Heat transfer in flat-plate boundary layers: a correlation for laminar, transitional, and turbulent flow

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</table>
Heat Transfer in Flat-Plate Boundary Layers: A Correlation for Laminar, Transitional, and Turbulent Flow

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1 Introduction

Simplified treatments of boundary layer heat transfer split the boundary layer into an upstream laminar section and downstream turbulent section. Correlations are then formed for these two sections, which are assumed to be separated by a distinct transition Reynolds number. An example is the correlation for average heat transfer coefficient proposed by Whitaker [1]: the heat transfer coefficient is averaged over the length, using results from laminar theory and correlation for turbulent flow. A transition Reynolds number of 200,000 is assigned, and single equation depending on the overall Reynolds number is obtained.

The actual transition process has long been known to include a lengthy transition region, in which the heat transfer coefficient rises smoothly. Indeed, data show this region to have a length similar to the laminar region (see Sec. 4). In contrast to Whitaker, Churchill [2] proposed a continuous correlation to predict the local Nusselt number, Nu, from laminar flow, through the transition region, and into the turbulent region. He used an algebraic combining formula to smoothly join a laminar correlation, Nu_lam, a transitional correlation, Nu_trans, and a turbulent correlation, Nu_turb. He also included an additive constant, Nu_0, for very low Reynolds numbers.

\[
Nu = Nu_0 + \left[ Nu_{lam} + (Nu_{trans} + Nu_{turb})^{\frac{x}{\Pr}} \right]^{1/3}
\]  

(1)

In Eq. (1), all four terms are included irrespective of the value of Re, Each term will become dominant in the appropriate range, as ensured by the exponents used and the terms' differing dependence on Re. Churchill’s values of \( p = -5 \) and \( s = 2 \) produce fairly gradual transitions between the three indicated ranges. To account for the variability of transition, his result must be matched to any particular dataset by fitting the value of Reynolds number at the upper limit of the transition region, Re_t, into his formula for Nu_trans.

Churchill expressed the three variable terms as functions of a single parameter, \( \phi(Re, \Pr) = RePr^{2/3} [1 + (d/Pr)^{2/3}]^{-1/2} \) with different values of d for uniform wall temperature (UWT) or for uniform heat flux (UHF).

This function was based upon his and Ozeno’s laminar flow curve-fits spanning all values of Pr [2,4]. As a consequence, he had to make an approximation for the turbulent Nusselt number. The full statement of Churchill’s equation is given in Appendix A.

Churchill compared his model to aggregated data for the local and average heat transfer coefficient, with partial agreement. As seen in Fig. 1, the overall agreement with local data is not close. The transition away from laminar flow begins too soon, the slope in the transition region does not match, and the values in the fully turbulent range are slightly high. Therefore, a more accurate fit is desirable.

Here, we develop a correlation that accurately captures the lengthy transition region. The available data sets are reviewed together with key theoretical points so that differences among the measurements will be more clear. Churchill’s basic concept is retained, but its primary elements are reevaluated. The resultant equation is compared to a large number of independent measurements. The measurements clearly indicate that the transition Nusselt number rises more rapidly with Reynolds number when transition begins at a higher Reynolds number. We offer a straightforward formula for calculating the local (and average) heat transfer coefficient, which has good agreement with data in the literature.

In any situation for which both the laminar and turbulent regimes must be considered, so should be the transitional region.
This fact is particularly important if local values of the heat transfer coefficient are required.

2 Theoretical Considerations and Assessment of Available Data Sets

The theory of boundary layer heat transfer is today very highly developed. To understand the differences in the data sets, a few fundamental points need to be in mind. These ideas are highlighted briefly, with no intention to give a comprehensive account.

2.1 Laminar Boundary Layers. Most differences between the laminar measurements used herein and the laminar theory can be attributed to differences in the wall boundary condition and/or in the upstream initial conditions.

The heat transfer coefficient in a laminar boundary layer is strongly sensitive to the upstream history of the flow, including both the leading-edge configuration and any variation in wall temperature. Most of the experimental reports have taken care to describe leading-edge conditions. In some studies, upstream boundary layers were suctioned away through a slot ahead of the test piece. Most of the experiments had an unheated initial length or an initial section in which temperatures were not well controlled, and most of those experimentalists applied analytical correction factors to make their reported data appear as if the initial length were uniform (Blair [5] is notable for giving a clear account of the unheated region and reporting it without adjustment; see Sec. 4).

The local heat transfer coefficient in a flat-plate boundary layer is higher for UHF than for UWT, by the factor of 0.453/0.332 = 1.36 [6]. While that knowledge, as an experimental result reported that A.P. Colburn came to a similar opinion after they wrote to him. Likewise, Kestin et al. [10] were very critical of the transient measurements of Sugawara et al. [11] for this same reason. Yet, Sugawara’s laminar measurements (and correlation) agree well with the analytical formula for a uniform wall heat flux: and the 1951 Japanese-language version of their work [12] explicitly considers a wall temperature varying as $x^{1/2}$ (as it would for a uniform heat flux), contrasting that result to Pohlhausen’s formula.

2.2 Freestream Turbulence. Freestream turbulence is normally manipulated by placing a grid of bars upstream of the test plate; the formation and decay of grid turbulence have been very well understood in the wind-tunnel literature since the late 1930s [13,14].

Experimentalists concluded early on that freestream turbulence at levels up to 5% or so had little or no effect on the laminar heat transfer coefficient [7,15]. Experiments also clearly showed that increasing freestream turbulence strongly reduced the transition Reynolds number and that, with extremely low turbulence, the onset of transition could be delayed to $Re_c \approx 2.8 \times 10^8$ [16]. However, the effect of freestream turbulence on heat transfer in the turbulent boundary layer was extensively debated during the mid-twentieth century [7,10,11,15,17]. Mainly, these early investigations concluded that freestream turbulence had little effect at zero pressure gradient (the work of Sugawara et al. [11] is an exception). Eventually, more precise measurements encompassing higher turbulence levels showed that freestream turbulence can modestly increase the turbulent heat transfer coefficient. Blair [5], for example, measured up to an 18% increase for a nominal freestream turbulence level of 6%.

In what follows, we consider data with reported freestream turbulence levels below 5% (and mostly below 3%).

The various experimental reports differ significantly on how turbulence was characterized. Blair, for example, used multicomponent hot-wire anemometer measurements to fully describe the turbulence levels below 5% (and mostly below 3%).

In modern notation, Pohlhausen’s equation is $Nu_{x0} = 0.332 Re^{1/2} Pr^{1/3}$. Sugawara et al. measured the transient heating and cooling of a plate in a wind tunnel at a dozen positions along the plate. The plate temperature response, locally, was treated as lumped. The plate would cool faster near the leading edge.
streamwise variation of the turbulence in each case. In other cases, experimentalists have reported only, say, “about 0.3%” without elaboration [18]. The length and time scales of turbulence in the various experimental systems differ and cannot productively be compared using only the reported value of $u'/u_c$. However, variation of turbulence within a single experimental system is clearly comparable within that system. We note the level of freestream turbulence of the data sets whenever it has been reported.

2.3 Turbulent Boundary Layers. Heat transfer coefficients in turbulent boundary layers are not sensitive to slow streamwise variations in the wall temperature, and the value of $h$ for UHF is only about 4% greater than for UWT [19].

The near-wall velocity distribution in the fully turbulent boundary layer is essentially a function of the local wall shear stress, as embodied by the universal velocity profile or “law-of-the-wall” [20,21]. Similarly, the local heat transfer coefficient for nonmetallic fluids is mainly determined by the local shear stress and the Prandtl number, as shown by the Reynolds–Colburn analogy [22,23] and its more accurate generalizations based on boundary-layer structure [6,24,25]. For the internal flow case, shear stress is represented by the Darcy friction factor, proportional to $f/8$, while the external flow case uses the skin friction coefficient, as $C_f/2$.

Data for external, turbulent boundary-layer heat transfer have mainly been acquired in air flows. Boundary layer data have been measured, apparently, only by Zukauskas and Slončiauskas [26] and by Hollingsworth [27]. Data for other liquids are even more limited, amounting to measurements in “transformer oil,” again by Zukauskas and coworkers. This situation could seem discouraging until one recalls that a vast body of data exists for turbulent internal flow (in pipes), spanning an enormous range of Prandtl numbers, and for unheated starting lengths and other geometric differences, so we will not consider them in any detail.

The relevant generalization of the Reynolds–Colburn analogy is

$$St = \frac{Nu}{RePr} = \frac{C_f/2}{a_1 + a_2(Pr^{a_3} - 1)^{1/2}}$$

Various authors have proposed values for the coefficients $\{a_1, a_2, a_3\}$. For example, Prandtl [28] chose $\{1; 8.77; 1\}$ (pipe flow); Zukauskas and Slončiauskas [26] proposed $\{0.93; 12.5; 2/3\}$ (flat plate); White [25] gave $\{1; 12.8; 0.68\}$ (flat plate); and Petukhov [29] gave $\{1.07; 12.7; 2/3\}$ (pipe flow). Gnielinski [30] examined thousands of data points in pipe flow, spanning $0.6 \leq Pr \leq 10^3$, leading him to suggest $\{1; 12.7; 2/3\}$. Gnielinski’s values capture 90% of the liquid data to $\pm 20\%$, with even better agreement for gases. In what follows, we adopt Gnielinski’s coefficients.

Theoretical considerations have suggested the use of virtual origins or of the enthalpy thickness in correlating boundary layer data [19,27,31], but here we focus on the local Reynolds number because that is how the vast majority of the data have been presented.

2.4 Variable Property Effects. For boundary layers with large temperature differences, property variation across the boundary may be important. For turbulent gas flows, property corrections are often made by evaluating a correlation at $T_w$ and multiplying the result by the absolute temperature ratio, $(T_w/T_c)^n$, with $n$ between 0.25 and 0.4 [26,32]. (For pipe flow, Gnielinski recommends $n = 0.45$ [30].) For the air data discussed here, temperature ratios are less than 1.11, and property variations are either negligible or small ($\pm 5\%$). No corrections are applied in plotting the correlations for air data in what follows.

For liquids, corrections are generally based on a ratio of viscosities or Prandtl numbers, and most data are for turbulent pipe flows. For external boundary layers, the most comprehensive experimental study is due to Zukauskas and Slončiauskas [26], who recommend evaluating Nusselt number correlations using properties at $T_w$ and multiplying the result by $(Pr/Tc)^{0.25}$. For turbulent pipe flow, Gnielinski recommends an exponent of 0.11. Typical values of the Prandtl number for the fluids discussed here are given in Appendix D.

For the water data used herein, the property corrections are modest, ranging from <2% to 6%, with a single run (discussed below) reaching 10%. No corrections have applied to the water correlations in the plots shown.

For transformer oil, however, the corrections can exceed 20%; these will be addressed in context.

2.5 Data Sets Examined. For this study, measurements of the local heat transfer coefficient through transition region are of primary interest. The data sets for local heat transfer coefficients are described in Tables 1 and 2. These data are used in examining the variation of the heat transfer coefficient in the transition region.

Data sets for average heat transfer coefficients are summarized in Table 3 because they were used by Churchill [2], Whitaker [1], and others in developing correlations. Apart from the laminar study of Fage and Falkner [7], those data sets are strongly affected by unheated starting lengths and other geometric differences, so we will not consider them in any detail.

3 Reconstituting Churchill’s Formula. The term $Nulab$ in Eq. (1) is intended to account for heat transfer at very low Reynolds numbers, as for creeping flow. Churchill selected $Nulab = 0.45$ on the basis of numerical simulations for $Pr = 1$ by Dennis and Smith [40] and without experimental support. The contribution of $Nulab$ is negligible for Reynolds numbers greater than 1000 or so in nonmetallic liquids. We shall restrict our attention to nonmetallic liquids, with $Pr \geq 0.6$ and $Re_l \geq 1000$, and omit $Nulab$.

For the laminar boundary layer, Churchill used his and Ozoe’s fits over all $Pr$ [2,4], introducing the function $\phi(Re_l, Pr)$ mentioned in Sec. 1. Having limited the range to $Pr \geq 0.6$, we only need the common results for laminar boundary layers on sharp-edged flat plates

$$Nu_{lamb} = a Re_l^{1/2} Pr^{1/3} \quad \text{for } Pr \geq 0.6$$

where $a = 0.332$ for UWT or 0.453 for UHF [6].

In the transition region, Churchill proposed a curve fit

$$Nu_{trans} = b Re_l^c$$

The value of $b$ is fixed by either the value of $Nu_{lamb}$ at the Reynolds number where transition starts, $Re_l$, or the value of $Nu_{trans}$ at the upper end where transition ends, $Re_u$. Churchill selected the latter; we shall use the former because the laminar equations are better defined. Thus

$$Nu_{trans} = Nu_{lamb}(Re_u Pr) (Re_u/Re_l)^c$$

Churchill suggested a fixed exponent of $c = 3/2$, which seemed to fit some of Zukauskas and Slončiauskas’ data sets [26]; the latter authors had suggested 1.4. In contrast, our comparison to data in Sec. 4 shows $c$ is usually larger than this and that the value $c$ rises as $Re_l$ rises. Our approach will be to treat $c$ as a function of $Re_l$ in fitting the data.
Table 1  Data sets for local heat transfer coefficients in air. All are wind tunnel measurements. Boundary condition (B.C.), r.m.s. freestream turbulence ($u_r/u_{∞}$), and temperature difference across boundary layer ($ΔT$) are as indicated.

<table>
<thead>
<tr>
<th>Authors and years</th>
<th>B.C.</th>
<th>Range of Re</th>
<th>$u_r/u_{∞}$</th>
<th>$ΔT$ (K)</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>Blair, 1982/3 [5,33,34]</td>
<td>UHF</td>
<td>$1.1 \times 10^4 - 4.5 \times 10^6$</td>
<td>0.2–9.0%</td>
<td>&lt;20</td>
<td>Upstream grid produced freestream turbulence, with thorough documentation. “Nominal” $u_r/u_{∞}$ from 0.25 to 6%, decaying along plate length. Includes 4.3 cm unheated starting length, without data corrections and in excellent agreement with laminar theory. Uncertainties (2σ): $St \pm 2.5%$; $Re_r \pm 1%$.</td>
</tr>
<tr>
<td>Junkhan and coworkers, 1964/7 [17,35]</td>
<td>UWT</td>
<td>$3.8 \times 10^4 - 3.6 \times 10^5$</td>
<td>0.4–1.8%</td>
<td>~17</td>
<td>Upstream grid produced freestream turbulence: no grid, 0.4–0.8%; with grid, 1.3–1.8%. Data corrected for the unheated starting length. Temperature controlled by rows of electrically heated strips. Uncertainties (2σ): Nu $\pm 5.2%$; $Re_r \pm 2.5%$.</td>
</tr>
<tr>
<td>Kestin et al., 1961 [10]</td>
<td>UWT</td>
<td>$3.5 \times 10^4 - 6.0 \times 10^5$</td>
<td>0.7–3.8%</td>
<td>35–44</td>
<td>Upstream screen produced freestream turbulence. Low turbulence set has $u_r/u_{∞}$ of 0.75–1.6%; high turbulence set has 3.1–3.8% in front section and 2.4–3.0% in back section. Plate temperature was steam controlled; data corrected (~5%) for leading-edge temperature variation. Uncertainties described in Ref. [10].</td>
</tr>
<tr>
<td>Reynolds et al., 1958 [32,36]</td>
<td>UWT</td>
<td>$9.2 \times 10^4 - 3.5 \times 10^6$</td>
<td>2.0–4.8%</td>
<td>10–14</td>
<td>Fully turbulent flow, b.l. tripped at the leading edge. $u_r/u_{∞}$ typically 2–3% over most of plate, but exceeded 4.5% for highest speeds. Temperature controlled by rows of electrically heated strips. Uncertainties (1σ): $St \pm 3%$; $Re_r \pm 1%$. Includes one data set for “natural transition” (low speed, $u_r/u_{∞} = 2–2.5%$).</td>
</tr>
<tr>
<td>Seban and Doughty, 1956 [18]</td>
<td>UHF</td>
<td>$1.1 \times 10^5 - 4 \times 10^6$</td>
<td>~0.3%</td>
<td>&lt;14</td>
<td>Wind tunnel with low freestream turbulence. Data were corrected for an unheated nose piece ahead of the thicker test section. The authors state that these corrections contributed some scatter to the transition region data. Uncertainties not clearly stated, although the experiments were carefully designed. Measurement of $u_r/u_{∞}$ not described.</td>
</tr>
<tr>
<td>Sugawara et al., 1951/8 [11,12]</td>
<td>$\sqrt{\alpha}$</td>
<td>$4.1 \times 10^3 - 3 \times 10^5$</td>
<td>0.7–7.4%</td>
<td>?</td>
<td>Grids at two distances upstream. Heat transfer coefficients increased 55% at $u_r/u_{∞}$ of 7%. Transient cooling of plate; nonuniform $T_w$ distribution coincidentally like that for UHF, but minimal detail reported. Blunt leading edge caused transition at $Re_w$ of 10,000–20,000 in NACA version [11]. Uncertainties not described.</td>
</tr>
<tr>
<td>Žukauskas and Šlanciauskas (1987 summary of older work) [26]</td>
<td>UHF</td>
<td>$1.1 \times 10^4 - 3.2 \times 10^6$</td>
<td>~0.3%</td>
<td>&lt;34</td>
<td>Data of Fig. 4(c) are from experiments around transition region, three runs with varying $ΔT$. Figure 4(d) shows separate experiments focused on higher $Re_w$, with significant scatter in laminar flow and Nu $r$ for turbulent flow about 15% higher than most reports. Book summarizes older works. Uncertainty in heat transfer stated as ±5%.</td>
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Table 2  Data sets for local heat transfer coefficients in liquids. Boundary condition (B.C.) and r.m.s. freestream turbulence ($u'_u/u_\infty$) are as indicated.

<table>
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<tr>
<th>Authors and years</th>
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<th>Range of Re</th>
<th>$u'<em>u/u</em>\infty$</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hollingsworth and coworkers, 1989 [27,31,37]</td>
<td>UWT</td>
<td>$\approx5 \times 10^5$%</td>
<td>0.5–1.5%</td>
<td>Data for Pr = 5.93 ± 0.13 with 4 K b.l. temperature difference. Boundary layer tripped ahead of plate, with a substantial unheated section; two locations measured in uncurved channel. Plate temperature controlled by rows of electrically heated strips. Data analysis focuses on enthalpy thickness Reynolds number. Data were used to calibrate a numerical model, leading to a correlation: $St = 0.02426 Re^{0.29} Pr^{-0.895}$ for $n = 0.1879 Pr^{-0.18}$ for 0.7 &lt; Pr &lt; 8. Measurement uncertainties (2σ): St ± 3.3%; $Re_t$ ± 1.6%.</td>
</tr>
<tr>
<td>Zukauskas and Šlanciauskas (1987 summary of older work) [26]</td>
<td>UHF</td>
<td>$4.0 \times 10^3$–$4.4 \times 10^6$</td>
<td>~1.2%</td>
<td>Data for Prandtl numbers of 2.95, 5.4, and 6.6. Stated uncertainty in heat transfer is up to ±10%, being largest for higher fluxes. Background turbulence measured at inlet to test section. For Pr = 5.4: Run 17, $\Delta T$ is 7–8 K for the turbulent and transitional data, with the laminar points rising to 17 K; Run 18, 6.5 ≤ $\Delta T$ ≤ 21.4 K; Table 25, $\Delta T$ is 7–8 K for the turbulent and transitional data, again rising to 15–17 K in the laminar region. For Pr = 6.6: Run 7, 1.3 ≤ $\Delta T$ ≤ 4.2 K; Run 8, 6.5 ≤ $\Delta T$ ≤ 21.4 K. For Pr = 6.95: 5.4 ≤ $\Delta T$ ≤ 8.6 K.</td>
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</table>

Table 3  Data sets for average heat transfer coefficients with r.m.s. freestream turbulence ($u'_u/u_\infty$) indicated when known.

<table>
<thead>
<tr>
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<th>$u'<em>u/u</em>\infty$</th>
<th>Comments</th>
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<tbody>
<tr>
<td>Fage and Falkner, 1931 [7]</td>
<td>$1.4 \times 10^3$ – $1.2 \times 10^4$</td>
<td>Laminar flow over electrically heated platinum strips with varied levels of turbulence. Average strip temperatures were measured from strip electrical resistance. The strips were placed into the centerline of long and short tubes, with or without a screen or perforated plate at the inlet; levels of turbulence were not reported. Freestream turbulence found not to affect laminar heat transfer rate. Temperature ratios were large ($T_{w,av}/T_\infty = 1.45$). Air thermal conductivity used was 12% less than current value.</td>
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<tr>
<td>Jakob and Dow, 1946 [9]</td>
<td>$4.5 \times 10^3$ – $1.5 \times 10^6$</td>
<td>Electrically heated copper section of a cylindrical tube was studied. Tube was placed axially in an open-air jet. Average surface temperature in the copper jacket recorded. Corrections were applied for unheated starting length of varied dimension. Average Nusselt number as a function of overall Reynolds number is reported. Transition to turbulence changed significantly with changes of shape of the unheated nose-piece. Freestream turbulence is not described.</td>
<td></td>
</tr>
<tr>
<td>Parmelee and Hueberscher, 1947 [38]</td>
<td>$1.9 \times 10^3$ – $9.3 \times 10^3$</td>
<td>Aluminum plate with four heated sections at center which were independently controlled. An &quot;average plate temperature&quot;, air speed, and $h$ were tabulated. Leading edge radius was observed to significantly affect $h$. No uncertainty analysis or freestream turbulence data. Transition is stated to occur at a Reynolds number of 150,000. Data were corrected for natural convection and guard heaters.</td>
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<tr>
<td>Edwards and Furber, 1956 [15]</td>
<td>$5 \times 10^3$ – $2.5 \times 10^6$</td>
<td>≤5%</td>
<td>Studied freestream turbulence effects on heat transfer in air. A 6 in. electrically heated, copper section was located 33 in. downstream of plate’s leading edge. The average temperature of the copper section was taken. Turbulence produced by an upstream grid; background turbulence not reported. Large corrections for the unheated starting length were applied (21–75%). Turbulence found to have only limited effect on heat transfer, but very strong effect on the location of transition.</td>
</tr>
<tr>
<td>Zukauskas and Ambrazyavichyus, 1961 [39]</td>
<td>$2 \times 10^3$ – $3 \times 10^7$</td>
<td>Plates up to 250 mm long were either electrically heated or water cooled. The same channel was used for air, water, and transformer oil. Average plate temperatures were found from thermocouples by integration. Heated portion of plate ranged from 11% to 80%; a simple power-law was used to correct the data for the unheated initial length. The average heat transfer coefficient over the heated length, $h_0$, is reported to be independent of the unheated length and correlated as: $Nu_0 = 0.037 Re_0^{0.65} Pr_0^{0.4} (Pr_0/P_\infty)^{0.25}$ in fully turbulent flow for 0.7 &lt; Pr &lt; 380. The scatter of the data about the curve fit is ±15% at a Reynolds number of 1,000,000 and greater for lower Re. Freestream turbulence is not reported.</td>
<td></td>
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</table>
In the turbulent region, Churchill improvised a bit, so as to adapt a power-law fit for Nuturb to his $\phi$ variable (see Appendix A). The resulting equation overpredicts most air data, other than Zukauskas and Slančiukas [26]; and the resulting Prandtl number dependence has no experimental support. Instead of working with $\phi$, we can directly employ the general form of the Reynolds–Colburn analogy for smooth walls, Eq. (2), as

$$Nu_{lum} = C_{0}Re_{x}Pr^{c_{l}}$$

for $Pr \geq 0.6$ (6)

The skin friction coefficient, $C_{f}$, may be evaluated using White’s formula [25,41] to an accuracy of 1–2%:

$$C_{f}(x) = \frac{0.455}{\ln(0.06Re_{x})^{2}}$$

White recommends this equation for zero-pressure-gradient turbulent boundary layers at any $Re_{x}$.

Equation (6) with (7) provides very good agreement with most data over a wide range of Prandtl numbers (see Sec. 4). For gases, the result is well approximated by the power-law proposed by Reynolds et al. [32]7

$$Nu_{lum} = 0.0296Re_{x}^{0.8}Pr^{0.6}$$

for gases (8)

To address the slow transition between regimes in Churchill’s fit, different exponents can be applied. We have tested several values. The rather large numbers $p = -10$ and $s = 5$ provide sharper transitions than Churchill’s values, better tracking the datasets examined. On this basis, Eq. (1) simplifies

$$Nu_{l}(Re_{x}, Pr) = \left[ Nu_{lum}^{5} + (Nu_{lum}^{10} + Nu_{lum}^{10})^{-1/2} \right]^{1/5}$$

(9)

7Reynolds et al. suggest multiplying by variable-properties correction factor of $(T_{w}/T_{x})^{-0.4}$, which amounted to about 2% for their data. Their equation fits their data to a standard deviation of $\pm 4\%$.

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4 Results

Equation (9) is compared to: data for air in Figs. 2–4; data for water in Figs. 6 and 7; and data for oil in Figs. 8 and 9. The transition region has been fitted with the values of $c$ and $Re_{x}$ shown in the figures. The agreement in the transition region is generally excellent.

4.1 Data for Air.

Figure 2 shows wind tunnel data including the transition region for UWT from Kestin et al. [10] and from Junkhan and Serovy [17]. Values of $c$ and $Re_{x}$ have been fitted to each of these four data sets, and the corresponding curve from Eq. (9) (with Eqs. (3), (5), (6), and (7)) is plotted. The value of $c$ trends higher as $Re_{x}$ rises. Also shown, for reference, are two sets of fully turbulent measurements from Reynolds et al. [32]; the latter are in excellent agreement with both Eqs. (6) and (8). Reynolds et al. reported six additional data sets, which are substantially the same as those shown in Fig. 2.

Figure 3 shows wind tunnel measurements by Blair for several carefully characterized levels of freestream turbulence with UHF. Blair’s test plate had an unheated starting length $x_{0} = 4.3$ cm, so that the laminar Nusselt number follows the theoretical prediction, $Nu_{lum} \times \left[ 1 - (x_{0}/x)^{1/4} \right]^{-1/3}$ (see Appendix B), which we use as the laminar term in Eq. (9) for this case. Agreement of the laminar theory with the data is excellent. Blair’s fully turbulent data converge to Eq. (6). Values of $c$ and $Re_{x}$ were fitted as before, and again the agreement with data is excellent. The value of $c$ becomes much larger for $Re_{x} > 10^{6}$. As noted earlier, Blair concluded that freestream turbulence could increase the turbulent heat transfer coefficient, an effect that may be discernible in the data for $u'_{l}/u_{\infty} \approx 2\%$. 

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Figure 4 shows fits through the transition region for four additional wind tunnel studies. Figure 4(a) shows data for seven different freestream speeds from Seban and Doughty [18]. The trend of these data for Reynolds numbers above $10^6$ is 7–8% below Eq. (6). In the laminar range, the data lie above Eq. (3); however, these data were analytically corrected for an unheated starting length, a process that the authors also note contributed to greater scatter in the transition range. Beyond these two differences, the fitted values of $c$ and $Re_l$ put Eq. (9) into good agreement with the transition data.

Figure 4(b) shows the one dataset in the study of Reynolds et al. [36] for which the boundary layer had not been intentionally tripped at the leading edge. Again using the fitted parameters, Eq. (9) has good agreement with the measurements. Equation (8), proposed by Reynolds et al. for fully turbulent flow, is also shown. The data can be observed to exceed the fully turbulent line just after transition. Reynolds et al. discussed and modeled this effect, which had also been described by Seban and Doughty. They attributed the effect to differences between the momentum and energy thicknesses of an already fully turbulent boundary layer and of the boundary layer formed at the end of transition (a similar, but less pronounced effect, might be perceived in Figs. 3 and 4(a)).

Figure 4(c) shows measurements from Zukauskas and Šlančiauskas [26] for a single plate at a fixed air speed with an increasing wall heat flux. The transition Reynolds number shifts slightly lower as the plate temperature increases. Equation (9) fits each case separately.

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Figure 4(d) shows additional measurements from Zukauskas and Slančiauskas from a different series of experiments. In the laminar range, the two data sets labeled Table 24 in Sec. 2.1 trend above laminar theory and have some significant scatter; the data labeled Table 25 have a large gap around the transition range are very scattered in the laminar range. The unusual laminar behavior might indicate a leading-edge influence of some unknown type (see Sec. 2.1). In the fully turbulent range, all three data sets lie in a fairly tight line about 15–25% above both Eq. (9) and measurements by other investigators; no reason for this difference is evident. However, the transition range of the Table 24 data sets is well represented by a slope $c = 2.5$, consistent with the other air data undergoing transition at similar Reynolds numbers.

Both Figs. 4(c) and 4(d) show the power-law fit proposed by Zukauskas and Slančiauskas (for all Pr, correlating their data to Eq. (6)). The sample standard deviation is 5.5%.

Figure 5. Fully turbulent data from Refs. [5, 10, 17, 18, 32], including six additional data sets (Runs 2–7) from Ref. [32]. 99.4% of the data are within ±15% of Eq. (6). The sample standard deviation is 5.5%.

$\text{Nu}_{\text{turb}} = 0.032 \text{Re}^{0.8} \text{Pr}^{0.43}$

Together with the relationship proposed by Reynolds et al. [32], Equation (10) is substantially higher than other correlations and data for air, presumably because it is based on the data of Fig. 4(d), which are also higher. Similarly, the value proposed by Zukauskas and Slančiauskas for the term $c_l$ in Eq. (2) is lower than in other investigations.

Zukauskas and Slančiauskas provide an additional dataset for high heat fluxes (not shown), with $T_w/T_x$ of 1.20–1.25, for which the variable properties correction would be 7–9%. Those data do not encompass the transition range. Their trend is similar to Fig. 4(d), but with somewhat more scatter.

To evaluate the performance of Eq. (6), all of the fully turbulent air data from Refs. [5], [10], [17], [18], and [32] are plotted in Fig. 5. This includes the data from Figs. 2–4(a) as well as six additional runs from Ref. [32] that are not in Fig. 2. The standard deviation of the data around Eq. (6) is ±5.5%, for a 95% confidence interval of ±11%. Figure 5 includes 328 data points; 326 of these (99.4%) are within ±15% of Eq. (6). The air data of Ref. [26] are not included because they are systematically high relative to all other studies (relative to Eq. (6), those data are 15–25% high, with a few points 30% high). Equation (8) is very close to Eq. (6) for air.

4.2 Data for Water. Figure 6 shows Zukauskas and Slančiauskas’ data for room temperature water. In Fig. 6(a), Runs 7 and 8 have the same velocity and freestream temperature, but the temperature differences in Run 8 are significantly greater (see Table 2). For Run 7, the property-ratio correction averages 2%, and for Run 8 it averages 10%. Neither correction has been applied to the correlations plotted. Both data sets show good agreement with the laminar theory, Eq. (3), and follow a transition curve with $c = 1.75$ and $Re_l = 68,000$. (The laminar correlation would be closer to the Run 8 data if the property correction were applied.)

Figure 6(b) shows data extending well into the turbulent range. The proposed correlation, Eq. (9), represents the transitional and turbulent data well. Also, shown here are the correlations of Hollingsworth [27] (see Table 2) and of Zukauskas and Slančiauskas. The turbulent data for Run 18 and Hollingsworth’s correlation are close one another, although about 10% below Eq. (6). The laminar data for these experiments are quite scattered. Property ratio corrections have not been applied. For Table 27, corrections average <5%, and for the Table 25 data, corrections average <6%.

Figure 7 shows Zukauskas and Slančiauskas’ data for $Pr = 2.95$, obtained using hot water in the fully turbulent regime.

4.3 Data for Transformer Oil. Figure 8 shows Zukauskas and Slančiauskas’ data for transformer oil at $Pr = 109$ and 257. For these data, property variations across the boundary layer are substantial. Therefore, the correction factor recommended by Zukauskas and Slančiauskas, $(Pr_{\infty}/Pr_a)^{0.25}$, has been applied to the proposed correlation, Eq. (9). A single average value (the average of Zukauskas and Slančiauskas’ pointwise values) has been used. With this adjustment, agreement between the data and Eq. (9) is remarkably good.

Figure 9(a) shows data for $Pr_{\infty} = 85$ compared to the proposed correlation. For these two data sets, the variable properties correction recommended by Zukauskas and Slančiauskas averages 12.7% and 17%. The proposed correlation is plotted both without adjustment and with a ±15% adjustment. The two curves bracket the data, in the turbulent regime and the later part of the transition regime; the data do not extend to laminar conditions.

Equation (6) is compared to Zukauskas and Slančiauskas’ data for fully turbulent flow of transformer oil at $Pr_{\infty} = 55$ in Fig. 9(b); 13 of 16 points are within 10% of the equation (15 of 16 are within 15%). Those data do not extend to laminar and transitional conditions. The average variable properties correction for these data is <3% and is not applied to correlation.

4.4 Recommended Values of $c$ and $Re_l$. The value of $c$ clearly rises as $Re_l$ rises. Figure 10 plots $c$ as a function of the corresponding value of $Re_l$ for each of the data sets fitted in this paper. A few data sets have poorly defined slopes in the transition region, as a result of either limited data or high scatter (Figs. 4(c) and 9(a), and Junkhan and Serovy’s low turbulence data in Fig. 2). The remaining data for $3 \times 10^4 < Re_l < 5 \times 10^5$ are all within ±8% of

$c = 0.9922 \log_{10} Re_l - 3.013$

and 80% are within ±5%. The value $c = 6$ for Blair’s low turbulence data does not lie on this curve, and more data for $Re_l > 5 \times 10^5$ would be needed to extend the curve fit to higher $Re_l$. The table numbers for Zukauskas and Slančiauskas’ data refer to the data tables in the appendix of their book [26].
Figures 4(b) and 6(a) display the intersection of the laminar correlation, Eq. (3), with the dotted line representing the transition correlation, Eq. (5). The intersection lies at Re\textsubscript{c}. The data and the fit begin to rise above the laminar value upstream of Re\textsubscript{c}; and, at Re\textsubscript{c}, the Nusselt number of Eq.(9) is 7% greater than the laminar Nusselt number.

Mayle [42] and Blair [34] discuss procedures for estimating the onset of transition from the local value of \( u'_{r}/u_{\infty} \). Most of the present datasets did not characterize \( u'_{r}/u_{\infty} \) locally. For the studies that did provide streamwise measurements of \( u'_{r}/u_{\infty} \) [10,17,32,34], Re\textsubscript{c} is within about a factor of two of Mayle’s empirical equation, which is equivalent to

\[
\text{Re}_{c} = (3.6 \times 10^{5})(100u'_{r}/u_{\infty})^{-5/4}
\]  

for a laminar boundary starting at the leading edge under zero pressure gradient. However, the laminar momentum boundary layers measured mostly differ from that condition and differ between the various studies. This upstream variability together with unclear local values of \( u'_{r}/u_{\infty} \) leaves Mayle’s equation with little predictive power for the heat transfer experiments summarized here. When experimental conditions are better defined, Mayle’s equation has good experimental support [42].
In working scenarios with significant disturbances ($u'_r/\nu_\infty \gtrsim 3\%$), transition is likely to begin in the range $4 \times 10^2 \leq \text{Re}_L \leq 10^5$. Only with extremely low levels of turbulence ($u'_r/\nu_\infty \leq 0.5\%$) can $\text{Re}_L$ exceed $10^5$; Schubauer and Skramstad [16] reported an asymptotic limit of $\text{Re}_L \approx 2.8 \times 10^8$ for $u'_r/\nu_\infty < 0.1\%$.

### 5 Average Nusselt Number

An average Nusselt number may be found by integration, if desired. Given that Eq. (9) provides rather sharp transitions between flow regimes, piecewise integration of the component terms seems sufficient. For uniform $T_w$:

$$\overline{N} = \frac{1}{L_0 T} \int_0^{L_0} q_w \, dx$$

$$= \frac{1}{L} \left[ \int_{x_l}^{x_u} \overline{h_{\text{laminar}}} \, dx + \int_{x_l}^{x_u} \overline{h_{\text{trans}}} \, dx + \int_{x_u}^{x_t} \overline{h_{\text{turbulent}}} \, dx \right]$$

where $x_l = (\nu/\nu_\infty) \text{Re}_L$ and $x_u = (\nu/\nu_\infty) \text{Re}_u$. The first two integrals can be evaluated by hand. The third integral is less simple analytically, unless a power law form is adopted for the turbulent region. For instance, if we consider gas flows, Eq. (8) will suffice

$$\overline{N}_L = \frac{1}{k} \overline{h}_L = 0.037 \nu^{0.6} (\text{Re}_L^{0.8} - \text{Re}_u^{0.8}) + 0.664 \text{Re}_L^{1/2} \nu^{0.13}$$

$$+ \frac{0.296}{k} \nu^{0.6} (\overline{h}_{\text{trans}})$$

for gases

$$\text{contribution of transition region}$$

As an example, we may consider a case corresponding to the low turbulence air data of Kestin et al. in Fig. 2: $\text{Re}_L = 140,000$, $c = 2$, $\text{Re}_u = 335,000$ with, say, $\text{Re}_L = 600,000$. Then

$$\overline{N}_L = 470.7 + 221.6 + 259.2 + 285.4 = 951.2$$

(16)

### 6 Conclusions

New equations are proposed for calculating the Nusselt number of a flat plate boundary layer from the laminar regime, through
the transition region, and into the fully turbulent region. These formulas are in good agreement with available data for air, water, and oil boundary layers spanning the ranges 0.7 \( \leq \text{Pr} \leq 257 \), 4000 \( \leq \text{Re}_x \leq 4 \times 300,000 \), with freestream turbulence levels up to 5%. The complete correlation is summarized in Table 4.

(1) The transition region has a length similar to the laminar region, as shown in Fig. 11. (2) Data show that the Nusselt number in the transition region varies as \( \text{Re}_c^c \) where the exponent \( c \) increases with an increase in \( \text{Re}_c \), the Reynolds number at the onset of transition. Through the examination of many independent experiments, values of \( c \) are fitted as a function of \( \text{Re}_c \). The transition regime is described by Eqs. (5) and (11).

Table 4 Summary of the proposed correlation. Data supporting these equations span 0.7 \( \leq \text{Pr} \leq 257 \) and 4000 \( \leq \text{Re}_x \leq 4 \times 300,000 \), with freestream turbulence levels up to 5%. The complete correlation is summarized in Table 4.

Combining formula

\[
\text{Nu}_w(\text{Re}_x, \text{Pr}) = \left[ \text{Nu}_{\text{lamin}} + (\text{Nu}_{\text{trans}} + \text{Nu}_{\text{turb}})^{1/2} \right]^{1/3}
\]

Laminar region

\[
\text{Nu}_{\text{lamin}}(\text{Re}_x, \text{Pr}) = \begin{cases} 0.332 \text{Re}_x^{1/2} \text{Pr}^{1/3} & \text{UWT} \\ 0.453 \text{Re}_x^{1/2} \text{Pr}^{1/3} & \text{UHF} \end{cases}
\]

With an unheated starting length of \( x_0 \), use

\[
\text{Nu}_{\text{lamin}}(\text{Re}_x, \text{Pr}) \times \left[ 1 - (\text{Re}_c/\text{Re}_x)^{1/4} \right]^{1/3}
\]

Transition region

\[
\text{Nu}_{\text{trans}}(\text{Re}_x, \text{Pr}) = \text{Nu}_{\text{lamin}}(\text{Re}_x, \text{Pr}) \times (\text{Re}_c/\text{Re}_x)^c
\]

\( \text{Re}_c \) is the Reynolds number at onset of transition

\[
c = 0.9922 \log_{10} \text{Re}_c - 3.013 \text{ for } \text{Re}_c < 5 \times 10^5
\]

Turbulent region

\[
\text{Nu}_{\text{turb}}(\text{Re}_x, \text{Pr}) = \frac{\text{Re}_c \text{Pr} (C_f/2)}{1 + 12.7 (\text{Pr}^{2/3} - 1) \sqrt{C_f/2}}
\]

\[
C_f(\text{Re}_x) = 0.455 \left[ \ln \left( \frac{0.06 \text{Re}_x^{1/6}}{1} \right) \right]^{1/4}
\]

For gases only, the following equation has similar accuracy

\[
\text{Nu}_{\text{turb}}(\text{Re}_x, \text{Pr}) = 0.0296 \text{Re}_x^{0.8} \text{Pr}^{0.6} \text{ for gases}
\]

(3) Churchill’s multiregion correlation is modified to accurately predict the local Nusselt number in the transition and turbulent ranges. The new correlation, Eq. (9), shows good agreement with a large number of experimental data drawn from many independent studies.

(4) Equation (6) correlates the fully turbulent air data to \( \pm 11\% \) (two standard deviations); 99.4\% of the data are within \( \pm 15\% \).

(5) The fully turbulent air data of Zukauskas and Šlančiauskas are systematically higher than air data from other investigators. Similarly, their correlation, Eq. (10), is higher for air than other correlations.

(6) The data of Fage and Falkner, for UHF laminar air flow, are in good agreement with corresponding analytical results, contrary to the suggestion of Jakob and Dow, down to an overall Reynolds number of 1400 (Appendix B).

(7) Very few measurements are available for heat transfer in turbulent liquid boundary layers.

(8) Older literature on convection contains a great deal of information that remains useful today.

Acknowledgment

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Nomenclature

Roman Letters

\( a = \) coefficient in Eq. (3)  
\( b = \) coefficient in Eq. (4)  
\( c = \) exponent on Reynolds number, Eq. (4)  
\( C_f = \) local skin friction coefficient  
\( d = \) coefficient in Churchill’s function  
\( f = \) Darcy friction factor  
\( h = \) local heat transfer coefficient, W m\(^{-2}\) K\(^{-1}\)  
\( \overline{h} = \) average heat transfer coefficient, W m\(^{-2}\) K\(^{-1}\)  
\( I(u_0) = \) integral defined by Eq. (B7)  
\( k = \) thermal conductivity of fluid, W m\(^{-1}\) K\(^{-1}\)  
\( \text{Nu}_L = \) local Nusselt number, \( h x / k \)  
\( \overline{\text{Nu}}_L = \) average Nusselt number, \( \overline{h} L / k \)  
\( p = \) exponent in Eq. (1)  
\( \text{Pr} = \) Prandtl number  
\( q_w = \) wall heat flux, W m\(^{-2}\)  
\( q_{u_0} = \) a uniform heat flux, W m\(^{-2}\)  
\( \text{Re}_c = \) local Reynolds number, \( u_{\infty} x / \nu \)  
\( \text{Re}_L = \) overall Reynolds number, \( u_{\infty} L / \nu \)  
\( s = \) exponent in Eq. (1)  
\( \text{St} = \) local Stanton number, \( \text{Nu}_w / \text{Re}_x \)  
\( T_w = \) wall temperature, K  
\( T_{\infty} = \) freestream temperature, K  
\( u_0 = 1 - (x_0 / x)^{1/4} \)  
\( u_{\infty} = \) mean freestream speed, m s\(^{-1}\)  
\( u'_r = \) r.m.s. fluctuation of freestream speed, m s\(^{-1}\)  
\( x = \) distance from leading edge, m  
\( x_0 = \) unheated starting length, m

Greek Symbols

\( B(n, m) = \) beta function  
\( \Gamma(x) = \) gamma function  
\( \Delta T = T_w - T_{\infty}, \) K  
\( \nu = \) kinematic viscosity of fluid, m\(^2\) s\(^{-1}\)  
\( \sigma = \) standard deviation  
\( \phi = \) Churchill’s function, Eq. (A2)

Subscripts

\( av = \) average over surface  
\( l = \) at lower Re of transition region
lam = local value in the laminar region  
trans = local value in the transition region  
urb = local value in the turbulent region  
u = at upper Re of transition region  
w = value at the wall  
∞ = value in freestream

Abbreviations

r.m.s. = root-mean-square  
UHF = uniform heat flux  
UWT = uniform wall temperature

Appendix A: Churchill’s Correlation in Full  
Churchill’s correlation [2] is

\[
\text{Nu}_e = 0.45 + (0.3387 \phi^{1/2}) \left\{ 1 + \left[ (\phi/2,600)^{1/5} \right] / \left[ 1 + (\phi_{\nu}/\phi)^{7/2} \right] \right\}^{1/2}
\]

(A1)

and \( \phi_{\nu} \) is a number typically between about 10^5 and 10^7. The actual value of \( \phi_{\nu} \) must be fit to each specific dataset. If the Reynolds number at the end of the transition region is \( \text{Re}_{\nu} \), an estimate is \( \phi_{\nu} \approx \phi(\text{Re}_{\nu} = \text{Re}_c) \).

The equation is for uniform \( T_w \). To adapt it to uniform \( q_w \), the constants 0.3387, 0.0468, and 2600 are replaced by 0.4637, 0.02052, and 7420, respectively. Churchill gave a similar equation for the average Nusselt number.

This equation embeds a transition slope of \( c = 1.5 \). At high Reynolds number (fully turbulent flow), the equation limits to

\[
\text{Nu}_e \rightarrow 0.032 \phi^{4/5} \text{Pr}^{1/5} \rightarrow 0.032 \text{Re}^{0.8} \text{Pr}^{2/15}
\]

(A3)

Appendix B: Laminar Boundary Layer With Uniform Heat Flux

The laminar similarity solution for uniform heat flux is not well known; even recently, some well-respected books report that there is no such solution [19]. This appendix provides that background, as well as sketching the UHF solution with an unheated starting length, which is less well known than that for UWT.

B.1 Uniform Flux Over Entire Plate. The similarity transformation for \( T^* = x^* \) was first discussed in 1931 by Fage and Falkner [7], who gave a perturbation solution, but the ordinary differential equation was not integrated until 1949 by Chapman and Rubesin [43]. The first numerical integrations for \( n = 1/2 \), corresponding to uniform wall heat flux, were due to Sugawara and Sato [44] in 1951 and Levy [45] in 1952. A linear regression on Levy’s four computed points for \( n = 1/2 \) (Pr of 0.7, 2, 10, 20) leads to

\[
\text{Nu}_e = 0.4542 \text{Re}^{1/2} \text{Pr}^{0.3301}
\]

(B1)

Various analytical results from this era are summarized by Imai [46]. Imai obtained an asymptotic solution for high Pr using the WKB method

\[
\frac{\text{Nu}_e}{\text{Re}_e^{1/2}} = \frac{\Gamma(2/3) A^{1/3}}{2^{1/2} 3^{2/3} \Gamma(4/3)} \text{ with } A = (1/2 + 2n)\sqrt{2(0.33206)\text{Pr}}
\]

(B2)

For \( n = 1/2 \)

\[
\text{Nu}_e = 0.4587 \text{Re}^{1/2} \text{Pr}^{1/3}
\]

(B3)

Imai reported 1% agreement with Levy’s numerical values down to \( \text{Pr} = 0.7 \), which is within the accuracy of Levy’s numerics.

Further, Fage and Falkner’s experimental values for air fit well to \( \text{Nu}_e = 0.75 \text{Re}^{1/2} \), based on a value of \( \text{Re}_{\text{mean}} \) that is 12% lower than modern property data. The temperature ratio, \( T_w/T_{\infty} = 1.45 \), is high enough that variable properties effects are not negligible. Without a variable properties correction, the experiments are within +9% of Imai’s formula. If a correction of \( (T_w/T_{\infty})^{-0.4} \) is applied, the agreement is ~6%.

B.2 Uniform Flux With Unheated Starting Length. Tribus and Klein [47,48] gave general equations for variable flux, working from the cubic integral-method solution for an unheated starting length

\[
T_w - T_{\infty} = A(x) \int_{0}^{s} \left[ 1 - \left( \frac{\xi}{x} \right)^{3/4} \right] q_w(\xi) \, d\xi
\]

(B4)

Here, \( A(x) = C/k \text{Re}_{\text{mean}}^{1/3} \text{Pr}_{1}^{2/3} \), where \( 1/C = 6\Gamma(4/3)\Gamma(5/3)c_1 \) with \( c_1 = (3/2)(1/20)^{1/3}(13/280)^{1/3} = 0.331293 \) (from the integral solutions) so that \( C = 0.624065 \). For an unheated starting length

\[
q_w(x) = \begin{cases} 0 & \text{for } x < x_0 \\ q_0 & \text{for } x \geq x_0 \end{cases}
\]

(B5)

and

\[
T_w - T_{\infty} = \frac{1}{A(x)q_0} \int_{x_0}^{\infty} \left[ 1 - \left( \frac{\xi}{x} \right)^{3/4} \right] \, d\xi
\]

(B6)

Setting \( u = 1 - (\xi/x)^{3/4} \), \( u_0 = 1 - (x_0/x)^{3/4} \), and \( s = u/u_0 \), some algebra leads to

\[
\frac{3(T_w - T_{\infty})}{4kA(x)q_0u_0^3} = I(u_0) \equiv \int_{0}^{1} s^{-2/3}(1 - u_0 s)^{1/3} \, ds
\]

(B7)

Hence

\[
\text{Nu}_e \equiv \frac{q_0x}{(T_w - T_{\infty})k} = \frac{3}{4kA(x)u_0^3} \int_{0}^{1} \frac{3 \text{Re}_{\text{mean}}^{1/3}}{4u_0^3} I(u_0)
\]

(B8)

We can easily bound the integral \( I(u_0) \). Setting \( u_0 = 0 \)

\[
I(u_0) \leq I(0) = \int_{0}^{1} s^{-2/3} \, ds = 3
\]

(B9)

Setting \( u_0 = 1 \) and identifying the beta and gamma functions

\[
I(u_0) \geq I(1) = \int_{0}^{1} s^{-2/3}(1 - s)^{1/3} \, ds = B(1/3, 4/3) = \frac{\Gamma(1/3)\Gamma(4/3)}{\Gamma(5/3)} = 2.649958 \cdots
\]

(B10)

These bounds are tight (2.649958/3 = 0.883). If we make the approximation \( I(u_0) \approx I(1) \), then,
The classical Reynolds–Colburn (or Chilton–Colburn) analogy, from 1933 (and 1934), has the form [22,23]

$$\text{Nu} = 4.0535 \frac{Re_{k}^{1/2} Pr_{k}^{1/3}}{[1 - (x_0/x)^{3/4}]^{1/3}}$$  \tag{B14}

What is key here is that the approximation is bounded within 12% right up to $x_0$.

The case $x_0 = 1$ corresponds to $x_0 = 0$, a uniform wall heat flux over the entire length, as commonly given in textbooks [6,19].

$$\text{Nu}_k = 0.4535 Re_{k}^{1/2} Pr_{k}^{1/3}$$ \tag{B15}

This result is within 1% of Imai’s formula.

### Appendix C: Comments on the Classical Colburn Analogy

The classical Reynolds–Colburn (or Chilton–Colburn) analogy, from 1933 and 1934, has the form [22,23]

$$\text{St} = \frac{C_{f}}{2} Pr^{-2/3}$$ \tag{C1}

By combining this expression with an equation for $C_{f}$ as power of the Reynolds number, the Nusselt number can be expressed as a product of powers of the Reynolds and Prandtl numbers. Such equations have commonly been used for turbulent boundary layers and turbulent pipe flows.

Unfortunately, these power laws cannot accurately span a wide range of Prandtl number: at high Reynolds numbers, the dependence of the Reynolds number exponent on the Prandtl number becomes significant so that a single, fixed exponent for Re and for Pr is inadequate to cover a wide range of conditions. Gnielinski [30] and Zukauskas and Slanciūskas [26] discuss this point with reference to experimental data for either pipe or boundary layer flows. Transport models that take account of the structure of the turbulent boundary layer result in expressions having the general form of Eq. (2) [24,26,28,29]. Those expressions are capable of covering the full range of Prandtl number [29,30]. For a narrow range of Pr (say, just for gases or for some range of liquids), power-law formulas can be quite accurate, as shown by Gnielinski.

Colburn considered a wide spectrum of data in his work, but neither Colburn [22] nor Chilton and Colburn [23] had much data for external boundary layers. And the data they had were only for gases. Chilton and Colburn included three data points for evaporation of water into laminar air flow. In the turbulent regime, Colburn referred to two somewhat scattered sets of air data. In the laminar range, Colburn used the UHF air data of Fage and Falkner [7], showing good agreement with Eq. (35); but he later confessed to Jakob and Dow that he had made a “slide-rule error” in plotting these data [9] and that the points should have been 21% higher. In fact, the Colburn analogy is incapable of discriminating the laminar thermal boundary condition.

### Appendix D: Typical Values of Prandtl Number

Values of the Prandtl number for the fluids discussed in this paper are given in Table 5.

Table 5 Values of Pr for the fluids discussed in this paper

<table>
<thead>
<tr>
<th>Fluid</th>
<th>Typical Pr</th>
<th>Fluid</th>
<th>Typical Pr</th>
</tr>
</thead>
<tbody>
<tr>
<td>Liquid metals</td>
<td>≤0.05</td>
<td>Air</td>
<td>0.70 to 0.74</td>
</tr>
<tr>
<td>Water</td>
<td>22</td>
<td>Transformer oil</td>
<td>30 to 222</td>
</tr>
<tr>
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<td>90 to 46.8</td>
</tr>
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Transformer oil data are from Ref. [26]; other values are from Ref. [6].

### References


Junkhan, G. H., 1964, “The Effects of Free-Stream Turbulence on Heat Trans-

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