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# A $(-q)$ -ANALOGUE OF WEIGHT MULTIPLICITIES

GEORGE LUSZTIG AND ZHIWEI YUN

**ABSTRACT.** We prove a conjecture in [L11] stating that certain polynomials  $P_{y,w}^\sigma(q)$  introduced in [LV11] for twisted involutions in an affine Weyl group give  $(-q)$ -analogues of weight multiplicities of the Langlands dual group  $\check{G}$ . We also prove that the signature of a naturally defined hermitian form on each irreducible representation of  $\check{G}$  can be expressed in terms of these polynomials  $P_{y,w}^\sigma(q)$ .

## 1. STATEMENT OF THE MAIN THEOREMS

**1.1. The  $P^\sigma$ -polynomials.** Let  $W$  be a Coxeter group with simple reflections  $S$ . Let  $\ell : W \rightarrow \mathbb{N}$  be the length function defined by the simple reflections  $S$ . In [KL79], for any two elements  $y, w \in W$ , a polynomial  $P_{y,w}(q) \in \mathbb{Z}[q]$  is attached. Consider the Hecke algebra  $\mathcal{H}$  over  $\mathcal{A} = \mathbb{Z}[q, q^{-1}]$  ( $q$  is an indeterminate) with basis  $\{T_w\}_{w \in W}$  and multiplication given by  $T_w T_{w'} = T_{ww'}$  if  $\ell(ww') = \ell(w) + \ell(w')$  and  $(T_s + 1)(T_s - q) = 0$  for all  $s \in S$ . Then  $\{\sum_{y \in W; y \leq w} P_{y,w}(q) T_y\}_{w \in W}$  is (up to a factor) the “new basis” of  $\mathcal{H}$  introduced in [KL79].

In [LV11] (for  $W$  a Weyl group) and [L11] (in general), the authors work in the situation of a triple  $(W, S, *)$  where  $(W, S)$  is as before and  $*$  is an involution of  $(W, S)$ . Let  $I_* = \{w \in W \mid w^* = w^{-1}\}$  be the  $*$ -twisted involutions in  $W$ . From the data  $(W, S, *)$ , a refined version  $P_{y,w}^\sigma(q) \in \mathbb{Z}[q]$  of  $P_{y,w}(q)$  is defined for  $y, w \in I_*$ . They also introduced a free  $\mathcal{A}$ -module  $M$  with basis  $\{a_w\}_{w \in I_*}$ , which carries a natural module structure over the Hecke algebra  $\mathcal{H}'$  with  $q$  replaced by  $q^2$ . Then  $\{\sum_{y \leq w, y \in I_*} P_{y,w}^\sigma(q) a_y\}_{w \in I_*}$  is (up to a factor) the “new basis” of  $M$  introduced in [LV11, Theorem 0.3] and [L11, Theorem 0.4].

**1.2. Affine Weyl group.** For the rest of the note we consider the setting of [L11, Section 6]:  $(W, S)$  is the Coxeter group associated to an untwisted connected affine Dynkin diagram. Let  $\Lambda \subset W$  be the subgroup of *translations*, i.e., those elements which have finite conjugacy classes. This is a free abelian subgroup of  $W$  of finite index. Let  $\overline{W} = W/\Lambda$ . We shall use additive notation for the group law in  $\Lambda$ . The conjugation action of  $w \in \overline{W}$  on  $\Lambda$  is denoted by  $\lambda \mapsto {}^w \lambda$ .

Fix a hyperspecial vertex  $s_0 \in S$  (i.e., a vertex in  $S$  with Dynkin label equal to 1). Then the finite Weyl group  $W_J$  generated by  $J = S - \{s_0\}$  is a section of the natural projection  $W \rightarrow \overline{W}$ , and we henceforth identify  $W_J$  with  $\overline{W}$ . Let  $w_J$  be the longest element of  $W_J$ .

An element  $\lambda \in \Lambda$  is *dominant* if  $\ell(\lambda w_J) = \ell(\lambda) + \ell(w_J)$ . Let  $\Lambda^+$  denote the set of dominant translations. The set of double cosets  $W_J \backslash W / W_J$  is in bijection with  $\Lambda^+$ : each  $W_J$ -double coset in  $W$  contains a unique  $\lambda \in \Lambda^+$ . For  $\lambda \in \Lambda^+$ , let  $d_\lambda = \lambda w_J$  be the longest element in the double coset  $W_J \lambda W_J$ .

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Let  $*$  be the automorphism of  $W$  defined by

$$(1.1) \quad \begin{aligned} w^* &:= w_J w w_J, \text{ for } w \in W_J; \\ \lambda^* &:= -{}^{w_J} \lambda \text{ for } \lambda \in \Lambda. \end{aligned}$$

This  $*$  is an involution which stabilizes  $S$  and fixes  $s_0$ . In fact, if  $w_J$  acts by  $-1$  on  $\Lambda$ , then  $*$  is the identity; otherwise  $*$  has order two. As shown in [L11, Proposition 8.2], every element  $d_\lambda$  belongs to  $I_*$ . Therefore we may consider the polynomials  $P_{d_\mu, d_\lambda}^\sigma(q)$ .

The following theorem is the main result of this note, which was conjectured by the first author in [L11, Conjecture 6.4].

**1.3. Theorem.** *Notation as above. Then for any  $\lambda, \mu \in \Lambda^+$ , we have*

$$P_{d_\mu, d_\lambda}^\sigma(q) = P_{d_\mu, d_\lambda}^\sigma(-q).$$

The proof of the theorem will be given in Section 3, after some preparation regarding the geometric Satake equivalence in Section 2. In Section 6, we give a generalization of the above theorem to other involutions  $\diamond$  of  $(W, S)$  which are closely related to  $*$ .

In [L11, Proposition 8.6], the first author proves special cases of this result by pure algebra.

It is proved in [L83, 6.1] that  $P_{d_\mu, d_\lambda}(q)$  is a  $q$ -analogue of the  $\mu$ -weight multiplicity in the irreducible representation  $V_\lambda$  of an algebraic group  $\check{G}$  (see the discussion in Section 2.3). Therefore, we may interpret the above theorem as saying that  $P_{d_\mu, d_\lambda}^\sigma(q)$  is a  $(-q)$ -analogue of weight multiplicities, hence the title of this note.

**1.4. The  $Z^\sigma$ -polynomials.** The polynomials  $P_{y,w}(q)$  is the Poincaré polynomial of the local intersection cohomology of an affine Schubert variety indexed by  $w$ ; the Poincaré polynomial of the global intersection cohomology of the same affine Schubert variety is given by

$$Z_w(q) = \sum_{y \in W; y \leq w} P_{y,w}(q) q^{\ell(y)} \in \mathbb{Z}[q].$$

Algebraically, consider the  $\mathcal{A}$ -algebra homomorphism  $\chi : \mathcal{H} \rightarrow \mathcal{A}$  given by  $\chi(T_w) = q^{\ell(w)}$  for all  $w \in W$ . Then  $Z_w(q)$  is the value of the new basis  $\sum_{y \leq w} P_{y,w}(q) T_y$  under the homomorphism  $\chi$ .

We want to define some polynomials  $Z_w^\sigma(q) \in \mathbb{Q}(q)$  which play the same role with respect to  $Z_w(q)$  as  $P_{y,w}^\sigma(q)$  plays with respect to  $P_{y,w}(q)$ . To do, we replace  $\chi : \mathcal{H} \rightarrow \mathcal{A}$  by the following  $\mathcal{A}$ -linear map introduced in [L11, 5.7]

$$(1.2) \quad \zeta : M \rightarrow \mathbb{Q}(q)$$

$$(1.3) \quad a_w \mapsto q^{\ell(w)} \left( \frac{q-1}{q+1} \right)^{\phi(w)} \text{ for all } w \in I_*$$

Here  $\phi : I_* \rightarrow \mathbb{N}$  is defined in [L11, 4.5]. Concretely, for  $w \in I_*$  with image  $\bar{w} \in \bar{W}$ ,  $\phi(w) = e(\bar{w}^*) - e(\bar{w})$ , where  $e(\bar{w})$  (resp.  $e(\bar{w}^*)$ ) is the dimension of the  $(-1)$ -eigenspace of the involution  $t \mapsto t^*$  (resp.  $t \mapsto w(t^*)$ ) on  $\Lambda_{\mathbb{Q}} = \Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$ .

For  $w \in I_*$  we let  $Z_w^\sigma(q)$  be the image of the new basis of  $M$  under  $\zeta$ :

$$(1.4) \quad Z_w^\sigma(q) = \zeta \left( \sum_{y \in I_*; y \leq w} P_{y,w}^\sigma(q) a_y \right) = \sum_{y \in I_*; y \leq w} P_{y,w}^\sigma(q) q^{\ell(y)} \left( \frac{q-1}{q+1} \right)^{\phi(y)} \in \mathbb{Q}(q)$$

We also set

$$(1.5) \quad \tilde{Z}_{d_\lambda}^\sigma(q) = Z_{d_\lambda}(q) Z_{w_J}(q)^{-1} \in \mathbb{Q}(q), \quad \tilde{Z}_{d_\lambda}^\sigma(q) = Z_{d_\lambda}^\sigma(q) Z_{w_J}^\sigma(q)^{-1} \in \mathbb{Q}(q).$$

Our second main result is

**1.5. Theorem.** *For any  $\lambda \in \Lambda^+$  we have  $\tilde{Z}_{d_\lambda}^\sigma(q) = \tilde{Z}_{d_\lambda}^\sigma(-q)$ . In particular,  $\tilde{Z}_{d_\lambda}^\sigma(q) \in \mathbb{Z}[q]$ .*

We will present two proofs of the theorem, one geometric in Section 4 which is based on a cohomological interpretation of  $Z_w^\sigma(q)$ , and one algebraic in Section 5. Both proofs rely on Theorem 1.3.

It is also observed in [L83] that  $\tilde{Z}_{d_\lambda}(q)$  is a  $q$ -analogue of the dimension of the irreducible representation  $V_\lambda$  of the group  $\check{G}$ . We will show in Section 5.6 that  $\tilde{Z}_{d_\lambda}^\sigma(q)$  is a  $q$ -analogue of the signature of  $V_\lambda$  under a naturally defined hermitian form introduced in [L97].

**1.6. Gelfand's trick.** It is interesting to notice the relation between the involution  $*$  and ‘‘Gelfand's trick’’ in proving that the spherical Hecke algebra is commutative. In fact, for a split simply-connected almost simple group  $G$  over a local field  $F$  with Weyl group  $W_J$ , the double coset  $G(\mathcal{O}_F) \backslash G(F) / G(\mathcal{O}_F)$  is in bijection with  $W_J \backslash W / W_J$ . The spherical Hecke algebra  $\mathcal{H}^{\text{sph}}$  consists of compactly supported bi- $G(\mathcal{O}_F)$ -invariant functions on  $G(F)$  with the algebra structure given by convolution. There is an involution  $g \mapsto g^*$  of  $G$  which stabilizes a split maximal torus  $T$  and acts by  $-w_J$  on  $\mathbb{X}_*(T) = \Lambda$ . The induced action on the affine Weyl group  $W$  is the same as the one given in Section 1.2. The anti-involution  $\tau : g \mapsto (g^*)^{-1}$  induces an anti-involution on  $\mathcal{H}^{\text{sph}}$  while fixing each double coset  $W_J \backslash W / W_J$ , hence acting by identity on  $\mathcal{H}^{\text{sph}}$ . This implies the commutativity of  $\mathcal{H}^{\text{sph}}$ . Roughly speaking, the main theorem is a categorification of Gelfand's trick: it explains what  $\tau$  does to the Satake category (categorification of  $\mathcal{H}^{\text{sph}}$ ) beyond the level of isomorphism classes of objects (on which it acts by identity).

**1.7. Notation and conventions.** By a tensor category, we mean a monoidal category with a commutativity constraint compatible with the associativity constraint.

For an algebraic torus  $T$ , let  $\mathbb{X}_*(T)$  (resp.  $\mathbb{X}^*(T)$ ) denote the group of cocharacters (resp. characters) of  $T$ . For a cocharacter  $\lambda : \mathbb{G}_m \rightarrow T$ , we use  $x^\lambda$  to mean the image of  $x \in \mathbb{G}_m$  under  $\lambda$ ; for a character  $\alpha : T \rightarrow \mathbb{G}_m$ , we use  $z^\alpha$  to denote the image of  $z \in T$  under  $\alpha$ . Note that  $(x^\lambda)^\alpha = x^{(\alpha, \lambda)} \in \mathbb{G}_m$ .

By an involution in a group, we mean an element of order at most two.

All algebraic varieties in this note are over  $\mathbb{C}$ ; all complexes of sheaves are with  $\mathbb{Q}$ -coefficients.

For an algebraic variety  $X$  of dimension  $n$ , let  $\text{IH}^\bullet(X)$  denote its intersection cohomology groups with  $\mathbb{Q}$ -coefficients. We normalize it so that  $\text{IH}^i(X) = 0$  unless  $0 \leq i \leq 2n$ .

## 2. GEOMETRIC DEFINITION OF THE $P^\sigma$ -POLYNOMIALS

**2.1. Affine flag variety.** In this section we give a geometric definition of the polynomials  $P_{x,y}^\sigma(q)$ . In fact, in the case of finite Weyl groups with  $*$  = id, such a geometric definition is given in [LV11, Section 3] using the geometry of flag varieties. It is remarked in [LV11, Section 7.1-7.2] that such a geometric definition works for affine Weyl groups and general  $*$ , with the flag varieties replaced by affine flag varieties. This section is an elaboration of this remark.

Let  $G$  be the simply-connected almost simple group over  $\mathbb{C}$  whose extended Dynkin diagram is the one we started with in Section 1.2, so that the usual Dynkin diagram of  $G$  is given by removing the vertex  $s_0$ . Fix a pinning for  $G$ ; in particular, fix a maximal torus  $T \subset G$ , and a Borel  $B$  containing  $T$ . We may identify  $(W_J, S - \{s_0\})$  with the Weyl group  $N_G(T)/T$  together with the simple reflections determined by  $B$ . We may also identify  $\Lambda$  with the cocharacter lattice  $\mathbb{X}_*(T)$ , which is also the coroot lattice of  $G$ .

Let  $G((t))$  be the loop group associated to  $G$ : it is the ind-scheme representing the functor  $R \mapsto G(R((t)))$  for any  $\mathbb{C}$ -algebra  $R$ . Let  $G[[t]] \subset G((t))$  be the subscheme representing the functor  $R \mapsto G(R[[t]])$ . The affine Weyl group  $W$  may be identified with the  $\mathbb{C}$ -points of  $N_{G((t))}(T((t)))/T[[t]]$ .

For each  $w \in W$ , we choose a lifting  $\dot{w}$  of it in  $N_{G((t))}(T((t)))$ . For example, if  $\lambda \in \Lambda$ , we may choose  $\dot{\lambda}$  to be the point  $t^\lambda \in T((t))$ .

An Iwahori subgroup of  $G((t))$  is one which is conjugate to  $\mathbf{I} = \pi^{-1}(B) \subset G[[t]]$  where  $\pi : G[[t]] \rightarrow G$  is the mod  $t$  reduction morphism. Let  $\text{Fl} = G((t))/\mathbf{I}$  be the affine flag variety of  $G$  classifying Iwahori subgroups of the loop group  $G((t))$ . This is a (locally finite) infinite union of projective varieties over  $\mathbb{C}$  of increasing dimensions. The group scheme  $\mathbf{I}$  acts on  $\text{Fl}$  from the left with orbits  $\text{Fl}_w = \mathbf{I}\dot{w}\mathbf{I}/\mathbf{I}$  indexed by  $w \in W$ . Each orbit  $\text{Fl}_w$  is isomorphic to an affine space of dimension  $\ell(w)$  (with respect to the simple reflections  $S$ ). Let  $\text{Fl}_{\leq w}$  be the closure of  $\text{Fl}_w$ , which is the union of  $\text{Fl}_y$  for  $y \leq w$ .

Consider the derived category  $D_{\mathbf{I}}(\text{Fl}) = \varinjlim_{w \in W} D_{\mathbf{I}}(\text{Fl}_{\leq w})$  of  $I$ -equivariant  $\mathbb{Q}$ -complexes which are supported on the  $\text{Fl}_{\leq w}$  for some  $w \in W$ . Note that for fixed  $w$ , the  $I$ -action on  $\text{Fl}_{\leq w}$  factors through a quotient group scheme  $\mathbf{I}_w$  of finite type such that  $\ker(\mathbf{I} \rightarrow \mathbf{I}_w)$  is pro-unipotent. We therefore understand  $D_{\mathbf{I}}(\text{Fl}_{\leq w})$  as the category of  $\mathbf{I}_w$ -equivariant derived category of  $\mathbb{Q}$ -complexes on the projective variety  $\text{Fl}_{\leq w}$  in the sense of [BL94].

**2.2. Geometric interpretation of the  $P^\sigma$ -polynomials.** Let  $*$  denote the pinned automorphism of  $G$  such that  $\lambda \mapsto ({}^w \lambda)^*$  acts by  $-1$  on  $\Lambda$ . This involution induces an involution on the affine Weyl group  $(W, S)$  which coincides with the  $*$  defined in (1.1). The involution  $*$  also induces an involution on  $G((t))$  preserving the Iwahori  $\mathbf{I}$ , so that it induces an involution on  $\text{Fl}$  which we still denote by  $*$ .

Consider the anti-involution  $\tau$  of  $G((t))$  defined as

$$\tau(g) = (g^*)^{-1}.$$

We would like to define a functor:

$$\tau^* : D_{\mathbf{I}}(\text{Fl}) \rightarrow D_{\mathbf{I}}(\text{Fl})$$

given by pull-back along the map  $\tau$ . We may identify each object of  $D_{\mathbf{I}}(\text{Fl})$  as a complex on  $G((t))$  equivariant under the left and right translation by  $\mathbf{I}$ . Since each  $\mathbf{I}$ -double coset  $\mathbf{I}\dot{w}\mathbf{I} \subset G((t))$  is sent to another double coset  $\mathbf{I}(\dot{w}^*)^{-1}\mathbf{I}$ , pull-back by  $\tau$  preserves bi- $\mathbf{I}$ -equivariance, and defines the functor  $\tau^*$ .

For each object  $\mathbf{K} \in D_{\mathbf{I}}(\text{Fl})$  and  $y \in W$ , the restriction of  $\mathbf{K}$  to  $\text{Fl}_y$  is a constant complex by  $\mathbf{I}$ -equivariance. We therefore have a vector space  $\mathcal{H}_y^i \mathbf{K}$ , which is canonically isomorphic to the  $i$ -th cohomology of the stalk of  $\mathbf{S}_w$  at any point of  $\text{Fl}_y$ .

For each  $w \in W$ , one has the (shifted) intersection cohomology complex  $\mathbf{S}_w \in D_{\mathbf{I}}(\text{Fl})$  of  $\text{Fl}_{\leq w}$ , which we normalize so that  $\mathbf{S}_w|_{\text{Fl}_w} \cong \mathbb{Q}$ . If  $w \in I_*$  (i.e.,  $(w^*)^{-1} = w$ ), we have a canonical isomorphism

$$(2.1) \quad \Phi_w : \tau^* \mathbf{S}_w \xrightarrow{\sim} \mathbf{S}_w$$

whose restriction to  $\text{Fl}_w$  is the identity map for the constant sheaf  $\mathbb{Q}$ . For each  $y \in I_*$ ,  $y \leq w$ , the restriction of  $\Phi_w$  induces an involution:

$$\mathcal{H}_y^i \Phi_w : \mathcal{H}_y^i \mathbf{S}_w = \tau^* \mathcal{H}_y^i (\tau^* \mathbf{S}_w) \rightarrow \mathcal{H}_y^i \mathbf{S}_w$$

where the first equality comes from the definition of  $\tau^*$ . Then

$$(2.2) \quad P_{y,w}^\sigma(q) = \sum_{i \in \mathbb{Z}} \text{tr}(\mathcal{H}_y^i \Phi_w, \mathcal{H}_y^i \mathbf{S}_w) q^{i/2}.$$

It is known that  $\mathcal{H}_y^i \mathbf{S}_w = 0$  for odd  $i$  (see [KL80, Theorem 4.2] for the case  $W$  finite, [KL80, Theorem 5.5] for the case  $W$  affine; see also [G01, A.7] for the affine case), therefore  $P_{y,w}^\sigma \in \mathbb{Z}[q]$ .

**2.3. Affine Grassmannian and the geometric Satake equivalence.** Let  $\mathrm{Gr} = G((t))/G[[t]]$  be the affine Grassmannian of  $G$ , which is also a locally finite union of projective varieties of increasing dimensions. The left translation by  $G[[t]]$  on  $\mathrm{Gr}$  has orbits indexed by  $W_J$ -orbits on  $\Lambda$ . For each dominant coweight  $\lambda \in \Lambda^+$ , there is a unique  $G[[t]]$ -orbit  $\mathrm{Gr}_\lambda$  containing  $t^\lambda$  (which also contains  $t^{\lambda'}$  for any  $\lambda'$  in the same  $W_J$ -orbit of  $\lambda$ ). The dimension of  $\mathrm{Gr}_\lambda$  is  $\langle 2\rho, \lambda \rangle$ , where  $2\rho$  is the sum of positive roots of  $G$ .

Let  $\mathcal{S} = P_{G[[t]]}(\mathrm{Gr})$  be the category of  $G[[t]]$ -equivariant perverse sheaves on  $\mathrm{Gr}$  which are supported on finitely many  $G[[t]]$ -orbits. This abelian category carries a convolution product  $\odot : \mathcal{S} \times \mathcal{S} \rightarrow \mathcal{S}$  (implicit in [L83], see [G95, Proposition 2.2.1]), which is equipped with an obvious associativity constraint and a less obvious commutativity constraint (based on ideas of Drinfeld, see an exposition in [MV07, Section 5]) making  $(\mathcal{S}, \odot)$  a tensor category (the convolution product is usually denoted by  $*$  in literature, and we change it to  $\odot$  to avoid confusion with the involution  $*$ ). Let  $\mathrm{Vec}^{\mathrm{gr}}$  be the category of finite dimensional graded  $\mathbb{Q}$ -vector spaces (the commutativity constraint is *not* adjusted by the Koszul sign convention, so  $\mathrm{Vec}^{\mathrm{gr}} \cong \mathrm{Rep}(\mathbb{G}_m)$  as tensor categories). Consider the functor

$$\begin{aligned} \mathbf{H}^\bullet &: \mathcal{S} \rightarrow \mathrm{Vec}^{\mathrm{gr}} \\ \mathbf{K} &\mapsto \bigoplus_{i \in \mathbb{Z}} \mathbf{H}^i(\mathrm{Gr}, \mathbf{K}). \end{aligned}$$

This functor carries a tensor structure (see [G95, Proposition 3.4.1] and [MV07, Proposition 6.3], note that the commutativity constraint of  $\mathcal{S}$  is adjusted by a sign in [MV07, Paragraph after Remark 6.2] in order to make  $\mathbf{H}^\bullet$  a tensor functor).

Composing  $\mathbf{H}^\bullet$  with the forgetful functor  $\mathrm{Vec}^{\mathrm{gr}} \rightarrow \mathrm{Vec}$  (the category of finite dimensional vector spaces), we get a fiber functor  $\mathbf{H}$  of the tensor category  $\mathcal{S}$ , hence an algebraic group  $\check{G} = \mathrm{Aut}^\otimes(\mathbf{H})$  over  $\mathbb{Q}$ . In [G95, Theorem 3.8.1] (with the corrected commutativity constraint by Drinfeld and based on results of [L83]), it is proved that  $\check{G}$  is a connected split reductive group over  $\mathbb{Q}$  whose root datum is dual to  $G$ . The proof in [MV07, Theorem 7.3] in fact equips  $\check{G}$  with a maximal torus  $\check{T}$  with a canonical identification  $\mathbb{X}^*(\check{T}) = \Lambda = \mathbb{X}_*(T)$ . In fact, the functor  $\mathbf{H}^\bullet$  factors as

$$\mathbf{H}^\bullet : \mathcal{S} \xrightarrow{\bigoplus_{\lambda \in \Lambda} F_\lambda} \mathrm{Vec}^\Lambda \xrightarrow{\langle 2\rho, - \rangle} \mathrm{Vec}^{\mathrm{gr}}$$

Here the first arrow is the sum of *weight functors* introduced in [MV07, Theorem 3.6]; the second functor turns a  $\Lambda$ -graded vector space  $\bigoplus_\lambda V^\lambda$  into a  $\mathbb{Z}$ -graded one  $V^i := \bigoplus_{\langle 2\rho, \lambda \rangle = i} V^\lambda$ . Under the identification  $\mathcal{S} \xrightarrow{\sim} \mathrm{Rep}(\check{G})$ , the functor  $\mathbf{H}^\bullet$  then factors as

$$\mathrm{Rep}(\check{G}) \rightarrow \mathrm{Rep}(\check{T}) \rightarrow \mathrm{Rep}(\mathbb{G}_m)$$

induced by the homomorphisms  $2\rho : \mathbb{G}_m \rightarrow \check{T} \hookrightarrow \check{G}$ .

**2.4. Geometric interpretation of  $P_{d_\mu, d_\lambda}^\sigma(q)$ .** For each  $\lambda \in \Lambda^+$ , let  $\mathbf{C}_\lambda$  be the shifted intersection cohomology complex of the closure  $\mathrm{Gr}_{\leq \lambda}$  of  $\mathrm{Gr}_\lambda$ , such that  $\mathbf{C}_\lambda|_{\mathrm{Gr}_\lambda} = \mathbb{Q}$ . The involution  $\tau$  of  $G((t))$  again induces a functor

$$(2.3) \quad \tau^* : \mathcal{S} \rightarrow \mathcal{S}.$$

One can similarly define the stalks  $\mathcal{H}_\mu^i \mathbf{C}_\lambda$  for  $\mu \leq \lambda \in \Lambda^+$ , which again vanishes for odd  $i$ . Each double coset  $G[[t]]t^\lambda G[[t]]$  is sent to  $G[[t]]t^{-\lambda^*} G[[t]]$ . By the definition of  $*$ , we have  $-\lambda^* = {}^w_J \lambda$ , hence  $G[[t]]t^{-\lambda^*} G[[t]] = G[[t]]t^\lambda G[[t]]$ , i.e., each  $G[[t]]$ -double coset in  $G((t))$  is stable under  $\tau$  (this is equivalent to saying that the longest element in each  $W_J$ -double coset belongs to the set  $I_*$  of

\*-twisted involutions). This means one can fix an isomorphism

$$(2.4) \quad \Psi_\lambda : \tau^* \mathbf{C}_\lambda \xrightarrow{\sim} \mathbf{C}_\lambda$$

which is the identity when restricted to  $\mathrm{Gr}_\lambda$ . This isomorphism similarly induces an involution:

$$\mathcal{H}_\mu^i \Psi_\lambda : \mathcal{H}_\mu^i \mathbf{C}_\lambda = \mathcal{H}_\mu^i (\tau^* \mathbf{C}_\lambda) \rightarrow \mathcal{H}_\mu^i \mathbf{C}_\lambda.$$

We have a projection map  $\pi : \mathrm{Fl} \rightarrow \mathrm{Gr}$ . For each  $\lambda \in \Lambda^+$ , the preimage  $\pi^{-1}(\mathrm{Gr}_{\leq \lambda}) = \mathrm{Fl}_{\leq d_\lambda}$  (recall  $d_\lambda \in W_J \lambda W_J$  is the longest element). Since  $\mathrm{Fl}_{\leq d_\lambda} \rightarrow \mathrm{Gr}_{\leq \lambda}$  is smooth, we have an isomorphism  $\phi_\lambda : \pi^* \mathbf{C}_\lambda \cong \mathbf{S}_{d_\lambda}$ , which can be made canonical by requiring its restriction to  $\mathrm{Fl}_{d_\lambda}$  to be the identity map on the constant sheaf. Moreover, the isomorphism  $\phi_\lambda$  clearly intertwines  $\Psi_\lambda$  and  $\Phi_{d_\lambda}$ . Using  $\phi_\lambda$ , we get a commutative diagram

$$\begin{array}{ccc} \mathcal{H}_\mu^j \mathbf{C}_\lambda & \xrightarrow{\mathcal{H}_\mu^j \phi_\lambda} & \mathcal{H}_{d_\mu}^j \mathbf{S}_{d_\lambda} \\ \mathcal{H}_\mu^j \Psi_\lambda \downarrow & & \downarrow \mathcal{H}_\mu^j \Phi_\lambda \\ \mathcal{H}_\mu^j \mathbf{C}_\lambda & \xrightarrow{\mathcal{H}_\mu^j \phi_\lambda} & \mathcal{H}_{d_\mu}^j \mathbf{S}_{d_\lambda} \end{array}$$

Therefore, from (2.2) we get

$$(2.5) \quad P_{d_\mu, d_\lambda}^\sigma(q) = \sum_{j \in \mathbb{Z}} \mathrm{tr}(\mathcal{H}_\mu^{2j} \Psi_\lambda, \mathcal{H}_\mu^{2j} \mathbf{C}_\lambda) q^j.$$

**2.5. Loop group of a compact form.** At certain points in the proof of the main theorem, it is convenient to take an alternative point of view of the affine Grassmannian  $\mathrm{Gr}$ , namely the space of polynomial loops on the compact form of  $G$ . We remark that the switch of viewpoint is not necessary for the proof, but it makes the idea of the proof more transparent.

Let  $K \subset G(\mathbb{C})$  be a compact real form which is stable under  $*$  (for example, we may define  $K$  using the Cartan involution  $w_J^*$ , for any lifting of  $w_J$  of  $w_J$  to  $N_G(T)$ ). Let  $\Omega = \Omega_{\mathrm{pol}} K$  be the space of polynomial loops on  $K$  based at the identity element  $1 \in K$  (see [PS86, §3.5]). By [PS86, Theorem 8.6.3], there is a homeomorphism

$$\iota : \Omega \xrightarrow{\sim} G(\mathbb{C}((t))) \xrightarrow{p} \mathrm{Gr}(\mathbb{C}).$$

The stratification of  $\mathrm{Gr}$  by  $\{\mathrm{Gr}_\lambda\}_{\lambda \in \Lambda^+}$  gives a Whitney stratification of  $\Omega$ . We denote the strata by  $\Omega_\lambda$  with closure  $\Omega_{\leq \lambda}$ . Let  $D^b(\Omega) = \varinjlim_\lambda D^b(\Omega_{\leq \lambda})$ . Let  $\mathcal{S}_K$  be the full subcategory of  $D^b(\Omega)$  consisting of perverse sheaves which are locally constant along each strata  $\Omega_\lambda$ .

Let  $m_K : \Omega \times \Omega \rightarrow \Omega$  be the multiplication map. This is stratified in the sense that  $m_K(\Omega_{\leq \lambda} \times \Omega_{\leq \mu}) = \Omega_{\leq \lambda + \mu}$  for  $\lambda, \mu \in \Lambda^+$ . Define

$$\begin{aligned} \odot_K & : D^b(\Omega) \times D^b(\Omega) \rightarrow D^b(\Omega) \\ & (\mathbf{K}_1, \mathbf{K}_2) \mapsto m_{K!}(\mathbf{K}_1 \boxtimes \mathbf{K}_2). \end{aligned}$$

Let

$$\mathbf{H}^\bullet : \mathcal{S}_K \rightarrow \mathrm{Vec}^{\mathrm{gr}}$$

be the functor of taking total cohomology.

The involution  $\tau : k \mapsto (k^*)^{-1}$  on  $K$  induces an involution  $\tau_K$  on  $\Omega$ , which gives the pullback functor

$$\tau_K^* : D^b(\Omega) \rightarrow D^b(\Omega).$$

**2.6. Lemma.**

- (1) The functor  $\odot_K$  has image in  $\mathcal{S}_K$ , and there is a natural associativity constraint making  $(\mathcal{S}_K, \odot_K)$  a monoidal category;  $\mathbf{H}^\bullet : \mathcal{S}_K \rightarrow \text{Vec}^{\text{gr}}$  is naturally a monoidal functor.
- (2) The pull-back functor  $\iota^*$  gives a monoidal equivalence  $\iota^* : (\mathcal{S}, \odot) \rightarrow (\mathcal{S}_K, \odot_K)$ .
- (3) There is a natural isomorphism of monoidal functors  $\mathbf{H}^\bullet \circ \iota^* \cong \mathbf{H}^\bullet : \mathcal{S} \rightarrow \text{Vec}^{\text{gr}}$ .
- (4) The functor  $\tau_K^*$  sends  $\mathcal{S}_K$  to  $\mathcal{S}_K$ ;  $\tau^*$  and  $\tau_K^*$  are naturally intertwined under  $\iota^*$ .

*Proof.* (1)(2) The functor  $\iota^*$  identifies  $\mathcal{S}_K$  with the category of perverse sheaves on  $\text{Gr}$  locally constant along the strata  $\text{Gr}_\lambda$ . By [MV07, Proposition A.1] the latter category is canonically equivalent to  $\mathcal{S}$ . To prove (1) and (2), it suffices to give  $\iota^*$  a monoidal structure. Recall that the convolution product  $\odot$  on  $\mathcal{S}$  is defined as

$$\mathbf{K}_1 \odot \mathbf{K}_2 = m_!(\mathbf{K}_1 \square \mathbf{K}_2)$$

Here  $m : G((t)) \times^{G[[t]]} \text{Gr} \rightarrow \text{Gr}$  is the multiplication map,  $\mathbf{K}_1 \square \mathbf{K}_2$  is the perverse sheaf on  $G((t)) \times^{G[[t]]} \text{Gr}$  characterized by

$$(2.6) \quad p'^* \mathbf{K}_1 \square \mathbf{K}_2 = p^* \mathbf{K}_1 \boxtimes \mathbf{K}_2 \text{ on } G((t)) \times \text{Gr},$$

where  $p : G((t)) \rightarrow \text{Gr}, p' : G((t)) \times \text{Gr} \rightarrow G((t)) \times^{G[[t]]} \text{Gr}$  are the projections. To give  $\iota^*$  a tensor structure, we need to give a canonical isomorphism

$$m_{K!}(\iota^* \mathbf{K}_1 \boxtimes \iota^* \mathbf{K}_2) \cong \iota^* m_!(\mathbf{K}_1 \square \mathbf{K}_2)$$

for any  $\mathbf{K}_1, \mathbf{K}_2 \in \mathcal{S}$ . Note that we have commutative diagram

$$(2.7) \quad \begin{array}{ccc} \Omega \times \Omega & \xrightarrow{\iota_2} & G((t)) \times^{G[[t]]} \text{Gr} \\ m_K \downarrow & & \downarrow m \\ \Omega & \xrightarrow{\iota} & \text{Gr} \end{array}$$

where  $\iota_2$  is given by the composition

$$\Omega \times \Omega \xrightarrow{\tilde{\iota} \times \iota} G((t)) \times \text{Gr} \xrightarrow{p'} G((t)) \times^{G[[t]]} \text{Gr}.$$

It is easy to see that  $\iota_2$  is also a homeomorphism, so (2.7) is a Cartesian diagram. Therefore we have a canonical isomorphism

$$\begin{aligned} \iota^* m_!(\mathbf{K}_1 \square \mathbf{K}_2) &\cong m_{K!} \iota_2^*(\mathbf{K}_1 \square \mathbf{K}_2) \\ &= m_{K!}(\tilde{\iota} \times \iota)^* p'^*(\mathbf{K}_1 \square \mathbf{K}_2) \stackrel{(2.6)}{=} m_{K!}(\tilde{\iota} \times \iota)^*(p^* \mathbf{K}_1 \boxtimes \mathbf{K}_2) \\ &= m_{K!}(\tilde{\iota}^* p^* \mathbf{K}_1 \boxtimes \iota^* \mathbf{K}_2) = m_{K!}(\iota^* \mathbf{K}_1 \boxtimes \iota^* \mathbf{K}_2) \end{aligned}$$

It is easy to check these isomorphisms are compatible with the associativity constraints.

(3) is obvious.

(4) For each  $\mathbf{K} \in \mathcal{S}$ , we need to give a functorial isomorphism

$$\iota^* \tau^* \mathbf{K} \xrightarrow{\sim} \tau_K^* \iota^* \mathbf{K}.$$

Recall  $\iota$  factors as  $\Omega \xrightarrow{\tilde{\iota}} G((t)) \xrightarrow{p} \text{Gr}$  and  $\tilde{\iota} \tau_K = \tau \tilde{\iota}$ , where  $\tau : g \mapsto (g^*)^{-1}$  is the anti-automorphism of  $G((t))$ . Therefore

$$\iota^* \tau^* \mathbf{K} = \tilde{\iota}^* p^* \tau^* \mathbf{K} = \tilde{\iota}^* \tau^* p^* \mathbf{K} = \tau_K^* \tilde{\iota}^* p^* \mathbf{K} = \tau_K^* \iota^* \mathbf{K}.$$

This gives the desired isomorphism.  $\square$



Using part (2) of Lemma 2.6, one can transfer the commutativity constraint of  $(\mathcal{S}, \odot)$  to  $(\mathcal{S}_K, \odot_K)$  making the latter a tensor category. Part (3) of Lemma 2.6 then gives the functor  $\mathbf{H}^\bullet$  a tensor (in addition to monoidal) structure.

### 3. PROOF OF THEOREM 1.3

For a monoidal category  $(\mathcal{C}, \otimes)$ , we let  $(\mathcal{C}, \otimes^\sigma)$  be the same category equipped with a new functor  $\otimes^\sigma : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$  given by  $X \otimes^\sigma Y := Y \otimes X$ . It is easy to check that  $(\mathcal{C}, \otimes^\sigma)$  also carries a monoidal structure.

**3.1. Lemma.** *The functor  $\tau^* : \mathcal{S} \rightarrow \mathcal{S}$  carries a natural structure of a monoidal functor*

$$\tau^* : (\mathcal{S}, \odot) \rightarrow (\mathcal{S}, \odot^\sigma).$$

*Proof.* Using Lemma 2.6(2) and (4), it suffices to construct the monoidal structure of  $\tau_K^*$ . Let  $\sigma : \Omega \times \Omega \rightarrow \Omega \times \Omega$  be the involution which interchanges two factors. Since  $\tau_K$  is an anti-involution, we have a Cartesian diagram

$$(3.1) \quad \begin{array}{ccc} \Omega \times \Omega & \xrightarrow{\tau_K \times \tau_K} & \Omega \times \Omega \\ \downarrow m_K \circ \sigma & & \downarrow m_K \\ \Omega & \xrightarrow{\tau_K} & \Omega \end{array}$$

Therefore by proper base change, for any  $\mathbf{K}_1, \mathbf{K}_2 \in \mathcal{S}_K$ , we have a canonical isomorphism

$$\tau_K^* m_{K!}(\mathbf{K}_1 \boxtimes \mathbf{K}_2) \cong (m_K \circ \sigma)_!(\tau_K^* \mathbf{K}_1 \boxtimes \tau_K^* \mathbf{K}_2) = m_{K!}(\tau_K^* \mathbf{K}_2 \boxtimes \tau_K^* \mathbf{K}_1).$$

By the definition of  $\odot_K$ , we get a canonical isomorphism

$$\tau_K^*(\mathbf{K}_1 \odot_K \mathbf{K}_2) \xrightarrow{\sim} \tau_K^* \mathbf{K}_2 \odot_K \tau_K^* \mathbf{K}_1.$$

It is easy to check that these isomorphisms are compatible with the associativity constraint and the unit objects of  $(\mathcal{S}_K, \odot_K)$  and  $(\mathcal{S}_K, \odot_K^\sigma)$ . This finishes the proof of the lemma.  $\square$

Let  $\mathbf{H}^{\bullet, \sigma} : (\mathcal{S}, \odot^\sigma) \rightarrow (\text{Vec}^{\text{gr}}, \otimes)$  be the same functor as  $\mathbf{H}^\bullet$ , except that we change its monoidal structure to the one of  $\mathbf{H}^\bullet$  composed with the commutativity constraint of  $\otimes$  for  $\text{Vec}^{\text{gr}}$ , so that  $\mathbf{H}^{\bullet, \sigma}$  is also a tensor functor.

**3.2. Lemma.** *There is a natural isomorphism  $\gamma : \mathbf{H}^{\bullet, \sigma} \circ \tau^* \xrightarrow{\sim} \mathbf{H}^\bullet$ , which preserves the monoidal structures of both functors.*

*Proof.* Using Lemma 2.6, it suffices to give a natural isomorphism  $\gamma_K : \mathbf{H}^\bullet \circ \tau_K^* \xrightarrow{\sim} \mathbf{H}^{\bullet, \sigma}$  between functors  $\mathcal{S}_K \rightarrow \text{Vec}^{\text{gr}}$ , which preserves the monoidal structures. Since  $\tau_K$  is an automorphism of  $\Omega$ , we have a canonical isomorphism  $\mathbf{H}^\bullet(\Omega, \tau_K^* \mathbf{K}) \xrightarrow{\sim} \mathbf{H}^\bullet(\Omega, \mathbf{K})$ , which gives the desired  $\gamma_K$ . It remains to check that  $\gamma$  preserves the monoidal structures. But this is also obvious from the natural monoidal structure of  $\mathbf{H}^\bullet : \mathcal{S}_K \rightarrow \text{Vec}^{\text{gr}}$ .  $\square$

Suppose we have two Tannakian categories  $(\mathcal{C}, \otimes)$  and  $(\mathcal{D}, \otimes)$  equipped with fiber functors  $\omega_{\mathcal{C}}$  and  $\omega_{\mathcal{D}}$  into  $\text{Vec}_k$  respectively ( $k$  is a field). Let  $F : (\mathcal{C}, \otimes) \rightarrow (\mathcal{D}, \otimes)$  be a monoidal functor equipped with a monoidal isomorphism  $\phi : \omega_{\mathcal{D}} \circ F \xrightarrow{\sim} \omega_{\mathcal{C}}$ . Then  $\phi$  induces a homomorphism of algebraic groups over  $k$ :

$$(F, \phi)^\# : \text{Aut}^\otimes(\omega_{\mathcal{D}}) \rightarrow \text{Aut}^\otimes(\omega_{\mathcal{C}})$$

$$(\omega_{\mathcal{D}} \xrightarrow{h} \omega_{\mathcal{D}}) \mapsto (\omega_{\mathcal{C}} \xrightarrow{\phi^{-1}} \omega_{\mathcal{D}} \circ F \xrightarrow{\text{hoid}_F} \omega_{\mathcal{D}} \circ F \xrightarrow{\phi} \omega_{\mathcal{C}}).$$

Note that the tensor morphisms between tensor functors only uses their structures as monoidal functors, therefore the above definition makes sense even if  $F$  is only a monoidal functor. More generally, if  $\omega_{\mathcal{C}}$  and  $\omega_{\mathcal{D}}$  take values in another Tannakian category  $\mathcal{V}$  equipped with a fiber functor  $\omega : \mathcal{V} \rightarrow \text{Vec}$ , then  $F$  induces a homomorphism of algebraic groups  $(F, \phi)^{\#} : \text{Aut}^{\otimes}(\omega \circ \omega_{\mathcal{D}}) \rightarrow \text{Aut}^{\otimes}(\omega \circ \omega_{\mathcal{C}})$  making the following diagram commutative

$$\begin{array}{ccc} & \text{Aut}^{\otimes}(\omega) & \\ \omega_{\mathcal{D}}^{\#} \swarrow & & \searrow \omega_{\mathcal{C}}^{\#} \\ \text{Aut}^{\otimes}(\omega \circ \omega_{\mathcal{D}}) & \xrightarrow{(F, \phi)^{\#}} & \text{Aut}^{\otimes}(\omega \circ \omega_{\mathcal{C}}) \end{array}$$

We apply the above remarks to the situation

$$\begin{array}{ccc} (\mathcal{S}, \odot) & \xrightarrow{\tau^*} & (\mathcal{S}, \odot^{\sigma}) \\ \searrow \mathbf{H}^{\bullet} & \cong & \swarrow \mathbf{H}^{\bullet, \sigma} \\ & \text{Vec}^{\text{gr}} & \end{array}$$

and get a commutative diagram of algebraic groups over  $\mathbb{Q}$ :

$$\begin{array}{ccc} & \mathbb{G}_m & \\ 2\rho \swarrow & & \searrow 2\rho \\ \check{G} & \xrightarrow{(\tau^*, \gamma)^{\#}} & \check{G} \end{array}$$

In other words,  $(\tau^*, \gamma)^{\#}$  is an automorphism of  $\check{G}$  commuting with elements in the torus  $2\rho(\mathbb{G}_m)$ . Since  $\tau^*$  does not change the isomorphism classes of irreducible objects in  $\mathcal{S}$ , this automorphism must be inner. Therefore  $(\tau^*, \gamma)^{\#}$  determines an element  $g \in \check{T}$  (note that  $\check{G}$  is of adjoint form).

Using the commutative constraint of  $(\mathcal{S}, \odot)$ , the identity functor gives a monoidal equivalence

$$\text{id}_{\mathcal{S}}^{\sigma} : (\mathcal{S}, \odot) \xrightarrow{\sim} (\mathcal{S}, \odot^{\sigma}).$$

There is a unique natural isomorphism of monoidal functors  $\Theta : \tau^* \xrightarrow{\sim} \text{id}_{\mathcal{S}}^{\sigma}$  making

$$\text{id}_{\mathbf{H}^{\bullet, \sigma}} \circ \Theta = \gamma : \mathbf{H}^{\bullet, \sigma} \circ \tau^* \rightarrow \mathbf{H}^{\bullet, \sigma} \circ \text{id}_{\mathcal{S}}^{\sigma} = \mathbf{H}^{\bullet}.$$

In fact, identifying  $\mathcal{S}$  with  $\text{Rep}(\check{G})$ , the functor  $\tau^*$  sends  $V \in \text{Rep}(\check{G})$  (with the action  $\alpha : \check{G} \rightarrow \text{Aut}(V)$ ) to the same vector space  $V$  with the new action  $\check{G} \xrightarrow{\text{Ad}(g)} \check{G} \xrightarrow{\alpha} \text{Aut}(V)$ . Then the effect of the natural isomorphism  $\Theta$  on  $V$  is given by  $\alpha(g^{-1}) : V \rightarrow V$ .

### 3.3. Lemma.

- (1) The element  $g \in \check{T}(\mathbb{Q})$  is  $(-1)^{\rho}$ , the image of  $-1$  under the cocharacter  $\rho : \mathbb{G}_m \rightarrow \check{T}$  (note that  $\check{G}$  is of adjoint type, so  $\rho$  is a cocharacter of  $\check{T}$ ).
- (2) The effect of the natural isomorphism  $\Theta$  on the intersection complex  $\mathbf{C}_{\lambda}[\langle 2\rho, \lambda \rangle] \in \mathcal{S}$  is  $(-1)^{\langle \rho, \lambda \rangle} \Psi_{\lambda}$ .
- (3) The action of the involution  $\tau_K^*$  on  $\text{IH}^{2j}(\Omega_{\leq \lambda})$  is by  $(-1)^j$ .

*Proof.* Let  $\lambda \in \Lambda^+$ . The action of  $g^{-1}$  on  $\mathbf{H}^{\bullet}(\Omega, \mathbf{C}_{\lambda})[\langle 2\rho, \lambda \rangle] = \text{IH}^{\bullet}(\Omega_{\leq \lambda})[\langle 2\rho, \lambda \rangle] = V_{\lambda} \in \text{Rep}(\check{G})$  is given by the composition

$$\text{IH}^{\bullet}(\Omega_{\leq \lambda}) \xrightarrow{\tau_K^*} \text{IH}^{\bullet}(\Omega_{\leq \lambda}) = \mathbf{H}^{\bullet}(\Omega, \tau_K^* \mathbf{C}_{\lambda}) \xrightarrow{\mathbf{H}^{\bullet}(\Omega, \Theta_{\lambda})} \mathbf{H}^{\bullet}(\Omega, \mathbf{C}_{\lambda}) = \text{IH}^{\bullet}(\Omega_{\leq \lambda}).$$

where the first arrow is the pull-back along the anti-involution  $\tau_K$  of  $\Omega_{\leq \lambda}$  and  $\Theta_{\lambda} : \tau_K^* \mathbf{C}_{\lambda} \rightarrow \mathbf{C}_{\lambda}$  is induced from the effect of  $\Theta$  on  $\mathbf{C}_{\lambda}[\langle 2\rho, \lambda \rangle] \in \mathcal{S}$ . Since the only automorphisms of  $\mathbf{C}_{\lambda}$  are

scalars, the isomorphisms  $\Theta_\lambda$  and  $\Psi_\lambda$  must be related by  $\Theta_\lambda = c_\lambda \Psi_\lambda$  for some constant  $c_\lambda \in \mathbb{Q}^\times$ : the restriction of  $\Theta_\lambda$  on  $\Omega_\lambda$  is given by multiplication by  $c_\lambda$  on the constant sheaf.

The stratum  $\Omega_\lambda$  homotopy retracts to the  $K$ -orbit of  $t^\lambda$ , which is a partial flag variety  $G/P_\lambda = K/P_\lambda \cap K$  (see [MV07, Top of page 100]). The action of  $\tau_K$  on  $\text{Ad}(K)t^\lambda \cong K/P_\lambda \cap K$  is given by

$$kt^\lambda k^{-1} \mapsto (k^* t^\lambda k^{*, -1})^{-1} = k^* t^{-\lambda} k^{*, -1} = k^* \dot{w}_J t^\lambda \dot{w}_0^{-1} k^{*, -1}.$$

Therefore the induced action of  $\tau_K^*$  on  $K/P_\lambda \cap K$  is given by  $k \bmod P_\lambda \cap K \mapsto k^* w_J \bmod P_\lambda \cap K$  (any lifting  $\dot{w}_J \in N_{T \cap K}(K)$  normalizes  $P_\lambda \cap K$ , hence the right translation makes sense).

Let  $j_\lambda : \Omega_\lambda \hookrightarrow \Omega_{\leq \lambda}$  be the inclusion. We have a commutative diagram

$$(3.2) \quad \begin{array}{ccccc} \mathrm{IH}^i(\Omega_{\leq \lambda}) & \xrightarrow{j_\lambda^*} & \mathrm{H}^i(\Omega_\lambda) & \xrightarrow{\sim} & \mathrm{H}^i(K/P_\lambda \cap K) \\ \downarrow c_\lambda^{-1} g^{-1} & & \downarrow \tau_K^* & & \downarrow w_{J^*} \\ \mathrm{IH}^i(\Omega_{\leq \lambda}) & \xrightarrow{j_\lambda^*} & \mathrm{H}^i(\Omega_\lambda) & \xrightarrow{\sim} & \mathrm{H}^i(K/P_\lambda \cap K) \end{array}$$

When  $i \leq 2$ , the horizontal restriction maps are isomorphisms. In fact, from the stratification  $\Omega_{\leq \lambda}$  by the open  $\Omega_\lambda$  and the closed complement  $z : \Omega_{< \lambda} \hookrightarrow \Omega_{\leq \lambda}$ , we get an exact sequence

$$(3.3) \quad \mathrm{H}^i(\Omega_{< \lambda}, z^! \mathbf{C}_\lambda) \rightarrow \mathrm{IH}^i(\Omega_{\leq \lambda}) \rightarrow \mathrm{H}^i(\Omega_\lambda) \rightarrow \mathrm{H}^i(\Omega_{< \lambda}, z^! \mathbf{C}_\lambda)$$

Since  $\dim \Omega_{< \lambda} \leq \langle 2\rho, \lambda \rangle - 2$  and  $z^! \mathbf{C}_\lambda[\langle 2\rho, \lambda \rangle]$  lies in perverse degree  $\geq 1$ ,  $z^! \mathbf{C}_\lambda$  lies in the usual cohomological degree  $\geq 3$ . This implies  $\mathrm{H}^i(\Omega_{< \lambda}, z^! \mathbf{C}_\lambda) = 0$  for  $i \leq 2$  hence the isomorphism follows from the exact sequence (3.3).

We claim that the action  $\tau_K^* : k \mapsto k^* w_J$  on the partial flag variety  $K/P_\lambda \cap K$  induces  $-1$  on  $\mathrm{H}^2(K/P_\lambda \cap K)$ . In fact,  $\mathrm{H}^2(K/P_\lambda \cap K, \mathbb{Q}) \hookrightarrow \mathrm{H}^2(K/T, \mathbb{Q}) \cong \mathbb{X}^*(T)_\mathbb{Q}$  by pull-back along the projection  $K/T \cap K \rightarrow K/P_\lambda \cap K$ , and this map is equivariant under the  $(W \rtimes \text{Out}(G))_\lambda$ -actions (subscript  $\lambda$  means stabilizer of  $\lambda$  under the  $W \rtimes \text{Out}(G)$ -action on  $\Lambda = \mathbb{X}_*(T)$ ). Since  $*w_J = w_{J^*} \in W \rtimes \text{Out}(G)$  acts on  $\Lambda$  by  $-1$  by definition, the claim follows.

Since  $\tau_K^*$  induces the identity action on  $\mathrm{H}^0(K/P_\lambda \cap K)$ ,  $c_\lambda^{-1} g^{-1}$  acts by identity on  $\mathrm{IH}^0(\Omega_{\leq \lambda})$  by diagram (3.2). Since  $\tau_K^*$  acts by  $-1$  on  $\mathrm{H}^2(K/P_\lambda \cap K)$  by the above claim,  $c_\lambda^{-1} g^{-1}$  acts on  $\mathrm{IH}^2(\Omega_{\leq \lambda})$  by multiplication by  $-1$  by diagram (3.2).

Recall that the grading on  $\mathrm{IH}^\bullet(\Omega_{\leq \lambda})[\langle 2\rho, \lambda \rangle] \cong V_\lambda$  comes from the action of the cocharacter  $2\rho : \mathbb{G}_m \rightarrow \check{T}$  on  $V_\lambda$ . Let  $V_\lambda(\mu)$  be the weight space of weight  $\mu$  under the  $\check{T}$ -action, we have

$$\mathrm{IH}^i(\Omega_{\leq \lambda}) = \bigoplus_{\langle 2\rho, \mu \rangle = i} V_\lambda(\mu).$$

In particular,

$$\begin{aligned} \mathrm{IH}^0(\Omega_{\leq \lambda}) &= V_\lambda(w_J \lambda); \\ \mathrm{IH}^2(\Omega_{\leq \lambda}) &= \bigoplus V_\lambda(w_J \lambda + \alpha_i^\vee) \end{aligned}$$

where the sum over simple roots  $\alpha_i^\vee$  of  $\check{G}$ . Therefore, the previous paragraph implies

$$(3.4) \quad \begin{aligned} c_\lambda^{-1} g^{-w_J \lambda} &= 1; \\ c_\lambda^{-1} g^{-w_J \lambda - \alpha_i^\vee} &= -1 \text{ for all simple roots } \alpha_i^\vee. \end{aligned}$$

Comparing these two equations we conclude that  $g^{\alpha_i^\vee} = -1$  for all simple roots  $\alpha_i^\vee$  of  $\check{G}$ . On the other hand,  $(-1)^\rho$  also has this property. Since  $\check{G}$  is adjoint, an element in  $\check{T}$  is determined by its image under simple roots, therefore  $g = (-1)^\rho$ . This proves (2). Plugging this back into (3.4),

we conclude that  $c_\lambda = ((-1)^\rho)^{-wJ\lambda} = (-1)^{\langle \rho, -wJ\lambda \rangle} = (-1)^{\langle \rho, \lambda \rangle}$ . This proves (1). Now (3) follows easily from (1) and (2).  $\square$

**3.4. Completion of the proof.** By (2.5), it suffices to show that  $\mathcal{H}_\mu^{2j}\Psi_\lambda$  acts on  $\mathcal{H}_\mu^{2j}\mathbf{C}_\lambda$  by  $(-1)^j$ .

We extend the partially ordered set  $\{\mu \in \Lambda^+, \mu \leq \lambda\}$  into a totally ordered one, and denote the total ordering still by  $\leq$ . For any  $\mu \leq \lambda$ , let  $\Omega_{[\mu, \lambda]} = \Omega_{\leq \lambda} - \Omega_{< \mu}$ . Similarly  $\Omega_{(\mu, \lambda]} = \Omega_{\leq \lambda} - \Omega_{\leq \mu}$ . Then we have a long exact sequence

$$\cdots \rightarrow H_c^i(\Omega_{(\mu, \lambda]}, \mathbf{C}_\lambda) \rightarrow H_c^i(\Omega_{[\mu, \lambda]}, \mathbf{C}_\lambda) \rightarrow \bigoplus_{a+b=i} H_c^a(\Omega_\mu) \otimes \mathcal{H}_\mu^b \mathbf{C}_\lambda \rightarrow \cdots$$

Since  $\Omega_\mu = \text{Gr}_\mu$  is an affine space bundle over a partial flag variety  $G/P_\mu$ , we have that  $H_c^\bullet(\Omega_\mu) \cong H^\bullet(G/P_\mu)[- \langle 2\rho, \mu \rangle + \dim G/P_\mu]$  which is concentrated in even degrees. We also know that  $\mathcal{H}_\mu^b \mathbf{C}_\lambda$  vanishes for odd  $b$ . Therefore the third term in the above exact sequence vanishes for odd  $i$ . Using decreasing induction for  $\mu$  (starting with  $\lambda$ ), we conclude that each  $H_c^\bullet(\Omega_{[\mu, \lambda]}, \mathbf{C}_\lambda)$  is concentrated in even degrees, and the above long exact sequence becomes a short one for even  $i$ . This gives a canonical decreasing filtration

$$F^{\geq \mu} \text{IH}^\bullet(\Omega_{\leq \lambda}) := H_c^\bullet(\Omega_{[\mu, \lambda]}, \mathbf{C}_\lambda)$$

with associated graded pieces

$$(3.5) \quad \text{gr}_F^\mu \text{IH}^\bullet(\Omega_{\leq \lambda}) = H_c^\bullet(\Omega_\mu) \otimes \mathcal{H}_\mu^\bullet \mathbf{C}_\lambda.$$

The action of  $\tau_K^*$  preserves each  $F^{\geq \mu}$ , and the induced action on the associated graded pieces takes the form

$$(3.6) \quad \text{gr}_F^\mu \tau_K^* = (\tau_K|_{\Omega_\mu})^* \otimes \mathcal{H}_\mu^\bullet \Psi_\lambda : H_c^\bullet(\Omega_\mu) \otimes \mathcal{H}_\mu^\bullet \mathbf{C}_\lambda \rightarrow H_c^\bullet(\Omega_\mu) \otimes \mathcal{H}_\mu^\bullet \mathbf{C}_\lambda.$$

By Lemma 3.3(3), the action of  $\tau_K^*$  on the top-dimensional cohomology  $H_c^{2\langle 2\rho, \mu \rangle}(\Omega_\mu) \cong \text{IH}^{2\langle 2\rho, \mu \rangle}(\Omega_{\leq \mu})$  is via multiplication by  $(-1)^{\langle 2\rho, \mu \rangle} = 1$ ; the action of  $\tau_K^*$  on  $H_c^{2\langle 2\rho, \mu \rangle}(\Omega_\mu) \otimes \mathcal{H}_\mu^{2j} \mathbf{C}_\lambda \subset \text{gr}_F^\mu \text{IH}^{2j+2\langle 2\rho, \lambda \rangle}(\Omega_{\leq \lambda})$ , as a subquotient of  $\text{IH}^{2j+2\langle 2\rho, \lambda \rangle}(\Omega_{\leq \lambda})$ , is via multiplication by  $(-1)^{j+\langle 2\rho, \lambda \rangle} = (-1)^j$ . Therefore, by (3.6),  $\mathcal{H}_\mu^{2j}\Psi_\lambda$  acts on  $\mathcal{H}_\mu^{2j}\mathbf{C}_\lambda$  via multiplication by  $(-1)^j$ . This finishes the proof of Theorem 1.3.

#### 4. GEOMETRIC PROOF OF THEOREM 1.5

The proof of Theorem 1.5 will become transparent once we give the cohomological interpretation of the  $Z^\sigma$ -polynomials.

**4.1. Affine flag variety via a compact form.** We already see that  $\Omega = \Omega K \xrightarrow{\sim} \text{Gr}(\mathbb{C})$  is a homeomorphism. We need analogous statement for the affine flag variety. Let  $T_c = K \cap T$  be the maximal torus in  $K$ . Then the inclusion  $K \subset G(\mathbb{C})$  induces a homeomorphism  $K/T_c \xrightarrow{\sim} (G/B)(\mathbb{C})$ . The multiplication  $(g, kT_c) \mapsto gk\mathbf{I}$  gives a continuous map  $\iota_{\text{Fl}} : \Omega \times K/T_c \rightarrow \text{Fl}(\mathbb{C})$  making the following diagram commutative

$$(4.1) \quad \begin{array}{ccc} \Omega \times K/T_c & \xrightarrow{\iota_{\text{Fl}}} & \text{Fl}(\mathbb{C}) \\ \downarrow \text{pr}_\Omega & & \downarrow \pi \\ \Omega & \xrightarrow{\iota} & \text{Gr}(\mathbb{C}) \end{array}$$

It is easy to check that  $\iota_{\text{Fl}}$  is bijective on points, hence a homeomorphism because it is a continuous map from a compact space to a Hausdorff one. Moreover,  $\iota_{\text{Fl}}$  is  $T_c$ -equivariant, where  $T_c$  acts on  $\Omega \times K/T_c$  diagonally by conjugation and left translation, and it acts on  $\text{Fl}(\mathbb{C})$  by left translation.

Let  $\Xi = K/T_c \times \Omega \times K/T_c$ , on which  $K$  acts diagonally via left translation on  $K/T_c$  and via conjugation on  $\Omega$ . The space  $\Xi$  also admits an involution  $\tilde{\tau} : (k_1 T_c, g, k_2 T_c) \mapsto (k_2^* T_c, (g^*)^{-1}, k_1^* T_c)$ , which intertwines the original diagonal  $K$ -action and the that action pre-composed with  $*$ . We may rewrite  $[T_c \backslash (\Omega \times K/T_c)]$  as  $[K \backslash \Xi]$ , so that the homeomorphism  $\iota_{\mathbf{Fl}}$  can be rewritten as a map of topological stacks

$$(4.2) \quad \begin{aligned} [K \backslash \Xi] &\xrightarrow{\tilde{\iota}_{\mathbf{Fl}}} T_c \backslash \mathbf{Fl} \rightarrow \mathbf{I} \backslash \mathbf{Fl} \\ (k_1 T_c, g, k_2 T_c) &\mapsto T_c k_1^{-1} g k_2 \mathbf{I} \mapsto \mathbf{I} k_1^{-1} g k_2 \mathbf{I}. \end{aligned}$$

which intertwines the involutions  $\tilde{\tau}$  and  $\tau$ . Since  $\iota_{\mathbf{Fl}}$  is a homeomorphism, so is  $\tilde{\iota}_{\mathbf{Fl}}$  (by which we mean that it comes from a  $K$ -equivariant homeomorphism of topological spaces). Via (4.2), we may define  $\Xi_w$  (resp.  $\Xi_{\leq w}$ ) as the preimage of  $\mathbf{I} \backslash \mathbf{Fl}_w$  (resp.  $\mathbf{I} \backslash \mathbf{Fl}_{\leq w}$ ) for each  $w \in W$ . Then  $\Xi_w$  is  $K$ -equivariantly homeomorphic to the twisted product  $K \times^{T_c} \mathbf{Fl}_w$ .

Recall from (2.1) we have an isomorphism  $\Phi_w : \tau^* \mathbf{S}_w \xrightarrow{\sim} \mathbf{S}_w$  in the category  $D_{\mathbf{I}}(\mathbf{Fl})$  for  $w \in I_*$ , where  $\mathbf{S}_w$  is the shifted intersection cohomology sheaf of  $\mathbf{Fl}_{\leq w}$ . This induces an involution on  $\mathbf{I}$ -equivariant cohomology

$$\tau^* = \mathbf{H}_{\mathbf{I}}^{\bullet}(\mathbf{Fl}, \Phi_w) : \mathbf{IH}_{\mathbf{I}}^{\bullet}(\mathbf{Fl}_{\leq w}) \xrightarrow{\sim} \mathbf{IH}_{\mathbf{I}}^{\bullet}(\mathbf{Fl}_{\leq w}).$$

**4.2. Lemma.** *Let  $r$  be the rank of  $G$ . Then*

$$\sum_{j \in \mathbb{Z}} \mathrm{tr} \left( \tau^*, \mathbf{IH}_{\mathbf{I}}^{2j}(\mathbf{Fl}_{\leq w}) \right) q^j = q^{\ell(w)} (1 - q)^{e(*) - r} (1 + q)^{-e(*)} Z_w^{\sigma}(q^{-1})$$

as elements in  $\mathbb{Z}[[q]]$ . Here  $e(*)$  is the dimension of the  $(-1)$ -eigenspace of  $*$  :  $\Lambda_{\mathbb{Q}} \rightarrow \Lambda_{\mathbb{Q}}$ .

*Proof.* By (4.2),  $\mathbf{IH}_{\mathbf{I}}^{\bullet}(\mathbf{Fl}_{\leq w}) \cong \mathbf{IH}_K^{\bullet}(\Xi_{\leq w})$ . We think of  $\mathbf{S}_w$  as the intersection complex of  $\Xi_{\leq w}$  which is the constant sheaf on  $\Xi_w$ . The stratification of  $\Xi$  by  $\Xi_{\leq w}$  gives a spectral sequence with the  $E_2$ -page consisting of  $\mathbf{H}_K^{\bullet}(\Xi_y, i_y^! \mathbf{S}_w)$  abutting to  $\mathbf{IH}_K^{\bullet}(\Xi_{\leq w})$ . Here  $i_y : \Xi_y \hookrightarrow \Xi$  is the inclusion. Since  $i_y^! \mathbf{S}_w$  is a sum of constant sheaves on  $\Xi_y$  concentrated on even degrees, and  $\mathbf{H}_K^{\bullet}(\Xi_y) \cong \mathbf{H}_T^{\bullet}(\mathrm{pt})$  is also concentrated in even degrees, the spectral sequence necessarily degenerates at  $E_2$ . Therefore  $\mathbf{IH}_K^{\bullet}(\Xi_{\leq w})$  admits an increasing filtration indexed by  $\{y \leq w\}$  with  $\mathrm{gr}_y \mathbf{IH}_K^{\bullet}(\Xi_{\leq w}) = \mathbf{H}_K^{\bullet}(\Xi_y, i_y^! \mathbf{S}_w)$ . The involution  $\tilde{\tau}^*$  on  $\mathbf{IH}_K^{\bullet}(\Xi_{\leq w})$  maps  $\mathrm{gr}_y$  to  $\mathrm{gr}_{(y^*)^{-1}}$ , therefore its trace is the sum of traces on  $\mathrm{gr}_y$  for  $y \in I_*$ , i.e.,

$$(4.3) \quad \sum_j \mathrm{tr}(\tilde{\tau}^*, \mathbf{IH}_K^{2j}(\Xi_{\leq w})) q^j = \sum_{y \leq w, y \in I_*} \sum_{j \in \mathbb{Z}} \mathrm{tr}(\tilde{\tau}^*, \mathbf{H}_K^{2j}(\Xi_y, i_y^! \mathbf{S}_w)) q^j.$$

Verdier duality gives an isomorphism in  $D_K(\Xi_y)$  commuting with the involutions induced by  $\Phi_w$

$$i_y^! \mathbf{S}_w \cong \bigoplus_k \mathcal{H}^{2\ell(w) - 2\ell(y) - 2k} \mathbf{S}_w[-2k].$$

Hence

$$(4.4) \quad \begin{aligned} &\sum_{j \in \mathbb{Z}} \mathrm{tr}(\tilde{\tau}^*, \mathbf{H}_K^{2j}(\Xi_y, i_y^! \mathbf{S}_w)) q^j \\ &= \sum_{k \in \mathbb{Z}} \mathrm{tr}(\mathcal{H}_y^{2\ell(w) - 2\ell(y) - 2k} \Phi_w, \mathcal{H}_y^{2\ell(w) - 2\ell(y) - 2k} \mathbf{S}_w) q^k \sum_{j \in \mathbb{Z}} \mathrm{tr}(\tilde{\tau}^*, \mathbf{H}_K^{2j}(\Xi_y)) q^j \\ &= q^{\ell(w) - \ell(y)} P_{y,w}^{\sigma}(q^{-1}) \sum_{j \in \mathbb{Z}} \mathrm{tr}(\tilde{\tau}^*, \mathbf{H}_K^{2j}(\Xi_y)) q^j. \end{aligned}$$

Since  $[K \setminus \Xi_y] \cong [T_c \setminus \text{Fl}_y]$ , it has the same cohomology as  $[T \setminus T \dot{y} T / T]$ , where  $\dot{y}$  is a lifting of  $y$  to  $G((t))$ . Let  $T_y = \{(\bar{y}(t^{-1}), t), t \in T\} \subset T \times T$  ( $\bar{y}$  is the image of  $y$  in  $\overline{W}$ ) be the stabilizer of the  $T \times T$ -action on  $T \dot{y} T$  via left and right translations. The involution  $\tau$  on  $[T \setminus T \dot{y} T / T] \cong [\text{pt} / T_y]$  is then induced by the involution  $(\bar{y}(t^{-1}), t) \mapsto ((t^*)^{-1}, \bar{y}(t^*))$  of  $T_y$ . We identify  $T_y$  with  $T$  via the second projection, then the involution of  $[\text{pt} / T_y] = [\text{pt} / T]$  induced by  $\tau^*$  comes from  $t \mapsto \bar{y}(t^*)$ . This involution gives a decomposition  $\mathfrak{t} = \mathfrak{t}_+ \oplus \mathfrak{t}_-$  of the Lie algebra  $\mathfrak{t} \cong \Lambda_{\mathbb{C}}$  of  $T$  into  $(+1)$  and  $(-1)$ -eigenspaces, with dimensions  $r - e(\bar{y}^*)$  and  $e(\bar{y}^*)$  respectively (see remarks following (1.2) for notations). Since

$$\mathbf{H}_K^\bullet(\Xi_y) \cong \mathbf{H}_{T_y}^\bullet(\text{pt}) \cong \text{Sym}(\mathfrak{t}^\vee[-2]) \cong \text{Sym}(\mathfrak{t}_+^\vee[-2]) \otimes \text{Sym}(\mathfrak{t}_-^\vee[-2]),$$

therefore

$$\sum_{j \in \mathbb{Z}} \text{tr}(\tau^*, \mathbf{H}_K^{2j}(\Xi_y)) q^j = \sum_{j \geq 0} \dim \text{Sym}^j(\mathfrak{t}_+^\vee) q^j \sum_{k \geq 0} \dim \text{Sym}^k(\mathfrak{t}_-^\vee) (-q)^k = (1 - q)^{e(\bar{y}^*) - r} (1 + q)^{-e(\bar{y}^*)}.$$

Plugging this into (4.4) and then into (4.3), we get

$$\begin{aligned} & \sum_j \text{tr}(\tau^*, \mathbf{IH}_K^{2j}(\Xi_{\leq w})) q^j \\ &= \sum_{y \leq w, y \in I_*} q^{\ell(w) - \ell(y)} P_{y,w}^\sigma(q^{-1}) (1 - q)^{e(\bar{y}^*) - r} (1 + q)^{-e(\bar{y}^*)} \\ &= q^{\ell(w)} (1 - q)^{-r} \sum_{y \leq w, y \in I_*} P_{y,w}^\sigma(q^{-1}) q^{-\ell(y)} \left( \frac{q^{-1} - 1}{q^{-1} + 1} \right)^{e(\bar{y}^*)} \\ &= q^{\ell(w)} (1 - q)^{-r} \left( \frac{q^{-1} - 1}{q^{-1} + 1} \right)^{e(\bar{y}^*)} Z_w^\sigma(q^{-1}) = q^{\ell(w)} (1 - q)^{e(\bar{y}^*) - r} (1 + q)^{-e(\bar{y}^*)} Z_w^\sigma(q^{-1}). \end{aligned}$$

□

In the situation of the affine Grassmannian, the isomorphism (2.4) induces an involution on the global sections  $\tau_K^* : \mathbf{IH}^\bullet(\Omega_{\leq \lambda}) \xrightarrow{\sim} \mathbf{IH}^\bullet(\Omega_{\leq \lambda})$ .

**4.3. Lemma.** *For  $\lambda \in \Lambda^+$ , we have*

$$(4.5) \quad \sum_{i \in \mathbb{Z}} \text{tr}(\tau_K^*, \mathbf{IH}^{2i}(\Omega_{\leq \lambda})) q^i = \tilde{Z}_{d_\lambda}^\sigma(q).$$

*Proof.* The map (4.2) induces an isomorphism on intersection cohomology commuting with the relevant involutions:

$$(4.6) \quad \mathbf{IH}_K^\bullet(\text{Fl}_{\leq d_\lambda}) \cong \mathbf{IH}_K^\bullet(\Xi_{\leq d_\lambda}) \cong \mathbf{IH}_K^\bullet(\Omega_{\leq \lambda}) \otimes_{\mathbf{H}_K^\bullet(\text{pt})} \mathbf{H}_K^\bullet(K/T_c \times K/T_c),$$

where the last equality comes from the degeneration of the Leray spectral sequence at  $E_2$  since all the relevant cohomology groups are concentrated in even degrees. Note that in (4.6), the involution on  $\mathbf{H}_K^\bullet(K/T_c \times K/T_c)$  is induced by  $(k_1 T_c, k_2 T_c) \mapsto (k_2^* T_c, k_1^* T_c)$ , and the involution on  $\mathbf{H}_K^\bullet(\text{pt})$  is induced by the involution  $*$  of  $K$ .

Another spectral sequence argument shows that we have an isomorphism

$$\mathbf{IH}_K^\bullet(\Omega_{\leq \lambda}) \cong \mathbf{H}_K^\bullet(\text{pt}) \otimes \mathbf{IH}^\bullet(\Omega_{\leq \lambda})$$

commuting with the obvious involutions (the one on  $\mathbf{H}_K^\bullet(\text{pt})$  is again induced by  $*$ , and the ones involving  $\Omega_{\leq \lambda}$  are given by  $\tau_K^*$ ). Combining this with (4.6) we get an isomorphism

$$\mathbf{IH}_K^\bullet(\text{Fl}_{\leq d_\lambda}) \xrightarrow{\sim} \mathbf{IH}^\bullet(\Omega_{\leq \lambda}) \otimes \mathbf{H}_K^\bullet(K/T_c \times K/T_c)$$

intertwining the involutions on both sides which we specified before. The special case  $\lambda = 0$ ,  $d_\lambda = w_J$  gives  $\mathrm{IH}_1^\bullet(\mathrm{Fl}_{\leq w_J}) \cong \mathrm{H}_K^\bullet(K/T_c \times K/T_c)$ . Therefore

$$\mathrm{IH}_1^\bullet(\mathrm{Fl}_{\leq d_\lambda}) \xrightarrow{\sim} \mathrm{IH}^\bullet(\Omega_{\leq \lambda}) \otimes \mathrm{IH}_1^\bullet(\mathrm{Fl}_{\leq w_J})$$

commuting with the relevant involutions. Taking the Poincaré polynomials with respect to the traces of these involutions, and using Lemma 4.2, we get

$$\begin{aligned} & q^{\ell(d_\lambda)}(1-q)^{e(*)-r}(1+q)^{-e(*)} Z_{d_\lambda}^\sigma(q^{-1}) \\ = & q^{\ell(w_J)}(1-q)^{e(*)-r}(1+q)^{-e(*)} Z_{w_J}^\sigma(q^{-1}) \sum_{j \in \mathbb{Z}} \mathrm{tr}(\tau_K^*, \mathrm{IH}^{2j}(\Omega_{\leq \lambda})) q^j. \end{aligned}$$

In view of the definition of  $\tilde{Z}_{d_\lambda}^\sigma(q)$  in (1.5), we get

$$\sum_{j \in \mathbb{Z}} \mathrm{tr}(\tau_K^*, \mathrm{IH}^{2j}(\Omega_{\leq \lambda})) q^j = q^{\ell(d_\lambda) - \ell(w_J)} \tilde{Z}_{d_\lambda}^\sigma(q^{-1}).$$

Let  $Q_\lambda(q)$  denote the left side. Substituting  $q^{-1}$  for  $q$  in the above, we get

$$Q_\lambda(q^{-1}) = q^{-\ell(d_\lambda) + \ell(w_J)} \tilde{Z}_{d_\lambda}^\sigma(q).$$

Poincaré duality for  $\mathrm{IH}^\bullet(\Omega_{\leq \lambda})$  (which has dimension  $\langle 2\rho, \lambda \rangle$ ) implies  $Q_\lambda(q) = q^{\langle 2\rho, \lambda \rangle} Q_\lambda(q^{-1})$ . Therefore

$$Q_\lambda(q) = q^{\langle 2\rho, \lambda \rangle} Q_\lambda(q^{-1}) = q^{\langle 2\rho, \lambda \rangle + \ell(w_J) - \ell(d_\lambda)} \tilde{Z}_{d_\lambda}^\sigma(q).$$

Since  $\ell(d_\lambda) = \ell(w_J) + \langle 2\rho, \lambda \rangle$  (i.e.,  $\dim \mathrm{Fl}_{\leq d_\lambda} = \dim G/B + \dim \mathrm{Gr}_{\leq \lambda}$ ), the above equality implies (4.5).  $\square$

**4.4. Completion of the proof.** By Lemma 3.3(3), the involution  $\tau_K^*$  acts on  $\mathrm{IH}^{2j}(\Omega_{\leq \lambda})$  via  $(-1)^j$ . Therefore, by Lemma 4.3, we have

$$(4.7) \quad \tilde{Z}_{d_\lambda}^\sigma(q) = \sum_{j \in \mathbb{Z}} (-1)^j \dim \mathrm{IH}^{2j}(\Omega_{\leq \lambda}) q^j = \sum_{j \in \mathbb{Z}} \dim \mathrm{IH}^{2j}(\Omega_{\leq \lambda}) (-q)^j.$$

On the other hand, the argument using the filtration (3.5) shows that  $Z_{d_\lambda}(q)$  is the Poincaré polynomial for  $\mathrm{IH}^\bullet(\mathrm{Fl}_{\leq d_\lambda})$ :

$$Z_w(q) = \sum_{j \in \mathbb{Z}} \dim \mathrm{IH}^{2j}(\mathrm{Fl}_{\leq w}) q^j.$$

Using the homeomorphism  $\iota_{\mathrm{Fl}}$  and the diagram (4.1), we have  $\mathrm{IH}^\bullet(\mathrm{Fl}_{\leq d_\lambda}) \cong \mathrm{IH}^\bullet(\Omega_{\leq \lambda}) \otimes \mathrm{H}^\bullet(K/T_c) \cong \mathrm{IH}^\bullet(\Omega_\lambda) \otimes \mathrm{IH}^\bullet(\mathrm{Fl}_{\leq w_J})$ . Therefore  $Z_{d_\lambda}(q)$  is the product of  $Z_{w_J}(q)$  with the Poincaré polynomial of  $\mathrm{IH}^\bullet(\Omega_{\leq \lambda})$ . By the definition of  $\tilde{Z}_{d_\lambda}(q)$  in (1.5), we have

$$(4.8) \quad \tilde{Z}_{d_\lambda}(q) = Z_{d_\lambda}(q) Z_{w_J}(q)^{-1} = \sum_{j \in \mathbb{Z}} \dim \mathrm{IH}^{2j}(\Omega_{\leq \lambda}) q^j.$$

The theorem now follows by comparing (4.7) and (4.8).

## 5. ALGEBRAIC PROOF OF THEOREM 1.5

Now we start the algebraic proof of Theorem 1.5. Using [L11, 3.6(f)] and Theorem 1.3 we see that

$$\begin{aligned}\tilde{Z}_{d_\lambda}^\sigma(q) &= \sum_{\mu \in \Lambda^+; d_\mu \leq d_\lambda} P_{d_\mu, d_\lambda}^\sigma(q) \zeta \left( \sum_{y \in W_J \mu W_J; y \in I_*} a_y \right) Z_{w_J}^\sigma(q)^{-1} \\ &= \sum_{\mu \in \Lambda^+; d_\mu \leq d_\lambda} P_{d_\mu, d_\lambda}(-q) \zeta \left( \sum_{y \in W_J \mu W_J; y \in I_*} a_y \right) Z_{w_J}^\sigma(q)^{-1}.\end{aligned}$$

On the other hand

$$\tilde{Z}_{d_\lambda}(-q) = \sum_{\mu \in \Lambda^+; d_\mu \leq d_\lambda} P_{d_\mu, d_\lambda}(-q) \sum_{y \in W_J \mu W_J} (-q)^{\ell(y)} Z_{w_J}(-q)^{-1}.$$

Hence to prove Theorem 1.5 it is enough to show that for any double coset  $W_J \mu W_J$  we have

$$(5.1) \quad \zeta \left( \sum_{y \in W_J \mu W_J \cap I_*} a_y \right) Z_{w_J}^\sigma(q)^{-1} = \sum_{y \in W_J \mu W_J} (-q)^{\ell(y)} Z_{w_J}(-q)^{-1}.$$

We fixed such a double coset  $W_J \mu W_J$  for the rest of this section, where  $\mu \in \Lambda^+ \cap W_J \mu W_J$  is the unique dominant translation. Let  $d = d_\mu$  (resp.  $b$ ) be the element of maximal (resp. minimal) length in  $W_J \mu W_J$ .

We shall be interested also in some parabolic analogues of  $Z_w(q), Z_w^\sigma(q)$ . For any  $H \subsetneq S$  let  $W_H$  be the subgroup of  $W$  generated by  $H$  so that  $(W_H, H)$  is a finite Coxeter group; let  $w_H$  be the longest element of  $W_H$ . We also set  $\mathbf{P}_H = \sum_{x \in W_H} q^{\ell(x)} \in \mathbb{N}[q]$  so that  $Z_{w_H}(q) = \mathbf{P}_H(q)$ . Recall that  $J = S - \{s_0\}$ , and our previous notation  $W_J, w_J$  is consistent with the new notation.

If in addition we are given an involution  $\tau : W_H \rightarrow W_H$  leaving  $H$  stable, we set (as in [L11, 5.1])  $\mathbf{P}_{H, \tau} = \sum_{x \in W_H; \tau(x)=x} q^{\ell(x)} \in \mathbb{N}[q]$ . By [L11, 5.9] we have  $Z_{w_J}^\sigma(q) = \mathbf{P}_J(q^2) \mathbf{P}_{J, *}(q)^{-1}$  (we use also that  $P_{y, w_J}^\sigma(q) = 1$  for any  $y \in W_J$ , see [L11, 3.6(f)]).

Let  $H = J \cap b J b^{-1}$ . Let  $\epsilon : W_{H^*} \rightarrow W_{H^*}$  be the involution  $y \mapsto b^{-1} y^* b$  ( $H^*$  is the image of  $H$  under  $*$ ). From [L11, 5.10] we have

$$\zeta \left( \sum_{y \in W_J \mu W_J \cap I_*} a_y \right) = \zeta(a_b) \mathbf{P}_J(q^2) \mathbf{P}_{H^*, \epsilon}(q)^{-1}.$$

Similarly,

$$\sum_{y \in W_J \mu W_J} q^{\ell(y)} = q^{\ell(b)} \mathbf{P}_J(q^2) \mathbf{P}_{H^*}(q)^{-1}.$$

We see that (5.1) is equivalent to the following statement:

$$(5.2) \quad \zeta(a_b) \mathbf{P}_{J, *}(q) \mathbf{P}_{H^*, \epsilon}(q)^{-1} = (-q)^{\ell(b)} \mathbf{P}_J(-q) \mathbf{P}_{H^*}(-q)^{-1}.$$

**5.1. Lemma.** *The involution  $\epsilon$  on  $W_{H^*}$  is the same as  $\text{Ad}(w_{H^*})$ ; i.e.,  $b^{-1} y^* b = w_{H^*} y w_{H^*}$  for all  $y \in W_{H^*}$ .*

*Proof.* We shall denote the inverse of  $\mu$  by  $\mu^{-1}$  instead of  $-\mu$  as before. We have  $\mu w_J = d = w_J w_H b w_J$ . Hence  $\mu = w_J w_H b$ . Now  $W_J \times W_J$  acts transitively on  $W_J \mu W_J$  by left and right multiplication and the isotropy group of  $\mu$  is isomorphic to  $W_\mu := \{w \in W_J; w \mu = \mu\}$ . Hence  $|W_J \mu W_J| = |W_J|^2 / |W_\mu|$ . By [L11, 1.1] we have also  $|W_J \mu W_J| = |W_J|^2 / |W_H|$  hence  $|W_\mu| = |W_H|$ . We show that  $W_\mu \subset W_{H^*}$ . We have  $W_H = W_J \cap b W_J b^{-1}$ ; applying  $*$  we deduce  $W_{H^*} =$



$W_J \cap b^{-1}W_Jb$ . Hence it is enough to show that  $W_\mu \subset b^{-1}W_Jb$ . Since  $\mu = w_Jw_Hb$  we have  $\mu^{-1}W_J\mu = b^{-1}w_Hw_JW_Jw_Jw_Hb = b^{-1}W_Jb$ . If  $w \in W_\mu$  then  $w\mu = \mu w$  hence  $\mu w\mu^{-1} = w \in W_J$ ; thus  $W_\mu \subset \mu^{-1}W_J\mu = b^{-1}W_Jb$ . We have shown that  $W_\mu \subset W_{H^*}$ . Since the last two groups have the same order we see that  $W_\mu = W_{H^*}$ . Hence to prove (a) it is enough to show that for any  $y \in W_\mu$  we have  $b^{-1}y^*b = w_{H^*}yw_{H^*}$  that is (after applying  $*$ )  $byb^{-1} = w_Hy^*w_H$ . Since  $b = w_Hw_J\mu$ , it is enough to show that for  $y \in W_\mu$  we have  $w_J\mu y\mu^{-1}w_J = y^*$ , or, using  $\mu y = y\mu$ , that  $w_Jyw_J = y^*$ . This follows from the definition of  $*$  in Section 1.2. This proves the lemma.  $\square$

**5.2. Lemma.** *If  $L \subsetneq S$  and  $\text{Ad}(w_L)$  is the conjugation by  $w_L$  on  $W_L$ , then*

$$\mathbf{P}_{L, \text{Ad}(w_L)}(q) = \mathbf{P}_L(-q) \left( \frac{1+q}{1-q} \right)^{n_L},$$

where  $n_L$  is the number of odd exponents of  $W_L$ .

*Proof.* Let  $e_i (i \in X)$  be the exponents of  $W_L$ . We have  $X = X' \sqcup X''$  where  $X' = \{i \in X; e_i \text{ is odd}\}$ ,  $X'' = \{i \in X; e_i \text{ is even}\}$ . It is well known that

$$\mathbf{P}_L(q) = \prod_{i \in X} \frac{q^{e_i+1} - 1}{q - 1}.$$

It follows that

$$(5.3) \quad \mathbf{P}_L(-q) = \prod_{i \in X'} \frac{q^{e_i+1} - 1}{-q - 1} \prod_{i \in X''} \frac{q^{e_i+1} + 1}{q + 1}$$

We have

$$(5.4) \quad \mathbf{P}_{L, \text{Ad}(w_L)}(q) = \prod_{i \in X'} \frac{q^{e_i+1} - 1}{q - 1} \prod_{i \in X''} \frac{q^{e_i+1} + 1}{q + 1}.$$

Here, the left hand side evaluated at a prime power  $q$  is the number of  $\mathbb{F}_q$ -rational Borel subgroups of a semisimple algebraic group defined over  $\mathbb{F}_q$  which is twisted according to the opposition involution. This can be computed from the known formula for the number of rational points of such an algebraic groups given in [S67, §11]. Now the lemma follows from (5.3) and (5.4).  $\square$

**5.3.** Using Lemma 5.1 and 5.2 (applied to  $L = H^*$ ) and the definition of  $\zeta$  we see that the desired equality (5.2) is equivalent to

$$q^{\ell(b)} \left( \frac{q-1}{q+1} \right)^{\phi(b)} \left( \frac{1+q}{1-q} \right)^{n_J - n_{H^*}} = (-q)^{\ell(b)},$$

that is, to the equality

$$(5.5) \quad \phi(b) = n_J - n_{H^*}.$$

Here we use that  $\phi(w) = \ell(w) \pmod{2}$  for any  $w \in I_*$ , see [L11, 4.5].

Define  $\phi' : \{z \in W_{H^*}; \epsilon(z) = z^{-1}\} \rightarrow \mathbb{N}$  in terms of  $(W_{H^*}, \epsilon)$  in the same way as  $\phi$  was defined in terms of  $W$  and  $*$  in [L11, 4.5] (using the difference of the dimension of the  $(-1)$ -eigenspaces of  $w \in W_{H^*}$  and  $w w_{H^*}$  on the reflection representation of  $W_{H^*}$ ). We show:

**5.4. Lemma.** *For any  $z \in W_{H^*}$  such that  $\epsilon(z) = z^{-1}$ , we have  $\phi(bz) = \phi'(z) + \phi(b)$ .*

*Proof.* We argue by induction on  $\ell(z)$ . If  $z = 1$  the result is clear. Now assume that  $z \neq 1$ . We can find  $s \in H^*$  such that  $\ell(sz) < \ell(z)$ . Assume first that  $sz \neq z\epsilon(s)$ . Then  $\ell(sze(s)) = \ell(z) - 2$  hence by the induction hypothesis we have  $\phi(bsz\epsilon(s)) = \phi'(sz\epsilon(s)) + \phi(b)$ . By definition,  $\phi'(sze(s)) = \phi'(z)$ . We have  $bsz\epsilon(s) = bsb^{-1}bz\epsilon(s) = \epsilon(s)^*bz\epsilon(s)$  and hence, by definition,

$\phi(bsz\epsilon(s)) = \phi(\epsilon(s)*bz\epsilon(s)) = \phi(bz)$ . Thus  $\phi(bz) = \phi'(z) + \phi(b)$ . Next we assume that  $sz = z\epsilon(s)$ . Then  $\ell(sz\epsilon(s)) = \ell(z) - 1$  hence by the induction hypothesis we have  $\phi(bsz\epsilon(s)) = \phi'(sz\epsilon(s)) + \phi(b)$ . By definition,  $\phi'(sz\epsilon(s)) = \phi'(z) - 1$  and  $\phi(bsz\epsilon(s)) = \phi(\epsilon(s)*bz\epsilon(s)) = \phi(bz) - 1$ . Thus  $\phi(bz) = \phi'(z) + \phi(b)$ . This completes the proof of the lemma.  $\square$

**5.5. Completion of the proof.** From Lemma 5.4 we deduce

$$(5.6) \quad \phi(bw_{H^*}) = \phi'(w_{H^*}) + \phi(b).$$

We have  $d = cbw_{H^*}c^{*-1}$  where  $c = w_Jw_H$  (see [L11, §1.2]). From the definition of  $\phi$  we see that  $\phi(d) = \phi(bw_{H^*})$  hence, using (5.6), we have

$$\phi(d) = \phi'(w_{H^*}) + \phi(b).$$

Hence (5.5) is equivalent to

$$(5.7) \quad \phi(d) - \phi'(w_{H^*}) = n_J - n_{H^*}.$$

For any linear map  $A : \Lambda_{\mathbb{Q}} \rightarrow \Lambda_{\mathbb{Q}}$  (where  $\Lambda_{\mathbb{Q}} = \Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$ ), recall  $e(A)$  is the dimension of the  $(-1)$ -eigenspace of  $A$ . We claim that

$$(5.8) \quad \phi(d) = e(w_J).$$

In fact, if  $w \in I_*$  with image  $\bar{w} \in \overline{W}$ , we have  $\phi(w) = e(\bar{w}*) - e(*)$ . Since the action of  $*$  is given by  $x \mapsto -w_J(x)$ , we have  $\phi(w) = e(-\bar{w}w_J) - e(-w_J)$ . If  $w = d$  then  $d = tw_J$  ( $t$  is the dominant translation) hence  $\bar{w} = w_J \in \overline{W} \cong W_J$  and  $\phi(d) = e(-\text{id}) - e(-w_J)$ , which is equal to  $e(w_J)$ . This proves (5.8).

Now let  $R'$  be the reflection representation of  $W_{H^*}$ . For any linear map  $A : R' \rightarrow R'$  we denote by  $e'(A)$  the dimension of the  $(-1)$ -eigenspace of  $A$ . We claim that

$$(5.9) \quad \phi'(w_{H^*}) = e'(w_{H^*}).$$

In fact, from the definition we have  $\phi'(w_{H^*}) = e'(w_{H^*}\epsilon) - e'(\epsilon)$ . Note that both  $w_{H^*}$  and  $\epsilon$  act naturally on  $R'$ ; the action of  $\epsilon$  is given by  $x \mapsto -w_{H^*}x$  by Lemma 5.1. Thus we have  $\phi'(w_{H^*}) = e'(-\text{id}) - e'(-w_{H^*}) = e'(w_{H^*})$ . This proves (5.9).

Using (5.8) and (5.9) we see that the desired equality (5.7) is equivalent to

$$(5.10) \quad e(w_J) - e'(w_{H^*}) = n_J - n_{H^*}.$$

Now for any finite Weyl group, the dimension of the  $(-1)$ -eigenspace of the longest element acting on the reflection representation is equal to the number of odd exponents of that Weyl group, as one easily verifies. It follows that  $e(w_J) = n_J$ ,  $e'(w_{H^*}) = n_{H^*}$ . Thus (5.10) is proved. This completes the proof of Theorem 1.5.

**5.6. Signature of a hermitian form.** Let  $\check{G}$  be the Langlands dual of  $G$  as before, with dual Cartan and Borel  $\check{T} \subset \check{B}$ . We identify the Weyl group of  $\check{G}$  with  $W_J$ . Let  $\lambda \in \Lambda^+$ , viewed as a dominant weight of  $\check{G}$ , and let  $V_\lambda$  be the corresponding irreducible representation of  $\check{G}$  with highest weight  $\lambda$ . In [L97] a hermitian form  $h_\lambda$  on  $V_\lambda$  is constructed in terms of a semisimple element  $s \in \check{T}$  with  $s^2 = 1$ . Here we shall take  $s = (-1)^\rho$ . The hermitian form  $h_\lambda$  is invariant under a real form of  $\check{G}$  which can be shown to be quasi-split (for our choice of  $s$ ) and admits a compact Cartan subgroup. Moreover, by [L97, 2.9], the signature of  $h_\lambda$  is given by

$$(5.11) \quad \text{Signature}(h_\lambda) = (-1)^{\langle \rho, \lambda \rangle} \text{tr}((-1)^\rho, V_\lambda).$$

Recall the following results from [L83]. First, it is shown in [L83, 6.1] that the multiplicity of the weight  $\mu$  in  $V_\lambda$  is equal to  $P_{d_\mu, d_\lambda}(1)$ . Second, we have the formula (see [L83, (8.10)] and its proof)

$$(5.12) \quad \tilde{Z}_{d_\lambda}(q) = q^{\langle \rho, \lambda \rangle} \sum_{\mu \in \Lambda^+; d_\mu \leq d_\lambda} P_{d_\mu, d_\lambda}(1) \sum_{\mu \in W_J \mu} q^{\langle \rho, \mu \rangle}.$$

Setting  $q = 1$  we obtain that  $\tilde{Z}_{d_\lambda}(1) = \dim V_\lambda$ . Setting  $q = -1$  in (5.12), we obtain

$$(5.13) \quad \tilde{Z}_{d_\lambda}(-1) = (-1)^{\langle \rho, \lambda \rangle} \text{tr}((-1)^\rho, V_\lambda).$$

We may also obtain (5.13) from Lemma 3.3(3) and Lemma 4.3. Combining (5.13) with Theorem 1.5 and (5.11), we obtain

$$(5.14) \quad \text{Signature}(h_\lambda) = \tilde{Z}_{d_\lambda}^\sigma(1).$$

Thus, while  $\tilde{Z}_{d_\lambda}(q)$  is a  $q$ -analogue of the dimension of  $V_\lambda$ ,  $\tilde{Z}_{d_\lambda}^\sigma(q) = \tilde{Z}_{d_\lambda}^\sigma(-q)$  is the  $q$ -analogue of the signature of the hermitian form  $h_\lambda$  on  $V_\lambda$ .

**5.7. Remark.** We expect that the hermitian form  $h_\lambda$  on  $V_\lambda$  is the complexification of the sum of the polarization Hodge structures  $\text{IH}^{2p}(\text{Gr}_{\leq \lambda})$  (which only has  $(p, p)$ -classes). By the Riemann-Hodge bilinear relation, this pairing is positive (resp. negative) definite on  $\text{IH}^{2p}(\text{Gr}_{\leq \lambda})$  when  $p$  is even (resp. odd). Therefore the signature on the total intersection cohomology  $\text{IH}^\bullet(\text{Gr}_{\leq \lambda})$  (which is also the signature of the Poincaré duality pairing) is also calculated by  $\tilde{Z}_{d_\lambda}(-1) = \tilde{Z}_{d_\lambda}^\sigma(1)$ .

## 6. GENERALIZATION

**6.1. More involutions in affine Weyl groups.** In Section 1.2, we fixed a hyperspecial vertex  $s_0 \in S$  in the Dynkin diagram of  $(W, S)$ . Let  $A = \text{Aut}(W, S)$ . Then  $A$  has a subgroup

$$A_\Lambda := \{a \in \text{Aut}(W, S) \mid \text{there exists } w \in W_J \text{ such that } a(\lambda) = {}^w \lambda \text{ for all } \lambda \in \Lambda\}.$$

One may identify  $A_\Lambda$  with the affine automorphisms fixing the standard alcove corresponding to  $S$ . It is easy to see that  $A_\Lambda$  is normal in  $A$ . Let  $\bar{A} := A/A_\Lambda$ . The stabilizer of  $s_0$  under  $A$  is  $A_J = \text{Aut}(W_J, J)$ , which projects isomorphically to  $\bar{A}$ .

We recall the extended affine Weyl group is the semi-direct product  $\tilde{W} = W \rtimes A_\Lambda$ , and it fits into an exact sequence

$$1 \rightarrow \tilde{\Lambda} \rightarrow \tilde{W} \rightarrow \bar{W} \rightarrow 1$$

where  $\tilde{\Lambda}$  is a lattice containing  $\Lambda$  such that the projection  $\tilde{\Lambda} \hookrightarrow \tilde{W} \rightarrow A_\Lambda$  induces an isomorphism  $\tilde{\Lambda}/\Lambda \cong A_\Lambda$ .

**6.2. Lemma.** *Recall we have an involution  $*$  in  $A_J$  defined in (1.1).*

- (1) *Every element in the coset  $A_\Lambda * = *A_\Lambda \subset A$  is an involution.*
- (2) *For any hyperspecial vertex  $s_1 \in S$ , there is a unique  $a \in A_\Lambda *$  which sends  $s_0$  to  $s_1$ .*

*Proof.* (1) The group  $A_J$  acts on  $A_\Lambda$  by conjugation. This action can be seen explicitly as follows:  $W_J \rtimes A_J$  acts on  $\tilde{\Lambda}$  by the reflection action stabilizing  $\Lambda$ . The action of  $A_J$  on the quotient  $A_\Lambda = \tilde{\Lambda}/\Lambda$  is then induced from this reflection action. In particular, the action of  $*$  in  $A_J$  on  $\tilde{\Lambda}$  is via  $\lambda \mapsto -{}^{w_J} \lambda$ , which is congruent to  $-\lambda$  modulo  $\Lambda$ . Therefore  $*$  acts on  $A_\Lambda$  by inversion, hence every element  $a* \in A_\Lambda *$  satisfies  $(a*)^2 = a(*a*) = aa^{-1} = 1$ .

(2) It is well-known that  $A_\Lambda$  permutes the hyperspecial vertices simply transitively. Then for any  $a \in A_\Lambda$ , we have  $(a*)(s_0) = a(s_0)$  which exhaust all hyperspecial vertices exactly once as  $a$  runs over  $A_\Lambda$ .  $\square$

Let  $s_1$  be another hyperspecial vertex in  $S$ . Let  $\diamond \in A_\Lambda^*$  be the unique involution taking  $s_0$  to  $s_1$ , hence taking  $J$  to  $J^\diamond = S - \{s_1\}$ . Let  $I_\diamond = \{w \in W \mid w^\diamond = w^{-1}\}$  be the  $\diamond$ -twisted involutions in  $W$ . To avoid complicated subscripts, we denote  $W_{J^\diamond}$  by  $W_J^\diamond$  instead.

The following theorem generalizes Theorem 1.3.

### 6.3. Theorem.

- (1) *Each double coset  $W_J \backslash W / W_J^\diamond$  in  $W$  is stable under the anti-involution  $w \mapsto (w^\diamond)^{-1}$ . In particular, the longest element in each  $(W_J, W_J^\diamond)$ -double coset belongs to  $I_\diamond$ .*
- (2) *For longest representatives  $d_1$  and  $d_2$  of  $(W_J, W_J^\diamond)$ -double cosets in  $W$ , we have*

$$P_{d_1, d_2}^{\sigma, \diamond}(q) = P_{d_1, d_2}(-q).$$

Here the polynomials  $P_{y, w}^{\sigma, \diamond}(q)$  ( $y, w \in I_\diamond$ ) are the ones defined in [L11] in terms of  $(W, S, \diamond)$ .

**6.4. Sketch of proof.** We only indicate how to modify the proof of Theorem 1.3 to give the proof of this theorem.

The anti-involution  $w \mapsto (w^*)^{-1}$  extends to an anti-involution on  $\widetilde{W}$  by the same formula (1.1) (except that  $\lambda$  now is any element in  $\widetilde{\Lambda}$ ). Again each double coset  $W_J \backslash \widetilde{W} / W_J$  is stable under this anti-involution. Write  $\diamond = a^*$  for  $a \in A_\Lambda$ , then  $W_J^\diamond = a(W_J)$ . Multiplication by  $a$  on the right gives a bijection

$$W_J \backslash W / W_J^\diamond \leftrightarrow W_J \backslash W \cdot a / W_J \subset W_J \backslash \widetilde{W} / W_J.$$

This shows part (1) of Theorem 6.3.

In the situation of Section 2.1,  $G$  is a simply-connected group. Let  $G^{\text{ad}}$  be the adjoint form of  $G$ , with maximal torus  $T^{\text{ad}} = T/Z(G)$ . Then we have a natural isomorphism  $\widetilde{\Lambda} \cong \mathbb{X}_*(T^{\text{ad}})$ . The connected components of the affine Grassmannian  $\text{Gr}^{\text{ad}}$  for  $G^{\text{ad}}$  are indexed by  $\widetilde{\Lambda}/\Lambda$ . The  $G^{\text{ad}}[[t]]$ -orbits on  $\text{Gr}^{\text{ad}}$  are indexed by  $\widetilde{\Lambda}/W_J$ , and the natural projection  $\widetilde{\Lambda}/W_J \rightarrow \widetilde{\Lambda}/\Lambda$  indicates which orbit belongs to which connected component. Identifying  $A_\Lambda$  with  $\widetilde{\Lambda}/\Lambda$ , we denote the corresponding component of  $\text{Gr}^{\text{ad}}$  by  $\text{Gr}_a^{\text{ad}}$  ( $a \in A_\Lambda$  such that  $\diamond = a^*$ ). We may similarly define the Satake category  $\mathcal{S}^{\text{ad}}$  for  $G^{\text{ad}}$  with simple objects  $\mathbf{C}_\lambda[2\rho, \lambda]$ ,  $\lambda \in \widetilde{\Lambda}^+$  (dominant coweights of  $G^{\text{ad}}$ ). Via the fiber functor  $\mathbf{H}^\bullet$ ,  $\mathcal{S}^{\text{ad}}$  is equivalent to  $\text{Rep}(\check{G}^{\text{sc}})$ , where  $\check{G}^{\text{sc}}$  is the simply-connected form of  $\check{G}$ . The same anti-involution  $\tau^*$  defines a functor  $(\mathcal{S}^{\text{ad}}, \odot) \rightarrow (\mathcal{S}^{\text{ad}}, \odot^\sigma)$ , and there is an isomorphism  $\Psi_\lambda : \tau^* \mathbf{C}_\lambda \xrightarrow{\sim} \mathbf{C}_\lambda$  normalized to be the identity on  $\text{Gr}_\lambda^{\text{ad}}$ , which induces an involution  $\mathcal{H}_\mu^i \Psi_\lambda$  on the stalks  $\mathcal{H}_\mu^i \mathbf{C}_\lambda$  for  $\mu \leq \lambda \in \widetilde{\Lambda}^+$ . Note that in the partial ordering of  $\widetilde{\Lambda}$ , two elements are comparable only if they are congruent modulo  $\Lambda$ .

Let  $\dot{a} \in N_{G^{\text{ad}}((t))}(T^{\text{ad}}((t)))$  be a lifting of  $a \in A_\Lambda < \widetilde{W}$ , then  $\dot{a}G^{\text{ad}}[[t]]\dot{a}^{-1}$  is a hyperspecial parahoric subgroup of  $G^{\text{ad}}((t))$  corresponding to the vertex  $s_1 = \diamond(s_0)$ . Let  $\mathbf{P} \subset G((t))$  be the hyperspecial parahoric subgroup (containing  $\mathbf{I}$ ) corresponding to  $s_1$ . Right multiplication by  $\dot{a}$  induces an isomorphism

$$(6.1) \quad G((t))/\mathbf{P} \xrightarrow{\sim} G^{\text{ad}}((t))/\dot{a}G^{\text{ad}}[[t]]\dot{a}^{-1} \xrightarrow{\sim} \text{Gr}_a^{\text{ad}}$$

which is equivariant under the left actions by  $G[[t]]$ . The double coset  $G[[t]] \backslash G((t))/\mathbf{P}$  is in bijection with  $W_J \backslash W / W_J^\diamond$ . As in (2.5), the coefficients of the polynomials  $P_{d_1, d_2}^{\sigma, \diamond}(q)$  are expressible as the traces of an involution on the stalks of the intersection cohomology complexes on  $G[[t]]$ -orbits of  $G((t))/\mathbf{P}$ . Under the isomorphism (6.1), we have the following formula generalizing (2.5):

$$P_{d_1, d_2}^{\sigma, \diamond}(q) = \sum_{j \in \mathbb{Z}} \text{tr}(\mathcal{H}_\mu^{2j} \Psi_\lambda, \mathcal{H}_\mu^{2j} \mathbf{C}_\lambda) q^j.$$

Here  $\mu \leq \lambda \in \tilde{\Lambda}^+$  have image equal to  $a$  in  $\tilde{\Lambda}/\Lambda$ , and  $d_1$  (resp.  $d_2$ ) is the longest element in the double coset  $W_J \mu a^{-1} W_J^\diamond$  (resp.  $W_J \lambda a^{-1} W_J^\diamond$ ).

So in order to prove Theorem 6.3(2), it suffices to show that  $\mathcal{H}_\mu^{2j} \Psi_\lambda$  acts on  $\mathcal{H}_\mu^{2j} \mathbf{C}_\lambda$  via multiplication by  $(-1)^j$  for any  $\mu \leq \lambda \in \tilde{\Lambda}^+$ . The argument in Section 3 works up to Lemma 3.2. The pair  $(\tau^*, \gamma)$  again determines the element  $g = (-1)^\rho \in \tilde{T} < \text{Aut}(\check{G}^{\text{sc}})$ . However, a monoidal isomorphism  $\Theta : \tau^* \xrightarrow{\sim} \text{id}_{\mathcal{S}^{\text{ad}}}$  is the same as the choice of an element  $\tilde{g} \in \tilde{T}^{\text{sc}}$  lifting  $(-1)^\rho$ : the effect of  $\Theta$  on  $V \in \text{Rep}(\check{G}^{\text{sc}}) \cong \mathcal{S}^{\text{ad}}$  is the action of  $\tilde{g}^{-1}$ . Lemma 3.3(2) should say that the effect of  $\Theta$  (or  $\tilde{g}^{-1}$ ) on  $\mathbf{C}_\lambda[\langle 2\rho, \lambda \rangle]$  is  $\tilde{g}^{-w_J \lambda} \tau_K^*$ . In the rest of the argument, we use (3.6). The piece  $\text{H}_c^{2\langle 2\rho, \mu \rangle}(\Omega_\mu) \otimes \mathcal{H}_\mu^{2j} \mathbf{C}_\lambda$  appears in degree  $2\langle 2\rho, \mu \rangle + 2j - \langle 2\rho, \lambda \rangle$  in  $\text{IH}^\bullet(\Omega_{\leq \lambda})[\langle 2\rho, \lambda \rangle] \cong V_\lambda$ , hence it appears as a subquotient of  $\bigoplus_\nu V_\lambda(\nu)$ , where  $\nu \in \tilde{\Lambda}$  has the same image as  $\lambda$  and  $\mu$  in  $\tilde{\Lambda}/\Lambda$  and

$$(6.2) \quad \langle 2\rho, \nu \rangle = 2\langle 2\rho, \mu \rangle + 2j - \langle 2\rho, \lambda \rangle, \text{ or } j = \langle \rho, \nu + \lambda - 2\mu \rangle.$$

We write  $\text{H}_c^{2\langle 2\rho, \mu \rangle}(\Omega_\mu) \otimes \mathcal{H}_\mu^{2j} \mathbf{C}_\lambda = \bigoplus_\nu (\text{H}_c^{2\langle 2\rho, \mu \rangle}(\Omega_\mu) \otimes \mathcal{H}_\mu^j \mathbf{C}_\lambda)_\nu$  according to the weight decomposition. Therefore  $\tilde{g}^{-1}$  or  $\tilde{g}^{-w_J \lambda} \tau_K^*$  acts on  $(\text{H}_c^{2\langle 2\rho, \mu \rangle}(\Omega_\mu) \otimes \mathcal{H}_\mu^{2j} \mathbf{C}_\lambda)_\nu$  by  $\tilde{g}^{-\nu}$ . Specializing to  $\lambda = \mu = \nu$ ,  $\tilde{g}^{-w_J \mu} \tau_K^*$  acts on  $\text{H}_c^{2\langle 2\rho, \mu \rangle}(\Omega_\mu) = \text{IH}^{2\langle 2\rho, \mu \rangle}(\Omega_{\leq \mu})$  by  $\tilde{g}^{-\mu}$ . Therefore, by (3.6), the action of  $\mathcal{H}_\mu^j \Psi_\lambda$  on  $\mathcal{H}_\mu^j \mathbf{C}_\lambda$  is given by

$$(6.3) \quad \tilde{g}^{-\nu + w_J \lambda} \cdot (\tilde{g}^{-\mu + w_J \mu})^{-1} = \tilde{g}^{-\nu + \mu + w_J(\lambda - \mu)}.$$

Since  $-\nu + \mu \in \Lambda$ , we have  $\tilde{g}^{-\nu + \mu} = g^{-\nu + \mu} = (-1)^{\langle \rho, -\nu + \mu \rangle}$ . Since  $\lambda - \mu \in \Lambda$ , we also have  $\tilde{g}^{w_J(\lambda - \mu)} = g^{w_J(\lambda - \mu)} = (-1)^{\langle \rho, w_J(\lambda - \mu) \rangle} = (-1)^{\langle -\rho, \lambda - \mu \rangle}$ . Taking these two facts together we conclude that the expression (6.3) is equal to

$$(-1)^{\langle \rho, -\nu + \mu \rangle} (-1)^{\langle -\rho, \lambda - \mu \rangle} = (-1)^{\langle \rho, -\nu - \lambda + 2\mu \rangle},$$

which is equal to  $(-1)^j$  by (6.2). This finishes the proof of Theorem 6.3.

**6.5. Remark.** The results of Section 4 and 5 can also be extended to the setup in Section 6. Thus  $\check{G}$  can be replaced by the corresponding simply connected group whose irreducible finite dimensional representations carry a natural hermitian form as in [L97] with signature expressible in terms analogous to (5.14). We omit the details.

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