*Arnold#Thom Gradient Conjecture for the Arrival Time*

# ARNOLD-THOM GRADIENT CONJECTURE FOR THE ARRIVAL TIME

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Abstract. We prove conjectures of René Thom and Vladimir Arnold for $C^2$ solutions to the degenerate elliptic equation that is the level set equation for motion by mean curvature.

We believe these results are the first instances of a general principle: Solutions of many degenerate equations behave as if they are analytic, even when they are not. If so, this would explain various conjectured phenomena.

## 0. Introduction

By a classical result, solutions of analytic elliptic PDEs, like the Laplace equation, are analytic. Many important equations are degenerate elliptic and solutions have much lower regularity. Still, one may hope that solutions share properties of analytic functions. On the surface, such properties seem to be purely analytic; however, they turn out to be closely connected to important open problems in geometry.

For an analytic function, Lojasiewicz, [L1], proved that any gradient flow line with a limit point has finite length and, thus, limits to a unique critical point. This result has since been known as *Lojasiewicz's theorem*. The proof relied on two *Lojasiewicz inequalities* for analytic functions that had also been used to prove two conjectures around 1960: Laurent Schwarz's division conjecture in 1959 in [L3] and a conjecture of Whitney about singularities in 1963 in [L4]. Around the same time, in 1958, Hörmander proved a special case of Schwarz's division conjecture by establishing Lojasiewicz's first inequality for polynomials, [Hö].

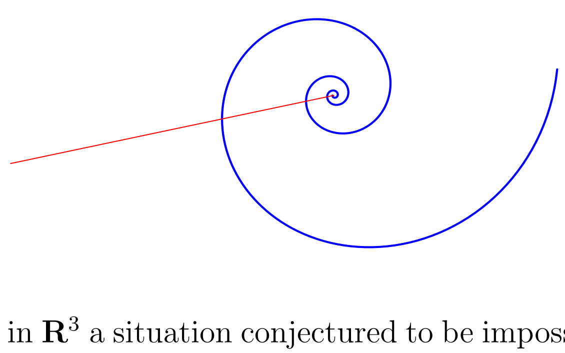

**Figure** illustrates in $\mathbf{R}^3$ a situation conjectured to be impossible. The Arnold-Thom conjecture asserts that a blue integral curve does not spiral as it approaches the critical set (illustrated in red, orthogonal to the plane where the curve spirals).

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Around 1972, Thom, [T], [L2], [Ku], [A], [G], conjectured a strengthening of Lojasiewicz's theorem, asserting that each gradient flow line of an analytic function approaches its limit from a unique limiting direction:

**Conjecture 0.1.** If a gradient flow line $x(t)$ for an analytic function has a limit point, then the limit of secants $\lim_{t\to\infty} \frac{x(t)-x_\infty}{|x(t)-x_\infty|}$ exists.

This conjecture arose in Thom's work on catastrophe theory and singularity theory and became known as *Thom's gradient conjecture*. The conjecture was finally proven in 2000 by Kurdyka, Mostowski, and Parusinski in [KMP], but the following stronger conjecture remains open (see page 282 in Arnold's problem list, [A]):

**Conjecture 0.2.** If a gradient flow line $x(t)$ for an analytic function has a limit point, then the limit of the unit tangents $\frac{x'(t)}{|x'(t)|}$ exists.

It is easy to see that if $\lim_{t\to\infty} \frac{x'(t)}{|x'(t)|}$ exists, then so does $\lim_{t\to\infty} \frac{x(t)-x_\infty}{|x(t)-x_\infty|}$. It follows that the *Arnold-Thom conjecture* 0.2 implies Thom's gradient conjecture 0.1. Easy examples show that the Lojasiewicz theorem, the Lojasiewicz inequalities, and both Conjectures 0.1 and 0.2 fail for general smooth functions; see, e.g., fig. 3.5 in [Si] or fig. 1 in [CM8].

Analytic functions play an important role in differential equations since solutions of analytic elliptic equations are themselves analytic. In many instances, the properties that come from being analytic are more important than analyticity itself. We will show that solutions of an important degenerate elliptic equation have analytic properties even though solutions are not even $C^3$. Namely, we will show that Conjectures 0.1, 0.2 hold for solutions of the classical degenerate elliptic equation, known as the *arrival time equation*,

$$-1 = |\nabla u| \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) . \tag{0.3}$$

Here $u$ is defined on a compact connected subset of $\mathbf{R}^{n+1}$ with smooth mean convex boundary. Equation (0.3) is the prototype for a family of equations, see, e.g., [OsSe], used for tracking moving interfaces in complex situations. These equations have been instrumental in applications, including semiconductor processing, fluid mechanics, medical imaging, computer graphics, and material sciences.

Even though solutions of (0.3) are a priori only in the viscosity sense, they are always twice differentiable by [CM5], though not necessarily $C^2$; see [CM6], [H2], [I], [KS]. Even when a solution is $C^2$, it still might not be $C^3$, Sesum, [S], let alone analytic as in Lojasiewicz's theorem. However, solutions behave like analytic functions are expected to:

**Theorem 0.4.** The Arnold-Thom conjecture holds for $C^2$ solutions of (0.3).

The geometric meaning of (0.3) is that the level sets $u^{-1}(t)$ are mean convex and evolve by mean curvature flow. One says that $u$ is the *arrival time* since $u(x)$ is the time the hypersurfaces $u^{-1}(t)$ arrive at $x$ under the mean curvature flow; see Chen-Giga-Goto, [ChGG], Evans-Spruck, [ES], Osher-Sethian, [OsSe], and [CM3]. Geometrically, singular points for the flow correspond to critical points for $u$.

We conjecture that even for solutions that are not $C^2$, but merely twice differentiable, the Arnold-Thom conjecture holds:

**Conjecture 0.5.** Lojasiewicz's inequalities and the Arnold-Thom conjecture hold for all solutions of (0.3).

If this conjecture holds, then the gradient Lojasiewicz inequality would imply that the flow is singular at only finitely many times as has been conjectured, [W3], [AAG], [Wa], [M].

One of the important ingredients in the proof of Theorem 0.4 is an essentially sharp rate of convergence for the rescaled mean curvature flow; this will be given in Proposition 2.4 below. This rate is not fast enough to directly show the convergence of unit tangents, which is closely related to the existence of a non-integrable kernel of the linearized operator. However, we overcome this by a careful analysis of this kernel.

We believe that the principle that solutions of degenerate equations behave as though they are analytic, even when they are not, should be quite general. For instance, there should be versions for other flows, including Ricci flow; cf. [CM9].

## 1. Lojasiewicz theorem for the arrival time

A function $v$ satisfies a gradient Lojasiewicz inequality near a point $y$ (see, e.g., [CM8]) if there exist $p > 1$, $C$ and a neighborhood of $y$ (all depending on $v$ and $y$) so that

$$|v - v(y)| \leq C\,|\nabla v|^p\,. \tag{1.1}$$

This is nontrivial only if $y$ is a critical point. If $\nabla v(y) = 0$ and $v$ satisfies (1.1), then $v(y)$ is the only critical value in this neighborhood (this applies for any $p > 0$).

In this section, we show (1.1) with $p = 2$ for a $C^2$ solution $u$ of (0.3). When $u$ is not $C^2$, then (1.1) can fail for any fixed $p > 1$. Namely, for any odd integer $m \geq 3$, Angenent and Velázquez construct rotationally symmetric examples in [AV] where $|u - u(y)| \approx |\nabla u|^{\frac{m}{m-1}}$ for a sequence of points tending to $y$. The examples in [AV] were constructed to analyze so-called type II singularities that were previously observed by Hamilton and proven rigorously to exist by Altschuler-Angenent-Giga, [AAG]; cf. also [GK].

From now on, $u$ will be $C^2$. To prove (1.1), we first recall the properties that we will use. Namely, if $\mathcal{S} = \{x \mid \nabla u(x) = 0\}$ denotes the critical set,[1] then [CM5] and [CM6] give:

- ($\mathcal{S}$1) $\mathcal{S}$ is a closed embedded connected $k$-dimensional $C^1$ submanifold whose tangent space is the kernel of $\text{Hess}_u$. Moreover, $\mathcal{S}$ lies in the interior of the region where $u$ is defined.
- ($\mathcal{S}$2) If $q \in \mathcal{S}$, then $\text{Hess}_u(q) = -\frac{1}{n-k}\,\Pi$ and $\Delta u(q) = -\frac{n+1-k}{n-k}$, where $\Pi$ is orthogonal projection onto the orthogonal complement of the kernel.

After subtracting a constant, we can assume that $\sup u = 0$.

Using these properties, the next theorem gives the gradient Lojasiewicz inequality.

**Theorem 1.2.** We have that $u(\mathcal{S}) = 0$ and

$$\frac{|\nabla u|^2}{-u} \to \frac{2}{n-k} \text{ as } u \to 0\,. \tag{1.3}$$

In particular, there exists $C > 0$ so that $C^{-1}\,|\nabla u|^2 \leq -u \leq C\,|\nabla u|^2$.

[1]The flow is smooth away from the singular set $\mathcal{S}$ consisting of cylindrical singularities; see, [W1], [W2], [H1], [HS1], [HS2], [HaK], [An]; cf. [B], [CM1]. See also [CM4].

*Proof.* The boundary of the domain is smooth and mean convex, so $\nabla u \neq 0$ on the boundary. The normalization $\sup u = 0$ implies that $u = 0$ at any maximum. Thus, there is at least one point in $\mathcal{S}$ with $u = 0$. By ($\mathcal{S}1$), $u$ is constant on $\mathcal{S}$ and we conclude that $u(\mathcal{S}) = 0$.

Given $\epsilon > 0$, choose $\delta > 0$ so that $|p - q| < \delta$ implies that $|u_{ij}(p) - u_{ij}(q)| < \epsilon$ and, moreover, so that the $\delta$-tubular neighborhood of $\mathcal{S}$ does not intersect the boundary of the domain. Let $q$ be any point with $\text{dist}(q, \mathcal{S}) < \delta$ and then let $p$ be a point in the compact set $\mathcal{S}$ that minimizes the distance to $q$ (note that $p$ might not be unique). Since $\mathcal{S}$ is $C^1$, the minimizing property implies that the vector $q - p$ is orthogonal to the tangent space to $\mathcal{S}$. In particular, ($\mathcal{S}2$) implies that

$$\text{Hess}_u(p)(q-p) = -\frac{q-p}{n-k}\,. \tag{1.4}$$

Given $t \in (0, 1]$, the fundamental theorem of calculus gives

$$\nabla u(p + t(q-p)) = \int_0^t \text{Hess}_u(p + s(q-p))(q-p)\, ds\,. \tag{1.5}$$

Combining this with (1.4) and the continuity of the Hessian gives

$$\left| \nabla u(p + t(q-p)) + t\, \frac{q-p}{n-k} \right| \leq \epsilon\, t\, |q-p|\,. \tag{1.6}$$

Using this at $t = 1$ gives

$$\left| \nabla u(q) + \frac{q-p}{n-k} \right| \leq \epsilon\, |q-p|\,. \tag{1.7}$$

Using the fundamental theorem of calculus on $u$ this time, (1.6) gives that

$$\left| u(q) + \frac{|p-q|^2}{2(n-k)} \right| \leq \int_0^1 \left| \langle \nabla u(p + t(q-p)) + t\, \frac{q-p}{n-k}, q-p \rangle \right| dt \leq \frac{\epsilon}{2}\, |p-q|^2\,. \tag{1.8}$$

Since $\epsilon > 0$ is arbitrary, combining the last two inequalities gives (1.3).

The last claim follows from (1.3) since $\{u = 0\} = \{|\nabla u| = 0\} = \mathcal{S}$. □

The next theorem shows that the gradient flow lines of $u$ have finite length (this is the Lojasiewicz theorem for $u$), converge to points in $\mathcal{S}$, and approach $\mathcal{S}$ orthogonally. The first claims follow immediately from the gradient Lojasiewicz inequality of Theorem 1.2. Let $\Pi_{\text{axis}}$ denote orthogonal projection onto the kernel of $\text{Hess}_u$.

**Theorem 1.9.** Each flow line $\gamma$ for $\nabla u$ has finite length and limits to a point in $\mathcal{S}$. Moreover, if we parametrize $\gamma$ by $s \geq 0$ with $|\gamma_s| = 1$ and $\gamma(0) \in \mathcal{S}$, then

$$u(\gamma(s)) \approx \frac{-s^2}{2(n-k)}\,, \tag{1.10}$$

$$|\nabla u(\gamma(s))|^2 \approx \frac{s^2}{(n-k)^2}\,, \tag{1.11}$$

$$\Pi_{\text{axis}}(\gamma_s) \to 0\,. \tag{1.12}$$

In particular, for $s$ small, we have that $\gamma(s) \subset B_{2n\sqrt{-u(\gamma(s))}}(\gamma(0))$.

*Proof.* Each point lies on a flow line where $u$ is increasing and limits to 0, so $\gamma$ limits to $\mathcal{S}$. If we parametrize $\gamma$ by time $t$ (so that $u \circ \gamma(t) = t$ and $|\gamma_t| = \frac{1}{|\nabla u|}$), then the length is

$$\int_T^0 \frac{1}{|\nabla u|}\, dt \approx \sqrt{\frac{n-k}{2}} \int_T^0 \frac{1}{\sqrt{-u}}\, dt = \sqrt{\frac{n-k}{2}} \int_T^0 \frac{1}{\sqrt{-t}}\, dt = \sqrt{2\,(k-n)\,T}\,, \tag{1.13}$$

where the approximation used (1.3). In particular, the flow lines starting from $u = T$ have finite length approximately equal to $\sqrt{2\,(k-n)\,T}$. It follows that $\gamma$ has a limit $\gamma(0) \in \mathcal{S}$ as $t \to 0$ and we get the approximation (1.10). Combining (1.10) and (1.3) gives (1.11).

For $s > 0$, the arrival time equation (0.3), continuity of $\Delta u$, and ($\mathcal{S}2$) give that

$$\mathrm{Hess}_u(\gamma_s, \gamma_s) = \frac{\mathrm{Hess}_u(\nabla u, \nabla u)}{|\nabla u|^2} = \Delta u(\gamma(s)) + 1 \to \Delta u(\gamma(0)) + 1 = -\frac{1}{n-k}\,. \tag{1.14}$$

Since $\mathrm{Hess}_u \to -\frac{1}{n-k}\,\Pi$, we conclude that $\Pi_{\text{axis}}(\gamma_s) \to 0$, giving the third claim. Finally, the last claim follows from (1.10) and $|\gamma_s| = 1$. □

## 2. Reducing Theorem 0.4 to an estimate for rescaled MCF

In this section, we will reduce the Arnold-Thom conjecture to an estimate for rescaled mean curvature flow.

A one-parameter family of hypersurfaces $M_\tau$ evolves by *mean curvature flow* (or *MCF*) if each point $x(\tau)$ evolves by $\partial_\tau x = -H\,\mathbf{n}$. Here $H$ is the mean curvature and $\mathbf{n}$ a unit normal. The rescaled MCF $\Sigma_t = \frac{1}{\sqrt{-u}}\,\{x \,|\, u(x) = -\mathrm{e}^{-t}\}$ is equivalent to simultaneously running MCF and rescaling space, up to reparameterizations of time and the hypersurfaces. A one-parameter family of hypersurfaces $\Sigma_t$ flows by the *rescaled MCF* if

$$\partial_t x = -\left(H - \frac{1}{2}\,\langle x, \mathbf{n}\rangle\right)\mathbf{n}\,. \tag{2.1}$$

It will be convenient to set $\phi = H - \frac{1}{2}\,\langle x, \mathbf{n}\rangle$. The fixed points for rescaled MCF are shrinkers where $\phi = 0$; the most important examples are cylinders $\mathcal{C} = \mathbf{S}^{n-k}_{\sqrt{2(n-k)}} \times \mathbf{R}^k$ where $k = 0, \ldots, n-1$. Below, $\Pi : \mathbf{R}^{n+1} \to \mathbf{R}^{n-k+1}$ is orthogonal projection on the orthogonal complement of the axis $\mathbf{R}^k$ of the cylinder $\mathcal{C}$. The rescaled MCF is the negative gradient flow for the Gaussian area

$$F(\Sigma) \equiv \int_\Sigma \mathrm{e}^{-\frac{|x|^2}{4}}\,. \tag{2.2}$$

In particular, $F(\Sigma_t)$ is non-increasing. Define the sequence $\delta_j$ by

$$\delta_j = \sqrt{F(\Sigma_{j-1}) - F(\Sigma_{j+2})}\,. \tag{2.3}$$

As in [CM1], the entropy $\lambda(\Sigma)$ is $\sup_{t_0>0, x_0 \in \mathbf{R}^{n+1}} F(t_0 \Sigma + x_0)$. We will use the Gaussian $L^p$ norm given by $\|g\|^p_{L^p(\Sigma_t)} \equiv \int_{\Sigma_t} |g|^p\, \mathrm{e}^{-\frac{|x|^2}{4}}$.

2.1. **Summability of $\delta_j$.** As we will see in (2.18) below, $\delta_j$ bounds the distance that $\Sigma_t$ evolves from $j$ to $j+1$. Existence of $\lim_{t\to\infty}\Sigma_t$ is proven in [CM2] by showing that $\sum \delta_j < \infty$. We will need that $\delta_j$ is summable even after being raised to a power less than one:

**Proposition 2.4.** There exists $\bar{\beta} < 1$ so that

$$\sum_{j=1}^{\infty} \delta_j^{\bar{\beta}} < \infty \, . \tag{2.5}$$

*Proof.* By (6.21) and lemma 6.9 in [CM2], there exists $\rho > 1$ and $C$ so that

$$\sum_{k=j}^{\infty} \delta_k^2 \leq 3\left(F(\Sigma_{j-1}) - \lim_{t\to\infty} F(\Sigma_t)\right) \leq C\, j^{-\rho} \, . \tag{2.6}$$

Moreover, lemma 6.9 in [CM2] shows that this implies that $\sum \delta_j < \infty$.

We will show next that if $0 < q < \rho$, then

$$\sum \delta_j^2\, j^q < \infty \, . \tag{2.7}$$

To prove this, set $b_j = j^q$ and $a_j = \sum_{i=j}^{\infty} \delta_i^2$, then $a_j - a_{j+1} = \delta_j^2$ and

$$b_{j+1} - b_j = (j+1)^q - j^q \leq c\, j^{q-1} \, , \tag{2.8}$$

where $c$ depends on $q$ and we used that $j \geq 1$. Summation by parts gives

$$\begin{aligned} \sum_{j=k}^{N} \delta_j^2 j^q = \sum_{j=k}^{N} b_j(a_j - a_{j+1}) &= b_k a_k - b_N a_{N+1} + \sum_{j=k}^{N-1} a_{j+1}(b_{j+1} - b_j) \\ &\leq k^q \sum_{j=k}^{\infty} \delta_j^2 + C \sum_{j=k}^{\infty} j^{-\rho}\, j^{q-1} \, . \end{aligned} \tag{2.9}$$

This is bounded independently of $N$ since $q < \rho$, giving (2.7).

Suppose that $a > 0$. The Hölder inequality gives $\sum \delta_j^{\beta} = \sum \left(\delta_j^{\beta}\, j^a\right) j^{-a} < \infty$ if

$$\sum \delta_j^2\, j^{\frac{2a}{\beta}} + \sum j^{-\frac{2a}{2-\beta}} < \infty \, . \tag{2.10}$$

To get (2.5), we need $\beta < 1$ and $a$ so that both sums in (2.10) are finite. By (2.7), the first is finite if $\frac{2a}{\beta} < \rho$. The second is finite if $2 - \beta < 2a$. To satisfy both, we must have

$$2 - \beta < 2a < \rho\, \beta \, . \tag{2.11}$$

This is possible as long as $2 < (1+\rho)\,\beta$. Since $1 < \rho$, we can choose such a $\beta < 1$. □

2.2. **Cylindrical approximation.** The rescaled MCF $\Sigma_t$ converges to a limiting cylinder $\mathcal{C}$ by [CM2]. Thus, for each large integer $j$, $\Sigma_j$ is well-approximated by $\mathcal{C}$.

In the next proposition, we will bound the distance from $\Sigma_t$ to some cylinder $\mathcal{C}_t$ that is allowed to change with $t$. We will let $\Pi_t$ denote the projection orthogonal to axis of $\mathcal{C}_t$. The operator $\mathcal{L}$ will be the drift Laplacian on the cylinder $\mathcal{C}_t$. Property (1) collects a priori estimates for the graph function $w$, (2) shows that $w$ almost satisfies the linearized equation, (3) shows the approximating cylinders converge, and (4) gives a priori bounds on higher derivatives. We will only use (3) in this section; (1), (2) and (4) will be used later.

**Proposition 2.12.** Given $0 < \epsilon_1$ and $\beta < 1$, there exist a constant $C$ and a sequence of radii $R_j$ and cylinders $\mathcal{C}_j$ satisfying:

(1) For $t \in [j, j+1]$, $\Sigma_t$ is a graph over $B_{R_j} \cap \mathcal{C}_{j+1}$ of a function $w$ with $\|w\|_{C^4} \leq \frac{\epsilon_1}{R_j}$,

$$\|w\|^2_{W^{3,2}} + \|\phi\|_{W^{3,2}(B_{R_j})} + \mathrm{e}^{-\frac{R_j^2}{4}} \leq C\, \delta_j^{\beta}\,,$$

$$w^2 + |\nabla w|^2 + |\mathrm{Hess}_w|^2 + |\nabla \mathrm{Hess}_w|^2 + |\nabla^2 \mathrm{Hess}_w|^2 \leq C\, \delta_j^{\beta}\, \mathrm{e}^{\frac{|x|^2}{4}}\,,$$

$$\phi^2 + |\nabla \phi|^2 + |\mathrm{Hess}_\phi|^2 \leq C\, \delta_j^{2\beta}\, \mathrm{e}^{\frac{|x|^2}{4}}\,.$$

(2) The function $w$ and its Euclidean partial derivatives $w_i$ and $w_{ij}$ on $\mathcal{C}_t$ satisfy

$$|\phi - (\mathcal{L}+1)\, w| \leq C(1+R_j)(w^2 + |\nabla w|^2) + C(|w| + |\nabla w|)\, |\mathrm{Hess}_w|\,,$$

$$\left|\phi_i - \left(\mathcal{L} + \frac{1}{2}\right) w_i\right| \leq C(1+R_j)\left(|w| + |\nabla w| + |\mathrm{Hess}_w| + |\nabla \mathrm{Hess}_w|\right)\left(|w| + |\nabla w| + |\nabla w_i|\right)\,,$$

$$|\phi_{ij} - \mathcal{L}\, w_{ij}| \leq C(1+R_j)\, \left(|w| + |\nabla w| + |\mathrm{Hess}_w| + |\nabla \mathrm{Hess}_w| + |\nabla^2 \mathrm{Hess}_w|\right)^2\,.$$

(3) $|\Pi_j - \Pi_{j+1}| \leq C\, \delta_j^{\beta}$.

(4) Given any $\ell$, there exists $C_\ell$ with $|\nabla^\ell w| + |\nabla^\ell \phi| \leq C_\ell$.

*Proof.* Let $\epsilon_0 > 0$ and $\alpha$ be fixed as in the definition of $r_\ell$ on page 261 in [CM2]. We will initially find a radius $R'_j$ so that every estimate (1), (2), (3) and (4) holds except for the $C^4$ bound in (1) which we replace by $\|w\|_{C^{2,\alpha}} \leq \epsilon_0$. We will then use (1) to get the $C^4$ bound on a slightly smaller $R_j < R'_j$ with the other bounds still holding.

As in (5.2) in [CM2], define $R'_j$ by $\mathrm{e}^{-\frac{(R'_j)^2}{2}} = \delta_j^2$. Since $\Sigma_t \to \mathcal{C}$, we can assume that $\Sigma_t$ is fixed close to $\mathcal{C}$ on a large set. Theorem 5.3 in [CM2] gives $C$ and $\mu > 0$ and a cylinder $\mathcal{C}_{j+1}$ so that $B_{(1+2\mu)\, R'_j - C} \cap \Sigma_t$, for $t \in [j, j+1]$, is a graph over $\mathcal{C}_{j+1}$ of a function $w$ with $\|w\|_{C^{2,\alpha}} \leq \epsilon_0$ and, moreover, (4) holds. Furthermore, lemma 5.32 in [CM2] gives $C$ so that

$$\int_{B_{(1+\mu)R'_j} \cap \Sigma_t} |\phi|^2\, \mathrm{e}^{-\frac{|x|^2}{4}} \leq C\, \delta_j^2\,. \tag{2.13}$$

Using theorem 0.24 from [CM2], we get for any $\beta_1 < 1$ that

$$\|w\|^2_{L^2} \leq C_{\beta_0, \beta_1}\, \delta_j^{\beta_1}\,. \tag{2.14}$$

Using the higher derivative bound from (4) and interpolation (e.g., lemma $B$.1 in [CM2]), we get for any $\beta_2 < \beta_1$ that

$$\|w\|^2_{W^{3,2}} \leq C_{\beta_0, \beta_1, \beta_2}\, \delta_j^{\beta_2}\,. \tag{2.15}$$

We have now established the first part of (1). Similarly, the second two parts of (1) follow from the first part, (4) and interpolation again.

We turn next to property (2). Lemma 4.6 in [CM2] computes the nonlinear graph equation for shrinkers; using $p$ for points in $\mathcal{C}_t$, this gives

$$\phi = \hat{f}(w, \nabla w) + \langle p, V(w, \nabla w)\rangle + \langle \Phi(w, \nabla w), \mathrm{Hess}_w\rangle\,, \tag{2.16}$$

where $\hat{f}(s,y)$, $V(s,y)$ and $\Phi(s,y)$ are smooth functions for $|s|$ small. Moreover, since $|A|^2 = \frac{1}{2}$ on $\mathcal{C}_t$, the operator $\mathcal{L}+1$ is the linearized operator for the shrinker equation and lemma 4.10 in [CM2] gives that

$$|\phi - (\mathcal{L}+1)\, w| \leq C_1(w^2 + |\nabla w|^2) + C_2(|w| + |\nabla w|)\ |\text{Hess}_w|\ , \tag{2.17}$$

where $C_1 \leq C(1+|p|)$ and $C_2$ is bounded. This gives the first claim in (2). Differentiating (2.16) in a Euclidean direction $x_i$ and arguing similarly gives the second claim. Finally, differentiating (2.16) again gives the remaining claim in (2).

We next prove (3) by bounding the Gaussian distance from $\Sigma_j$ to $\Sigma_{j+1}$ by $C\,\delta_j$ and showing that $\mathcal{C}_j$ is Lipschitz in $\Sigma_j$. The first part follows since $|x_t| = |\phi|$ and

$$\int_j^{j+1} \|\phi\|_{L^1}\, dt \leq C \int_j^{j+1} \|\phi\|_{L^2}\, dt \leq C \left( \int_j^{j+1} \|\phi\|_{L^2}^2\, dt \right)^{\frac{1}{2}} \leq C\,\delta_j\,. \tag{2.18}$$

To see that $\mathcal{C}_j$ is Lipschitz in $\Sigma_j$, we need to slightly modify the proof of theorem 0.24 in [CM2]. The choice of the cylinder in [CM2] occurs on page 240 in step 1 of the proof of proposition 2.1 there. There, the $\mathbf{R}^k$ factor is determined to be the approximate kernel of $A$ at any point $p$ in a fixed ball $B_{2\sqrt{2n}}$. In [CM2], $p$ is left arbitrary – it does not effect the bounds in (1), (2) and (4) – and the $\mathbf{R}^k$ factor given by choosing any $p$ would work (all that is needed are (2.22)–(2.24) there). To make $\mathcal{C}_j$ Lipschitz in $\Sigma_j$, we will choose the $\mathbf{R}^k$ factor by averaging over the approximate kernel of $A$ for each point in the ball $B_{2\sqrt{2n}}$. The resulting $\mathbf{R}^k$ factor, and thus the cylinder, is then Lipschitz in $\Sigma_j$ as desired.

Finally, we will fix $R_j \leq R_j'$ where $\|w\|_{C^4} \leq \frac{\epsilon_1}{R_j}$ and we still have $\mathrm{e}^{-\frac{R_j^2}{4}} \leq C\,\delta_j^{\beta}$ where $C$ now also depends on $\epsilon_1$. This follows from the pointwise $C^4$ bounds in (1) and $\mathrm{e}^{-\frac{(R_j')^2}{4}} = \delta_j$. □

2.3. **Reduction.** The next theorem reduces Theorem 0.4 to an estimate for rescaled MCF.

**Theorem 2.19.** Theorem 0.4 holds if every rescaled MCF $\Sigma_t$ with $\lambda(\Sigma_t) < \infty$ that goes to a cylinder as $t \to \infty$ satisfies

$$\sum_{j=1}^{\infty} \int_j^{j+1} \left( \sup_{B_{2n} \cap \Sigma_t} |\Pi_{j+1}(\nabla H)| \right) dt < \infty\,. \tag{2.20}$$

We will prove Theorem 2.19 here and (2.20) in Section 5. Suppose, therefore, that the function $u$ and reparameterized gradient flow line $\gamma(s)$ are as in Section 1. In particular, $\gamma(s)$ is defined on $[0,\ell]$ with $|\gamma_s| = 1$ and $\gamma(0) \in \mathcal{S}$. We will show that $\gamma_s$ has a limit as $s \to 0$. The derivative of $\gamma_s = -\frac{\nabla u}{|\nabla u|}$ is

$$\gamma_{ss} = -\frac{1}{|\nabla u|}\, \left(\text{Hess}_u(\gamma_s) - \gamma_s\, \langle \text{Hess}_u(\gamma_s), \gamma_s \rangle\right) = -\frac{\left(\text{Hess}_u(\gamma_s)\right)^T}{|\nabla u|} = \nabla^T \log |\nabla u|\,, \tag{2.21}$$

where $(\cdot)^T$ is the tangential projection onto the level set of $u$.

The simplest way to prove that $\lim \gamma_s$ exists would be to show that $\int |\gamma_{ss}| < \infty$, which is related to the rate of convergence for an associated rescaled MCF. While this rate fails to give integrability of $|\gamma_{ss}|$, it does give the following:

**Lemma 2.22.** Given any $\Lambda > 1$, we have $\lim_{s \to 0} \int_s^{\Lambda\, s} |\gamma_{ss}|\, ds = 0$.

*Proof.* Using Theorem 1.9 and the fact that $\text{Hess}_u \to -\frac{1}{n-k}\,\Pi$, (2.21) implies that $s\,|\gamma_{ss}| \to 0$. The lemma follows immediately from this. □

To get around the lack of integrability, we will decompose $\gamma_s$ into two pieces - the parts tangent and orthogonal to the axis - and deal with these separately. The tangent part goes to zero by (1.12) in Theorem 1.9. We will use (2.20) to control the orthogonal part.

*Proof of Theorem 2.19.* Translate so that $\gamma(0) = 0$ and let $\bar{H} = \frac{1}{|\nabla u|}$ be the mean curvature of the level set of $u$. The mean curvature $H$ of $\Sigma_t$ at time $t = -\log(-u)$ is given by

$$\bar{\nabla} \log \bar{H} = \frac{\nabla \log H}{\sqrt{-u}} \approx \frac{\sqrt{2(n-k)}}{s}\, \nabla \log H\,. \tag{2.23}$$

Note that $u(\gamma(s))$ is decreasing and Theorem 1.9 gives $t(s) \approx -2\,\log s + \log(2(n-k))$ and

$$t'(s) = -\partial_s\, (\log(-u(\gamma(s))) = \frac{-\partial_s u(\gamma(s))}{u(\gamma(s))} \approx -\frac{2}{s}\,. \tag{2.24}$$

Given a positive integer $j$, define $s_j$ so that $t(s_j) = j$. Note that $\left|\log \frac{s_{j+1}}{s_j}\right|$ is uniformly bounded. Therefore, by Lemma 2.22, it suffices to show that $\gamma_{s_j}$ has a limit.

We can write $\gamma_{s_j} = \Pi_{\text{axis},j}(\gamma_{s_j}) + \Pi_j(\gamma_{s_j})$. We have $\Pi_{\text{axis},j}(\gamma_{s_j}) \to 0$ since $\Pi_{\text{axis},j} \to \Pi_{\text{axis}}$ and $\Pi_{\text{axis}}(\gamma_s) \to 0$. Thus, we need that $\lim_{j\to\infty} \Pi_j(\gamma_{s_j})$ exists; this will follow from

$$\sum_j \left|\Pi_j(\gamma_{s_j}) - \Pi_{j+1}(\gamma_{s_{j+1}})\right| < \infty\,. \tag{2.25}$$

Theorem 1.9 gives (for $s$ small) that $\gamma(s) \subset B_{2n\,\sqrt{-u(\gamma(s))}}$ and, thus, (2.21) gives

$$\begin{aligned} \left|\Pi_{j+1}(\gamma_{s_j}) - \Pi_{j+1}(\gamma_{s_{j+1}})\right| &\leq \int_{s_{j+1}}^{s_j} |\Pi_{j+1}(\gamma_{ss})|\, ds = \int_{s_{j+1}}^{s_j} \left|\Pi_{j+1}\left(\bar{\nabla} \log \bar{H}(\gamma(s))\right)\right|\, ds \\ &\leq C \int_{s_{j+1}}^{s_j} \sup_{B_{2n\,\sqrt{-u(\gamma(s))}}} \left|\Pi_{j+1}(\nabla \log \bar{H})\right|(\cdot, -u)\, ds\,. \end{aligned} \tag{2.26}$$

Using (2.23) and (2.24) in (2.26) and then applying Theorem 2.19 gives

$$\sum_j \left|\Pi_{j+1}(\gamma_{s_j}) - \Pi_{j+1}(\gamma_{s_{j+1}})\right| \leq C \sum_j \int_j^{j+1} \sup_{B_{2n}\cap\Sigma_t} |\Pi_{j+1}(\nabla H)|\, dt < \infty\,. \tag{2.27}$$

On the other hand, $\sum_j \left|\Pi_j(\gamma_{s_j}) - \Pi_{j+1}(\gamma_{s_j})\right| < \infty$ by (3) in Proposition 2.12 and Proposition 2.4. Therefore, the triangle inequality gives (2.25), completing the proof. □

## 3. Approximate eigenfunctions on cylinders

The key remaining point is summability of $\Pi_{j+1}(\nabla H)$. The bound for $w^2$ in (1) from Proposition 2.12 is summable by Proposition 2.4, but the bound for $w$ is not. In particular, (1) gives a bound for $\nabla H$ that is not summable. This bound for $\nabla H$ cannot be improved due to slowly growing Jacobi fields. However, these Jacobi fields do not contribute to $\Pi_{j+1}(\nabla H)$. We will show that the remainder of $w$, after we subtract these Jacobi fields, is small.

In this section, we will show that if an approximate eigenfunction $w$ on a cylinder $\mathcal{C}$ begins to grow, then it must grow rapidly. The key tool is the frequency function for the drift

Laplacian as in [CM7]; the difficulty here is handling error terms. Let $x \in \mathbf{R}^k$ be coordinates on the Euclidean factor, $f = \frac{|x|^2}{4}$, $\mathcal{L}$ the drift Laplacian $\mathcal{L} = \Delta_{\mathcal{C}} - \frac{1}{2}\nabla_x = \Delta_\theta + \mathcal{L}_{\mathbf{R}^k}$, where $\Delta_\theta$ is the Laplacian on $\mathbf{S}^{n-k}_{\sqrt{2(n-k)}}$, and $\operatorname{div}_f = \operatorname{div} - \langle \frac{x}{2}, \cdot \rangle$ the drift divergence.

In applications, $w$ will be given by Proposition 2.12 and, thus, will satisfy (1), (2) and (4) there. Thus, we will assume that $w$ is a function on $\{|x| < R\} \subset \mathcal{C}$ satisfying:

$$|(\mathcal{L}+1)w - \phi| \leq \epsilon\,(|w| + |\nabla w|) \text{ where } \phi \text{ is a function and } \frac{8}{9} < (1-3\epsilon)^3 \,, \tag{3.1}$$

$$\sup_{|x|<4n} |(\mathcal{L}+1)w| \leq \mu \,. \tag{3.2}$$

Equation (3.1) arises from $w$ satisfying a nonlinear equation $\mathcal{M}w = \phi$ and $\mathcal{L}+1$ is the linearization of $\mathcal{M}$. We will also assume that $\mu > 0$ is small and

$$\|w\|^2_{W^{3,2}} + \|\phi\|_{W^{3,2}} + \mathrm{e}^{-\frac{R^2}{4}} \leq \mu \,, \tag{3.3}$$

$$w^2(x) + |\nabla w(x)|^2 + |\mathrm{Hess}_w(x)|^2 \leq \mu\, \mathrm{e}^{\frac{|x|^2}{4}} \,. \tag{3.4}$$

We will assume that the Euclidean first derivatives $w_i$ and second derivatives $w_{ij}$ satisfy

$$\left| \left(\mathcal{L} + \frac{1}{2}\right) w_i \right| \leq |\phi_i| + \epsilon\,(|w| + |\nabla w| + |\nabla w_i|) \,, \tag{3.5}$$

$$\sup_{|x|<r} |\mathcal{L}\, w_{ij}| \leq C_r\, \mu \text{ where } C_r \text{ depends on } r \,. \tag{3.6}$$

By lemma 3.26 in [CM2], the kernel of $\mathcal{L}+1$ on the weighted Gaussian space on $\mathcal{C}$ consists of quadratic polynomials and "infinitesimal rotations" of the form

$$\tilde{w} = \sum_i a_i (x_i^2 - 2) + \sum_{i<j} a_{ij} x_i x_j + \sum_k x_k h_k(\theta) \,, \tag{3.7}$$

where $a_i, a_{ij}$ are constants and each $h_k$ is a $\Delta_\theta$-eigenfunction with eigenvalue $\frac{1}{2}$.

The next theorem quadratically approximates $w$ in $|x| \leq 3n$ by $\tilde{w}$ as in (3.7). Namely, while (3.4) gives $|w| \leq C\,\mu^{\frac{1}{2}}$, the next theorem gives $|w - \tilde{w}| \leq C\,\mu^\nu$ with $\nu \approx 1$.

**Theorem 3.8.** Given $\nu < 1$, there exists $C$, $\bar{\epsilon}$, and $\mu_0 > 0$ so that if $w$ satisfies (3.1)–(3.6) with $\mu_0 > \mu$ and $\bar{\epsilon} > \epsilon$, then there is a function $\tilde{w}$ as in (3.7) with

$$\sup_{|x|\leq 3n} |w - \tilde{w}| \leq C\,\mu^\nu \,. \tag{3.9}$$

## 3.1. **First reduction.**

**Lemma 3.10.** If $w$ satisfies (3.1)–(3.6), then there is a function $\tilde{w}$ as in (3.7) so that $v = w - \tilde{w}$ satisfies (3.1)–(3.6) and

- (A1) Each Euclidean second derivative $v_{ij}$ has $\int_{x=0} v_{ij} = 0$.
- (A2) Each Euclidean first derivative $v_i$ has $\int_{x=0} v_i\, h = 0$ for any $h$ with $\Delta_\theta h = -\frac{1}{2}\, h$.
- (A3) We have $\left|\int_{x=0} v\right| \leq \mu\, \mathrm{Vol}(x=0)$.

*Proof.* Given $\tilde{w}$ as in (3.7), the Euclidean first and second derivatives are given at $x = 0$ by

$$\tilde{w}_i = h_i(\theta),\ \tilde{w}_{ii} = 2a_i,\ \tilde{w}_{ij} = a_{ij} \text{ for } i < j \,. \tag{3.11}$$

To arrange (A1), define $a_i$ and $a_{ij}$ by

$$2a_i \int_{x=0} 1 = \int_{x=0} w_{ii} \text{ and } a_{ij} \int_{x=0} 1 = \int_{x=0} w_{ij} \text{ for } i < j\,. \tag{3.12}$$

Similarly, for (A2), let $h_i$ be the projection of $w_i$ onto the $\frac{1}{2}$-eigenspace of $\Delta_\theta$ at $x = 0$. Claim (A3) follows by integrating $(\mathcal{L} + 1)w$ at $x = 0$ and using (3.2) and (A1). □

For a function $v$, we let $\text{Hess}_v^x = \frac{\partial^2 v}{\partial x_i \partial x_j}$ denote its Euclidean Hessian.

**Corollary 3.13.** Given $\beta > 2$, there exists $C$ so that if $v$ satisfies (3.1)–(3.6) and (A1)–(A3), then

$$\sup_{|x|<3n} \left( |v|^\beta + |\nabla_{\mathbf{R}^k} v|^\beta + |\text{Hess}_v^x|^\beta \right) \leq C\,\mu^2 + C \int_{|x|<4n} |\text{Hess}_v^x|^2\,. \tag{3.14}$$

Similarly, given $\beta > 2$ and $r > 3n$, there exists $C_{\beta,r}$ so that

$$\sup_{|x|<r} \left( |v|^\beta + |\nabla_{\mathbf{R}^k} v|^\beta + |\text{Hess}_v^x|^\beta \right) \leq C_{\beta,r}\,\mu^2 + C_{\beta,r} \int_{|x|<r+1} |\text{Hess}_v^x|^2\,. \tag{3.15}$$

*Proof.* We will prove (3.14); (3.15) follows similarly. Set $\delta^2 = \mu^2 + \int_{|x|<4n} |\text{Hess}_v^x|^2$. Since we have uniform higher derivative bounds on $v$, interpolation implies that all norms are equivalent if we go to any worse power. Thus, given any $\beta_1 < 1$, (3.2) gives

$$\|\text{Hess}_v^x\|_{C^2} + \|(\mathcal{L} + 1)v\|_{C^2} \leq C_1\,\delta^{\beta_1}\,, \tag{3.16}$$

where $C_1$ depends on $\beta_1$. It follows that $|(\Delta_\theta + 1)v(\theta, 0)| \leq C_1\,\delta^{\beta_1}$. Since $\Delta_\theta + 1$ is invertible (lemma 2.5 in [CM2]), this (and interpolation again) gives for any $\beta_2 < \beta_1$ that

$$|v(\theta, 0)| \leq C_2\,\delta^{\beta_2}\,. \tag{3.17}$$

Given a Euclidean first derivative $v_i$, (3.16) gives that

$$\left| \left( \Delta_\theta + \frac{1}{2} \right) v_i(\theta, 0) \right| \leq C_1\,\delta^{\beta_1}\,. \tag{3.18}$$

The operator $\left(\Delta_\theta + \frac{1}{2}\right)$ is not invertible, but (A2) implies that $v_i(\theta, 0)$ is orthogonal to the kernel so we get (using interpolation again) that $|v_i(\theta, 0)| \leq C_2\,\delta^{\beta_2}$. The bound on $v_i$ at $x = 0$ and the Hessian bound give a bound on $v_i$ everywhere. Integrating this and using (3.17) gives the desired pointwise bound on $v$, completing the proof. □

## 3.2. **The frequency.**

Given a function $u$ on $\mathcal{C}$, define $I$ and $D$ by[2]

$$I(r) = r^{1-k} \int_{|x|=r} u^2\,, \tag{3.19}$$

$$D(r) = r^{2-k} \int_{|x|=r} u\,u_r = \mathrm{e}^{\frac{r^2}{4}}\, r^{2-k} \int_{|x|<r} \left( |\nabla u|^2 + u \mathcal{L}\, u \right) \mathrm{e}^{-f}\,. \tag{3.20}$$

Here $u_r$ denotes the normal derivative of $u$ on the level set $|x| = r$. Note that $f$ is proper. It is easy to see that $I' = \frac{2D}{r}$ and $(\log I)' = \frac{2U}{r}$, where the frequency $U = \frac{D}{I}$; cf. [Be], [CM7].

[2] When $k = 1$ and the sphere is disconnected, let $r$ be signed distance and set $I(|r|) = \int_{x=r} u^2 + \int_{x=-r} u^2$.

The next theorem shows that if the growth of an approximate eigenfunction hits a certain threshhold, then it grows very rapidly. The theorem is stated for eigenvalue 1, but generalizes easily to other eigenvalues. The case $(\mathcal{L}+1)v=0$, where $\epsilon=\phi=0$, follows from [CM7].

**Theorem 3.21.** Given $r_1 > \max\{9n, 4n+64\sqrt{2}\}$, there exist $\bar{R}=\bar{R}(n,r_1)$, $C=C(n,r_1)$ so that if $v$ is a function on $\{|x|\leq R\}$ satisfying (3.1), where $\bar{R}\leq R$, and

$$2\int_{|x|<9n} v^2\,\mathrm{e}^{-f} \leq \int_{|x|<r_1} v^2\,\mathrm{e}^{-f} \text{ and } \frac{r_1^2}{16} < U(r_1)\,, \tag{3.22}$$

then for any $\Lambda\in(0,1/3)$

$$\int_{|x|<4n} v^2\,\mathrm{e}^{-f} \leq \Lambda^{-2}\,\|\phi\|_{L^2}^2 + C\,I(R)\,R^{2n+68}\,\mathrm{e}^{-\frac{(1-3\epsilon-\Lambda)\,R^2}{2(1+3\epsilon+\Lambda)^2}}\,. \tag{3.23}$$

To prove Theorem 3.8, we will find a scale $r_1$ where Theorem 3.21 applies to give that $w$ is bounded by $\mu^\nu$. To do this, we will find a long stretch where $\text{Hess}_w^x$ must grow and, thus, $w$ must also have grown. Note that $\text{Hess}_w^x$ is easier to work with since each $\mathbf{R}^k$ derivative lowers the eigenvalue by 1/2 and, thus, lowers the threshold for growth (cf. [CM7]).

The proof of Theorem 3.21 uses a modified version of the frequency. Define $E$ and $U_E$ by

$$E(r) = r^{2-k}\,\mathrm{e}^{\frac{r^2}{4}}\int_{|x|<r}\left\{|\nabla v|^2 - v^2\right\}\mathrm{e}^{-f} = D(r) - r^{2-k}\,\mathrm{e}^{\frac{r^2}{4}}\int_{|x|<r}\left(v\mathcal{L}v + v^2\right)\mathrm{e}^{-f}\,, \tag{3.24}$$

$$U_E(r) = \frac{E(r)}{I(r)}\,. \tag{3.25}$$

**Lemma 3.26.** If $E(r)>0$, then

$$\left(\log U_E\right)'(r) \geq \frac{2-k}{r} + \frac{r}{2} - \frac{r}{U_E} + \frac{U(r)}{r}\left(\frac{D(r)}{E(r)} - 2\right)\,. \tag{3.27}$$

*Proof.* The Cauchy-Schwarz inequality $\left(\int u u_r\right)^2 \leq \int u^2 \int |\nabla u|^2$ gives

$$E'(r) = \frac{2-k}{r}\,E + \frac{r}{2}\,E + r^{2-k}\int_{|x|=r}(|\nabla v|^2 - v^2) \geq \frac{2-k}{r}\,E + \frac{r}{2}\,E + \frac{UD}{r} - r\,I\,. \tag{3.28}$$

The lemma follows from this since $\frac{I'}{I} = \frac{2U}{r}$. □

The next lemma is valid for any function $v$.

**Lemma 3.29.** If $r > \bar{r} > 3n$ and

$$\int_{|x|<\bar{r}} v^2\,\mathrm{e}^{-f} \leq \int_{\bar{r}<|x|<r} v^2\,\mathrm{e}^{-f}\,, \tag{3.30}$$

then

$$\int_{|x|<r} v^2\,\mathrm{e}^{-f} \leq \frac{32}{\bar{r}^2 - 8n}\int_{|x|<r} |\nabla v|^2\,\mathrm{e}^{-f}\,. \tag{3.31}$$

*Proof.* Since $\mathcal{L}\,|x|^2 = 2k - |x|^2$, we have $\operatorname{div}_f\,(v^2\,x) = 2\,v\langle x, \nabla v\rangle + v^2(k - |x|^2/2)$. Using the absorbing inequality $2\,|v\langle x, \nabla v\rangle| \leq |x|^2 v^2/4 + 4\,|\nabla v|^2$, the divergence theorem gives

$$\int_{|x|<r} \left( \frac{|x|^2}{2} - k \right) v^2\,\mathrm{e}^{-f} = -r\,\mathrm{e}^{-\frac{r^2}{4}} \int_{|x|=r} v^2 + 2 \int_{|x|<r} v\,\langle x, \nabla v\rangle\,\mathrm{e}^{-f}$$

$$\leq -r\,\mathrm{e}^{-\frac{r^2}{4}} \int_{|x|=r} v^2 + \int_{|x|<r} \left( v^2\,\frac{|x|^2}{4} + 4\,|\nabla v|^2 \right)\,\mathrm{e}^{-f}\,. \tag{3.32}$$

It follows that

$$\left( \frac{\bar{r}^2}{4} - k \right) \int_{\bar{r}<|x|<r} v^2\,\mathrm{e}^{-f} - k \int_{|x|<\bar{r}} v^2\,\mathrm{e}^{-f} \leq \int_{|x|<r} \left( \frac{|x|^2}{4} - k \right) v^2\,\mathrm{e}^{-f} \leq 4 \int_{|x|<r} |\nabla v|^2\,\mathrm{e}^{-f}\,.$$

Bringing in the assumption (3.30) gives

$$\left( \frac{\bar{r}^2}{4} - 2k \right) \int_{\bar{r}<|x|<r} v^2\,\mathrm{e}^{-f} \leq 4 \int_{|x|<r} |\nabla v|^2\,\mathrm{e}^{-f}\,.$$

The lemma follows from this and using the assumption again. □

*Proof of Theorem 3.21.* We can assume that $\|\phi\|_{L^2}^2 \leq \Lambda^2 \int_{|x|<4n} v^2 \mathrm{e}^{-f}$ since otherwise we get (3.23) immediately. Therefore, given any $r \geq 4n$, (3.1) gives

$$|D - E|\,(r) \leq r^{2-k}\,\mathrm{e}^{\frac{r^2}{4}} \int_{|x|<r} \left( (\epsilon + \Lambda) v^2 + \epsilon |\nabla v|^2 \right)\,\mathrm{e}^{-f}\,. \tag{3.33}$$

Suppose now that some $r \geq 4n$ satisfies

($\star$1) $\int_{|x|<r} v^2\,\mathrm{e}^{-f} \leq \frac{1}{2} \int_{|x|<r} |\nabla v|^2\,\mathrm{e}^{-f}$.

We will use ($\star$1) to show that $D(r)$ and $E(r)$ are comparable, get a differential inequality for $U_E(r)$ and bound the ratio of the derivatives of quantities in ($\star$1). Namely, ($\star$1) gives

$$\frac{1}{2}\,r^{2-k}\,\mathrm{e}^{\frac{r^2}{4}} \int_{|x|<r} |\nabla v|^2\,\mathrm{e}^{-f} \leq r^{2-k}\,\mathrm{e}^{\frac{r^2}{4}} \int_{|x|<r} \left( |\nabla v|^2 - v^2 \right)\,\mathrm{e}^{-f} = E(r)\,. \tag{3.34}$$

Similarly, using (3.33), ($\star$1) and (3.34) gives that

$$|D - E|(r) \leq \left( \frac{3\epsilon}{2} + \frac{\Lambda}{2} \right)\,r^{2-k}\,\mathrm{e}^{\frac{r^2}{4}} \int_{|x|<r} |\nabla v|^2\,\mathrm{e}^{-f} \leq (3\epsilon + \Lambda)\;E(r)\,. \tag{3.35}$$

We conclude that $D(r)$, and thus also $I'(r)$, are also positive and

$$|U - U_E|\,(r) \leq (3\epsilon + \Lambda)\;U_E(r)\,, \tag{3.36}$$

$$(1 - 3\epsilon - \Lambda)\;U_E \leq U(r) \leq (1 + 3\epsilon + \Lambda)\;U_E\,. \tag{3.37}$$

Using this in Lemma 3.26 gives the differential inequality at $r$

$$(\log U_E)' \geq \frac{2-k}{r} + \frac{r}{2} - \frac{r}{U_E} - (1 + 3\epsilon + \Lambda)^2\;\frac{U_E}{r}\,. \tag{3.38}$$

From (3.37), the definition of $U$, and the Cauchy-Schwarz inequality, we get at $r$ that

$$(1 - 3\epsilon - \Lambda)^2\;U_E^2\,I^2 \leq U^2\,I^2 = D^2 \leq I\,r^{3-k} \int_{|x|=r} |\nabla v|^2\,. \tag{3.39}$$

Noting that $3\epsilon + \Lambda < \frac{1}{2}$, we get

$$\frac{U_E^2(r)}{4\,r^2} \leq (1-3\epsilon-\Lambda)^2 \, \frac{U_E^2(r)}{r^2} \leq \frac{\int_{|x|=r} |\nabla v|^2}{\int_{|x|=r} v^2} \, . \tag{3.40}$$

We will also need a second property (the first part is the strict form of ($\star$1)):

($\star$2) $\int_{|x|<r} v^2 \, \mathrm{e}^{-f} < \frac{1}{2} \int_{|x|<r} |\nabla v|^2 \, \mathrm{e}^{-f}$ and $\frac{r^2}{32} < U_E(r)$.

Set $r_0 = 4n + 64\sqrt{2}$. We will show that if ($\star$2) holds for some $r \geq r_0$, then it holds for all $s \geq r$. We will argue by contradiction, so suppose that $s > r$ is the first time ($\star$2) fails. Note that ($\star$2) is equivalent to $U_E(r) > \frac{r^2}{32}$ and $F(r) > 0$ where $F(r) = \int_{|x|<r} \left(\frac{1}{2}\,|\nabla v|^2 - v^2\right) \mathrm{e}^{-f}$. Since $s$ is the first time, we have $F(s) \geq 0$ (i.e., ($\star$1)), $U_E(s) - \frac{s^2}{32} \geq 0$ and

$$F(t) > 0 \text{ and } U_E(t) - \frac{t^2}{32} > 0 \text{ for all } t \in [r,s) \, . \tag{3.41}$$

We also have that at least one of $F(s)$ and $U_E(s) - \frac{s^2}{32}$ is zero. Suppose first that $F(s) = 0$ and, thus, $F'(s) \leq 0$. However, (3.40) and $U_E(s) - \frac{s^2}{32} \geq 0$ give that

$$\left(\frac{s}{64}\right)^2 \leq \frac{U_E^2(s)}{4\,s^2} \leq \frac{\int_{|x|=s} |\nabla v|^2}{\int_{|x|=s} v^2} \, . \tag{3.42}$$

However, this implies that $F'(s) > 0$ as long as $s > 64\sqrt{2}$, giving the desired contradiction in the first case. Suppose now that $U_E(s) = \frac{s^2}{32}$ and, thus, $U_E'(s) \leq \frac{s}{16}$ and

$$(\log U_E)'(s) \leq \frac{2}{s} \, . \tag{3.43}$$

On the other hand, (3.38) gives that

$$(\log U_E)'\,(s) \geq \frac{2-k}{s} + \frac{s}{2} - \frac{32}{s} - (1+3\epsilon+\Lambda)^2 \, \frac{s}{32} \geq \frac{3s}{8} - \frac{k+30}{s} \, . \tag{3.44}$$

This contradicts (3.43) since $s \geq r_0$, completing the proof of the claim.

We will now show that ($\star$2) holds for $r_1$. Using the first part of (3.22), we can apply Lemma 3.29 (with $\bar{r} = 9n$) to get

$$\int_{|x|<r_1} v^2 \, \mathrm{e}^{-f} \leq \frac{32}{81n^2 - 8n} \int_{|x|<r_1} |\nabla v|^2 \, \mathrm{e}^{-f} < \frac{1}{2} \int_{|x|<r_1} |\nabla v|^2 \, \mathrm{e}^{-f} \, , \tag{3.45}$$

where the last inequality used that $81n^2 > 8n + 64$. This gives the first part of ($\star$2); in particular, ($\star$1) holds and (3.37) gives that

$$U(r_1) \leq (1+3\epsilon+\Lambda) \; U_E(r_1) \leq \frac{3}{2}\, U_E(r_1) \, . \tag{3.46}$$

Since $\frac{r^2}{16} \leq U(r_1)$ by the second part of (3.22), the second part of ($\star$2) also holds.

We have established that ($\star$2) holds for all $r \geq r_1$, so we get the differential inequality (3.38) for $U_E$ and the equivalence (3.36) between $U$ and $U_E$. This will give the desired growth of $U$ and, thus also, $I$. We do this next. Set $\kappa = (3\epsilon + \Lambda)$. We claim that there exists $\bar{R} = \bar{R}(k, r_1) \geq r_1$ so that for all $r \geq \bar{R}$ we have

$$U_E(r) > \frac{r^2 - 2k - 68}{2(1+\kappa)^2} \, . \tag{3.47}$$

The key is that if (3.47) fails for some $r \geq r_1$, then (3.38) implies that

$$(\log U_E)' \geq \frac{r}{2} - \frac{k+30}{r} - (1+\kappa)^2 \, \frac{U_E}{r} \geq \frac{4}{r} \, . \tag{3.48}$$

On the other hand, for $r \geq 4k$, we have

$$\left( \log \frac{r^2 - 2k - 68}{2(1+\kappa)^2} \right)' = \frac{2r}{r^2 - 2k - 68} < \frac{3}{r} \, , \tag{3.49}$$

where the last inequality used that $6k + 204 < r_0^2$. Integrating (3.48) and (3.49) and using that $U_E \geq \frac{r^2}{32}$, gives an upper bound for the maximal interval where (3.47) fails. The first derivative test, (3.48), and (3.49) imply that once (3.47) holds for some $R \geq r_1$, then it also holds for all $r \geq R$. This gives the claim. Using (3.36) and (3.47), we get for $r \geq \bar{R}$ that

$$U(r) \geq (1-\kappa) U_E(r) > \frac{(1-\kappa)}{(1+\kappa)^2} \left( \frac{r^2}{2} - k - 34 \right) \, . \tag{3.50}$$

Integrating this from $\bar{R}$ to $R$ gives that

$$\log \frac{I(R)}{I(\bar{R})} = 2 \int_{\bar{R}}^{R} \frac{U(r)}{r} \, dr \geq \frac{(1-\kappa)}{(1+\kappa)^2} \left( \frac{R^2 - \bar{R}^2}{2} - (2k+68) \log \frac{R}{\bar{R}} \right) \, . \tag{3.51}$$

Since $\bar{R}$ depends only on $k$ and $r_1$, exponentiating gives $C = C(k, r_1)$ so that

$$\sup_{r_1 \leq r \leq \bar{R}} I(r) = I(\bar{R}) \leq C \, I(R) \, R^{(2k+68) \frac{1-\kappa}{(1+\kappa)^2}} \, \mathrm{e}^{-\frac{(1-\kappa)}{2(1+\kappa)^2} R^2} \, . \tag{3.52}$$

Choose $r_2 \in [r_1, 2r_1]$ that achieves the minimum of $D$ on $[r_1, 2r_1]$. Since $I' = \frac{2D}{r}$, it follows that $D(r_2) \leq I(2r_1)$. Therefore, since ($\star$1) holds for $r_2$, we have

$$r_2^{2-k} \, \mathrm{e}^{\frac{r_2^2}{4}} \int_{|x|<r_2} v^2 \, \mathrm{e}^f \leq E(r_2) \leq \frac{D(r_2)}{(1-\kappa)} \leq 2 \, I(2r_1) \, . \tag{3.53}$$

Finally, combining (3.52) and (3.53) gives (3.23). □

## 4. General frequency

In this section, we will prove Theorem 3.8 by showing that either we already have the bound on $w$ or (3.22) holds and Theorem 3.21 bounds $w$. Throughout this section, we will assume that $w$ satisfies (3.1)–(3.6) and (A1)–(A3).

The main task left is to prove the following proposition:

**Proposition 4.1.** Given $r_+ \geq 9n$, there exist $\lambda > r_+$ and $\zeta_2$ so that if $\zeta \geq \zeta_2$ and

$$\int_{|x|<4n} |\mathrm{Hess}_w^x|^2 \geq \zeta \, \mu^2 \, , \tag{4.2}$$

then there exists $r_1 \in (r_+, \lambda)$ satisfying (3.22).

Throughout this section, $C_r$ will be a constant that depends on $r$ (but not on $w$ or $\mu$) that will be allowed to change from line to line.

### 4.1. **Proof of Theorem 3.8 assuming Proposition 4.1.**

**Lemma 4.3.** Given $r > 0$, there exists $C_r$ so that

$$\left| \int_{|x|<r} \text{Hess}_w^x \right| + \left| \int_{|x|<r} w \right| \leq C_r \, \mu \, . \tag{4.4}$$

Furthermore, given any $h$ on $\mathbf{S}^{n-k}$ with $\Delta_\theta h = -\frac{1}{2}\, h$ and $\int h^2(\theta)\, d\theta = 1$, we have

$$\left| \int_{|x|<r} h \, \nabla^{\mathbf{R}^k} w \right| \leq C_r \, \mu \, . \tag{4.5}$$

*Proof.* Let $w_{ij}$ be a Euclidean second derivative and define the spherical average

$$J_{ij}(r) = r^{1-k} \int_{|x|=r} w_{ij} \, . \tag{4.6}$$

By (A1), we have $J_{ij}(0) = 0$. Note that $|\mathcal{L}\, w_{ij}| \leq C_r \, \mu$, so we have

$$|J_{ij}'|(r) \leq r^{1-k} \, \mathrm{e}^{\frac{r^2}{4}} \int_{|x|<r} |\mathcal{L}\, w_{ij}| \, \mathrm{e}^{-f} \leq C_r \, \mu \, . \tag{4.7}$$

Thus, we get that $|J_{ij}(r)| \leq C_r \, \mu$. Integrating this gives the integral bound on $\text{Hess}_w$ in (4.4) and, thus, the same bound on $\left| \int_{|x|<r} \Delta_{\mathbf{R}^k} w \right|$. The bound on $\left| \int_{|x|<r} w \right|$ follows similarly by setting $J(r) = r^{1-k} \int_{|x|=r} w$. Namely, (A3) bounds $J(0)$ and we bound $J'(r)$ by using that $\Delta_\theta w$ integrates to zero over each sphere and $\left| \int_{|x|<r} \Delta_{\mathbf{R}^k} w \right| \leq C_r \, \mu$.

To get the last claim, define a vector-valued function $J_h(r)$ by

$$J_h(r) = r^{1-k} \int_{|x|=r} h \, \nabla^{\mathbf{R}^k} w \, , \tag{4.8}$$

so that $J_h(0) = 0$ by (A2). Arguing as above and using the integral bound on the Euclidean Hessian bounds $J_h(r)$ and integrating this gives the last claim. □

**Corollary 4.9.** Given $\bar{r} > 4n$, there exist $C_{\bar{r}}$ so that

$$\int_{|x|<\bar{r}} |\text{Hess}_w^x|^2 \leq C_{\bar{r}} \, \mu^2 + C_{\bar{r}} \int_{\bar{r}<|x|<\bar{r}+1} |\text{Hess}_w^x|^2 \, . \tag{4.10}$$

*Proof.* Set $\mathcal{A} = \{\bar{r} < |x| < \bar{r}+1\}$. Let $w_{ij}$ be a Euclidean second derivative and $\eta$ a cutoff function that is one for $|x| < \bar{r}$, zero for $\bar{r}+1 < |x|$, and $|\nabla \eta| \leq 2$. Given $\delta > 0$, we get

$$\begin{aligned} \text{div}_f \, \left( \eta^2 w_{ij} \, \nabla w_{ij} \right) &= \eta^2 \left( |\nabla w_{ij}|^2 + w_{ij} \, \mathcal{L} w_{ij} \right) + 2\eta \, w_{ij} \langle \nabla w_{ij}, \nabla \eta \rangle \\ &\geq \eta^2 \left( \frac{1}{2} \, |\nabla w_{ij}|^2 - \delta \, w_{ij}^2 - \frac{1}{4\delta} \, |\mathcal{L} w_{ij}|^2 \right) - 2 |\nabla \eta|^2 \, w_{ij}^2 \, . \end{aligned} \tag{4.11}$$

We get that

$$\begin{aligned}\int_{|x|<\bar r} |\nabla w_{ij}|^2 \, \mathrm{e}^{-f} &\leq 2\,\delta \int_{|x|<\bar r} w_{ij}^2 \, \mathrm{e}^{-f} + \frac{1}{2\delta} \int_{|x|<\bar r+1} |\mathcal{L} w_{ij}|^2 \, \mathrm{e}^{-f} + (8+2\delta) \int_{\mathcal{A}} w_{ij}^2 \, \mathrm{e}^{-f} \\ &\leq 2\,\delta \int_{|x|<\bar r} w_{ij}^2 \, \mathrm{e}^{-f} + C_{\bar r} \frac{\mu^2}{\delta} + (8+2\delta) \int_{\mathcal{A}} w_{ij}^2 \, \mathrm{e}^{-f} \, . \end{aligned} \tag{4.12}$$

On the other hand, Lemma 4.3 and the Neumann Poincaré inequality give $C_{\bar r}$ so that

$$\int_{|x|<\bar r} w_{ij}^2 \, \mathrm{e}^{-f} \leq C_{\bar r} \left( \mu^2 + \int_{|x|<\bar r} |\nabla w_{ij}|^2 \, \mathrm{e}^{-f} \right) . \tag{4.13}$$

Using this to bound the first term on the right in (4.12) and taking $\delta > 0$ small enough (depending on $\bar r$), this can be absorbed. Finally, summing over $i, j$ gives the corollary. □

*Proof of Theorem 3.8.* Lemma 3.10 gives $\tilde w$ as in (3.7) so that $v = w - \tilde w$ satisfies (3.1)–(3.6) and (A1), (A2) and (A3). By Corollary 3.13, it suffices to get $\int_{|x|<4n} v^2 \leq C\, \mu^\beta$ with $\nu < \beta$.

Proposition 4.1 gives $\lambda$ and $\zeta_2$ (depending just on $n$) so that if (4.2) holds with $\zeta \geq \zeta_2$, then there exists $r_1$ satisfying (3.22) with

$$r_1 \in (\max\{9n, \sqrt{8n+256}\}, \lambda) \, . \tag{4.14}$$

We can assume that (4.2) holds with $\zeta \geq \zeta_2$ since the theorem otherwise follows from Corollary 3.13. Therefore, Theorem 3.21 applies and we get $\bar R = \bar R(n, r_1)$ and $C = C(n, r_1)$ so that for any $\Lambda \in (0, 1/3)$

$$\begin{aligned}\int_{|x|<4n} v^2 \, \mathrm{e}^{-f} &\leq \Lambda^{-2} \, \|\phi\|_{L^2}^2 + C\, I(R)\, R^{2n+68} \, \mathrm{e}^{-\frac{(1-3\epsilon-\Lambda)\, R^2}{2(1+3\epsilon+\Lambda)^2}} \\ &\leq \Lambda^{-2}\, \mu^2 + C\, R^{2n+68} \left( \mathrm{e}^{-\frac{R^2}{2}} \right)^{\frac{(1-3\epsilon-\Lambda)}{(1+3\epsilon+\Lambda)^2}} \, . \end{aligned} \tag{4.15}$$

This required $R \geq \bar R$; if $\bar R > R$, then there is a positive lower bound for $\mu$ and the theorem holds trivially. Since $\mathrm{e}^{-\frac{R^2}{2}} \leq \mu^2$, the theorem follows by taking $\epsilon, \Lambda > 0$ small enough that

$$(1 - 3\epsilon - \Lambda) > \nu \, (1 + 3\epsilon + \Lambda)^2 \, . \tag{4.16}$$

□

4.2. **Proof of Proposition 4.1.** We will get a positive lower bound for the frequency $U_2$ for $\mathrm{Hess}_w^x$ that will force $\mathrm{Hess}_w^x$ to grow very rapidly. We will then combine Poincaré and reverse Poincaré inequalities to show that $w$ itself grows rapidly as claimed. To do this, define quantities $I_2$, $D_2$ and $U_2$ for $\mathrm{Hess}_w^x$ by

$$I_2(r) = r^{1-k} \int_{|x|=r} |\mathrm{Hess}_w^x|^2 \, , \tag{4.17}$$

$D_2 = \frac{r}{2}\, I_2'$, and $U_2 = \frac{D_2}{I_2}$ so that $(\log I_2)' = \frac{2\, U_2}{r}$. Define $\psi = (\mathcal{L} + 1)\, w$ so that $\mathcal{L} w_{ij} = \psi_{ij}$. Differentiating $I_2$, we see that

$$D_2(r) = r^{2-k} \sum_{i,j} \int_{|x|=r} w_{ij} \partial_r w_{ij} = r^{2-k} \, \mathrm{e}^{\frac{r^2}{4}} \int_{|x|<r} \left( |\nabla \mathrm{Hess}_w^x|^2 + \sum_{i,j} w_{ij} \psi_{ij} \right) \mathrm{e}^{-f} \, . \tag{4.18}$$

The next two lemmas give a differential inequality for $U_2$ when $U_2 > 0$ and then establish that $U_2(r)$ is positive on an interval.

**Lemma 4.19.** If $U_2(r) > 0$, then

$$(\log U_2)'(r) \geq \frac{2-k}{r} + \frac{r}{2} - \frac{U_2}{r} - \frac{r}{U_2} \left( \frac{r^{1-k} \int_{|x|=r} \sum_{i,j} \psi_{ij}^2}{I_2} \right)^{\frac{1}{2}} . \tag{4.20}$$

*Proof.* Differentiating $D_2$ gives that

$$D_2'(r) = \frac{2-k}{r} \, D_2 + \frac{r}{2} \, D_2 + r^{2-k} \int_{|x|=r} \left( |\nabla \mathrm{Hess}_w^x|^2 + \sum_{i,j} w_{ij} \psi_{ij} \right) . \tag{4.21}$$

The first equality in (4.18) and the Cauchy-Schwarz inequality give that

$$D_2^2(r) \leq I_2(r) \, r^{3-k} \int_{|x|=r} \sum_{i,j} (\partial_r w_{ij})^2 \leq I_2(r) \, r^{3-k} \int_{|x|=r} |\nabla \mathrm{Hess}_w^x|^2 . \tag{4.22}$$

Since $D_2(r) > 0$ (by assumption), using (4.22) in (4.21) and dividing by $D_2(r)$ gives

$$(\log D_2)'(r) \geq \frac{2-k}{r} + \frac{r}{2} + \frac{U_2}{r} - \frac{r^{2-k}}{D_2} \int_{|x|=r} \left| \sum_{i,j} w_{ij} \psi_{ij} \right| . \tag{4.23}$$

The lemma follows from this and the Cauchy-Schwartz inequality since $(\log I_2)' = \frac{2U_2}{r}$. $\square$

**Lemma 4.24.** Given $\lambda > 4n$, there exists $\zeta_0$ so that if (4.2) holds for $\zeta \geq \zeta_0$, then for each $r \in (4n, 2\lambda)$ we have $U_2(r) \geq 0$ and, moreover, for $r \in (4n, 2\lambda - 1)$ there exists $c_r > 0$ so

$$\max \{ U_2(s) \, | \, s \in [r, r+1] \} > c_r . \tag{4.25}$$

*Proof.* Given $r \in (4n, 2\lambda)$, the Neumann Poincaré inequality, (4.4) and (4.2) give

$$\int_{|x|<r} |\mathrm{Hess}_w^x|^2 \leq C_r \, \mu^2 + C_r \int_{|x|<r} |\nabla \mathrm{Hess}_w^x|^2 \leq \frac{C_r}{\zeta} \int_{|x|<4n} |\mathrm{Hess}_w^x|^2 + C_r \int_{|x|<r} |\nabla \mathrm{Hess}_w^x|^2 .$$

If $\zeta$ is large enough (depending on $\lambda$), we can absorb the first term on the right to get

$$\int_{|x|<r} |\mathrm{Hess}_w^x|^2 \leq C_r \int_{|x|<r} |\nabla \mathrm{Hess}_w^x|^2 . \tag{4.26}$$

To bound the error term in (4.18), use the absorbing inequality to get for any $\delta > 0$

$$\sum_{i,j} \int_{|x|<r} |\psi_{ij} w_{ij}| \mathrm{e}^{-f} \leq \delta \int_{|x|<r} |\mathrm{Hess}_w^x|^2 \, \mathrm{e}^{-f} + \frac{C_r \, \mu^2}{\delta} . \tag{4.27}$$

Taking $\delta > 0$ small and then $\zeta$ even larger, the last two inequalities give that

$$\sum_{i,j} \int_{|x|<r} |\psi_{ij} w_{ij}| \mathrm{e}^{-f} \leq \frac{1}{2} \int_{|x|<r} |\nabla \mathrm{Hess}_w^x|^2 \mathrm{e}^{-f} . \tag{4.28}$$

Using this in (4.18), we conclude that

$$D_2(r) \geq \frac{r^{2-k}}{2} \, \mathrm{e}^{\frac{r^2}{4}} \int_{|x|<r} |\nabla \mathrm{Hess}_w^x|^2 \, \mathrm{e}^{-f} \geq C_r \int_{|x|<r} |\mathrm{Hess}_w^x|^2 = C_r' \int_0^r s^{k-1} \, I_2(s) \, ds . \tag{4.29}$$

In particular, $U_2(r) \geq 0$. Moreover, we get for $r \in (4n, 2\lambda - 1)$ that

$$D_2(r+1) \geq C_r \, I_2(r)\,. \tag{4.30}$$

Note that Corollary 4.9 implies that $I_2(r) > 0$. We have either $I_2(r+1) \leq 2\, I_2(r)$ or $2\, I_2(r) < I_2(r+1)$; the claim (4.25) follows from (4.30) in each case, completing the proof. □

The next lemma gives $r_n$ so that $U_2(r) \geq \frac{r^2}{3}$ when $r \geq r_n$ as long as (4.2) holds for a large $\zeta$ that depends on $\lambda$. It will be crucial that $r_n$ does not depend on $\lambda$.

**Lemma 4.31.** Given $\lambda > 4n$, there exists $\zeta_1$ so that if (4.2) holds for $\zeta \geq \zeta_1$, then for each $r \in (4n, 2\lambda)$ we have $U_2(r) > 0$ and, moreover,

$$U_2(r) \geq \frac{r^2}{3} \text{ for } r \in (r_n, 2\lambda), \text{ where } r_n \text{ depends only on } n\,. \tag{4.32}$$

*Proof.* We will choose $\zeta_1$ even greater than the $\zeta_0$ given by Lemma 4.24. Thus, Lemma 4.24 gives that $I_2'(r) \geq 0$ for $r \in (4n, 2\lambda)$ and (4.25) holds. Let $c_{4n}$ be the constant from (4.25) with $r = 4n$, so that there exists $s \in (4n, 4n+1)$ with

$$U_2(s) \geq c_{4n} > 0\,. \tag{4.33}$$

Corollary 4.9 and the monotonicity of $I_2$ give $C_0$ so that $C_0\, \zeta\, \mu^2 \leq I_2(r)$ and, thus,

$$\left( \frac{r^{1-k} \int_{|x|=r} \sum_{i,j} \psi_{ij}^2}{I_2} \right)^{\frac{1}{2}} \leq \frac{C_\lambda}{\zeta}\,. \tag{4.34}$$

Using this in Lemma 4.19 gives for $r \in (4n, 2\lambda)$ that

$$(\log U_2)'(r) \geq \frac{2-k}{r} + \frac{r}{2} - \frac{U_2}{r} - \frac{C_\lambda}{\zeta U_2}\,. \tag{4.35}$$

Now choose $\zeta_1 > \zeta_0$ so that $\frac{C_\lambda}{\zeta_1\, c_{4n}} \leq \frac{1}{4}$. Thus, if $c_{4n} \leq U_2(r) \leq \frac{11r^2}{30}$ and $r \in (4n, 2\lambda)$, then

$$(\log U_2)'(r) \geq \frac{r^2 - 2\, U_2(r)}{2r} - \frac{1}{2} \geq \frac{4r - 15}{30} \geq \frac{r}{24} > 0\,. \tag{4.36}$$

Combining this with (4.33), we see that $U_2 \geq c_{4n}$ for $r \in (4n+1, 2\lambda)$. Arguing as in the proof of (3.47), (4.36) gives that

- There exists $r_n$ depending on $n$ so that there is $r_1 \in [4n+1, r_n]$ with $U_2(r_1) > \frac{r_1^2}{3}$.
- There cannot be a first $r \in (r_1, 2\lambda)$ with $U_2(r) = \frac{r^2}{3}$.

□

The next lemma uses reverse Poincaré inequalities to bound the Euclidean Hessian in terms of the $L^2$ norm of the function on a larger set.

**Lemma 4.37.** If $r > 4n$ and (4.2) holds for $\zeta \geq 84$, then

$$\int_{|x|<r} |\mathrm{Hess}_w^x|^2 \, \mathrm{e}^{-f} \leq 204 \int_{|x|<r+2} w^2 \, \mathrm{e}^{-f}\,. \tag{4.38}$$

*Proof.* Given a compactly supported function $\eta(x)$ and a Euclidean partial derivative $w_i$, we have $\left(\mathcal{L}+\frac{1}{2}\right) w_i = \psi_i$ and, thus,

$$\operatorname{div}_f \left(\eta^2 w_i \nabla w_i\right) = \eta^2 \left(|\nabla w_i|^2 - \frac{1}{2} w_i^2 + w_i \psi_i\right) + 2\eta w_i \langle \nabla \eta, \nabla w_i \rangle . \tag{4.39}$$

The divergence theorem and Cauchy-Schwarz and absorbing inequalities give

$$\int \eta^2 \, |\nabla w_i|^2 \, \mathrm{e}^{-f} \leq \int \left((4|\nabla \eta|^2 + 2\eta^2) w_i^2 + \eta^2 \psi_i^2\right) \, \mathrm{e}^{-f} . \tag{4.40}$$

Taking $\eta \leq 1$ identically one for $|x| < r$ and cutting off linearly for $r < |x| < r+1$, we get

$$\int_{|x|<r} |\nabla w_i|^2 \, \mathrm{e}^{-f} \leq \int_{|x|<r+1} |\psi_i|^2 \, \mathrm{e}^{-f} + 6 \int_{|x|<r+1} |w_i|^2 \, \mathrm{e}^{-f} . \tag{4.41}$$

Since (3.5) gives that $|\psi_i| \leq |\nabla \phi| + \epsilon \left(|w| + |\nabla w| + |\nabla w_i|\right)$ and $\|\nabla \phi\|_{L^2} \leq \mu$, we get

$$\int_{|x|<r} |\mathrm{Hess}^x_w|^2 \, \mathrm{e}^{-f} \leq 2\,\mu^2 + \int_{|x|<r+1} \left(2w^2 + 10\,|\nabla w|^2\right) \, \mathrm{e}^{-f} . \tag{4.42}$$

We will now argue similarly to bound the right-hand side of (4.42) in terms of $w$ itself. We will again let $\eta$ be a cutoff function (on a different set). We have

$$\operatorname{div}_f \left(\eta^2 w \nabla w\right) = \eta^2 \left(|\nabla w|^2 - w^2 + w\psi\right) + 2\eta w \left\langle \nabla w, \nabla \eta \right\rangle . \tag{4.43}$$

Using the absorbing inequality $|2\eta w \left\langle \nabla w, \nabla \eta\right\rangle| \leq \eta^2 |\nabla w|^2/2 + 2|\nabla \eta|^2 w^2$ and the Cauchy-Schwarz inequality on the $w\psi$ term, the divergence theorem gives that

$$\int \eta^2 \, |\nabla w|^2 \, \mathrm{e}^{-f} \leq \int (3\eta^2 + 4|\nabla \eta|^2) \, w^2 \, \mathrm{e}^{-f} + \|\eta \, \psi\|^2_{L^2} . \tag{4.44}$$

Equation (3.1) gives that $\psi^2 \leq 2\phi^2 + 2\epsilon(w^2 + |\nabla w|^2)$, so we get

$$\int \eta^2 \, |\nabla w|^2 \, \mathrm{e}^{-f} \leq 2\,\mu^2 + 2\epsilon \int \eta^2 |\nabla w|^2 \, \mathrm{e}^{-f} + \int \left((3+2\epsilon) w^2 + |\nabla \eta|^2\right) w^2 \, \mathrm{e}^{-f} . \tag{4.45}$$

Since $\epsilon < \frac{1}{9}$, we can absorbe the $|\nabla w|^2$ term. Thus, taking $\eta \leq 1$ identically one for $|x| < r+1$ and cutting off linearly for $r+1 < |x| < r+2$, we get

$$\int_{|x|<r+1} |\nabla w|^2 \, \mathrm{e}^{-f} \leq 4\,\mu^2 + 10 \int_{|x|<r+2} w^2 \, \mathrm{e}^{-f} . \tag{4.46}$$

Combining this with (4.42) gives that

$$\int_{|x|<r} |\mathrm{Hess}^x_w|^2 \, \mathrm{e}^{-f} \leq 42\,\mu^2 + 102 \int_{|x|<r+2} w^2 \, \mathrm{e}^{-f} , \tag{4.47}$$

The lemma follows since (4.2) implies that $42\,\mu^2 \leq \frac{1}{2} \int_{|x|<4n} |\mathrm{Hess}^x_w|^2 \, \mathrm{e}^{-f}$. $\square$

Given a function $u$ on the cylinder, $y \in \mathbf{R}^k$, and $\lambda \in \mathbf{R}$, let $\Psi_{\lambda,u,y}$ be the norm squared of the projection of $u$ on the $\lambda$ eigenspace of $\Delta_\theta$ on the sphere $x = y$. Let $B^k_R$ be the ball in $\mathbf{R}^k$.

**Lemma 4.48.** Given $\lambda \in \mathbf{R}$, there exists $C$ depending on $\lambda, k, n$ so that

$$\int_{|x|<R} u^2 \leq \int_{B^k_R} \Psi_{\lambda,u,x} + C\,R^2 \int_{|x|<R} |\nabla^{\mathbf{R}^k} u|^2 + C \int_{|x|<R} \left\{ ((\mathcal{L}+\lambda)u)^2 + |\mathrm{Hess}^x_u|^2 \right\} \tag{4.49}$$

*Proof.* Since $\mathcal{L} + \lambda = (\Delta_\theta + \lambda) + \mathcal{L}_{\mathbf{R}^k}$, we get for each $y \in B^k_R$ that

$$\int_{x=y} u^2 \leq \Psi_{\lambda,u,y} + c \int_{x=y} ((\Delta_\theta + \lambda)u)^2 \leq \Psi_{\lambda,u,y} + 2\,c \int_{x=y} ((\mathcal{L} + \lambda)u)^2 + 2\,c \int_{x=y} (\mathcal{L}_{\mathbf{R}^k} u)^2$$

$$\leq \Psi_{\lambda,u,y} + 2\,c \int_{x=y} ((\mathcal{L} + \lambda)u)^2 + 4\,k\,c \int_{x=y} |\mathrm{Hess}^x_u|^2 + 4\,c\,R^2 \int_{x=y} \left|\nabla^{\mathbf{R}^k} u\right|^2 . \tag{4.50}$$

Integrating this over $B^k_R$ gives the lemma. □

The next lemma is a Poincaré inequality bounding $w$ by $\mathrm{Hess}^x_w$.

**Lemma 4.51.** We have

$$\int_{|x|<R} w^2 \leq C_R \int_{|x|<R+1} |\mathrm{Hess}^x_w|^2 + C_R\,\mu^2 \tag{4.52}$$

*Proof.* Lemma 4.48 with $u = w$ and $\lambda = 1$, so that $\Psi_{\lambda,u,x} \equiv 0$ and $|(\mathcal{L} + 1)w| \leq C_r\,\mu$, gives

$$\int_{|x|<R} w^2 \leq C\,R^2 \int_{|x|<R} |\nabla^{\mathbf{R}^k} w|^2 + C \int_{|x|<R} |\mathrm{Hess}^x_w|^2 + C_R\,\mu^2 . \tag{4.53}$$

We need to absorb the first term on the right side. Let $w_i$ be a Euclidean derivative of $w$. Applying Lemma 4.48 with $u = w_i$ and $\lambda = \frac{1}{2}$ gives

$$\int_{|x|<R} w_i^2 \leq \int_{B^k_R} \Psi_{\frac{1}{2},w_i,x} + C\,R^2 \int_{|x|<R} |\mathrm{Hess}^x_w|^2 + C \int_{|x|<R} |\mathrm{Hess}^x_{w_i}|^2 + C_R\,\mu^2 . \tag{4.54}$$

Let $\{h_j\}$ be an $L^2$-orthonormal basis of $\frac{1}{2}$-eigenfunctions for $\Delta_\theta$ and define $\Psi_{ij}(x) = \int h_j(\theta) w_i(x,\theta)\,d\theta$. It follows that

$$\Psi_{\frac{1}{2},w_i,x} = \sum_j (\Psi_{ij}(x))^2 \ . \tag{4.55}$$

By Lemma 4.3, $\left(\int_{|x|<r} \Psi_{ij}(x)\right)^2 \leq C_r\,\mu^2$. The Poincaré inequality on $B^k_R$ gives

$$\int_{B^k_R} (\Psi_{ij}(x))^2 \leq C_r\,\mu^2 + C_R \int_{B^k_R} \left|\nabla^{\mathbf{R}^k} \Psi_{ij}\right|^2 \leq C_r\,\mu^2 + C_R \int_{|x|<R} |\mathrm{Hess}^x_w|^2 . \tag{4.56}$$

Putting this together gives

$$\int_{|x|<R} w^2 \leq C_R \int_{|x|<R} |\nabla^{\mathbf{R}^k} \mathrm{Hess}^x_w|^2 + C_R \int_{|x|<R} |\mathrm{Hess}^x_w|^2 + C_R\,\mu^2 . \tag{4.57}$$

Finally, to complete the proof, we use $|\mathcal{L}\mathrm{Hess}^x_w| \leq C_r\,\mu$ and the reverse Poincaré inequality to bound the $|\nabla \mathrm{Hess}^x_w|$ term. □

*Proof of Proposition 4.1.* We will fix $\lambda$ at the end depending just on $n$ and $r_+$ and then choose $\zeta_2$. Given $r > 4n$, Lemma 4.51 and (4.2) give

$$\int_{|x|<r} w^2 \leq C'_r \int_{|x|<r+1} |\mathrm{Hess}^x_w|^2 + C_r\,\mu^2 \leq C_r \int_{4n<|x|<r+1} |\mathrm{Hess}^x_w|^2 . \tag{4.58}$$

Let $r_n$ be given by Lemma 4.31, so that $U_2(r) \geq \frac{r^2}{3}$ for $r \geq r_n$. If $r_n \leq r$ and $r+1 \leq s \leq \lambda - 2$, then (4.58) and Lemma 4.31 give that

$$\int_{|x|<r} w^2 \leq C_r\, I_2(r+1) \leq C_r\, I_2(s)\, \mathrm{e}^{-\frac{s^2}{3}} \,. \tag{4.59}$$

On the other hand, Lemma 4.37 gives that

$$\int_0^s t^{k-1}\, I_2(t)\, \mathrm{e}^{-\frac{t^2}{4}}\, dt = \int_{|x|<s} |\mathrm{Hess}^x_w|^2\, \mathrm{e}^{-f} \leq 204 \int_{|x|<s+2} w^2\, \mathrm{e}^{-f} \,. \tag{4.60}$$

It follows that we can choose $r_a$, depending just on $n$, so that $2 \int_{|x|<9n} w^2\, \mathrm{e}^{-f} \leq \int_{|x|<r_a} w^2\, \mathrm{e}^{-f}$. In particular, the first part of (3.22) holds for any $r_1 \geq r_a$.

Let $I_0$ and $U_0$ be the quantities $I$ and $U$ for $w$. We repeat the argument starting from $r = \max\{r_a, r_+\}$ using $U_2 \geq \frac{r^2}{3}$ to force $I_0$ to grow. For $\lambda$ large, depending on $n$ and $r_+$, this gives $r_1 \in (\max\{r_a, r_+\}, \lambda)$ with $U_0(r_1) \geq \frac{r^2}{16}$ (we could do this for any rate below $\frac{r^2}{3}$). Finally, choose $\zeta_2 > 84$ larger than the $\zeta_1$ from Lemma 4.31 with this $\lambda$. □

## 5. Proving the estimate for rescaled MCF

We will now prove (2.20) and, thus, complete the proof of Theorem 0.4. From now on, $\Sigma_t \subset \mathbf{R}^{n+1}$ is a rescaled MCF with $\lambda(\Sigma_t) < \infty$ and $\Sigma_t$ converges as $t \to \infty$ to a cylinder $\mathcal{C} = \mathbf{S}^{n-k}_{\sqrt{2(n-k)}} \times \mathbf{R}^k$. The sequence $\delta_j$ is defined in (2.3).

**Proposition 5.1.** (2.20) holds.

*Proof.* We will assume that $k \geq 1$ as the case $k = 0$ follows similarly, but much more easily. Let $\bar{\beta} < 1$ be given by Proposition 2.4. The proposition will follow once we show

$$\sup_{t\in[j,j+1]} \sup_{B_{2n}\cap\Sigma_t} \left|\Pi_{j+1}(\nabla \bar{H})\right| \leq C\, \delta_j^{\bar{\beta}} \,, \tag{5.2}$$

where $C$ does not depend on $j$.

We next explain how the parameters will be chosen. First, since $\bar{\beta} < 1$, we can choose $\nu, \beta < 1$ so that $\bar{\beta} < \nu\,\beta$. Next, given this $\nu$, Theorem 3.8 gives $\bar{\epsilon} > 0$. Finally, we choose the constant $\epsilon_1 > 0$ in Proposition 2.12 to ensure that (3.1) holds with $\bar{\epsilon}$.

Proposition 2.12 with $\epsilon_1$ and $\beta \in (\bar{\beta}, 1)$ as above gives constants $R_j, C$ and cylinders $\mathcal{C}_j$ so that $B_{R_j} \cap \Sigma_t$ is a graph over $\mathcal{C}_{j+1}$ of a function $w$ for each $t \in [j, j+1]$. Moreover, (1), (2) and (4) in Proposition 2.12 give (3.1)–(3.6) with $\epsilon < \bar{\epsilon}$ and $\mu = C\,\delta_j^{\beta}$. Theorem 3.8 now applies with our choice of $\nu < 1$ above. Thus, we get a constant $C_\nu$ and function

$$\tilde{w} = \sum_i a_i(x_i^2 - 2) + \sum_{i<k} a_{ik}\, x_i x_k + \sum_k x_k h_k(\theta) \,, \tag{5.3}$$

where $a_i, a_{ik}$ are constants and each $h_k(\theta)$ is a $\frac{1}{2}$-eigenfunction for $\Delta_\theta$, and we have

$$\sup_{|x|\leq 3n} |w - \tilde{w}| \leq C_\nu\, \mu^\nu = C\, \delta_j^{\beta\,\nu} \,. \tag{5.4}$$

The $\mathbf{R}^k$ unit vector fields $\partial_{x_i}$ on $\mathcal{C}_{j+1}$ push forward to vector fields on $\Sigma_t$ that we still denote $\partial_{x_i}$. Since $\{|x| \leq 3n\} \cap \Sigma_t$ is the graph over $\mathcal{C}_{j+1}$ of $w$ with $\|w\|^2_{C^2(|x|\leq 3n)} \leq C\,\delta_j^{\beta}$, it follows that $|\nabla \bar{H}| \leq C\,\delta_j^{\frac{\beta}{2}}$ and $|\Pi_{j+1}(\partial_{x_i})| \leq C\,\delta_j^{\frac{\beta}{2}}$ on $|x| \leq 3n$. Therefore, (5.2) follows from

$$\sup_{|x|\leq 3n} |\nabla_\theta \bar{H}| \leq C\,\delta_j^{\bar{\beta}}\,, \tag{5.5}$$

where $\bar{H}$ is now regarded as a function on $\mathcal{C}_{j+1}$ itself. It remains to establish (5.5).

The mean curvature $\bar{H}$ of the graph of $w$ is given at each point explicitly as a function of $w$, $\nabla w$ and $\text{Hess}_w$; see corollary $A.30$ in [CM2]. We can write this as the first order part (in $w, \nabla w, \text{Hess}_w$) plus a quadratic remainder

$$\bar{H} = H_{\mathcal{C}} + \left(\Delta_\theta + \Delta_x + \frac{1}{2}\right) w + O(w^2)\,. \tag{5.6}$$

Here $O(w^2)$ is a term that depends at least quadratically on $w, \nabla w, \text{Hess}_w$ and the constant $H_{\mathcal{C}} = \frac{\sqrt{n-k}}{\sqrt{2}}$ is the mean curvature of $\mathcal{C}$. We will show that the $\theta$ derivative of each of the terms in (5.6) is bounded by $C\,\delta_j^{\bar{\beta}}$. This is obvious for the constant term. It is also obvious for the quadratic term $O(w^2)$ using interpolation and $\|w\|^2_{C^2(|x|\leq 3n)} \leq C\,\delta_j^{\beta}$. Similarly, since $\beta\,\nu > \bar{\beta}$, the estimate (5.4) and interpolation give that

$$\left|\nabla \left(\Delta_\theta + \Delta_x + \frac{1}{2}\right)(w - \tilde{w})\right| \leq C\,\delta_j^{\bar{\beta}}\,. \tag{5.7}$$

The proposition follows from the above since applying the linearized operator to $\tilde{w}$ gives

$$\nabla_\theta \left(\Delta_\theta + \Delta_x + \frac{1}{2}\right)\tilde{w} = \nabla_\theta \sum_k \left(\Delta_\theta + \Delta_x + \frac{1}{2}\right) x_k\, h_k = 0\,. \tag{5.8}$$

□

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