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Citation: Asada, H. Harry and Filippos E. Sotiropoulos, "Dual Faceted Linearization of Nonlinear Dynamical Systems Based on Physical Modeling Theory." Journal of Dynamic Systems, Measurement, and Control 141, 2 (February 2019): no. 021002 doi. 10.1115/1.4041448 ©2019 Authors

As Published: https://dx.doi.org/10.1115/1.4041448

Publisher: ASME International

Persistent URL: https://hdl.handle.net/1721.1/126131

Version: Final published version: final published article, as it appeared in a journal, conference proceedings, or other formally published context

Dual Faceted Linearization of Nonlinear Dynamical Systems Based on Physical Modeling Theory

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A new approach to modeling and linearization of nonlinear lumped-parameter systems based on physical modeling theory and a data-driven statistical method is presented. A nonlinear dynamical system is represented with two sets of differential equations in an augmented space consisting of independent state variables and auxiliary variables that are nonlinearly related to the state variables. It is shown that the state equation of a nonlinear dynamical system having a bond graph model of integral causality is linear, if the space is augmented by using the output variables of all the nonlinear elements as auxiliary variables. The dynamic transition of the auxiliary variables is investigated as the second set of differential equations, which is linearized by using statistical linearization. It is shown that the linear differential equations of the auxiliary variables inform behaviors of the original nonlinear system that the first set of state equations alone cannot represent. The linearization based on the two sets of linear state equations, termed dual faceted linearization (DFL), can capture diverse facets of the nonlinear dynamics and, thereby, provide a richer representation of the nonlinear system. The two state equations are also integrated into a single latent model consisting of all significant modes with no collinearity. Finally, numerical examples verify and demonstrate the effectiveness of the new methodology. [DOI: 10.1115/1.4041448]

Keywords: linearization, physical system modeling, bond graph, statistical linearization, augmented state space, dual faceted linearization, nonlinear dynamical systems

1 Introduction

A nonlinear system behaves more linearly when it is recast in a larger space. While the system exhibits pronounced nonlinearities, it appears nearly or completely linear when represented with additional variables or in an augmented space. This intriguing statement motivates us to explore a new methodology for linearizing nonlinear systems.

In the literature, similar statements can be found in several isolated fields, including machine learning, system dynamics, and mathematics. In his seminal paper on pattern recognition, Thomas Cover revealed that complex patterns can be made linearly separable by augmenting variables with new ones replacing nonlinear terms [1]. In subspace methods, it has been known that some nonlinear effects can be modeled effectively by increasing the number of state variables [2,3]. Various types of kernel functions and locally tunable functions have been used for representing a nonlinear dynamical system as a linear combination of nonlinear basis-functionals. Ranging from the classical Volterra series expansion [4,5] to locally weighted kernel functions [6], radial basis functions [7], and wavelets [8], nonlinear behaviors are recast in a larger space, which allows us to apply effective tools for system analysis and synthesis as well as for learning and identification. Furthermore, the Carleman embedding technique reduces a nonlinear dynamic equation to a linear differential equation in an infinite dimensional space [9,10]. This linearization is exact, although the dimension is infinite.

These methods differ in functionality, theoretical basis and principle, depending on the individual field of use. However, in all of these methods, recasting a nonlinear system in a larger space, one can find a linear or approximately linear representation of the original nonlinear system in a global sense. This is strikingly different from the standard linearization method based on Taylor expansion and small perturbations, which are valid only in the vicinity of an operating point. The objective of the current work is to establish a new type of linearization of complex nonlinear systems that are valid in a global sense. Furthermore, additional variables for augmenting the state space should have a clear physical sense and should not be too large in dimension. The use of kernels and locally tunable functions tends to increase the dimension dramatically, leading to the curse of dimensionality for higher order systems. Increased state variables in subspace methods neither provide a clear physical sense nor manifest the mechanism by which the nonlinearity is handled. In this paper, we aim to establish a systematic method for properly augmenting the state space. In finding physically meaningful variables to augment the space and linearizing the nonlinear system in the augmented space, we explore the following two fundamental principles and concepts.

First, we explore "natural" linearity in physical system modeling. Considering nonlinear lumped-parameter systems, elements of the system (mass, spring, and damper, or capacitor, inductor, and resister) are linearly connected in formulating governing equations. Kirchhoff's laws, for example, state that these components are linearly connected, although constitutive laws of individual elements may be nonlinear. Newton's law combined with d'Alembert's principle dictates that all the forces acting on a mass sum to zero, which is a linear relationship. In these systems, nonlinearity comes from constitutive laws of individual elements. Replacing these nonlinear terms by new variables, called *auxiliary variables*,[2] and augmenting the state space with these auxiliary variables, we can expect to obtain a linear expression,

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Contributed by the Dynamic Systems Division of ASME for publication in the Journal of Dynamic Systems, Measurement, and Control. Manuscript received December 24, 2017; final manuscript received August 28, 2018; published online October 5, 2018. Assoc. Editor: Dumitru I. Caruntu.

[2]Auxiliary variables are formally defined in Sec. 3.

highlighting the linear connectivity of the system. Compared to the existing methods described earlier, this method, which is based on fundamental principles and philosophy of physical system modeling, provides us with a clear physical sense; each auxiliary variable is associated with a nonlinear element's constitutive law, rather than merely a mathematical expression.

Second, we exploit multiple representations of a dynamical system. Consider two equations of motion representing the same dynamical system with two different variables. If the transformation of one variable to the other is linear and invertible, the two equations are identical and, thereby, no benefit is expected in representing it with two variables. However, it may not be the case if the two variables are nonlinearly related. Although the two represent the same nonlinear system, they delineate the nonlinear behaviors from two different coordinate systems. When each of the equations is linearized individually, they represent two facets of the system behaviors, each making distinct contributions to capturing the nonlinear dynamics. Combining the two facets of system descriptions, we can expect to attain a more accurate and richer representation of the nonlinear system.

In this paper, the above two concepts are integrated into a new linearization method with two sets of linear dynamic equations: one by exploiting the natural linearity of elements' connectivity and the other by formulating a linear differential equation predicting the dynamic transition of auxiliary variables that are nonlinearly related to the original state variables. While the two sets of variables are related to each other as nonlinear algebraic equations, they do not show up explicitly in the dynamic equations. Instead both contribute to the dynamic transition with two sets of linear dynamic equations, which facilitate the application of linear system analysis and synthesis. They can provide a more powerful representation of the nonlinear behaviors than a single set of linear state equations. The method is referred to as dual faceted linearization (DFL).

In the following, the physical system modeling based on bond graph [12,13] will be used for addressing the natural linearity of lumped-parameter systems, followed by the principle and algorithm of dual faceted linearization. Numerical examples demonstrate that the new method can deal with highly nonlinear systems and predict nonlinear behaviors significantly more accurately than traditional counterparts.

2 Dual-Faceted Linearization: The Method in a Nutshell

Consider a simple nonlinear system consisting of two elements: a mass and a spring. The governing equations are Newton's equation of motion, $F = m\ddot{x}$, and a constitutive law of spring, $F = -\Phi(x)$. First, if the spring's constitutive law is linear, $F = -kx$, we can write the equation of motion in terms of x as $m\ddot{x} + kx = 0$, or in terms of F as $(m/k)\ddot{F} + F = 0$. These two expressions are identical, so we gain nothing by writing the equation of motion in two ways. However, that is not the case when the system is nonlinear. Suppose that the spring's constitutive law is

$$F = -ax - bx^3, \quad a > 0,\ b > 0 \tag{1}$$

We can write the equation of motion in two ways: one in terms of x

$$m\ddot{x} + ax + bx^3 = 0 \tag{2}$$

and the other in terms of F. Using the inverse of the spring's constitutive law: $x = -g(F)$ and its derivatives, $g' = dg/dF$, $g'' = d^2g/dF^2$, we can write

$$mg'\ddot{F} + mg''\dot{F}^2 + F = 0 \tag{3}$$

Both Eqs. (2) and (3) represent the same nonlinear system, but their expressions are different. Linearizing these two equations may lead to two state equations representing two facets of the nonlinear system (see Fig. 1).

There are a few significant observations and arguments on the dual faceted representation illustrated earlier.

2.1 Intrinsic Linearity in Elements Connectivity. Using both the x and F variables, we can augment the state space where the two facets of the nonlinear dynamics can be represented. Although F is a function of x, we treat F as an additional state variable with a state transition equation. If we use both the x and F variables, the first equation of motion (2) appears to be linear: $m\ddot{x} - F = 0$. As shown in Sec. 3, the state transition equation of independent state variables $x \in \Re^n$ can be written as a linear differential equation in an augmented state space for an arbitrary nonlinear, lumped parameter system under mild conditions.

These linear state equations are not an approximation, but are exact equations. Let $\eta \in \Re^{n_a}$ be a vector of auxiliary variables, and $u \in \Re^r$ be a vector of inputs. The first state equation is given by

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}_x\mathbf{x} + \mathbf{A}_\eta\eta + \mathbf{B}_x\mathbf{u} \tag{4}$$

where $\mathbf{A}_x$, $\mathbf{A}_\eta$, and $\mathbf{B}_x$ are constant, parameter matrices of consistent dimensions. We will prove this linearity based on physical modeling theory, in particular, using bond graph. In bond graph, a nonlinear system is graphically represented as a network of elements, such as mass, spring, and damper or inductor, capacitor, and resistor. The connectivity of elements is governed by fundamental physical laws, such as Kirchhoff's laws and Newton's equations of motion. It should be noted that Kirchhoff's voltage law, for example, dictates that the voltages of all the elements directly involved in a loop sum to zero, which is in fact a linear relationship. Kirchhoff's current law and Newton's equation, too, are linear expressions, where relevant variables sum to zero. Therefore, by treating those voltages, currents, and forces as additional state variables, called auxiliary variables η, we can obtain a linear state equation as given earlier.

2.2 Algebraic Versus Dynamic Linearization. The traditional approach to linearizing a nonlinear system is to algebraically linearize the auxiliary variables η using the first-order Taylor expansion or other similar methods

$$\eta \cong \bar{\eta} + J(\bar{\mathbf{x}})\,(\mathbf{x} - \bar{\mathbf{x}}) \tag{5}$$

where $J(\bar{\mathbf{x}}) \in \Re^{n_a \times n}$ is the Jacobian matrix evaluated at a reference point $\bar{\mathbf{x}}$. The validity of such linearization is limited to a local

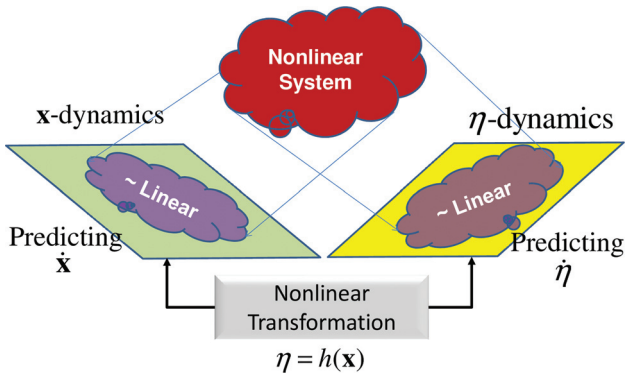

Fig. 1 Conceptual diagram of DFL. A nonlinear dynamical system is viewed from an augmented space consisting of independent state variables and auxiliary variables, which are nonlinearly related to each other.

region near the point $\bar{\mathbf{x}}$. Here, we take a different approach. We treat η as an auxiliary state variable that possesses a state transition equation. Furthermore, we aim to obtain a linear regression for predicting the time rate of change to η in the augmented space. This leads to another linear differential equation

$$\frac{d\eta}{dt} = \mathbf{H}_x\mathbf{x} + \mathbf{H}_\eta\eta + \mathbf{H}_\eta\mathbf{u} \tag{6}$$

where $\mathbf{H}_x, \mathbf{H}_\eta$, and $\mathbf{H}_\eta$ are regression parameter matrices of consistent dimensions.

It is important to note that the time derivative of η based on the Taylor expansion does not provide any new information

$$\dot{\eta} \cong \bar{J} \cdot \dot{\mathbf{x}} \tag{7}$$

This implies that $\dot{\eta}$ and $\dot{\mathbf{x}}$ are collinear; the state equation of $\dot{\eta}$ is basically the same as the first state equation of $\dot{\mathbf{x}}$. This argument applies to any traditional linearization using a fixed matrix $\bar{J}$. No matter how the matrix $\bar{J}$ is determined, the differential equations of $\dot{\eta}$ and $\dot{\mathbf{x}}$ are collinear and thus redundant. In contrast, the time derivative $\dot{\eta}$ in the linear regression (6) is not collinear with $\dot{\mathbf{x}}$ in general ($\dot{\eta} \neq \bar{J} \cdot \dot{\mathbf{x}}$), but provides a different facet of the original system that supplements the dynamics represented by the first state equation (see Fig. 1).

2.3 Latent Modeling. One drawback of dual faceted linearization is the increase of state variables. This is a serious problem when dealing with a complex, high-dimensional system, such as a biological system. For example, collective cells interacting through extra-cellular matrix have over 2000 independent state variables. Adding auxiliary variables, the total number of variables becomes several thousands. The state transition equations of those variables may contain similar dynamic modes and almost collinear relationships. Those modes can be eliminated by using latent analysis [11].

Combining state variables $\mathbf{x}$ and auxiliary variables η, we define an Augmented State Space with variables: $\zeta = (\mathbf{x}^T\ \eta^T)^T \in \Re^{k\times 1}$, $k = n + n_a$. Without loss of generality, let us assume that the augmented state variables are mean-centered. The covariance of the augmented state variables can be decomposed to

$$\mathbf{C} = E[\zeta\zeta^{\mathrm{T}}] = \mathrm{T}\Sigma\mathbf{T}^{\mathrm{T}},\ \text{where}\ \mathbf{\Sigma} = \begin{pmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_m \end{pmatrix}$$
$$\text{and}\ \mathbf{T} = \begin{pmatrix} \mathbf{V} \\ \mathbf{W} \end{pmatrix} \in \Re^{k\times m} \tag{8}$$

where $m \leq k$ is the rank of the covariance matrix, $\lambda_1 \geq \cdots \geq \lambda_m > 0$ are eigenvalues, and $\mathbf{V} \in \Re^{n\times m}$ and $\mathbf{W} \in \Re^{n_a\times m}$ are blocks of the orthonormal matrix $\mathbf{T}$ associated with state variables $\mathbf{x}$ and auxiliary variables η, respectively. With these matrices, the augmented state space can be converted to latent space with latent variables given by

$$\mathbf{z} = \mathbf{V}^{\mathrm{T}}\mathbf{x} + \mathbf{W}^{\mathrm{T}}\eta \tag{9}$$

where $\mathbf{z} \in \Re^{m\times 1}$ can be truncated to a lower-dimensional vector, $m^* < m$, if the principal components beyond m^* have negligibly small eigenvalues, $1 \gg \lambda_{m^*+1} \geq \cdots \geq \lambda_m$. It is expected that the combination of the state variables and the auxiliary variables in the latent space may better capture significant dynamics of the original system even in linearized state equation form. The following is to derive state transition equations in the latent space, which subsumes dynamic equations of both state and auxiliary variables.

By taking the time derivative of Eq. (9) and substituting Eqs. (4) and (6) into it, we can obtain a state equation in the latent space

$$\dot{\mathbf{z}} = \mathbf{V}^{\mathrm{T}}\dot{\mathbf{x}} + \mathbf{W}^{\mathrm{T}}\dot{\eta} = \mathbf{A}\mathbf{z} + \mathbf{B}\mathbf{u} \tag{10}$$

where

$$\mathbf{A} = [\mathbf{V}^{\mathrm{T}}\mathbf{A}_x + \mathbf{W}^{\mathrm{T}}\mathbf{H}_x]\mathbf{V} + [\mathbf{V}^{\mathrm{T}}\mathbf{A}_\eta + \mathbf{W}^{\mathrm{T}}\mathbf{H}_\eta]\mathbf{W}$$
$$\mathbf{B} = \mathbf{V}^{\mathrm{T}}\mathbf{B}_x + \mathbf{W}^{\mathrm{T}}\mathbf{H}_u \tag{11}$$

Note that, from Eq. (8), we have $\mathbf{x} = \mathbf{V}\mathbf{z}$ and $\eta = \mathbf{W}\mathbf{z}$, which are used in the above derivation.

Dual faceted linearization briefly presented earlier raises a number of important questions. What is the rigorous definition of Auxiliary variables? Why does the state Eq. (4) become linear? To which class of nonlinear systems does this apply? Why does the second differential Eq. (6) provide a different facet of the original nonlinear system? How can we determine the parameters of the linear regression? Are they unique? And what is the limitation to the original formulation of DFL? Sections 3, 4, and 5 address these questions based on physical modeling theory and estimation methods. First, we will show that the linearity of the first state equation (4) stems from the connectivity of elements that is intrinsically linear.

3 Natural Linearity in Element Connectivity

Consider a nonlinear, lumped-parameter system that can be modeled with bond graph. As illustrated in Fig. 2, the system consists of elements connected by power bonds. Attached to each bond are an effort variable (force, voltage, pressure, etc.) and a flow variable (velocity, current, flow rate, etc.), the product of which represents power flowing through the bond. Connections among elements are governed by physical laws in individual energy domains. In generic term, "1" junction represents Kirchhoff's voltage law, Newton's equation of motion, etc., where all the effort variables associated with the bonds connected to the 1 junction sum to zero with a proper sign convention. Similarly, "0" junction is a generic representation of Kirchhoff's current law and others, where all the flow variables associated with all the bonds connected to the 0 junction sum to zero. Both junction conditions are intrinsically linear. On the other hand, constitutive laws of individual elements may be nonlinear. The constitutive law of a resistive element is expressed as a functional relationship between effort e and flow f: $e = \Phi_R(f)$ or its inverse function, the constitutive law for a capacitive element is expressed as a functional relationship between effort e and displacement $q = \int f dt$: $e = \Phi_C(q)$ and that of an inertial element is by $f = \Phi_I(q)$, where the variable p is momentum $p = \int e dt$. As illustrated in Fig. 3, these constitutive laws may be nonlinear.

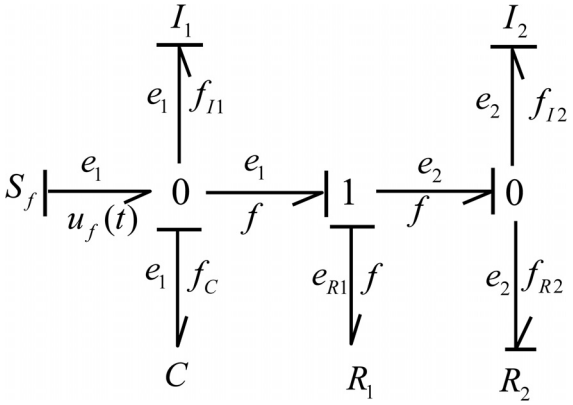

Fig. 2 Example of bond graph

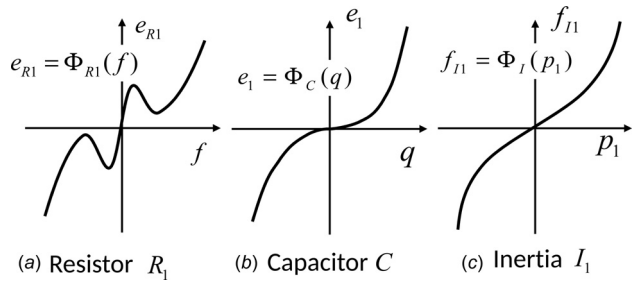

Fig. 3 Examples of constitutive laws of elements: (*a*) resistor R_1, (*b*) capacitor C, and (*c*) inertia I_1

A complete bond graph may include transformers and gyrators, which convert energy from one form to other with characteristic conversion rates. We make the following assumptions in the bond graph modeling:

- All the conversions of effort and flow variables associated with transformers and gyrators have been completed, so that the resultant bond graph does not include any transformer and gyrator;
- any causal conflict among the elements has been removed, so that the system is of integral causality [12,13]; and
- the system is time-invariant.

Through causality analysis, bond graph allows us to find causal relationships among all the elements and obtain a computable procedure for determining state transitions. The small vertical or horizontal bars attached to one end of each power bond indicate causal relations between the two elements or subsystems connected by the power bond.[3] Following the causal strokes, state transition can be determined, given initial conditions and inputs. In the bond graph modeling theory, displacement q and momentum p are often used as state variables associated with capacitive and inertial elements, respectively. State equations can be derived from a Bond Graph starting with each of these energy storage elements and following a sequence of causal relationship guided by causal strokes.

In following the causal input–output sequence across a bond graph, we encounter nonlinear elements for which we define Auxiliary variables.

DEFINITION. *A set of variables is called auxiliary variables if they are outputs of all the nonlinear elements connected to a lumped parameter model with integral causality*.

Using these auxiliary variables, we can show that a nonlinear, lumped-parameter system can be decomposed to a linear dynamic model and a nonlinear algebraic model.

THEOREM 1. *State equations of a nonlinear lumped-parameter system that possesses a bond graph of integral causality and that contains n independent energy storage elements, n_a nonlinear elements, and r sources or exogenous inputs can be expressed as a linear equation in terms of n state variables, $\mathbf{x} \in \Re^n$, n_a auxiliary variables, $\eta \in \Re^{n_a}$, and inputs, $\mathbf{u} \in \Re^r$, as*

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}_x\mathbf{x} + \mathbf{A}_\eta\eta + \mathbf{B}_x\mathbf{u} \tag{4}$$

where $\mathbf{A}_x \in \Re^{n\times n}$, $\mathbf{A}_\eta \in \Re^{n\times n_a}$, and $\mathbf{B}_x \in \Re^{n\times r}$ are constant matrices with consistent dimensions.

This theorem can be proven by construction. Examining the causal relationship among all the elements, we can construct state equations in the form of Eq. (4). Appendix A shows the proof.

[3]For example, the power bond connecting the inertial element I in Fig. 2 has a causal stroke on the element side, that is, the opposite side of the 0-junction. This means that the effort variable e is the input to the inertial element, while the flow variable f_I is the output. The capacitive element C has a causal stroke on the 0-junction side, meaning that the flow is input and the effort is output.

The following examples demonstrate this process and important properties:

Example 1. Consider a third-order system represented with a bond graph in Fig. 2. Momenta p_1 and p_2 associated with inertial elements I_1 and I_2, respectively, and displacement q associated with the capacitive element C can be used as state variables. Following the causal strokes, we can obtain the following differential equations of state variables (see Appendix B for more details)

$$\frac{dp_1}{dt} = e_1 = \Phi_C(q) \tag{12}$$

$$\begin{aligned}\frac{dq}{dt} &= f_C = u_f - f_{I1} - f \\ &= u_f - f_{I1} - f_{I2} - f_{R2} \\ &= u_f - \Phi_{I1}(p_1) - \Phi_{I2}(p_2) - \Phi_{R2}(e_2)\end{aligned} \tag{13}$$

$$\frac{dp_2}{dt} = e_2 = e_1 - e_{R1} = \Phi_C(q) - \Phi_{R1}\left[\Phi_{I2}(p_2) + \Phi_{R2}(e_2)\right] \tag{14}$$

where $e_1 = \Phi_C(q)$ is a nonlinear constitutive law of the capacitive element, $f_{I1} = \Phi_{I1}(p_1)$ and $f_{I2} = \Phi_{I2}(p_2)$ are, respectively, nonlinear constitutive laws of the two inertial elements, and $e_{R1} = \Phi_{R1}(f)$ and $f_{R2} = \Phi_{R2}(e_2)$ are the ones of the resistive elements R_1 and R_2, respectively. There are five nonlinear elements involved in this system. Therefore, the auxiliary variables are the outputs of these five nonlinear elements

$$\eta = \begin{pmatrix} e_1 \\ f_{I1} \\ f_{I2} \\ e_{R1} \\ f_{R2} \end{pmatrix} \begin{matrix} \cdots C \\ \cdots I_1 \\ \cdots I_2 \\ \cdots R_1 \\ \cdots R_2 \end{matrix} \tag{15}$$

The above state Eqs. (12)–(14) are nonlinear due to the nonlinear constitutive laws of the elements. However, these equations are linear in terms of the output variables of the individual nonlinear elements and input variables from the sources. Namely, if we leave the equations to the ones containing auxiliary variables, i.e., the outputs of nonlinear elements, the resultant equations are linear.

In case some linear elements are involved in the system, the outputs of the linear elements are excluded from the auxiliary variables. If I_1 and R_1, for example, are linear, $\Phi_{I1}(p_1) = p_1/m$ and $\Phi_{R1}(f_{R1}) = Rf_{R1}$, the output variables of these linear elements, f_{I1} and e_{R1}, are excluded so that the auxiliary variables reduce to $\eta = (e_1\ f_{I2}\ f_{R2})^{\mathrm{T}}$ and the state equations can be written as

$$\begin{pmatrix} \dot{q} \\ \dot{p}_1 \\ \dot{p}_2 \end{pmatrix} = \begin{bmatrix} 0 & -1/m & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} q \\ p_1 \\ p_2 \end{pmatrix} + \begin{bmatrix} 0 & -1 & -1 \\ 1 & 0 & 0 \\ 1 & R & R \end{bmatrix} \begin{pmatrix} e_1 \\ f_{I2} \\ f_{R2} \end{pmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u_f \tag{16}$$

which is in the form of Eq. (4).

The combination of independent state variables and auxiliary variables are sufficiently informing the underlying nonlinear dynamical system [14]. In other words, no more variables are needed.

As demonstrated in Example 1, state equations can be derived from a bond graph in a systematic manner by following causal propagation paths [12,13]. A critical issue, however, is the case where a propagation path forms a "loop." This incurs an algebraic loop problem. The following example illustrates this issue and a

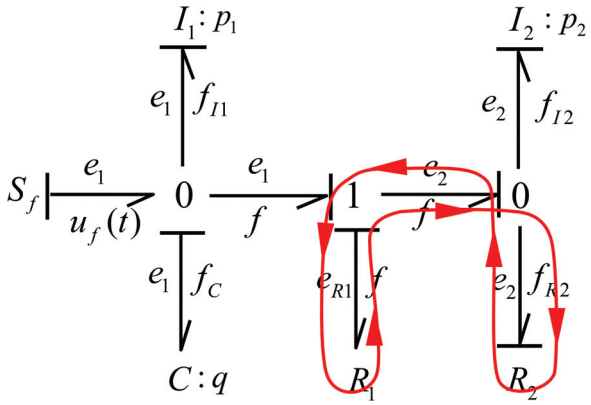

Fig. 4 Algebraic loop involved in the bond graph in Fig. 2

way of getting rid of the algebraic loop. The bottom line is that the algebraic problem does not occur in the DFL formulation:

Example 2. Equations (13) and (14) are not complete state equations due to an algebraic loop involved. The same bond graph as Example 1 is redrawn in Fig. 4 with a loop that goes through the two resistive elements R_1, R_2, the right 0-junction, and the 1-junction. Backtracking the causal path finds the existence of a loop: $e_2 \mapsto e_{R1} \mapsto f \mapsto f_{R2} \mapsto e_2$. Accordingly, the state equation (14) for $dp_2/dt = e_2$ contains a nonlinear term that depends on e_2 in the last expression: $\Phi_{R1}[\Phi_{I2}(p_2) + \Phi_{R2}(e_2)]$. If resistive element R_1 is nonlinear, this whole term is replaced by the output variable e_{R1} in the DFL formulation, and thereby the causal path backtracking is terminated there. If R_1 is linear, the causal path continues until it reaches another nonlinear element R_2, but its output variable f_{R2} replaces $\Phi_{R2}(e_2)$, and thereby the causal path backtracking terminates there.

An algebraic loop is formed only when both resistive elements are linear in the state equation formulation using auxiliary variables. Consider the following linear resistive constitutive laws: Resistor $R_1 : e_{R1} = R_1 f$ and Resistor $R_2 : f_{R2} = (1/R_2)e_2$. Substituting these into the algebraic condition in Eq. (14): $e_2 = e_1 - \Phi_{R1}[f_{I2} + \Phi_{R2}(e_2)]$, and solving it for e_2, we can find the state equation given by

$$\frac{dp_2}{dt} = \frac{R_2}{R_1 + R_2} e_1 - \frac{R_1 R_2}{R_1 + R_2} f_{I2} \tag{17}$$

Elimination of algebraic loops is an important property and a consequence of the use of auxiliary variables. This holds in general and stated in the following Corollary.

COROLLARY. *For a lumped-parameter, nonlinear system possessing a bond graph of integral causality, elements along any algebraic loop involved in the system are linear resistive elements. Therefore, algebraic conditions are linear and thereby solved explicitly*.

This can be proven based on the definition of auxiliary variables and properties of causal bond graph. In any algebraic loop, if it exists, no energy storage element can be on the loop, because the constitutive law of the energy storage element is a function of a state variable, and the causal path backtracking terminates at the energy storage element. Along the same line, if a nonlinear resistive element is involved, the backtracking terminates at the output of the nonlinear element, because an auxiliary variable is defined for the output of the nonlinear resistive element. Therefore, only linear resistive elements are on the loop. The causal path backtracking continues through the linear resistive element by converting it between effort and flow variables. Therefore, if a complete algebraic loop exists, only linear resistive elements are involved in the loop. The algebraic relation can be solved easily for linear resistive elements.

This property stated in the above Corollary is important. Algebraic loops often result in differential-algebraic equations [13]. The use of auxiliary variables can solve this problem. The linear differential equation (4) is obtained without an algebraic loop.

4 Linear Regression Based on Statistical Linearization

From Theorem 1, an arbitrary nonlinear, lumped-parameter system represented with a causal bond graph can be separated into a linear dynamical system and a nonlinear algebraic relation. The simplest method for linearizing such a nonlinear algebraic equation is to take the first-order Taylor expansion. The validity of such a local, point-wise linearization may be limited to a specific region in the state space and may fail to represent significant properties. For example, linearizing the capacitive element shown in Fig. 3(*b*) at the origin yields only zero voltage ($e_c \equiv 0$), since the slope at the origin is zero. In the case of a mechanical system, this implies zero stiffness, although the spring is a hard spring. A better alternative is based on a global, statistical linearization. In the literature, stochastic/statistic linearization was originally developed for solving nonlinear random vibration problems [15,16]. Instead of linearizing at a particular point in space, a set of samples, i.e., a data set, is used for linearization, so that the linearized model may have the least mean prediction error. The result, however, is still an algebraic linearization in which auxiliary variables are related to independent state variables through an algebraic map with a constant matrix. Therefore, the structure is same as the Taylor expansion. The nonlinear properties of η are all averaged out and confined in the linearized expression (5), while the auxiliary variables exhibit more profound dynamics. We can capture these dynamics to better predict the state transition. Here, we treat the auxiliary variables as a type of state variables and form another set of dynamic state equations representing the transition of auxiliary variables.

In general, auxiliary variables depend on state variables $\mathbf{x}$ and inputs $\mathbf{u}$: $\eta = \eta(\mathbf{x}, \mathbf{u})$. Therefore, we cannot simply take time derivatives of the auxiliary variables. The differential equation includes the time derivative of input $\mathbf{u}$, which makes the state transition non-causal. Here, we first consider the case where auxiliary variables do not depend on input $\mathbf{u}$, $\partial\eta/\partial\mathbf{u} = 0$. Under this condition, the time derivatives of auxiliary variables do not contain the derivative of input $\dot{\mathbf{u}}$ and can be written as

$$\dot{\eta} = g(\eta, \mathbf{x}, \mathbf{u}) \tag{18}$$

Here, we consider a linear regression for predicting the transition of auxiliary variables and apply statistical linearization to the dynamical equation

$$\frac{d\hat{\eta}}{dt} = \mathbf{H}_x \mathbf{x} + \mathbf{H}_\eta \eta + \mathbf{H}_u \mathbf{u} \tag{19}$$

where $\hat{\eta}$ is the prediction of η, and $\mathbf{H}_x \in \Re^{n_a \times n}$, $\mathbf{H}_\eta \in \Re^{n_a \times n_a}$, and $\mathbf{H}_u \in \Re^{n_a \times r}$ are parameter matrices to be tuned based on statistical linearization. For brevity, we combine these parameter matrices into $\mathbf{H} \triangleq (\mathbf{H}_x, \mathbf{H}_\eta, \mathbf{H}_u) \in \Re^{n_a \times l}$ and variables into $\xi \triangleq (\mathbf{x}^{\mathrm{T}}\, \eta^{\mathrm{T}}\, \mathbf{u}^{\mathrm{T}}) \in \Re^{l \times 1}$, where $\ell = n + n_a + r$. The parameter matrix $\mathbf{H}$ can be optimized so that the mean squared error of predicting $\dot{\eta}$ may be minimized

$$\mathbf{H}^0 = \arg\min_{\mathbf{H}} E[|\hat{\dot{\eta}} - \dot{\eta}|^2] \tag{20}$$

where $\dot{\eta}$ is the true derivative from the original nonlinear system and $\hat{\dot{\eta}}$ is the approximated one based on the linearized model (19). If the standard least squares estimate is used, the solution is given by

$$\mathbf{H}^0 = E[\dot{\eta}\xi^{\mathrm{T}}](E[\xi\xi^{\mathrm{T}}])^{-1} \tag{21}$$

assuming that the system is persistently excited and that there is no state feedback, i.e., $\mathbf{u}$ is not collinear with $\mathbf{x}$. Note that matrix $E[\xi\xi^{\mathrm{T}}]$ is nonsingular, since all the elements associated with the auxiliary variables are nonlinear. If some elements would have linear constitutive laws, the auxiliary variables would be collinear with the state variables, making $E[\xi\xi^{\mathrm{T}}]$ singular. It is essential that auxiliary variables are defined for all the nonlinear elements and only for the nonlinear elements, and that samples used for computing Eq. (21) are taken broadly from the true nonlinear system.

Remark. If the auxiliary variables are nonlinear functions of state variables alone: $\eta = \eta(\mathbf{x})$, and its time derivative is approximated to $\dot{\eta} = \mathbf{J}(\mathbf{x}) \cdot \dot{\mathbf{x}} \cong \bar{\mathbf{J}} \cdot \dot{\mathbf{x}}$ with some fixed matrix $\bar{\mathbf{J}}$, then the time derivatives $\dot{\mathbf{x}}$ and $\dot{\eta}$ are completely collinear and, thereby, the second set of state equations does not provide any new information. This is true no matter which method is used for approximating the state-dependent matrix $\mathbf{J}(\mathbf{x})$ to a fixed one $\bar{\mathbf{J}}$. Both Taylor expansion and statistic linearization methods fail to create a new state equation that is not collinear with the first equation (4).

The linear regression (19) can provide us with a different facet of the system dynamics, which is not collinear with Eq. (4). The tuning of the parameter matrix $\mathbf{H}$ is performed in a broader parameter space than that of the fixed parameter matrix $\bar{\mathbf{J}}$, and, in fact, the prediction accuracy is better. The theorem below guarantees this.

THEOREM 2. *The linear regression for predicting the dynamic transition of auxiliary variables $\eta \in \Re^{n_a \times 1}$ given by Eq. (19) does not underperform the one with a fixed matrix $\bar{\mathbf{J}}$ in mean squared error*

$$\min_{\mathbf{H} \in \Re^{n_a \times l}} E[|\dot{\eta} - \mathbf{H}\xi|^2] \leq \min_{\mathbf{H} \in \Re^{n_a \times n}} E[|\dot{\eta} - \bar{\mathbf{J}}\dot{\mathbf{x}}|^2] \tag{22}$$

The proof is simple. From Eq. (4), the prediction based on the fixed matrix, $\hat{\dot{\eta}} = \bar{\mathbf{J}}\dot{\mathbf{x}} = [\bar{\mathbf{J}}\mathbf{A}_x \quad \bar{\mathbf{J}}\mathbf{A}_\eta \quad \bar{\mathbf{J}}\mathbf{B}_x]\xi$ is a special case of general $\mathbf{H}$

$$\{\bar{\mathbf{H}} | \bar{\mathbf{H}} = [\bar{\mathbf{J}}\mathbf{A}_x \quad \bar{\mathbf{J}}\mathbf{A}_\eta \quad \bar{\mathbf{J}}\mathbf{B}_x] \in \Re^{n_a \times l}, \bar{\mathbf{J}} \in \Re^{n_a \times n}\} \subset \{\mathbf{H} | \mathbf{H} \in \Re^{n_a \times l}\}$$

Equation (22) directly follows from this.

The real data of ξ and $\dot{\eta}$ exhibit more profound properties than the one connected with a fixed $\bar{\mathbf{J}}$, thus making the auxiliary state equation a meaningful contributor and surrogate of the nonlinear dynamics.

With this auxiliary state equation, the total state equation can be expressed as

$$\frac{d}{dt}\begin{pmatrix}\mathbf{x}\\ \eta\end{pmatrix} = \begin{bmatrix}\mathbf{A}_x & \mathbf{A}_\eta\\ \mathbf{H}_x & \mathbf{H}_\eta\end{bmatrix}\begin{pmatrix}\mathbf{x}\\ \eta\end{pmatrix} + \begin{bmatrix}\mathbf{B}_x\\ \mathbf{H}_u\end{bmatrix}\mathbf{u} \tag{23}$$

5 Causal Auxiliary Variables

There is a fundamental impediment in formulating state equations for auxiliary variables. The time derivative of the auxiliary variables may be anti-causal, if the auxiliary variables depend on input $\mathbf{u}$. Namely, the time derivative of $\eta(\mathbf{x}, \mathbf{u})$ includes the derivative of input

$$\dot{\eta}(\mathbf{x}, \mathbf{u}) = \frac{\partial \eta}{\partial \mathbf{x}}\dot{\mathbf{x}} + \frac{\partial \eta}{\partial \mathbf{u}}\dot{\mathbf{u}} \tag{24}$$

Note, however, that this anti-causal situation occurs only for a specific type of nonlinear elements. The constitutive laws of independent energy storage elements, i.e., capacitors and inertias, are nonlinear functions of state variables, i.e., displacement and momentum, as shown in Figs. 3(*b*) and 3(*c*). If auxiliary variables are associated only with energy storage elements, they are functions of state variables alone and do not contain input $\mathbf{u}$. The following Lemma can be proven:

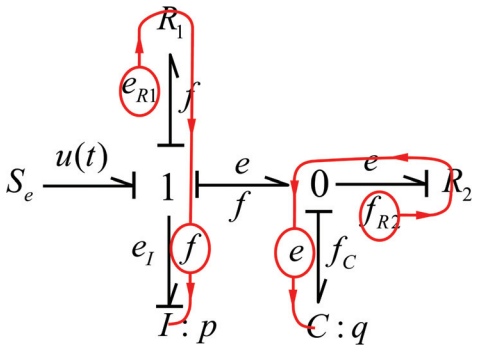

Fig. 5 Example of causal auxiliary variables where input *u*(*t*) is not involved in each causal path

LEMMA. *Auxiliary variables associated with nonlinear energy storage elements do not contain input $\mathbf{u} \in \Re^{r \times 1}$. Let η_C be auxiliary variables associated with capacity elements and η_I be the ones associated with inertia elements. Then*

$$\frac{\partial \eta_C}{\partial \mathbf{u}} = 0, \quad \frac{\partial \eta_I}{\partial \mathbf{u}} = 0 \tag{25}$$

Any component involved in the vectors, η_C and η_I, is a constitutive law, either $e = \Phi_C(q)$ or $f = \Phi_I(p)$, where q and p are independent state variables that do not depend on input $\mathbf{u}$. While the input variables drive the state variables through differential relations, the inputs cannot algebraically enter the constitutive laws of energy storage elements: $(\partial \Phi_C / \partial \mathbf{u}) = 0$ and $(\partial \Phi_I / \partial \mathbf{u}) = 0$.

Example 3. Consider a second-order system represented by the Bond Graph in Fig. 5. The constitutive laws of the inductance, capacitance, and the two resistances are all nonlinear functions. Therefore, the auxiliary variables are the output variables of these nonlinear elements: $\eta = (f \quad e \quad e_{R1} \quad f_{R2})^{\mathrm{T}}$. In these auxiliary variables, f and e are associated with energy storage elements (inductance and capacitance, respectively) and, therefore, functions of independent state variables alone, and do not include input $\mathbf{u}$. The above Lemma implies that resistive auxiliary variables may or may not include input $\mathbf{u}$. The auxiliary variable e_{R1} associated with resistance R_1 is a nonlinear function of flow f: $e_{R1} = \Phi_{R1}(f)$, and flow f is a nonlinear function of state variable p. Input $u(t)$ is not involved in these relationships. This can easily be found in the causal path shown in red in the figure. Backtracking the causal path from e_{R1}, the path goes through the resistance R_1 and the 1-junction, reaches flow f that is the input to the inductance I, and is terminated at the state variable p. Similarly, the causal path starting at the auxiliary variable f_{R2} goes through resistance R_2 and the 0-junction, reaches e that is the input of the Capacitor C, and is terminated at the state variable q. No input is involved in these causal paths, and the auxiliary variable is a nonlinear function of the state variable alone. The differential equations associated with these auxiliary variables are causal.

Example 4. Now consider another second-order system shown in Fig. 6, where a single resistance is connected to the 1-junction and the inductance is now connected to the 0-junction. If all three elements are nonlinear, the auxiliary variables are $\eta = (e \quad f_I \quad f)^{\mathrm{T}}$ in which f_I and e are directly determined by the state variables through the element constitutive laws, $f_I = \Phi_I(p)$ and $e = \Phi_C(q)$. However, the auxiliary variable f associated with the resistor R is connected to the input $u(t)$. As shown in the causal path in the figure, the path from f branches out to both input $u(t)$ and effort e at the 1-junction, which is governed by the junction condition, $u = e + e_R$. From this, we obtain

$$f = \Phi_R[u(t) - \Phi_C(q)] \tag{26}$$

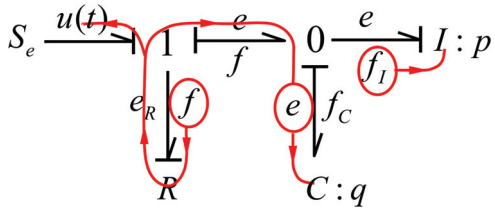

Fig. 6 Example of an anti-causal auxiliary variable where input $u(t)$ is involved in the causal path. Backtracking the causal path terminated at auxiliary variable f leads to input $u(t)$.

Therefore, this auxiliary variable is not causal. Its differential equation contains the time derivative of input $u(t)$ and, thereby, it is anti-causal.

5.1 An Algorithm for Reducing the Influence of Input on Auxiliary Variables. Suppose that there are n_{Ru} nonlinear resistive elements where input $\boldsymbol{u}$ is involved. All the auxiliary variables associated with these input-dependent resistive elements are placed in a vector $\eta_{Ru}(\mathbf{x},\mathbf{u}) \in \Re^{n_{Ru}\times 1}$. To solve the anti-causality problem, a two-step method is considered. First, we apply algebraic statistical linearization to the input-dependent resistive auxiliary variables, and then redefine the auxiliary variables so that their time derivatives do not depend on input's derivative.

First, we decompose the input-dependent auxiliary variables into a linear input term and the rest of the part

$$\eta_{Ru}(\mathbf{x},\mathbf{u}) = \tilde{\eta}_{Ru}(\mathbf{x},\mathbf{u}) + \mathbf{D}_{Ru}\mathbf{u} \tag{27}$$

where $\mathbf{D}_{Ru} \in \Re^{n_{Ru}\times r}$ is a coefficient matrix to be tuned statistically, so that the second term $\mathbf{D}_{Ru}\mathbf{u}$ can absorb most of the input-dependent component from $\eta_{Ru}(\mathbf{x},\mathbf{u})$. This implies that $E[|\mathbf{D}_{Ru}\mathbf{u} - \eta_{Ru}|^2]$ be minimized, or the residuals $E[|\tilde{\eta}_{Ru}|^2]$ be minimized

$$\mathbf{D}^o_{Ru} = \arg\min_{\mathbf{D}_{Ru}} E[|\mathbf{D}_{Ru}\mathbf{u} - \eta_{Ru}|^2] \tag{28}$$

Next, we define causal auxiliary variables from the residuals: $\tilde{\eta}_{Ru}(\mathbf{x},\mathbf{u}) = \eta_{Ru}(\mathbf{x},\mathbf{u}) - \mathbf{D}_{Ru}\mathbf{u}$. Namely, we extract only the state-dependent portion of the residuals by taking partial derivatives

$$\dot{\eta}^*_{Ru}(\mathbf{x},\mathbf{u}) \triangleq \frac{\partial \eta_{Ru}(\mathbf{x},\mathbf{u})}{\partial \mathbf{x}}\dot{\mathbf{x}} \tag{29}$$

where it is assumed that the input-dependent resistive elements have differentiable constitutive laws. Note that the new auxiliary variables $\eta^*_{Ru}(\mathbf{x}) \in \Re^{n_{Ru}\times 1}$ do not include the derivative of inputs and, thereby, is causal. The auxiliary state equation (18) can be obtained in the same manner as Eq. (21), where n_{Ru} components in $\dot{\eta}$ must be replaced by $\dot{\eta}^*_{Ru}(\mathbf{x},\mathbf{u})$ in Eq. (29). Namely

$$\dot{\eta} \leftarrow \dot{\eta}^* = \begin{pmatrix} \dot{\eta}_C \\ \dot{\eta}_I \\ \dot{\eta}_R \\ \dot{\eta}^*_{Ru} \end{pmatrix} \tag{30}$$

where η_R is a vector of auxiliary variables associated with nonlinear resistive elements that do not depend on input $\mathbf{u}$, like the ones in Example 3.

The auxiliary state equation is then modified to

$$\hat{\dot{\eta}}^* = \mathbf{H}^*_x\mathbf{x} + \mathbf{H}^*_\eta\eta + \mathbf{H}^*_u\mathbf{u} \tag{31}$$

where the coefficient matrices are obtained by minimizing the mean squared error for predicting $\dot{\eta}^*$ given by Eq. (29). Using Eq. (30), the auxiliary variables are

$$\eta \cong \eta^* + \mathbf{D}\mathbf{u} \tag{32}$$

where

$$\mathbf{D} = \begin{bmatrix} \mathbf{0}_{(n_a - n_{Ru})\times r} \\ \mathbf{D}^o_{Ru} \end{bmatrix} \in \Re^{n_a \times r} \tag{33}$$

Substituting Eq. (32) into Eq. (23) yields

$$\frac{d}{dt}\begin{pmatrix} \mathbf{x} \\ \eta* \end{pmatrix} = \begin{bmatrix} \mathbf{A}_x & \mathbf{A}_\eta \\ \mathbf{H}^*_x & \mathbf{H}^*_\eta \end{bmatrix}\begin{pmatrix} \mathbf{x} \\ \eta* \end{pmatrix} + \begin{bmatrix} \mathbf{B}_x + \mathbf{A}_\eta\mathbf{D} \\ \mathbf{H}^*_u + \mathbf{H}^*_\eta\mathbf{D} \end{bmatrix}\mathbf{u} \tag{34}$$

This provides causal augmented state equations.

The conversion and truncation in the latent space can be performed in the same way as before

$$\frac{d\mathbf{z}^*}{dt} = \mathbf{T}^*\begin{bmatrix} \mathbf{A}_x & \mathbf{A}_\eta \\ \mathbf{H}^*_x & \mathbf{H}^*_\eta \end{bmatrix}\mathbf{T}^{*\mathrm{T}}\mathbf{z}^* + \mathbf{T}^*\begin{bmatrix} \mathbf{B}_x + \mathbf{A}_\eta\mathbf{D} \\ \mathbf{H}^*_u + \mathbf{H}^*_\eta\mathbf{D} \end{bmatrix}\mathbf{u} \tag{35}$$

where $\mathbf{T}^* \in \Re^{k\times m}$ is the orthonormal matrix associated with the transformation to the new latent variables

$$\mathbf{z}^* = \mathbf{T}^{*\mathrm{T}}\begin{pmatrix} \mathbf{x} \\ \eta^* \end{pmatrix} \tag{36}$$

6 Numerical Examples

6.1 Systems With All Causal Auxiliary Variables

6.1.1 System. The above theory and method can be applied to various systems, where data are obtained from experiments, simulations, or a mixture of the two. Figure 7 is a simplified bond graph model inspired by an earthmoving robotic system consisting of a hydraulic actuator, an arm, and an end-effector. The dynamic interaction between the end-effector and the environment, e.g., soil and rock, is complex and nonlinear, but its behavior can be modeled as a combination of nonlinear spring (C_2), mass (I_2), and damper (R). Furthermore, the high-pressure hydraulic line exhibits a hard-spring nonlinear compliance (C_1) due to cavitation and oil compressibility as well as pipe and structural compliance. Their constitutive laws manifest pronounced nonlinearities, as shown in Fig. 8. The mass reflected to the actuator (I_1), on the other hand, is deemed linear. Various sensors, including pressure, position, velocity, and acceleration sensors, can be attached to the system to monitor not only state variables, but also auxiliary variables. Detailed finite-element simulation software is also available for simulating the system under realistic conditions. All variables are normalized and mean-centered.

The system contains four independent energy storage elements, I_1, I_2, C_1, and C_2, and therefore, it is a fourth-order system with

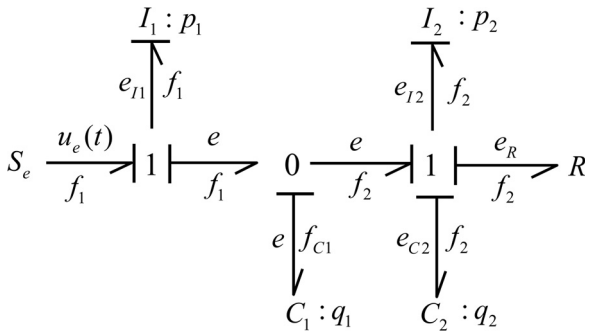

Fig. 7 Bond graph of a dynamical system inspired by an excavator powered by a hydraulic system

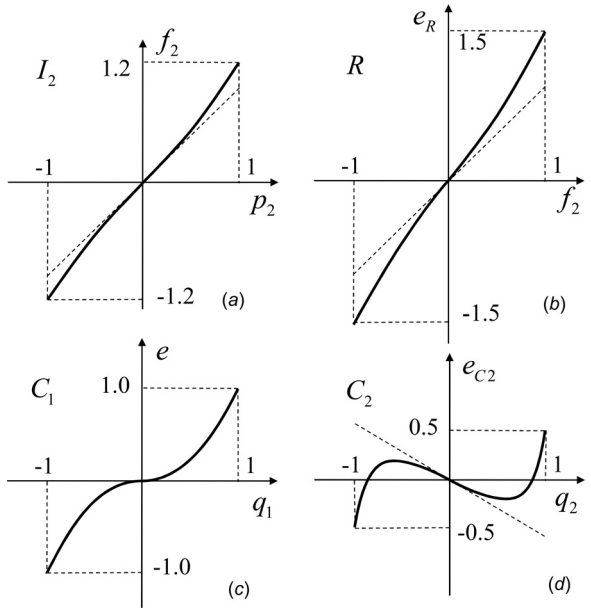

Fig. 8 Constitutive laws of the system in Fig. 7

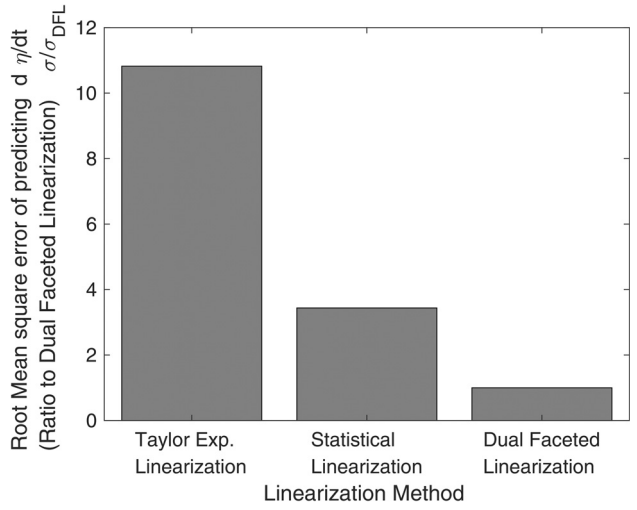

Fig. 9 Comparison of three linearization methods, Taylor expansion, statistical linearization, and DFL, in terms of the root mean square of predicting $\dot{\eta}$. The bar chart is normalized by the root-mean-square (RMS) of DFL: σ_{DFL} .

four state variables, p_1, p_2, q_1, and q_2. The system also contains four nonlinear elements, C_1, C_2, I_2, and R, and therefore, requires four auxiliary variables

$$\eta = (\, e \quad e_{C2} \quad f_2 \quad e_R \,)^{\mathrm{T}} \tag{37}$$

These are outputs of the four nonlinear elements

$$e = \Phi_{C1}(q_1), \quad e_{C2} = \Phi_{C2}(q_2), \quad f_2 = \Phi_{I2}(p_2), \quad e_R = \Phi_R(f_2) \tag{38}$$

These auxiliary variables do not contain input u_e, and therefore, they are causal auxiliary variables for which time derivatives exist.

6.1.2 Linearization Comparison. Figure 9 shows comparison of the three linear approximation methods, (a) Taylor expansion, (b) statistical linearization, and (c) dual faceted linearization, in terms of root mean square error in predicting the transition of auxiliary variables, $\dot{\eta}$. Note that the state equation of $\mathbf{x}$ is exact; the error comes from the transition of the auxiliary variables. The bar chart in the figure indicates that the dual faceted linearization is approximately three times more accurate than the Statistical Linearization and over ten times more accurate than the Taylor expansion.

Actual trajectories of the linearized models inevitably deviate from the original nonlinear system, as illustrated in Fig. 10. However, the deviation speed is different depending on the linearization method. In other words, the time horizon in which the linearized model stays within an acceptable error limit is different. Here, we evaluate accuracy of each linearization method in terms of the error for a given time horizon.

Figure 11 shows the deviation envelope, that is, the maximum deviation of each linearized model for a time horizon of 0.3 s after departing from each point on the trajectory of the original nonlinear model. The envelope was computed by simulating each linearized model with initial conditions taken from every point on the true trajectory. The maximum deviation for the following 0.3 s was plotted at each time slice along the true trajectory. The traditional Taylor expansion method significantly deviates from the correct trajectory in 0.3 s, as indicated by the large envelope. The Statistical Linearization method can keep track of the correct trajectory with a decent accuracy. The dual faceted linearization, on the other hand, is significantly better, tracking the true trajectories of both independent state and auxiliary variables with high accuracy.

While Fig. 11 shows the deviation envelopes of only one state variable and one auxiliary variable, similar results were obtained for other variables. Figure 12 shows the total mean squared error of all the four state variables over diverse time horizons. Again, the dual faceted linearization shows a significantly better result than the other two methods.

6.2 Systems Containing Noncausal Auxiliary Variables. Consider the system shown in Fig. 13, consisting of two resistive elements R_1 and R_2, one capacitive element with state variable q and one inertial element with state variable p. Elements I and R_2 are assumed linear, while C and R_1 are nonlinear with which auxiliary variables f_1 and e_1 are associated. As shown in the figure, input $u_e(t)$ sneaks in the auxiliary variable f_1 through the left 1-junction

$$\begin{aligned} e_1 &= \Phi_C(q) \\ f_1 &= \Phi_{R_1}(u_e - \Phi_C(q)) \end{aligned} \tag{39}$$

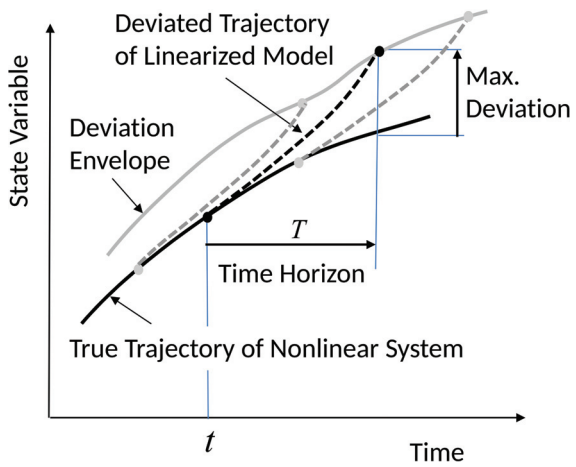

Fig. 10 Schematic of deviated trajectories of linearized models from the exact nonlinear model

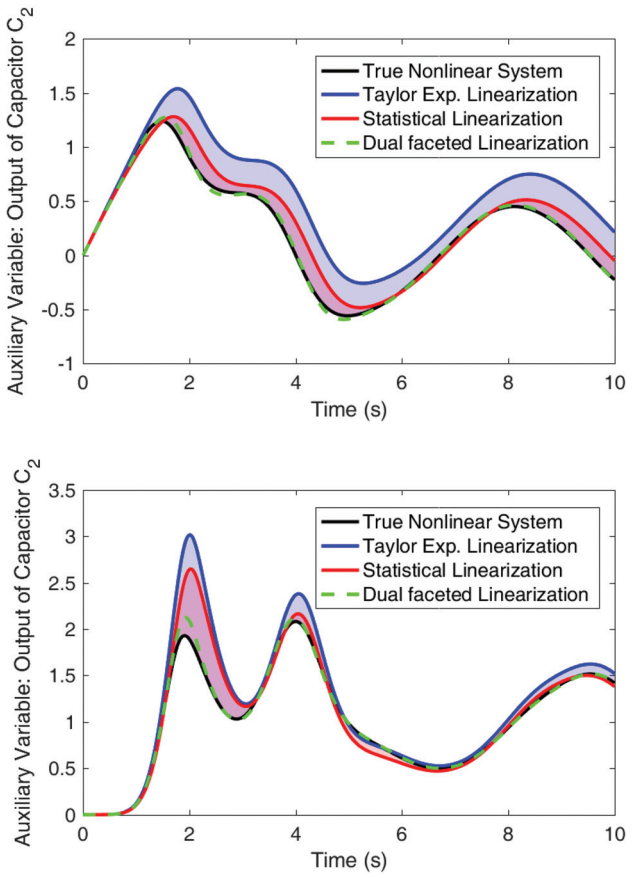

Fig. 11 Deviation envelopes of different linearization models. The maximum deviation from the exact nonlinear model over the time horizon of 0.3 s is shown by the envelope.

The state equations can be expressed as

$$\frac{dq}{dt} = f_1 - \frac{1}{m}p$$
$$\frac{dp}{dt} = e_1 - \frac{R_2}{m}p \tag{40}$$

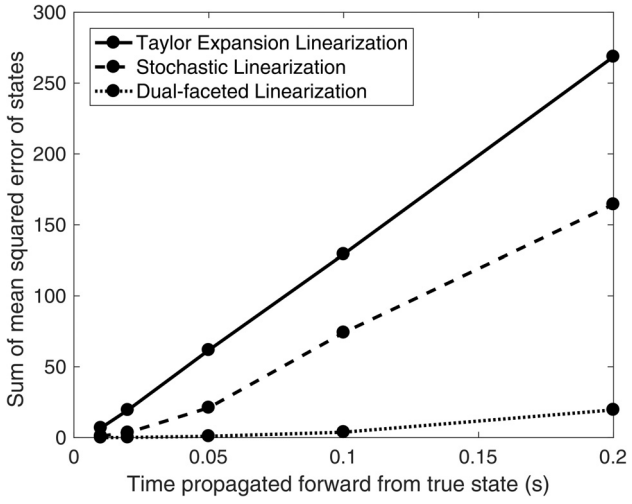

Fig. 12 Sum of mean squared errors of all the state variables. Comparison of the three linearization models against time horizon.

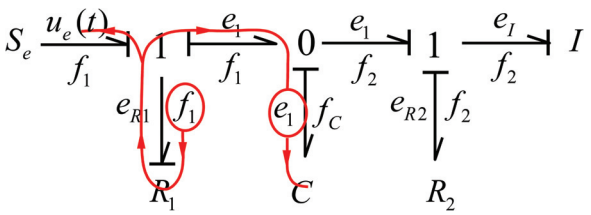

Fig. 13 Bond graph of a system with anti-causal auxiliary variable

Using the method presented in Sec. 5, the influence of input is maximally absorbed by a linear term $D_{Ru}u_e$. The optimal value of D_{Ru} is obtained by minimizing the expectation of $|D_{Ru}u_e - f_1|^2$. This is given by calculating $E[f_1 u_e](E[u_e^2])^{-1}$. For the sample data used for simulations, the optimal value was found to be $D_{Ru} = 0.348$.

Using the optimal D_{Ru}, the modified state equation (34) was formed, and the linearization accuracy was compared to other algebraic linearization methods (Taylor series expansion and Statistical Linearization) in the same way as before. Figure 14 shows typical responses to a sinusoidal input. Again, the DFL outperforms the Taylor series expansion and the Statistical Linearization. In terms of the overall root mean square error in predicting the transition of the auxiliary variables, the DFL method is approximately six times better than the Taylor series expansion and twice better than the Statistical Linearization.

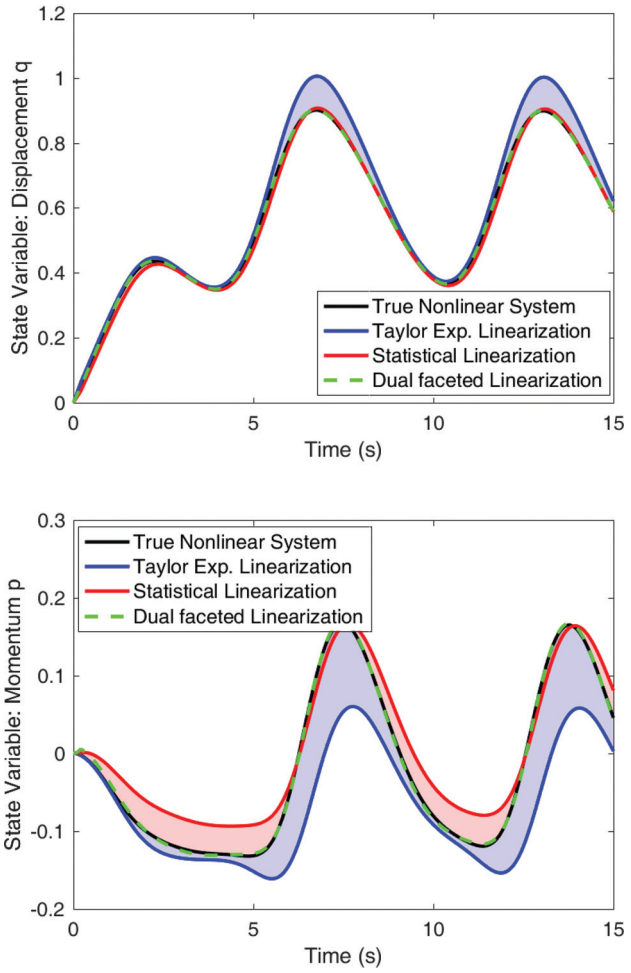

Fig. 14 Deviation envelopes of simulations using the causal auxiliary variables $\eta*$

6.3 Discussion. The salient feature of the dual faceted linearization is to formulate two state equations for each of nonlinear energy storage elements. In the first numerical example, the nonlinear inertial element I_2 having the nonlinear constitutive law $f_2 = \Phi_{I2}(p_2)$ created two state equations: one with respect to p_2 and the other with f_2. The former is in the form of $dp_2/dt = e_{I2}$, where variable e_{I2} is expanded by following the causal path across the bond graph in Fig. 7, leading to the matrices $\mathbf{A}_x, \mathbf{A}_\eta$, and $\mathbf{B}_x$. The latter state equation is in terms of the output of the nonlinear inertial element: $df_2/dt = \cdots$. The linearized state equation with coefficient matrices $\mathbf{H}_x, \mathbf{H}_\eta, \mathbf{H}_u$ delineates a different characteristic of the system dynamics viewed from the auxiliary variable f_2. Similarly, associated with the nonlinear capacitive element C_1 with constitutive law $e = \Phi_{C1}(q_1)$, two state equations, $dq_1/dt == f_{C1} = \cdots$ and $de/dt = \cdots$, were used. On the other hand, the linear inertial element I_1 had only one state equation, since its constitutive law is linear, $f_1 = p_1/m$, yielding the same state equation for both p_1 and f_1. The use of dual state variables for nonlinear elements is more informative and can predict the system's behaviors more accurately, as demonstrated in the numerical examples.

Care must be taken, however, since the two state equations may be collinear. If the traditional algebraic linearization methods, e.g., Taylor expansion and statistical linearization, are used, the second set of state equations does not add any new information, as long as the nonlinear constitutive laws are approximated to linear algebraic relations with a fixed coefficient matrix $\bar{\mathbf{J}}$. No matter which method is used for optimal selection of $\bar{\mathbf{J}}$, the resultant two sets of state equations are completely collinear and redundant.

It is an important challenge to make all the auxiliary variables causal, so that the governing equations of dynamic transition may not contain the time derivative of input. The method presented in Sec. 5 is in a sense a hybrid method combining the traditional algebraic statistical linearization and DFL. The former is used for extracting the influence of input from the input-dependent auxiliary variables with the linear term $D_{Ru}u_e$ and the latter is used for all the other input-free auxiliary variables. As such, the approximation accuracy relative to the traditional algebraic linearization reduces, as demonstrated by the numerical example.

The dual faceted linearization has the potential to make a highly complex nonlinear control problem to an order-of-magnitude simpler problem. DFL can predict accurate dynamic responses for limited yet long-enough intervals. Model predictive control (MPC) of nonlinear stochastic systems, for example, can be solved effectively with DFL. Although the original system is highly complex and real-time computation of MPC is infeasible, DFL provides a linear model that is valid within a certain time horizon and, thereby, reduces the nonlinear MPC to linear MPC. Although care must be taken with the valid time horizon, control decisions to be made will be valid within the time horizon of tolerance error. Added benefits include the simplification of nonlinear stochastic dynamics. Although nonlinear stochasticity is often too complex to deal with, the DFL formulation can reduce the nonlinear stochasticity to linear stochasticity in the augmented state space [17].

Both statistical linearization and dual faceted linearization require data sets for parameter tuning. In the numerical example presented earlier, a simple grid method was used for generating data sets. As the order of a system increases, such an exhaustive grid point method becomes impractical. Effective sampling techniques will be required for representing a high-order system with fewer data points.

7 Conclusion

A new approach to modeling and linearization of nonlinear lumped-parameter systems has been presented. Nonlinear state equations are recast in an augmented state space by adding a set of auxiliary variables that sufficiently informs the nonlinear dynamics. Two major results have been obtained:

- Without causing any nonlinear algebraic loop problem, a linear state equation can be derived from a bond graph of integral causality by using auxiliary variables, i.e., the outputs of all the nonlinear elements involved in the system.
- A set of linear differential equations predicting the dynamic transition of auxiliary variables has been obtained. For this second set of dynamic equations, it has been shown: (i) the second dynamic equations do not add any new information if an algebraic linearization with a fixed coefficient matrix $\bar{\mathbf{J}}$ relating the auxiliary variables to the independent state variables is used for linear approximation and (ii) the proposed linear regressor can predict the transition of the auxiliary variables more accurately than the one with a fixed algebraic linearization. The two sets of linear dynamic equations can capture diverse facets of the nonlinear system and, thereby, represent the true behaviors more precisely. The two sets of state equations have also been combined using a latent modeling method.

Furthermore, it has been shown that a class of auxiliary variables associated with nonlinear resistive elements may depend on inputs. This makes the state equations of the auxiliary variables anti-causal. To alleviate the problem, causal auxiliary variables with minimum influence of inputs have been introduced. Finally, numerical examples have verified the theoretical results and demonstrated the effectiveness of the proposed methods compared to Taylor expansion and stochastic linearization methods. Future research issues will include the application of DFL to more complex systems, such as biological systems, where thousands of state variables and nonlinear elements are involved. Combined with the latent modeling method, DFL can provide a compact, linear dynamical model that is amenable for analysis and control design.

Acknowledgment

The authors thank Dr. Michaelle Mayalu of Cal Tech, Dr. Sampriti Bhattacharyya of Hydroswam, and Professors Neville Hogan and Kamal Youcef-Toumi of MIT for their valuable comments and discussion on this work.

Funding Data

- The National Science Foundation (NSF), Science and Technology Center (STC), and Emergent Behaviors in Integrated Cellular Systems (EBICS) Grant CBET-0939511.
- The National Research Foundation Singapore through the Singapore MIT Alliance for Research and Technology's BioSyM IRG Research Program.
- Komatsu, Ltd.
- Ford Foundation.

Appendix A: Proof of Theorem 1

The theorem can be proven by showing that backtracking any causal propagation of variables from any of energy storage elements, $dp/dt = e$ or $dq/dt = f$, leads to either state variables, or auxiliary variables, or inputs only through linear relations. In each backtracking step of causal propagation, a bond is connected to either:

(1) A source (or a Sink) with input u_e for an effort source, or u_f for a flow source;
(2) an energy storage element with a constitutive law

$$f = \Phi_I(p) \text{ or } e = \Phi_C(q) \tag{A1}$$

(3) another junction, 1 or 0 junction; or
(4) a resistive element with a constitutive law

$$e_{\text{out}} = \Phi_R(f_{\text{in}}) \text{ or } f_{\text{out}} = \Phi_{\bar{R}}(e_{\text{in}}) \tag{A2}$$

In case 1, the causal propagation originates at input u_e or u_f. In case 2, if the energy storage element is nonlinear, the backtracking of the causal path also terminates with the output of the element, either $f = \Phi_I(p)$ or $e = \Phi_C(q)$, both of which are auxiliary variables. If the energy storage element is linear, then the causal path reaches one of the state variables, either $e = q/C$ or $f = p/m$ where C and m are constant parameters. In case 3, the backtracking of the causal path continues to a single bond connected to the junction or spreads out to all the bonds connected. Whichever the case, the propagated variable coming to the junction is a linear function of the outgoing variables. In case 4, if the resistive element is nonlinear, then the backtracking terminates at the element with auxiliary variable $e_{\text{out}} = \Phi_R(f_{\text{in}})$ or $f_{\text{out}} = \Phi\tilde{R}(e_{\text{in}})$. If the resistive element is linear, the variable is converted between effort and flow, and continues to propagate with $e_{\text{out}} = Rf_{\text{in}}$ or $f_{\text{out}} = e_{\text{in}}/R$. In case linear resistive elements form a loop, the algebraic equation can be solved explicitly.

Therefore, all the possible continuing propagations are expressed as linear relations, and the variable associated with any terminated propagation is an input, state variable, or auxiliary variable. Any algebraic loop of this propagation is a series of all linear resistive elements from which a linear relation among inputs, state variables, and auxiliary variables can be obtained. ■

Appendix B: Derivation of the Equations of Motion From the Bond Graph in Fig. 2

By definition, the time-derivative of momentum p_1 associated with inertial element I_1 is effort variable e_1 : $(dp_1/dt) = e_1$. Note that the inertial element I_1 is connected to the left 0-junction in Fig. 2, where the four power bonds are connected, sharing the common effort e_1. Among the four power bonds, the capacitive element has the causal stroke on the junction side, indicating that this element determines effort e_1 with its own constitutive law $e_1 = \Phi_C(q)$. Substitution of this constitutive law to the above definition equation gives Eq. (12).

By definition, the state transition of displacement q associated with the capacitive element C is given by $(dq/dt) = f_C$. Note that the capacitive element C is connected to the left 0-junction where the four flow variables sum to zero. This gives the junction condition: $u_f(t) - f_C - f - f_{I1} = 0$. Thereby, f_C is replaced by $u_f(t) - f - f_{I1}$. The flow variable f is coming from the right 0-junction, where the three flow variables sum to zero: $f - f_{I2} - f_{R2} = 0$, which allows us to extend f to $f_{I2} + f_{R2}$. The inertial element I_1 determines the flow f_{I1} as a function of its state $f_{I1} = \Phi_{I1}(p_1)$, and so does $f_{I2} = \Phi_{I2}(p_2)$ for I_2. Note that $f_{R2} = \Phi_{R2}(e)$ is a function of effort e, which is determined by the capacitive element connected to the left 0-junction, $e = \Phi_C(q)$. Substitution of these yields Eq. (13).

Equation (14) can be derived in the same manner by backtracking the causal propagation path from the inertial element I_2 through the right 0-junction and the 1-junction, as directed by the causal strokes.

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