

In §5.3.4, we consider a case of a stacked fibration with another fiber type, in §5.3.5 we illustrate a case of a fibered polytope where the mirror is not fibered, and in §5.3.6 we give an example of the factorization of mirror symmetry for a Calabi-Yau fourfold constructed as a generic elliptic fibration over a toric threefold base. These examples simply illustrate some of the directions in which the framework developed here can be extended, and a more detailed analysis of these directions is left for the future.
5.3.1 Tunings of Generic Fibrations (Example: Tuning an $SU(2)$ on $\mathbb{P}^2$)

We consider tuning an $SU(2)$ on one of the $+1$-curves in the $\mathbb{P}^2$ base of the generic fibration model in §5.2.2. The corresponding $\Delta$ polytope of the tuned model can be constructed by reducing the set of lattice points in the $\Delta$ polytope of the generic model (detailed examples of constructing tuned polytope models can be found in §3.3.2); this corresponds to the polytope in the case database with data: $M:316$ $N:11$ $H:3,231$. The new Hodge numbers match with the prediction from F-theory physics of tuning the $SU(2)$ on a $+1$-curve $(2, 272) + (1, -41) = (3, 231)$. The standard stacking polytope $\nabla$ has vertices

\{v_i\} = \{(0, 1; -3, -2), (-1, -1; -3, -2), (1, 0; -3, -2), (1, 0; -2, -1), (0, 0; 1, 0), (0, 0; 0, 1)\}.

Now there is a non-trivial top over the base divisor associated with the ray $(1, 0)$ due to the $SU(2)$ tuning over the divisor, which we can interpret in terms of a Tate tuning in $\Delta$. $\nabla$ has the usual $(0, 0; v^F)$ form of $F_{10}$ fiber as the only 2D subpolytope $\nabla_2$. The dual polytope $\Delta$ has vertices

\{w_i\} = \{(-4, -6; -1, -1), (-4, 10; -1, -1), (-2, -2; -1, 1), (-2, 4; -1, 1), (12, -6; -1, -1), (0, 0; -1, 2), (0, 0; 1, -1)\} (5.41)

The $\Delta$ polytope has multiple distinct fibrations, consisting of four $F_{10}$ fibers, three $F_{13}$ fibers, and one $F_{16}$ fiber. In particular, there is a fiber $\Delta_2$ dual to $\nabla_2$, which has lattice points also in the form $(0, 0; w^F)$. This gives the mirror fibration of the $\Delta$ polytope $\check{X}$ when viewed as a fan polytope.\(^{13}\) We can calculate the dual base $\check{B}$ by

\(^{13}\)Note that $\Delta$ does not project onto the other fibers (the projection strictly contains the other fibers); otherwise $\nabla$ would have had more than one fiber subpolytope.
equation (5.6) for the polytope $\Delta$, giving the self-intersection sequence

\[
\hat{B} \rightarrow \quad \langle -12, \quad -12, \quad -11, \quad -12, \quad -12, \quad -1, \quad -2, \quad -3, \quad -1, \quad -5, \quad -1, \quad -3, \quad 6242 \\
-1, \quad -8, \quad -1, \quad -2, \quad -3, \quad -2, \quad -1, \quad -8, \quad -1, \quad -2, \quad -3, \quad -2, \quad -1, \quad -8, \quad -1, \quad -2, \quad -3, \quad -2, \\
-1, \quad -8, \quad -1, \quad -2, \quad -3, \quad -1, \quad -5, \quad -1, \quad -3, \quad -2, \quad -2, \quad -1 \rangle.
\]

This is exactly the base over which the generic elliptic fibration has Hodge numbers $(231, 3)$. Therefore, we know $\Delta$ is the fan polytope associated with this generic fibration model. This generic fibration is thus the mirror fibration to the tuned $SU(2)$ fibration model.

Note that this base has a smaller toric diagram than the base described in (5.17). There are additional constraints on the rays $w$ in (5.12) in this case coming from base rays $v_B$ over points in the fiber $V_2$ other than $v_s$. We can see how this works explicitly as an example of (5.35). Tuning an $SU(2)$ gauge factor using a Tate tuning over the divisor in $\mathbb{P}^2$ associated with $v_B = (1,0)$ requires tuning the coefficients in (5.13) to vanish to orders $([a_1], [a_2], [a_3], [a_4], [a_6]) = (0, 0, 1, 1, 2)$ on this divisor. Thus, the rays in $\hat{B}$ are restricted to a subset of those in $\mathcal{O}(-6K_B)$, which satisfy the additional condition $w \cdot v_3^B \geq -4$, so $w_1 \geq -4$. This is already no stronger than the constraint on any of the other sections $\mathcal{O}(-nK_B), n \leq 4$, so we can use (5.35) and there are no other contributions from the more general formula (5.37). From Figure 5-3 we see that the base $\hat{B}$ is thence associated with the primitive rays in the polytope with vertices $(-4,10), (-4,-6), (12,-6)$. This matches perfectly with the projection from (5.41).

As $\Delta$ is also a generic elliptic fibration associated with a standard stacking polytope with stacking point $w_s = (-1,-1)$, we can calculate $B \sim \mathcal{O}(-6K_{\hat{B}})$, and confirm that the monomials in $\mathcal{O}(-6K_{\hat{B}})$ are all the lattice points of the form $\langle \cdot, \cdot, -3, -2 \rangle$ in the original polytope $\nabla$.

It is interesting to note that in this case while there is a tuning of the fibration on the $\nabla$ side, corresponding to a reduction in the size of $\hat{B}$ on the $\Delta$ side, the mirror is still a generic elliptic fibration over the new base. We now consider a case where
there are tunings on both sides.

5.3.2 Tunings of Generic Fibrations over Base $B$ and Mirror Base $\tilde{B}$

We now consider an example where both the fibration and the mirror fibration are tuned models. There is only one polytope associated with a CY3 with Hodge numbers $(6, 248)$, which is a standard stacking polytope. The polytope $\nabla$ has vertices

$$\{v_i\} = \{(1, 0; -3, -2), (0, 1; -3, -2), (-1, -3; -3, -2), (0, -1; -1, 0), (0, -2; -3, -2), (0, 0; 1, 0), (0, 0; 0, 1)\}. \quad (5.43)$$

The obvious $F_{10}$ fiber in the plane $(0, 0; \cdot, \cdot)$ is the only fiber of $\nabla$. The associated CY3 is a Tate tuned model over the base $F_3$ with $\mathfrak{so}(7)$ gauge symmetry enhanced on the $-3$-curve. The dual polytope $\Delta$ has vertices

$$\{w_i\} = \{(24, -6; -1, -1), (0, 2; -1, -1), (-6, -6; -1, -1), (-6, 2; -1, -1), (-2, 2; -1, 0), (-4, 2; -1, 0)(0, 0; -1, -1), (0, 0; -1, 2)\}. \quad (5.44)$$

The dual polytope $\Delta$ has three $F_{10}$ fibers and one $F_{13}$ fiber. The dual fiber with lattice points in the form $\{0, 0; \cdot, \cdot\}$ gives the mirror fibration with Hodge numbers $(248, 6)$. $\Delta$ is a standard stacking polytope with stacking point $w_s = (-1, -1)$ with respect to the fiber. The mirror base, by direct calculation (taking the rays of $\tilde{B}$ to be the primitive rays of all the projected 4D lattice points in $\Delta$ using (5.6) on $\Delta$) is

$$\tilde{B} \rightarrow [[-4, -1, -4, -1, -3, -2, -2, -1, -12// -11// -12// -12// -12// -12// -11// -12, -1, -2, -2, -3, -1]]. \quad (5.45)$$

The generic fibration over $\tilde{B}$ has Hodge numbers $(247, 7)$, so this is a tuned model. By explicit analysis of the Weierstrass model over $\tilde{B}$ associated with the polytope $\Delta$, we know there are enhanced gauge symmetries $\mathfrak{so}(9) \oplus \mathfrak{sp}(1) \oplus \mathfrak{so}(9)$ on the first
three curves $-4, -1, -4$. The Hodge number shifts calculated from F-theory physics (with shared matter representations carefully considered) are $(1, -1)$, which agrees with the Hodge numbers associated with $\Delta (247, 7) + (1, -1) = (248, 6)$.

Unlike the previous example, in this case we have tunings both on the original fan polytope ($\nabla$) and on the mirror ($\Delta$) side. Nonetheless, the tunings of gauge symmetries in $X$ and $\tilde{X}$ allow us to use equation (5.35) to calculate the bases in both cases:\footnote{We can safely use equation (5.35) without going to the general formula (5.37) in the absence of Tate tunings of gauge symmetries $SU(n), n \geq 6$ and $SO(n), n \geq 13$.}

\begin{equation}
\{w_i^{(B)}\} = V^B(a_6) \text{ and } \{v_i^{(B)}\} = V^{\tilde{B}}(\tilde{a}_6),
\end{equation}

where $a_6$ is associated with lattice points in $\Delta$ of the form $(\cdot, \cdot, -1, -1)$, and where $\tilde{a}_6$ is associated with lattice points in $\nabla$ of the form $(\cdot, \cdot, -3, -2)$ (cf. equations 5.39 and 5.40.) All the examples we have considered so far are cases in which equation (5.35) applies to the calculation of both $B$ and $\tilde{B}$. We now consider a case in which the more general formula (5.37) is required.

### 5.3.3 Standard Stacking $F_{10}$-fibered $\nabla$ vs. Non-standard $F_{10}$-fibered $\Delta$

We now consider tuning an $SU(6)$ on one of the $+1$-curves in the $\mathbb{P}^2$ base of the generic CY elliptic fibration model in §5.2.2. The tuned model corresponds to the polytope M:207 11 N:15 8 H:7,154 in the KS database. The new Hodge numbers match with the prediction from F-theory physics of tuning the $SU(6)$ on a $+1$-curve: $(2, 272) + (5, -118) = (7, 231)$. The standard stacking polytope $\nabla$ has vertices

\begin{equation}
\{v_i\} = \{(0, 0; 1, 0), (0, 1; -3, -2), (-1, -1; -3, -2), (1, 0; -1, -1), (1, 0; -3, -2), (0, 0; 0, 1), (1, 0; 0, 1), (1, 0; 0, 0)\}.
\end{equation}
The dual polytope $\Delta$ has vertices

$$\{w_i\} = \{(-1,-4;-1,0), (-1,-2;-1,1), (-1,-1;0,0), (-1,2;0,0), (-1,3;-1,1), (-1,5;-1,0),
(0,-6;-1,-1), (0,0;-1,2), (0,0;1,-1), (0;-1,-1), (12,-6;-1,-1)\}. \quad (5.48)$$

This is however not a standard stacking $\mathbb{P}^{2,3,1}$ polytope, which can be seen from the feature that there is more than one monomial in the coefficients of $x^3$ or $y^2$ in the Tate form as discussed in §3.6.2: two lattice points $(0,0,0,1)$ and $(1,0,0,1)$ in (5.47) contribute to the $x^3$ terms. The set of rays in the dual base $\tilde{B}$ is given by equation (5.37), and in this case

$$\bigcup_{n \in \{1,2,3,4,6\}} V^B(a_n) = V^B(a_4) \cup V^B(a_6), \quad (5.49)$$

which gives a 2D toric fan with the self-intersection numbers

$$\tilde{B} \to [[-10/ / -12/ / -11/ / -12/ / -10, -1, -2, -2, -2, -2, -2, -2, -2, -2, -2, -1]].$$

The generic elliptic fibration over $\tilde{B}$ has Hodge numbers $(143, 23)$, so $\tilde{X}$ is a tuned model. As $\Delta$ is not a standard stacking $\mathbb{P}^{2,3,1}$ polytope, we do not have a Weierstrass model from a Tate form for this polytope. Nonetheless, the Weierstrass model can be obtained with the trick of “treating $\Delta$ as a Bl$_2\mathbb{P}^{1,1,2}$-fibered polytope” as described in §3.6.2. This is a tuned Weierstrass model with tunings of gauge symmetries $\mathfrak{su}(2) \oplus \mathfrak{su}(3) \oplus \mathfrak{su}(3) \oplus \mathfrak{su}(3) \oplus \mathfrak{su}(2)$ enhanced on $-2, -2, -2, -1, -2, -2, -2, -2, -2, -2, -2, -2, -1$, and also an $\mathfrak{su}(2)$ gauge symmetry enhanced on a non-toric 0-curve intersecting the two $-10$-curves. The polytope $\Delta$ gives a non-flat elliptic fibration model; we can however find an equivalent flat elliptic fibration description with the same tuning of gauge symmetries over a resolved base. In the resolved base, the original $-10$-curves are resolved to $-12$-curves through four successive blowups, and the non-toric 0-curve is replaced by curves $-1, -2, -2, -2, -1$ where the two $-1$-curves intersect with the two $-12$-curves, respectively (see figure 5-6). The $\mathfrak{su}(2)$ gauge symmetry that was enhanced on the non-toric 0-curve is now enhanced on the middle $-2$-curve in the
blowup sequence; this is therefore a flat elliptic fibration. The Hodge number shifts of the flat fibration model calculated from the tunings match exactly with the polytope model: \((11, -16) + (1, -5) = (154, 7) - (143, 23)\).

### 5.3.4 Other Toric Fibers (Example: Vertex Stacking on Fiber \(F_2 = \mathbb{P}^1 \times \mathbb{P}^1\))

For the other examples we have considered so far we have restricted attention to fibrations with the fiber \(F_{10}\) and the “standard stacking” form. The mirror symmetry structure also factorizes with other fiber types, though the physics of the corresponding F-theory models is more complicated and does not follow from standard Tate tuning structures. We give one example here of another toric fiber type, and leave further exploration of mirror symmetry with other fiber structures to further work.

As a simple example of a mirror pair of elliptically fibered CY3s associated with fibered polytopes with different fiber types, we start with an \(F_2\)-fibered polytope with base \(B = \mathbb{P}^2\). We again use the “stacked” form where all the rays of the fan of \(B\) are embedded within \(\nabla\) in the form \((v_i^B, v_s), v_s \in \nabla_2\). Let the stacking point be one of
the $F_2$ vertices $v_s = (1, 0)$. Therefore, the polytope $\nabla$ has vertices

$$\{v_i\} = \{(-1, -1; 1, 0), (1, 0; 1, 0), (0, 1; 1, 0), (0, 0; 0, 1), (0, 0; 0, -1), (0, 0; -1, 0)\}.$$  

(5.50)

This corresponds to the polytope given by the data M:117 8 N:8 6 H:4,94 [-180] in the KS database.

The mirror polytope also has a simple structure. The dual polytope $\Delta$ has vertices

$$\{w_i\} = \{(-2, -2; 1, 1), (4, -2; 1, 1), (-2, 4; 1, 1), (-2, 4; 1, -1), (-2, -2; 1, -1), (4, -2; 1, -1), (0, 0; -1, 1), (0, 0; -1, -1)\}.$$  

(5.51)

The mirror fiber $\tilde{F} = F_{15}$ has the vertices

$$\{(0, 0, -1, -1), (0, 0, 1, -1), (0, 0, 1, 1), (0, 0, -1, 1)\}.$$  

(5.52)

Over the points in the mirror fiber, we have points associated with the monomials in $\mathcal{O}(-2K_B)$ over all the points $(1, y), y = -1, 0, 1$, and points associated with $\mathcal{O}(-K_B)$ over the points $(0, y), y = -1, 0, 1$, and only the points $(0, 0; -1, y)$ over the remaining points in the fiber. From this we see that the dual base $\tilde{B}$ is given by the toric surface with the self-intersection sequence that can be read off from $\mathcal{O}(-2K_B)$,

$$\tilde{B} \rightarrow [[-1, -4, -1, -3, -1, -4, -1, -4, -1, -3, -1, -4, -1, -4, -1, -3, -1, -4]].$$

While the factorization of mirror symmetry is equally clear in this example to the others considered here, the F-theory interpretation is more subtle. A full analysis involves considerations using methods like those of [40, 71]. We outline the analysis on the $\nabla$ side in this case and leave further work in this direction to the future. The polytope $\nabla$ has the single obvious fiber $F_2$, which does not provide a section, so this

\[\text{Note that } \Delta \text{ has many distinct fibrations; the numbers of each of the 16 fiber types of } \Delta \text{ are } \{6, 0, 6, 6, 6, 6, 0, 12, 6, 9, 0, 0, 9, 6, 4, 1\}. \text{ We can immediately read off the mirror fibration however from the form of } \nabla, \text{ and none of the other fibrations has a corresponding projection since the fiber of } \nabla \text{ is unique.}\]
is a genus one fibration. We can analyze the gauge group and matter structure of the corresponding Jacobian fibration, which is relevant for F-theory [50]. The Weierstrass model of the Jacobian fibration has no nonabelian gauge symmetries. From the Hodge numbers we expect a nontrivial Mordell-Weil group of rank 2,

\[ G = U(1) \times U(1). \]  

(5.53)

Codimension two singularities in the Weierstrass model suggest \( 36 + 72 + 72 \) matter fields charged in various ways under the U(1) factors in \( G \), which is in accordance with the expected Hodge numbers \( (4, 94) = (2, 272) + (2, 2 - (36 + 72 + 72)) \).\(^{16}\)

### 5.3.5 Elliptic Fibration with a Non-fibered Mirror

As a final Calabi-Yau threefold example we consider a case where a CY threefold has an elliptic fibration associated with a 2D reflexive subpolytope, but the mirror has no fibration. In such cases, there cannot be a projection onto the fiber of \( \Delta \), as discussed in §5.1.3.

We consider the polytope from the KS database associated with the Calabi-Yau threefold with Hodge numbers \( (h^{1,1}, h^{2,1}) = (149, 1) \). There is a coordinate system in which this polytope \( \nabla \) has vertices

\[ \{v_i\} = \{(0, 0; 1, 0), (-2, 8; -3, -2), (-2, 0; -3, -2), (6, 0; -3, -2), (-2, -8, 5, 6)\} . \]

(5.54)

In this coordinate system there is an \( F_{10} \) reflexive 2D fiber in the standard form \( (0, 0; v^F) \). The polytope does not satisfy the condition needed for the mirror to have a fibration, however, since the last vertex \( v = (v^{(I)}; v^{(II)}) \), \( v^{(II)} = (5, 6) \) does not satisfy \( v^{(II)} \in \nabla_2 \). Furthermore, it is straightforward to see that no linear transformation that preserves \( \nabla_2 \) can move all the vertices to satisfy this condition. In particular,

\(^{16}\)Note that from [71], the F-theory models associated with the Jacobian fibrations of \( F_2 \) fibered polytopes should generically have toric Mordell-Weil group \( U(1) \times \mathbb{Z}_2 \). The stacked form we have here over a vertex of \( F_2 \) gives a degeneration that may enhance the \( \mathbb{Z}_2 \) to a U(1) factor. (Thanks to Paul Oehlmann for explaining this to us.)

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the second and last vertices will always have \( v^{(II)} \) values that differ by a vector of the form

\[
v_5^{(II)} - v_2^{(II)} = (8, 8) + 16(x, y), \quad x, y \in \mathbb{Z}.
\] (5.55)

Thus, the mirror can never satisfy the fibration condition. This is not surprising since the mirror has Hodge numbers \((1, 149)\) and cannot have a genus one or elliptic fibration since the Shioda-Tate-Wazir would give \( h^{1,1}(\bar{X}) \geq h^{1,1}(\bar{B}) + 1 \geq 2 \). One can also check explicitly that the lattice points in the mirror polytope \( \Delta \) do not contain a linearly embedded \( \mathbb{P}^{2,3,1} \) fiber.

### 5.3.6 Elliptic Calabi-Yau Fourfolds (Example: Generic Elliptic Fibration over \( \mathbb{P}^3 \))

The factorization structure that we have described for elliptic toric hypersurface Calabi-Yau threefolds can occur in much the same fashion for higher-dimensional Calabi-Yau manifolds realized as toric hypersurfaces. We leave a more detailed investigation of higher-dimensional Calabi-Yau mirror symmetry for future work, but note here simply that the simplest cases of generic elliptic fibrations over toric bases have a natural generalization to arbitrary higher dimension. We present here only a single simple case, the Calabi-Yau fourfold given by the generic elliptic fibration over the base \( \mathbb{P}^3 \). The polytope \( \nabla \) in this case is the simple generalization of the polytope with vertices given by Eq. (5.14), and has vertices

\[
\{v_i\} = \{(0, 0, 0; 1, 0), (0, 0, 0; 0, 1), (1, 0, 0; -3, -2), (0, 1, 0; -3, -2), (5.56) \\
(0, 0, 1; -3, -2), (-1, -1, -1; -3, -2)\}.
\] (5.57)

The Hodge numbers of the corresponding Calabi-Yau fourfold are \( h^{1,1}(X) = 2, h^{3,1}(X) = 3878 \). The dual polytope \( \Delta \) is again \( \mathbb{P}^{2,3,1} \) fibered, with a base \( \bar{B} \) given by the set of toric rays within the polytope having vertices

\[
\{v_i^{(B)}\} = \{(-6, -6, -6), (18, -6, -6), (-6, 18, -6), (-6, -6, 18)\}.
\] (5.58)
There are a tremendous number of ways of triangulating this 5D polytope to give a complete toric fan. Independent of the triangulation, however, the generic elliptic fibration over the toric base $\hat{B}$ will have a non-Higgsable gauge group

$$G = E_8^{34} \times F_4^{56} \times G_2^{256} \times SU(2)^{384}. \quad (5.59)$$

This can be seen from the numbers of primitive rays in $\hat{B}$ with specific minimum values of the inner products with the vertices in (5.57). For each triangulation, the mirror Calabi-Yau will have Hodge numbers $h^{1,1}(\hat{X}) = 3878, h^{3,1}(\hat{X}) = 2$. An interesting feature of the mirror Calabi-Yau fourfolds associated with the polytope $\Delta$ is that it is one of the "attractive" endpoints reached by blowing up $\mathbb{P}^3$ as far as possible consistent with a base that supports an elliptic fibration [84]. This may be associated with the large number of triangulations that are possible in this case. The other attractive endpoints also are factorizable mirrors over simple bases that are 3D toric varieties with few rays. We leave further exploration of the many interesting questions associated with mirror symmetry and Calabi-Yau fourfolds to further work.

5.4 Conclusions and Further Questions

5.4.1 Summary of Results

Building on the work in Chapter 4 in which we showed that most toric hypersurface Calabi-Yau threefolds have a manifest genus one or elliptic fibration, we have found that many of these elliptic fibrations exhibit a mirror symmetry that factorizes, in the sense that the fiber $F$ of $X$ is a mirror Calabi-Yau 1-fold to the fiber $\hat{F}$ of $\hat{X}$. This connects with a number of directions of earlier research related to aspects of such mirror fibers [68, 95, 96, 71, 102, 103, 97]. We have furthermore found that the structure of the mirror base $\hat{B}$ is determined in a clear and well-defined way from the base $B$ and fibration structure of $X$. In many cases of interest the mirror base takes a simple form in terms of the toric geometry of a line bundle over $B$. In particular, for generic CY elliptic fibrations over toric base surfaces, the mirror base $\hat{B}$ has a
toric fan that is built from the primitive rays in the set of sections of the line bundle $\mathcal{O}(-6K_B)$. For tuned Tate models over a toric base surface, there is a slightly more complicated expression for the rays in $\hat{B}$, given by (5.35) when $\Delta$ is also a standard stacking polytope, or (5.37) when $\Delta$ fails to maintain the standard stacking structure due to excessive removal of points associated with monomials in $\mathcal{O}(-6K_B)$. As shown in Chapter 3, almost all the Hodge number pairs in the KS database with $h^{1,1} \geq 240$ or $h^{2,1} \geq 240$ are realized by generic or tuned elliptic fibrations with the standard stacking structure.

We have explored some simple examples of this factorized mirror symmetry, particularly for some cases of Calabi-Yau threefolds with large Hodge numbers, and generic and tuned elliptic fibrations over simple bases such as $\mathbb{P}^2$ and the Hirzebruch surfaces $\mathbb{F}_n$.

With growing evidence that most known Calabi-Yau threefolds admit a genus one or elliptic fibration, the results we have found here suggest that there may be a very general way of understanding mirror symmetry in terms of fibration by smaller-dimensional Calabi-Yau fibers. Further work is clearly needed to explore the details of the mirror dictionary for different bases and fibers, in higher dimensions, and the extent to which the factorization structure identified here can be extended beyond the toric hypersurface framework.

5.4.2 Further Questions and Directions

We list here some specific open questions that may be of interest for further research.

- We have given a variety of examples here where mirror symmetry between a pair of elliptic Calabi-Yau threefolds factorizes between the base and the fiber of the fibrations. From these examples it is clear that there are regular local structures that could be used to begin to form a dictionary relating structure on the mirror base $\hat{B}$ to structure of the base $B$ and fibration structure of the original elliptic Calabi-Yau $X$. For example, a curve of self-intersection +1, such as was encountered in the example in §5.2.2, naturally corresponds to a sequence of toric curves of self-intersections // −12/> −12// in the mirror base $\hat{B}$. Similarly, in Table 5.2, we see a pattern where
blowing up a non-toric vs. toric point in the base $B$ corresponds to blowing up a toric
vs. non-toric point in the mirror base $\tilde{B}$. It would be interesting to try to system-
atically develop this kind of structure, ideally including the additional reductions on
the base $\tilde{B}$ that are imposed by different Tate tunings on the generic elliptic fibration
over a toric $B$, which can be understood through additional constraints arising from
the associated “tops”.

- We know that at large Hodge numbers many of the Calabi-Yau threefolds in the
Kreuzer-Skarke database are generic or tuned elliptic fibrations over toric bases that
can be constructed from polytopes with a fiber $F_{10}$, and these exhibit the simplest
forms of mirror factorization studied in §5.2, §5.3. It would be interesting to study
further what fraction of the KS database exhibit mirror factorization and what other
types of structures arise frequently or in isolated cases at smaller Hodge numbers.

- We have focused here primarily on the simplest cases where the toric 2D fiber
is the self-dual fiber $F_{10}$, corresponding to generic elliptic fibrations. It would be
interesting to study in more detail the structure of the other 2D toric fibers. In
particular, it was found in [71] that the dual fibers $F_i, \tilde{F}_i$ exhibit some interesting
structure, including identical numbers of sections associated with toric Mordell-Weil
rank, and a matching between Mordell-Weil torsion on one side and discrete sym-
metries associated with the Tate-Shafarevich/Weil-Chatalet group on the dual side.
It would be interesting to understand better how these features of the fibers can be
used in understanding mirror symmetry of the full Calabi-Yau threefolds $X, \tilde{X}$ with
the different dual fiber types.

- One natural way of trying to extend the analysis here is to look at complete
intersection Calabi-Yau varieties. A large class of complete intersection fibers were
analyzed in [102], and the properties of mirror fibers in these more general cases were
noted in this paper and studied more thoroughly in [103]. It would be interesting to
investigate these structures further in the context of full elliptic Calabi-Yau threefolds
(and fourfolds).

- One of the most powerful approaches to mirror symmetry that has been used
in earlier work is the Strominger-Yau-Zaslow (SYZ) picture [104], in which mirror
symmetry is realized by T-duality on a 3-torus fiber over a real threefold base. While this picture has led to some powerful insights into mirror symmetry, it is incompatible with the algebro-geometric structure of Calabi-Yau manifolds, and the 3-torus fibration of a general Calabi-Yau is extremely singular. The factorization structure identified here seems to match more naturally with ideas from algebraic geometry and involve more controlled singularity structures. It would be interesting to understand whether there is a way of relating the SYZ picture to the factorization structure found here.

- As described in a simple example in §5.3.6, the factorization structure explored here should be equally valid for higher-dimensional Calabi-Yau varieties, and particularly for Calabi-Yau fourfolds. It would be interesting to explore further the structures that arise for mirror symmetry of elliptic Calabi-Yau fourfolds.

- The mirror symmetry identified here between elliptic fibrations suggests that there may be an interesting corresponding duality in F-theory. This would be interesting to explore further.

- It would be interesting to connect this factorization structure of mirror symmetry to other aspects of mirror symmetry research and Calabi-Yau geometry. For example, it would be interesting to understand how the form of Calabi-Yau periods and the structure of the moduli space, or recent progress on the all genus amplitudes of topological string theory on elliptic Calabi-Yau threefolds [105] fits into this factorized picture.

- There are several contexts in which this structure of mirror symmetry may have an interesting physical interpretation or consequences for physics. Beyond the possibility mentioned above of a new duality in F-theory, for more conventional type II mirror symmetry the existence of an elliptic fibration on both sides of the duality suggests additional structure that may lead to physical insights. We have not explored this direction significantly in this work but it should be a very interesting direction for further study.
Appendix A

Classification of 2D Fiber Subpolytopes

A.1 The 16 Reflexive 2D Fiber Polytopes $\nabla_2$

We list here the 16 reflexive 2D polytopes $\nabla_2$. The dual polytopes $\Delta_2$ are listed in Appendix A.2. The mirror of fiber $F = F_i$ is the fiber $\tilde{F} = F_{17-i}$ for $i < 7, i > 10$; the fibers $F_i$ for $i = 7, 8, 9, 10$ are self-mirror up to linear transformations. For each lattice point $v^F$, we include the maximum value of $1 + v^F \cdot m^{(II)}, m^{(II)} \in \Delta_2$. With each polytope we also provide the value $I_{\text{max}}$ associated with the maximum possible value of $v \cdot m$ where $v \in \nabla_2, m \in \Delta_2$. As discussed in the main text, the three fibers $F_1 = \mathbb{P}^2, F_2 = \mathbb{P}^1 \times \mathbb{P}^1 = F_0, F_4 = F_2$ have no $-1$ curves, associated with divisors that give global sections; all other fibers have $-1$ curves and correspond to elliptic fibers.
of the Calabi-Yau threefold.
A.2 The 16 Dual Polytopes $\Delta_2$

The dual polytopes $\Delta_2$ for the 16 reflexive 2D fiber polytopes listed in the previous Appendix. For each fiber type $\nabla_2$ in Appendix A.1, a lattice point $v^{(F)} \in \nabla_2$ is given such that a fibration built from the stacking construction (§4.1.3) over the point $v^{(F)}$ allows the most negative curve self-intersection in the base among all stackings with that fiber.
A.3 Distribution of Polytopes with Each Fiber Type

The figures in this Appendix depict the distribution of Hodge numbers for the Calabi-Yau threefolds associated with the polytopes that have each type of reflexive 2D fiber. The largest values of $h^{1,1}$ and $h^{2,1}$ for Calabi-Yau threefolds associated with polytopes having each fiber type are shown in the figure. In each figure, the density scale at the right indicates the color coding according to the total number of fibrations at each Hodge number pair, which results both from the multiplicity of the fibers of a given polytope and from the multiplicity of the polytopes at each Hodge number pair.
A.4 Polytopes with Non-trivial Fibration Orbits in the Regions $h^{1,1}, h^{2,1} \geq 140$

The following table indicates the difference between the total number of fibrations and the number of inequivalent fibration classes under automorphisms in the relevant 16 cases.
<table>
<thead>
<tr>
<th>polytope data (in the format of the KS database)</th>
<th># fibrations for each of the 16 fibers</th>
<th># fibrations modulo the automorphism symmetry group</th>
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<tr>
<td>M:12 5 N:348 5 H:251,5 [[492]]</td>
<td>0 0 0 0 0 0 0 0 0 3 0 0 1 0 0 0</td>
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Bibliography


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