SYNTHESIS OF A FINITE TWO-TERMINAL NETWORK WHOSE PHASE ANGLE IS A PRESCRIBED FUNCTION OF FREQUENCY

With Application to the Design of Symmetrical Networks

by

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SUBMITTED IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE OF
MASTER OF SCIENCE

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

1948

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ABSTRACT

The increased use of modern "exact" procedures in the field of electrical network design has revealed the need for research in the problem of synthesizing networks with prescribed phase angle characteristics. This thesis seeks to provide the general solution to the phase synthesis problem in the case of a two-terminal passive network.

In order to avoid the transcendental relationships associated with phase angles, attention is centered on the tangent of the angle rather than the angle itself. This point of view leads naturally to simple algebraic relationships. It is shown that there exists a network solution if, and only if, the prescribed tangent function is an odd rational function of frequency. Moreover, general synthesis methods for obtaining the networks are worked out.

Although the discussion is carried on mainly from the standpoint of network theory with emphasis on

existence theorems, the needs of the engineer who must design the networks are not overlooked. Detailed design procedures are developed and are illustrated with a variety of numerical examples.

The kernel of the argument is the fact that the imaginary axis of the complex frequency plane is a sorting boundary for roots of the expression formed by adding numerator and denominator of the tangent function. When properly sorted these roots are recombined to form the impedance function. Much of the discussion revolves about the proof of this statement, roots missing due to cancellation, and the correct allocation of roots on the imaginary axis.

As an application of phase synthesis a design method for symmetrical filter networks is presented. It is shown that the usual computational labor can often be materially reduced by the exploitation of a network partitioning theorem which relates the desired transfer impedance to the phase angle of half of the network. The method is extended to cover certain non-symmetrical networks. Examples of these applications are given.

The methods used in obtaining the results cannot be said to follow any particular pattern other than a general search for relationships which might be suitable for synthesis. The search was conducted by starting with the general positive real driving-point impedance expression, forming from it the expression for the tangent of the phase angle, and noting the results of the latter operation to see if it could be worked in reverse. In the discovery of the root-sorting process, which is fundamental, it is probable that no small part was played by luck.

ACKNOWLEDGMENT

I am deeply grateful to Professor E. A. Guillemin for his help and encouragement during the course of this work. His surprisingly unvarying confidence in the outcome in spite of many early failures left me no alternative but to pursue the work diligently to a successful conclusion.

I am especially indebted to Professor M. F. Gardner, who four years ago inspired the program which is here culminated, and who, together with Professor K. L. Wildes, patiently guided my efforts.

Nothing, however, could have been accomplished without the kind and patient help of my wife. Any thanks I give here are small measure indeed for her understanding.

Finally, my grateful thanks are due to Mrs. C. Ross for her careful handling of the manuscript.

Part I

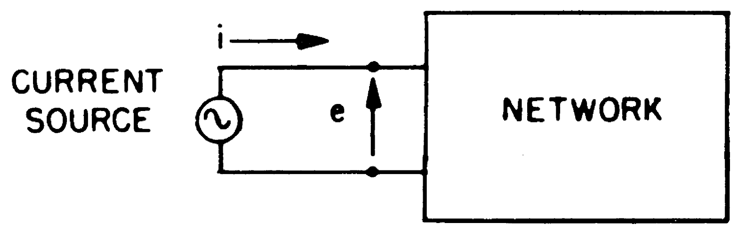
INTRODUCTION

The problem of phase synthesis is introduced and, several applications are described. The structure of the ensuing argument is presented.

1. DESCRIPTION OF THE PROBLEM

Among the variety of characteristics which are frequently found useful in describing or specifying the dynamic behavior of a two-terminal electrical network, an important role is played by the concept of the phase angle of the network impedance.

Implied in the definition of this concept is first, the convention that the driving excitation is to vary sinusoidally with time, and second, the well-known result of linearity that the network response also varies sinusoidally with time and at the same frequency. Since the most general passive two-terminal network can be completely characterized by the relationship that exists between the voltage and the current at its terminals, it is sufficient for the study of the network impedance to assign, for example, a driving force consisting of a sinusoidal current source of unit amplitude, and then to examine the amplitude and phase displacement of the voltage appearing across the terminals as the driving frequency is varied. The physical situation is shown in Fig. I-1 and the usual vector representation in the complex plane in Fig. I-2.



$i(t) = 1 \sin \omega t$

$e(t) = E_0(\omega) \sin(\omega t - \theta(\omega))$

FIG. I-1

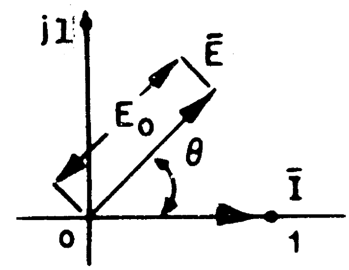


FIG. I-2

In this thesis, we are particularly concerned with the phase displacement, $\theta(\omega)$, and with the possibility and methods of constructing two-terminal networks whose impedances have phase displacements which are prescribed functions of frequency.

Necessary and sufficient conditions on the prescribed functions are presented, and, in addition, workable synthesis techniques are developed. It turns out that from the phase function alone we can determine the whole impedance. From this the network can then be synthesized.

2. APPLICATIONS

A useful application of phase synthesis lies in the design of feedback amplifier networks. Design methods currently in use involve the synthesis of a transfer network with a prescribed phase angle characteristic. We can readily obtain such a network by inserting a properly designed driving-point impedance in the plate circuit of a vacuum tube. In order to apply this procedure, however, we must have available a synthesis technique for the driving-point impedance, which permits accurate and reliable control of the phase angle tolerances over the frequency spectrum. Our method seeks to provide such a technique, and it achieves its goal within the usual limitations of practical network components and economic considerations of size and complexity.

Feedback networks are ordinarily designed from an amplitude vs. frequency curve which is first calculated from the desired phase vs. frequency curve. In practice this amplitude curve must be approximated by an algebraic expression before synthesis can proceed. Herein lies the difficulty in maintaining an allowable phase tolerance. It turns out that the relation is very obscure between the degrees of approximation to the two associated curves, and hence some uncertainty enters into the accuracy of the resulting phase when its tolerance is indirectly controlled by the amplitude tolerance.

On the other hand, the use of phase synthesis permits direct control by allowing the approximation to be made at once to the phase curve so that deviations are known at the start.

Another application of phase synthesis is presented in a later portion of this paper, where it is shown that the computational labor in the design of symmetrical filter networks can be very materially reduced.

3. PLAN OF THE DISCUSSION

In Part II we collect sufficient background material to promote continuity of the argument.

Part III deals with the existence of solutions. We show that there exists a physically realizable network for any phase function $\theta(\omega)$ whose tangent is an odd rational function of ω with real coefficients.

In Part IV we develop synthesis procedures which enable the designer to construct the impedance function and hence the network corresponding to any realizable phase function.

In Part V we present a network theorem which relates the transfer impedance of a suitable network to the impedances of its parts. We combine this theorem with the results of the phase synthesis discussion to develop a simple method for obtaining the design of symmetrical filter networks.

In the Appendix we present formulas for the determination of certain roots associated with odd Tchebycheff polynomials for use in filter design.

In addition, we demonstrate a curious result involving alternation of these roots.

Part II

BACKGROUND MATERIAL

Since much of the discussion which follows is based upon a mathematical treatment which uses a variety of concepts, theorems, and definitions, some of which do not appear in the literature, it seems desirable to summarize at the outset certain background material in order to preserve continuity of the argument. We present this material briefly and without formal proof but with references where possible.

1. THE CONCEPT OF COMPLEX FREQUENCY

Expressed in its usual form as a function of real impressed frequency ω, the impedance of a lumped constant network turns out to be a rational algebraic function of the pure imaginary variable $j\omega$. On a purely formal basis, we can generalize the impedance function by replacing $j\omega$ with the complex variable $\lambda = \sigma + j\omega$. This is equivalent to adopting the viewpoint that for real frequencies we are dealing with a restricted set of values of λ, namely those on the imaginary axis.

We make no attempt to ascribe a physical meaning to a complex value of frequency, but we justify the generalization by the fact that it enables us to apply all the power of function theory to study the properties and peculiarities of impedance functions that are associated with physical networks.

An excellent discussion of the meaning and use of the complex frequency concept has been presented by Bode[1] (Chapter II).

2. IMPEDANCE DEFINITIONS

The complex impedance function with the independent variable λ is expressed variously as

$$Z(\lambda) = \frac{P(\lambda)}{Q(\lambda)}, \tag{2-1}$$

or

$$Z(\lambda) = \frac{m_1(\lambda) + n_1(\lambda)}{m_2(\lambda) + n_2(\lambda)}, \tag{2-2}$$

with

$$P(\lambda) = m_1(\lambda) + n_1(\lambda), \tag{2-3}$$

and

$$Q(\lambda) = m_2(\lambda) + n_2(\lambda) \tag{2-4}$$

---

1 For footnote references see bibliography.

Both $P(\lambda)$ and $Q(\lambda)$ are polynomials in λ, while the m's and n's are respectively the even and odd parts of these polynomials. For example, m_1 consists of all terms of $P(\lambda)$ involving even powers of λ. We note that

$$P(-\lambda) = m_1(\lambda) - n_1(\lambda) \tag{2-5}$$

$$Q(-\lambda) = m_2(\lambda) - n_2(\lambda) \tag{2-6}$$

All of the polynomials have real coefficients.

The above notation follows that given in the M.I.T. Staff Text[2] on Circuit Analysis (Chapter VI, Art. 26).

3. ZEROS OF THE POLYNOMIALS

For brevity we discuss $P(\lambda)$ with the understanding that the following remarks apply with appropriate modifications to $Q(\lambda)$.

$P(\lambda)$ is a Hurwitz polynomial, that is, none of its zeros lie in the right half of the complex plane; they are restricted to the left half-plane or the imaginary axis.

The zeros of $m_1(\lambda)$ and $n_1(\lambda)$ lie only on the imaginary axis. Furthermore, the zeros of $m_1(\lambda)$ alternate with those of $n_1(\lambda)$ along this axis.

When $Z(\lambda)$ is in its lowest terms, any zeros of $P(\lambda)$ which lie on the imaginary axis must be simple.

An explanation of the reasons for these requirements is contained in the Staff Text[2] (Chapter VI, Art. 25).

4. POSITIVENESS OF THE REAL PART OF $Z(\lambda)$

If $Z(\lambda)$ is not a pure reactance function, its real part will not be identically zero for all real frequencies. Where the real part differs from zero it must be positive.

When $Z(\lambda)$ is rationalized it turns out that its real part can be expressed as

$$Re\left[Z(j\omega)\right] = \left.\frac{m_1 m_2 - n_1 n_2}{m_2^2 - n_2^2}\right]_{\lambda = j\omega} \qquad (2\text{-}7)$$

Gewertz[3] has shown that this expression yields a criterion for the required positiveness of $Re\left[Z(j\omega)\right]$ in the form of a requirement that the even polynomial $m_1 m_2 - n_1 n_2$, when expressed as a polynomial in λ^2, must not have λ^2-zeros of odd multiplicity on the negative real axis.

5. POSITIVE REAL FUNCTIONS

By definition, a positive real function is one which has the following properties:

a) It takes on real values when the independent variable is real, and

b) Its real part is positive when the real part of the independent variable is positive.

Brune[4] has shown that if there is to exist a physically realizable network corresponding to an arbitrary impedance function, it is both necessary and sufficient that the function be positive real.

There are various equivalent methods for demonstrating the positive real character of an arbitrary function. For our purposes, it is sufficient to show that:

a) $P(\lambda)$ and $Q(\lambda)$ are Hurwitz polynomials,

b) $Re\,[Z(j\omega)] \geqslant 0$ for all ω, and

c) All poles of $Z(\lambda)$ on the imaginary axis are simple and have positive real residues.

6. RELATION BETWEEN TRANSFER AND DRIVING-POINT IMPEDANCE

Darlington[5] has shown that any positive real driving-point impedance function can be synthesized in the form of a lossless two terminal-pair network terminated in a one ohm resistance.

The transfer impedance of such a network has been shown by Guillemin[6] to be related to the driving-point impedance by the following relationship.

$$\left| Z_{12} \right|^2_{j\omega} = Re\left[Z(j\omega) \right] \tag{2-8}$$

We use as notation for the lossless network with the one ohm termination removed from the output

$\mathfrak{z}_{11}$ - the open-circuit driving-point impedance at the input end,

$\mathfrak{z}_{22}$ - the open-circuit driving-point impedance at the output end, and

$\mathfrak{z}_{12}$ - the open-circuit transfer impedance of the network.

Part III

EXISTENCE OF SOLUTIONS

In order to investigate the interrelationship that exists between a phase angle function and its associated impedance function, we start with a general impedance expression and examine the resulting phase. This procedure yields the necessary conditions for physical realizability. We show that these are also sufficient and thus lay the groundwork for the attack on the synthesis problem.

1. THE TANGENT FUNCTION $T(\lambda)$

Because of the transcendental nature of the relationship between an impedance and its phase angle, it is difficult to correlate their behavior.

For this reason it is found convenient to deal not with the phase angle directly, but instead with its tangent. This procedure allows the mathematical discussion to proceed on a purely algebraic basis with consequent simplicity. In addition it turns out that the tangent function appears explicitly in a rather elegant manner in the customary expression for the loss function of a symmetrical filter network. The multi-valued character of the inverse tangent need not cause any ambiguity in synthesis because the phase angles considered lie only in the region $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$ in order to apply to realizable driving-point impedances. In this domain the phase angle and its tangent have a one-to-one relationship.

In order to form the tangent function we first rationalize the impedance (in its lowest terms) in the usual manner to obtain a separation of its real and imaginary parts.

$$
\begin{aligned}
Z(\lambda) &= \frac{m_1 + n_1}{m_2 + n_2} \cdot \frac{m_2 - n_2}{m_2 - n_2} \\
&= \frac{m_1 m_2 - n_1 n_2}{m_2^2 - n_2^2} + \frac{m_2 n_1 - m_1 n_2}{m_2^2 - n_2^2} .
\end{aligned}
\tag{3-1}
$$

For imaginary values of λ (i.e. $\lambda = j\omega$) the first term on the right is $Re[Z(j\omega)]$ and the second is $j\,Im[Z(j\omega)]$. To emphasize this, Eq. (3-1) is rewritten

$$
Z(j\omega) = \left.\frac{m_1 m_2 - n_1 n_2}{m_2^2 - n_2^2}\right]_{\lambda = j\omega} + j \left.\frac{\frac{1}{j}(m_2 n_1 - m_1 n_2)}{m_2^2 - n_2^2}\right]_{\lambda = j\omega} , \tag{3-2}
$$

from which we obtain

$$
\tan\theta = \left.\frac{\frac{1}{j}(m_2 n_1 - m_1 n_2)}{m_1 m_2 - n_1 n_2}\right]_{\lambda = j\omega} . \tag{3-3}
$$

That this expression is real, as it necessarily must be for θ to be a real angle, is evident from the odd functional character of the numerator, which allows a cancellation of the j when the function is evaluated at $\lambda = j\omega$.

In order to forestall any confusion arising from repeated manipulation of the j in the ensuing discussion, we define the following function, which we call the tangent function:

$$T(\lambda) = \frac{m_2 n_1 - m_1 n_2}{m_1 m_2 - n_1 n_2}, \qquad (3\text{-}4)$$

and we allow λ to range over the complex plane, keeping in mind that whenever λ lies on the j-axis, $T(j\omega)$ is equal to $j \tan \theta$.

The numerator and denominator on the right are necessarily simple polynomials with real coefficients because the positive real character of the original impedance requires this property for the m's and n's. In addition $T(\lambda)$ is seen to be an odd function.

2. THE FUNCTION TO BE SYNTHESIZED

In general the synthesis problem starts with the examination of an arbitrary function of the real frequency variable ω, which purports to be the tangent of the phase angle of an impedance. Reference to Eq. (3-4) shows that the only allowable functions are necessarily in the form of rational algebraic fractions. This fact may be expressed as

$$\tan \theta = \frac{A(\omega)}{B(\omega)}, \tag{3-5}$$

where A and B are polynomials.

To convert the expression on the right into a form more suitable for synthesis operations, it is readily changed to $T(\lambda)$ by substituting $-j\lambda$ in place of ω and multiplying the whole expression by j. This yields

$$T(\lambda) = j\,\frac{A(-j\lambda)}{B(-j\lambda)} = \frac{a(\lambda)}{b(\lambda)}. \tag{3-6}$$

A comparison of Eq. (3-6) with Eq. (3-4) reveals that a necessary condition for realizability requires a and b to be simple polynomials with real coefficients, and their ratio to be odd in λ.

3. THE CANCELLATION PROBLEM

Further than the above conclusions it is difficult to generalize at present for the following reason.

The given fraction a/b of Eq. (3-6) is, of course, in its lowest terms because one would not ordinarily specify a function which has common factors in its numerator and denominator. At any rate, the value of the ratio is not changed if common factors are cancelled, and we define a and b as the resulting polynomials after the cancellation of any accidental common factors.

On the other hand, the expression

$$\frac{m_2 n_1 - m_1 n_2}{m_1 m_2 - n_1 n_2}, \tag{3-7}$$

from Eq. (3-4), which is obtained by forming $T(\lambda)$ from $Z(\lambda)$, may not be in its lowest terms. Hence there is no assurance that a is equal to $m_2 n_1 - m_1 n_2$ nor that b is equal to $m_1 m_2 - n_1 n_2$ separately.

It is shown presently that the synthesis procedure requires a knowledge of the full expression for each of the polynomials $m_2 n_1 - m_1 n_2$ and $m_1 m_2 - n_1 n_2$. It is necessary, therefore, to investigate fully at this time the conditions under which cancellation can occur, the types of factors which can be cancelled, and, most important, a technique for examining a/b to find out if cancellation has occurred; so that the cancelled factors can be restored.

It turns out that concealed in the polynomial $b(\lambda)$ there is unmistakable evidence as to whether or not cancellation has occurred and further, that a simple operation regains all of the cancelled factors.

In explaining how this comes about, it is convenient to distinguish two separate cases; where cancellation has not occurred and where it has. It is found that the synthesis procedure requires additional operations in the second case.

Because the results to be obtained are quite simple and useful, it is felt desirable to provide the remainder of the discussion with sufficient rigor to insure full generality in spite of the somewhat lengthy mathematical argument.

4. THE SUM FUNCTION $S(\lambda)$

In order to study the conditions of cancellation, we introduce a new function $S(\lambda)$ defined as the sum of the numerator and denominator of $T(\lambda)$ in its uncancelled form.

$$S(\lambda) = (m_1 m_2 - n_1 n_2) + (m_2 n_1 - m_1 n_2). \tag{3-8}$$

All factors which cancel in $T(\lambda)$ are factors of $S(\lambda)$.

Since

$$T(\lambda) = \frac{\lambda\ \text{(even polynomial)}}{\text{(even polynomial)}}, \tag{3-9}$$

cancellable factors containing λ are restricted to be either a single λ or such factors as are peculiar to even polynomials. Obviously we can also cancel a constant.

Finally we note that $S(\lambda)$ can be factored into

$$S(\lambda) = (m_1 + n_1)(m_2 - n_2), \tag{3-10}$$

in which the first factor contains no zeros in the right half plane and the second none in the left half plane. Either or both may contain zeros on the imaginary axis.

5. CLASSIFICATION OF ROOT FACTORS

There is no restriction on the cancellation of the single factor λ. This occurs when the constant term is absent in $m_1 m_2 - n_1 n_2$.

In order to investigate the even polynomial factors of $S(\lambda)$, it should be recalled that any polynomial $P(\lambda)$ of degree n may be expressed in factored form as the product of n root-factors as follows:

$$P(\lambda) = c(\lambda - \lambda_1)(\lambda - \lambda_2) \ldots (\lambda - \lambda_n), \tag{3-11}$$

where c is a constant, and the λ_ν are the zeros of $P(\lambda)$, that is, the roots of the equation $P(\lambda) = 0$.

Further if $P(\lambda)$ has real coefficients, any zeros which are not real occur in conjugate pairs and, hence, each associated root-factor $(\lambda - \lambda_\nu)$ is accompanied by another root-factor $(\lambda - \bar{\lambda}_\nu)$, where $\bar{\lambda}_\nu$ is the conjugate of λ_ν.

If, in addition, $P(\lambda)$ is an even polynomial, it may be regarded as a reduced polynomial $p(\lambda^2)$, in the variable λ^2. In this case the zeros of $P(\lambda)$ are restricted to certain configurations in the λ-plane which determine whether cancellation is possible in $T(\lambda)$. The λ^2-roots of $p(\lambda^2) = 0$ may be classified as follows.

(a) Positive Real λ^2-Roots

Such a root leads to a pair of λ-roots of $P(\lambda) = 0$, one positive real and the other negative real of equal magnitude. If $m_1 m_2 - n_1 n_2$ and $m_2 n_1 - m_1 n_2$ have such a common λ^2-root factor, the previous discussion shows that one of the roots belongs to $m_1 + n_1$ and the other to $m_2 - n_2$. But this situation cannot exist, for then $m_1 + n_1$ and $m_2 + n_2$ would have a common factor, which contradicts the hypothesis that $Z(\lambda)$ was in its lowest terms. Hence it may be stated that in the formation of $T(\lambda)$ from $Z(\lambda)$ there can be no cancellation of root-factors involving non-zero real λ-roots.

(b) Pairs of Conjugate Complex λ^2-Roots

Such a pair leads to a symmetrical quadruplet of roots in the λ-plane. In this case again $m_1 + n_1$ and $m_2 + n_2$ have a common factor, and by the previous argument cancellation of these root-factors is impossible in $T(\lambda)$.

(c) λ^2-Roots at the Origin

This leads to a pair of λ-roots at the origin; again a situation where cancellation is impossible for the following reason. If one of these roots belongs to $m_1 + n_1$ and one to $m_2 - n_2$, there is a contradiction as before; while if both roots belong to either one of the polynomials $m_1 + n_1$ or $m_2 - n_2$, that polynomial would have a double zero on the j-axis, a situation which is not allowed by the positive real requirement on $Z(\lambda)$.

(d) Negative Real λ^2-Roots

Such a root leads to a pair of conjugate pure imaginary λ-roots. Both of these roots together must be factors either of $m_1 + n_1$ or of $m_2 - n_2$ because the coefficients of these polynomials must be real. Hence in this case there appears to be no restriction on cancellation.

The classification is summarized in Fig. 3-1.

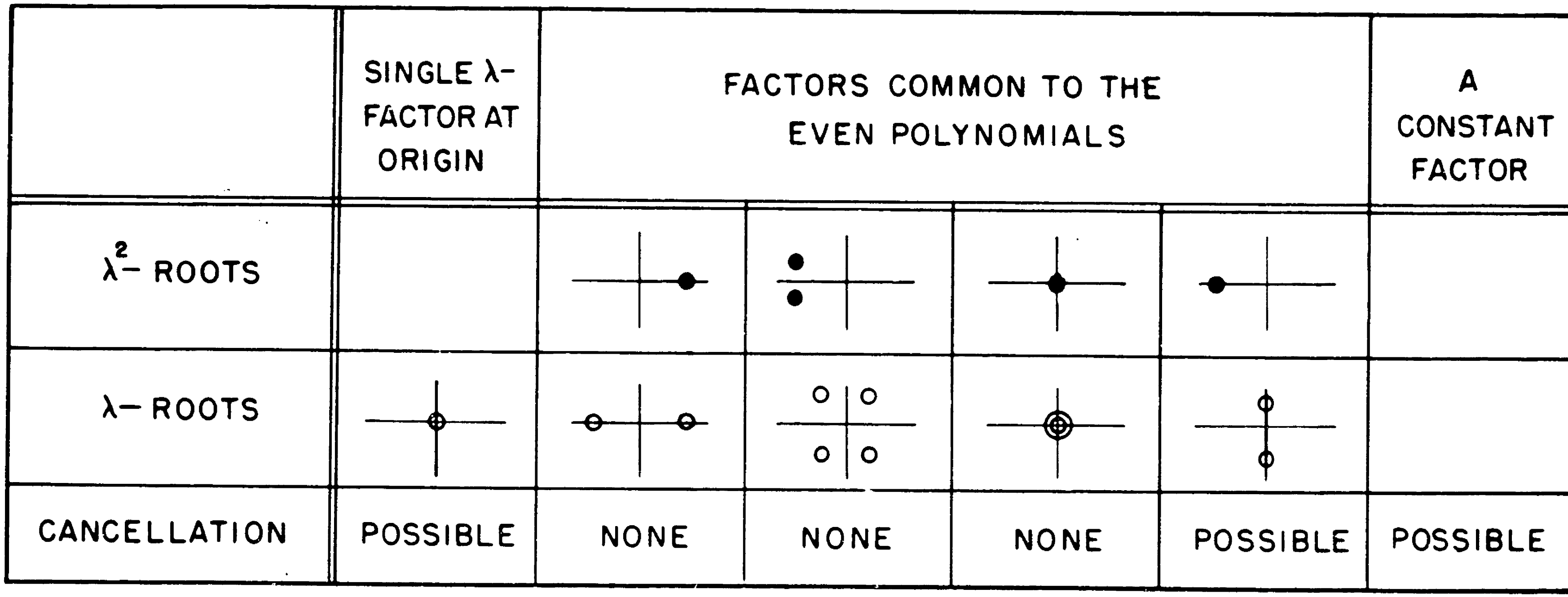


FIG. 3-1 SUMMARY OF CLASSIFICATION

6. SOLUTION OF THE CANCELLATION PROBLEM

We now see that the only possible factors common to $m_2 n_1 - m_1 n_2$ and $m_1 m_2 - n_1 n_2$ (other than a constant) can be a single root-factor at the origin or pairs of conjugate imaginary root-factors. The forms of these are λ and $\lambda^2 + \omega_v^2$, respectively. For brevity both types are called j-axis factors. We note that a j-axis factor of $m_2 - n_2$ is also a j-axis factor of $m_2 + n_2$.

It is now apparent that in the formation of the tangent function $T(\lambda)$ from the impedance function $Z(\lambda)$, a necessary condition on cancelled root-factors is that they be j-axis root-factors of $m_1 + n_1$ or $m_2 + n_2$.

Because $m_1 + n_1$ and $m_2 + n_2$ are Hurwitz polynomials, it is not necessary to make a classification of their factors in order to discuss common factors of the m's and the n's. It is easily shown by an argument similar to the above that any j-axis factor of $m_1 + n_1$ is also a factor of m_1 and n_1 separately.

When the function $S(\lambda)$ is examined term by term, it is found that m_1 or n_1 appears in every term. Hence, a j-axis root-factor of $m_1 + n_1$ will surely cancel in $T(\lambda)$, and likewise for $m_2 + n_2$.

We summarize these results as follows:

In the formation of the tangent function

$$T(\lambda) = \frac{m_2 n_1 - m_1 n_2}{m_1 m_2 - n_1 n_2},$$

from the positive real impedance function

$$Z(\lambda) = \frac{m_1 + n_1}{m_2 + n_2},$$

(a) the necessary and sufficient condition that $m_2 n_1 - m_1 n_2$ and $m_1 m_2 - n_1 n_2$ have common factors is that $Z(\lambda)$ have zeros or poles on the imaginary axis, and

(b) the highest common factor is the product of the aggregate of root-factors corresponding to all such zeros and poles.

We designate this product as $f(\lambda)$ and note that if a/b turns out to be a realizable tangent function,

$$af = m_2 n_1 - m_1 n_2,$$

and

$$bf = m_1 m_2 - n_1 n_2.$$

7. THE NECESSARY FORM OF THE FUNCTION TO BE SYNTHESIZED

From the preceding results we may now establish additional restrictions on the form of a/b as follows.

One of the collateral results of the Hurwitz requirement on $m_1 + n_1$ and $m_2 + n_2$ requires that all the zeros of m_1, n_1, m_2, and n_2 lie on the j-axis. As a consequence it may easily be shown that the cancellation of conjugate j-axis factors in $T(\lambda)$ does not disturb the oddness nor the evenness of $m_2 n_1 - m_1 n_2$ and $m_1 m_2 - n_1 n_2$, respectively. The cancellation of a single j-axis factor at the origin interchanges the oddness and evenness of the two polynomials. Hence, it may be deduced that a necessary condition that a/b be the tangent function of a physical impedance is that it be of the form

$$\frac{odd}{even} \quad \text{or} \quad \frac{even}{odd} .$$

It is interesting to note that this result could alternatively be obtained directly as a consequence of the necessary oddness of $T(\lambda)$, as the following considerations show.

An arbitrary function can always be expressed as the sum of an even and an odd part. A requirement that the function be odd can be set up mathematically by equating the even part to zero. In the case of a rational function in its lowest terms,

$$F(\lambda) = \frac{M_1(\lambda) + N_1(\lambda)}{M_2(\lambda) + N_2(\lambda)} ,$$

the oddness requirement becomes

$$\tfrac{1}{2}\left[F(\lambda) + F(-\lambda)\right] = 0 ,$$

yielding

$$\frac{M_1+N_1}{M_2+N_2} + \frac{M_1-N_1}{M_2-N_2} = 0,$$

from which

$$M_1 M_2 - N_1 N_2 = 0,$$

which is satisfied for a function whose numerator and denominator exist if, and only if, either $M_1 \equiv N_2 \equiv 0$ or $M_2 \equiv N_1 \equiv 0$.

8. EVIDENCE OF CANCELLATION

As a result of the requirement that a physical impedance have a non-negative real part at all frequencies, we know that the equation $m_1 m_2 - n_1 n_2 = 0$ must not have negative real λ^2-roots of odd multiplicity. This, of course, requires that such roots be present at least twice.

On the other hand, $m_2 n_1 - m_1 n_2$ cannot contain these root-factors more than once; for if it did, $S(\lambda)$ would contain them at least twice, and we would then be confronted either with the contradiction that $Z(\lambda)$ was not in its lowest terms or with the untenable conclusion that $Z(\lambda)$ had a double zero or pole at a real frequency.

Hence cancellation of a negative real λ^2-root factor always changes the even multiplicity of that factor in $m_1 m_2 - n_1 n_2$ into odd multiplicity in b .

This is the cancellation criterion we are seeking. From it we are able to deduce that the common root-factors (whose product is f) missing from a and b by cancellation are precisely those required to restore even multiplicity to all of the j-axis zeros of b including the possible zero at the origin.

Now that we are able to restore the cancelled factors, we can investigate the final necessary realizability condition on the function to be synthesized. This condition comes from the additional requirement that $m_1 m_2 - n_1 n_2$ must be shown to be definitely positive at some frequency which we can select. For later convenience we choose infinite frequency. As a result it is necessary that after restoration of the missing factors $bf > 0$ at $-\lambda^2 = \infty$.

If this condition is not fulfilled by the given function, we can obviously regard the constant -1 as a cancelled factor and we can obtain fulfillment by changing signs in numerator and denominator. Hence it turns out that this requirement offers no additional restrictions.

9. SUFFICIENT CONDITIONS FOR REALIZABILITY

At this point we are in possession of a number of necessary conditions on the form of a/b. Further reflection on the nature of these conditions and on the possibility of satisfying some of them by restoring cancelled factors reveals that they are all equivalent to the single necessary condition that a/b must be a real odd rational function of λ. (A "real" function of λ takes on real values for real values of λ.) We now show that this condition is also sufficient to guarantee that a/b is the tangent function of a realizable impedance. This we do by reversing our original procedure; and starting with an expression a/b which conforms to the necessary condition, we derive a positive real function whose tangent function is equal to a/b.

We consider first the case where cancellation is absent. This, of course, is indicated by the fact that b is even, it has no negative real λ^2-zeros of odd multiplicity, and it remains > 0 as $-\lambda^2 \to \infty$. Evidently the impedance we are seeking is to have no j-axis zeros or poles. Furthermore, its component portions are related to a and b as follows:

$$f(\lambda) = 1,$$

$$a(\lambda) = m_2 n_1 - m_1 n_2,$$

$$b(\lambda) = m_1 m_2 - n_1 n_2,$$

$$T(\lambda) = \frac{a}{b},$$

$$S(\lambda) = a + b.$$

If we again consider the factored form of

$$S(\lambda) = (m_1 + n_1)(m_2 - n_2),$$

it becomes apparent that the imaginary axis may be used as a boundary to sort out the zeros of $m_1 + n_1$ from those of $m_2 - n_2$. Thus, if we set

$$a + b = 0,$$

those roots which lie in the left half-plane can be assigned to the polynomial $m_1 + n_1$ and those in the right to $m_2 - n_2$. There will be none on the axis because cancallation is absent. We can convert $m_2 - n_2$ into the Hurwitz polynomial $m_2 + n_2$ by reversing the signs of its odd terms.

If we now form the impedance

$$Z(\lambda) = C\,\frac{m_1 + n_1}{m_2 + n_2},$$

where C is an arbitrary positive constant, it is evident from the following considerations that $Z(\lambda)$ is positive real.

(a) Both $m_1 + n_1$ and $m_2 + n_2$ are Hurwitz polynomials with real coefficients because of their method of formation.

(b) $Re[Z(j\omega) \geqslant 0$ because $m_1 m_2 - n_1 n_2 \geqslant 0$ by the necessary conditions.

(c) $Z(\lambda)$ has no j-axis poles.

The case where a/b is in a reduced form due to cancellation is indicated by the presence of negative real λ^2-zeros of odd multiplicity in b or by the possibility that b is odd or both. In any instance we use our knowledge of the cancelled factors to make the necessary realizability correction on the sign of $m_1 m_2 - n_1 n_2$ at $-\lambda^2 = \infty$.

Here the relationships are

$$af = m_2 n_1 - m_1 n_2 ,$$
$$bf = m_1 m_2 - n_1 n_2 ,$$
$$T(\lambda) = \frac{af}{bf} = \frac{a}{b} ,$$
$$S(\lambda) = f(a+b) .$$

In this case again

$$S(\lambda) = (m_1 + n_1)(m_2 - n_2) ,$$

but this time the imaginary axis cannot be used as a sorting boundary for all the roots because those contained in f are "on the fence," and there is no way as yet to determine with which half-plane they are associated. That is, we do not know whether they are to belong to the numerator or to the denominator of $Z(\lambda)$.

We resolve this difficulty presently by an appeal to the requirement that the residues of $Z(\lambda)$ at its j-axis poles must be real and positive. In the meantime we note that the roots of the equation

$$S(\lambda) = 0$$

are the roots of

$$a + b = 0 ,$$

and

$$f(\lambda) = 0 .$$

The first of these equations is still valid as a means of sorting out the zeros of $m_1 + n_1$ and $m_2 - n_2$ that do not lie on the imaginary axis.

10. RESIDUE DETERMINATION

In attempting to determine the residues of $Z(\lambda)$ at its poles, we find a fresh difficulty. We must assume an arbitrary allocation of root-factors of $f(\lambda)$ to the numerator and denominator of $Z(\lambda)$ before we can proceed to evaluate any residues. If $f(\lambda)$ contains a large number of root-factors, the number of possible combinations can cause this procedure to become a tedious task.

We obviate such computational work by recognizing that we do not need to know the magnitude of the residue; our purposes will be served if we know its argument. The latter we find readily by the use of certain principles taken from function theory.

It is well known that when a function $Z(\lambda)$ has simple poles, λ_1, λ_2, ... , λ_h, the value of the residue k_ν at any pole can be determined from

$$k_\nu = \lim_{\lambda \to \lambda_\nu} (\lambda - \lambda_\nu) Z(\lambda) .$$

Since we are dealing with simple poles;

$$k_\nu = \lim_{\lambda \to \lambda_\nu} \frac{(\lambda - \lambda_\nu)(m_1 + n_1)}{m_2 + n_2} .$$

Rationalizing gives

$$k_\nu = \lim_{\lambda \to \lambda_\nu} \frac{(\lambda - \lambda_\nu)(m_2 n_1 - m_1 n_2)}{m_2^{\,2} - n_2^{\,2}} + \lim_{\lambda \to \lambda_\nu} \frac{(\lambda - \lambda_\nu)(m_1 m_2 - n_1 n_2)}{m_2^{\,2} - n_2^{\,2}} .$$

The two limits are the real and imaginary parts respectively of k_ν. We note that these limits exist and that

$$Re\,[k_\nu] \neq 0 ,$$

while

$$Im\,[k_\nu] = 0 ,$$

since the factor $\lambda - \lambda_\nu$ is contained twice in $m_2^{\,2} - n_2^{\,2}$, once in $m_2 n_1 - m_1 n_2$, and at least twice in $m_1 m_2 - n_1 n_2$. Hence the residue exists and is real.

For $\lambda_\nu \neq 0$, m_2 and n_2 both contain $\lambda - \lambda_\nu$ as a simple factor, and in addition they contain its conjugate $\lambda + \lambda_\nu$ (also simple). Let

$$M_2 = \frac{m_2}{\lambda^2 - \lambda_\nu^{\,2}}$$

and

$$N_2 = \frac{n_2}{\lambda^2 - \lambda_\nu^{\,2}} .$$

Then

$$k_v = \lim_{\lambda \to \lambda_v} \frac{(\lambda-\lambda_v)(m_2 n_1 - m_1 n_2)}{(M_2^2 - N_2^2)(\lambda^2 - \lambda_v^2)^2} ,$$

$$= \lim_{\lambda \to \lambda_v} \frac{1}{M_2^2 - N_2^2} \times \lim_{\lambda \to \lambda_v} \frac{1}{(\lambda + \lambda_v)^2} \times \lim_{\lambda \to \lambda_v} \frac{m_2 n_1 - m_1 n_2}{\lambda - \lambda_v} ,$$

$$= \quad c_1 \quad \times \quad c_2 \quad \times \quad \frac{d}{d\lambda} (m_2 n_1 - m_1 n_2) \Big]_{\lambda = \lambda_v} ,$$

where c_1 is a positive constant since $M_2^2 - N_2^2$ is the square of an absolute magnitude, and c_2 is a negative constant.

For $\lambda_v = 0$, m_2 and n_2 both contain λ as a simple factor. Let

$$M_2 = \frac{n_2}{\lambda} ,$$

and

$$N_2 = \frac{m_2}{\lambda} .$$

Then

$$k_0 = \lim_{\lambda \to 0} \frac{\lambda (m_2 n_1 - m_1 n_2)}{(N_2^2 - M_2^2)\, \lambda^2} ,$$

$$= c_3 \frac{d}{d\lambda} (m_2 n_1 - m_1 n_2) \Big]_{\lambda = 0} ,$$

where c_3 is a negative constant.

It is now apparent that the correct allocation of each j-axis factor to the numerator or denominator of $Z(\lambda)$

can be made from the sign of the derivative of $m_2 n_1 - m_1 n_2$ evaluated at the point in question. If this derivative is negative, the factor is assigned to $m_2 + n_2$; otherwise it is assigned to $m_1 + n_1$.

If we invert the impedance obtained by this rule and evaluate the residues at the j-axis zeros of $m_1 + n_1$, it is seen that they turn out to have the proper positive sign, for now the sign-determining function is $\frac{d}{d\lambda}(-m_2 n_1 + m_1 n_2)$.

We remark at this time that the synthesis discussion presently reveals the interesting and useful result that the signs of the residues of $T(\lambda)$ at its simple j-axis poles yield the same information. It turns out that $T(\lambda)$ and $Z(\lambda)$ have precisely the same set of simple j-axis poles with positive real residues.

If we now set

$$Z(\lambda) = C \frac{m_1 + n_1}{m_2 + n_2} ,$$

it is apparent that $Z(\lambda)$ is positive real, since we have satisfied the same requirements as in the previous case, and in addition the residues at the j-axis poles are real and positive. This completes the proof of sufficiency.

11. CONDITIONS FOR EXISTENCE OF NETWORKS

From the preceding results we are now in a position to formulate an existence theorem for networks whose impedances are to have prescribed phase angles vs. frequency.

We state that given $\theta(\omega)$, the necessary and sufficient condition that there exists a finite two-terminal network with driving-point impedance Z such that

$$\arg Z(j\omega) = \theta(\omega),$$

is that tan θ be a real odd rational function of ω.

PART IV

SYNTHESIS PROCEDURES

We exploit the mathematical results of Part III in the development of a set of workable synthesis procedures. It turns out that the chief difficulty in application is the root-finding process, which is an annoying feature common to several recent "exact" methods. We illustrate the procedures with several examples.

1. THE APPROXIMATION PROBLEM

The first step in design is the development of the tangent of the phase angle expressed as

$$\tan\theta = \frac{A(\omega)}{B(\omega)},$$

with the realizability conditions satisfied.

Frequently the requirements of a physical problem are specified as a curve of θ vs. ω. Evidently, we can obtain the proper form by first replotting to get a curve of tan θ vs. ω and then approximating the latter curve with an appropriate algebraic expression. We might, for example, select a set of $2n+1$ points with ordinates t_ν on the tangent curve at corresponding frequencies ω_ν. To satisfy the required conditions we write

$$t_\nu = \frac{C_1 \omega_\nu + C_3 \omega_\nu^3 + \dots + C_{2n+1} \omega_\nu^{2n+1}}{C_0 + C_2 \omega_\nu^2 + \dots + C_{2n} \omega_\nu^{2n}}, \quad (\nu = 1, 2, \cdots, 2n+1),$$

from which we obtain $2n+1$ simultaneous linear equations in the same number of unknown coefficients C_ν. The solution of this set enables us to form A/B. If the values of θ from this result are plotted over the original curve, the deviations in phase angle resulting from the approximation are apparent at sight.

It should be noted that cancellation is absent if $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, that is, if $\tan\theta$ is finite at real frequencies. In this case, it is seen that B has no zeros at real frequencies and we are not later required to check further into its roots. The resulting network will be minimum reactive and minimum susceptive (except possibly at infinity).

At each real frequency where θ approaches closely to $\pm\frac{\pi}{2}$ it is seen that the tangent curve has a large peak. To obtain a good approximation to such a curve, we find it necessary to use a large number of points t_ν. Various schemes can be used to reduce the computational labor. For example, B might be set up initially to contain the factor

$(\omega^2 - \omega_v^2)^2 + \varepsilon$ where the smallness of ε is a measure of the closeness of θ to $\pm\frac{\pi}{2}$ at the frequency ω_v. Points of zero phase angle ω_v can be set up in a similar manner as factors $\omega^2 - \omega_v^2$ of A.

In the case where the impedance is required to be a pure reactance at a specified frequency ω_v, we put $\omega^2 - \omega_v^2$ as a factor in B. Actually the only requirement to keep in mind in the formation of A/B is that one polynomial must be odd and the other one even.

Assuming now that we have tan θ expressed as A/B either from graphical analysis or from other known requirements, we make the transformation to $T(\lambda)$ by forming

$$T(\lambda) = j \frac{A(-j\lambda)}{B(-j\lambda)} = \frac{a(\lambda)}{b(\lambda)}$$

2. DETERMINATION OF $f(\lambda)$

The second step is the determination of the cancelled factors $f(\lambda)$. These are the factors required to restore even multiplicity to every j-axis factor of $b(\lambda)$. A j-axis factor exists at the origin if b is odd in which case f contains the factor λ. The other j-axis factors are the negative real λ^2-factors of $b(\lambda)$. We can learn of the existence of such factors by replacing λ^2 with x to obtain the reduced polynomial $b_1(x)$ and by using Sturms

Theorem to study the negative real roots of $b_1(x) = 0$. In addition, since we must have $bf > 0$ as $\lambda^2 \to -\infty$, we assign the factor -1 to f, if necessary in order to meet this requirement.

3. PARTITIONING OF $f(\lambda)$

The function $f(\lambda)$ is, of course, composed of factors such as λ and $\lambda^2 - \lambda_v^2$ all of single multiplicity with each λ_v a pure imaginary quantity. We separate these factors into two groups, $f_1(\lambda)$ and $f_2(\lambda)$, according to the following rule. For each λ_v we determine the sign of

$$k(\lambda_v) = a \left.\frac{df}{d\lambda}\right]_{\lambda=\lambda_v}$$

The factor $\lambda^2 - \lambda_v^2$ (or λ) is assigned to f_2 if $k(\lambda_v) < 0$ (or $k(0) < 0$) ; otherwise the factor is assigned to f_1.

That the expression for $k(\lambda_v)$ is consistent with

$$\left.\frac{d}{d\lambda}(m_2 n_1 - m_1 n_2)\right]_{\lambda=\lambda_v}$$

of Part III is evident when we evaluate the latter.

$$\frac{d}{d\lambda}\left[m_2 n_1 - m_1 n_2\right]_{\lambda=\lambda_v} = \frac{d}{d\lambda}(af) = a'f + af'\Big]_{\lambda=\lambda_v} = af'\Big]_{\lambda=\lambda_v},$$

since $f(\lambda_v) = 0$.

ı the special case, which is the usual one in practice, where the negative real λ^2-zeros of $m_1 m_2 - n_1 n_2$ have double multiplicity, we can obtain a more useful alternate form of the sign-determining function.

First we show that in this case

$$\frac{d^2}{d\lambda^2}(m_1 m_2 - n_1 n_2)\Big]_{\lambda=\lambda_v} < 0 .$$

Factoring out the double root factor $(\lambda - \lambda_v)$ yields

$$m_1 m_2 - n_1 n_2 = g(\lambda)(\lambda - \lambda_v)^2 .$$

Since on the j-axis

$$(\lambda - \lambda_v)^2 \leq 0,$$

and

$$m_1 m_2 - n_1 n_2 \geqslant 0,$$

while

$$g(\lambda_v) \neq 0,$$

we see that

$$g(\lambda_v) < 0.$$

If we evaluate the second derivative at λ_v we obtain

$$\frac{d^2}{d\lambda^2}\left[g(\lambda)(\lambda-\lambda_v)^2\right]_{\lambda=\lambda_v}$$

$$=\left[2g+4g'(\lambda-\lambda_v)+g''(\lambda-\lambda_v)^2\right]_{\lambda=\lambda_v}=2g(\lambda_v)<0.$$

On the other hand we also have

$$m_1 m_2 - n_1 n_2 = b(\lambda)\, f(\lambda),$$

where each factor on the right contains a single $(\lambda - \lambda_v)$. If we evaluate the second derivative of this expression, we obtain

$$\frac{d^2}{d\lambda^2}(bf)\Big]_{\lambda=\lambda_v}=\left[bf''+2b'f'+b''f\right]_{\lambda=\lambda_v}=2b'f'\Big]_{\lambda=\lambda_v},$$

and hence

$$b'(\lambda_v)\, f'(\lambda_v) < 0.$$

If we divide this expression into the sign-determining function, we obtain

$$\frac{af'}{b'f'}=\frac{a}{b'} \qquad at\ \lambda=\lambda_v,$$

which is equally valid for determining the allocation of root factors except that we reverse the sign rule. But the expression

$$\frac{a(\lambda_\nu)}{b'(\lambda_\nu)}$$

is the residue of $T(\lambda)$ at its simple pole λ_ν, as may be seen from an application of L'Hôpital's rule to

$$\lim_{\lambda \to \lambda_\nu} \frac{(\lambda - \lambda_\nu)\, a(\lambda)}{b(\lambda)} .$$

Hence we obtain the interesting result that $T(\lambda)$ and $Z(\lambda)$ have in common all of their simple j-axis poles with positive real residues.

4. FORMATION OF $Z(\lambda)$

In the next step we obtain the polynomial $p(\lambda)$ representing the aggregate of the left-half plane root-factors of

$$a + b = 0 .$$

The number of such roots can be obtained with the help of Routh's stability criterion*. Their location unfortunately requires considerable computation. We may use Graffe's root-squaring method* or various other approximation procedures or mechanical root-finders*. In this connection it is helpful to note that we are not required to find individual left half-plane roots; if by any means we can factor out large portions of $a + b$ containing only left half-plane zeros, our labor is reduced.

We next obtain by simple division the polynomial $g(-\lambda)$ which contains all of the right half-plane zeros of $a + b$. Thus

$$g(-\lambda) = \frac{a + b}{p(\lambda)} ,$$

and we set

$$Z(\lambda) = c \, \frac{f_1 \, p}{f_2 \, g} = \frac{P}{Q} ,$$

where c is an arbitrary positive constant.

$Z(\lambda)$ can now be developed as a two-terminal network by any one of several methods such as that of Brune or Darlington.

5. <u>SUMMARY OF PROCEDURE</u>

We group the rules as follows:

(a) Obtain $\tan \theta = \dfrac{A(\omega)}{B(\omega)}$.

(b) Transform to $T(\lambda) = j \, \dfrac{A(-j\lambda)}{B(-j\lambda)} = \dfrac{a(\lambda)}{b(\lambda)}$

(c) Form $f(\lambda)$ as the minimum factor required to secure even multiplicity of j-axis factors of $f(\lambda) \, b(\lambda)$.

(d) Partition $f(\lambda)$ into $f_1(\lambda)$ and $f_2(\lambda)$ using sign of $a(\lambda_\nu) f'(\lambda_\nu)$ (or $\frac{a(\lambda)}{b'(\lambda_\nu)}$ in special cases) to determine assignment.

(e) Obtain $p(\lambda)$ as the product of left half-plane root-factors of $a + b$.

(f) Obtain $q(-\lambda) = \frac{a+b}{p}$.

(g) Set $Z(\lambda) = c \, \frac{f_1(\lambda)\, p(\lambda)}{f_2(\lambda)\, q(\lambda)}$.

(h) Synthesize network from $Z(\lambda)$.

6. EXAMPLES OF PHASE SYNTHESIS

The following examples illustrate the mechanics of the processes described in Parts III and IV.

The first three examples are quite elementary, but they serve to clarify first principles. They are shown in pairs to illustrate the effect of changing signs. It is, of course, obvious that changing the sign of the tangent function should reciprocate the impedance function, and the examples demonstrate this result.

The fourth example shows the effect of a cancelled pole at the origin and also a cancelled zero. It illustrates the rule that $T(\lambda)$ and $Z(\lambda)$ share each other's simple poles with positive real residues.

Example 5 is included as a further illustration of the method of forming $Z(\lambda)$ from $S(\lambda)$, and also the use of the arbitrary constant to produce a network with a one ohm termination.

Example 6 shows the method of obtaining the network when there is cancellation of a pole or zero at a real frequency.

In Example 7 we illustrate the replacement of two cancelled poles and also show the interesting result, not previously mentioned, that when $T(\lambda)$ is itself positive real, then $Z(\lambda) = 1 + T(\lambda)$.

We present Example 8 as an interesting case where $Z(\lambda)$ has both a pole and a zero at real frequencies. It is seen that again the imaginary axis sorts out for numerator and denominator of $Z(\lambda)$ those zeros not on the axis; while the residue rule sorts out those on the axis.

Example 9 illustrates the method when the prescribed phase angle is expressed as a Tchebycheff polynomial multiplied by an arbitrary constant.

$$\tan \phi = \mathcal{E} V_n(\omega)$$

The resulting network is to be used in Part V as an example of the use of phase synthesis in filter design. It is there shown that we merely need connect two of these networks "back to back" in order to obtain the transfer impedance.

$$\left| Z_{12} \right|^2 = \frac{1}{1 + \varepsilon^2 V_n^2(\omega)} .$$

The apparently difficult root-solving process would not ordinarily be carried out as shown in the example. In the appendix we show how the polynomials can be obtained by means of a simple direct calculation.

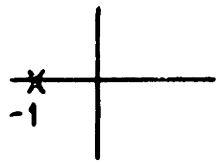
Example 1

(a)

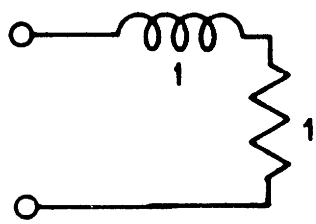
$\tan\theta = \omega \quad (\text{i.e. } \frac{\omega}{1})$

$T(\lambda) = \lambda \quad (\text{i.e. } \frac{\lambda}{1})$

$S(\lambda) = (\lambda + 1)\ 1$



$Z(\lambda) = \frac{\lambda + 1}{1}$

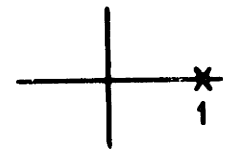


(b)

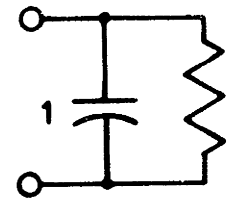
$\tan\theta = -\omega$

$T(\lambda) = \frac{-\lambda}{1}$

$S(\lambda) = 1\ (1 - \lambda)$



$Z(\lambda) = \frac{1}{\lambda + 1}$



Example 2

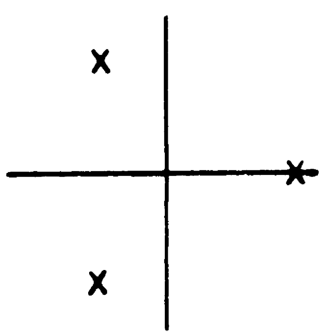
(a)

$\tan\theta = \omega^3$

$T(\lambda) = -\lambda^3$

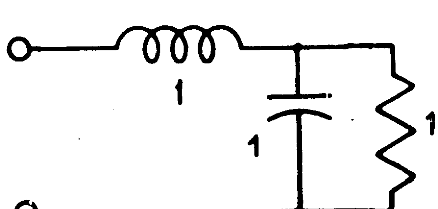
$S(\lambda) = 1 - \lambda^3$

$= (\lambda^2 + \lambda + 1)(1 - \lambda)$



$Z(\lambda) = \frac{\lambda^2 + \lambda + 1}{\lambda + 1}$

$= \lambda + \frac{1}{\lambda + 1}$



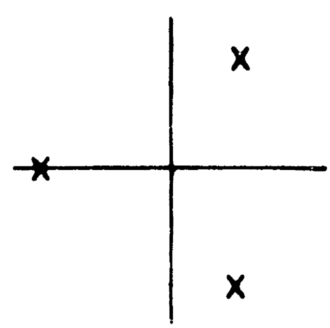
2 (b)

$\tan\theta = -\omega^3$

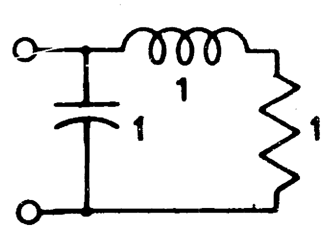
$T(\lambda) = \lambda^3$

$S(\lambda) = 1 + \lambda^3$

$= (\lambda + 1)\ (\lambda^2 - \lambda + 1)$



$Z(\lambda) = \frac{\lambda + 1}{\lambda^2 + \lambda + 1}$



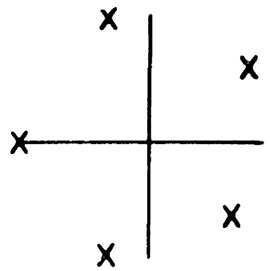
(a)

$$\tan\theta = \omega^5$$

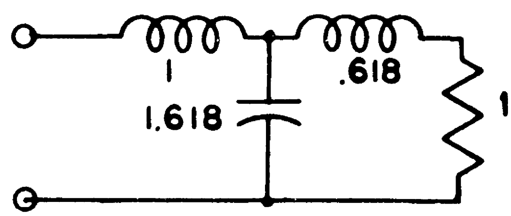
$$T(\lambda) = \lambda^5$$

$$S(\lambda) = \lambda^5 + 1$$

$$= (\lambda^3 + 1.618\lambda^2 + 1.618\lambda + 1)(\lambda^2 - 1.618\lambda + 1)$$



$$Z(\lambda) = \frac{\lambda^3 + 1.618\lambda^2 + 1.618\lambda + 1}{\lambda^2 + 1.618\lambda + 1}$$



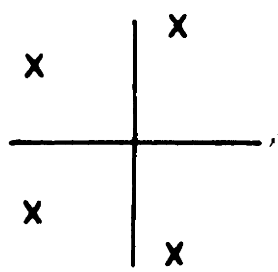
3 (b)

$$\tan\theta = -\omega^5$$

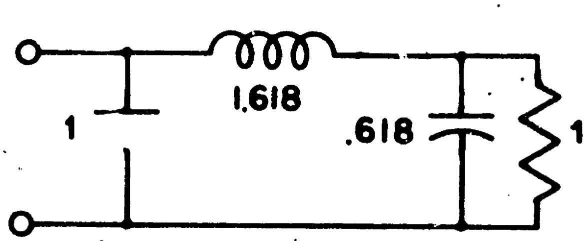
$$T(\lambda) = -\lambda^5$$

$$S(\lambda) = 1 - \lambda^5$$

$$= (\lambda^2 + 1.618\lambda + 1)(-\lambda^3 + 1.618\lambda^2 - 1.618\lambda + 1)$$



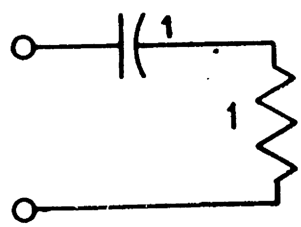
$$Z(\lambda) = \frac{\lambda^2 + 1.618\lambda + 1}{\lambda^3 + 1.618\lambda^2 + 1.618\lambda + 1}$$



Example 4

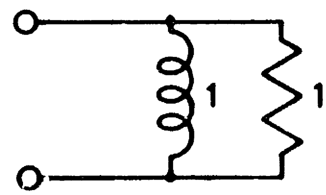
(a)

$$\tan\theta = \frac{1}{\omega}$$
$$T(\lambda) = \frac{1}{\lambda}$$
$$a(\lambda) + b(\lambda) = 1 + \lambda$$
$$f(\lambda) = -\lambda$$
$$S(\lambda) = -\lambda(1 + \lambda)$$
$$Z(\lambda) = \frac{\lambda + 1}{\lambda}$$



(b)

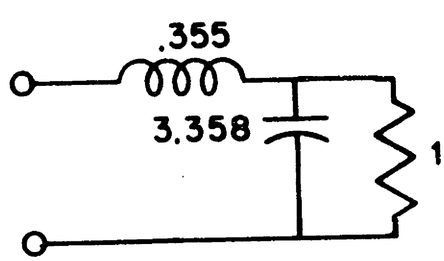
$$\tan\theta = \frac{1}{-\omega}$$
$$T(\lambda) = \frac{1}{-\lambda}$$
$$a(\lambda) + b(\lambda) = 1 - \lambda$$
$$f(\lambda) = \lambda$$
$$S(\lambda) = \lambda(1 - \lambda)$$
$$Z(\lambda) = \frac{\lambda}{\lambda + 1}$$



Example 5

$$\tan\theta = 4\omega^3 - 3\omega$$
$$T(\lambda) = -4\lambda^3 - 3\lambda$$
$$S(\lambda) = -4\lambda^3 - 3\lambda + 1$$
$$= (\lambda - .298)\,(4\lambda^2 + 1.192\,\lambda + 3.358)$$

$$Z(\lambda) = c\,\frac{4\lambda^2 + 1.92\lambda + 3.358}{\lambda + .298}$$
$$= \frac{.355\lambda^2 \text{ to } 106\lambda + .298}{\lambda + .298}$$



Example 6

(a)

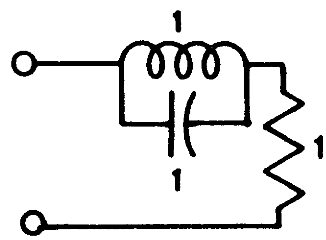
$$\tan\theta = \frac{\omega}{1-\omega^2}$$

$$T(\lambda) = \frac{\lambda}{\lambda^2+1}$$

$$a(\lambda) + b(\lambda) = \lambda^2 + \lambda + 1$$

$$f(\lambda) = \lambda^2 + 1$$

$$Z(\lambda) = \frac{\lambda^2+\lambda+1}{\lambda^2+1}$$



(b)

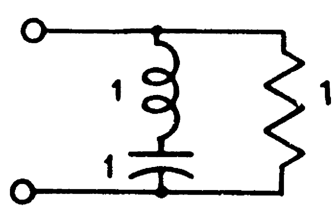
$$\tan\theta = \frac{\omega}{\omega^2-1}$$

$$T(\lambda) = \frac{\lambda}{-\lambda^2-1}$$

$$a(\lambda) + b(\lambda) = -\lambda^2 + \lambda - 1$$

$$f(\lambda) = -(\lambda^2+1)$$

$$Z(\lambda) = \frac{\lambda^2+1}{\lambda^2+\lambda+1}$$



Example 7

$$\tan\theta = \frac{\omega(2-\omega^2)}{(1-\omega^2)(3-\omega^2)}$$

$$T(\lambda) = \frac{\lambda(\lambda^2+2)}{(\lambda^2+1)(\lambda^2+3)}$$

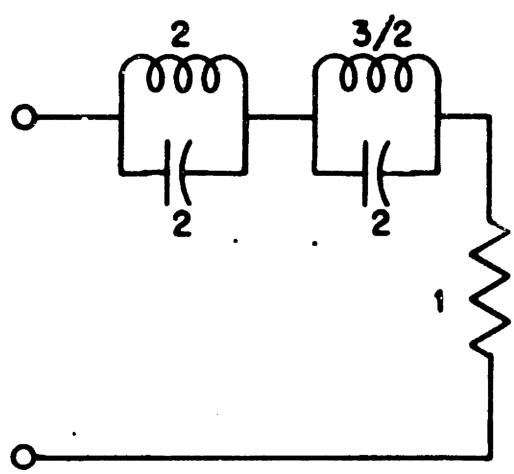
$$f(\lambda) = (\lambda^2+1)(\lambda^2+3)$$

$$a(\lambda) + b(\lambda) = \lambda(\lambda^2+2) + (\lambda^2+1)(\lambda^2+3)$$

$$Z(\lambda) = \frac{\lambda(\lambda^2+2) + (\lambda^2+1)(\lambda^2+3)}{(\lambda^2+1)(\lambda^2+3)}$$

$$= 1 + \frac{\lambda(\lambda^2+2)}{(\lambda^2+1)(\lambda^2+3)}$$

$$= 1 + \frac{(1/2)\lambda}{(\lambda^2+1)} + \frac{(1/2)\lambda}{(\lambda^2+3)}$$



Example 8

$$\tan\theta = \frac{3\omega^5 + 8\omega^2 - 5\omega}{(1-\omega^2)(4-\omega^2)}$$

$$T(\lambda) = \frac{3\lambda^5 - 8\lambda^3 - 5\lambda}{(\lambda^2+1)(\lambda^2+4)}$$

$$f(\lambda) = (\lambda^2+1)(\lambda^2+4)$$

$$f_1(\lambda) = \lambda^2 + 1$$

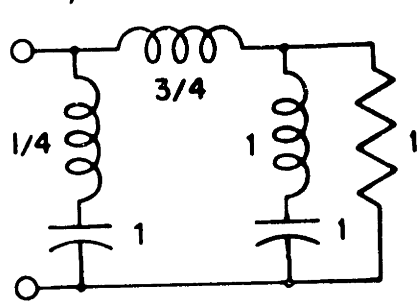
$$f_2(\lambda) = \lambda^2 + 4$$

$$S(\lambda) = \left[(3\lambda^5 - 8\lambda^3 - 5\lambda) + (\lambda^2+1)(\lambda^2+4)\right]\left[(\lambda^2+1)(\lambda^2+\right.$$

$$= \left[(3\lambda^3 + 7\lambda^2 + 3\lambda + 4)(\lambda^2 - 2\lambda + 1)\right]\left[(\lambda^2+1)(\lambda^2+\right.$$

$$Z(\lambda) = c\,\frac{(\lambda^2+4)(3\lambda^3+7\lambda^2+3\lambda+4)}{(\lambda^2+1)(\lambda^2+2\lambda+1)}$$

Let c = 1/16



Example 9

$$\operatorname{Tan}\theta = .331\, V_7(\omega) = .331\,(64\omega^7 - 112\omega^5 + 56\omega^3 - 7\omega)$$

$$T(\lambda) = -.331\lambda\,(64\lambda^6 + 112\lambda^4 + 56\lambda^2 + 7)$$

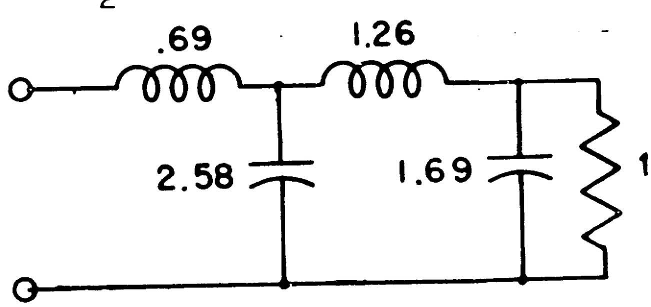
$$S(\lambda) = 1 - .331\lambda\,(64\lambda^6 + 112\lambda^4 + 56\lambda^2 + 7)$$

$$= (\lambda^4 + .595\lambda^3 + 1.332\lambda^2 + .519\lambda + .261)\,(-\lambda^3 + .594\lambda^2 - .769\lambda +$$

$$Z(\lambda) = c\,\frac{\lambda^4 + .595\lambda^3 + 1.332\lambda^2 + .519\lambda + .261}{\lambda^3 + .594\lambda^2 + .769\lambda + .18}$$

$$= \frac{\lambda^4 + .595\lambda^3 + 1.332\lambda^2 + .519\lambda + .261}{1.45\lambda^3 + .863\lambda^2 + 1.118\lambda + .261}$$

$$z_{11} = \frac{m_1}{n_2} = \frac{\lambda^4 + 1.332\lambda^2 + .261}{1.45\lambda^3 + 1.118\lambda}$$



Part V

APPLICATION TO SYMMETRICAL FILTERS

In this section we present a theorem which is useful in expressing the transfer impedance of a general passive network in terms of the impedances of simpler portions of the network. We exploit this theorem by partitioning several common networks to show how certain well-known relations may be more easily demonstrated, and, in addition, we derive some new relationships.

One of these relationships intimately associates the transfer impedance of a symmetrical network with the phase angle of the driving-point impedance seen from the center of the network. We use this result to formulate a new process for designing certain symmetrical networks.

1. A PARTITIONING THEOREM

Consider the network shown in Fig. 5-1.

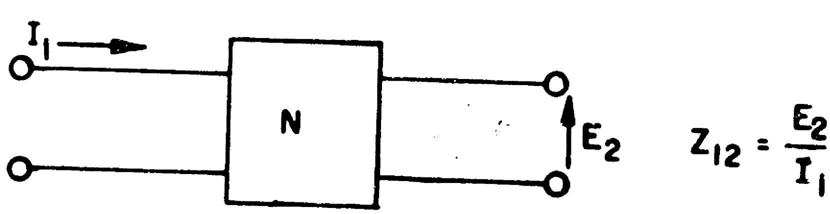


FIG.5-1

If we partition this network into two smaller networks through any section which is traversed by two conducting paths, we obtain the structure shown in Fig. 5-2.

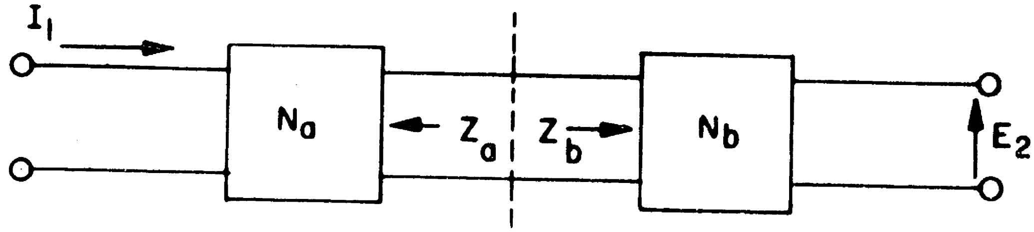


FIG.5-2

In order to establish relationships among the various impedances, we first apply Thevenin's Theorem to replace the left-hand network N_a by a voltage source E_{oc} in series with an impedance Z_s, Fig. 5-3.

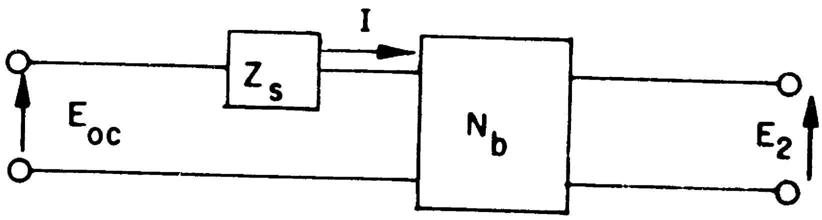


FIG.5-3

In this application, it is noted that I_1 is a current source and hence is left on open-circuit in determining the series impedance. We obtain, Fig. 5-4,

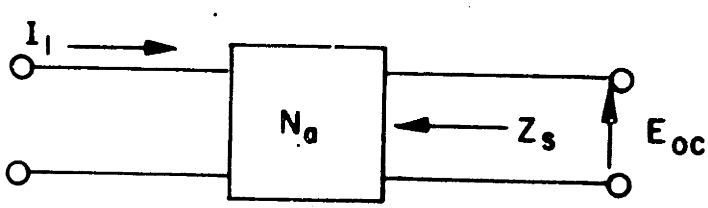


FIG.5-4

$$E_{oc} = Z_{12a} I_1, \tag{5-1}$$

and

$$Z_s = Z_a . \tag{5-2}$$

Hence

$$I = \frac{E_{oc}}{Z_s + Z_b},$$

$$= \frac{Z_{12a} I_1}{Z_a + Z_b}, \tag{5-3}$$

but

$$Z_{12b} = \frac{E_2}{I}, \tag{5-4}$$

which allows us to write

$$\frac{E_2}{I_1} = \frac{Z_{12a} Z_{12b}}{Z_a + Z_b},$$

or

$$Z_{12} = \frac{Z_{12a} Z_{12b}}{Z_a + Z_b} . \tag{5-5}$$

Hence we may state that if a two terminal-pair network is partitioned into two separate two terminal-pair networks in cascade, the transfer impedance of the original network is equal to the product of the transfer impedances of the separate parts divided by the impedance around the center loop.

2. RESULTS OF THE PARTITIONING THEOREM

a) Output Loading

As a first example we consider the case of a lossless two terminal-pair network terminated in a resistance, Fig. 5-5.

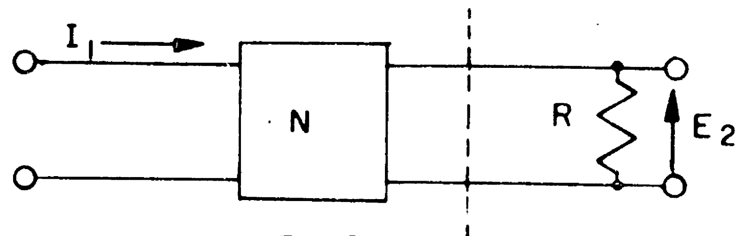


FIG. 5-5

Denoting the input, output, and transfer impedances of the lossless network alone by $\mathfrak{z}_{11}$, $\mathfrak{z}_{22}$, and $\mathfrak{z}_{12}$, respectively, we find at once that when the network is partitioned as shown

$$\frac{E_2}{I_1} = Z_{12} = \frac{R\mathfrak{z}_{12}}{R + \mathfrak{z}_{22}} \qquad (5\text{-}6)$$

a well-known result.

b) Loading at Input and Output

If we partition a network having resistance loading at both ends of a lossless network as shown in Fig. 5-6, we obtain

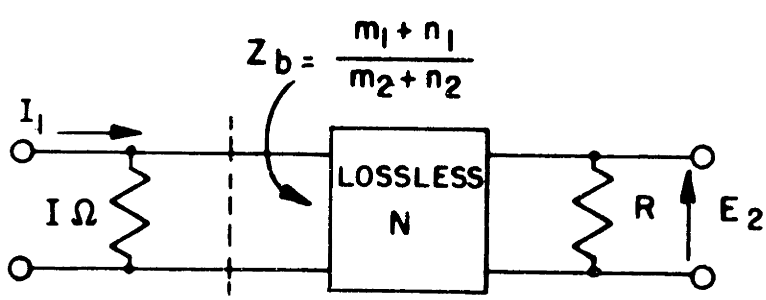


FIG. 5-6

$$Z_{12} = \frac{Z_{12b}}{1+Z_b} = \frac{\sqrt{m_1 m_2 - n_1 n_2}}{m_1 + n_1 + m_2 + n_2}, \tag{5-7}$$

from which we easily derive the insertion loss formulae used in synthesizing this type of network,

$$\left|2 Z_{12}\right|^2_{i\omega} = \frac{4(m_1 m_2 - n_1 n_2)}{(m_1+m_2)^2 - (n_1+n_2)^2}, \tag{5-8}$$

and

$$\left|w\right|^2 = 1 - \left|2 Z_{12}\right|^2 = \frac{(m_1-m_2)^2 - (n_1-n_2)^2}{(m_1+m_2)^2 - (n_1+n_2)^2}. \tag{5-9}$$

c) Symmetrical Networks

In the case of a network which can be partitioned into two separate identical halves symmetric about the partition line, we obtain several interesting results, Fig. 5-7.

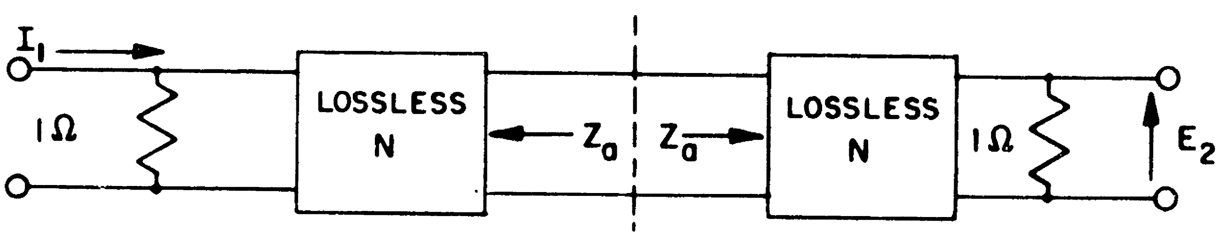


FIG.5-7

The partitioning theorem here yields

$$Z_{12} = \frac{Z_{12a}^{2}}{2 Z_a}, \tag{5-10}$$

or

$$\left| \frac{2E_2}{I_1} \right|^2 = \left[\frac{|Z_{12a}|^2}{|Z_a|} \right]^2 ,$$

$$= \left[\frac{Re\,[Z_a]}{|Z_a|} \right]^2 ,$$

$$= \cos^2 \theta \quad . \tag{5-11}$$

where θ is the phase angle of Z_a. But

$$\cos^2 \theta = \frac{1}{1 + \tan^2 \theta} ; \tag{5-12}$$

hence

$$\left| \frac{2E_2}{I_1} \right|^2 = \frac{1}{1 + \tan^2 \theta} . \tag{5-13}$$

Since tan θ is an odd rational function of λ as shown in Part III, we deduce that a necessary condition on the expression $\left| \frac{2E_2}{I_1} \right|^2$ corresponding to the transfer impedance of a symmetrical network is that it be of the form

$$\left| \frac{2E_2}{I_1} \right|^2 = \frac{1}{1 + (\text{odd rational function})^2} \tag{5-14}$$

Of more importance to the synthesis problem is the conclusion that this condition is also sufficient, a statement that follows from our demonstration in Part III that we can obtain a driving-point impedance (Z_a in this case) corresponding to any odd rational function which is prescribed as the tangent of a phase angle.

3. SYNTHESIS OF SYMMETRICAL NETWORKS

From the discussion above it is apparent that we can obtain the design for a symmetrical network merely by identifying the odd rational portion of the expression for $\left| \frac{2E_2}{I_1} \right|^2$ with the tangent of the phase angle of one-half of the network; then synthesizing the network half by means of the procedure set up in Part IV; and finally connecting two of these network halves "back to back." The synthesis of the network half must, of course, be made on the basis of a lossless network terminated in a one ohm resistance in order to satisfy the requirement $\left| Z_{12a} \right|^2 = Re[Z_a]$, which is one of the essential steps in the reasoning. That this configuration is always possible is well-known. Moreover our ability to select at will the arbitrary multiplier C for $Z(\lambda)$ (see Example 5, Part IV) eliminates the possible need for an ideal transformer to obtain the one ohm termination.

Many of the common filter networks are represented in the form of Equation 5-14, and for most of these the above procedure can be used to effect a large saving in computational labor.

It should be noted, however, that one of the most common types of symmetrical filters, which is usually represented in the form

$$\left|\frac{2E_2}{I_1}\right|^2 = \frac{1}{1+\left[S_0\,\frac{\omega(\omega^2-\omega_1^2)\ldots(\omega^2-\omega_n^2)}{(1-\omega_1^2\omega^2)\ldots(1-\omega_n^2\omega^2)}\right]^2} \qquad (5\text{-}15)$$

does not enjoy the advantage of reduced computational labor and, in general, leads to networks which have superfluous elements. This situation arises because each λ^2-root in the denominator is present as a factor of odd multiplicity, and this we recognize as evidence of cancellation. The restoration of these missing factors causes the degree of the impedance expression to increase. In physical terms we can explain the situation by noting that the given transfer impedance has single order zeros at the real frequencies.

In a ladder development, each of these zeros would ordinarily be obtained by a shunt branch exhibiting series resonance. In filters developed by the phase angle process just described, each half would have its own shunt branch, thus giving the network the appearance of presenting double instead of single zeros. How this apparent anomaly is resolved is seen when the transfer impedance is computed from the element values. These values group into factors which cancel and leave the required single zeros.

On the other hand, symmetrical filters having all of the zeros of their transfer impedance at infinity are particularly amenable to phase synthesis. A common type of filter having this property is characterized by Tchebycheff behavior in the pass band and monotonic behavior in the attenuation band. The design of such a filter is carried out in Example 1 below, using one of the examples already worked out for the network half.

It should be noted that when any of the networks shown in the examples of Part IV are connected symmetrically in cascade, the over-all transfer impedance becomes related to $T(\lambda)$ by

$$\left| 2 Z_{12} \right|^2 = \frac{1}{1 - T^2}, \quad \lambda = j\omega .$$

4. EXTENSION TO NON-SYMMETRICAL NETWORKS

With a slight generalization we can increase the usefulness of our results by extending them to include non-symmetrical networks. This extension is obtained without any increase in the root-finding computation, and it covers, without the use of ideal transformers, the case where the input and output loading differ.

Let us consider the effect on the transfer impedance if, instead of connecting the two identical networks together as described, we employ the arbitrary constant C to modify the termination of the right-hand network from unity to R ohms, while leaving the termination on the input side at one ohm. It is obvious that this change does not affect Z_a or Z_{12a}; while it changes Z_b and Z_{12b} only by constant multipliers. Hence Z_{12} is changed only by a constant.

As a result of this reasoning we see that even when the input and output loading are different, the design can still proceed on the basis of the phase angle of half of a symmetrical network. After the impedance function $Z(\lambda)$ is found, it is synthesized differently for the left and right halves to accomodate the proper loading conditions. The resulting network is, of course, not symmetrical.

5. EXAMPLES SHOWING THE USE OF PHASE SYNTHESIS IN FILTER DESIGN

Example 1

The design of a prototype low-pass filter with resistance loading at both ends is desired. The filter is to have Tchebycheff behavior in the pass band with voltage ripple not in excess of five per cent. In the stop band the behavior is to be monotonic and at least

40 db of attenuation is to be reached at the point $\omega = 1.5$.

Using the relation

$$\left| 2 Z_{12} \right|^2 = \frac{1}{1 + \varepsilon^2 V_n^2(\omega)}, \tag{5-16}$$

as a basis for design we note that a symmetrical filter will result when n is odd. By applying the specifications to Equation 5-16, we obtain

$$\varepsilon = .331,$$
$$n = 7.$$

We next obtain a two terminal network with the phase angle

$$\tan\theta = .331\, V_7(\omega), \tag{5-17}$$

in the form of a lossless network terminated in a one ohm resistance. This network is synthesized in Example 9, Part IV. The solution to the design problem results from the connection of a pair of these networks as shown in Fig. 5-8.

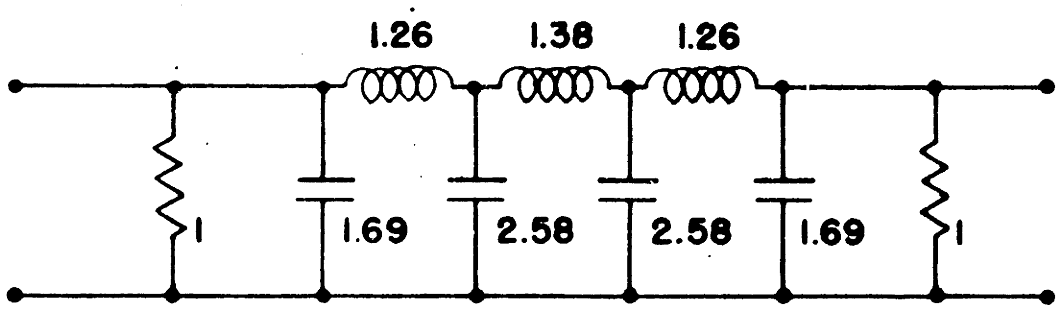


FIG. 5-8

We note that Equation 5-17 could as well be written

$$\tan \theta = -.331\, V_7(\omega) \tag{5-17a}$$

because of the square in Equation 5-16. As a result, the half-network driving-point impedance would be inverted yielding the alternate solution shown in Fig. 5-9.

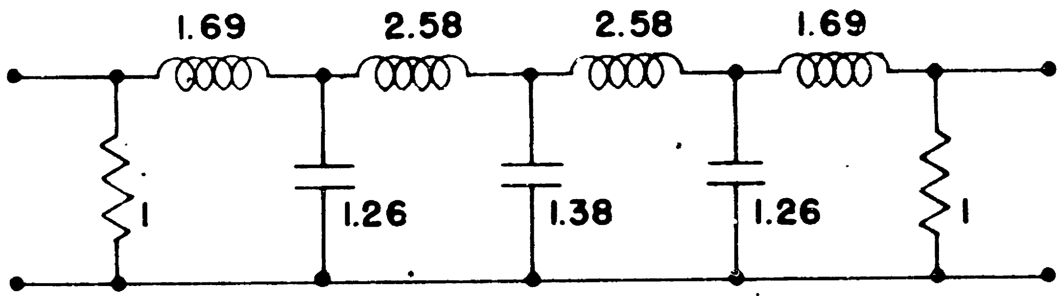


FIG. 5-9

Example 2

In this example we modify the symmetrical result of Example 1 to handle an output loading of 10 ohms. This is done by raising the impedance level of the right-hand half.

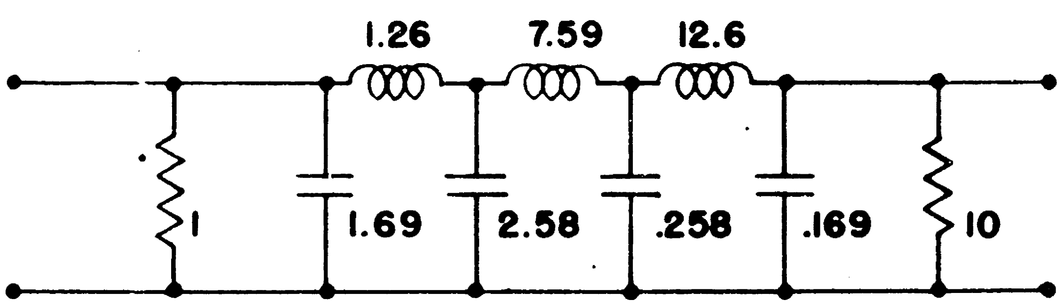


FIG. 5-10

APPENDIX

1. ROOTS OF TCHEBYCHEFF POLYNOMIALS FOR PHASE SYNTHESIS

In order to apply the method of synthesis presented in this thesis, we must determine the grouping of left and right half-plane roots of

$$S(\lambda) = 0. \tag{6-1}$$

This computation can be materially reduced in the common case where the tangent function is of the form

$$\tan\theta = \mathcal{E} V_n(\omega), \tag{6-2}$$

as in Example 9 of Part IV. $V_n(\omega)$ is the n'th order odd Tchebycheff polynomial $\cos(n \cos^{-1}\omega)$ and $\mathcal{E}$ is a positive real constant.

In this case Equation 6-1 becomes, after the manipulation to form $T(\lambda)$ and the adding of numerator and denominator,

$$V_n(-j\lambda) = \frac{j}{\mathcal{E}}. \tag{6-3}$$

The left half-plane roots of Equation 6-3, which combine to make $p(\lambda)$, are readily shown to be

$$\lambda_v = -\sin\frac{v\pi}{2n}\sinh\phi_2 \pm j\cos\frac{v\pi}{2n}\cosh\phi_2,$$

$$\left(v = 1, 5, 9, \ldots \begin{smallmatrix} n \text{ or} \\ n-2 \end{smallmatrix}\right), \tag{6-4}$$

where

$$\left.\begin{matrix}\cosh \phi_2 \\ \sinh \phi_2\end{matrix}\right\} = \frac{1}{2}\left[\left(\sqrt{1+\frac{1}{\mathcal{E}_2}}+\frac{1}{\mathcal{E}}\right)^{\frac{1}{n}} \pm \left(\sqrt{1+\frac{1}{\mathcal{E}_2}}+\frac{1}{\mathcal{E}}\right)^{-\frac{1}{n}}\right] \qquad (6\text{-}5)$$

while the right half-plane roots, which combine to make $g(-\lambda)$, are

$$\lambda_\nu = \sin\frac{\nu\pi}{2n}\sinh\phi_2 \pm j\cos\frac{\nu\pi}{2n}\cosh\phi_2,$$

$$\left(\nu = 3, 7, 11, \ldots \begin{matrix} n \text{ or} \\ n-2\end{matrix}\right). \qquad (6\text{-}6)$$

2. ALTERNATION OF ROOTS

It is interesting to compare the results just obtained with the insertion loss method of designing a symmetrical filter such as the one illustrated in Example 1, Part V. The latter procedure would require solving for the roots of

$$V_n(\omega) = \pm\frac{j}{\mathcal{E}}.$$

Here the presence of the $\pm$ sign yields values for the left half-plane λ-roots

$$\lambda_\nu = -\sin\frac{\nu\pi}{2n}\sin\phi_2 \pm j\cos\frac{\nu\pi}{2n}\cosh\phi_2,$$

$$(\nu = 1, 3, 5, \ldots n). \qquad (6\text{-}7)$$

A comparison of the signs and the running subscript of this formula with those of Equations 6-4 and 6-6 reveals that the root distribution in the two cases is that shown in Fig. 6-1,

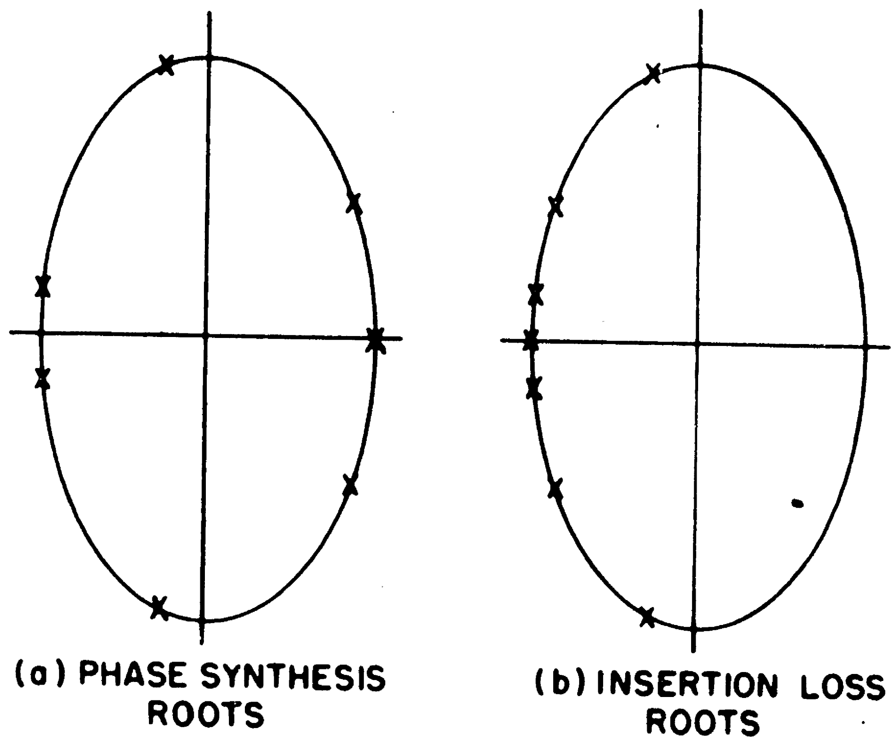


FIG.6-1

We substantiate from this distribution the validity of a method of design obtained heuristically by Professor E. A. Guillemin. He pointed out that there was some evidence to indicate that the impedance function $Z(\lambda)$ for half of a symmetrical network could be obtained by selecting alternate roots from the distribution of Fig. 6-1(b) for the numerator of $Z(\lambda)$ and the intervening roots for the denominator. Our analysis shows why his method gave correct results in every case which he checked.

In conclusion, it may be of interest to the reader to learn that this modest result, which is quite overshadowed by the general solution to the phase synthesis problem, was the original goal of this thesis. At the start, it was not suspected that this deceptively simple problem was intimately associated with phase synthesis.

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