18.785 Number Theory I **Fall 2017**
Problem Set #7

Description

These problems are related to the material covered in Lectures 13–15. Your solutions are to be written up in latex (you can use the latex source for the problem set as a template) and submitted as a pdf-file via e-mail to the instructor on the due date. Collaboration is permitted/encouraged, but you must identify your collaborators, and any references you consulted. If there are none, write "**Sources consulted: none**" at the top of your problem set. The first person to spot each nontrivial typo/error in any of the problem sets or lecture notes will receive 1–5 points of extra credit.

Instructions: First do the warm up problems, then pick a set of Problems 1–6 that sum to 96 points (if you have taken 18.783 and solved Problem 4 in that course, please do not choose it again). Finally, complete the survey problem (worth 4 points).

Problem 0.

These are warm up problems that do not need to be turned in.

(a) Prove that a cubic field K is Galois if and only if D_K is a perfect square.

(b) Prove that our two definitions of a lattice Λ in $V \simeq \mathbb{R}^n$ are equivalent: Λ is a $\mathbb{Z}$-submodule generated by an $\mathbb{R}$-basis for V if and only if it is a discrete cocompact subgroup of V.

(c) Let $n \in \mathbb{Z}_{>0}$ and assume n^2-1 is squarefree. Prove $n+\sqrt{n^2-1}$ is the fundamental unit of $\mathbb{Q}(\sqrt{n^2-1})$.

Problem 1. Classification of global fields (64 points)

Let K be a field and let M_K be the set of places of K (equivalence classes of nontrivial absolute values). We say that K has a (strong) *product formula* if M_K is nonempty for each $v \in M_K$ there is an absolute value $|\ |_v$ in its equivalence class and a positive real number m_v such that for all $x \in K^\times$ we have

$$\prod_{v \in M_K} |x|_v^{m_v} = 1,$$

where all but finitely many factors in the product are equal to 1. Equivalently, if we fix *normalized absolute values* $\|\ \|_v := |x|_v^{m_v}$ for each $v \in M_K$, then for all $x \in K^\times$ we have

$$\prod_{v \in M_K} \|x\|_v = 1,$$

with $\|x\|_v = 1$ for all but finitely many $v \in M_K$.

Definition. A field K is a *global field* if it has a product formula and the completion K_v of K at each place $v \in M_K$ is a local field.

In Lectures 10 and 13 we proved every finite extension of $\mathbb{Q}$ and $\mathbb{F}_q(t)$ is a global field. In this problem you will prove the converse, a result due to Artin and Whaples [1].

Let K be a global field with normalized absolute values $\| \, \|_v$ for $v \in M_K$ that satisfy the product formula. As we defined in lecture, an M_K-*divisor* is a sequence of positive real numbers $c = (c_v)$ indexed by $v \in M_K$ with all but finitely many $c_v = 1$ such that for each $v \in M_K$ there is an $x \in K_v^\times$ for which $c_v = \|x\|_v$. For each M_K-divisor c we define the set

$$L(c) := \{x \in K : \|x\|_v \le c_v \text{ for all } v \in M_K\}.$$

(a) Let E/F be a finite Galois extension. Prove E is a global field if and only if F is.

(b) Extend your proof of (a) to all finite extensions E/F.

(c) Prove that M_K is infinite but contains only finitely many archimedean places.

(d) Assume K has an archimedean place. Prove that $L(c)$ is finite for every M_K-divisor c (we proved this in class for number fields, but here K is a global field as defined above).

(e) Extend your proof of (e) to the case where K has no archimedean places.

(f) Prove that if M_K contains an archimedean place then K is a finite extension of $\mathbb{Q}$ (hint: show $\mathbb{Q} \subseteq K$ and use (b) to show that $K/\mathbb{Q}$ is a finite extension).

(g) Prove that if M_K does not contain an archimedean place then K is a finite extension of $\mathbb{F}_q(t)$ for some finite field $\mathbb{F}_q$ (hint: by choosing an appropriate M_K-divisor c, show that $L(c)$ is a finite field $k \subseteq K$ and that every $t \in K - k$ is transcendental over k; then show that K is a finite extension of $k(t)$).

(h) In your proofs of (a)-(g) above, where did you use the fact that the completions of K are local fields? Show that if K has a product formula and K_v is a local field for any place $v \in M_K$ then K_v is a local field for every place $v \in M_K$ (so we could weaken our definition of a global field to only require one K_v to be a local field). Are there fields with a product formula for which no completion is a local field?

Problem 2. A non-solvable quintic extension (32 points)

Let $f(x) := x^5 - x + 1$, let $K := \mathbb{Q}[x]/(f) =: \mathbb{Q}[\alpha]$ and let L be the splitting field of f.

(a) Prove that f is irreducible in $\mathbb{Q}[x]$, thus K is number field. Determine the number of real and complex places of K, and the structure of $\mathcal{O}_K^\times$ as a finitely generated abelian group (both torsion and free parts).

(b) Prove that the ring of integers of K is $\mathcal{O}_K := \mathbb{Z}[\alpha]$ and compute $\operatorname{disc} \mathcal{O}_K$, which you should find is squarefree. Use this to prove that for each prime p dividing $\operatorname{disc} \mathcal{O}_K$ exactly one of $\mathfrak{q}|p$ is ramified, and it has ramification index $e_\mathfrak{q} = 2$ and residue field degree $f_\mathfrak{q} = 1$. Conclude that $K/\mathbb{Q}$ is tamely ramified.

(c) Using the fact that any extension of local fields has a unique maximal unramified subextension, prove that for any monic irreducible polynomial $g \in \mathbb{Z}[x]$ the splitting field of g is unramified at all primes that do not divide the discriminant of g. Conclude that $L/\mathbb{Q}$ is unramified away from primes dividing $\operatorname{disc}\mathcal{O}_K$ and tamely ramified everywhere, and show that every prime dividing $\operatorname{disc}\mathcal{O}_K$ has ramification index 2. Use this to compute $\operatorname{disc}\mathcal{O}_L$.

(d) Show that $\mathcal{O}_K$ has no ideals of norm 2 or 3 and use this to prove that the class group of $\mathcal{O}_K$ is trivial and therefore $\mathcal{O}_K$ is a PID.

(e) Prove that $\operatorname{Gal}(L/\mathbb{Q}) \simeq S_5$, and that it is generated by the Frobenius elements σ_2 and σ_5 (S_5 is nonabelian, so these are conjugacy class representatives).

Problem 3. Some applications of the Minkowski bound (32 points)

For a number field K, let

$$m_K := \frac{n!}{n^n}\left(\frac{4}{\pi}\right)^s \sqrt{|D_K|}$$

denote the Minkowski constant and let $h_K := \#\operatorname{cl}\mathcal{O}_K$ denote the class number. You may wish to use a computer to help with some of the calculations involved in this problem, but if you do so, please describe your computations (preferably in words or pseudo-code).

(a) Prove that if O_K contains no prime ideals $\mathfrak{p}$ of prime norm $\mathrm{N}(\mathfrak{p}) \leq m_K$ then $h_K = 1$, and that when K is an imaginary quadratic field then the converse holds.

(b) Let K be an imaginary quadratic field. Show that if $h_K = 1$ then $|D_K|$ is a power of 2 or a prime congruent to 3 mod 4, and then determine all imaginary quadratic fields K of class number one with $|D_K| < 200$ (this is in fact all of them).

(c) Prove that there are no totally real cubic fields of discriminant less than 20 and that every real cubic field K with $D_K < M$ can be written as $K = \mathbb{Q}(\alpha)$, where α is an algebraic integer with minimal polynomial $x^3 + ax^2 + bx + c$ whose coefficients satisfy $|a| < \sqrt{M} + 2$, $|b| < 2\sqrt{M} + 1$, and $|c| < \sqrt{M}$.

(d) Prove that for any prime p there is at most one totally real cubic field K that is ramified only at p. Determine the primes $p < 10$ for which this occurs and give a defining polynomial for each field that arises. You may wish to use the formula

$$\operatorname{disc}(x^3 + ax^2 + bx + c) = -4a^3c + a^2b^2 + 18abc - 4b^3 - 27c^2.$$

(e) Prove that a totally real cubic field ramified at only one prime is Galois if and only if it is totally ramified at that prime.

Problem 4. Binary quadratic forms (32 points)

A *binary quadratic form* is a homogeneous polynomial of degree 2 in two variables:

$$f(x, y) = ax^2 + bxy + cy^2,$$

which we identify by the triple (a, b, c). We are interested in a specific set of binary quadratic forms, namely, those that are *integral* ($a, b, c \in \mathbb{Z}$), *primitive* ($\gcd(a, b, c) = 1$), and *positive definite* ($b^2 - 4ac < 0$ and $a > 0$). To simplify matters, in this problem we shall use the word *form* to refer to an integral, primitive, positive definite, binary quadratic form.

The *discriminant* of a form is the integer $D := b^2 - 4ac < 0$; although this is not necessary, for the sake of simplicity we restrict our attention to *fundamental discriminants* D, those for which D is the discriminant of $\mathbb{Q}[x]/(f(x,1)) = \mathbb{Q}(\sqrt{D})$.

We define the (principal) *root* $\tau := \tau(f)$ of a form $f = (a,b,c)$ to be the unique root of $f(x,1)$ in the upper half plane $\mathbb{H} := \{z \in \mathbb{C} : \operatorname{im} z > 0\}$:

$$\tau = \frac{-b + \sqrt{D}}{2a}.$$

Let $F(D)$ denote the set of forms with fundamental discriminant D, let $K = \mathbb{Q}(\sqrt{D})$, and let $\mathcal{O}_K$ be the ring of integers of K.

(a) For each form $f = (a,b,c) \in F(D)$ with root τ, define $I(f) := a\mathbb{Z} + a\tau\mathbb{Z}$. Prove that $\mathcal{O}_K = \mathbb{Z} + a\tau\mathbb{Z}$ and that $I(f)$ is a nonzero $\mathcal{O}_K$-ideal of norm a. Show that every nonzero fractional ideal J lies in the ideal class of $I(f)$ for some $f = (a,b,c) \in F(D)$.

(b) For each $\gamma = \left(\begin{smallmatrix} s & t \\ u & v \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and $f(x,y) \in F(D)$ define

$$f^\gamma(x,y) := f(sx + ty,\ ux + vy).$$

Show that $f^\gamma \in F(D)$, and that this defines a right group action of $\mathrm{SL}_2(\mathbb{Z})$ on the set $F(D)$ (this means $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ acts trivially and $f^{(\gamma_1\gamma_2)} = (f^{\gamma_1})^{\gamma_2}$ for all $\gamma_1, \gamma_2 \in \mathrm{SL}_2(\mathbb{Z})$).

Call two forms $f, g \in F(D)$ *equivalent* if $g = f^\gamma$ for some $\gamma \in \mathrm{SL}_2(\mathbb{Z})$.

(c) Prove that two forms $f, g \in F(D)$ are equivalent if and only if $I(f)$ and $I(g)$ represent the same ideal class in $\mathrm{cl}(\mathcal{O}_K)$.

Recall that $\mathrm{SL}_2(\mathbb{Z})$ acts on the upper half plane $\mathbb{H}$ (on the left) via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau := \frac{a\tau + b}{c\tau + d},$$

and that the set

$$\mathcal{F} = \left\{\tau \in \mathbb{H} : \operatorname{re}(\tau) \in [-1/2, 0] \text{ and } |\tau| \geq 1\right\} \cup \left\{\tau \in \mathbb{H} : \operatorname{re}(\tau) \in (0, 1/2) \text{ and } |\tau| > 1\right\}$$

is a fundamental region for $\mathbb{H}$ modulo the $\mathrm{SL}_2(\mathbb{Z})$-action. A form $f = (a,b,c)$ is said to be *reduced* if

$$-a < b \leq a < c \qquad \text{or} \qquad 0 \leq b \leq a = c.$$

(d) Prove that two forms are equivalent if and only if their roots lie in the same $\mathrm{SL}_2(\mathbb{Z})$-orbit, and that a form is reduced if and only if its root lies in $\mathcal{F}$. Conclude that each equivalence class in $F(D)$ contains exactly one reduced form.

(e) Prove that if f is reduced then $a \leq \sqrt{|D|/3}$; conclude that $\#\,\mathrm{cl}(\mathcal{O}_K) \leq |D|/3$.

Remark. One can define (as Gauss did) a composition law for forms corresponding to multiplication of ideals; the product of reduced forms need not be reduced, so one also needs an algorithm to reduce a given form, but this is straight-forward. This makes it possible to compute the group operation in $\mathrm{cl}(\mathcal{O}_K)$ using composition and reduction of forms. One can then use generic group algorithms (such as the baby-step giant-step method) to compute $\#\,\mathrm{cl}(\mathcal{O}_K)$ much more efficiently than by simply enumerating reduced forms; one can also compute the group structure of $\mathrm{cl}(\mathcal{O}_K)$ not just its cardinality.

Problem 5. Unit groups of real quadratic fields (64 points)

A (simple) *continued fraction* is a (possibly infinite) expression of the form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}}$$

with $a_i \in \mathbb{Z}$ and $a_i > 0$ for $i > 0$. They are more compactly written as $(a_0; a_1, a_2, \ldots)$. For any $t \in \mathbb{R}_{>0}$ the *continued fraction expansion* of t is defined recursively via

$$t_0 := t, \qquad a_n := \lfloor t_n \rfloor, \qquad t_{n+1} := 1/(t_n - a_n),$$

where the sequence $a(t) := (a_0; a_1, a_2, \ldots)$ terminates at a_n if $t_n = a_n$, in which case we say that $a(t) = (a_0; a_1, \ldots, a_n)$ is *finite*, and otherwise call $a(t) = (a_0; a_1, a_2, \ldots)$ *infinite*. If $a(t)$ is infinite and there exists $\ell \in \mathbb{Z}_{>0}$ such that $a_{n+\ell} = a_n$ for all sufficiently large n, we say that $a(t)$ is *periodic* and call the least such integer $\ell := \ell(t)$ the *period* of $a(t)$.

Given a continued fraction $a(t) := (a_0; a_1, a_2, \ldots)$ define the sequences of integers (P_n) and (Q_n) by

$$\begin{aligned} P_{-2} &= 0, & P_{-1} &= 1, & P_n &= a_n P_{n-1} + P_{n-2}; \\ Q_{-2} &= 1, & Q_{-1} &= 0, & Q_n &= a_n Q_{n-1} + Q_{n-2}. \end{aligned}$$

(a) Prove that $a(t)$ is finite if and only if $t \in \mathbb{Q}$, in which case $t = a(t)$.

(b) Prove that if $a(t) = (a_0; a_1, a_2, \ldots)$ is infinite then $(a_0; a_1, \ldots, a_n) = P_n/Q_n$ and $t_n = (a_n; a_{n+1}, a_{n+2}, \ldots)$ for all $n \geq 0$; conclude that $t = \lim_{n\to\infty} P_n/Q_n = a(t)$.

(c) Prove that $a(t)$ is periodic if and only if $\mathbb{Q}(t)$ is a real quadratic field.

Now let $D > 0$ be a squarefree integer that is not congruent to $1 \bmod 4$ and let $K = \mathbb{Q}(\sqrt{D})$. As shown on previous problem sets, $\mathcal{O}_K = \mathbb{Z}[\sqrt{D}]$, and it is clear that $(\mathcal{O}_K^\times)_{\mathrm{tors}} = \{\pm 1\}$. Every $\alpha = x + y\sqrt{D} \in \mathcal{O}_K^\times$ has $N(\alpha) = \pm 1$, and (x, y) is thus an (integer) solution to the *Pell equation*

$$X^2 - DY^2 = \pm 1 \tag{1}$$

(d) Prove that if (x_1, y_1) and (x_2, y_2) are solutions to (1) with $x_1, y_1, x_2, y_2 \in \mathbb{Z}_{>0}$ then $x_1 + y_1\sqrt{D} < x_2 + y_2\sqrt{D}$ if and only if $x_1 < x_2$ and $y_1 \leq y_2$. Conclude that the fundamental unit $\epsilon = x + y\sqrt{D}$ of $\mathcal{O}_K^\times$ is the unique solution (x, y) to (1) with $x, y > 0$ and x minimal.

(e) Let $a(\sqrt{D}) = (a_0; a_1, a_2, \ldots)$, and define t_n, P_n, Q_n as above. Prove that

$$P_{n-1}Q_{n-2} - P_{n-2}Q_{n-1} = \pm 1 \qquad \text{and} \qquad \frac{t_n P_{n-1} + P_{n-2}}{t_n Q_{n-1} + Q_{n-2}} = \sqrt{D}$$

for all $n \geq 0$. Use this to show that $(P_{k\ell-1}, Q_{k\ell-1})$ is a solution to (1) for all $k \geq 0$, where $\ell := \ell(\sqrt{D})$. Conclude that $\epsilon = P_{\ell-1} + Q_{\ell-1}\sqrt{D}$.

(f) Compute the fundamental unit ϵ for each of the real quadratic fields $\mathbb{Q}(\sqrt{19})$, $\mathbb{Q}(\sqrt{570})$, and $\mathbb{Q}(\sqrt{571})$; in each case give the period $\ell(\sqrt{D})$ as well as ϵ.

Problem 6. S-class groups and S-unit groups (32 points)

Let K be a number field with ring of integers $\mathcal{O}_K$, and let S be a finite set of places of K including all archimedean places. Define the *ring of S-integers* $\mathcal{O}_{K,S}$ as the set

$$\mathcal{O}_{K,S} := \{x \in K : v_{\mathfrak{p}}(x) \geq 0 \text{ for all } \mathfrak{p} \notin S\}.$$

(a) Prove that $\mathcal{O}_{K,S}$ is a Dedekind domain containing $\mathcal{O}_K$ with the same fraction field.

(b) Define a natural homomorphism between $\operatorname{cl}\mathcal{O}_{K,S}$ and $\operatorname{cl}\mathcal{O}_K$ (it is up to you to determine which direction it should go) and use it to prove that $\operatorname{cl}\mathcal{O}_{K,S}$ is finite.

(c) Prove that there is a finite set S for which $\mathcal{O}_{K,S}$ is a PID and give an explicit upper bound on $\#S$ that depends only on $n = [K : \mathbb{Q}]$ and $|\operatorname{disc}\mathcal{O}_K|$.

(d) Prove the *S-unit theorem*: $\mathcal{O}_{K,S}^{\times}$ is a finitely generated abelian group of rank $\#S-1$.

Problem 7. Survey (4 points)

Complete the following survey by rating each problem you attempted on a scale of 1 to 10 according to how interesting you found it (1 = "mind-numbing," 10 = "mind-blowing"), and how difficult you found it (1 = "trivial," 10 = "brutal"). Also estimate the amount of time you spent on each problem to the nearest half hour.

	Interest	Difficulty	Time Spent
Problem 1			
Problem 2			
Problem 3			
Problem 4			
Problem 5			
Problem 6			

Please rate each of the following lectures that you attended, according to the quality of the material (1="useless", 10="fascinating"), the quality of the presentation (1="epic fail", 10="perfection"), the pace (1="way too slow", 10="way too fast", 5="just right") and the novelty of the material to you (1="old hat", 10="all new").

Date	Lecture Topic	Material	Presentation	Pace	Novelty
10/25	Minkowski bound				
10/29	Dirichlet's unit theorem				
11/3	The prime number theorem				

Please feel free to record any additional comments you have on the problem sets and the lectures, in particular, ways in which they might be improved.

References

[1] Emil Artin and George Whaples, *Axiomatic characterization of fields by the product formula for valuations*, Bull. Amer. Math. Soc. **51** (1945), 469–492.