APPROXIMATE SOLUTIONS OF THE POINCARÉ PROBLEM

by

STEVEN GERO

B. Eng. McGill University (1965)

SUBMITTED IN PARTIAL FULFILLMENT

OF THE REQUIREMENTS FOR THE

DEGREE OF MASTER OF SCIENCE

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

January, 1968

Signature of Author ....

Department of Mathematics, January 28, 1968

Certified by..

Thesis Supervisor

APPROXIMATE SOLUTIONS OF THE POINCARÉ PROBLEM

Steven Gero

Submitted to the Department of Mathematics on February 1, 1968, in partial fulfillment of the requirements for the degree of Master of Science.

ABSTRACT

The inviscid modes of oscillation of a contained rotating fluid (the Poincaré problem) are considered. Approximate ways of solution, the Galerkin method and the method of small perturbations are used to find eigenvalues for container shapes that cannot be dealt with analytically. The geometries investigated are the cylinder, the parallelepiped, the right circular cone, and a cylinder with a conical bottom.

Thesis Supervisor: Harvey P. Greenspan
Title: Professor of Mathematics

INTRODUCTION

Our aim is to solve the Poincaré problem, that is to say, to find the inviscid modes of oscillation of a contained rotating fluid. The work can be done analytically for only two geometries, namely the cylinder and the sphere. (1) Approximate methods of solutions have to be used for other geometries of interest. In this paper the Galerkin method (3) is used to find approximate eigenvalues. The geometries investigated are the cylinder, for which the exact solutions are available, the parallelepiped, and the cone. In each case the axis of rotation coincides with the axis of symmetry of the container.

The boundary value problem to be solved is

$$\nabla^2\Phi - \frac{4}{\lambda^2}(\hat{k}.\nabla)^2\Phi = 0$$

with the boundary condition

$$\hat{n}.\nabla\Phi - \frac{2}{i\lambda}\hat{n}.\hat{k}\times\nabla\Phi - \frac{4}{\lambda^2}\hat{n}.\hat{k}(\hat{k}.\nabla\Phi) = 0$$

on the bounding surface S

The unit vector $\hat{k}$ is to the rotation axis, $\hat{n}$ is the normal to the bounding surface. Φ is the potential and λ the eigenvalue (s) for which nontrivial solutions Φ exist.

The Cylinder

Let $\Phi = \psi_{mk}(r,z)e^{i(k\theta + \lambda_{mk}t)}$

then the boundary conditions become

(i) $\frac{\partial \psi_{mk}}{\partial z} = 0$ on $z = 0, 1$

(ii) $\left(\frac{\partial}{\partial r} + \frac{2k}{\lambda_{mk} r}\right) \psi_{mk} = 0$ on $r = r_o$

using the standard notation of Kudlick's thesis (4).

The exact solutions of the problem are the functions

$\psi_{mk} = J_m (\alpha_{mk} r) \cos m\pi z$

Boundary condition (i) is satisfied, by inspection, by

$\psi = \phi (r) \cos m\pi z$

Now using the Galerkin method of approximate solution (3), we take

$$u(r) = \Sigma\, a_i (r - r_o)^{i+2}$$

$$i = 0, 1, 2, \ldots$$

$$= \Sigma\, a_i \phi_i$$

this function satisfies the second boundary condition on the curved surface, for

$$\Sigma\, a_i\, (i+2)(r-r_o)^{i+2} + \frac{2k}{\lambda_{mn}} \Sigma\, a_i \frac{(r-r_o)^{i+2}}{r} = 0$$

on $r = r_o$

Now, in cylindrical coordinates the governing equation is

$$\left[r \frac{\partial}{\partial r} r \frac{\partial}{\partial r} - k^2 - \left(1 - \frac{4}{\lambda_{mn}^2}\right) m^2\pi^2 r^2\right] \phi = 0$$

$$\text{let } \Lambda = \left(\frac{4}{\lambda_{mn}^2} - 1\right) m^2\pi^2$$

in the Galerkin method we must make

$$\int L(u)\, \phi_j = 0 \qquad \text{where} \quad u = \Sigma\, a_i \phi_i$$

that is to say, we must make

$$\Sigma\, a_i \left\{ \int_o^{r_o} \left[r \frac{\partial}{\partial r}\, r \frac{\partial}{\partial r}\, \phi_i + \wedge r^2 \phi_i \right] \phi_j \right.$$

$$\left. - k^2 \int_o^{r_o} \phi_i \phi_j \right\} = 0$$

A homogeneous system of n equations in n unknowns

$$\Delta = |\alpha_{ij} - k^2 \beta_{ij}| = 0$$

evaluating the integrals involved we get

$$\alpha_{ij} = \frac{r_o^{\,i+j+s} (-1)^{i+j}}{i+j+5} \left[\frac{(i+2)(-1)}{i+j+4} + \frac{2(i+1)(i+2)}{(i+j+4)(i+j+3)} \right.$$

$$\left. + 2 \frac{\Lambda r_o^{\,2}}{(i+j+7)(i+j+6)} \right]$$

$$\beta_{ij} = \frac{(-1)^{i+5} r_o^{\,i+j+5}}{i+j+5}$$

Using the one-term approximation $u = (r-r_o)^2$ corresponding to $i = j = 0$, we get

$$\frac{1}{2} + \frac{1}{3} + 2 \frac{\wedge r_o^{\,2}}{4z} - k^2 = 0$$

which defines λ as a function of k, r_o, m.

Evaluating, for instance, for $r_o = 1$, $k = 0$, $m = 1$ we get $\lambda = 1.40$. This is in good agreement with the exact value of $\lambda = 1.708$.

Using the two-term approximation

$$u = a_o\, (r-r_o)^2 + a_1\, (r-r_o)^3$$

we get a quadratic in Λ, which for $r_o = 1$, $k = 0$, $m = 1$ yields the two roots

$\lambda_1 = 1.52$, $\lambda_2 = 1.96$ which are in good agreement with the true values of 1.708 and 1.913.

It should be noted that since this is not a variational method, successive approximations do not yield necessarily uniformly better results. (See Appendix).

The Parallelepiped

$$\psi_{xx} + \psi_{yy} + \psi_{zz} - \frac{4}{\lambda^2}\psi_{zz} = 0$$

Boundary conditions

(i) $\frac{\partial\psi}{\partial z} = 0$ at $z = 0, 1$

(ii) $\frac{\partial\psi}{\partial x} - \frac{\partial\psi}{\partial y}\frac{2}{i\lambda} = 0$ at $x = \pm a$

(iii) $\frac{\partial\psi}{\partial y} - \frac{\partial\psi}{\partial x}\frac{2}{i\lambda} = 0$ at $y = \pm b$

It should be noted that the mixed boundary conditions will make the finding of trial functions that satisfy the boundary conditions more difficult.

Let $\psi = \phi\,(x, y)\cos m\pi z$ this satisfies the b.c. at $z = 0, 1$ then $\phi_{xx} + \phi_{yy} + \left(\frac{4}{\lambda^2} - 1\right) m^2\pi^2\phi = 0$

$$\frac{\partial\phi}{\partial x} - \frac{\partial\phi}{\partial y}\frac{2}{i\lambda} = 0 \quad \text{at} \quad x = \pm a$$

$$\frac{\partial\phi}{\partial y} - \frac{\partial\phi}{\partial y}\frac{2}{i\lambda} = 0 \quad \text{at} \quad y = \pm b$$

$$\phi = (x^2-a^2)^{j+2}(y^2-b^2)^{j+2} \qquad j \geq 0$$

satisfies these boundary conditions; in fact it makes

$$\frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial y} = 0 \quad \text{at} \quad x = \pm a \ , \ y = \pm b$$

This function then is more restrictive than one which would give balance between two non-zero terms. However, even after several prolonged efforts, such functions could not be found by the author.

Now according to the Galerkin's method we must make

$$\det \left| \int \mathcal{L}(\phi_i)\, \phi_j \right| = 0$$

Let $i = j = 0$

Performing the necessary integrations we get

$$\Lambda = \frac{4(a^2+b^2)}{a^2b^2} \times \frac{.3048}{.406}$$

now for $a = b = \frac{1}{2}$ this yields $\lambda = 1.07$ which compares tolerably with the experimental value of $\lambda = .87$ (D. Fultz, oral communication with Prof. H. P. Greenspan).

Putting more "structure" into the solution we next tried the functions

$$\phi = x\,(x^2-a^2)^2\,(y^2-b^2)^2$$

$$\phi = y\,(y^2-b^2)^2\,(x^2-a^2)^2$$

both of which, for a cube yielded $\lambda = 0.91$

The trial function $\phi = xy\,(x^2-a^2)^2\,(y^2-b^2)^2$ gave $\lambda = 0.71$.

Adding odd terms to the first order approximation did not improve it; but using the next highest even term

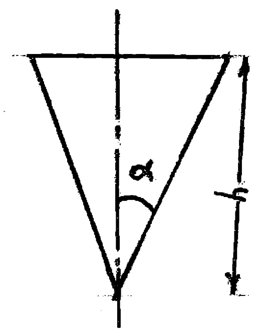
$$\emptyset = a_1 (x^2-a^2)^2 (y^2-b^2)^2 + a_2 (x^2-a^2)^2 (y^2-b^2)^2 x^2y^2$$

we obtain

$\lambda = .95, 1.17$ The first λ agrees reasonably with the experimental value.

This finishes the work for the cylinder and the cube. For the sphere the exact solution is in fact in terms of Legendre polynomials: so that if a polynomial approximation is used in the Galerkin method, it should yield the form of the exact answer.

The Cone



The boundary conditions become

$$\frac{\partial \emptyset}{\partial r} + \frac{2}{i\lambda} \frac{\partial \emptyset}{r \partial \theta} + \sin \alpha \frac{\partial \emptyset}{\partial z} \left(\frac{4}{\lambda^2} - 1\right) = 0$$

on $r = z \tan \alpha$

$$\frac{\partial \emptyset}{\partial z} = 0 \quad \text{on} \quad z = h$$

$\emptyset = (r - z \tan \alpha)^2 \sin^2 m\pi (z-h) e^{ik\theta}$ makes $\frac{\partial \emptyset}{\partial z} = 0$ on $z = h$ and $\frac{\partial \emptyset}{\partial r} = \frac{\partial \emptyset}{\partial z} = \emptyset = 0$ on $r = z \tan \alpha$ hence boundary conditions are satisfied.

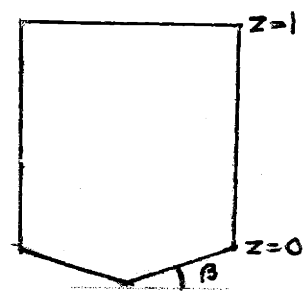
Let $\Lambda = \frac{4}{\lambda^2} - 1$ as before, we get

$$\Lambda = \frac{1}{\tan^2\alpha} \frac{\int_{z=0}^{h} \int_{\xi=0}^{z} \phi\, \xi \frac{\partial}{\partial\xi}\, \xi \frac{\partial}{\partial\xi}\, \phi\, \xi \; d\xi \; dz}{\int_{z=0}^{h} \int_{\xi=0}^{z} \frac{d^2\phi}{dz^2}\, \phi\, \xi^3 \; d\xi \; dz}$$

substituting the trial function, after some fairly involved integrations we get that $\Lambda = 0$ implying that $\lambda = 2$ as for the two parallel plates!!

This problem of the cone merits much more extended investigation: it is not certain, indeed, whether a solution exists at all. Of course, if it does not, it is useless to try to find an approximate solution. Both experimental and theoretical work are needed.

To explore further the problem of the cone, a cylindrical container with a conical bottom of small slope was considered, and the shape viewed as a small perturbation of the cylinder.



Now

$$i\lambda\bar{Q} + 2\hat{k} \times \bar{Q} = -\nabla\Phi$$

$$\nabla \cdot \bar{Q} = 0$$

$$\bar{Q} \cdot \hat{n} = 0$$

where $\bar{Q}$ is the velocity vector, Φ the potential, $\hat{n}$ the vector normal to the surface.

Let $\lambda = \lambda_o + \beta\lambda_1 + \ldots$

$$Q = \bar{Q}o + \beta Q_1 + \ldots$$

$$n = n_o + \beta n_1 + \ldots$$

λ_o, $\bar{Q}_o$ correspond to the known solution of the cylinder problem. To find $\bar{Q}_1$ is a difficult task; however, theoretically it is possible to find λ_1, the first order perturbation of the eigenvalue, solely in terms of the zeroth order quantities, without knowing Q_1.

Order α^o

$$\text{(i)} \quad i\lambda_o \bar{Q}_o + 2\hat{k} \times \bar{Q}_o = - \nabla\Phi_o$$

$$\nabla \cdot \bar{Q}_o = 0$$

$$\bar{Q}_o \cdot \hat{n}_o = 0 \quad \text{on the cylinder}$$

Order α

$$\text{(ii)} \quad i\lambda_o \bar{Q}_1 + 2k \times Q_1 = - \nabla\Phi - i\lambda_1 Q_o$$

$$\nabla \cdot \bar{Q}_1 = 0$$

$$\bar{Q}_1 \cdot \hat{n}_o = - Q_o \cdot \hat{n}_1 \quad \text{on } S$$

Taking the dot product of the conjugate of (i) with $\bar{Q}_1$, and of (ii) with $\bar{Q}_1^*$, the complex conjugate of Q_1, we get

$$- i\lambda \bar{Q}_o^* \cdot \bar{Q}_1 + 2 \bar{Q}_1 \cdot \hat{k} \times \bar{Q}_o = - \nabla\varnothing_o^* \cdot \bar{Q}_1$$

$$i\lambda \bar{Q}_1 \cdot \bar{Q}_o^* + 2 \bar{Q}_o^* \cdot \hat{k} \times \bar{Q}_1 = - \nabla\varnothing \cdot \bar{Q}_o^*$$
$$- i\lambda_1 \bar{Q}_o \cdot \bar{Q}_o^*$$

But $\bar{Q}_1 \cdot \hat{k} \times \bar{Q}_o^* = - \bar{Q}_o^* \cdot \hat{k} \times \bar{Q}_1$ hence, adding, we get

$$0 = -\nabla\Phi_o^* \cdot \bar{Q}_1 - \nabla\Phi_1 \cdot \bar{Q}_o^*$$
$$- i\lambda_1 \bar{Q}_o \cdot \bar{Q}_o^*$$

integrating over the volume of the container

$$\int dV \, \nabla \cdot (\phi_o^* \bar{Q}_1) + \int dV \, \nabla \cdot (\phi_1 \bar{Q}_o^*)$$
$$= - i\lambda_1 \int \bar{Q}_o \cdot \bar{Q}_o^* \, dV$$

the first integral can be rewritten as

$$\int dS \, \hat{n} \cdot (\phi_o^* \bar{Q}_1)$$

and the second as

$$\int dS \, \hat{n} \cdot (\phi_1 Q_o^* \bar{Q}_1)$$

but $\hat{n} \cdot \bar{Q}_o = 0$ and $\hat{n}_o \cdot \hat{Q}_1 = - \bar{Q}_o \cdot \hat{n}_1$

hence

$$- \int dS \, \phi_o^* \hat{n}_1 \cdot \bar{Q}_o = - i\lambda_1 \int \bar{Q}_o \cdot \bar{Q}_o^* \, dV$$

hence

$$\lambda_1 = \frac{\int \phi_o^* \bar{n}_1 \cdot \bar{Q}_o \, dS}{i \int \bar{Q}_o \cdot \bar{Q}_o^* \, dV}$$

Now $\hat{n}_1 \cdot \bar{Q}_o = 0$ everywhere except at the bottom where $\hat{n}_1 = \hat{r}$, the radial unit vector.

Thus the numerator is $\int \phi_o^* u_r \, dS$, u_r is the radial velocity component

$$\int \bar{Q}_o \cdot \bar{Q}_o^* \, dV = \int \left(u_r^2 + u_\theta^2 + u_z^2\right) dV$$

is a positive definite quantity.

The velocity components and ϕ are given in Kudlick, p.72.

It is not possible to perform all the integrations analytically.

However, in certain cases it is possible to determine whether λ_1 is positive or negative.

λ_1 - (A Pos. Def. quantity). I

where

$$I = \int_{r=o}^{r_o} \alpha_{mk} \lambda_{mk} J_k (\alpha_{mk} r) J_k' (\alpha_{mk} r) r \, dr$$

$$+ 2k \int [J_k (\alpha_{mk} r)]^2 dr$$

Now, by integration by parts

$$\int_o^{r_o} J_k (\alpha r) J_k' (\alpha r) = \frac{1}{2\alpha} \left\{ \xi \left[J_k (\xi)\right]^2 \Big|_o^{\alpha r_o} - \int_o^{\alpha r_o} \left[J_k (\xi)\right]^2 d\xi \right.$$

hence

$$\lambda_1 = - (\text{Pos. Def.}) . \left[\frac{\lambda}{2} \left[J_k (\alpha r_o)\right]^2 + \left(\frac{2k}{\alpha} - \frac{\lambda}{2}\right) \int_o^{\alpha r_o} \left[J_k (\xi)\right]^2 d\xi \right]$$

if $\frac{2k}{\alpha_{mi}} - \frac{\lambda_{mk}}{2} \geq 0$, then λ_1 is negative which is equivalent to

$$\lambda_{mk} \geq \sqrt{4 - \frac{16 k^2}{m^2 \pi^2}}$$

e.g. for m=1, k=2, a possible λ is 1.9442

but $1.9442 > \sqrt{4 - \frac{32}{\pi^2}}$

when, however $\frac{2k}{\alpha} - \frac{\lambda}{2} < 0$, the two terms $[J_k (\alpha r_o)]^2$ and the integral have to be evaluated and their magnitudes compared. The integral has to be done numerically.

e.g. for $k = 0$, it can be shown that, the smallest value of αr_o being 3.7, the first zero of J_1, the integral is larger than $[J_o\ (\alpha r_o)]^2$ and hence λ_1 is again negative. The work has to be done separately for each Bessel function and the particular value of αr_o in question.

Since for the problem of the cone the Galerkin method yielded $\lambda = 2$, the conical perturbation to the bottom of the cylinder would have been expected to yield λ_1 positive. In general, for most cases, it seems λ_1 is negative; however a 45° cone can hardly be considered a small perturbation to a right cylinder; the results of the perturbation analysis are not really very relevant to the full cone problem.

Conclusion

The Galerkin method yields eigenvalues for the Poincaré problem without excessive labour for shapes that cannot be dealt with analytically. However, since it is not necessarily yield successively better results. It is seemingly very difficult to find trial functions which satisfy the mixed boundary conditions in a not overly restrictive manner.

The problem of the cone is not clear, nor the significance of the answers obtained. In fact it is not at all certain that solutions exist; therefore the use of approximate methods will not necessarily yield meaningful results.

Further, numerical, work has to be done in the perturbation analysis, to determine the sign of λ_1, for all cases. But as said before, the results of a perturbation analysis are not really carried over to the full problem of the cone.

Appendix

What are the natural boundary conditions of the Galerkin method viewed as a variational principle?

$$\Lambda = \frac{\int L(\phi)\ \phi}{\int \phi^2} = \frac{I}{J}$$

to make this an extremum

$$\delta\Lambda = \delta I - \Lambda\delta J = 0 \quad \text{according to (2)}$$

$$\delta I = \delta \iint \phi \left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right) dx\ dy$$

Let $p = \frac{\partial\phi}{\partial x} \qquad q = \frac{\partial\phi}{\partial y}$

$$\delta J = \iint 2\ \phi\ \delta\ \phi\ dx\ dy$$

$$\int_R\int \phi \left(\frac{\partial p}{\partial x} + \frac{\partial q}{\partial y}\right) dx\ dy - \Lambda \iint \phi^2\ dx\ dy$$

$$= \int_R\int \frac{\partial}{\partial x}(p\phi) + \frac{\partial}{\partial y}(q\phi) - (p^2 + q^2)\ dx\ dy$$
$$- \Lambda \iint \phi^2\ dx\ dy$$

$$= \int_B p\ \phi\ dy - \int q\ \phi\ dx - \iint (p^2 + q^2)\ dx\ dy$$
$$- \Lambda \iint \phi^2\ dx\ dy$$

Hence

$$\delta \int p\ \phi\ dy = \int p\ \delta\ \phi\ dy + \int \delta\ p\ \phi\ dy = 0$$

Now if we make $p = 0$ on the boundary we get $\delta p = 0$ and variation is zero; similarly for $\int q\ \phi\ dx$.

Thus the natural boundary conditions are

$$p = q = 0 \quad \text{on the boundary}$$

or $\frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial y} = 0$

which are not the boundary conditions of the Poincaré problem.

BIBLIOGRAPHY

1. Greenspan, H. P., On The General Theory of Contained Rotating Fluid Motions, J.F.M. 22 (1965), pp. 449-462.

2. Irving, I., and Mullineux, N., Mathematics in Physics and Engineering, N.Y. 1959.

3. Kantorovich, L.V., and Krylov, V.I., Approximate Methods of Higher Analysis, Groningen 1958.

4. Kudlick, M.D., On Transient Motions in a Contained, Rotating Fluid, Ph.D. Thesis, M.I.T. Department of Mathematics, 1965 (unpublished).

This research was partially supported by the Office of Scientific Research of the U. S. Air Force, Contract F44620-67-C-0007.