ELASTIC WAVE PROPAGATION IN HOMOGENEOUS TRANSVERSELY ISOTROPIC MEDIUM WITH SYMMETRY AXIS PARALLEL TO THE FREE SURFACE

by

Chi-fung Wang

B.S., National Taiwan University

(1966)

M.A., State University of New York at Stoney Brook

(1970)

SUBMITTED IN
PARTIAL FULFILLMENT
OF THE REQUIREMENTS FOR THE
DEGREE OF MASTER OF SCIENCE
at the
MASSACHUSETTS INSTITUTE OF
TECHNOLOGY
September, 1973

Signature of Author.......................................
Department of Earth and Planetary ...nces

Certified by....................................... 8/13/73
...sis Supervisor

Accepted by....................................... 8/13/73
...mental Committee
on Graduate Students

ELASTIC WAVE PROPAGATION IN HOMOGENEOUS TRANSVERSELY ISOTROPIC MEDIUM WITH SYMMETRY AXIS PARALLEL TO THE FREE SURFACE

by
Chi-fung Wang

Submitted to the Department of Earth and Planetary Sciences on September , 1973 in partial fulfillment of the requirement for the degree of Master of Science

ABSTRACT

Phase velocities of elastic wave propagation in a homogeneous transversely isotropic medium with symmetry axis parallel to the free surface of a half space is investigated. Approximate solutions of the problem of phase velocities of Rayleigh, horizontally propagating P and SH waves is obtained by means of perturbation method on the assumption that the deviation of the elastic coefficients from isotropy is small. In the case of horizontally propagating SV waves an exact solution is obtained. The vertical lamination model approximating fracture zones and the Olivine model based on Francis' hypothesis have been tested. The results derived from fracture zone model showing small anisotropy fail to explain the observed data. The Olivine model showing large azimuthal variation of P and Rayleigh waves needed some modification in such a way that the a axis of Olivine crystal will be distributed diffusely. Suitable choice of weighting functions averaging the orientation of a a axis will give close agreement between both observed and predicted P and Rayleigh waves.

Thesis Supervisor: Keiiti Aki
Title: Professor of Geophysics

TABLE OF CONTENTS

Introduction

A great deal of attention has been given to the anisotropies of the propagation velocities of seismic waves in connection with the investigation of the structure of the Earth's crust and upper mantle. An anisotropic medium is characterized by the change of its elastic properties with the direction. In seismology, the transverse isotropy in which the elastic properties remain invariant in the plane perpendicular to the symmetry axis has been of the greatest interest.

Stoneley (1949) is the first seismologist who discussed surface and body wave propagation in a homogeneous transversely isotropic half space with the symmetry axis normal to the free surface. Synge (1957) showed that the propagation of undamped Rayleigh waves do not exist unless the symmetry axis is either parallel or perpendicular to the free surface. Bulchwald (1961) discussed the waves radiated from a time-harmonic source and used the method of Fourier double integrals to obtain the equation for the velocity of Rayleigh waves which is the same as that obtained by Stoneley.

A rapid variation of elastic constants with depth may apparently have the same effect as an anisotropic medium on the propagation of long waves. Postma (1955) gave the explicit formula for the five elastic constants of homogeneous transversely isotropic medium which is equivalent to a periodic, isotropic two-layered medium in the long wave limit under the restriction that the Lame's Constants are positive. Backus (1962) applied the averaging technique to the constitutive relation and equation of motion to approximate an inhomogeneous isotropic and transversely isotropic horizontally layered medium by a long wave equivalent, but more slowly varying transversely isotropic inhomogeneous medium in the direction perpendicular to the layers. Their theories will motivate us to set up a model to approximate the crust in the Nazca Plate discussed in this paper.

The anisotropy of the oceanic uppermantle beneath the Mohorovicic discontinuity is first suggested by Hess (1964). In the light of the observed results of the measurements of P_n velocities showing that low velocities perpendicular to the fracture zones and high velocities parallel to them in the region near the ridge axis in the East and North East Pacific, he claimed that seismic anisotropy of the upper mantle beneath the oceans results from a preferential alignment of Olivine crystals and predicted that the b axis that tends to be perpendicular to the fracture zones could explain the observed anisotropy. Francis (1969) suggested, on the other hand, that the a axis of olivine crystals tend to point away from the ridge axis and the b and c axes will be randomly oriented in the vertical plane parallel to the ridge axis at the time that the oceanic lithosphere was produced.

Backus (1965) used perturbation technique to derive the general form of the azimuthal dependence of the phase velocity of the P_n waves as a function of the azimuth of the wave number vector. He showed that correct to the first order in perturbation, the expression for the P_n wave phase velocity can be written as

$$C_p(\theta) = A_1 + A_2 \cos 2\theta + A_3 \sin 2\theta + A_4 \cos 4\theta + A_5 \sin 4\theta$$

where the five coefficients A_i are functions of the elastic constants of the wave medium. Later on, all the investigators utilized the formula derived by Backus to interpret their observed results in seismic refraction measurements (Raitt, 1969; Morris, 1969; Keen and Barret, 1971; Raitt, Shor, and Morris, 1971).

Forsyth (1972) observed that there was a 2% azimuthal variation of Rayleigh wave phase velocity in the region of Nazca plate. Smith and Dahlen (1972) combined Rayleigh's principle and Backus' harmonic tensor decomposition to discuss the effect of small anisotropy on the propagation of surface waves of Rayleigh and Love type with the motivation of explaining Forsyth's observed value. They also obtained the general form of the dependence of surface wave dispersion on the azimuth

θ of the horizontal wave number vector.

Much more detailed study of the propagation of surface and body waves will be required in order to predict the degree of anisotropy of the earth's crust and upper mantle. However, it is worth adopting some simple models to compare the theoretically calculated results for comparison with observation. In this paper, two such simple models will be set up to explain Forsyth's observation and previously observed P_n wave anisotropies. For simplicity, we shall assume that the media in both are vertically homogeneous. The first model will be a laminated mantle in which soft vertical layers representing fracture zones are sandwiched alternately between hard layers. The second model will be a homogeneous half space made of Olivine crystals in which the orientations of the b and c axes are uniformly distributed over the vertical plane and the a axis distributes with a certain distribution function with respect to the normal to the ridge axis (Francis, 1969). Both models can be reduced to a homogeneous transversely isotropic half space with symmetry axis parallel to the free surface. In this case, the problem of the surface waves of Rayleigh type can be solved by means of the separation of the equations of motion and boundary conditions under the restriction of neglecting the displacement of SH type which is of the second order in perturbation. The powerful procedure analogous to that of Stoneley will be used in both the lamination model and the olivine upper mantle model. In order to relate the P_n wave anisotropy and Rayleigh wave anisotropy, body waves will also be considered in the equivalent homogeneous transversely isotropic medium derived from the second model and some of the necessary averaging technique will be taken into account. The purpose of this paper is to present the theory based on these two models to explain the observed P_n wave anisotropy and Rayleigh wave anisotropy simultaneously.

2. Statement of the problem and equation of motion

Our analysis is primarily concerned with the propagation of surface waves of Rayleigh type in two simple earth models: one is the model of 'fracture zone' in which soft vertical layers forming the fracture zones are sandwiched in the normal oceanic lithosphere, and the other half-space composed of olivine crystal in which the a axis of the olivine crystal tends to become perpendicular to the ridge axis and parallel to the horizontal plane, and the b and c axes orient randomly in the plane perpendicular to the a axis. In either case, the problem is reduced to finding the phase velocity of the surface waves of Rayleigh type in a homogeneous transversely isotropic medium, whose axis of symmetry is perpendicular to the ridge axis and parallel to the free surface.

Let $\{\xi_1, \xi_2, \xi_3\}$ be an orthonormal coordinate frame oriented in the right-hand sense at a point of the free surface with ξ_3 pointing in the direction of the axis of symmetry, directed vertically downward into the medium. To obtain expressions convenient for comparison with observations, we have to choose another orthonormal coordinate frame $\{\xi_1', \xi_2', \xi_3'\}$ obtained by rotating $\{\xi_1, \xi_2, \xi_3\}$ about the vertical axis in such a way that ξ_1' become parallel to a vertical plane containing the wave number vector.

In the notation of Grant and West (page 27), the stress components and strain components in a transversely isotropic body are related by the following constitutive relation.

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{13} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} (\lambda_\perp + 2\mu_\perp) & \lambda_\perp & 0 & 0 & 0 & 0 \\ \lambda_\perp & (\lambda_{\parallel} + 2\mu_{\parallel}) & 0 & 0 & 0 & 0 \\ \lambda_\perp & \lambda_{\parallel} & (\lambda_{\parallel} + 2\mu_{\parallel}) & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\mu_{\parallel} & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\nu & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\nu \end{bmatrix} \begin{bmatrix} e_{11} \\ e_{22} \\ e_{33} \\ e_{23} \\ e_{13} \\ e_{12} \end{bmatrix} \qquad (2\text{-}1)$$

Because of the existence of strain-energy function, there exists the symmetry of the elastic coefficients $C_{ijkl} = C_{jikl} = C_{ijlk} = C_{klij}$ and hence the elastic constant tensor of the medium in our problem can be written as the linear combination of tetradics (Morse and Fashbach, page 72) in the following form,

$$\overleftrightarrow{C}=\sum_{i_1,i_2,i_3,i_4=1}^{3} C_{i_1i_2i_3i_4}\bar{e}_{i_1}\bar{e}_{i_2}\bar{e}_{i_3}\bar{e}_{i_4}=\sum_{j_1,j_2,j_3,j_4=1}^{3} C'_{j_1j_2j_3j_4}\bar{e}'_{j_1}\bar{e}'_{j_2}\bar{e}'_{j_3}\bar{e}'_{j_4}$$

$$=(\lambda_\perp+2\mu_\perp)\bar{e}_1\bar{e}_1\bar{e}_1\bar{e}_1+(\lambda_\parallel+2\mu_\parallel)(\bar{e}_2\bar{e}_2\bar{e}_2\bar{e}_2+\bar{e}_3\bar{e}_3\bar{e}_3\bar{e}_3)$$

$$+\lambda_\perp(\bar{e}_1\bar{e}_1\bar{e}_2\bar{e}_2+\bar{e}_2\bar{e}_2\bar{e}_1\bar{e}_1+\bar{e}_1\bar{e}_1\bar{e}_3\bar{e}_3+\bar{e}_3\bar{e}_3\bar{e}_1\bar{e}_1)$$

$$+\lambda_\parallel(\bar{e}_2\bar{e}_2\bar{e}_3\bar{e}_3+\bar{e}_3\bar{e}_3\bar{e}_2\bar{e}_2)+\mu_\parallel(\bar{e}_2\bar{e}_3+\bar{e}_3\bar{e}_2)(\bar{e}_2\bar{e}_3+\bar{e}_3\bar{e}_2)$$

$$+\nu(\bar{e}_1\bar{e}_3+\bar{e}_3\bar{e}_1)(\bar{e}_1\bar{e}_3+\bar{e}_3\bar{e}_1)+\nu(\bar{e}_1\bar{e}_2+\bar{e}_2\bar{e}_1)(\bar{e}_1\bar{e}_2+\bar{e}_2\bar{e}_1) \qquad (2\text{-}2)$$

Suppose that the departure from isotropy is sufficiently small, we may assume that the elastic constant tensor $\overleftrightarrow{C}$ of the medium is perturbed from an isotropic tensor $\overleftrightarrow{C}_0$ to $\overleftrightarrow{C}=\overleftrightarrow{C}_0+\delta\overleftrightarrow{C}$ where $\delta\overleftrightarrow{C}$ is small compared to $\overleftrightarrow{C}_0$. Hence we take

$$\overleftrightarrow{C}_0=(\lambda_\parallel+2\mu_\parallel)(\bar{e}_1\bar{e}_1\bar{e}_1\bar{e}_1+\bar{e}_2\bar{e}_2\bar{e}_2\bar{e}_2+\bar{e}_3\bar{e}_3\bar{e}_3\bar{e}_3)$$

$$+\lambda_\parallel(\bar{e}_1\bar{e}_1\bar{e}_2\bar{e}_2+\bar{e}_2\bar{e}_2\bar{e}_1\bar{e}_1+\bar{e}_1\bar{e}_1\bar{e}_3\bar{e}_3+\bar{e}_3\bar{e}_3\bar{e}_1\bar{e}_1+\bar{e}_2\bar{e}_2\bar{e}_3\bar{e}_3+\bar{e}_3\bar{e}_3\bar{e}_2\bar{e}_2)$$

$$+\mu_\parallel\left[(\bar{e}_1\bar{e}_2+\bar{e}_2\bar{e}_1)(\bar{e}_1\bar{e}_2+\bar{e}_2\bar{e}_1)+(\bar{e}_1\bar{e}_3+\bar{e}_3\bar{e}_1)(\bar{e}_1\bar{e}_3+\bar{e}_3\bar{e}_1)+(\bar{e}_2\bar{e}_3+\bar{e}_3\bar{e}_2)(\bar{e}_2\bar{e}_3+\bar{e}_3\bar{e}_2)\right] \qquad (2\text{-}3)$$

$$\delta\overleftrightarrow{C}=(\delta\lambda+2\delta\mu)\bar{e}_1\bar{e}_1\bar{e}_1\bar{e}_1+\delta\lambda(\bar{e}_1\bar{e}_1\bar{e}_2\bar{e}_2+\bar{e}_2\bar{e}_2\bar{e}_1\bar{e}_1+\bar{e}_1\bar{e}_1\bar{e}_3\bar{e}_3+\bar{e}_3\bar{e}_3\bar{e}_1\bar{e}_1)$$

$$+\delta\nu\left[(\bar{e}_1\bar{e}_3+\bar{e}_3\bar{e}_1)(\bar{e}_1\bar{e}_3+\bar{e}_3\bar{e}_1)+(\bar{e}_1\bar{e}_2+\bar{e}_2\bar{e}_1)(\bar{e}_1\bar{e}_2+\bar{e}_2\bar{e}_1)\right] \qquad (2\text{-}4)$$

where $\delta\lambda=\lambda_\perp-\lambda_\parallel$, $\delta\mu=\mu_\perp-\mu_\parallel$ and $\delta\nu=\nu-\mu_\parallel$

Next, we shall introduce the infinitesimal displacement $\bar{u}(\bar{x},t)=u_1\bar{e}'_1+u_2\bar{e}'_2+u_3\bar{e}'_3$ arising from the wave propagation with the wave number vector $\bar{k}=k_1\bar{e}'_1+k_3\bar{e}'_3$;

then the strain dyadic tensor is

$$\overline{\overline{\varepsilon}}(\overline{x},t) = \frac{1}{2}\left[\nabla_{\overline{x}}\overline{u} + (\nabla_{\overline{x}}\overline{u})^{T}\right] \; ; \quad e_{ij} = \frac{1}{2}\left[\partial_{x_i} u_j + \partial_{x_j} u_i\right] \tag{2-5}$$

where T denotes the transpose, and the stress dyadic tensor $\overline{\overline{\sigma}}$ is given by

$$\overline{\overline{\sigma}}(\overline{x},t) = \overleftrightarrow{C} \cdot\cdot\, \overline{\overline{\varepsilon}}(\overline{x},t) = \left[\dot{\lambda}_3(\nabla_{\overline{x}})\overleftrightarrow{C}\right] \cdot \overline{u}(\overline{x},t) \tag{2-6}$$

or $$\sigma_{ij} = \sum_{i,j,k,l=1}^{3} C_{ijkl}\, e_{kl} = \sum_{i,j,k,l=1}^{3} C_{ijkl}\, \partial_{x_k}(\overline{u}\cdot\overline{e}_l) \tag{2-6'}$$

where $\left[\dot{\lambda}_3(\nabla_{\overline{x}})\overleftrightarrow{C}\right]$ is the operator which transforms the displacement fields $\overline{u}(\overline{x},t) = u_1\overline{e}_1' + u_2\overline{e}_2' + u_3\overline{e}_3'$ into the stress fields. $\sum_{i_1,i_2,i_3,i_4=1}^{3} C_{i_1 i_2 i_3 i_4}\, \partial_{x_{i_3}}(\overline{u}\cdot\overline{e}_{i_4})\, \overline{e}_{i_1}\overline{e}_{i_2}$.

In the discussion of the propagation of surface waves of Rayleigh type, we are mostly concerned with the regions devoid of sources. The initial approach is to obtain the source free equation of motion appropriate to the medium of our problem in terms of the spatial derivatives of the displacement components with respect to the orthonormal coordinate frame $\{\overline{e}_1, \overline{e}_2, \overline{e}_3\}$ and then use the method of displacement potentials for plane waves and the prescribed boundary condition at free surface to obtain the modified Rayleigh's equation and determine whether surface waves of Rayleigh type can or cannot exist in such a medium. A simple means to reach our goal is to express every vector and tensor in terms of their components with respect to the frame $\{\bar{e}_1', \bar{e}_2', \bar{e}_3'\}$.

In homogeneous source-free regions, for time-harmonic displacement fields with $\exp(-i\omega t)$ dependence, and assume that plane wave solutions of the type $\exp\left[-i(\omega t - \bar{k}\cdot\bar{x})\right]$ are admissible, then in the equation of motion, ∂_t can be replaced by $-i\omega$ and $\nabla_{\bar{x}} = \partial_1 \bar{e}_1' + \partial_3 \bar{e}_3'$ can be replaced by $i\bar{k} = i(k_1\bar{e}_1' + k_3\bar{e}_3')$ where $\bar{x} = \sum_{i=1}^{3} x_i \bar{e}_i = \sum_{j=1}^{3} x_j' \bar{e}_j'$ is the position vector and $\partial_i = \partial x_i'$ $(i = 1, 2, 3)$

The equation of motion in a medium of density ρ on the assumption of infinitesimal deformation, and in the absence of body force is

$$\rho \partial_t^2 \bar{u} = \sum_{i_1, i_2, i_3, i_4 = 1}^{3} C'_{i_1 i_2 i_3 i_4} (\bar{e}_{i_2}' \cdot \nabla_{\bar{x}}) (\bar{e}_{i_3}' \cdot \nabla_{\bar{x}}) (\bar{u} \cdot \bar{e}_{i_4}') \tag{2-7}$$

$$= \sum_{i_1, i_2, i_3, i_4 = 1}^{3} \left(C'_{i_1 i_2 i_3 i_4} \partial_{i_2} \partial_{i_3} u_{i_4}\right) \bar{e}_{i_1}'$$

or

$$\rho \omega^2 \bar{u} = \sum_{i_1, i_2, i_3, i_4 = 1}^{3} C'_{i_1 i_2 i_3 i_4} k_{i_2} k_{i_3} u_{i_4} \bar{e}_{i_1}' \tag{2-7$'$}$$

Our final step in obtaining the equation of motion is to express $C'_{i_1 i_2 i_3 i_4}$ in terms of five elastic constants $\lambda_{\parallel}, \mu_{\parallel}, \lambda_{\perp}, \mu_{\perp}, \nu$ through the coordinate transformation

$$
\begin{aligned}
\vec{e}_1 &= \cos\theta\,\vec{e}_1{}' - \sin\theta\,\vec{e}_2{}' \\
\vec{e}_2 &= \sin\theta\,\vec{e}_1{}' + \cos\theta\,\vec{e}_2{}' \\
\vec{e}_3 &= \vec{e}_3{}'
\end{aligned}
\tag{2-8}
$$

and

$$
\begin{aligned}
\vec{e}_1{}' &= \cos\theta\,\vec{e}_1 + \sin\theta\,\vec{e}_1 \\
\vec{e}_2{}' &= -\sin\theta\,\vec{e}_1 + \cos\theta\,\vec{e}_2 \\
\vec{e}_3{}' &= \vec{e}_3
\end{aligned}
\tag{2-9}
$$

Note that the tensor $\overleftrightarrow{C}_0$ in $\overleftrightarrow{C} = \overleftrightarrow{C}_0 + \delta\overleftrightarrow{C}$ is an isotropic tensor, which can be written as

$$
\begin{aligned}
\overleftrightarrow{C}_0 = {} & (\lambda_{11} + 2\mu_{11})(\vec{e}_1{}'\vec{e}_1{}'\vec{e}_1{}'\vec{e}_1{}' + \vec{e}_2{}'\vec{e}_2{}'\vec{e}_2{}'\vec{e}_2{}' + \vec{e}_3{}'\vec{e}_3{}'\vec{e}_3{}'\vec{e}_3{}') \\
& + \lambda_{11}(\vec{e}_1{}'\vec{e}_1{}'\vec{e}_2{}'\vec{e}_2{}' + \vec{e}_2{}'\vec{e}_2{}'\vec{e}_1{}'\vec{e}_1{}' + \vec{e}_1{}'\vec{e}_1{}'\vec{e}_3{}'\vec{e}_3{}' + \vec{e}_3{}'\vec{e}_3{}'\vec{e}_1{}'\vec{e}_1{}' + \\
& \vec{e}_2{}'\vec{e}_2{}'\vec{e}_3{}'\vec{e}_3{}' + \vec{e}_3{}'\vec{e}_3{}'\vec{e}_2{}'\vec{e}_2{}') \\
& + \mu_{11}\Big[(\vec{e}_1{}'\vec{e}_2{}' + \vec{e}_2{}'\vec{e}_1{}')(\vec{e}_1{}'\vec{e}_2{}' + \vec{e}_2{}'\vec{e}_1{}') + (\vec{e}_1{}'\vec{e}_3{}' + \vec{e}_3{}'\vec{e}_1{}')(\vec{e}_1{}'\vec{e}_3{}' + \vec{e}_3{}'\vec{e}_1{}') \\
& + (\vec{e}_2{}'\vec{e}_3{}' + \vec{e}_3{}'\vec{e}_2{}')(\vec{e}_2{}'\vec{e}_3{}' + \vec{e}_3{}'\vec{e}_2{}')\Big]
\end{aligned}
\tag{2-10}
$$

In case of $\overleftrightarrow{C} = \overleftrightarrow{C}_0$ the equation of motion is reduced of course to the equation for an isotropic body

$$
\rho\partial_t^2 \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} (\lambda_{11}+2\mu_{11})\partial_1^2 + \mu_{11}\partial_3^2 & 0 & (\lambda_{11}+\mu_{11})\partial_1\partial_3 \\ 0 & \mu_{11}(\partial_1^2+\partial_3^2) & 0 \\ (\lambda_{11}+\mu_{11})\partial_1\partial_3 & 0 & \mu_{11}\partial_1^2 + (\lambda_{11}+2\mu_{11})\partial_3^2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}
\tag{2-11}
$$

In a homogeneous transversely isotropic medium, the departure from isotropy is due to the existence of three parameters $\delta\lambda$, $\delta\mu$ and $\delta\nu$. The absence of symmetry of $\delta\overleftrightarrow{C}$ in the horizontal plane requires some complicated algebra to obtain the right hand side of (2-7) in terms of the five elastic constants $\lambda_{\parallel}$, $\mu_{\parallel}$, $\lambda_{\perp}$, $\mu_{\perp}$ and ν in the coordinate frame $\{\vec{e}_1{}', \vec{e}_2{}', \vec{e}_3{}'\}$. Using (2-8), (2-9) and

$$\begin{aligned}\vec{u} &= u_1\vec{e}_1{}' + u_2\vec{e}_2{}' + u_3\vec{e}_3{}' \\ &= (\cos\theta\, u_1 - \sin\theta\, u_2)\vec{e}_1 + (\sin\theta\, u_1 + \cos\theta\, u_2)\vec{e}_2 + u_3\vec{e}_3 \qquad (2\text{-}12)\end{aligned}$$

then after some lengthy calculation, (2-7) becomes

$$\rho\partial_t^2\vec{u} = (\vec{e}_1{}' \; \vec{e}_2{}' \; \vec{e}_3{}') \cdot \begin{bmatrix} A_{11}^0(\nabla_x) & A_{12}^0(\nabla_x) & A_{13}^0(\nabla_x) \\ A_{21}^0(\nabla_x) & A_{22}^0(\nabla_x) & A_{23}^0(\nabla_x) \\ A_{31}^0(\nabla_x) & A_{32}^0(\nabla_x) & A_{33}^0(\nabla_x) \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} +$$

$$(\vec{e}_1{}' \; \vec{e}_2{}' \; \vec{e}_3{}') \cdot \begin{bmatrix} \delta A_{11}(\nabla_x) & \delta A_{12}(\nabla_x) & \delta A_{13}(\nabla_x) \\ \delta A_{21}(\nabla_x) & \delta A_{22}(\nabla_x) & \delta A_{23}(\nabla_x) \\ \delta A_{31}(\nabla_x) & \delta A_{32}(\nabla_x) & \delta A_{33}(\nabla_x) \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \qquad (2\text{-}13)$$

where

$$\begin{bmatrix} A_{11}^0(\nabla_x) & A_{12}^0(\nabla_x) & A_{13}^0(\nabla_x) \\ A_{21}^0(\nabla_x) & A_{22}^0(\nabla_x) & A_{23}^0(\nabla_x) \\ A_{31}^0(\nabla_x) & A_{32}^0(\nabla_x) & A_{33}^0(\nabla_x) \end{bmatrix} = \begin{bmatrix} (\lambda_{\parallel}+2\mu_{\parallel})\partial_1^2+\mu_{\parallel}\partial_3^2 & 0 & (\lambda_{\parallel}+\mu_{\parallel})\partial_1\partial_3 \\ 0 & \mu_{\parallel}(\partial_1^2+\partial_3^2) & 0 \\ (\lambda_{\parallel}+\mu_{\parallel})\partial_1\partial_3 & 0 & \mu_{\parallel}\partial_1^2+(\lambda_{\parallel}+2\mu_{\parallel})\partial_3^2 \end{bmatrix} \qquad (2\text{-}14)$$

and $[\delta A_{ij}(\nabla_{\bar{x}})]$ is a 3x3 symmetric matrix with elements

$$\delta A_{11}(\nabla_{\bar{x}}) = \cos^2\theta\left\{\left[\delta\lambda\,(1+\sin^2\theta) + 2\delta\mu\cos^2\theta + 4\delta\nu\sin^2\theta\right]\partial_1^2 + \delta\nu\,\partial_3^2\right\}$$

$$\delta A_{22}(\nabla_{\bar{x}}) = \left[\delta\nu\cos^2 2\theta - (\delta\lambda - 2\delta\mu)\sin^2\theta\cos^2\theta\right]\partial_1^2 + \delta\nu\sin^2\theta\,\partial_3^2$$

$$\delta A_{33}(\nabla_{\bar{x}}) = \delta\nu\cos^2\theta\,\partial_1^2$$

$$\delta A_{12}(\nabla_{\bar{x}}) = \delta A_{21}(\nabla_{\bar{x}}) = (-\delta\lambda\sin^3\theta\cos\theta - 2\delta\mu\sin\theta\cos^3\theta + \delta\nu\sin 2\theta\cos 2\theta)\partial_1^2$$
$$- \delta\nu\sin\theta\cos\theta\,\partial_3^2$$

$$\delta A_{13}(\nabla_{\bar{x}}) = \delta A_{31}(\nabla_{\bar{x}}) = (\delta\lambda + \delta\nu)\cos^2\theta\,\partial_1\partial_3$$

$$\delta A_{23}(\nabla_{\bar{x}}) = \delta A_{32}(\nabla_{\bar{x}}) = -\sin\theta\cos\theta\,(\delta\lambda + \delta\nu)\,\partial_1\partial_3 \qquad (2\text{-}15)$$

Equation (2-13) constitutes a set of three coupled simultaneous linear partial differential equations in three unknowns u_1, u_2 and u_3. To solve this set, we first represent u_1, u_2 and u_3 as consisting of part due to $\overleftrightarrow{\tau} = \overleftrightarrow{\tau}_0$ plus a perturbed part. The displacement in (2-13) becomes $\bar{u} = \bar{u}^{(0)} + \bar{u}^{(1)} + \bar{u}^{(2)} + \dots + \bar{u}^{(N)} + \cdots$ (2-16)

The zero subscript denotes the value of $\bar{u}$ in isotropic medium with λ_{11} and μ_{11} as Lame's constants, and the i(i>0) subscript denotes the perturbed portion of $\bar{u}$ which is of i-th order in $\delta\lambda$, $\delta\mu$ and $\delta\nu$. If we assume that $\bar{u}\cdot\bar{e}_2' = 0$, then every term in $u_2 = \bar{u}\cdot\bar{e}_2' = \bar{u}^{(1)}\cdot\bar{e}_2' + \bar{u}^{(2)}\cdot\bar{e}_2' + \dots$ is of order higher than zero in $\delta\lambda$, $\delta\mu$ and $\delta\nu$. We note also that the elements $A^{o}_{12}(\nabla_{\bar{x}})+\delta A_{12}(\nabla_{\bar{x}})=A^{o}_{21}(\nabla_{\bar{x}})+\delta A_{21}(\nabla_{\bar{x}})$, $A^{o}_{13}(\nabla_{\bar{x}})+\delta A_{13}(\nabla_{\bar{x}})=A^{o}_{31}(\nabla_{\bar{x}})+\delta A_{31}(\nabla_{\bar{x}})$ and $A^{o}_{23}(\nabla_{\bar{x}})+\delta A_{23}(\nabla_{\bar{x}})=A^{o}_{32}(\nabla_{\bar{x}})+\delta A_{32}(\nabla_{\bar{x}})$. Neglecting all terms involving u_2 which are of order higher than one in $\delta\lambda$, $\delta\mu$ and $\delta\nu$, we get

$$\rho\partial_t^2 u_1 = \left(A^{o}_{11}(\nabla_{\bar{x}})+\delta A_{11}(\nabla_{\bar{x}})\right)u_1 + \left(A^{o}_{13}(\nabla_{\bar{x}})+\delta A_{13}(\nabla_{\bar{x}})\right)u_3$$

$$\rho\partial_t^2 u_2 = \delta A_{21}(\nabla_{\bar{x}})u_1 + \left(A^{o}_{22}(\nabla_{\bar{x}})+\delta A_{22}(\nabla_{\bar{x}})\right)u_2 + \delta A_{23}(\nabla_{\bar{x}})u_3$$

$$\rho\partial_t^2 u_3 = \left(A^{o}_{31}(\nabla_{\bar{x}})+\delta A_{31}(\nabla_{\bar{x}})\right)u_1 + \left(A^{o}_{33}(\nabla_{\bar{x}})+\delta A_{33}(\nabla_{\bar{x}})\right)u_3 \tag{2-17}$$

The second equation in (2-17) offers explanation how horizontal transverse displacement u_2 can be generated by u_1 and u_3 through the perturbation of the elastic properties of the medium. On the other hand, the motion governing u_1 and u_3 can be discussed in two dimensions (x_1', x_3') — the vertical plane containing the wave number vector. Therefore, only the first and third equations of (2-17) are needed to discuss the propagation of P-waves, SV-waves and surface waves of Rayleigh type; whereas, in considering the propagation of SH-waves, u_1 and u_3 are regarded at least of the first order in perturbation and the second equation of (2-17) becomes

$$\rho\partial_t^2 u_2 = \left(A^{o}_{22}(\nabla_{\bar{x}}) + \delta A_{22}(\nabla_{\bar{x}})\right)u_2 \tag{2-18}$$

3. Boundary condition

The boundary condition of our problem is the vanishing of all the components of stress working at free surface $x_3'=0$ The combination of this condition and the condition at infinity: $\overline{u}=0$ at $x_3'=\infty$ will determine the property of Free Rayleigh waves. The stress working on the horizontal plane is

$$\overline{e}_3'\cdot\overline{\overline{\sigma}} = \overline{e}_3'\cdot\left\{\left[\dot{\lambda}_3(\nabla_X)\cdot\overleftrightarrow{C}\right]\cdot\overline{u}\right\}$$

$$= \overline{e}_3'\cdot\left\{\left[\dot{\lambda}_3(\nabla_X)\overleftrightarrow{C}_0\right]\cdot\overline{u}\right\} + \overline{e}_3'\cdot\left\{\left[\dot{\lambda}_3(\nabla_X)\delta\overleftrightarrow{C}\right]\cdot\overline{u}\right\} \tag{3-1}$$

For the isotropic part, we have the well known result:

$$\overline{e}_3'\cdot\left\{\left[\dot{\lambda}_3(\nabla_X)\overleftrightarrow{C}_0\right]\cdot\overline{u}\right\} = \overline{e}_3'\cdot\left\{\lambda_{11}(\mathrm{div}_X\overline{u})\left(\sum_{i=1}^{3}\overline{e}_i\overline{e}_i\right)+\mu_{11}\left[\nabla_X\overline{u}+(\nabla_X\overline{u})^T\right]\right\}$$

$$= \lambda_{11}(\mathrm{div}_X\overline{u})\overline{e}_3' + \mu_{11}\left[(\partial_1 u_3+\partial_3 u_1)\overline{e}_1' + \partial_3 u_2\overline{e}_2' + 2(\partial_3 u_3)\overline{e}_3'\right]$$

$$= \lambda_{11}(\partial_1 u_3+\partial_3 u_1)\overline{e}_1' + \mu_{11}(\partial_3 u_2)\overline{e}_2' + \left[\lambda_{11}\partial_1 u_1 + (\lambda_{11}+2\mu_{11})\partial_3 u_3\right]\overline{e}_3' \tag{3-2}$$

Next, we shall consider the perturbation term of (3-1). From (2-4) and (2-6), we have

$$\overline{e}_3'\cdot\left\{\left[\dot{\lambda}_3(\nabla_X)\delta\overleftrightarrow{C}\right]\cdot\overline{u}\right\}$$

$$= \delta\lambda\,\partial_{x_1}(\overline{u}\cdot\overline{e}_1)\overline{e}_3 + \delta\nu\left[\partial_{x_1}(\overline{u}\cdot\overline{e}_3) + \partial_{x_3}(\overline{u}\cdot\overline{e}_1)\right]\overline{e}_1 \tag{3-3}$$

Applying (2-9) and (2-12) to (3-3), we finally find

$$\bar{e}_3'\cdot\left\{\left[\lambda_3(\nabla_x)\delta\overleftrightarrow{C}\right]\cdot\pi\right\} = \cos\theta\,\delta\nu\left(\cos\theta\,\partial_3 u_1 - \sin\theta\,\partial_3 u_2 + \cos\theta\,\partial_1 u_3\right)\bar{e}_1'$$
$$- \delta\nu\,\sin\theta\left(\cos\theta\,\partial_3 u_1 - \sin\theta\,\partial_3 u_2 + \cos\theta\,\partial_1 u_3\right)\bar{e}_2'$$
$$+ \delta\lambda\left(\cos^2\theta\,\partial_1 u_1 - \sin\theta\cos\theta\,\partial_1 u_2\right)\bar{e}_3' \qquad (3\text{-}4)$$

When we are interested in the motion described by u_1 and u_3 only, $\delta\lambda u_2$ and $\delta\lambda\, u_2$ may be neglected for the reason discussed in relation to (2-17) and hence (3-4) can be written approximately as

$$\bar{e}_3'\cdot\left\{\left[\lambda_3(\nabla_x)\delta\overleftrightarrow{C}\right]\cdot\pi\right\}$$
$$= \delta\nu\cos^2\theta\left(\partial_3 u_1 + \partial_1 u_3\right)\bar{e}_1' - \delta\nu\,\sin\theta\cos\theta\left(\partial_3 u_1 + \partial_1 u_3\right)\bar{e}_2'$$
$$+ \delta\lambda\cos^2\theta\,\partial_1 u_1\,\bar{e}_3' \qquad (3\text{-}5)$$

Then the boundary condition at free surface $x_3=0$ becomes

$$(\lambda_{11}+2\mu_{11})\partial_3 u_3 + (\lambda_{11}+\delta\lambda\cos^2\theta)\partial_1 u_1 = 0$$

$$\partial_1 u_3 + \partial_3 u_1 = 0$$

$$\partial_3 u_2 = 0 \qquad (3\text{-}6)$$

This relation plus the condition at infinity that the energy will decay to zero is sufficient to determine the motion of free surface waves. When $\overleftrightarrow{C} = \overleftrightarrow{C}_0$, (3-6) reduces to the boundary condition for isotropic body.

$$\lambda_{11}\partial_1 u_1 + (\lambda_{11}+2\mu_{11})\partial_3 u_3 = 0$$

$$\partial_1 u_3 + \partial_3 u_1 = 0 \tag{3-7}$$

4. Theory of the surface waves of Rayleigh type

In order to obtain the phase velocity of waves analogous to Rayleigh waves, we shall apply the method of plane waves due to Stoneley (1949) to our modified equation of motion and boundary condition at free surface.

Recall that in Section 2 the first and the third equations in (2-17) are sufficient to discuss P-waves, SV-waves and surface waves of Rayleigh type approximately and the second equation will be ignored for the present so that the equations of motion expressed in terms of the components in the coordinate frame $\{\mathfrak{e}_1', \mathfrak{e}_2'\ \mathfrak{e}_3'\}$ are

$$\rho\partial_t^2 u_1 = (A\partial_1^2 + L\partial_3^2)u_1 + (F+L)\partial_1\partial_3 u_3$$

$$\rho\partial_t^2 u_3 = (F+L)\partial_1\partial_3 u_1 + (L\partial_1^2 + C\partial_3^2)u_3 \tag{4-1}$$

where

$$A = (\lambda_{11}+2\mu_{11}) + \cos^2\theta\left[\delta\lambda(1+\sin^2\theta) + 2\delta\mu\cos^2\theta + 4\delta\nu\sin^2\theta\right]$$

$$F = \lambda_{11} + \delta\lambda\cos^2\theta$$

$$L = \mu_{11} + \delta\nu\cos^2\theta$$

$$C = \lambda_{11} + 2\mu_{11} \tag{4-2}$$

Now we set

$$u_1\vec{e}_1' + u_3\vec{e}_3' = \nabla_\chi\phi - \nabla_\chi\times\vec{\chi} = \partial_1\phi\,\vec{e}_1' + \partial_3\phi\,\vec{e}_3' - \begin{vmatrix} \vec{e}_1' & \vec{e}_2' & \vec{e}_3' \\ \partial_1 & 0 & \partial_3 \\ 0 & \chi & 0 \end{vmatrix} \tag{4-3}$$

or

$$u_1 = \partial_1\phi + \partial_3\chi \quad ; \quad u_3 = \partial_3\phi - \partial_1\chi \tag{4-3)'$$

Assume that waves are plane of sinusoidal disturbance, the displacement must have exp$(-i\omega t)$ dependence, where ω is the circular frequency. Then we can write (4-1) as

$$-\rho\omega^2(\partial_1\phi+\partial_3\chi) = A(\partial_1^3\phi+\partial_1^2\partial_3\chi) + F(\partial_1\partial_3^2\phi-\partial_1^2\partial_3\chi) + L(2\partial_1\partial_3^2\phi+\partial_3^3\chi-\partial_3\partial_1^2\chi)$$

$$-\rho\omega^2(\partial_3\phi-\partial_1\chi) = L(2\partial_1^2\partial_3\phi+\partial_1\partial_3^2\chi-\partial_1^3\chi) + C(\partial_3^3\phi-\partial_1\partial_3^2\chi) + F(\partial_1^2\partial_3\phi+\partial_1\partial_3^2\chi) \tag{4-4}$$

Note that we no longer have in general purely compressional waves or purely rotational waves as in the isotropic medium. The details will be described in Sections 7 and 8. In order to investigate the surface waves of Rayleigh type, it is convenient to derive the modified Rayleigh's equation by means of displacement potentials.

Considering homogeneous plane waves with the phase velocity c, we assume that

$$\phi = \phi(x_3')\, e^{ik_1(x_1' - ct)}$$

$$\chi = \chi(x_3')\, e^{ik_1(x_1' - ct)} \tag{4-5}$$

and insert them into (4-1) and (4-3) to get the following equations.

$$ik_1\left[(\rho c^2 - A)k_1^2\phi + (F+2L)\partial_3^2\phi\right] + \left[L\partial_3^2\chi + (\rho c^2 - A + F + L)k_1^2\partial_3\chi\right] = 0$$

$$\left[(\rho c^2 - 2L - F)k_1^2\partial_3\phi + C\partial_3^3\phi\right] + ik_1\left[(L - C + F)\partial_3^2\chi + (L - \rho c^2)k_1^2\chi\right] = 0 \tag{4-6}$$

or $\theta = 0.5\pi$ $\delta\lambda = \delta\mu = \delta\nu = 0,$ the equations of motion are reduced to those on unperturbed isotropic medium with Lame's constants λ_{11} and μ_{11} , and (4-6) becomes

$$ik_1\left\{\left[\rho c^2 - (\lambda_{11} + 2\mu_{11})\right]k_1^2\phi + (\lambda_{11} + 2\mu_{11})\partial_3^2\phi\right\} + \left[\mu_{11}\partial_3^3\chi + (\rho c^2 - \mu_{11})k_1^2\partial_3\chi\right] = 0$$

$$\left\{\left[\rho c^2 - (\lambda_{11} + 2\mu_{11})\right]k_1^2\partial_3\phi + (\lambda_{11} + 2\mu_{11})\partial_3^3\phi\right\} + ik_1\left[-\mu_{11}\partial_3^2\chi + (\mu_{11} - \rho c^2)k_1^2\chi\right] = 0 \tag{4-7}$$

which correspond to equations $\phi'' + k_1^2\left(\frac{c^2}{c_p^2} - 1\right)\phi = 0$ and $\chi'' + k_1^2\left(\frac{c^2}{c_s^2} - 1\right) = 0$ for inhomogeneous plane waves in isotropic medium, where $c_p = \sqrt{\frac{\lambda_{11} + 2\mu_{11}}{\rho}}$ and $c_s = \sqrt{\frac{\mu_{11}}{\rho}}$

We assume further that

$$\phi(x_3') = \phi_0 \exp(k_1 \gamma x_3') \qquad \text{and} \qquad \chi(x_3') = \chi_0 \exp(k_1 \gamma x_3')$$

where ϕ_0, χ_0, γ are constants. From (4-7) and (4-8) we have

$$i\phi_0\left[\rho c^2 - A + \gamma^2(F+2L)\right] + \gamma\chi_0\left(\rho c^2 - A + F + L + L\gamma^2\right) = 0\,;$$

$$\gamma\phi_0\left(\rho c^2 - 2L - F + C\gamma^2\right) - \chi_0\left[L - \rho c^2 + \gamma^2(L - C + F)\right] = 0 \qquad (4\text{-}9)$$

In order that inhomogeneous plane waves may exist, there must be non-trivial solutions ϕ_0 and χ_0 of (4-9). This condition is

$$(1-\gamma^2)\Big\{ LC\gamma^4 + \left[(\rho c^2 - A)C\gamma^4 + L(\rho c^2 - L) + (F+L)^2\right]\gamma^2 + (\rho c^2 - A)(\rho c^2 - L)\Big\} = 0 \qquad (4\text{-}10)$$

In equation (4-10), the solutions $\gamma = \pm 1$ have nothing to do with the motion of surface waves of Rayleigh type. We can solve for γ^2 in (4-10) to give the values

$$\gamma_1^2 = \frac{-\Gamma + \left[\Gamma^2 - 4LC(\rho c^2 - A)(\rho c^2 - L)\right]^{\frac{1}{2}}}{LC}$$

$$\gamma_2^2 = \frac{-\Gamma - \left[\Gamma^2 - 4LC(\rho c^2 - A)(\rho c^2 - L)\right]^{\frac{1}{2}}}{LC} \tag{4-11}$$

where $\Gamma = C(\rho c^2 - A) + L(\rho c^2 - L) + (F+L)^2$

For $\theta = 0.5\pi$ or isotropic medium with λ_{11}, μ_{11} as Lame's constants, they reduce to

$$\gamma_1^2 = \left(1 - \frac{\rho c^2}{(\lambda_{11} + 2\mu_{11})}\right) \quad ;$$

$$\gamma_2^2 = \left(1 - \frac{\rho c^2}{\mu_{11}}\right)$$

There are two roots γ_1^2 and γ_2^2 of equation (4-10), and hence the solutions of (4-7) must be of the form,

$$\phi(x_3') = \phi_1 \exp(k_1 \gamma_1 x_3') + \phi_2 \exp(k_1 \gamma_2 x_3') \;;$$

$$\chi(x_3') = \chi_1 \exp(k_1 \gamma_1 x_3') + \chi_2 \exp(k_1 \gamma_2 x_3') \tag{4-12}$$

According to (4-9), (4-12) can be written as

$$\phi(x_3') = \phi_1 \exp(k_1 \gamma_1 x_3') + \phi_2 \exp(k_1 \gamma_2 x_3')$$

$$\chi(x_3') = \left(\frac{i m_1 \phi_1}{\gamma_1}\right) \exp(k_1 \gamma_1 x_3') + \left(\frac{i m_2 \phi_2}{\gamma_2}\right) \exp(k_1 \gamma_2 x_3') \tag{4-13}$$

in which both γ_1 and γ_2 must be negative in order to satisfy the condition at infinity, where

$$m_i = m_i(\gamma_i) = -\frac{\left[\rho c^2 - A + \gamma_i^2 (F+2L)\right]}{\rho c^2 - A + F + L + L\gamma_i^2} \quad (i=1,2)$$

Now that we have (4-13) which expresses the characteristic of surface waves, we are going to put them into the boundary conditions at free surface given by (3-6) to obtain the modified Rayleigh's equation. Since we are not interested in [illegible] for the time being, we only write the condition in the following way.

$$(\lambda_{11}+2\mu_{11})\partial_3 u_3 + (\lambda_{11}+\delta\lambda\cos^2\theta)\partial_1 u_1 = 0$$

$$\partial_1 u_3 + \partial_3 u_1 = 0 \qquad (4\text{-}14)$$

In terms of compressional and rotational potentials, (4-14) becomes

$$\lambda_{11}(\partial_1^2+\partial_3^2)\phi + 2\mu_{11}(\partial_3^2\phi - \partial_3\partial_1\chi) + \delta\lambda\cos^2\theta\,\partial_1(\partial_1\phi + \partial_3\chi) = 0\,;$$

$$2\partial_1\partial_3\phi + \partial_3^2\chi - \partial_1^2\chi = 0 \qquad (4\text{-}15)$$

On substituting (4-13) into (4-15), we obtain

$$\phi_1 \exp(k_1 q_1 x_3')\left[(\lambda_{11}+2\mu_{11})q_1^2 + 2\mu_{11} m_1 - \lambda_{11} + \delta\lambda \cos^2\theta\,(1+m_1)\right]$$
$$+\phi_2 \exp(k_1 q_2 x_3')\left[(\lambda_{11}+2\mu_{11})q_2^2 + 2\mu_{11} m_2 - \lambda_{11} - \delta\lambda \cos^2\theta\,(1+m_2)\right] = 0$$
$$\phi \exp(k_1 q_1 x_3')\left(2q_1 + m_1 + \frac{m_1}{q_1}\right) + \phi_2 \exp(k_2 q_2 x_3')\left(2q_2 + m_2 + \frac{m_2}{q_2}\right) = 0 \qquad (4\text{-}16)$$

For non-vanishing ϕ_0 and ϕ_1 to exist, the following condition must be satisfied.

$$\det\begin{bmatrix} (\lambda_{11}+2\mu_{11})(q_1^2+m_1) - (\lambda+\delta\lambda\cos^2\theta)(1+m_1), & (\lambda_{11}+2\mu_{11})(q_2^2+m_2)(\lambda_{11}+\delta\lambda\cos^2\theta)(1+m_2) \\ \frac{q_1^2+m_1}{q_1} + q_1(1+m_1) & \frac{q_2^2+m_2}{q_2} + q_2(1+m_2) \end{bmatrix} = 0 \qquad (4\text{-}17)$$

which is the modified Rayleigh's equation. It can be shown that in an isotropic medium, this equation is reduced to the ordinary Rayleigh's equation $\left(2-\frac{c^2}{c_s^2}\right)^2 = 16\left(1-\frac{c^2}{c_s^2}\right)\left(1-\frac{c^2}{c_p^2}\right)$ for arbitrary Poisson's ratio where c_p and c_s are phase velocities of P wave and S wave respectively.

Let us look at the elements of (4-17) more explicitly and find that

$$m_i = m_i(\xi_i) = -\frac{(\rho c^2 - A) + \xi_i^2\left[\lambda_{11} + 2\mu_{11} + (\delta\lambda + 2\delta\nu)\cos^2\theta\right]}{\left[\rho c^2 - A + \lambda_{11} + \mu_{11} + (\delta\lambda + \delta\nu)\cos^2\theta + (\mu_{11} + \delta\nu\cos^2\theta)\xi_i^2\right]}$$

$$1 + m_i = \frac{(1 - \xi_i^2)\left[\lambda_{11} + \mu_{11} + (\delta\lambda + \delta\nu)\cos^2\theta\right]}{\left[\rho c^2 - A + \lambda_{11} + \mu_{11} + (\delta\lambda + \delta\nu)\cos^2\theta + (\mu_{11} + \delta\nu\cos^2\theta)\xi_i^2\right]}$$

$$\xi_i^2 + m_i = \frac{(\xi_i^2 - 1)\left[(\rho c^2 - A) + \xi_i^2(\mu_{11} + \delta\nu\cos^2\theta)\right]}{\left[\rho c^2 - A + \lambda_{11} + \mu_{11} + (\delta\lambda + \delta\nu)\cos^2\theta + (\mu_{11} + \delta\nu\cos^2\theta)\xi_i^2\right]}$$

$$\frac{m_i + \xi_i^2}{\xi_i} + \xi_i(1 + m_i)$$

$$= \frac{1}{\xi_i}\left[\frac{(\rho c^2 - A) - \xi_i^2(\lambda_{11} + \delta\lambda\cos^2\theta)}{\rho c^2 - A + \lambda_{11} + \mu_{11} + (\delta\lambda + \delta\nu)\cos^2\theta + (\mu_{11} + \delta\nu\cos^2\theta)\xi_i^2}\right]$$

(4-18)

where

$$A = \lambda_{11} + 2\mu_{11} + \delta A$$
$$= \lambda_{11} + 2\mu_{11} + \cos^2\theta\left[\delta\lambda(1 + \sin^2\theta) + 2\delta\mu\cos^2\theta + 4\delta\nu\sin^2\theta\right]$$

is indicated in (4-2)

From the above results, we find a common factor

$$\frac{\xi_i^2-1}{(\rho c^2-A)+\lambda_{11}+\mu_{11}+(\delta\lambda+\delta\nu)\cos^2\theta+(\mu_{11}+\delta\nu\cos^2\theta)\,\xi_i^2}$$

in the column of (4-17). From the expression of $m_i = m_i(\xi_i)$ we see that the matrix in (4-17) has rank 1 if we set $\xi_1=\xi_2$ and hence there must be a common factor $(\xi_1-\xi_2)$ on the left hand side of (4-17). After removing the factor

$$\frac{(\xi_1^2-1)(\xi_2^2-1)(\xi_1-\xi_2)}{\left[(\rho c^2-A)+F+L+L\xi_1^2\right]\left[(\rho c^2-A)+F+L+L\xi_2^2\right]}$$

where A, F and L are expressed in (4-2), we get

$$\begin{aligned}
&(\lambda_{11}+2\mu_{11})(\mu_{11}+\delta_L)(\lambda_{11}+\delta_F)\,\xi_1^2\xi_2^2\\
&-\left[\begin{aligned}&(\lambda_{11}+\delta_F)(\lambda_{11}+2\mu_{11})(\rho c^2-A)+(\lambda_{11}+\delta_F)^2(\lambda_{11}+\mu_{11}+\delta_F+\delta_L)\\&+(\mu_{11}+\delta_L)(\lambda_{11}+2\mu_{11})(\rho c^2-A)\end{aligned}\right]\xi_1\xi_2\\
&-(\lambda_{11}+2\mu_{11})(\mu_{11}+\delta_L)(\rho c^2-A)(\xi_1^2+\xi_2^2)-(\lambda_{11}+2\mu_{11})(\rho c^2-A)^2\\
&-(\lambda_{11}+\delta_F)(\lambda_{11}+\mu_{11}+\delta_F+\delta_L)(\rho c^2-A)=0
\end{aligned}\qquad\text{(4-19)}$$

where

$$\delta_F = \delta\lambda \cos^2\theta$$

$$\delta_L = \delta\nu \cos^2\theta$$

From (4-11), we recall that

$$\gamma_1^2 + \gamma_2^2 = -\frac{\Gamma}{LC}$$

$$= \frac{-\left[\left(c^2 - \frac{\lambda_{11}+2\mu_{11}}{\rho} + \frac{\delta_A}{\rho}\right)\left(\frac{\lambda_{11}+2\mu_{11}}{\rho}\right) + \left(c^2 - \frac{\mu_{11}}{\rho} - \frac{\delta_L}{\rho}\right)\left(\frac{\mu_{11}}{\rho} + \frac{\delta_L}{\rho}\right) + \left(\frac{\lambda_{11}+\mu_{11}}{\rho} + \frac{\delta_F+\delta_L}{\rho}\right)^2\right]}{\left(\frac{\mu_{11}}{\rho} + \frac{\delta_L}{\rho}\right)\left(\frac{\lambda_{11}+2\mu_{11}}{\rho}\right)} \tag{4-20}$$

$$\gamma_1^2 \gamma_2^2 = \frac{\left(\frac{\lambda_{11}+2\mu_{11}}{\rho} + \frac{\delta_A}{\rho} - c^2\right)\left(\frac{\mu_{11}}{\rho} + \frac{\delta_L}{\rho} - c^2\right)}{\left(\frac{\mu_{11}}{\rho} + \frac{\delta_L}{\rho}\right)\left(\frac{\lambda_{11}+2\mu_{11}}{\rho}\right)} \tag{4-21}$$

Substituting (4-20) and (4-21) into (4-19) and after some arrangement, the modified Rayleigh's equation is finally obtained.

$$F(X) \equiv X$$

$$+ \sqrt{\frac{X-\mu_{11}-\delta_L}{X-\lambda_{11}-2\mu_{11}-\delta_A}} \; \frac{(\lambda_{11}+2\mu_{11})(X-\lambda_{11}-2\mu_{11}-\delta_A)+(\lambda_{11}+\delta_F)^2}{(\lambda_{11}+2\mu_{11})^{\frac{1}{2}}\,(\mu_{11}+\delta_L)^{\frac{1}{2}}} \tag{4-22}$$

where $X = \rho c^2$

For small anisotropy, $\mu_{11}+\delta_L > 0$ and $(\lambda_{11}+\delta_F)^2 < (\lambda_{11}+2\mu_{11})(\lambda_{11}+2\mu_{11}+\delta_A)$ in the medium under consideration. We have $F(0)<0$ and $F(\mu_{11}+\delta_L)>0$ and hence there must be a root of $F(X)=0$ in the open interval $(0,\ \mu_{11}+\delta_L)$. For real roots, we must have $(X-\mu_{11}-\delta_L)(X-\lambda_{11}-2\mu_{11}-\delta_A)$ >0 which is equivalent to $X>\mu_{11}+\delta_L$, $X>\lambda_{11}+2\mu_{11}+\delta_A$ and $X<\mu_{11}+\delta_L$, $X<\lambda_{11}+2\mu_{11}+\delta_A$ but $F(X)$ only takes real roots for the second case.

Rationalizing equation (4-22), we obtain a polynomial equation in $Y^2 = \frac{c^2}{\left(\frac{\mu_{11}}{\rho}\right)}$ of degree 3,

$$\left[\left(\frac{\lambda_{11}}{\mu_{11}}+2\right)^2 - \left(2+\frac{\lambda_{11}}{\mu_{11}}\right)\left(1+\frac{\delta_L}{\mu_{11}}\right)\right](Y^2)^3$$

$$+ \left\{\begin{array}{l} 2\left(2+\frac{\lambda_{11}}{\mu_{11}}\right)J - \left(1+\frac{\delta_L}{\mu_{11}}\right)\left(2+\frac{\lambda_{11}}{\mu_{11}}\right)^2 \\ \left(2+\frac{\lambda_{11}}{\mu_{11}}\right)\left(1+\frac{\delta_L}{\mu_{11}}\right)\left(\frac{\delta_A}{\mu_{11}}+2+\frac{\lambda_{11}}{\mu_{11}}\right) \end{array}\right\}(Y^2)^2$$

$$+J\left[J-2\left(2+\frac{\lambda_{11}}{\mu_{11}}\right)\left(1+\frac{\delta L}{\mu_{11}}\right)\right]Y^2 - J^2\left(1+\frac{\delta L}{\mu_{11}}\right)=0 \qquad (4\text{-}23)$$

where $J=\left(\frac{\lambda_{11}}{\mu_{11}}+\frac{\delta E}{\mu_{11}}\right)^2-\left(\frac{\lambda_{11}}{\mu_{11}}+2\right)\left(2+\frac{\lambda_{11}}{\mu_{11}}+\frac{\delta A}{\mu_{11}}\right)$

for the poisson solid satisfying $\lambda_{11}=\mu_{11}$, we have

$$\begin{aligned}&\left[9-3\left(1+\frac{\delta L}{\mu_{11}}\right)\right](Y^2)^3\\&+\left[6J-9\left(1+\frac{\delta L}{\mu_{11}}\right)+3\left(1+\frac{\delta L}{\mu_{11}}\right)\left(3+\frac{\delta A}{\mu_{11}}\right)\right](Y^2)^2\\&+J\left[J-6\left(1+\frac{\delta L}{\mu_{11}}\right)\right]Y^2-\left(1+\frac{\delta L}{\mu_{11}}\right)J^2=0.\end{aligned} \qquad (4\text{-}24)$$

where $J=\left(1+\frac{\delta E}{\mu_{11}}\right)^2-3\left(3+\frac{\delta A}{\mu_{11}}\right)$

Both of (4-23) and (4-24) are the modified Rayleigh's equation from which we can determine the phase velocity as a function of the azimuth of wave number vector. In the following sections, we shall calculate the ratio of the phase velocity to $\sqrt{\frac{\mu_{11}}{\rho}}$ from (4-23) and (4-24) in ten directions corresponding to $\theta = 0^\circ, 10^\circ, \cdots,$ and 90° for the two models; the fracture zone model and the olivine model discussed in the introduction.

5. Transverse isotropy as the limiting case of a laminated material

The motivation of our choice of approximating the oceanic lithosphere by a laminated model comes from the conjecture that the fracture zones may be regarded as the soft thin layers sandwiched in the Normal Oceanic lithosphere and in welded contact with it. Postma (1955) has shown that in the long wave limit, a medium consisting of alternating plane parallel isotropic layers of two different elastic properties can be approximated by a homogeneous transversely isotropic medium. Since Forsyth found the anisotropy for Rayleigh waves in the Nazca plate up to wave lengths of 400 km, the long wave assumption may be justified.

Consider a laminated half space, in which vertical thin layers with thickness h_1 and Lame's constants λ_1, μ_1 are sandwiched in a half space with Lame's constants λ_0, μ_0 and thickness h_0 as shown in Figure 1. Let us choose an orthonormal coordinate frame $\{\bar{e}_1, \bar{e}_2, \bar{e}_3\}$ at some point on the free surface such that $\bar{e}_1$ is the horizontal unit vector normal to the plane of lamination, and $\bar{e}_3$ is the unit vector directed vertically downward into the medium. The geometry of the frame is also indicated in Figure 1. Following Postma (1955), we choose a fraction of their thickness as the weighting function to average all the stress and strain components in these two different isotropic layers, finally the averaged stress and strain relation is obtained. Comparing this

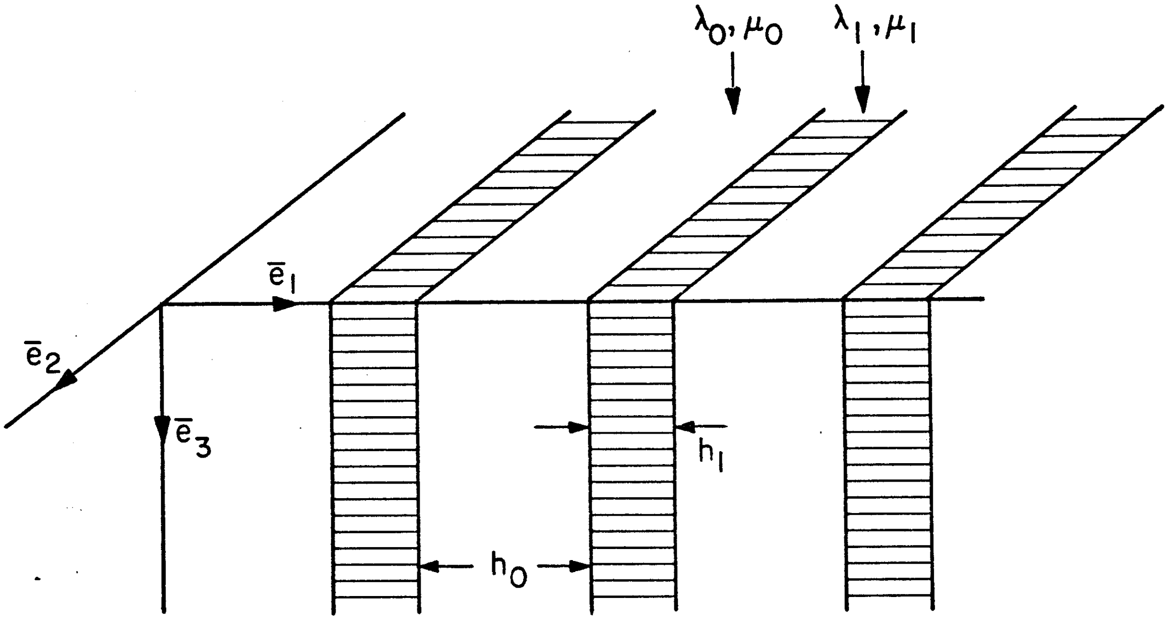

Figure 1. Coordinate frame $\{\bar{e}_1, \bar{e}_2, \bar{e}_3\}$ attached to a laminated half-space. $\bar{e}_1$ is the horizontal unit vector normal to the interfaces. $\bar{e}_3$ is the vertical unit vector and $\bar{e}_2 = -\bar{e}_1 \times \bar{e}_3$.

result with the constitutive relation in homogeneous transversely isotropic medium with symmetry axis parallel to $\mathbf{e}_1$ we find

$$\lambda_{\parallel} = \left(\frac{1}{1+\alpha}\right)\left[\lambda_0 + \alpha\lambda_1 - \frac{\alpha(\lambda_1-\lambda_0)^2}{(\lambda_0+a\mu_0) + \alpha(\lambda_1+2\mu_1}\right]$$

$$\lambda_{\perp} = \left\{\lambda_1 - \frac{\alpha(\lambda_1-\lambda_0)(\lambda_1+2\mu_1)}{(\lambda_0+2\mu_0)+\alpha(\lambda_1+2\mu_1)}\right\}$$

$$\mu_{\parallel} = \frac{\mu_1+\mu_0}{1+a}$$

$$\nu = \frac{(1+\alpha)\mu_1\mu_0}{\mu_0+a\mu_1}$$

$$\lambda_{\perp}+2\mu_{\perp} = \frac{(\lambda_1+2\mu_1)(\lambda_0+2\mu_0)(1+\alpha)}{(\lambda_0+2\mu_0)+\alpha(\lambda_1+2\mu_1)} \tag{5-1}$$

where $\alpha - \frac{h_0}{h_1}$

There are five elastic constants of the transversely isotropic body equivalent to the laminated body at long wave

limit. For simplicity, we shall assume $\lambda_0 = \mu_0$ and $\lambda_1 = \mu_1$
Then the ratio $\Upsilon(\theta) = \frac{C(\theta)}{\sqrt{\frac{\mu_{11}}{\rho}}}$ in equation (4-24) was computed for $0^\circ \leq \theta \leq 90^\circ$ at 10° interval for several cases of $\alpha = \frac{h_0}{h_1}$ and a fixed rigidity ratio. The result is given in Table 1 for $\frac{\mu_0}{\mu_1} = \frac{21.16}{18}$. For $\theta = 90^\circ$, $\Upsilon(90^\circ) = 0.919402$ which is the value of $\frac{C}{C_s}$ in the isotropic Poisson's solid.

We see from Table 1 that the variation of $\Upsilon(\theta) = \frac{C(\theta)}{\sqrt{\frac{\mu_{11}}{\rho}}}$ with the azimuth of the propagation is small compared with the 2% anisotropy observed by Forsyth for Rayleigh waves in the Nazca plate. The set of parameters $\frac{\mu_0}{\mu_1}$ and $\frac{h_0}{h_1}$ which give the required 2% anisotropy were obtained by the following procedure.

For Poisson's solid, we assume that $\lambda_0 = \mu_0$ and $\lambda_1 = \mu_1$
Then equation (5-1) becomes

$$\lambda_{\parallel} = \left(\frac{1}{1+\alpha}\right)\left[\mu_1 + 2\mu_0 - \frac{\alpha(\mu_1-\mu_0)^2}{3(\mu_0+\alpha\mu_1)}\right]$$

$$\lambda_{\perp} = \left[\mu_1 - \frac{\alpha(\mu_1-\mu_0)\mu_1}{\mu_0+\alpha\mu_1}\right]$$

$$\mu_{\parallel} = \frac{\mu_1+\alpha\mu_0}{1+\alpha}$$

$$\nu = \frac{(1+\alpha)\mu_0\mu_1}{\mu_0+\alpha\mu_1}$$

$$\lambda_{\perp} + 2\mu_{\perp} = \frac{3(1+\alpha)\mu_0\mu_1}{\mu_0+\alpha\mu_1} \qquad (5\text{-}2)$$

Table 1. $\Upsilon(\theta) = C(\theta)/\sqrt{\mu_1/\rho}$ calculated for the laminated model in which $\frac{\lambda_0}{\lambda_1} = \frac{\mu_0}{\mu_1}$ = 21.16/16

$a = \frac{h_0}{h_1}$	θ (Degree)	$\Upsilon(\theta)$
5	0	0.9131
5	10	0.9132
5	20	0.9136
5	30	0.9143
5	40	0.9153
5	50	0.9164
5	60	0.9175
5	70	0.9185
5	80	0.9191
5	90	0.9194
10	0	0.9169
10	10	0.9170
10	20	0.9171
10	30	0.9174
10	40	0.9178
10	50	0.9182
10	60	0.9187
10	70	0.9190
10	80	0.9193
10	90	0.9194
15	0	0.9176
15	10	0.9178
15	20	0.9178
15	30	0.9180
15	40	0.9182
15	50	0.9185
15	60	0.9189
15	70	0.9191
15	80	0.9193
15	90	0.9194

from which we have

$$\lambda_\perp + 2\mu = \lambda_\perp + 2\mu_\perp = \frac{3(1+\alpha)\mu_0\mu_1}{\mu_0 + \alpha\mu_1}$$

$$\nu = \mu_\perp$$

$$\nu - \mu_{\parallel} = \mu_\perp - \mu_{\parallel} = \frac{(1+\alpha)\mu_0\mu_1}{\mu_0 + \alpha\mu_1} - \frac{\alpha\mu_0 + \mu_1}{1+\alpha}$$

$$= -\left(\frac{\alpha}{1+\alpha}\right)\frac{(\mu_0 - \mu_1)^2}{(\mu_0 + \alpha\mu_1)}$$

$$\lambda_\perp - \lambda_{\parallel} = \mu_1 - \frac{\alpha(\mu_0 - \mu_1)\mu_1}{\mu_0 + \alpha\mu_1} - \left(\frac{1}{1+\alpha}\right)\left[\mu_0 + \alpha\mu_1 - \left(\frac{1}{3}\right)\frac{\alpha(\mu_0 - \mu_1)^2}{\mu_0 + \alpha\mu_1}\right]$$

$$= -\frac{2}{3}\left(\frac{\alpha}{1+\alpha}\right)\frac{(\mu_1 - \mu_0)^2}{(\mu_0 + \alpha\mu_1)} \tag{5-3}$$

and hence

$$\frac{\delta\mu}{\mu_{\parallel}} = \frac{\delta\nu}{\mu_{\parallel}} = 1.5\,\frac{\delta\lambda}{\mu_{\parallel}} = (-1)\,\alpha\,\frac{(\mu_0 - \mu_1)^2}{(\mu_0 + \alpha\mu_1)(\mu_1 + \alpha\mu_0)} \tag{5-4}$$

For given values of $\frac{\delta\nu}{\mu_{\parallel}} = \frac{\delta\mu}{\mu_{\parallel}} = 1.5\left(\frac{\delta\lambda}{\mu_{\parallel}}\right)$.

(5-4) is reduced to the homogeneous quadratic equation

$$\alpha\left(1 + \frac{\delta\mu}{\mu_{\parallel}}\right)\left(\frac{\mu_0}{\mu_1}\right)^2 + \left[(1+\alpha^2)\left(\frac{\delta\nu}{\mu_{\parallel}}\right) - 2\alpha\right]\left(\frac{\mu_0}{\mu_1}\right) \tag{5-5}$$

$$+ \alpha\left(1 + \frac{\delta\nu}{\mu_{\parallel}}\right) = 0$$

Table 2. $\gamma(\theta) = c(\theta)/\sqrt{\mu_1/\rho}$ calculated for laminated model compared with observed by Forsyth (1972).

θ (Degree)	Calculated	Observed
0	0.899	0.901
10	0.900	0.902
20	0.901	0.903
30	0.903	0.905
40	0.906	0.909
50	0.910	0.912
60	0.913	0.915
70	0.917	0.917
80	0.919	0.919
90	0.919	0.919

Table 3. Model parameters μ_0/μ_1 and h_0/h_1 which give the observed anisotropy for Rayleigh wave phase velocity.

h_0/h_1	μ_0/μ_1
1	1.599
2	1.634
3	1.704
4	1.780
5	1.854
6	1.930
7	2.000
8	2.070
9	2.140
10	2.208
12	2.342
15	2.538
20	2.854
30	3.460
40	4.048
50	4.625
60	5.195
70	5.760
80	6.321
90	6.800
100	7.440

in which the root $\left(\frac{\mu_0}{\mu_1}\right)$ must be greater than 1. On the other hand, from the requirement of 2% anisotropy, in Rayleigh velocity the values of $\frac{\delta\lambda}{\mu_{11}} = -0.025895$ $\frac{\delta\mu}{\mu_{11}} = \frac{\delta\nu}{\mu_{11}} = -0.038843$ are required from the first order perturbation of Rayleigh equation (4.24) in $\delta\lambda$, $\delta\mu$ and $\delta\nu$. By successive approximation, we finally obtain $\frac{\delta\lambda}{\mu_{11}} = -0.034527$, $\frac{\delta\mu}{\mu_{11}} = \frac{\delta\nu}{\mu_{11}} = 0.05173$ which gives nearly 2% anisotropy in Rayleigh wave phase velocity. The variation of $Y(\theta) = \frac{C(\theta)}{\sqrt{\frac{\mu_{11}}{\rho}}}$ with θ are given for the final solution together with the observed data as shown in Table 2.

Thus, the observed 2% anisotropy in Rayleigh waves require the anisotropy parameter $\left|\frac{\delta\nu}{\mu_{11}}\right|$ as much as 0.051730 From equation (5-4), one can find the combination of parameters for the fracture zone model which give this value for $\frac{\delta\nu}{\mu_{11}}$. The result is given in Table 3. For example, if the thickness ratio is 10:1, the rigidity ratio more than 2:1 is required. We consider the result rather unreasonable and therefore conclude that the fracture zone model is unacceptable.

6. Olivine model based on the Francis' suggestion

Observed anisotropy of the upper mantle in the Pacific Ocean appear to agree with the hypothesis that maximum velocity is in the direction of sea-floor spreading. Since Fracture zone model is inadequate, an alternative Olivine model will now be tested. In this model, we can include the measurements for short period P wave anisotropy simultaneously. In this section, we shall approximate the upper mantle by a half space made of an aggregate of olivine grains in which the crystallographic a axis lies horizontally in the direction perpendicular to the ridge axis, and the orientation of the b and c axes are randomly and uniformly distributed over a vertical plane parallel to the ridge axis.

Let $\{\vec{e}_1'', \vec{e}_2'', \vec{e}_3''\}$ be an orthonormal frame field which is parallel everywhere with $\vec{e}_1''$ in the direction of a axis, $\vec{e}_2''$ and $\vec{e}_3''$ in the directions of the b and c axes respectively of a single olivine crystal, then the constitutive equation for this grain is described by nine elastic constants as

$$\begin{bmatrix} \sigma_{11}'' \\ \sigma_{22}'' \\ \sigma_{33}'' \\ \sigma_{23}'' \\ \sigma_{13}'' \\ \sigma_{12}'' \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & 2C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & 2C_{66} \end{bmatrix} \begin{bmatrix} e_{11}'' \\ e_{22}'' \\ e_{33}'' \\ e_{23}'' \\ e_{13}'' \\ e_{12}'' \end{bmatrix} \qquad (6\text{-}1)$$

from which the elastic constant tensor of this single crystal can be written as

$$\overleftrightarrow{C} = c_{11}\bar{e}_1''\bar{e}_1''\bar{e}_1''\bar{e}_1'' + c_{22}\bar{e}_2''\bar{e}_2''\bar{e}_2''\bar{e}_2'' + c_{33}\bar{e}_3''\bar{e}_3''\bar{e}_3''\bar{e}_3''$$
$$+ c_{12}(\bar{e}_1''\bar{e}_1''\bar{e}_2''\bar{e}_2'' + \bar{e}_2''\bar{e}_2''\bar{e}_1''\bar{e}_1'') + c_{13}(\bar{e}_1''\bar{e}_1''\bar{e}_3''\bar{e}_3'' + \bar{e}_3''\bar{e}_3''\bar{e}_1''\bar{e}_1'')$$
$$+ c_{23}(\bar{e}_2''\bar{e}_2''\bar{e}_3''\bar{e}_3'' + \bar{e}_3''\bar{e}_3''\bar{e}_2''\bar{e}_2'') + c_{44}(\bar{e}_2''\bar{e}_3''+\bar{e}_3''\bar{e}_2'')(\bar{e}_2''\bar{e}_3''+\bar{e}_3''\bar{e}_2'')$$
$$+ c_{55}(\bar{e}_1''\bar{e}_3''+\bar{e}_3''\bar{e}_1'')(\bar{e}_1''\bar{e}_3''+\bar{e}_3''\bar{e}_1'') + c_{66}(\bar{e}_1''\bar{e}_2''+\bar{e}_2''\bar{e}_1'')(\bar{e}_1''\bar{e}_2''+\bar{e}_2''\bar{e}_1'') \quad (6\text{-}2)$$

where σ_{ij}'' and e_{ij}'' are the components of the stress dyadic $\bar{\bar{\sigma}} = \sum_{i,j=1}^{3} \sigma_{ij}''\bar{e}_i''\bar{e}_j''$ and strain dyadic $\bar{\bar{e}} = \sum_{i,j=1}^{3} e_{ij}''\bar{e}_i''\bar{e}_j''$ respectively.

Let $\{\bar{e}_1, \bar{e}_2, \bar{e}_3\}$ be another orthonormal coordinate frame at a point of the free surface of the half space such that $\bar{e}_1$ is the horizontal vector in the direction perpendicular to the ridge axis and is directed vertically downward into the medium. According to Francis' hypothesis, we may assume $\bar{e}_1 = \bar{e}_1''$ first and hence these two frames are related by

$$\bar{e}_1'' = \bar{e}_1$$

$$\bar{e}_2'' = \cos\alpha\,\bar{e}_2 + \sin\alpha\,\bar{e}_3$$

$$\bar{e}_3'' = -\sin\alpha\,\bar{e}_2 + \cos\alpha\,\bar{e}_3 \quad (6\text{-}3)$$

and

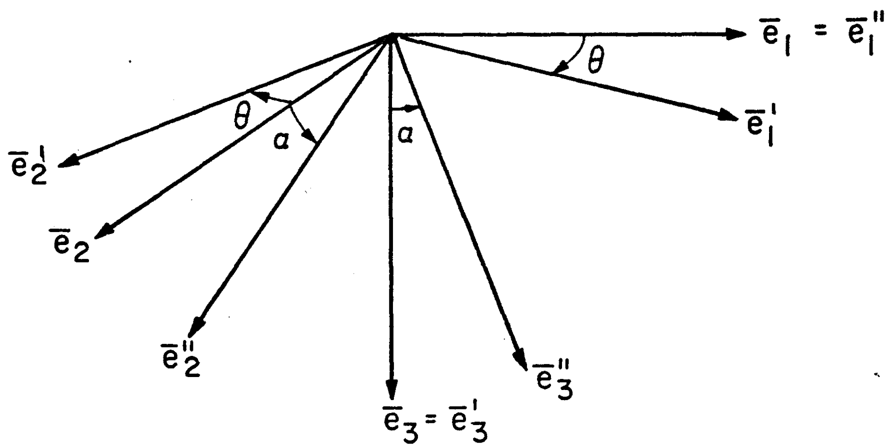

Figure 2. Coordinate frames $\{\bar{e}_1, \bar{e}_2, \bar{e}_3\}$ and $\{\bar{e}_1'', \bar{e}_2'', \bar{e}_3''\}$, $\bar{e}_1$ is parallel to the free surface and normal to the ridge axis. $\bar{e}_3$ is downward unit vector into the half-space. α is the angle between $\bar{e}_3$ and $\bar{e}_3''$. The plane spanned by $\bar{e}_2$ and $\bar{e}_3$ is the same as the plane spanned by $\bar{e}_2''$ and $\bar{e}_3''$ θ is the azimuthal angle measuring the rotation around the vertical $\bar{e}_3$.

$$\bar{e}_1 = \bar{e}_1''$$

$$\bar{e}_2 = \cos\alpha\,\bar{e}_2'' - \sin\alpha\,\bar{e}_3''$$

$$\bar{e}_3 = \sin\alpha\,\bar{e}_2'' + \cos\alpha\,\bar{e}_3'' \qquad (6-4)$$

The geometry of this transformation is shown in Figure 2.

A great deal of recent work indicates that preferred grain oreintation is probably the most important factor in producing anisotropy in a compact aggregate (e.g., Babuska, 1968; Kumazawa, 1964; Klima and Babuska, 1968; Crosson and Lin, 1971), and hence we shall neglect the interface structure between the grains in our model. In order to predict the elastic properties of the model in which the b and c axes orient randomly in the plane perpendicular to the a axis (or $\bar{e}_1$), we shall go through the procedure according to Voigt, Reuss and Voigt-Reuss-Hill approximation with the spatial average replaced by averaging the single crystal property with respect to rotation around the a axis.

If the components C_{ij} in (6-2) which form a tensor of fourth order are transformed into the components of $\overleftrightarrow{C}$ in the coordinate frame $\{\bar{e}_1, \bar{e}_2, \bar{e}_3\}$ and we write $\overleftrightarrow{C} = \sum_{i,j,k,l=1}^{3} \widehat{C}_{ijkl}\,\bar{e}_i\,\bar{e}_j\,\bar{e}_k\,\bar{e}_l$. Then the components $\widehat{C}_{ijkl} = \widehat{C}_{ijkl}(\alpha)$ are functions in the angular variable α

according to the transformation represented by the rotation (6-3). In the Voigt's scheme it is assumed that the strain field induced is uniform throughout the aggregate, then we consider an equivalent uniform elastic body which would produce the averaged stress for the given uniform strain. The stress-strain relation in this set of grains can be written as

$$\bar{\bar{\sigma}}(a) = \overleftrightarrow{C}(a) \cdot\cdot\, \bar{\bar{\epsilon}} \tag{6-5}$$

where $\overleftrightarrow{C}(a) = \sum_{i,j,k,l=1}^{3} \widetilde{C}_{ijkl}\, \bar{e}_i \bar{e}_j \bar{e}_k \bar{e}_l$ is the elastic constant tensor in (6-2). The averaged stress field in the whole aggregate is then given by

$$\langle \bar{\bar{\sigma}} \rangle = \left(\frac{\int_0^{2\pi} \overleftrightarrow{C}(a)\, da}{\int_0^{2\pi} da} \right) \cdot\cdot\, \bar{\bar{\epsilon}} = \langle \overleftrightarrow{C} \rangle^{V} \cdot\cdot\, \bar{\bar{\epsilon}} \tag{6-6}$$

The Voigt's averaged elastic constant tensor is $\langle \overleftrightarrow{C} \rangle^{V}$. After the evaluation of each integrals with components of $\overleftrightarrow{C}(a)$ as integrands

$$\begin{aligned} \langle \overleftrightarrow{C} \rangle^{V} = {} & C_{11}^{V}\, \bar{e}_1\bar{e}_1\bar{e}_1\bar{e}_1 + C_{22}^{V}\, \bar{e}_2\bar{e}_2\bar{e}_2\bar{e}_2 + C_{33}^{V}\, \bar{e}_3\bar{e}_3\bar{e}_3\bar{e}_3 \\ & + C_{12}^{V}(\bar{e}_1\bar{e}_1\bar{e}_2\bar{e}_2 + \bar{e}_2\bar{e}_2\bar{e}_1\bar{e}_1) + C_{13}^{V}(\bar{e}_1\bar{e}_1\bar{e}_3\bar{e}_3 + \bar{e}_3\bar{e}_3\bar{e}_1\bar{e}_1) \\ & + C_{23}^{V}(\bar{e}_2\bar{e}_2\bar{e}_3\bar{e}_3 + \bar{e}_3\bar{e}_3\bar{e}_2\bar{e}_2) + C_{44}^{V}(\bar{e}_2\bar{e}_3 + \bar{e}_3\bar{e}_2)(\bar{e}_2\bar{e}_3 + \bar{e}_3\bar{e}_2) \\ & + C_{55}^{V}(\bar{e}_1\bar{e}_3 + \bar{e}_3\bar{e}_1)(\bar{e}_1\bar{e}_3 + \bar{e}_3\bar{e}_1) + C_{66}^{V}(\bar{e}_1\bar{e}_2 + \bar{e}_2\bar{e}_1)(\bar{e}_1\bar{e}_2 + \bar{e}_2\bar{e}_1) \end{aligned} \tag{6-7}$$

where

$$C_{11}^{V} = C_{11}$$

$$C_{22}^{V} = C_{33}^{V} = \frac{3}{8}(C_{22} + C_{33}) + \frac{1}{4} C_{23} + \frac{1}{2} C_{55}$$

$$C_{12}^{V} = C_{13}^{V} = \frac{1}{2}(C_{12} + C_{13})$$

$$C_{23}^{V} = \frac{1}{8}(C_{22} + C_{33}) + \frac{3}{4} C_{23} - \frac{1}{2} C_{44}$$

$$C_{44}^{V} = \frac{1}{8}(C_{22} + C_{33}) - \frac{1}{4} C_{23} + \frac{1}{2} C_{44}$$

$$C_{55}^{V} = C_{66}^{V} = \frac{1}{2}(C_{55} + C_{66}) \tag{6-8}$$

According to equation (6-7), $\langle \overleftrightarrow{C} \rangle^{V}$ becomes the elastic constant tensor of a homogeneous transversely isotropic medium with $\vec{e}_1$ as the symmetry axis in which the matrix representation of this tensor is

$$\begin{bmatrix} C_{11}^{V} & C_{12}^{V} & C_{13}^{V} & 0 & 0 & 0 \\ C_{12}^{V} & C_{22}^{V} & C_{23}^{V} & 0 & 0 & 0 \\ C_{13}^{V} & C_{23}^{V} & C_{33}^{V} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2C_{44}^{V} & 0 & 0 \\ 0 & 0 & 0 & 0 & 2C_{55}^{V} & 0 \\ 0 & 0 & 0 & 0 & 0 & 2C_{66}^{V} \end{bmatrix}$$

For our convenience, we shall adopt the notation in (2-1) and write

$$\begin{bmatrix} C^V_{11} & C^V_{12} & C^V_{13} & 0 & 0 & 0 \\ C^V_{12} & C^V_{22} & C^V_{23} & 0 & 0 & 0 \\ C^V_{13} & C^V_{23} & C^V_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2C^V_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & 2C^V_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & 2C^V_{66} \end{bmatrix} = \begin{bmatrix} (\lambda^V_\perp+2\mu^\perp_\perp) & \lambda^V_\perp & \lambda^V_\perp & 0 & 0 & 0 \\ \lambda^V_\perp & (\lambda^V_\parallel+2\mu^V_\parallel) & \lambda^V_\parallel & 0 & 0 & 0 \\ \lambda^V_\perp & \lambda^V_\parallel & (\lambda^V_\parallel+2\mu^V_\parallel) & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\mu^V_\parallel & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\nu^V & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\nu^V \end{bmatrix} \tag{6-9}$$

or

$$\begin{aligned} \lambda^V_\parallel &= C^V_{23} \\ \lambda^V_\perp &= C^V_{12} = C^V_{13} \\ \mu^V_\parallel &= 0.5\,(C^V_{22}-C^V_{23}) \\ \mu^V_\perp &= 0.5\,(C^V_{11}-C^V_{12}) \\ \nu^V &= C^V_{55} = C^V_{66} \end{aligned} \tag{6-9'}$$

In the Reuss scheme, it is assumed that stress is uniform throughout the whole aggregate, then we consider an equivalent uniform elastic body which would produce the average strain for the given uniform stress. Hence

$$\bar{\bar{\epsilon}}(a) = \overleftrightarrow{S}(a) \cdot\cdot\, \bar{\bar{\epsilon}} \tag{6-10}$$

where the compliance tetradic $\overleftrightarrow{S}(a)$ is the inverse of tensor $\overleftrightarrow{C}(a)$ when we regard them as the symmetric operators on the 6-dimensional tensor space.

In the notation of matrix in the frame $\{\bar{e}''_1, \bar{e}''_2, \bar{e}''_3\}$, $\overleftrightarrow{S}(a)$ can be represented by the matrix

$$[S] = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{13} & S_{23} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & 2S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & 2S_{66} \end{bmatrix}$$

$$= \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & 2C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & 2C_{66} \end{bmatrix}^{-1} = [C]^{-1} \tag{6-11}$$

where $[C]^{-1}$ is the inverse of the matrix $[C]$ represented in (6-1) and hence in tensor notation, $\overleftrightarrow{S}(a)$ can be written as

$$\begin{aligned} \overleftrightarrow{S}(a) = {} & S_{11}\bar{e}''_1\bar{e}''_1\bar{e}''_1\bar{e}''_1 + S_{22}\bar{e}''_2\bar{e}''_2\bar{e}''_2\bar{e}''_2 + S_{33}\bar{e}''_3\bar{e}''_3\bar{e}''_3\bar{e}''_3 \\ & + S_{12}(\bar{e}''_1\bar{e}''_1\bar{e}''_2\bar{e}''_2 + \bar{e}''_2\bar{e}''_2\bar{e}''_1\bar{e}''_1) + S_{13}(\bar{e}''_1\bar{e}''_1\bar{e}''_3\bar{e}''_3 + \bar{e}''_3\bar{e}''_3\bar{e}''_1\bar{e}''_1) \\ & + S_{23}(\bar{e}''_2\bar{e}''_2\bar{e}''_3\bar{e}''_3 + \bar{e}''_3\bar{e}''_3\bar{e}''_2\bar{e}''_2) + \left(\frac{1}{4C_{44}}\right)(\bar{e}''_2\bar{e}''_3 + \bar{e}''_3\bar{e}''_2)(\bar{e}''_2\bar{e}''_3 + \bar{e}''_3\bar{e}''_2) \\ & + \left(\frac{1}{4C_{55}}\right)(\bar{e}''_1\bar{e}''_3 + \bar{e}''_3\bar{e}''_1)(\bar{e}''_1\bar{e}''_3 + \bar{e}''_3\bar{e}''_1) + \left(\frac{1}{4C_{66}}\right)(\bar{e}_1\bar{e}_2 + \bar{e}_2\bar{e}_1)(\bar{e}''_1\bar{e}''_2 + \bar{e}''_2\bar{e}''_1) \end{aligned} \tag{6-12}$$

Following the same procedure as that in Voigt's scheme, we find that the Reuss average compliance tensor becomes

$$\langle \underline{S} \rangle^R = S^R_{11}\underline{e}_1\underline{e}_1\underline{e}_1\underline{e}_1 + S^R_{22}\underline{e}_2\underline{e}_2\underline{e}_2\underline{e}_2 + S^R_{33}\underline{e}_3\underline{e}_3\underline{e}_3\underline{e}_3$$

$$+ S^R_{23}(\underline{e}_2\underline{e}_2\underline{e}_3\underline{e}_3 + \underline{e}_3\underline{e}_3\underline{e}_2\underline{e}_2) + S^R_{13}(\underline{e}_1\underline{e}_1\underline{e}_3\underline{e}_3 + \underline{e}_3\underline{e}_3\underline{e}_1\underline{e}_1)$$

$$+ S^R_{12}(\underline{e}_1\underline{e}_1\underline{e}_2\underline{e}_2 + \underline{e}_2\underline{e}_2\underline{e}_1\underline{e}_1) + S^R_{44}(\underline{e}_2\underline{e}_3 + \underline{e}_3\underline{e}_2)(\underline{e}_2\underline{e}_3 + \underline{e}_3\underline{e}_2)$$

$$+ S^R_{55}(\underline{e}_1\underline{e}_3 + \underline{e}_3\underline{e}_1)(\underline{e}_1\underline{e}_3 + \underline{e}_3\underline{e}_1) + S^R_{66}(\underline{e}_1\underline{e}_2 + \underline{e}_2\underline{e}_1)(\underline{e}_1\underline{e}_2 + \underline{e}_2\underline{e}_1) \tag{6-13}$$

in which

$$S^R_{11} = S_{11}$$

$$S^R_{22} = S^R_{33} = 0.375\,(S_{22} + S_{33}) + 0.125\,S_{23} + 0.5\,S_{44}$$

$$S^R_{12} = S^R_{13} = (S_{12} + S_{13})$$

$$S^R_{44} = 0.5\,(S^R_{22} - S^R_{23}) = 0.125\,(S_{22} + S_{33}) - 0.25\,S_{23} + 0.5\,S_{44}$$

$$S^R_{55} = S^R_{66} = 0.5\,(S_{55} + S_{66}) \tag{6-14}$$

The matrix representations of the elastic compliance tensor and stiffness tensor are related by (6-11) and hence the matrix representation of the Reuss average elastic stiffness tensor

$$\langle \overleftrightarrow{C} \rangle^R = C_{11}^R \bar{e}_1\bar{e}_1\bar{e}_1\bar{e}_1 + C_{22}^R \bar{e}_2\bar{e}_2\bar{e}_2\bar{e}_2 + C_{33}^R \bar{e}_3\bar{e}_3\bar{e}_3\bar{e}_3$$

$$+ C_{23}^R (\bar{e}_2\bar{e}_2\bar{e}_3\bar{e}_3 + \bar{e}_3\bar{e}_3\bar{e}_2\bar{e}_2) + C_{13}^R (\bar{e}_1\bar{e}_1\bar{e}_3\bar{e}_3 + \bar{e}_3\bar{e}_3\bar{e}_1\bar{e}_1)$$

$$+ C_{12}^R (\bar{e}_1\bar{e}_1\bar{e}_2\bar{e}_2 + \bar{e}_2\bar{e}_2\bar{e}_1\bar{e}_1) + C_{44}^R (\bar{e}_2\bar{e}_3 + \bar{e}_3\bar{e}_2)(\bar{e}_2\bar{e}_3 + \bar{e}_3\bar{e}_2)$$

$$+ C_{55}^R (\bar{e}_1\bar{e}_3 + \bar{e}_3\bar{e}_1)(\bar{e}_1\bar{e}_3 + \bar{e}_3\bar{e}_1) + C_{55}^R (\bar{e}_1\bar{e}_2 + \bar{e}_2\bar{e}_1)(\bar{e}_1\bar{e}_2 + \bar{e}_2\bar{e}_1) \qquad (6\text{-}15)$$

is obtained. For any values of C_{ij} in (6-2), we find that $\langle \overleftrightarrow{C} \rangle^R$ still represents the transverse isotropy with $\bar{e}_1$ as the symmetry axis. In the notation of the preceding sections also, we set

$$\lambda_{\parallel}^R = C_{23}^R$$

$$\lambda_{\perp}^R = C_{12}^R = C_{13}^R$$

$$\mu_{\parallel}^R = 0.5\,(C_{22}^R - C_{23}^R)$$

$$\mu_{\perp}^R = 0.5\,(C_{11}^R - C_{12}^R)$$

$$\nu^R = C_{55}^R = C_{66}^R \qquad (6\text{-}16)$$

The Voigt-Reuss-Hill (VRH) approximation, the arithmetic mean of the elastic constants calculated according to the Voigt and Reuss schemes, is given by $C_{ij}^{H} = 0.5(C_{ij}^{R} + C_{ij}^{V})$

Now we shall go back to the discussion of the phase velocities of the propagation of surface waves of Rayleigh type on the model with elastic constants calculated from these three schemes.

Let us choose Olivine (93% Fosterite) as an example, the density and the elastic constants are as follows:

density $\rho = 3.3110$ gm/cm^3
$C_{11} = 3.2370$ Mb.
$C_{22} = 1.9760$ Mb.
$C_{33} = 2.3510$ Mb.
$C_{12} = 0.6640$ Mb.
$C_{13} = 0.7160$ Mb.
$C_{23} = 0.7560$ Mb.
$C_{44} = 0.6462$ Mb.
$C_{55} = 0.7805$ Mb.
$C_{66} = 0.7904$ Mb.

According to the foregoing procedures, the five elastic constants of these three schemes are

Voigt:

$$\lambda_{\parallel}^{V} = 0.784775 \text{ Mb}$$
$$\mu_{\parallel}^{V} = 0.674975 \text{ Mb}$$
$$\lambda_{\perp}^{V} = 0.690000 \text{ Mb}$$
$$\mu_{\perp}^{V} = 1.273500 \text{ Mb}$$
$$\nu^{V} = 0.785450 \text{ Mb.} \qquad (6\text{-}17)$$

Table 4. $\gamma(\theta) = C(\theta)/\sqrt{C_{44}/\rho}$ computed for three schemes of average for the "Olivine Model" by Francis.

θ (Degree)	Voigt value	Reuss value	VRH value
0	1.0184	1.0254	1.0220
10	1.0153	1.0220	1.0187
20	1.0063	1.0124	1.0093
30	0.9924	0.9975	0.9950
40	0.9755	0.9793	0.9774
50	0.9580	0.9604	0.9592
60	0.9527	0.9435	0.9431
70	0.9313	0.9308	0.9311
80	0.9246	0.9231	0.9239
90	0.9224	0.9205	0.9215

Reuss:

$$\begin{aligned}
\lambda_{\parallel}^{R} &= 0.699500 \text{ Mb} \\
\mu_{\parallel}^{R} &= 0.661440 \text{ Mb} \\
\lambda_{\perp}^{R} &= 0.636386 \text{ Mb} \\
\mu_{\perp}^{R} &= 1.265040 \text{ Mb} \\
\nu^{R} &= 0.785419 \text{ Mb}
\end{aligned} \tag{6-18}$$

VRH:

$$\begin{aligned}
\lambda_{\parallel}^{H} &= 0.742137 \text{ Mb} \\
\mu_{\parallel}^{H} &= 0.668210 \text{ Mb} \\
\lambda_{\perp}^{H} &= 0.663193 \text{ Mb} \\
\mu_{\perp}^{H} &= 1.269270 \text{ Mb} \\
\nu^{H} &= 0.785434 \text{ Mb}
\end{aligned} \tag{6-19}$$

Substituting all the values of these three schemes into equation (4-23), we obtain the calculated values of the ratio $Y(\theta) = \dfrac{C(\theta)}{\sqrt{\frac{\mu_{\parallel}}{\rho}}}$ in ten directions corresponding to θ = 0°, 10°, 20°, ..., 90° as given in Table 4. The results show that the predicted anisotropy is much stronger than the observed 2%. In order to obtain agreement with the observation, we shall diffusely distribute the a axis around the normal to the vertical plane along the ridge. In the following sections, we shall find such a distribution of a axis that will explain both P and Rayleigh wave anisotropy simultaneously.

7. Perturbation technique for the P-wave velocity

Accumulating evidences from refraction measurements support strong anisotropy in the velocity of horizontally propagating P waves in the top of the Oceanic Upper Mantle. The propagation of body waves through an anisotropic media may be described by three phase velocities obtained from the dispersion relation. As already mentioned, purely longitudinal waves and purely transverse waves can no longer exist in a generally anisotropic medium. We shall, at this point, treat the case of small anisotropy analogous to the work of Backus (1965) by means of the perturbation technique in dealing with the coupled body waves.

We shall assume that the wave number vector is horizontal, and the model will be the half space composed of aggregate of Olivine grains discussed in Section 6 which can be specified as a homogeneous transversely isotropic medium having the symmetry axis in the horizontal direction $\hat{e}_1$. In considering the phase velocities of the propagating waves, plane wave approach could be an appropriate one.

If $\vec{u}(\vec{x},t)$ is a plane wave of dependence $\exp\{i(\vec{k}\cdot\vec{x}-\omega t)\}$ then it will satisfy equation (2-7). Define $\vec{\nu}=\frac{\vec{k}}{\|\vec{k}\|}$; let $c=\frac{\omega}{\|\vec{k}\|}$ be the phase velocity and $\overline{\overline{B}}(\vec{\nu})=\rho^{-1}\left[\left(\dot{\iota}_2(\vec{\nu})\otimes\dot{\iota}_3(\vec{\nu})\right)\overleftrightarrow{C}\right]$ be the positive definite symmetric tensor of rank 2, where $\left[\left(\dot{\iota}_2(\vec{\nu})\otimes\dot{\iota}_3(\vec{\nu})\right)\overleftrightarrow{C}\right]$ is the contraction of elastic constant tensor defined by

$$\left[\left(\hat{i}_2(\bar{\nu})\otimes\hat{i}_3(\bar{\nu})\right)\overleftrightarrow{C}\right]=\sum_{i_1,i_2,i_3,i_4=1}^{3} C_{i_1 i_2 i_3 i_4}\,(\bar{e}_{i_2}\cdot\bar{\nu})(\bar{e}_{i_3}\cdot\bar{\nu})\,\bar{e}_{i_1}\bar{e}_{i_4}$$

then equation (2-7) becomes

$$\bar{\bar{B}}(\bar{\nu})\cdot\bar{u}(\bar{\nu})=c^2(\bar{\nu})\,\bar{u}(\bar{\nu}) \tag{7-1}$$

The three eigenvalues of $\bar{\bar{B}}(\bar{\nu})$ which are squared phase velocities correspond to the polarizations of three body waves. It is clear that the tensor $\bar{\bar{B}}(\bar{\nu})$ can be separated into a part $\bar{\bar{B}}_0(\bar{\nu})$ which describes the propagation in isotropic medium with Lame's constants λ_{11}, μ_{11} and another part $\delta\bar{\bar{B}}(\bar{\nu})$ which describes the effect
due to the anisotropy, namely

$$\bar{\bar{B}}(\bar{\nu})=\bar{\bar{B}}_0(\bar{\nu})+\delta\bar{\bar{B}}(\bar{\nu}) \tag{7-2}$$

where

$$\bar{\bar{B}}_0(\bar{\nu})=\rho^{-1}\left[\left(\hat{i}_2(\bar{\nu})\otimes\hat{i}_3(\bar{\nu})\right)\overleftrightarrow{C_0}\right]$$

$$\delta\bar{\bar{B}}(\bar{\nu})=\rho^{-1}\left[\left(\hat{i}_2(\bar{\nu})\otimes\hat{i}_3(\bar{\nu})\right)\delta\overleftrightarrow{C}\right] \tag{7-2'}$$

From the isotropic tensor $\overleftrightarrow{C_0}$ given by (2-3), we can write

$$B_0(\nu) = \left[\left(\frac{\lambda_{11}+2\mu_{11}}{\rho}\right) - \left(\frac{\mu_{11}}{\rho}\right)\right]\nu\nu + \left(\frac{\mu_{11}}{\rho}\right)\left(\sum_{i=1}^{3} \hat{e}_i \hat{e}_i\right) \qquad (7\text{-}3)$$

Hence, we have the degenerate case for an isotropic medium in which the polarizations of the longitudinal waves and transverse waves are uncoupled.

In this section, we consider the propagation of P waves, SV, and SH waves will be discussed in the next section. Recall that the unit vector $\hat{e}_1'$ in the orthonormal frame defined in section 2 is precisely $\hat{\nu}$ for horizontally propagating body waves. As we have already seen that from $\delta\hat{c}$ given in (2-4), $\hat{e}_1'$ is not eigenvector of $\bar{\bar{B}}(\hat{e}_1')$ and hence the polarization is no longer longitudinal for P wave. However, the perturbation theory (Backus, 1965) which is nondegenerate for P waves provides us with the information on small anisotropy. If we write the eigenvalue of $\bar{\bar{B}}(\nu)$ as $\left(\frac{\lambda_{11}+2\mu_{11}}{\rho}\right) + \delta c_p^2$ then, correct to the first order in perturbation

$$\delta c_p^2 = \hat{e}_1' \cdot \delta \bar{\bar{B}}(\hat{e}_1') \cdot \hat{e}_1' \qquad (7\text{-}4)$$

from

$$\delta \bar{\bar{B}}(\bar{\nu}) \cdot \bar{u} = \rho^{-1}\left[\left(\hat{i}_2(\bar{\nu}) \otimes \hat{i}_3(\bar{\nu})\right) \delta \overleftrightarrow{C}\right] \cdot \bar{u}$$

$$= \rho^{-1}\left(\bar{e}_1' \bar{e}_2' \bar{e}_3'\right) \begin{bmatrix} \delta A_{11}(\nabla_x) & \delta A_{12}(\nabla_x) & \delta A_{13}(\nabla_x) \\ \delta A_{12}(\nabla_x) & \delta A_{22}(\nabla_x) & \delta A_{23}(\nabla_x) \\ \delta A_{13}(\nabla_x) & \delta A_{23}(\nabla_x) & \delta A_{33}(\nabla_x) \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \quad (7\text{-}5)$$

and (2-15) where $\delta A_{ij}(\bar{\nu})$ are given by (2-15) with ∇_x replaced by $\bar{\nu} = \bar{e}_1'$. We finally have

$$\delta C_p^2 = \rho^{-1} \cos^2\theta \left[\delta\lambda(1+\sin^2\theta) + 2\delta\mu \cos^2\theta + 4\delta\nu \sin^2\theta\right] \quad (7\text{-}6)$$

which is the deviation of the squared phase velocity as a function of the azimuth of P wave propagation, which is reducible to the general form provided by Backus.

In general, when we want to evaluate the phase velocity of the propagation of P waves in any weakly anisotropic medium, we may impose the condition that wave number vector is in the direction of the eigenvector (to be more precise, the polarization of the particle motion is along the direction of the wave number vector) regardless of its coupling with other modes.

The P_n velocities calculated by the formula

$$C_p = \left\{\left(\frac{\lambda_0 + 2\mu_0}{\rho}\right) + \left(\frac{1}{\rho}\right) \cos^2\theta \left[\delta\lambda(1+\sin^2\theta) + 2\delta\mu \cos^2\theta + 4\delta\nu \sin^2\theta\right]\right\}^{\frac{1}{2}} \quad (7\text{-}7)$$

Table 5A. P, SV and SH wave phase velocities computed for Voigt scheme for the "Olivine Model" by Francis.

θ (Degree)	C_P^V (Km/Sec)	C_{SV}^V (Km/Sec)	C_{SH}^V (Km/Sec)
0	9.8876	4.8705	4.8705
10	9.7984	4.8602	4.8933
20	9.5509	4.8303	4.9542
30	9.1994	4.7784	5.0146
40	8.8195	4.7269	5.0560
50	8.4856	4.6652	5.0560
60	8.2452	4.6065	5.0146
70	8.1065	4.5581	4.9542
80	8.0453	4.5262	4.8933
90	8.0295	4.5150	4.8705

Table 5B. P, SV, and SH wave phase velocities computed for Reuss Scheme for the "Olivine Model" by Francis.

θ (Degree)	C_P^R (Km/Sec)	C_{SV}^R (Km/Sec)	C_{SH}^R (Km/Sec)
0	9.6814	4.8705	4.8705
10	9.5918	4.8589	4.8859
20	9.3419	4.8253	4.9249
30	8.9836	4.7734	4.9688
40	8.5895	4.7090	4.9972
50	8.2330	4.6400	4.9972
60	7.9641	4.5731	4.9688
70	7.7975	4.5183	4.9249
80	7.7157	4.4822	4.8859
90	7.6925	4.4696	4.8705

Table 5C. P, SV, and SH wave phase velocities computed for VRH schemes for the "Olivine Model" by Francis.

θ (Degree)	C_P^H (Km/Sec)	C_{SV}^H (Km/Sec)	C_{SH}^H (Km/Sec)
0	9.7850	4.8752	4.8705
10	9.6957	4.8595	4.8896
20	9.4470	4.8280	4.9377
30	9.0921	4.7760	4.9917
40	8.7052	4.7180	5.0267
50	8.3453	4.6524	5.0267
60	8.1059	4.5898	4.9917
70	7.9535	4.5382	4.9377
80	7.8822	4.5042	4.8896
90	7.8628	4.4924	4.8705

for 10 directions θ = 0°, 10°, ..., 90° for the Olivine model with averaged elastic constants of three schemes mentioned in the last section are tabulated in Table 5. These values are all too large along the direction of ξ_1. We must conclude that the Olivine Upper Mantle model in which the orientation of the a axis of all the Olivine grains lies in the horizontal direction perpendicular to the ridge axis cannot explain the observed anisotropy. It is again necessary to diffusely distribute the orientation of the a axis.

8. Anisotropy of s waves

From (7-3) we see that the eigenvalue $\left(\frac{\mu_{11}}{\rho}\right)$ of $\overline{\overline{B}}_0(\xi_1')$ is degenerate with a two dimensional eignespace normal to ξ_1' generated by ξ_2' and ξ_3'. If we write the eignevalues of $\overline{\overline{B}}(\xi_1') = \overline{\overline{B}}_0(\xi_1') + \delta\overline{\overline{B}}(\xi_1')$ in terms of $\left(\frac{\mu_{11}}{\rho}\right) + \delta c_{SV}^2$, $\left(\frac{\mu_{11}}{\rho}\right) + \delta c_{SH}^2$ for waves of SV and SH type respectively, then again to the first order in perturbation by imposing the condition that ξ_3' and ξ_2' are the eigenvectors corresponding to the waves of SV and SP type respectively, we find, following Backus, that δc_{SH}^2 and δc_{SV}^2 are the eigenvalues of the following matrix

$$\begin{bmatrix} \xi_2' \cdot \delta\overline{\overline{B}}(\xi_1') \cdot \xi_2' & \xi_2' \cdot \delta\overline{\overline{B}}(\xi_1') \cdot \xi_3' \\ \xi_2' \cdot \delta\overline{\overline{B}}(\xi_1') \cdot \xi_3' & \xi_3' \cdot \delta\overline{\overline{B}}(\xi_1') \cdot \xi_3' \end{bmatrix} \qquad (8\text{-}1)$$

These two eigenvalues can be obtained by choosing two orthonormal vectors which generate the same eigenspace normal to $\vec{e}_1'$ and diagonalize the matrix in (8-1).

In the present case of transverse isotropy, the problem is much simplified. Synge (1957) has shown that in a homogeneous transversely isotropic medium, there is an eigenvector of $\bar{\bar{B}}(\vec{e}_1')$ normal to the symmetry axis and the wave number vector. Hence in the medium characterized by the elastic tensor $\overleftrightarrow{C} = \overleftrightarrow{C}_0 + \delta\overleftrightarrow{C}$ with $\overleftrightarrow{C}_0$ and $\delta\overleftrightarrow{C}$ expressed in the form of (2-11) and (2-4) respectively, $\vec{e}_3'$ is the eigenvector of $\bar{\bar{B}}(\vec{e}_1') = \left[\hat{n}_2(\vec{e}_1') \otimes \hat{n}_3(\vec{e}_1') \overleftrightarrow{C}\right]$ and the matrix in (8-1) is diagonal, namely, the polarization of SV waves is purely transverse. For the waves of the SH type, the polarization is no longer purely transverse. However, the approximation of obtaining (8-1) gives $\delta C_{SH}^2 = \vec{e}_2' \cdot \delta \bar{\bar{B}}(\vec{e}_1') \cdot \vec{e}_2'$ correct to the first order in perturbation. As long as it is permissible to impose the condition that the polarization of SH waves is transverse, the azimuthal dependence of the phase velocity C_{SH} will be obtained.

Summarizing the foregoing remarks, we have $\delta C_{SH}^2 = \vec{e}_2' \cdot \delta\bar{\bar{B}}(\vec{e}_1') \cdot \vec{e}_2'$ exactly for SV, and $\delta C_{SH}^2 = \rho^{-1}\left[\delta\nu(\cos^2\theta - \sin^2\theta)^2 - (\delta\lambda - 2\delta\mu)\sin^2\theta\cos^2\theta\right]$ correct to the first order in perturbation for SH waves. In other words, the phase velocities C_{SV} and C_{SH} can be written as

$$C_{SV} = \sqrt{\left(\frac{\mu_{11}}{\rho}\right) + \left(\frac{1}{\rho}\right)\delta\nu \cos^2\theta}$$

$$= \sqrt{\rho^{-1}(\nu \cos^2\theta + \mu_{11} \sin^2\theta)} \tag{8-2}$$

$$C_{SH} = \sqrt{\left(\frac{\mu_{11}}{\rho}\right) + \left(\frac{1}{\rho}\right)\left[\delta\nu \cos^2 2\theta - (\delta\lambda - 2\delta\mu) \sin^2\theta \cos^2\theta\right]}$$

$$= \sqrt{\rho^{-1}(\nu \cos^2 2\theta + \mu_{11} \sin^2\theta) - \rho^{-1}(\delta\lambda - 2\delta\mu) \sin^2\theta \cos^2\theta} \tag{8-3}$$

The phase velocities C_{SH} and C_{SV} given by (8-2) and (8-3) are also tabulated in Table 5 for the three averaging schemes in (6-17), (6-18) and (6-19). They show that the azimuthal dependence of C_{SH} is weaker than C_{SV}. P waves exhibit much greater azimuthal dependence than S waves. The phase velocity of SV is monotonically decreasing from $\theta = 0°$ to $\theta = 90°$, while that of SH waves is the smallest at $\theta = 0°$ and $90°$ and the largest at $\theta = 45°$. These features should be helpful for diagnosing the anisotropy of a transversely isotropic body.

9. Transversely isotropic medium with the orientation of symmetry axis diffusely distributed.

So far, the phase velocities of Rayleigh waves in 'fracture zone' model and 'Olivine' model computed in Section 5 and Section 6 were unable to explain the observed data; the latter gave too large azimuthal variation and the former too little. Also, the calculated values of the phase velocities of P waves did not fit the observation. In order to explain observed results, we shall modify the 'Olivine' model in such a way that the a-axis is no longer strictly perpendicular to the vertical plane along the ridge axis, but diffusely distributed around the direction. Since we found in Section 6 that the bounds given by Voigt and Reuss averages are narrow enough, we shall calculate only the Voigt average here assuming that the strain is uniform.

On the assumption of uniform strain throughout the whole medium, the averaging of the elastic constants is reduced to the averaging of the equation of motion. Before this average, we have to perform the transformation of the coordinate frames. Let $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$, $\{\vec{e}_1', \vec{e}_2', \vec{e}_3'\}$ and $\{\vec{e}_1''', \vec{e}_2''', \vec{e}_3'''\}$ be three orthonormal frames having the same origin at a point of the free surface of half space with the geometry described in the following way:

$\vec{e}_1$ is the horizontal unit vector normal to the ridge axis, $\bar{e}_3$ is the unit vector directed vertically into the medium and $\bar{e}_2$ is obtained by setting $\bar{e}_2 = \bar{e}_3 \times \bar{e}_1$ the frame $\{\bar{e}_1', \bar{e}_2', \bar{e}_3'\}$ is obtained by rotating frame $\{\bar{e}_1, \bar{e}_2, \bar{e}_3\}$ about $\bar{e}_3$ through an angle θ such that $\bar{e}_1'$ is the horizontal vector parallel to the vertical plane containing the wave number vector. The frame $\{\bar{e}_1''', \bar{e}_2''', \bar{e}_3'''\}$ is obtained by rotating the frame $\{\bar{e}_1, \bar{e}_2, \bar{e}_3\}$ about $\bar{e}_3$ through an angle γ first and then followed by the rotation of the rotated frame through an angle β about the axis $\bar{e}_2'''$ which is at an angle γ with $\bar{e}_2$ as shown in Figure 3. Any two pair of these three frames are related by

$$
\begin{aligned}
\bar{e}_1 &= \cos\theta\, \bar{e}_1' - \sin\theta\, \bar{e}_2' \\
\bar{e}_2 &= \sin\theta\, \bar{e}_1' + \cos\theta\, \bar{e}_2' \\
\bar{e}_3 &= \bar{e}_3'
\end{aligned}
\tag{9-1}
$$

$$
\begin{aligned}
\bar{e}_1''' &= \cos\beta \cos\gamma\, \bar{e}_1 + \cos\beta \sin\gamma\, \bar{e}_2 - \sin\beta\, \bar{e}_3 \\
\bar{e}_2''' &= -\sin\gamma\, \bar{e}_1 + \cos\gamma\, \bar{e}_2 \\
\bar{e}_3''' &= \sin\beta \cos\gamma\, \bar{e}_1 + \sin\beta \sin\gamma\, \bar{e}_2 + \cos\, \bar{e}_3
\end{aligned}
\tag{9-2}
$$

$$
\begin{aligned}
\bar{e}_1''' &= \cos\beta \cos(\theta-\gamma)\, \bar{e}_1' - \cos\beta \sin(\theta-\gamma)\, \bar{e}_2' - \sin\beta\, \bar{e}_3' \\
\bar{e}_2''' &= \sin(\theta-\gamma)\, \bar{e}_1' + \cos(\theta-\gamma)\, \bar{e}_2' \\
\bar{e}_3''' &= \sin\beta \cos(\theta-\gamma)\, \bar{e}_1' - \sin\beta \sin(\theta-\gamma)\, \bar{e}_2' + \cos\beta\, \bar{e}_3'
\end{aligned}
\tag{9-3}
$$

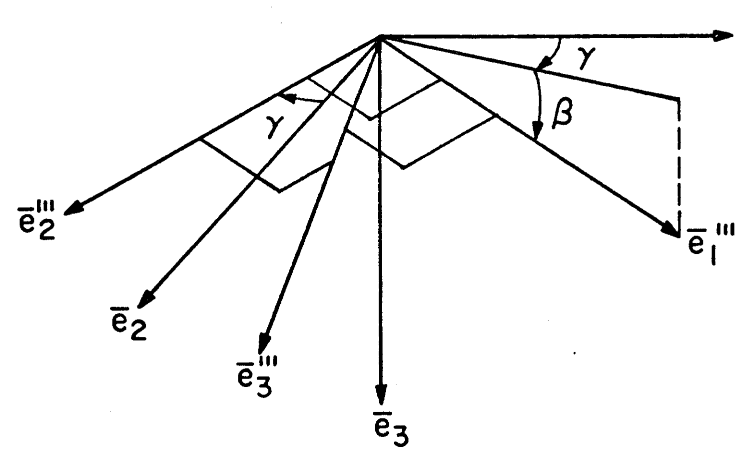

Figure 3. Coordinate frame $\{\bar{e}_1, \bar{e}_2, \bar{e}_3\}$ is specified in figure 2 γ is measured clockwise about $\bar{e}_3$. $\bar{e}_3''' = \bar{e}_1''' \times \bar{e}_2'''$, $\bar{e}_1'''$ is parallel to the symmetry axis of transversely isotropic medium, and at an angle β with the horizontal plane.

In particular, when $\beta = 0$, namely, $\bar{e}_1'''$ lies in a horizontal plane, (9-3) is reduced to

$$\begin{aligned}
\bar{e}_1''' &= \cos(\theta-\gamma)\,\bar{e}_1' - \sin(\theta-\gamma)\,\bar{e}_2' \\
\bar{e}_2''' &= \sin(\theta-\gamma)\,\bar{e}_1' + \cos(\theta-\gamma)\,\bar{e}_2' \\
\bar{e}_3''' &= \bar{e}_3'''
\end{aligned} \tag{9-4}$$

The displacement vector $\bar{u}$ in case of infinitesimal deformation can be written as

$$\begin{aligned}
\bar{u} &= u_1\bar{e}_1' + u_2\bar{e}_2' + u_3\bar{e}_3' \\
&= \left[\cos\beta\cos(\theta-\gamma)\,u_1 - \cos\beta\sin(\theta-\gamma)\,u_2 - \sin\beta\,u_3\right]\bar{e}_1''' \\
&\quad + \left[\sin(\theta-\gamma)\,u_1 + \cos(\theta-\gamma)\,u_2\right]\bar{e}_2''' \\
&\quad + \left[\sin\beta\cos(\theta-\gamma)\,u_1 - \sin\beta\sin(\theta-\gamma)\,u_2 + \cos\beta\,u_3\right]\bar{e}_3'''
\end{aligned} \tag{9-5}$$

Furthermore, if we assume that a symmetry axis of a homogeneous transverse isotropic medium with five elastic constants $\lambda_\perp$, $\mu_\perp$, $\lambda_\parallel$, $\mu_\parallel$ and ν is arbitratily oriented, then by suitable choice of the angular variables β and γ, $\bar{e}_1'''$ will lie in the direction of the symmetry axis and $\bar{e}_2'''$ in a horizontal plane.

The choice of this frame outlined to the transverse isotropy is reasonable and can simplify our calculation in deriving the equation of motion later-on, since in the plane normal to the symmetry axis, the frame can be arbitrarily rotated without affecting the expression of the elastic tensor. After the choice of the frame $\{\bar{e}_1''', \bar{e}_2''', \bar{e}_3'''\}$ is made, the elastic tensor can be expressed as

$$
\begin{aligned}
\overleftrightarrow{C}(\beta,\gamma) = {} & (\lambda_\perp + 2\mu_\perp)\,\bar{e}_1'''\bar{e}_1'''\bar{e}_1'''\bar{e}_1''' + (\lambda_\parallel + 2\mu_\parallel)(\bar{e}_2'''\bar{e}_2'''\bar{e}_2'''\bar{e}_2''' + \bar{e}_3'''\bar{e}_3'''\bar{e}_3'''\bar{e}_3''') \\
& + \lambda_\perp(\bar{e}_1'''\bar{e}_1'''\bar{e}_2'''\bar{e}_2''' + \bar{e}_2'''\bar{e}_2'''\bar{e}_1'''\bar{e}_1''' + \bar{e}_1'''\bar{e}_1'''\bar{e}_3'''\bar{e}_3''' + \bar{e}_3'''\bar{e}_3'''\bar{e}_1'''\bar{e}_1''') \\
& + \lambda_\parallel(\bar{e}_2'''\bar{e}_2'''\bar{e}_3'''\bar{e}_3''' + \bar{e}_3'''\bar{e}_3'''\bar{e}_2'''\bar{e}_2''') + \mu_\parallel(\bar{e}_2'''\bar{e}_3''' + \bar{e}_3'''\bar{e}_2''')(\bar{e}_2'''\bar{e}_3''' + \bar{e}_3'''\bar{e}_2''') \\
& + \nu(\bar{e}_1'''\bar{e}_3''' + \bar{e}_3'''\bar{e}_1''')(\bar{e}_1'''\bar{e}_3''' + \bar{e}_3'''\bar{e}_1''') + \nu(\bar{e}_1'''\bar{e}_2''' + \bar{e}_2'''\bar{e}_1''')(\bar{e}_1'''\bar{e}_2''' + \bar{e}_2'''\bar{e}_1''')
\end{aligned}
\tag{9-6}
$$

which can be decomposed into a part $\overleftrightarrow{C}_0$ in isotropic medium with Lame's constants $\lambda_\parallel$, $\mu_\parallel$ and a perturbed part $\delta\overleftrightarrow{C}(\beta,\gamma)$ given by

$$
\begin{aligned}
\overleftrightarrow{C}_0 = {} & (\lambda_\parallel + 2\mu_\parallel)(\bar{e}_1'''\bar{e}_1'''\bar{e}_1'''\bar{e}_1''' + \bar{e}_2'''\bar{e}_2'''\bar{e}_2'''\bar{e}_2''' + \bar{e}_3'''\bar{e}_3'''\bar{e}_3'''\bar{e}_3''') \\
& + \lambda_\parallel(\bar{e}_1'''\bar{e}_1'''\bar{e}_2'''\bar{e}_2''' + \bar{e}_2'''\bar{e}_2'''\bar{e}_1'''\bar{e}_1''' + \bar{e}_1'''\bar{e}_1'''\bar{e}_3'''\bar{e}_3''' + \bar{e}_3'''\bar{e}_3'''\bar{e}_1'''\bar{e}_1''' + \bar{e}_2'''\bar{e}_2'''\bar{e}_3'''\bar{e}_3''' + \bar{e}_3'''\bar{e}_3'''\bar{e}_2'''\bar{e}_2''') \\
& + \mu_\parallel\big[(\bar{e}_1'''\bar{e}_2''' + \bar{e}_2'''\bar{e}_1''')(\bar{e}_1'''\bar{e}_2''' + \bar{e}_2'''\bar{e}_1''') + (\bar{e}_1'''\bar{e}_3''' + \bar{e}_3'''\bar{e}_1''')(\bar{e}_1'''\bar{e}_3''' + \bar{e}_3'''\bar{e}_1''')
\end{aligned}
\tag{9-7}
$$

$$+ (\vec{e}_2''\vec{e}_3'' + \vec{e}_3''\vec{e}_2'')(\vec{e}_2''\vec{e}_3'' + \vec{e}_3''\vec{e}_2'')\Big] \tag{9-7}$$

$$\delta\hat{C} = (\delta\lambda + 2\delta\mu)\,\vec{e}_1''\vec{e}_1''\vec{e}_1''\vec{e}_1''$$

$$+ \delta\lambda(\vec{e}_1''\vec{e}_1''\vec{e}_2''\vec{e}_2'' + \vec{e}_2''\vec{e}_2''\vec{e}_1''\vec{e}_1'' + \vec{e}_1''\vec{e}_1''\vec{e}_3''\vec{e}_3'' + \vec{e}_3''\vec{e}_3''\vec{e}_1''\vec{e}_1'')$$

$$+ \delta\nu\Big[(\vec{e}_1''\vec{e}_3'' + \vec{e}_3''\vec{e}_1'')(\vec{e}_1''\vec{e}_3'' + \vec{e}_3''\vec{e}_1'') + (\vec{e}_1''\vec{e}_2'' + \vec{e}_2''\vec{e}_1'')(\vec{e}_1''\vec{e}_2'' + \vec{e}_2''\vec{e}_1'')\Big] \tag{9-8}$$

where $\delta\lambda = \lambda_\perp - \lambda_\parallel$, $\delta\mu = \mu_\perp - \mu_\parallel$ and $\delta\nu = \nu - \mu_\parallel$.

The choice of the frame $\{\vec{e}_1', \vec{e}_2', \vec{e}_3'\}$ makes $\partial_2 = \vec{e}_2' \cdot \nabla_{\bar{x}} = 0$ and hence the relation between the partial derivatives with respect to the coordinates in frame $\{\vec{e}_1', \vec{e}_2', \vec{e}_3'\}$ and frame $\{\vec{e}_1''', \vec{e}_2''', \vec{e}_3'''\}$ becomes

$$\partial_{x_1'''} = \cos\beta\cos(\theta-\gamma)\,\partial_1 - \sin\beta\,\partial_3$$

$$\partial_{x_2'''} = \sin(\theta-\gamma)\,\partial_1$$

$$\partial_{x_3'''} = \sin\beta\cos(\theta-\gamma)\,\partial_1 + \cos\beta\,\partial_3 \tag{9-9}$$

where $\partial_i = \partial_{x_i'}$ (i=1,2,3) and $\bar{x} = x_1\vec{e}_1 + x_2\vec{e}_2 + x_3\vec{e}_3 = x_1'\vec{e}_1' + x_2'\vec{e}_2' + x_3\vec{e}_3'$ is the position vector of the medium particle.

If we apply (9-3), (9-5) and (9-9) to the equation of motion $\rho\,\partial_t^2\bar{u} = \sum_{i_1,i_2,i_3,i_4=1}^{3} C'''_{i_1 i_2 i_3 i_4}\,\partial_{i_2}\partial_{i_3}u_{i_4}\vec{e}_{i_1}$

we find that

$$\rho\partial_t^2\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}=\left(\begin{bmatrix}\widetilde{A}^0_{11}(\nabla_T) & \widetilde{A}^0_{12}(\nabla_T) & \widetilde{A}^0_{13}(\nabla_T)\\ \widetilde{A}^0_{12}(\nabla_T) & \widetilde{A}^0_{22}(\nabla_T) & \widetilde{A}^0_{23}(\nabla_T)\\ \widetilde{A}^0_{13}(\nabla_T) & \widetilde{A}^0_{23}(\nabla_T) & \widetilde{A}^0_{33}(\nabla_T)\end{bmatrix}+\begin{bmatrix}\delta\widetilde{A}_{11}(\nabla_T) & \delta\widetilde{A}_{12}(\nabla_T) & \delta\widetilde{A}_{13}(\nabla_T)\\ \delta\widetilde{A}_{12}(\nabla_T) & \delta\widetilde{A}_{22}(\nabla_T) & \delta\widetilde{A}_{23}(\nabla_T)\\ \delta\widetilde{A}_{13}(\nabla_T) & \delta\widetilde{A}_{23}(\nabla_T) & \delta\widetilde{A}_{33}(\nabla_T)\end{bmatrix}\right)\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix} \tag{9-10}$$

where the 3x3 matrix operator $\left[\widetilde{A}^0_{ij}(\nabla_T)\right]$ is the same as that given in (2-12) and $\left[\delta\widetilde{A}_{ij}(\nabla_T)\right]$ is the perturbed part which contributes to the dependence of the motion on the angle between the direction of propagation vector and symmetry axis. Following the same procedures as those discussed in the previous sections, we shall calculate $\delta\widetilde{A}_{11}(\nabla_T)$, $\delta\widetilde{A}_{13}(\nabla_T)=\delta\widetilde{A}_{31}(\nabla_T)$ and $\delta\widetilde{A}_{33}(\nabla_T)$ when we deal with P, SV waves and surface waves of Rayleigh type and $\delta\widetilde{A}_{22}(\nabla_T)$ when dealing with SH waves. For the discussion of P and Rayleigh waves, the relevant motion in two dimensions (x_1', x_3') will be

$$\rho\partial_t^2\begin{bmatrix}u_1\\u_3\end{bmatrix}=\begin{bmatrix}\widetilde{A}^0_{11}(\nabla_T)+\delta\widetilde{A}_{11}(\nabla_T) & \widetilde{A}^0_{13}(\nabla_T)+\delta\widetilde{A}_{13}(\nabla_T)\\ \widetilde{A}^0_{31}(\nabla_T)+\delta\widetilde{A}_{33}(\nabla_T) & \widetilde{A}^0_{33}(\nabla_T)+\delta\widetilde{A}_{33}(\nabla_T)\end{bmatrix}\begin{bmatrix}u_1\\u_3\end{bmatrix} \tag{9-11}$$

where

$$\delta\widetilde{A}_{11}(\nabla_T)=\left\{\begin{array}{l}\cos^2\beta\cos^2(\theta-\gamma)\left[\delta\lambda\left(2-\cos^2\beta\cos^2(\theta-\gamma)\right)+2\delta\mu\cos^2\beta\cos^2(\theta-\gamma)\right]\\ +\delta\nu\left[\sin^2 2\beta\cos^4(\theta-\gamma)+\cos^2\beta\sin^2 2(\theta-\gamma)\right]\end{array}\right\}\partial_1^2$$

$$
+\,2\left[\begin{array}{l} -\delta\lambda \sin\beta\cos\beta\cos(\theta-\gamma) + (\delta\lambda - 2\delta\mu)\sin\beta\cos^3\beta\cos^3(\theta-\gamma) \\ \quad + \delta\nu\left(\begin{array}{l}\sin 2\beta\cos 2\beta\cos^3(\theta-\gamma) \\ -\sin\beta\cos\beta\sin(\theta-\gamma)\sin 2(\theta-\gamma)\end{array}\right)\end{array}\right]\partial_1\partial_3
$$

$$
+\left[\begin{array}{l}(2\delta\mu - \delta\lambda)\sin^2\beta\cos^2\beta\cos^2(\theta-\gamma) \\ + \delta\nu\cos^2 2\beta\cos^2(\theta-\gamma) + \delta\nu\sin^2\beta\sin^2(\theta-\gamma)\end{array}\right]\partial_3^2
$$

$$
\begin{aligned}
\delta\tilde{A}_{13}(\nabla_x) &= \delta\tilde{A}_{31}(\nabla_x) \\
&= \left\{\begin{array}{l} -\sin\beta\cos\beta\cos(\theta-\gamma)\left[\delta\lambda + (2\delta\mu - \delta\lambda)\cos^2\beta\cos^2(\theta-\gamma)\right] \\ + \delta\nu\left[\sin 2\beta\cos 2\beta\cos^3(\theta-\gamma) - \sin\beta\cos\beta\sin 2(\theta-\gamma)\sin(\theta-\gamma)\right]\end{array}\right\}\partial_1^2 \\
&\quad + \left\{\begin{array}{l} 2(\delta\mu - \delta\lambda)\sin^2\beta\cos^2\beta\cos^2(\theta-\gamma) + \delta\lambda\left[\sin^2\beta + \cos^2\beta\cos^2(\theta-\gamma)\right] \\ + \delta\nu\left[\sin^2\beta\sin^2(\theta-\gamma) + \cos 4\beta\cos^2(\theta-\gamma)\right]\end{array}\right\}\partial_1\partial_3 \\
&\quad - \left[\begin{array}{l}\delta\lambda\sin\beta\cos\beta\cos(\theta-\gamma) + \delta\nu\sin 2\beta\cos 2\beta\cos(\theta-\gamma) \\ + (\delta\lambda - 2\delta\mu)\sin^3\beta\cos\beta\cos(\theta-\gamma)\end{array}\right]\partial_3^2
\end{aligned}
$$

$$
\begin{aligned}
\delta\tilde{A}_{33}(\nabla_x) &= \left\{\begin{array}{l}(2\delta\mu - \delta\lambda)\sin^2\beta\cos^2\beta\cos^2(\theta-\gamma) \\ + \delta\nu\left[\cos^2 2\beta\cos^2(\theta-\gamma) + \sin^2\beta\sin^2(\theta-\gamma)\right]\end{array}\right\}\partial_1^2 \\
&\quad - 2\left[\begin{array}{l}\delta\nu\sin 2\beta\cos 2\beta\cos(\theta-\gamma) + \delta\lambda\sin\beta\cos\beta\cos(\theta-\gamma) \\ + (2\delta\mu - \delta\lambda)\sin^3\beta\cos\beta\cos(\theta-\gamma)\end{array}\right]\partial_1\partial_3 \\
&\quad + \left[(2\delta\mu - \delta\lambda)\sin^4\beta + \delta\lambda\sin^2\beta + \delta\nu\sin^2 2\beta\right]\partial_3^2
\end{aligned}
$$

(9-12)

In the particular case that symmetry axis is parallel to the free surface and at an angle γ with $\bar{e}_1$, equation (9-11) is reduced to equation (4-1) with the angle θ replaced by $\theta-\gamma$

The boundary condition that normal stress $\bar{e}_3'\cdot\bar{\bar{\sigma}}=0$ at the free surface $x_3=0$ can also be obtained by following the procedure of carrying out the transformation of the tensor components. Among the three components of $\bar{e}_3'\cdot\bar{\bar{\sigma}}=0$, the condition $(\bar{e}_3'\cdot\bar{\bar{\sigma}})\cdot\bar{e}_2'=0$ can be ignored. The remaining two are approximately

$$
\begin{aligned}
(\bar{e}_3'\cdot\bar{\bar{\sigma}})\cdot\bar{e}_1' = {} & \mu_{11}(\partial_1 u_3+\partial_3 u_1) \\
& + \left\{ \begin{bmatrix} -\delta\lambda \sin\beta\cos\beta\cos(\theta-\gamma) + (\delta\lambda-2\delta\mu)\sin\beta\cos^3\beta\cos^3(\theta-\gamma) \\ +\delta\nu\sin 2\beta\cos 2\beta\cos^3(\theta-\gamma) - \delta\nu\sin\beta\cos\beta\sin(\theta-\gamma)\sin 2(\theta-\gamma) \end{bmatrix}\partial_1 \right. \\
& \quad \left. + \begin{bmatrix} (2\delta\mu-\delta\lambda)\sin^2\beta\cos^2\beta\cos^2(\theta-\gamma) \\ +\delta\nu\cos^2 2\beta\cos^2(\theta-\gamma)+\delta\nu\sin^2\beta\sin^2(\theta-\gamma) \end{bmatrix}\partial_3 \right\} u_1 \\
& + \left\{ \begin{bmatrix} (2\delta\mu-\delta\lambda)\sin^2\beta\cos^2\beta\cos^2(\theta-\gamma) \\ +\delta\nu\sin^2\beta\sin^2(\theta-\gamma)+\delta\nu\cos^2 2\beta\cos^2(\theta-\gamma) \end{bmatrix}\partial_1 \right. \\
& \quad \left. - \begin{bmatrix} \delta\lambda\sin\beta\cos\beta\cos(\theta-\gamma)+\delta\nu\sin 2\beta\cos 2\beta\cos(\theta-\gamma) \\ +(2\delta\mu-\delta\lambda)\sin^3\beta\cos\beta\cos(\theta-\gamma) \end{bmatrix}\partial_3 \right\} u_3
\end{aligned}
$$

(9-13)

$$(\vec{\xi}_3' \cdot \bar{\bar{\sigma}}) \cdot \vec{\xi}_3' = (\lambda_{11} + 2\mu_{11}) \partial_3 u_3 + \lambda_{11} \partial_1 u_1$$

$$+ \left\{ \begin{array}{l} \left[\begin{array}{l} \delta\lambda \operatorname{Sin}^2\beta + (2\delta\mu - \delta\lambda) \operatorname{Sin}^2\beta \operatorname{Cos}^2\beta \operatorname{Cos}^2(\theta-\gamma) \\ - \delta\nu \operatorname{Sin}^2 2\beta \operatorname{Cos}^2(\theta-\gamma) + \delta\lambda \operatorname{Cos}^2\beta \operatorname{Cos}^2(\theta-\gamma) \end{array} \right] \partial_1 \\ - \left[\begin{array}{l} (2\delta\mu - \delta\lambda) \operatorname{Sin}^3\beta \operatorname{Cos}\beta \operatorname{Cos}(\theta-\gamma) \\ + \delta\nu \operatorname{Sin} 2\beta \operatorname{Cos} 2\beta \operatorname{Cos}(\theta-\gamma) + \delta\lambda \operatorname{Sin}\beta \operatorname{Cos}\beta \operatorname{Cos}(\theta-\gamma) \end{array} \right] \partial_3 \end{array} \right\} u_1$$

$$+ \left\{ \begin{array}{l} - \left[\begin{array}{l} (2\delta\mu - \delta\lambda) \operatorname{Sin}^3\beta \operatorname{Cos}\beta \operatorname{Cos}(\theta-\gamma) \\ + \delta\nu \operatorname{Sin} 2\beta \operatorname{Cos} 2\beta \operatorname{Cos}(\theta-\gamma) - \delta\lambda \operatorname{Sin}\beta \operatorname{Cos}\beta \operatorname{Cos}(\theta-\gamma) \end{array} \right] \partial_1 \\ + \left[(2\delta\mu - \delta\lambda) \operatorname{Sin}^4\beta + \delta\nu \operatorname{Sin}^2 2\beta + \delta\lambda \operatorname{Sin}^2\beta \right] \partial_3 \end{array} \right\} u_3 \qquad (9\text{-}13)$$

The operators $\widehat{A}^{0}_{ij}(\nabla_T) + \delta \widehat{A}_{ij}(\nabla_T)$ are a linear combination of ∂_1^2, ∂_3^2 and $\partial_1 \partial_3$. For sinusoidal disturbance, $\partial_1 \partial_3$ is replaced by $-k_1 k_3$ which is purely imaginary in the case of surface wave and hence we see from (9-11), (9-12), (9-13) and the assumption of the displacement potentials of the form (4-12) that the modified Rayleigh's equation will become a polynomial with complex coefficients on the assumption that $\beta \neq 0, 0.5\pi$, then the solution for phase velocity would be complex in general. It means that steady Rayleigh waves no longer exist in homogeneous transversely isotropic medium unless the symmetry axis is parallel to the free surface or vertical, confirming the result of Synge (1957).

Let us consider a unit hemisphere H: $x_1^2 + x_2^2 + x_3^2 = 1$ $(x_1 \geq 0)$ in which any point can be specified by two angular variables β, γ, then its differential area element would be $\cos\beta \, d\beta \, d\gamma$ Let $w(\beta,\gamma)$ be any weighting function that averages over H defined in the following way:

1). $w(\beta,\gamma) \geq 0$

2). $w(\beta,\gamma) = w(-\beta,\gamma) = w(\beta,-\gamma) = w(-\beta,-\gamma)$

3). $\int_H w(\beta,\gamma) \cos\beta \, d\beta \, d\gamma = 1.$

4). $w(\beta,\gamma)$ has maximum at $\beta = 0$, $\gamma = 0$.

Then if $f(\beta,\gamma)$ is any function

$$\langle f \rangle = \int_H f(\beta,\gamma) \, w(\beta,\gamma) \cos\beta \, d\beta \, d\gamma \tag{9-14}$$

is the average of f over the whole hemisphere.

We can extend the average of functions to the average of any partial differential operator $T = \sum_{n_i, n_j \geq 0} T_{ij} \, \partial_i^{n_i} \partial_j^{n_j}$

by setting $\langle T \rangle = \sum_{n_i, n_j \geq 0} \langle T_{ij} \rangle \, \partial_i^{n_i} \partial_j^{n_j}$ (9-15) then the averaged equation of motion becomes

$$\rho\partial_t^2\begin{bmatrix} u_1 \\ \\ u_3 \end{bmatrix} = \begin{bmatrix} \hat{A}_{11}^{0}(\nabla_{\bar{x}})+\langle\delta\hat{A}_{11}(\nabla_{\bar{x}})\rangle & \hat{A}_{13}^{0}(\nabla_{\bar{x}})+\langle\delta\hat{A}_{13}(\nabla_{\bar{x}})\rangle \\ \\ \hat{A}_{31}^{0}(\nabla_{\bar{x}})+\langle\delta\hat{A}_{31}(\nabla_{\bar{x}})\rangle & \hat{A}_{33}^{0}(\nabla_{\bar{x}})+\langle\delta\hat{A}_{33}(\nabla_{\bar{x}})\rangle \end{bmatrix}\begin{bmatrix} u_1 \\ \\ u_3 \end{bmatrix} \tag{9-16}$$

The average of Voigt's approach is that it can simplify our calculation. It is easy to see that for integers m, m_β, n, n_r, $\langle \sin^{m_\beta} m\beta \cos^{n_r} n(\theta-r)\rangle = 0$ if m_β is odd positive integer. Then equation (9-11) can be averaged to calcel out some terms to make (9-16) become a equation of motion for a homogeneous transversely isotropic medium with symmetry axis parallel to the free surface.

To avoid bulky expressions, the following abbreviation will be adopted.

$$\langle(\bar{e}_1'\cdot\bar{\bar{\sigma}})\cdot\bar{e}_1\rangle = \hat{A}\,\partial_1 u_1 + \hat{F}\,\partial_3 u_3$$

$$\langle(\bar{e}_3'\cdot\bar{\bar{\sigma}})\cdot\bar{e}_1\rangle = \hat{L}\,\partial_3 u_1 + \hat{L}\,\partial_1 u_3$$

$$\langle(\bar{e}_3'\cdot\bar{\bar{\sigma}})\cdot\bar{e}_3\rangle = \hat{C}\,\partial_3 u_3 + \hat{F}\,\partial_1 u_1 \tag{9-17}$$

Hence equation of motion becomes

$$\rho\,\partial_t^2 u_1 = \partial_1\left[\hat{A}\,\partial_1 u_1 + \hat{F}\,\partial_3 u_3\right] + \partial_3\left[\hat{L}\,\partial_3 u_1 + \hat{L}\,\partial_1 u_3\right]$$

$$\rho\,\partial_t^2 u_1 = \partial_1\left[\hat{L}\,\partial_3 u_1 + \hat{L}\,\partial_1 u_3\right] + \partial_3\left[\hat{C}\,\partial_3 u_3 + \hat{F}\,\partial_1 u_1\right] \tag{9-18}$$

in which

$$\hat{A} = (\lambda_{\parallel} + 2\mu_{\parallel}) + \left\langle \begin{array}{l} \cos^2\beta\cos^2(\theta-\gamma)\left[2\delta\lambda + \cos^2\beta\cos^2(\theta-\gamma)(2\delta\mu-\delta\lambda)\right] \\ +(\sin 2\beta)^2\cos^4(\theta-\gamma)\,\delta\nu + \delta\nu\cos^2\beta\,\sin^2 2(\theta-\gamma) \end{array} \right\rangle$$

$$\hat{F} = \lambda_{\parallel} + \left\langle \begin{array}{l} \delta\lambda\cos^2\beta\cos^2(\theta-\gamma) + \sin^2\beta\left[\delta\lambda + \cos^2\beta\cos^2(\theta-\gamma)(2\delta\mu-\delta\lambda)\right] \\ -(\sin 2\beta)^2\cos^2(\theta-\gamma)\,\delta\nu \end{array} \right\rangle$$

$$\hat{L} = \mu_{\parallel} + \left\langle \begin{array}{l} \sin^2\beta\cos^2\beta\cos^2(\theta-\gamma)(2\delta\mu-\delta\lambda) \\ +\cos^2 2\beta\cos^2(\theta-\gamma)\,\delta\nu + \sin^2\beta\,\sin^2(\theta-\gamma)\,\delta\nu \end{array} \right\rangle$$

$$\hat{C} = (\lambda_{\parallel} + 2\mu_{\parallel}) + \left\langle \sin^4\beta\,(2\delta\mu-\delta\lambda) + \sin^2 2\beta\,\delta\nu + \sin^2\beta\,\delta\lambda \right\rangle \tag{9-19}$$

Let $\widetilde{c}_R(\theta)$ be the Rayleigh wave phase velocity. Set $\gamma(\theta) = \dfrac{\widetilde{c}_R(\theta)}{\sqrt{\frac{\mu_{11}}{\rho}}}$ and combine the theory in Section 4 with (9-18) and the boundary condition at the free surface

$$\partial_1 u_3 + \partial_3 u_1 = 0$$

$$\widetilde{C}\,\partial_3 u_3 + \widetilde{F}\,\partial_1 u_1 = 0 \qquad (9\text{-}20)$$

the modified Rayleigh's equation becomes

$$(C'^2 - C'L')\gamma^3 + \left[(2F'^2C' - 2A'C'^2) - L'C'^2 + C'L'A'\right]\gamma^2$$

$$+\left[(C'^2A'^2 - 2F'^2C'A' + F'^4) - L'(2F'^2C' - 2A'C'^2)\right]\gamma$$

$$-L'(C'^2A'^2 - 2F'^2C'A' + F'^4) = 0 \qquad (9\text{-}21)$$

where

$$A' = \frac{\widetilde{A}}{\mu_{11}}$$
$$F' = \frac{\widetilde{F}}{\mu_{11}}$$
$$L' = \frac{\widetilde{L}}{\mu_{11}}$$
$$C' = \frac{\widetilde{C}}{\mu_{11}}$$

Let $\hat{c}_p$ and $\hat{c}_{sv}$ be the P and SV wave velocity respectively, then

$$\hat{c}_p = \rho^{-1}\hat{A} \tag{9-22}$$

$$\hat{c}_{sv} = \rho^{-1}\hat{L} \tag{9-23}$$

We shall consider two averaging schemes. In one scheme, we average the orientation of the symmetry axis over a segment of the horizontal great circle on the hemisphere H which is symmetric with respect to the axis parallel to $\vec{e}_1$ through the origin. In the other, the average is taken over a section of H enclosed by a cone which is also symmetric with respect to the axis parallel to $\vec{e}_1$ through the origin. For simplicity, we shall approximate the weighting function by two sequences of step functions defined in the following way:

For each positive integer n and real number $0 < \lambda < 1$, a closed interval $D_n^{(\lambda)}$ on the hemisphere is defined by

$$D_n^{(\lambda)} = \left\{ (\beta, \gamma) \;\middle|\; |\beta| \leq 0.5\lambda^n\pi, \quad |\gamma| \leq 0.5\lambda^n\pi \right\} \tag{9-24}$$

Let

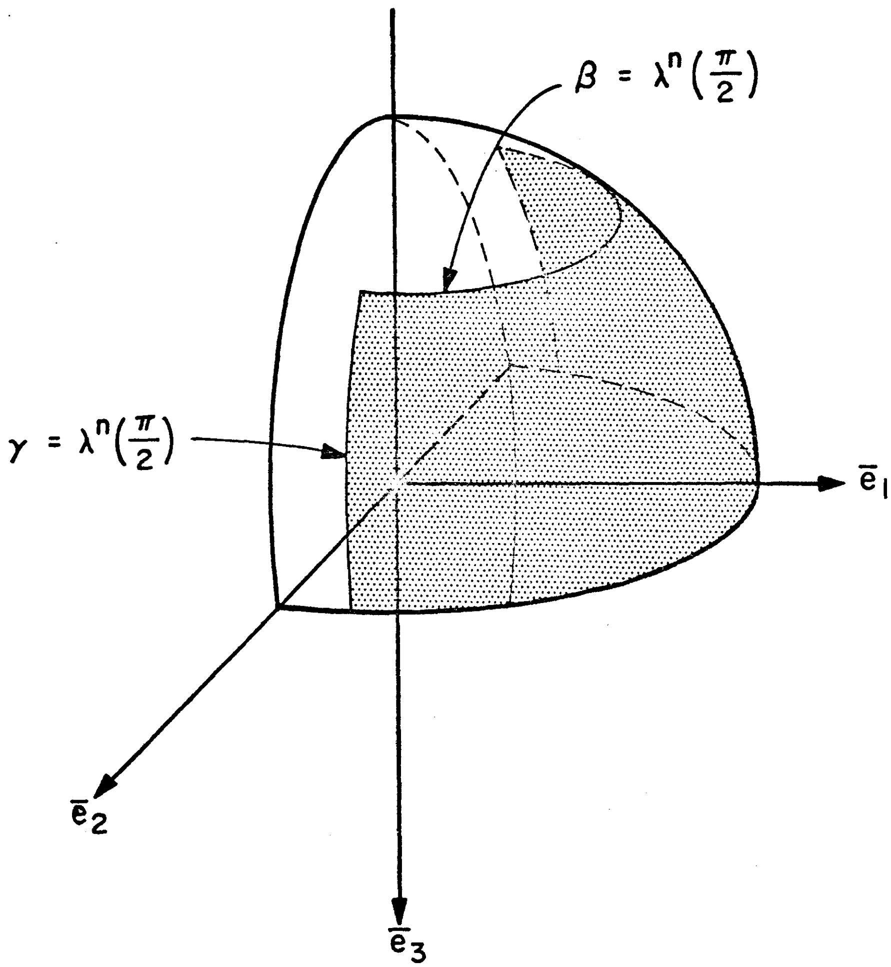

Figure 4. Upper half part of $D_n^{(\lambda)}$ (the shaded region) on hemisphere $H = \left\{ (x_1, x_2, x_3) \mid x_1^2 + x_2^2 + x_3^2 = 1 \, , \; x_1 \geq 0 \right\}$ Boundary of $D_n^{(\lambda)}$ is defined by $|\beta| = 0.5\lambda^n\pi$, $|\gamma| = 0.5\lambda^n\pi$

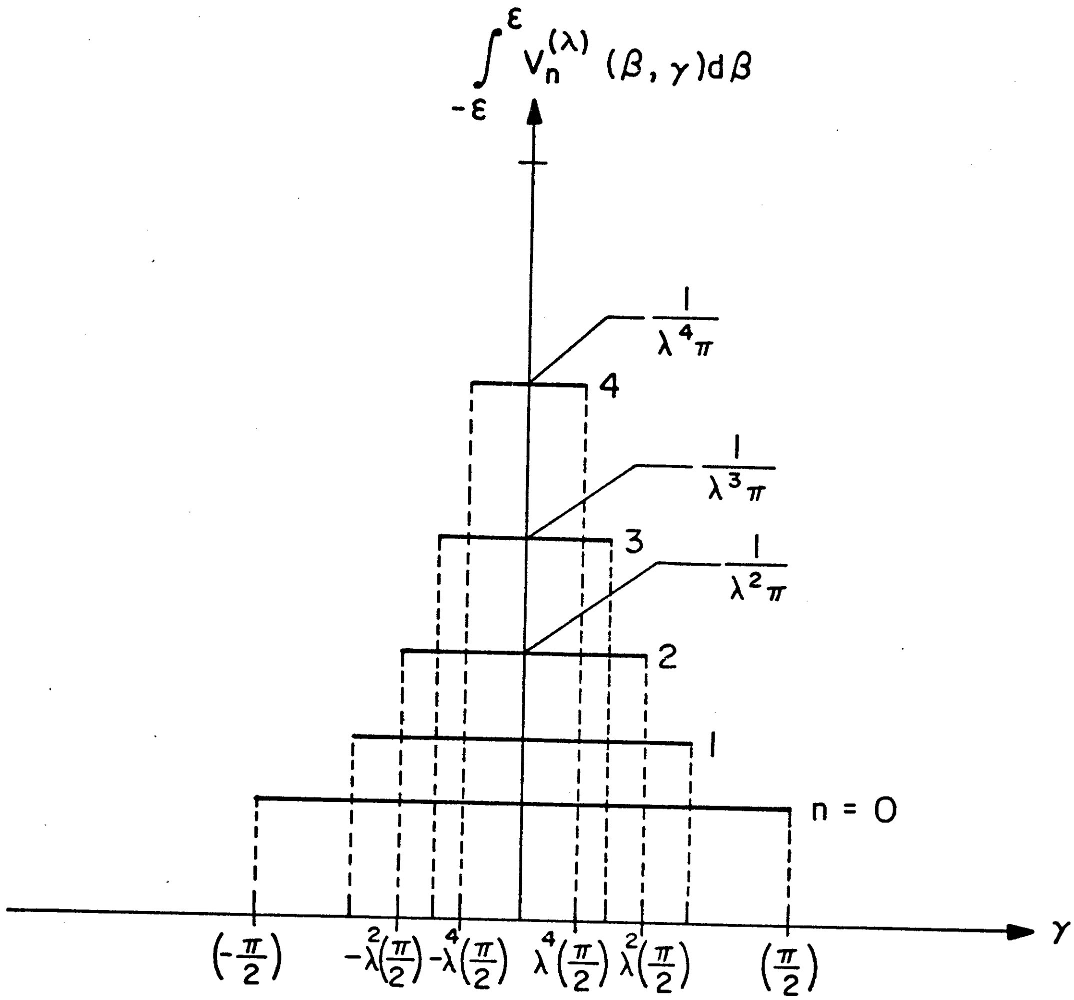

Figure 5. $\int_{-\varepsilon}^{\varepsilon} V_n^{(\lambda)}(\beta,\gamma)d\beta$ defined on closed interval as a function of γ, where $\varepsilon > 0$, the area below each $\int_{-\varepsilon}^{\varepsilon} V_n^{(\lambda)}(\beta,\gamma)d\beta$ is equal to unity.

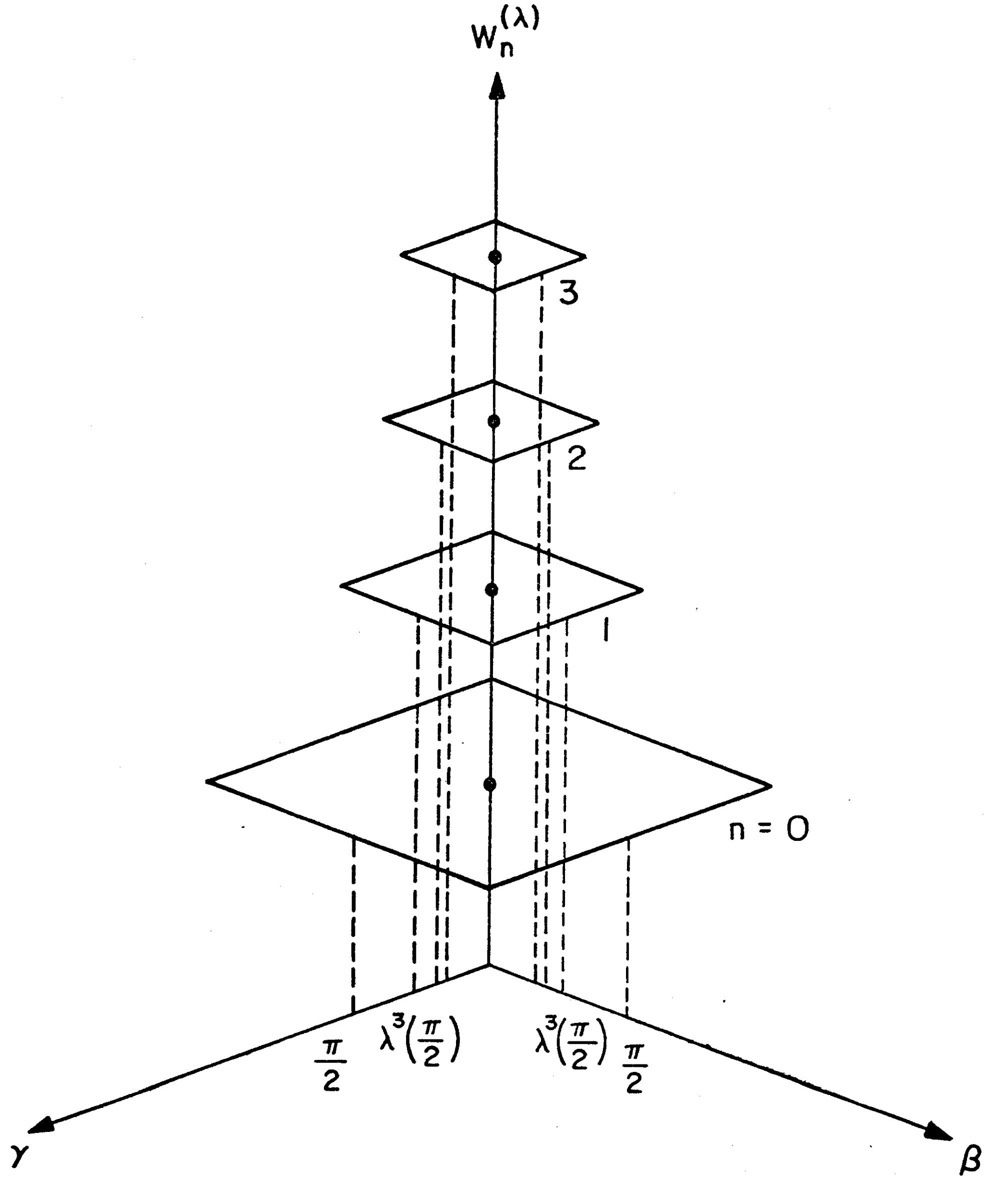

Figure 6. The sequence $\{W_n^{(\lambda)}\}_{n=0}^{\infty}$ plotted against β and γ. The volume under each $W_n^{(\lambda)}$ is equal to unity.

$$V_n^{(\lambda)}(\beta,\gamma) = \begin{cases} \left(\frac{1}{\lambda^n \pi}\right) & \text{if } (\beta,\gamma) \in D_n^{(\lambda)} \\ 0 & \text{if } (\beta,\gamma) \notin D_n^{(\lambda)} \end{cases} \tag{9-25}$$

where $\delta(\beta)$ is the Dirac Delta function.

$$\text{and} \quad W_n^{(\lambda)}(\beta,\gamma) = \begin{cases} \dfrac{1}{2\,\mathrm{Sin}\left(\lambda^n \frac{\pi}{2}\right)(\lambda^n \pi)} & \text{if } (\beta,\gamma) \in D_n^{(\lambda)} \\ 0 & \text{if } (\beta,\gamma) \notin D_n^{(\lambda)} \end{cases} \tag{9-26}$$

Their schematic figures are shown in Figure 4, 5 and 6. The significant difference between these two sequences is that $W_n^{(0.95)}$ gives uniform distribution of the orientation of the symmetry axis (or crystallographic a axis) within a cone enclosed by $D_n^{(0.95)}$ on H, while $V_n^{(0.75)}(\beta,\gamma)$ confines the uniform distribution of the symmetry axis to the horizontal great circle within $D_n^{(0.75)}$ due to the factor $\delta(\beta)$ in (9-25) which is the dirac delta function in angular variable β only. It is obvious that both $V_n^{(0.75)}$ and $W_n^{(0.95)}$ are weighting functions with Dirac Delta function in angular variables β and γ as their sequence limit.

It is noted that the delta function served as a weighting function will give rise to the same equations and results in a homogeneous transversely isotropic medium with ξ_1 as the symmetry axis and each $V_n^{(0.75)}$ and $W_n^{(0.95)}$ will give certain degree of uniform distribution of the orientation of

Table 6A. P, SV and SH phase velocity anisotropy calculated for Voigt scheme for the weighting function $V_n^{(0.75)}$

n	θ (Degree)	$\tilde{C}_P^V$ (Km/Sec.)	$\tilde{C}_{SV}^V$ (Km/Sec.)	$\tilde{C}_{SH}^V$ (Km/Sec.)
0	0	8.8267	4.6962	4.9670
0	90	8.8267	4.6962	4.9670
1	0	9.0680	4.7492	4.9873
1	90	8.4991	4.6426	4.9873
2	0	9.3167	4.7938	4.9774
2	90	8.2659	4.5965	4.9774
3	0	9.5218	4.8244	4.9500
3	90	8.1424	4.5643	4.9500
4	0	9.6655	4.8436	4.9230
4	90	8.0839	4.5439	4.9230
5	0	9.7571	4.8551	4.9028
5	90	8.0566	4.5317	4.9028
14	0	9.8868	4.8705	4.8708
14	90	8.0297	4.5152	4.8710

Table 6B. P, SV, and SH phase velocity anisotropy calculated for Reuss scheme for the weighting function $V_n^{(0.75)}$

n	θ (Degree)	$\tilde{C}_P^R$ (Km/Sec)	$\tilde{C}_{SV}^R$ (Km/Sec)	$\tilde{C}_{SH}^R$ (Km/Sec)
0	0	8.6853	4.6743	4.9362
0	90	8.6853	4.6743	4.9362
1	0	8.9442	4.7340	4.9501
1	90	8.3444	4.6138	4.9501
2	0	9.2036	4.7842	4.9433
2	90	8.0950	4.5620	4.9433
3	0	9.4127	4.8187	4.9246
3	90	7.9567	4.5254	4.9246
4	0	9.5576	4.8403	4.9061
4	90	7.8873	4.5023	4.9061
5	0	9.6493	4.8531	4.8924
5	90	7.8529	4.4884	4.8924
14	0	9.7785	4.8706	4.8704
14	90	7.8156	4.4697	4.8704

Table 6C. P, SV, and AH phase velocity anisotropy calculated for VRH scheme for the weighting function

n	θ (Degree)	$\tilde{c}_P^H$ (Km/sec.)	$\tilde{c}_{SV}^H$ (Km/sec.)	$\tilde{c}_{SH}^H$ (Km/sec.)
0	0	8.7562	4.6853	4.9517
0	90	8.7562	4.6853	4.9517
1	0	9.0023	4.7416	4.9687
1	90	9.4674	4.6282	4.9687
2	0	9.2603	4.7890	4.9604
2	90	8.1809	4.5792	4.9604
3	0	9.4674	4.8215	4.9474
3	90	8.0501	4.5449	4.9374
4	0	9.6117	4.8419	4.9145
4	90	7.9862	4.5231	4.9145
5	0	9.7034	4.8541	4.8976
5	90	7.9554	4.5101	4.8976
14	0	9.8328	4.8704	4.8707
14	90	7.9234	4.4925	4.8707

Table 7. P_n wave and Rayleigh wave phase velocity anisotropies calculated for three averaging schemes by choosing W_n (0.95) as the weight function.

n	θ(Degree)	Voigt $\tilde{C}_p^V$(Km/Sec)	$\tilde{C}_R^V(\theta)/\sqrt{\mu_{11}^V/\rho}$	Reuss $\tilde{C}_p^R$(Km/Sec)	$\tilde{C}_R^V(\theta)/\sqrt{\mu_{11}^R/\rho}$	VRH $\tilde{C}_p^H$(Km/Sec.)	$\tilde{C}_R^H(\theta)/\sqrt{\mu_{11}^H/\rho}$
0	0°	8.4940	0.9972	8.3355	0.9991	8.4151	0.9982
0	90°	8.4940	0.9972	8.3355	0.9991	8.4151	0.9982
1	0°	8.5453	1.0028	8.3926	1.0048	8.4693	1.0038
1	90°	8.4255	0.9913	8.2601	0.9930	8.3417	0.9922
2	0°	8.6034	1.0086	8.4567	1.0107	8.5304	1.0097
2	90°	8.3527	0.9853	8.1862	0.9868	8.2699	0.9861
3	0°	8.6673	1.0144	8.5264	1.0167	8.5971	1.0156
3	90°	8.2890	0.9795	8.1183	0.9809	8.2041	0.9802
4	0°	8.7355	1.0201	8.5997	1.0225	8.6678	1.0213
4	90°	8.2335	0.9741	8.0586	0.9752	8.1465	0.9747
5	0°	8.8060	1.0254	8.6750	1.0279	8.7407	1.0267
5	90°	8.1870	0.9691	8.0079	0.9700	8.0980	0.9696

the symmetry axis. The phase velocity anisotropies of the P, S waves and Rayleigh waves calculated from the average of the symmetry axis of the three schemes given in section 6 by means of the weighting functions will be given in Table 6 and Table 7. All these results will agree with the results obtained in Section 6-8 as n goes to infinity.
Each $V_n^{(\lambda)}$ of the sequence $\{V_n^{(0.75)}\}_{n=0}^{\infty}$ gives consistently large P wave velocity or higher P wave velocity anisotropies than the observed value. However, the sequence $\{W_n^{(0.95)}\}_{n=0}^{\infty}$ give $\hat{c}_p$ values that fit the observed data, and agree with the observed 2% anisotropy of Rayleigh wave phase velocity closely.

10. Conclusion

We have presented the detailed derivation of the modified Rayleigh's equation, and the body wave phase velocities for a homogeneous transversely isotropic medium in which the symmetry axis is parallel to the free surface. The motivation for the derivation is the observed anisotropies of the P_n wave and Rayleigh wave.

We started with the 'fracture zone model', in which the earth is considered as a lamination of fracture zone and normal crust-mantle, approximated by an equivalent homogeneous transversely isotropic medium in the long wave limit. The computed results for any reasonable values of the rigidities of the fracture zones give much weaker anisotropy than observed. On the other hand, the 'Olivine' model proposed by Francis (1969) gives much greater anisotropy than observed.

In order to obtain agreement with observation, we introduced a distribution of the symmetry axis of the transverse isotropy randomly oriented about the boundary surface of a circular cone which is symmetric with respect to the horizontal axis normal to the ridge. The averaging of elastic constants over distributed orientation was carried out by the use of a sequence of weighting function $\{W_n^{(\lambda)}\}_{n=0}^{\infty}$ which has the maximum in the horizontal direction normal to the ridge axis but slowly decreasing away from this direction.

The seismic refraction measurements show the observed magnitude of P wave anisotropies from 3% to 8%. Table 7 shows calculated values for the P wave and Rayleigh wave phase velocity anisotropies for the modified Olivine model averaged by the weighting sequence $\{W_n^{(\lambda)}\}_{n=0}^{\infty}$. The calculated $\hat{c}_p$ values from n=2 to n=5 fall within the range of all the measured values approximately; whereas, the corresponding magnitudes of Rayleigh wave phase velocity anisotropy range from 2.3% to 5.7% which is higher than observed value of 2% (Forsyth, 1972). The small discrepancy may be attributed to the fact that the Rayleigh velocities were measured over greater area than the P_n velocities, and anisotropy may be diminished when averaged over a larger area.

References

Backus, G.E., 1965, Possible forms of seismic anisotropy of the upper mantle under oceans, J. Geophys. Rev., 70, 3429-3439.

Backus, G.E., 1970, A geometrical picture of anisotropic elastic tensors, Rev. Geophys. Space Phys. 8, 633-671.

Bulchwald, V.T., 1961, Rayleigh waves in transversely isotropic media, Quart. Journ. Mech. and Applied Math., Vol. 14, pt. 3, 293-317.

Forsyth, D., 1972, The structural evolution of an oceanic plate, paper presented at the 9th symposium on Geophysical Theory and Computers, Banff, Alberta.

Francis, T.J.G., 1969, Generation of seismic anisotropy in the upper mantle along the mid-oceanic ridges, Nature, 221, 162.

Grand, F.S. and G.F. West, 1965, Interpretation theory in applied geophysics, McGraw-Hill Book Company.

Hess, H.H., 1964, Seismic anisotropy of the upper mantle under oceans, Nature 203, 629-631.

Crosson, R.S. and N.I. Christensen, 1969, Transverse isotropy of the upper mantle in the vicinity of Pacific fracture zones, Bull. Seism. Soc. Am., 59, 59-72.

Keen, C.E., and D.L. Barrett, 1971, A measurement of seismic anisotropy in the northeast Pacific, Can. J. Earth Sci., 8, 1056-1064.

Morris, G.B., R.W. Raitt, and G.G. Shor, 1969, Velocity anisotropy and delay time maps of the mantle near Hawaii,

Morse, P. and H. Feshbach, 1965, Method of Theoretical Physics, McGraw-Hill Book Company, Volume 1.

Postma, G.W., 1955, Wave propagation in stratified medium, Geophysics, 20, 780-806.

Raitt, R.W., G.G. Shor, T.J. Francis, and G.B. Morris, 1969, Anisotropy of the Pacific upper mantle, J. Geophys. Res., 74, 3095-3109.

Raitt, R.W., G.G. Shor, G.B. Morris, and H.K. Kirk, 1971, Mantle anisotropy in the Pacific Ocean, Tectonophysics, 12, 173-186.

Smith, M.L. and F.A. Dahlen, 1973, The azimuthal dependence of Love and Rayleigh wave propagation in a slightly anisotropic medium, J. Geophys. Res., 78, 3321-3333.

Stoneley, R., 1949, The seismological implications of aelotropy in continental structure, Monthly Notices Roy. Astron. Soc., Geophys. Suppl., 5, 222-232.

Synge, J.L., 1957, Elastic waves in anisotropic media, J. Math. Phys., 35, 323-335.

White, J.E. and F.A. Angona, 1955, Elastic wave velocities in laminated media, J. Acoustic Soc. Am., 22, 310-317.

Acknowledgements

This work was supported by the Advanced Research Project Agency, monitored by the Air Force Office of Scientific Research under contract F44620-71-C-0049.