ON THE STABILITY OF THE LAMINAR BOUNDARY LAYER BETWEEN PARALLEL STREAMS

by

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Dear Professor DenHartog:

I take pleasure in submitting herewith my thesis entitled, "On the Stability of the Laminar Boundary Layer between Parallel Streams", in partial fulfillment of the requirements for the degree of Doctor of Science at the Massachusetts Institute of Technology.

Sincerely yours,

Martin Lessen

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TABLE OF CONTENTS

		Page
I.	Introduction	1
II.	Abstract	3
III.	Historical Survey of Related Investigations	7
	A. The Stability of Laminar Flow	7
	B. The Laminar Boundary Layer	14
IV.	Theoretical Analysis	16
	A. Rotationally Symmetrical Flow	16
	B. The Laminar Boundary Layer Flow	19
	C. The Stability of the Flow Configuration	26
V.	Solution of the Boundary Layer and Stability Equations	31
	A. Solution of the Boundary Layer Equation	31
	B. Solution of the Orr-Sommerfeld Equation	36
	C. Discussion of Solution of the Orr-Sommerfeld Equation	42
VI.	Results, Conclusions and Suggestions for Further Work	45
VII.	References	49
VIII.	Appendix	54
	A. Solutions of the Boundary Layer Equation	54
	B. Method of Solution of Orr-Sommerfeld Equation with Digital-Type Computing Machine	59
	C. Biographical Note	62

LIST OF ILLUSTRATIONS

	Page
Figure 1	17
2	23
3	23
4	32
5	33
6	38
7	38

I. INTRODUCTION

The original problem that finally led to the present investigation was the problem of the transition from laminar to turbulent motion of a free jet of fluid immersed in a surrounding fluid. The problem, in original form, was very complex and certain simplifications had to be made before a successful analytical approach could be accomplished and methods evolved for dealing with the full problem.

The first simplification of the problem was to assume that the jet and the surrounding fluid were composed of the same fluid medium. It was then assumed that the fluid medium was incompressible. Finally, the problem of a jet with rotational symmetry was simplified to the corresponding two dimensional problem of two parallel streams both of semi-infinite extent contacting each other beginning at a point fixed in space.

It was considered that, under certain circumstances, the type of flow that would result between parallel streams would be of the "boundary layer" variety. The point of transition of flow in the boundary layer from laminar to turbulent would occur somewhere after the laminar flow configuration became unstable; the measure of stability being whether or not small disturbances introduced into the flow field would die out or become magnified with time.

In its present form, it is the object of this investigation to study the stability of the laminar, free boundary layer between two parallel streams of fluid in plane flow. First, it will be shown how the case of a jet with rotational symmetry reduces, in the first approximation, to the case of plane flow. The two dimensional problem of the free boundary layer between parallel streams will then be considered in detail. After the flow configuration of the laminar boundary layer is determined, the stability of said configuration will be investigated by considering the behavior (in relation to time) of small disturbances in the flow field. The investigation of stability will be carried out for only one case of flow, the case where one of the moving streams is considered at rest. Throughout the treatment of the problem, the fluid will be considered incompressible.

II. ABSTRACT

The problem of the stability of the surface of discontinuity between parallel streams has long been of much concern to mathematicians and physicists. In the latter part of the nineteenth century, Helmholtz and Rayleigh suggested it as a prize problem. Helmholtz conducted a few investigations of his own on the problem, but no positive results other than instability were forthcoming. It was because of the powerful mathematical tools and calculating machines such as the analogue and digital computers that were developed since the time of the early investigators that a direct attack on the problem with minimum approximation was made possible at this time.

The present investigation deals with the stability of the laminar boundary layer formed between uniform parallel streams flowing at different velocities. The streams are assumed to be semi-infinite in extent and to meet starting at a point fixed in space. The present problem differs mathematically from previous investigations in the stability of parallel flows in that the boundary conditions of the equations of motion for the present investigation can only be expressed for points at infinity (the boundary conditions are asymptotic in character). In previous investigations, Couette, Poiseuille, and Blasius type flows were investigated for stability. In Couette and Poiseuille flows, the boundary

conditions are expressed for finite points. Blasius-type flow has a boundary condition at a finite point and also one at infinity. Although the problem with boundary conditions at infinity would seem to be more complicated than previous problems, the analysis is, in some respects simplified.

The method of attack used in the analysis of the stability of the "free" boundary layer was to consider the effect of small disturbances superimposed on the steady-state incompressible laminar flow configuration. From these considerations, an equation for the disturbance is obtained and contained in this equation are parameters representing the velocity of propagation of the disturbance, its wave number, and the viscosity of the fluid medium. It is possible to demonstrate a necessary functional relationship between the parameters of the disturbance equation so that the boundary conditions imposed on the disturbance are satisfied; the problem then resolves itself into developing a method for solving for the relationship between the parameters. Throughout the treatment of the problem, the magnitude of the disturbances is considered small compared to the magnitude of the steady-state velocities, and, as a consequence, the disturbance equation can be linearized.

Before the stability equation can be solved, however, it is necessary to solve for the "steady-state" flow configuration. Although a certain amount of work was done in this direction by Görtler and Keulegan using approximate methods, it was considered advantageous to solve the boundary layer equation using the differential analyser to obtain continuous results especially in the higher derivatives of the velocity profile.

The results obtained include a full family of velocity profiles for the case of the boundary layer between parallel streams. A process for the solution of the eigenvalue problem of the disturbance equation is then developed and the calculations are carried forward for one case.

The conclusions arrived at from the results indicate that at large Reynolds numbers (small effect of viscosity) instability occurs with disturbance wave numbers (α) that are less than .4 approximately. In general the effect of Reynolds number on instability is small and the flow is highly unstable. The existence of a minimum Reynolds number for instability can, however, be demonstrated.

The differential-analyser proved useful in solving for the steady-state flow configuration and the first approximation of the solution of the disturbance equation. However, the differential analyser proved inadequate for complete solution of the problem. In the near future, therefore, more detailed and refined calculations will be performed on this problem with a digital-type computing machine.

In the course of investigating the disturbance equation, the details of integrating it over the complex-plane with a computing machine were developed and are herein presented.

As suggestions for further work in this general field, the author presents the following:-

1. Experimental exploration of the stability problem for the case under consideration using air at low Mach numbers as a working fluid and utilizing hot-wire measuring techniques. Either a vibrating ribbon or strong sound waves can be used to introduce controlled disturbances into the boundary layer.

2. Theoretical and experimental exploration of the stability problem of the boundary layer between parallel streams consisting of dissimilar incompressible fluids.

3. Theoretical and experimental exploration of the stability problem of the boundary layer between parallel streams consisting of compressible fluid.

III. HISTORICAL SURVEY OF RELATED INVESTIGATIONS

The Stability of Laminar Flow

The problem of the stability of laminar motion was first studied by Helmholtz and Reynolds[38] and, by the end of the nineteenth century, had already attracted considerable attention from the leading investigators of that time. Since then, not only have specialists in hydrodynamics been deeply interested in the problem but also a host of mathematicians and physicists.

From the outset, there seemed to be two distinct schools of thought concerning the cause of transition from steady to turbulent flow. One school of thought believed that the cause of the transition was that the flow configuration was definitely unstable, that infinitesimal disturbances would grow exponentially. The other school believed that the flow configuration was, in most cases, stable for infinitesimal disturbances, but that either suitable disturbances of a finite magnitude or a sufficiently large pressure gradient or a combination of both caused the transition to turbulence. However, both schools agreed that the fluid medium could be considered as incompressible and that the motion of the fluid was described by the Navier-Stokes equations of motion.

The formulation of the theory of finite disturbances was first due to Reynolds[38] and Kelvin[16]. Further developments were due to Schiller, Taylor and others. Since the solution of the non-linear equations satisfied by the finite disturbance is extremely difficult, mathematical considerations of the theory of finite disturbances are usually based on energy or disturbance vorticity considerations.

The development of the theory of infinitesimal disturbances proceeded in two directions. One method consisted of a consideration of the energy and the vorticity of the disturbance. The development of this line of thought can be traced from Orr(29) through Lorentz(23), von Karman(13), Synge(46)(47) and others. It is at present recognized that this method of approach can only yield sufficient conditions for stability. Since in this analysis all kinds of disturbances are allowed including disturbances that do not satisy the hydrodynamics equations of motion, a larger viscous decay is required to insure stability than when the nature of the disturbance is limited to one in which the equations of motion are satisfied. As a consequence of this, the limit of stability as obtained by these conditions is always much lower than is obtained by experiment. However, from the foregoing considerations, Synge(46) has arrived at a convenient form of a sufficient condition for stability of two-dimensional parallel flows with respect to two dimensional disturbances. The calculation of the neutral curve by Synge's criterion will be included in the present investigation.

In order to arrive at something more specific than the sufficient conditions for stability, it is necessary to solve the linearized equations satisfied by the disturbance. This line of attack was satisfactorily carried out by Taylor[49] for the specific case of Couette-type flow between concentric cylinders. Experimental verifications of Taylor's results were carried out by himself and others. Mathematical proof of a sufficient condition for stability of Couette flow was given by Synge[48]. Taylor's work was later extended by Görtler[7][8] to cover the case of boundary layer flow over a curved wall. It should be noted that generally in curved flows of this type centrifugal force plays a dominant role.

The most extensive investigation of hydrodynamic stability of laminar flows deals with the solution of the eigenvalue problem associated with the linearized equations governing the disturbance. This line of development can be traced in the work of Helmholtz and Rayleigh[35][36] right on through Orr[29], Sommerfeld[44], von Mises[24][25], Hopf[12], Prandtl[33], Tietjens[53], Heisenberg[10], Tollmein[54-56], Schlichting[41-43] and Lin[21][22]. Other contributors were Noether[28], Squire[45], Goldstein[5], Pekeris[31][32], and others.

The theory of infinitesimal disturbances deals mainly with two-dimensional wavy disturbances propagated along the direction of the main flow. Squire[45] demonstrated that two dimensional infinitesimal disturbances are more destablizing than three dimensional disturbances in an incompressible fluid. Most of the work before Squire was based on two dimensional disturbances, but Squire proved the validity of this assumption. The latest work in this field by C. C. Lin[22] is also based on the validity of the assumption of a two dimensional infinitesimal disturbance for an incompressible fluid.

The earliest study of two-dimensional hydrodynamic stability seems to have been made by Helmholtz who demonstrated the instability of wavy disturbances over the surface of discontinuity of two parallel streams having different velocities. Later, Lord Rayleigh[35] extended the analysis of Helmholtz to include continuous velocity distributions. Rayleigh approximated a continuous velocity profile by substituting in its stead a number of straight line velocity profiles joined end to end. It should be noted that since the vorticity is a constant for each straight line segment of the velocity profile, the vorticity distribution over the entire profile is discontinuous in nature. Rayleigh also investigated continuous vorticity distributions. All of this work was based on the assumption that the fluid medium was inviscid and incompressible. Rayleigh's investigations resulted in two important conclusions: (1) that instability (in an inviscid fluid) can only occur with velocity profiles having a point of inflection, (2) and that broken linear velocity profiles approximating actual velocity distributions having a point of inflection are also unstable. Rayleigh obtained an approximate condition determining stability as follows:

$$\int_{y_1}^{y_2} \frac{dy}{(w-c)^2} = 0$$

where

$w(y)$ - velocity distribution

y_1, y_2 - coordinates of solid boundaries

c - constant

The nature of c is that the real part of c represents the wave velocity whereas the imaginary part of c determines whether or not the disturbance is damped or amplified.

Investigations of linear velocity distributions including viscosity effects were carried out by von Mises[24][25], Hopf[12] and Rayleigh[36], the results of which indicated that only the condition of stability could exist. Prandtl and Tietjens[53] carried out an investigation on the stability of the boundary layer using Rayleigh's approximate method and including the effect of viscosity, but due to the fact that the analysis was inadequate, the results were inconclusive.

The first successful study of the stability of a variable continuous vorticity distribution was carried out by Heisenberg[10] who studied plane Poiseuille flow in particular. At the same time, Heisenberg critically examined Rayleigh's approximate method using broken linear profiles and showed it to be erroneous as an approximation of a continuous velocity distribution. Heisenberg's theory was not generally accepted. Later, Tollmien and Schlicting studied the cases of Blasius[54] and plane Couette[39] flow using Heisenberg's theory.

Heisenberg used the asymptotic expansions of the Orr-Sommerfeld equation to obtain his results. The general form of the differential equation that he dealt with was

$$\sum_{j=0}^{4} a_j y^{(4-j)} + \lambda^2 \sum_{k=0}^{2} b_k y^{(2-k)} = 0$$

where a_j, b_k are analytic functions of the complex variable x, λ is a large complex parameter with constant argument, $a_o = 1$ and b_o has a zero of first order at some point at say $x = x_o$. Since the above equation is of the fourth order, there exists a fundamental system of four independent solutions that form the complete solution of the equation. Two of the solutions may be obtained by the asymptotic expansion

$$y = \sum_{i=0}^{n} \frac{y_i}{2i}$$

The other two solutions are of the form

$$y = \exp \left(\int g dx\right)$$

Lin in his investigations of Poiseuille and Blasius flow in particular used this approach, the validity of which was recently demonstrated by Wasow[57].

In his investigations Tollmien[55] rigorously proved the instability of velocity distributions having a point of inflection for inviscid fluids. For a viscous fluid, Lin[21][22] demonstrated that instability depended on the general class of velocity distribution rather than on a point of inflection.

The most recent investigations in the field of stability of fluid flows have been performed by Lin[18] and Lees[19] and have been concerned with the stability of the boundary layer in a compressible fluid.

Up to this point, the historical development of the theoretical aspects of the stability of laminar flows have been discussed. Experimental investigations in the early nineteen-thirties failed to uncover any evidence of amplified disturbances of certain wave lengths in certain regions as predicted by the theory of infinitesimal disturbances. Hence, the view shared by the experimeters was one unfavorable to the instability theory.

In 1940 a program of research was undertaken at the Bureau of Standards to study transition of Blasius type flow in a low turbulence level wind tunnel. In the course of this research, Schubauer and Skramstad obtained records of velocity fluctuations in the boundary layer by means of a hot wire anemometer. The frequency of these fluctuations agreed quite well with the values predicted in the calculations of Tollmien and Schlichting using Heisenberg's theory. The reason why these fluctuations had not been observed before was because the high turbulence level prevalent in the previous tests had obscured the self-excited oscillations. Other tests were then performed by Schubauer and Skramstad to produce waves in the laminar boundary layer under controlled conditions. They observed where oscillations of a given wave length were neither damped nor amplified and so could determine a curve of neutral stability of wave length versus Reynolds number.

After the completion of the experimental investigation, C. C. Lin(21)(22) undertook a complete reorganization and revision of the stability theory (infinitesimal disturbances) of two-dimensional parallel flows. His results agreed remarkably well with experimental data.

The Laminar Boundary Layer

The concept of the boundary layer was introduced by Prandtl in 1904. It states that there is a narrow region adjacent to the surface of a body immersed in a fluid stream (the fluid being of small viscosity) in which the velocity of the fluid relative to the immersed body varies from zero at the surface of the body to the velocity that would have existed had the flow been frictionless.

The boundary layer type flow was first calculated for a flat plate in a uniform stream with no adverse pressure gradient. This first case was investigated by Blasius[1]. Later, flow over curved plates and with pressure gradients in the direction of flow were studied.

The concept of the boundary layer can be extended to the narrow transition region between two streams of different velocity contacting each other. The boundary layer can then be conceived as being "free" (without any solid boundary). One case of this type of flow was solved by Görtler[9] when he solved the turbulent mixing problem between parallel streams using a constant exchange coefficient. The assumption of a constant exchange coefficient made the problem mathematically analagous to the laminar boundary layer. An experimental check of Görtler's work was performed by Liepmann[20]. Another important contribution to the field of laminar boundary layer flow was the work done by Keulegan[17] who recently solved for the flow configuration at the interface of two different liquids. In the present investigation the free laminar boundary layer between two parallel streams will be solved for all cases of velocity ratio between the two streams.

IV. THEORETICAL ANALYSIS

Rotationally Symmetrical Flow (Jet in Surrounding Fluid)

First, we will show that under certain conditions, the problem of rotational symmetry can be approximated by a corresponding plane flow.

The symbols used in the following discussion are:

- x — positional coordinate in axial direction
- y — positional coordinate in radial direction (radial distance)
- t — time
- u — velocity component in x-direction
- v — velocity component in y-direction
- ρ — density
- μ — absolute viscosity
- $\nu = \frac{\mu}{\rho}$ — kinematic viscosity

If we leave out of our consideration the body forces in the x and y directions, the Navier-Stokes equations for rotationally symmetric incompressible flow can be stated as follows:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{1}{\rho}\frac{\partial p}{\partial x} = \nu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{1}{y}\frac{\partial u}{\partial y}\right]$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \frac{1}{\rho}\frac{\partial p}{\partial y} = \nu\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{1}{y}\frac{\partial v}{\partial y} - \frac{1}{y^2}v\right] \tag{1.0}$$

The equation of continuity is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{v}{y} = 0$$

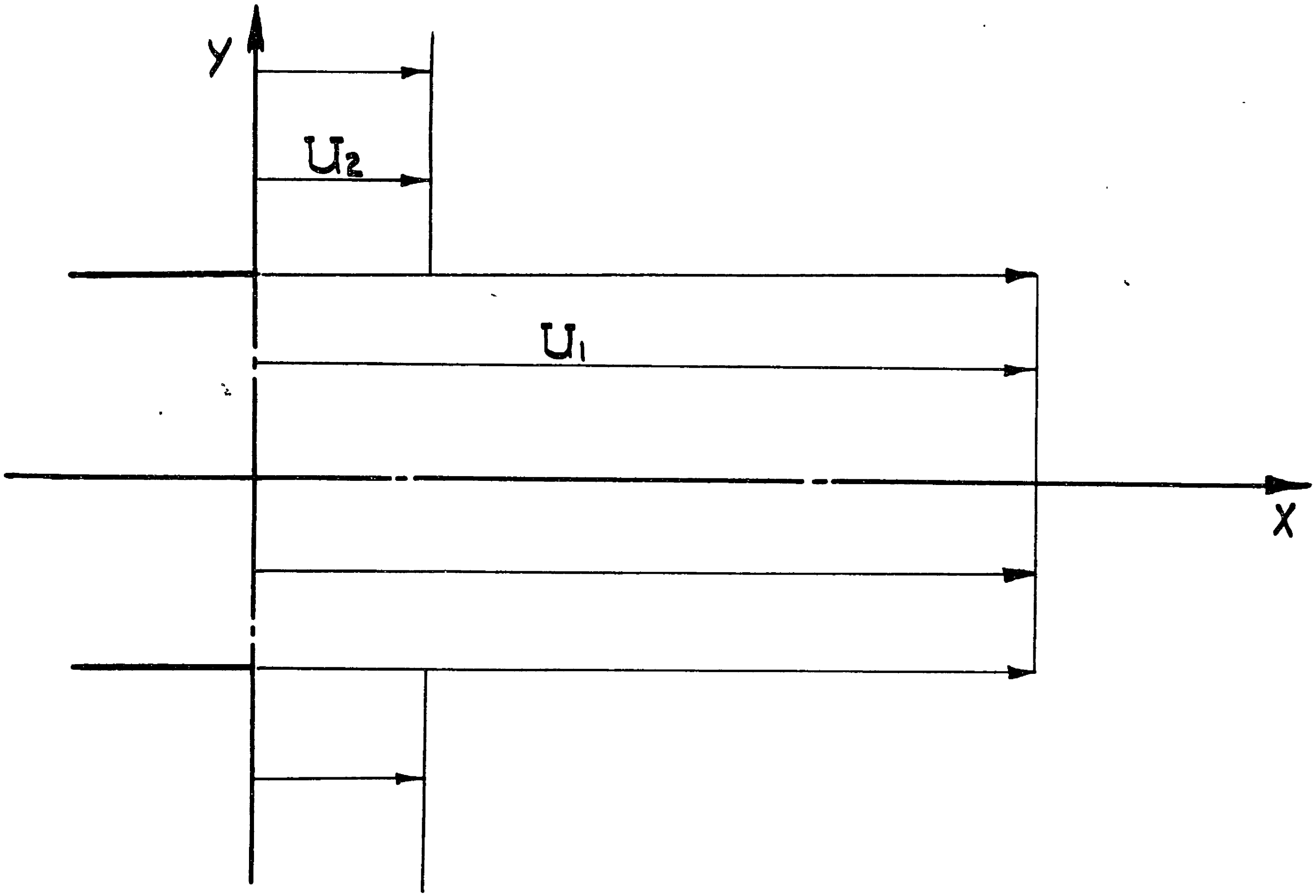

FIGURE 1 – SECTION THROUGH AXIS OF FREE JET WITH ROTATIONAL SYMMETRY

Let us now consider the direction of flow of the parallel streams to be in the x-direction and the characteristic measure of boundary layer thickness, δ, between the parallel flows to be in the y-direction (see Figure 1). We now specify that the velocity of the jet or inner stream is U_1 and the velocity of the surrounding fluid or outer stream is U_2.

At this point, we can define a Reynolds number, R, such that

$$R = \frac{U_1 \delta}{\nu}$$

Let us now introduce dimensionless variables as follows:

$$x = x'\delta \qquad u = u'U_1$$
$$y = y'\delta \qquad v = v'U_1$$
$$t = t'\frac{\delta}{U_1} \qquad p = p'\rho U_1^2$$

We can now rewrite the Navier-Stokes equations (1.0) in terms of the new variables.

$$\frac{\partial u'}{\partial t'} + u'\frac{\partial u'}{\partial x'} + v'\frac{\partial u'}{\partial y'} + \frac{\partial p'}{\partial x'} = \frac{1}{R}\left[\frac{\partial^2 u'}{\partial x'^2} + \frac{\partial^2 u'}{\partial y'^2} + \frac{1}{y'}\frac{\partial u'}{\partial y'}\right]$$

$$\frac{\partial v'}{\partial t'} + u'\frac{\partial v'}{\partial x'} + v'\frac{\partial v'}{\partial y'} + \frac{\partial p'}{\partial y'} = \frac{1}{R}\left[\frac{\partial^2 v'}{\partial x'^2} + \frac{\partial^2 v'}{\partial y'^2} + \frac{1}{y'}\frac{\partial v'}{\partial y'} - \frac{1}{y'^2}v'\right] \qquad (1.1)$$

$$\frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'} + \frac{v'}{y'} = 0$$

Let us now expand our dependent variables in terms of a power series in y' such that

$$u' = \sum_{i=0}^{\infty} u'_i \, y'^{-i}$$

$$v' = \sum_{i=0}^{\infty} v'_i \, y'^{-i}$$

$$p' = \sum_{i=0}^{\infty} p'_i \, y'^{-i}$$

Substituting the powers series expansions above into the dimensionless Navier-Stokes equations (1.1) and equating terms in like powers of y', we obtain the following set of equations for the first approximation..

$$\frac{\partial u_0'}{\partial t'} + u_0 \frac{\partial u_0'}{\partial x'} + v_0 \frac{\partial u_0'}{\partial y'} + \frac{\partial p_0'}{\partial x'} = \frac{1}{R}\left[\frac{\partial^2 u_0'}{\partial x'^2} + \frac{\partial^2 u_0'}{\partial y'^2}\right]$$

$$\frac{\partial v_0'}{\partial t'} + u_0 \frac{\partial v_0'}{\partial x'} + v_0 \frac{\partial v_0'}{\partial y'} + \frac{\partial p_0'}{\partial y'} = \frac{1}{R}\left[\frac{\partial^2 v_0'}{\partial x'^2} + \frac{\partial^2 v_0'}{\partial y'^2}\right] \qquad (1.2)$$

$$\frac{\partial u_0'}{\partial x'} + \frac{\partial v_0'}{\partial y'} = 0$$

However, the above equations are the Navier-Stokes equations of an incompressible fluid in two-dimensional flow. This first approximation of flow with rotational symmetry will be good at large values of y' or large values of radius compared to boundary layer thickness.

The Laminar Boundary Layer Flow

Let us now limit our consideration to the first approximation of flow with rotational symmetry; the case of plane flow. Dropping all subscripts, the dimensionless equations of two-dimensional viscous incompressible flow are:

$$\frac{\partial u'}{\partial t'} + u'\frac{\partial u'}{\partial x'} + v'\frac{\partial u'}{\partial y'} + \frac{\partial p'}{\partial x'} = \frac{1}{R}\left[\frac{\partial^2 u'}{\partial x'^2} + \frac{\partial^2 u'}{\partial y'^2}\right]$$

$$\frac{\partial v'}{\partial t'} + u'\frac{\partial v'}{\partial x'} + v'\frac{\partial v'}{\partial y'} + \frac{\partial p'}{\partial y'} = \frac{1}{R}\left[\frac{\partial^2 v'}{\partial x'^2} + \frac{\partial^2 v'}{\partial y'^2}\right] \qquad (2.0)$$

$$\frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'} = 0$$

Let us now transform the equations (2.0) to new variables z', w' such that

$$y' = z' R^{-1/2}$$

$$v' = w' R^{-1/2}$$

The Navier-Stokes equations then become:

$$\frac{\partial u'}{\partial t'} + u'\frac{\partial u'}{\partial x'} + w'\frac{\partial u'}{\partial z'} + \frac{\partial p'}{\partial x'} = \frac{1}{R}\frac{\partial^2 u'}{\partial x'^2} + \frac{\partial^2 u'}{\partial z'^2}$$

$$\frac{1}{R}\left(\frac{\partial w'}{\partial t'} + u'\frac{\partial w'}{\partial x'} + w'\frac{\partial w'}{\partial z'}\right) + \frac{\partial p'}{\partial z'} = \frac{1}{R^2}\frac{\partial^2 w'}{\partial x'^2} + \frac{1}{R}\frac{\partial^2 w'}{\partial z'^2} \qquad (2.1)$$

$$\frac{\partial u'}{\partial x'} + \frac{\partial w'}{\partial z'} = 0$$

Now, we expand our dependent variables in terms of power series in R such that

$$u' = \sum_{i=0}^{\infty} \frac{u'_i}{R^i} \qquad p' = \sum_{i=0}^{\infty} \frac{p'_i}{R^i}$$

$$w' = \sum_{i=0}^{\infty} \frac{w'_i}{R^i}$$

Then, substituting the above series into the transformed equations (2.1) and equating like powers of R, we obtain for the initial or boundary layer approximation

$$\frac{\partial u_0'}{\partial t'} + u_0'\frac{\partial u_0'}{\partial x'} + w_0'\frac{\partial u_0'}{\partial z'} + \frac{\partial p_0'}{\partial x'} = \frac{\partial^2 u_0'}{\partial z'^2}$$

$$\frac{\partial p_0'}{\partial z'} = 0 \qquad (2.2)$$

$$\frac{\partial u_0'}{\partial x'} + \frac{\partial w_0'}{\partial z'} = 0$$

The initial, or boundary layer approximation, will in fact be very close to the true state of affairs if the value of Reynolds number, R, is large. If we then specify that only flows with large Reynolds numbers will be under consideration, we can limit discussion to the boundary layer approximation.

Transforming equations (2.2) back to the original dimensional variables, we obtain the boundary layer equations.

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{1}{\rho}\frac{\partial p}{\partial x} = \nu\frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial p}{\partial y} = 0 \qquad (2.3)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

At this point, we specify that there is steady motion and that the pressure over the flow field is constant. The final boundary layer equations therefore are:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (2.4)$$

From continuity considerations, we can now introduce a stream function, ψ , as follows:

$$u = \frac{\partial \psi}{\partial y}$$

$$v = -\frac{\partial \psi}{\partial x}$$

The stream function, ψ , can in turn be defined in terms of a new variable, η , such that

$$\psi = \sqrt{\nu x U_1}\, f(\eta)$$

$$\eta = \frac{y}{\sqrt{\frac{\nu x}{U_1}}}$$

From the above, we can now calculate the various quantities in the boundary layer equations in terms of the new variable.

$$u = U_1 f'(\eta)$$

$$v = \frac{1}{2}\sqrt{\frac{\nu U_1}{x}}(\eta f' - f)$$

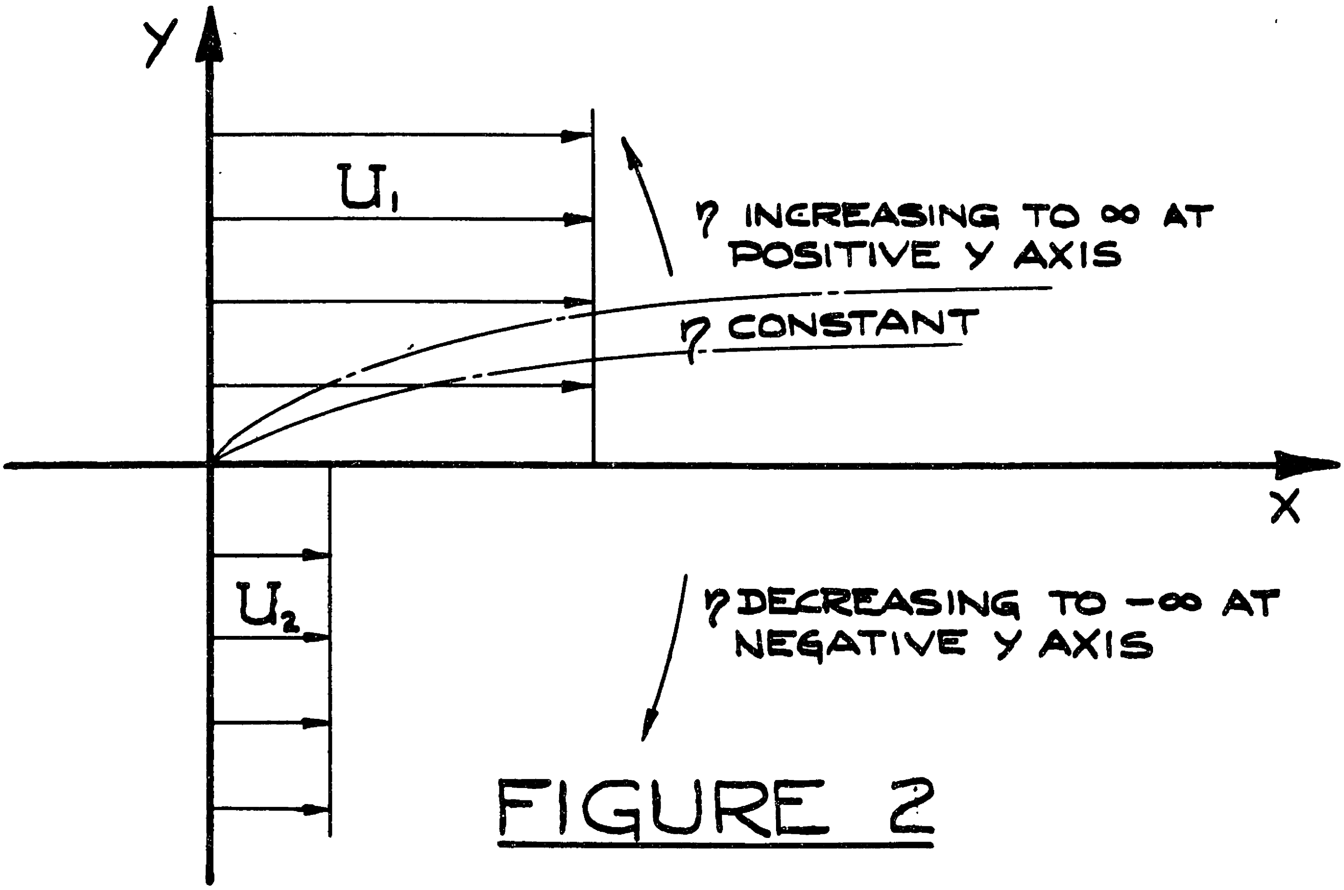

FIGURE 2

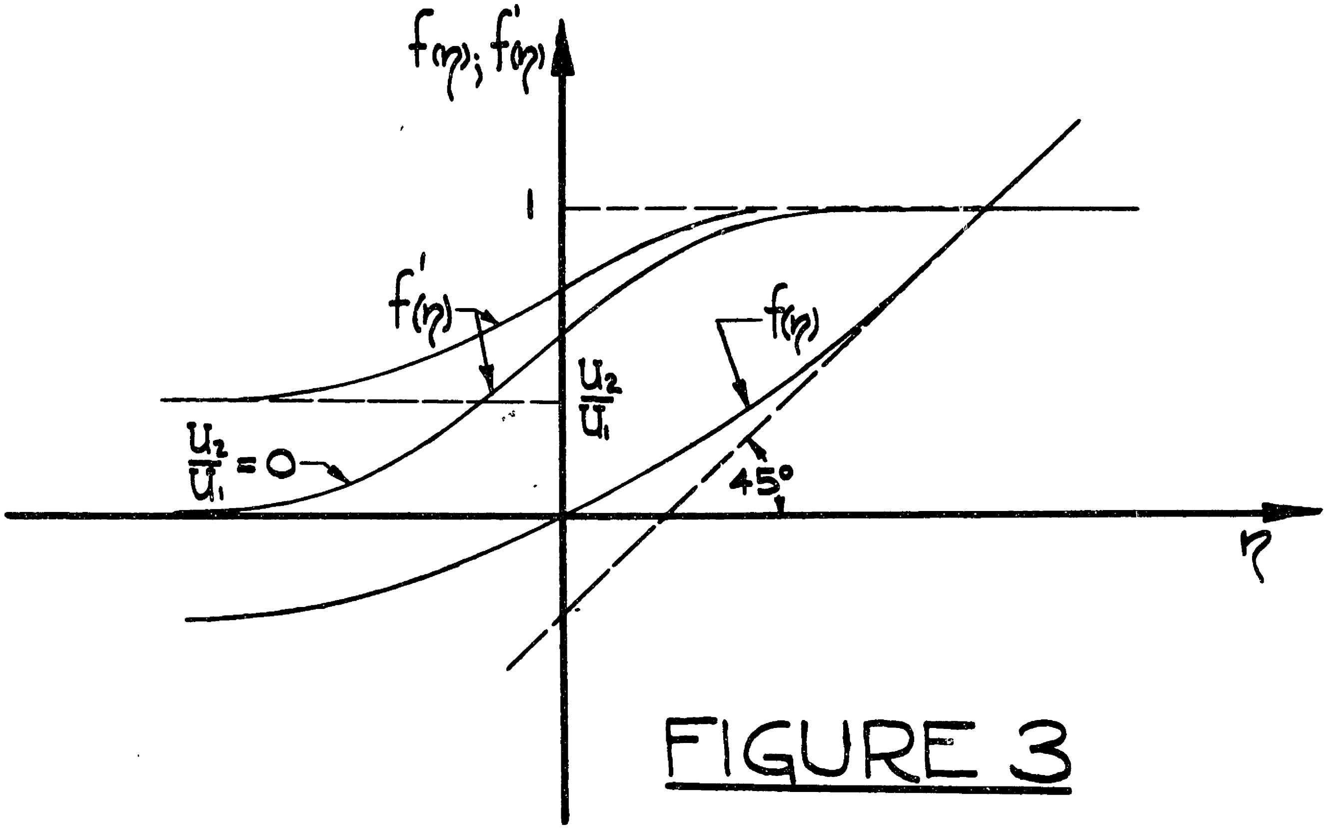

FIGURE 3

$$u\frac{\partial u}{\partial x} = -\frac{U_1^2}{2x}\eta f'f''$$

$$v\frac{\partial u}{\partial y} = \frac{U_1^2}{2x}f''(\eta f'-f)$$

$$\nu\frac{\partial^2 u}{\partial y^2} = 2\frac{U_1^2}{2x}f'''$$

Substituting the preceding quantities into equation (2.4) we obtain

$$-\frac{U_1^2}{2x}\eta f'f'' + \frac{U_1^2}{2x}f''(\eta f'-f) = 2\frac{U_1^2}{2x}f'''$$

And simplifying, we obtain the well known boundary layer equation in terms of η as independent variable.

$$ff'' + 2f''' = 0 \qquad (2.5)$$

Equation (2.5) is a non-linear, third order ordinary differential equation, and, from a well known theorem in the theory of ordinary differential equations, three independent boundary conditions are necessary in order to completely determine the solution. In order to ascertain the boundary conditions for our problem, we must now refer directly to a picture of the physical arrangement (see Figure 2).

Along the positive y axis

$u = U_1$ or $f' = 1$

when $\eta = \infty$

Along the negative y axis

$$u = U_2 \quad \text{or} \quad f' = \frac{U_2}{U_1}$$

$$\text{when} \quad \eta = -\infty$$

So far, we have only two boundary conditions for equation (2.5). Let us arbitrarily establish the third boundary condition as

$f = 0$

$\eta = 0$ (See Figure 3)

Let us now investigate where the boundary between the parallel streams lies.

We can consider a line of η = constant to be the boundary between the streams if the direction of flow at the line η = constant is in the direction of the line. Or stated mathematically

$$\left(\frac{dy}{dx}\right)_{\eta = \text{constant}} = \frac{v}{u} \tag{2.6}$$

However, from the original definition of η

$$\left(\frac{dy}{dx}\right)_{\eta = \text{constant}} = \frac{d}{dx}\eta\sqrt{\frac{\nu x}{U_1}} = \frac{\eta}{2}\sqrt{\frac{\nu}{U_1 x}}$$

Also

$$\frac{v}{u} = \frac{\frac{1}{2}\sqrt{\frac{\nu U_1}{x}}\,(\eta f' - f)}{U_1 f'}$$

Substituting into equation (2.6)

$$\frac{\eta}{2}\sqrt{\frac{\nu}{U_1 x}} = \frac{1}{2}\sqrt{\frac{\nu}{U_1 x}}\left(\eta - \frac{f}{f'}\right)$$

Since f' will always have a finite value, the equation will hold only if f = 0. The boundary between the two streams is therefore where f = 0.

The Stability of the Flow Configuration

After simplification of the rotationally symmetrical flow and consideration of the free, laminar boundary layer, we can now investigate the stability of the laminar flow configuration. The flow will be considered stable if a small periodic disturbance introduced into the flow field is damped out with time. If, however, the disturbance grows with time, the flow will be considered as unstable. Should the disturbance remain unchanged with time, the flow would then be considered as neutrally stable.

Let us now eliminate the pressure term from the dimensionless two-dimensional Navier-Stokes equations (2.0) and introduce a stream function, ψ , in the usual manner. The resulting equation is

$$\frac{\partial}{\partial t}\left(\frac{\partial^2\psi}{\partial y'^2}+\frac{\partial^2\psi}{\partial x'^2}\right)+\frac{\partial\psi}{\partial y'}\left(\frac{\partial^3\psi}{\partial x'\partial y'^2}+\frac{\partial^3\psi}{\partial x'^3}\right)-\frac{\partial\psi}{\partial x'}\left(\frac{\partial^3\psi}{\partial y'^3}+\frac{\partial^3\psi}{\partial x'^2\partial y'}\right)$$

$$=\frac{1}{R}\left(\frac{\partial^4\psi}{\partial x'^4}+2\frac{\partial^4\psi}{\partial x'^2\partial y'^2}+\frac{\partial^4\psi}{\partial y'^4}\right) \qquad (3.0)$$

It will now be assumed that since $v \ll u$ in the steady-state laminar flow configuration, the effect of a steady-state v can be left out of the stability relations for all practical considerations. We can therefore specify a stream function of the form

$$\psi = \Phi(y') + \varphi(y')\, e^{i\alpha(x'-ct')} \qquad (3.1)$$

where $\varphi \ll \Phi$. The form of ψ is such that it implies that the steady-state flow is parallel to the x axis. The perturbation stream function $\varphi(y')\, e^{i\alpha(x'-ct')}$ is of two-dimensional form. The use of a two-dimensional perturbation stream function of this form for two-dimensional parallel flows has been justified by Squire.

If we insert the stream function, equation (3.1), into equation (3.0) we obtain

$$\begin{aligned}
&-i\alpha c\varphi'' e^{i\alpha(x'-ct')} + i\alpha^3 c\varphi\, e^{i\alpha(x'-ct')} \\
&+\left[\Phi' + \varphi' e^{i\alpha(x'-ct')}\right]\left[i\alpha\varphi'' e^{i\alpha(x'-ct')} - i\alpha^3\varphi\, e^{i\alpha(x'-ct')}\right] \\
&-i\alpha\varphi e^{i\alpha(x'-ct')}\left[\Phi''' + \varphi''' e^{i\alpha(x'-ct')} - \alpha^2\varphi' e^{i\alpha(x'-ct')}\right] \\
&\qquad = \frac{1}{R}\left[-2\alpha^2\varphi'' e^{i\alpha(x'-ct')} + \Phi^{IV} + \varphi^{IV} e^{i\alpha(x'-ct')} + \alpha^4\varphi\, e^{i\alpha(x'-ct')}\right]
\end{aligned}$$

$$(3.2)$$

The equation of the steady-state terms alone in equation (3.2) will be satisfied. We can therefore eliminate all steady-state terms from equation (3.2). Eliminating all terms quadratic in the disturbance and simplifying, we obtain the Orr-Sommerfeld equation for the disturbance in an incompressible parallel flow of fluid.

$$(\Phi' - c)(\varphi'' - \alpha^2\varphi) - \Phi'''\varphi = -\frac{i}{\alpha R}(\varphi^{iv} - 2\alpha^2\varphi'' + \alpha^4\varphi)$$

(3.3)

If we choose δ, the characteristic length in the y' direction, properly, we can arrange matters so that

$$y' = \eta$$

η being defined by the following relationship with dimensional x and y:

$$\eta = \frac{y}{\sqrt{\frac{\nu x}{U_1}}}$$

We can therefore rewrite the Orr-Sommerfeld equation in terms of functions of η.

$$(f' - c)(\varphi'' - \alpha^2\varphi) - f'''\varphi = -\frac{i}{\alpha R}(\varphi^{iv} - 2\alpha^2\varphi'' + \alpha^4\varphi)$$

(3.4)

where $f = f(\eta)$

$\varphi = \varphi(\eta)$

c = dimensionless velocity of propagation of disturbance

α = dimensionless phase velocity, wavelength = $\frac{2\pi}{\alpha}$
(wave number)

α will always be taken as real and positive.

c may be complex.

From equation (3.1) it can easily be seen that the flow configuration will be unstable to small perturbations if the imaginary part of c is positive. If the imaginary part of c is negative, the flow will be stable whereas if the imaginary part of c is equal to zero, the flow configuration will be neutrally stable.

There remains to solve the characteristic or eigenvalue problem of equation (3.4). If equation (3.4) is to satisfy certain boundary conditions, there must be some functional relationship between the parameters, c, α, and R. This functional relationship can be demonstrated as follows.

Since equation (3.4) is of fourth order, there exists a fundamental set of four solutions of equation (3.4) that are analytic functions of the variable η and the parameters c, α, and R.

$$\varphi = C_1\varphi_1 + C_2\varphi_2 + C_3\varphi_3 + C_4\varphi_4 \tag{3.5}$$

where C_1, C_2, C_3, C_4 are constants.

Let us now specify boundary conditions on φ.

$$\varphi(\eta_1) = 0 \qquad \varphi(\eta_2) = 0$$
$$\varphi'(\eta_1) = 0 \qquad \varphi'(\eta_2) = 0$$

Substituting the boundary conditions into equation (3.5) we obtain

$$C_1\varphi_1(\eta_1) + C_2\varphi_2(\eta_1) + C_3\varphi_3(\eta_1) + C_4\varphi_4(\eta_1) = 0$$

$$C_1\varphi_1(\eta_2) + C_2\varphi_2(\eta_2) + C_3\varphi_3(\eta_2) + C_4\varphi_4(\eta_2) = 0$$

$$C_1\varphi_1'(\eta_1) + C_2\varphi_2'(\eta_1) + C_3\varphi_3'(\eta_1) + C_4\varphi_4'(\eta_1) = 0$$

$$C_1\varphi_1'(\eta_2) + C_2\varphi_2'(\eta_2) + C_3\varphi_3'(\eta_2) + C_4\varphi_4'(\eta_2) = 0$$

If the constants C_1, C_2, C_3 and C_4 are not all identically zero, then the following relations must exist.

$$\begin{vmatrix} \varphi_1(\eta_1) & \varphi_2(\eta_1) & \varphi_3(\eta_1) & \varphi_4(\eta_1) \\ \varphi_1(\eta_2) & \varphi_2(\eta_2) & \varphi_3(\eta_2) & \varphi_4(\eta_2) \\ \varphi_1'(\eta_1) & \varphi_2'(\eta_1) & \varphi_3'(\eta_1) & \varphi_4'(\eta_1) \\ \varphi_1'(\eta_2) & \varphi_2'(\eta_2) & \varphi_3'(\eta_2) & \varphi_4'(\eta_2) \end{vmatrix} = F(c,\alpha,R) = 0$$

We thusly obtain a necessary relationship between the parameters c, α , and R so that the boundary conditions will be satisfied.

The remaining problem is to solve the eigenvalue problem of equation (3.4) with boundary conditions as follows:-

when $\eta = -\infty$ $\qquad$ $\eta = \infty$

$\varphi = 0$ $\qquad$ $\varphi = 0$

Equation (3.4) is a fourth order equation and therefore four boundary conditions are necessary to completely determine the soltuion. Only two boundary conditions have been specified above. A discussion of the other two boundary conditions will be included in the solution of equations (2.5) and (3.4). The eigenvalue problem for the above boundary conditions will also be formulated later in the discussion of the solution of the Orr-Sommerfeld equation.

V. SOLUTION OF THE BOUNDARY LAYER AND STABILITY EQUATIONS

In order to accomplish as much as possible and to arrive most directly and painlessly at solutions, it was decided to utilize the differential analyser[2] to carry out as much of the calculations as it could. The proper technique of utilizing the analyser for some of these computations had to be developed before any results were forthcoming.

Solution of the Boundary Layer Equations

Before the stability or Orr-Sommerfeld equation can be solved, it is first necessary to solve for the flow configuration. Therefore, the first equation to be considered is the boundary-layer equation (equation 2.5) which is restated herewith.

$$f f'' + 2f''' = 0$$

where $f = f(\eta)$

The boundary conditions for this equation are

$\eta = -\infty$	$\eta = 0$	$\eta = \infty$
$f' = \frac{U_2}{U_1}$	$f = 0$	$f' = 1$

Since equation (2.5) is of the third order, the three boundary conditions stated above are sufficient to mathematically determine the solution. It is unfortunate, however, that the differential analyser cannot utilize boundary conditions at infinity. For analyser

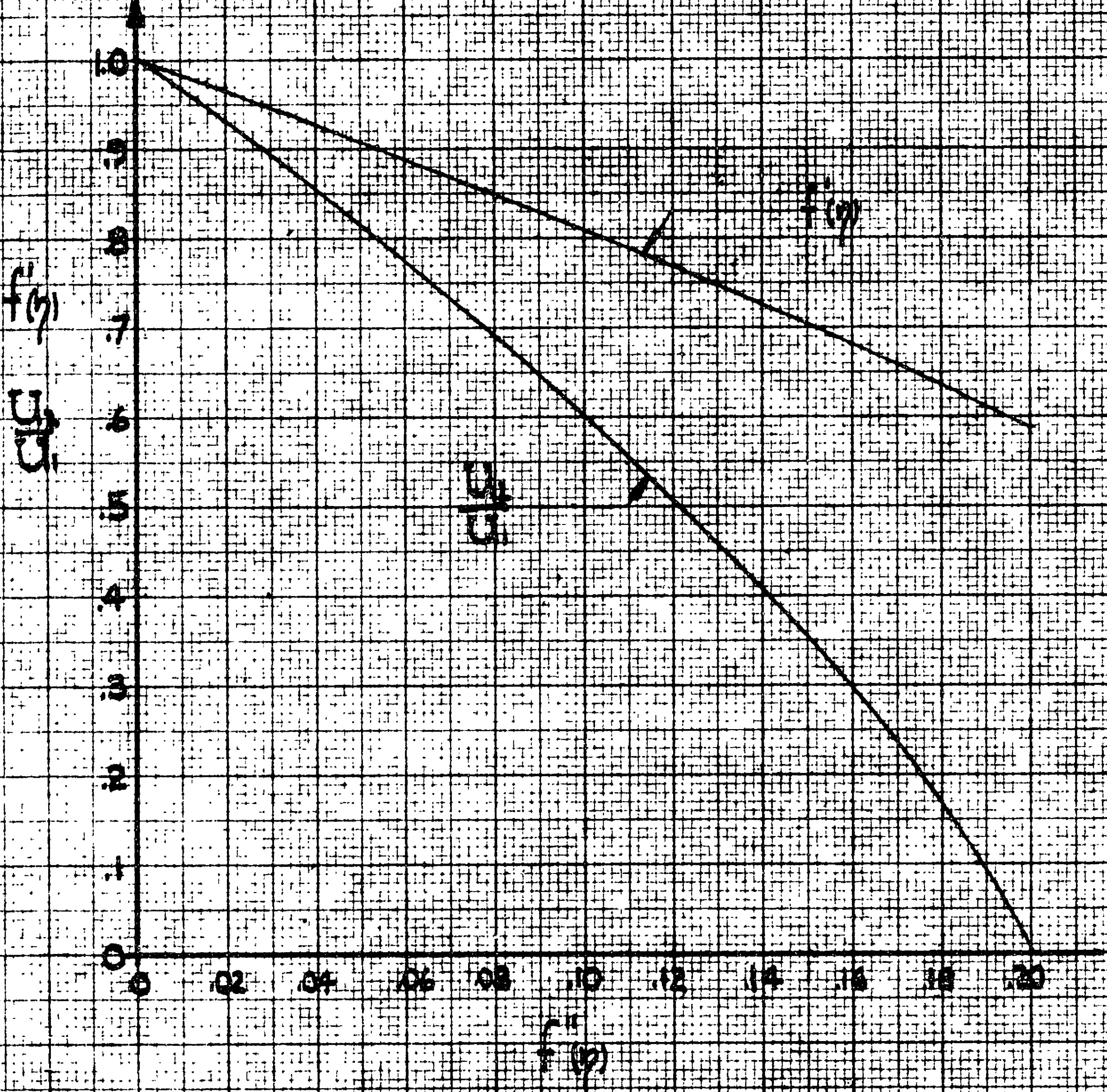

FIGURE 4 - BOUNDARY LAYER CHARACTERISTIC CURVES

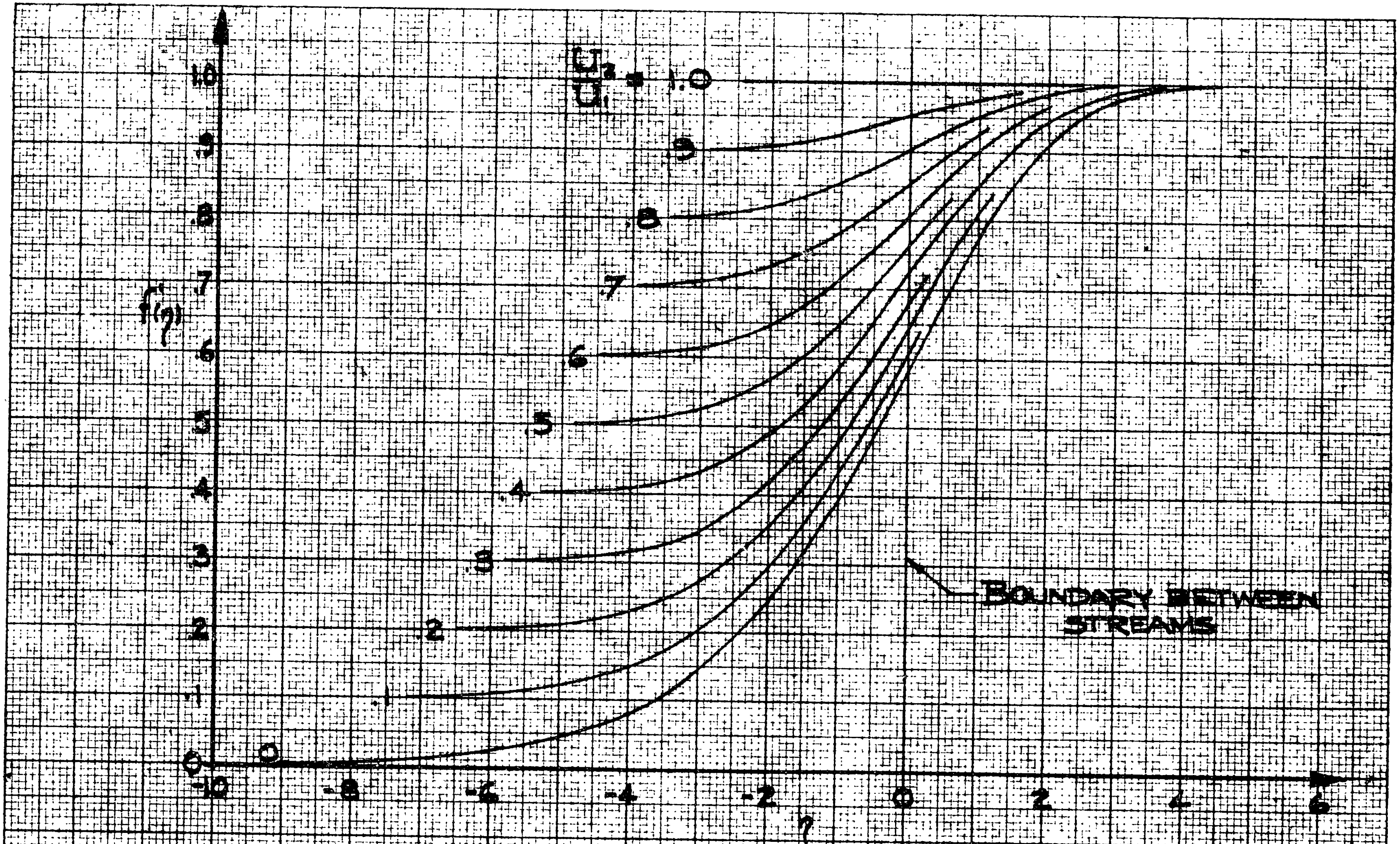

FIGURE 5 - VELOCITY PROFILE OF FREE BOUNDARY LAYER BETWEEN PARALLEL STREAMS

solution, it is necessary to specify all of the boundary conditions at a finite point. The procedure then to be followed is to guess at values of f' and f" where $\eta = 0$ until the boundary conditions at $\eta = \pm\infty$ are satisfied.

<u>Plan of trial and error solution of boundary layer equation on differential analyser</u>:-

1. At $\eta = 0$, f = 0, find values of f' for fixed values of f" such that the boundary condition f' = 1 at $\eta = \infty$ is satisfied. These points will form a curve in the f',f" plane (see Figure 4).

2. Find the values of $\frac{U_2}{U_1}$ satisfied when $\eta = -\infty$ by points along the f',f" curve satisfying the condition f' = 1 at $\eta = \infty$. Plot these values of $\frac{U_2}{U_1}$ versus f" (see Figure 4). Corresponding values of f" can now be obtained for any desired ratio of $\frac{U_2}{U_1}$.

3. From Figure 4, corresponding values of f' can be obtained for the previously obtained values of f".

4. Since the boundary conditions of f, f', and f" are now all known at $\eta = 0$ for a given ratio of $\frac{U_2}{U_1}$, the solutions can be run off on the differential analyser (see Figure 5 and tables 1 to 10).

So far, the solution of the boundary layer equation has been considered in the real domain. There are, however, complex solutions of the boundary layer equation and these are defined by analytic continuation of the real solution into the complex plane. It is advantageous at this point to reintegrate the original equation over the complex ζ plane.

Consider therefore

$$f f'' + 2f''' = 0$$

where $f = f(\zeta)$

and $\zeta = \eta + i\xi$

f, f', f'' and f''' can be broken into real and imaginary parts as follows:

$$f = f_r + if_i$$

$$f' = f'_r + if'_i$$

$$f'' = f''_r + if''_i$$

$$f''' = f'''_r + if'''_i$$

The boundary layer equation can therefore be re-written $f_r f''_r - f_i f''_i + if_r f''_i + if_i f''_r + 2f'''_r + i2f'''_i = 0.$

The foregoing equation can be separated into two real equations by considering real and imaginary parts separately.

Therefore;

$$f_r f''_r - f_i f''_i + 2f'''_r = 0$$

$$f_r f''_i + f_i f''_r + 2f'''_i = 0$$

It must be remembered that when integrating in the η direction

$$f''_r = \int f'''_r d\eta \qquad f''_i = \int f'''_i d\eta$$

$$f'_r = \int f''_r d\eta \qquad f'_i = \int f''_i d\eta$$

$$f_r = \int f'_r d\eta \qquad f_i = \int f'_i d\eta$$

and when integrating in the $i\xi$ direction,

$$f''_r = -\int f'''_i \, d\xi \qquad f''_i = \int f'''_r \, d\xi$$

$$f'_r = -\int f''_i \, d\xi \qquad f'_i = \int f''_r \, d\xi$$

$$f_r = -\int f'_i \, d\xi \qquad f_i = \int f'_r \, d\xi$$

The equation can be integrated in the foregoing manner over the complex plane from a point on the real axis where the real parts of f, f', f" are known and the imaginary parts of f, f', f" are zero.

Solution of the Orr-Sommerfeld Equation

Let us now consider equation (3.4). Dividing through by (f' - c) we obtain

$$\varphi'' - \left(\alpha^2 + \frac{f'''}{f'-c}\right)\varphi = -\frac{i}{\alpha R(f'-c)}\left(\varphi^{iv} - \alpha^2\varphi'' + \alpha^4\varphi\right) \tag{4.1}$$

If we expand φ in terms of a power series in αR as follows

$$\varphi = \sum_{K=0}^{n} \frac{\varphi_K}{(\alpha R)^K}$$

we obtain the following set of equations:

$$\varphi_K'' - \left(\alpha^2 + \frac{f'''}{f'-c}\right)\varphi_K = -\frac{i}{f'-c}\left(\varphi_{K-1}^{iv} - 2\alpha^2\varphi_{K-1}'' + \alpha^4\varphi_{K-1}\right) \tag{4.2}$$

Let us first consider the solution of (4.1) when $\alpha R \rightarrow \infty$ (the "inviscid" solution). Let us examine in particular the nature of the inviscid solution in the region of $|\eta| \rightarrow \infty$. In this region $f''' \rightarrow 0$ and $f' \rightarrow$ constant and if the limiting value of f' does not equal c then

$$\frac{f'''}{f' - c} \rightarrow 0 \quad \text{when} \quad |\eta| \rightarrow \infty$$

We can now consider the equation

$$\varphi'' - \alpha^2 \varphi = 0 \qquad (4.3)$$

The solutions of equation (4.3) can be stated as follows:

$$\varphi = c_1 e^{\alpha\eta} + c_2 e^{-\alpha\eta}$$

It can therefore be seen that the solution of equation (4.1) where $\alpha R \rightarrow \infty$ and $|\eta| \rightarrow \infty$ is exponential in nature. If we ask that φ also be asymptotic to zero when $|\eta| \rightarrow \infty$ then the nature of the solution must be

$$\varphi = c_1 e^{\alpha\eta} \quad \text{when} \quad \eta \rightarrow -\infty$$

$$\varphi = c_2 e^{-\alpha\eta} \quad \eta \rightarrow +\infty$$

Let us now reexamine equation (4.1) with $\alpha R \rightarrow \infty$ but with no restriction on η. The equation is of the form

$$\varphi'' - \left(\alpha_s^2 + \frac{f'''}{f' - c_s}\right)\varphi = 0 \qquad (4.4)$$

where α_s and c_s are eigenvalues of α and c corresponding to infinite R. The value of c_s has been shown by Rayleigh and Tollmien to be equal to the flow velocity at a point of inflection in the velocity profile. There remains but to find the value of α_s such that the boundary conditions are satisfied. These boundary conditions are $\varphi \rightarrow 0$ and $\varphi' \rightarrow \pm\alpha_s\varphi$ when $\eta \rightarrow \mp\infty$. α_s can then be found by a trial and error process on the

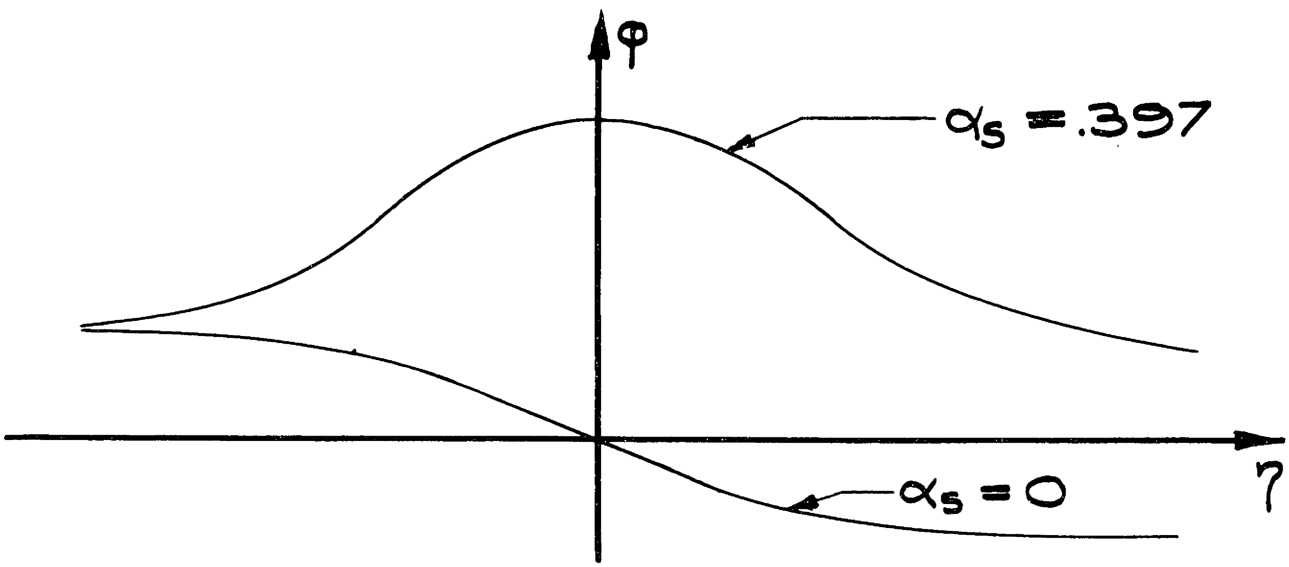

FIG. 6- THE INVISCID SOLUTION FOR φ

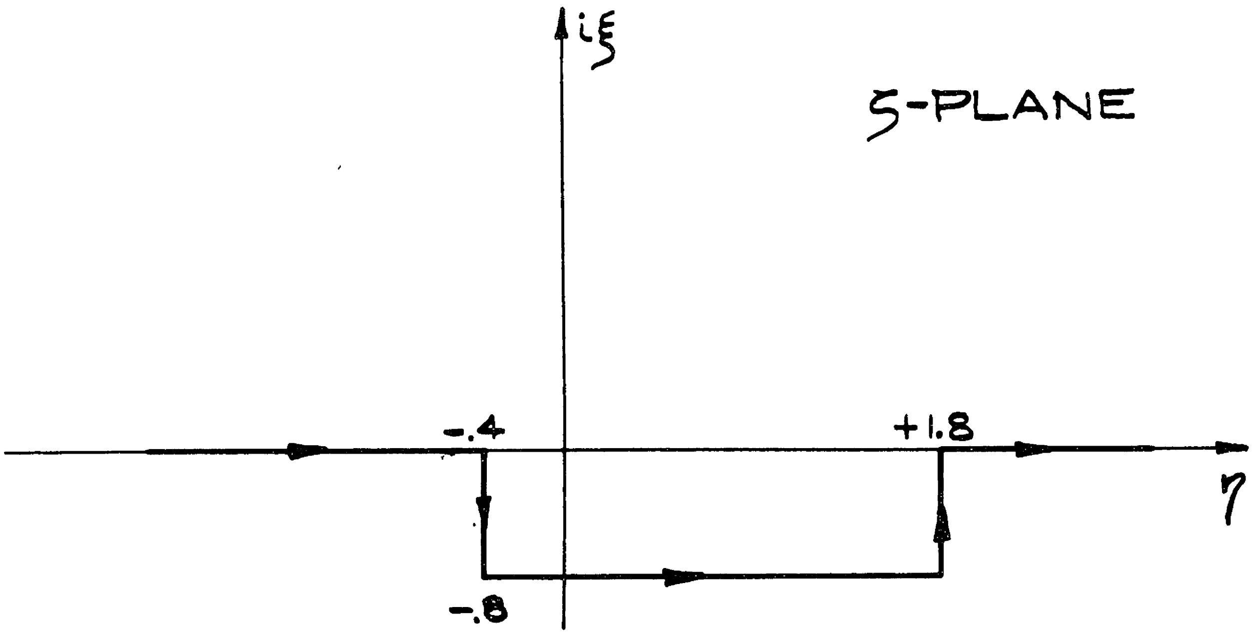

FIG. 7-PATH OF INTEGRATION THROUGH COMPLEX PLANE

differential analyser by starting at a large negative value of η and testing whether or not the boundary conditions are satisfied at a large positive value of η. One value of α_s comes out equal to zero in which case $\varphi \propto (f' - c_s)$. The other value of α_s is .397. For a qualitative picture of φ see figure 6.

It should be noted that the integration along the real η axis is permissible only if there are no singularities along the path of integration. In the case of the inviscid solution

$$f''' = 0$$

when $f' = c_s$

and $\dfrac{f'''}{f'-c_s}$ is bounded.

This, however, is not generally true in the case of finite R: there is usually a singularity at the point where $f' = c$. In order to avoid the singularity in the integration of the Orr-Sommerfeld equation, the integration is carried out along a path through the complex plane (see Figure 7). Before this can be done, it is necessary to evaluate $\dfrac{f'''}{f' - c}$ along this curve. This can be accomplished by integrating the boundary layer equation along the same path.

Let us now denote $\dfrac{f'''}{f' - c}$ by A and write out equation (4.2) for k = 0.

$$\varphi_{or}'' + i\varphi_{oi}'' - (\alpha^2 + A_r)\varphi_{or} + A_i\varphi_{oi} - iA_i\varphi_{or} - iA_r\varphi_{oi} = 0$$

where $\varphi_0'' = \varphi_{or}'' + i\,\varphi_{oi}''$

$$\varphi_0 = \varphi_{or} + i\,\varphi_{oi}$$

$$A = A_r + i\,A_i$$

separating real and imaginary quantities, we obtain

$$\begin{aligned} &\varphi_{or}'' - (\alpha^2 + A_r)\varphi_{or} + A_i\,\varphi_{oi} = 0 \\ &\varphi_{oi}'' - A_i\varphi_{or} - A_r\varphi_{oi} = 0 \end{aligned} \qquad (4.5)$$

The foregoing are the equations for the first approximation. The equations for the second approximation (k=1) are

$$\begin{aligned} &\varphi_{1r}'' - (\alpha^2 + A_r)\varphi_{1r} + A_i\,\varphi_{1i} = B_{or} \\ &\varphi_{1i}'' - A_i\,\varphi_{1r} - A_r\varphi_{1i} = B_{oi} \end{aligned} \qquad (4.6)$$

where

$$B_o = -\frac{i}{f'-c}(\varphi_0^{iv} - 2\alpha^2\varphi_0'' + \alpha^4\varphi_0) = B_{or} + i\,B_{oi}$$

By evaluating as many terms in the power series expansion of φ as are convergent, φ can be obtained with any degree of accuracy up to the accuracy of the methods of computation and the degree of convergence of the expansion.

It should here be noted that in regions where A is negligible compared to α^2, φ_0 will be exponential in character and B will equal zero. Using these facts, it is possible to solve the eigenfunction problem partly with the differential analyser.

The plan of solution is therefore as follows:

1. With a fixed value of c, integrate equations (4.5) along the proposed path through the complex plane starting at a large negative value of η where A is small compared to α^2, for various values of α. At the starting point the boundary conditions should be

$$\varphi_0 = \text{any constant}$$

$$\varphi_0' = \alpha\varphi_0$$

2. From the solution of equation (4.5), B can now be evaluated. Now integrate equation (4.6) through the complex plane along the same path as step 1 from the same starting point and with the same starting conditions.

Where η has a large positive value, the solution must satisfy the following conditions.

$$\varphi' = -\alpha\varphi$$

Having evaluated the first two terms of an expansion of φ, we can now write out the following approximate relation:

$$\varphi_0' + \frac{\varphi_1'}{\alpha R} = -\left(\alpha\varphi_0 + \frac{\varphi_1}{R}\right)$$

or solving for R

$$R = -\frac{1}{\alpha}\,\frac{\varphi_1' + \alpha\varphi_1}{\varphi_0' + \alpha\varphi_0}$$

R will generally be complex. If we limit our investigation to real c and real α, we will obtain the eigenvalue of α along the neutral curve when R comes out real.

In this manner we can obtain the neutral curve in a point by point manner to as great an accuracy as the differential analyser and the asymptotic expansion will permit.

Discussion of Solution of Orr-Sommerfeld Equation

It is to be noted here that the asymptotic expansion presented represents only two of the four independent solutions of the Orr-Sommerfeld equation. The reason for this is that the parameter of the expansion, $\frac{1}{\alpha R}$, occurs with the highest derivative of the Orr-Sommerfeld equation and therefore, in evaluating successive approximations to the solution, the order of the equation solved is that of the next highest term not multiplied by $\frac{1}{\alpha R}$. Thus, since the order of the next highest term is two, the number of independent solutions of the equation is two, when expanded in the foregoing manner.

In the interest of completeness, let us now discuss the nature of the other two independent solutions. If we set $\varphi = \exp\left[\int g\, d\eta\right]$ where $g = g(\eta)$ and transform the Orr-Sommerfeld equations accordingly, we obtain

$$(f'-c)\left[(g^2+g')-\alpha^2\right]-f'' = -\frac{i}{\alpha R}\left[g^4-6g^2g'+3g'^2+4gg''+g''' -2\alpha^2(g^2+g')+\alpha^4\right] \qquad (4.7)$$

The solution of equation (4.7) can be expanded in terms of a power series as follows.

$$g = \sum_{i=0}^{n} (\alpha R)^{\frac{1-i}{2}} g_i(\eta) \qquad (4.8)$$

Substituting equation (4.8) into equation (4.7), the following set of equations are obtained.

$$(f'-c)g_0^2 = -i\, g_0^4$$
$$(f'-c)(g_0' + 2g_0 g_1) = -i(4g_0^3 g_1 + 6g_0^2 g_0')$$
$$\text{---} \quad \text{---} \quad \text{---} \quad \text{---} \qquad \text{---} \quad \text{---} \quad \text{---} \quad \text{---}$$

from which we can obtain the successive approximations without integration. Thusly,

$$g_0 = \pm\sqrt{i(f'-c)}$$
$$g_1 = -\frac{5}{2}\,\frac{g_0'}{g_0} \tag{4.9}$$
$$\text{---} \quad \text{---} \quad \text{---} \qquad \text{---} \quad \text{---} \quad \text{-}$$

However, if (4.9) is substituted into (4.8), the two remaining asymptotic solutions are obtained.

$$\varphi_3(\eta) = (f'-c)^{-\frac{5}{4}} \exp\left[-\int_{\eta_0}^{\eta}\sqrt{i\alpha R(f'-c)}\,d\eta\right]$$
$$\varphi_4(\eta) = (f'-c)^{-\frac{5}{4}} \exp\left[+\int_{\eta_0}^{\eta}\sqrt{i\alpha R(f'-c)}\,d\eta\right]$$

where η_0 is that value of η where f' = c.

At this point, it should be noted that when $\eta \longrightarrow +\infty$, $\varphi_4 \longrightarrow \infty$ and when $\eta \longrightarrow -\infty$, $\varphi_3 \longrightarrow \infty$. If the boundary conditions

$$\varphi \longrightarrow 0 \qquad \text{when} \qquad \eta \longrightarrow \pm\infty$$

are to be satisfied, then C_3 and C_4 of equation (3.5) must both be equal to zero.

We can therefore leave out of our consideration the solutions φ_3 and φ_4 of the Orr-Sommerfeld equation for our particular boundary conditions.

The eigenvalue problem of equation (3.4) for the boundary conditions under consideration can, therefore, be formulated as follows:-

Since

$$\varphi = C_1\varphi_1 + C_2\varphi_2$$

and

$$\varphi'(-\infty) - \alpha\varphi(-\infty) = 0$$

$$\varphi'(+\infty) + \alpha\varphi(+\infty) = 0$$

therefore

$$\begin{vmatrix} \varphi_1'(-\infty) - \alpha\varphi_1(-\infty) & \varphi_2'(-\infty) - \alpha\varphi_2(-\infty) \\ \varphi_1'(+\infty) + \alpha\varphi_1(+\infty) & \varphi_2'(+\infty) + \alpha\varphi_2(+\infty) \end{vmatrix} = F(c, \alpha, R) = 0$$

VI. RESULTS, CONCLUSIONS AND SUGGESTIONS FOR FURTHER WORK

The first of the results obtained are the solutions of the boundary layer equation (2.5) for various ratios of U_2/U_1. These solutions are presented in Tables 1-10 and are plotted in Fig. 5. Secondly, the eigenvalues of the stability equation for the case of inviscid flow are obtained and these are as follows:

$R \rightarrow \infty$

$C_s = .5873$

$\alpha_s = .397$

The boundary layer and disturbance equations were then successfully integrated over the complex plane and zero order solutions of the Orr-Sommerfeld equation were obtained. However, the differential analyser was not sufficiently flexible or accurate to permit solution of the first order approximation to the disturbance functions. It, therefore, was impossible to identify a corresponding Reynolds number with sets of eigenvalues of wave number and velocity of wave propagation.

The dependence of R on the other parameters can be investigated by considering the nature of the disturbance function φ. As seen previously, the highly oscillating solutions, φ_3 and φ_4, were demonstrated to be physically impossible for boundary conditions at infinity. The effect of viscosity, though, is most pronounced on these solutions. The solutions of φ_1 and φ_2, however, have only a lower order dependence on viscosity since the effect of Reynolds number first enters into the second term of the power series expansion of the disturbance function. It is, therefore, reasonable to arrive at the conclusion that α does not depend greatly on R except for low values of R and that the solution is highly unstable. The existence of some minimum Reynolds number for instability has already been demonstrated by Synge.

It is concluded that the process developed in this investigation can successfully be utilized to explore the eigenvalue problem of the Orr-Sommerfeld equation provided that sufficient accuracy is maintained throughout the computations. The necessary computations, however, are so extensive that anything but high-speed digital machine computing can be considered futile.

Until the present time, only the surface of this interesting and important field has been touched. The principal analytical researches have been concerned with a very few cases of the classical types of flow and most of the work has been concerned with the incompressible fluid flow. It is, therefore, important that future investigators work along lines that are broader in nature and scope.

It is very important, at this point, to improve experimental techniques so that the highly philosophical results achieved in analytical research be fortified by actual fact. The analyst must, of necessity, make many simplifying assumptions in order to render even the simplest problem solvable that many times he finally solves a problem that is quite different from the physical situation that he started with. Not only must the consistancy of his reasoning be checked, therefore, but also the nature and validity of his simplifying assumptions.

In the present problem, it is suggested that some way be devised to bring two parallel streams together with no initial boundary layer between them. A promising method for doing this has been suggested by Professor E. S. Taylor, who proposes that the two streams flow along opposite sides of a thin wedge and that the boundary layer formed against the wedge be removed by suction into the wedge before the streams contact each other. The next step is to introduce a controlled disturbance into the boundary layer either by an oscillating ribbon or by intense sound waves. The resulting disturbance characteristics in the boundary layer can then be studied by means of a hot wire anemometer; it is suggested that the working fluid be air at low Mach numbers.

Another problem of interest is the theoretical and experimental investigation of the laminar boundary layer between parallel streams consisting of different fluids. Other problems that suggest themselves are the stability considerations of jets and wakes.

The author has arranged to have extensive computations performed by a high-speed, digital computing machine on his present problem. After these are completed, he hopes to extend the analysis to the case of the compressible boundary layer for the same type of flow.

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VIII. APPENDIX

Solutions of the Boundary Layer Equation

Presented herein in Tables 1-10 are solutions of the boundary layer equation (2.5) for the boundary conditions as enumerated in section IV-B.

The equation is hereby restated:

$$ff'' + 2f''' = 0$$

where

$$f = f(\eta)$$

$$\eta = \frac{y}{\sqrt{\frac{\nu x}{U_1}}}$$

The boundary conditions are

$$\eta \to -\infty \qquad\qquad \eta \to \infty \qquad\qquad \eta = 0$$

$$f' \to \frac{U_2}{U_1} \qquad\qquad f' \to 1 \qquad\qquad f = 0$$

TABLE I.

$\frac{U_2}{U_1} = 0.00$

η	f	f'	f"
-10.0	-1.224	.001	.001
- 9.6	-1.224	.001	.002
- 9.2	-1.223	.002	.002
- 8.8	-1.222	.003	.003
- 8.4	-1.221	.004	.003
- 8.0	-1.219	.005	.004
- 7.6	-1.216	.007	.006
- 7.2	-1.213	.010	.007
- 6.8	-1.208	.013	.009
- 6.4	-1.202	.018	.012
- 6.0	-1.194	.023	.015
- 5.6	-1.184	.030	.019
- 5.2	-1.170	.038	.024
- 4.8	-1.153	.049	.030
- 4.4	-1.131	.062	.038
- 4.0	-1.103	.079	.047
- 3.6	-1.062	.100	.059
- 3.2	-1.022	.127	.072
- 2.8	-0.965	.159	.088
- 2.4	-0.894	.198	.106
- 2.0	-0.806	.244	.126
- 1.6	-0.697	.299	.147
- 1.2	-0.565	.362	.167
- 0.8	-0.407	.432	.184
- 0.4	-0.219	.509	.196
- 0.0	-0.000	.587	.200
0.4	0.251	.667	.195
0.8	0.533	.742	.181
1.2	0.844	.810	.158
1.8	1.180	.867	.129
2.0	1.536	.913	.098
2.4	1.908	.946	.070
2.8	2.292	.969	.046
3.2	2.683	.984	.028
3.6	3.078	.992	.016
4.0	3.476	.997	.009
4.4	3.875	.999	.005
4.8	4.275	.999	.002

TABLE II.

$\frac{U_2}{U_1} = .100$

η	f	f'	f"
-7.0	-1.647	.001	
-6.5	-1.596	.104	
-6.0	-1.543	.108	
-5.5	-1.488	.113	
-5.0	-1.429	.121	
-4.5	-1.366	.133	
-4.0	-1.296	.149	
-3.5	-1.216	.171	
-3.0	-1.123	.201	
-2.5	-1.013	.241	
-2.0	-0.900	.293	
-1.5	-0.718	.357	
-1.0	-0.521	.433	
-0.5	-0.283	.520	
0.0	0.000	.613	.189
0.5	0.330	.706	
1.0	0.706	.793	
1.5	1.121	.865	
2.0	1.568	.919	
2.5	2.037	.956	
3.0	2.522	.979	
3.5	3.014	.991	
4.0	3.512	.997	
4.5	4.011	.999	

TABLE III.

$\frac{U_2}{U_1} = .200$

η	f	f'	
-7.0	-2.154	.201	
-6.5	-2.054	.202	
-6.0	-1.954	.203	
-5.5	-1.852	.205	
-5.0	-1.748	.210	
-4.5	-1.641	.218	
-4.0	-1.530	.230	
-3.5	-1.410	.248	
-3.0	-1.281	.272	
-2.5	-1.136	.306	
-2.0	-0.972	.352	
-1.5	-0.782	.409	
-1.0	-0.561	.479	
-0.5	-0.301	.560	f"
0.0	0.000	.646	.175
0.5	0.345	.733	
1.0	0.731	.812	
1.5	1.155	.878	
2.0	1.607	.928	
2.5	2.079	.961	
3.0	2.565	.981	
3.5	3.059	.991	
4.0	3.556	.996	
4.5	4.055	.999	

TABLE IV.

$\frac{U_2}{U_1} = .300$

η	f	f'	
-7.0	-2.694	.301	
-6.5	-2.545	.301	
-6.0	-2.396	.302	
-5.5	-2.246	.303	
-5.0	-2.095	.305	
-4.5	-1.942	.310	
-4.0	-1.786	.318	
-3.5	-1.625	.330	
-3.0	-1.455	.350	
-2.5	-1.273	.378	
-2.0	-1.075	.417	
-1.5	-0.854	.468	
-1.0	-0.605	.531	
-0.5	-0.321	.603	f"
0.0	0.000	.682	.160
0.5	0.361	.761	
1.0	0.760	.833	
1.5	1.192	.893	
2.0	1.650	.937	
2.5	2.127	.967	
3.0	2.615	.984	
3.5	3.110	.993	
4.0	3.608	.998	
4.5	4.107	.999	

TABLE V.

$\frac{U_2}{U_1} = .400$

η	f	f'	f"
-6.0	-2.863	.401	
-5.5	-2.665	.401	
-5.0	-2.465	.402	
-4.5	-2.265	-405	
-4.0	-2.062	.410	
-3.5	-1.856	.419	
-3.0	-1.643	.434	
-2.5	-1.422	.456	
-2.0	-1.186	.488	
-1.5	-0.931	.532	
-1.0	-0.652	.587	
-0.5	-0.343	.651	
0.0	0.000	.721	.142
0.5	0.378	.791	
1.0	0.790	.855	
1.5	1.231	.907	
2.0	1.695	.946	
2.5	2.175	.972	
3.0	2.664	.987	
3.5	3.159	.995	
4.0	3.657	.999	

TABLE VI.

$\frac{U_2}{U_1} = .500$

η	f	f'	f"
-6.0	-3.358	.501	
-5.5	-3.109	.501	
-5.0	-2.860	.502	
-4.5	-2.610	.504	
-4.0	-2.360	.506	
-3.5	-2.106	.513	
-3.0	-1.848	.523	
-2.5	-1.583	.540	
-2.0	-1.307	.566	
-1.5	-1.016	.602	
-1.0	-0.704	.648	
-0.5	-0.366	.703	
0.0	0.000	.763	.123
0.5	0.397	.824	
1.0	0.823	.878	
1.5	1.274	.923	
2.0	1.744	.956	
2.5	2.228	.977	
3.0	2.720	.989	
3.5	3.216	.996	
4.0	4.214	.999	

TABLE VII.

$\frac{U_2}{U_1} = .600$

η	f	f'	f"
-5.0	-3.269	.601	
-4.5	-2.970	.601	
-4.0	-2.670	.602	
-3.5	-2.368	.607	
-3.0	-2.063	.614	
-2.5	-1.754	.626	
-2.0	-1.436	.646	
-1.5	-1.106	.675	
-1.0	-0.759	.713	
-0.5	-0.392	.758	
0.0	0.000	.808	.101
0.5	0.417	.858	
1.0	0.857	.903	
1.5	1.319	.940	
2.0	1.796	.966	
2.5	2.283	.983	
3.0	2.777	.993	
3.5	3.275	.997	
4.0	3.775	.999	

TABLE VIII.

$\frac{U_2}{U_1} = .700$

η	f	f'	f"
-5.0	-3.696	.701	
-4.5	-3.346	.702	
-4.0	-2.995	.702	
-3.5	-2.644	.704	
-3.0	-2.291	.709	
-2.5	-1.935	.718	
-2.0	-1.572	.732	
-1.5	-1.201	.753	
-1.0	-0.817	.782	
-0.5	-0.418	.817	
0.0	0.000	.855	.078
0.5	0.437	.894	
1.0	0.893	.929	
1.5	1.364	.956	
2.0	1.846	.976	
2.5	2.337	.987	
3.0	2.833	.995	
3.5	3.331	.998	
4.0	3.830	.999	

TABLE IX.

$\frac{U_2}{U_1} = .800$

η	f	f'	f"
-5.0	-4.122	.801	
-4.5	-3.723	.801	
-4.0	-3.323	.802	
-3.5	-2.922	.803	
-3.0	-2.521	.805	
-2.5	-2.117	.810	
-2.0	-1.710	.819	
-1.5	-1.297	.833	
-1.0	-0.876	.852	
-0.5	-0.445	.876	
0.0	0.000	.902	.054
0.5	0.463	.928	
1.0	0.933	.952	
1.5	1.415	.971	
2.0	1.904	.984	
2.5	2.399	.992	
3.0	2.897	.997	
3.5	3.397	.999	

TABLE X.

$\frac{U_2}{U_1} = .900$

η	f	f'	f"
-5.0	-4.556	.901	
-4.5	-4.106	.901	
-4.0	-3.657	.901	
-3.5	-3.207	.902	
-3.0	-2.757	.902	
-2.5	-2.306	.904	
-2.0	-1.853	.908	
-1.5	-1.398	.915	
-1.0	-0.938	.925	
-0.5	-0.425	.939	
0.0	0.000	.951	.028
0.5	0.479	.964	
1.0	0.965	.975	
1.5	1.460	.985	
2.0	1.951	.992	
2.5	2.449	.996	
3.0	2.949	.998	
3.5	3.449	.999	

Method of Solution of Orr-Sommerfeld Equation with Digital-Type Computing Machine

In the course of this investigation, it was found impracticable to solve the complete eigenvalue problem on the differential analyser. However, the digital-type computer was found to be satisfactory for this purpose and the problem is hereby rearranged for solution on that type of machine.

Let us restate the boundary layer equation (A) and the first two equations in the power series expansion of the Orr-Sommerfeld equation (B) (C) in the form suitable for complex integration.

$$\left.\begin{aligned} &f_r f_r'' - f_i f_i'' + 2 f_r''' = 0 \\ &f_r f_i'' + f_i f_r'' + 2 f_i''' = 0 \end{aligned}\right\} \quad \text{(A)}$$

$$\left.\begin{aligned} &(f_r' - c)(\varphi_{0r}'' - \alpha^2 \varphi_{0r}) - f_i'(\varphi_{0i}'' - \alpha^2 \varphi_{0i}) - f_r''' \varphi_{0r} + f_i''' \varphi_{0i} = 0 \\ &(f_r' - c)(\varphi_{0i}'' - \alpha^2 \varphi_{0i}) + f_i'(\varphi_{0r}'' - \alpha^2 \varphi_{0r}) - f_r''' \varphi_{0i} - f_i''' \varphi_{0r} = 0 \end{aligned}\right\} \quad \text{(B)}$$

$$\left.\begin{aligned} &(f_r' - c)(\varphi_{1r}'' - \alpha^2 \varphi_{1r}) - f_i'(\varphi_{1i}'' - \alpha^2 \varphi_{1i}) - f_r''' \varphi_{1r} + f_i''' \varphi_{1i} = \\ &\qquad (\varphi_{0i}^{iv} - 2\alpha^2 \varphi_{0i}'' + \alpha^4 \varphi_{0i}) \\ &(f_r' - c)(\varphi_{1i}'' - \alpha^2 \varphi_{1i}) + f_i'(\varphi_{1r}'' - \alpha^2 \varphi_{1r}) - f_r''' \varphi_{1i} - f_i''' \varphi_{1r} = \\ &\qquad -(\varphi_{0r}^{iv} - 2\alpha^2 \varphi_{0r}'' + \alpha^4 \varphi_{0r}) \end{aligned}\right\} \quad \text{(C)}$$

The digital computer can most advantageously be utilized in integrating these equations if the path of integration chosen is a semi-circle through the complex ζ-plane.

The general rules of integration to be followed for integration along a circular path through the complex z-plane are as follows:-

$$F^{(n)} = \int F^{(n+1)} dz = \int F^{(n+1)} d(re^{i\theta}) = iR\int F^{(n+1)} e^{i\theta} d\theta$$

where $F^{(n)} = \dfrac{d^n F}{dz^n} = F^{(n)}_r + i\,F^{(n)}_i$

$$F^{(n+1)} = F^{(n+1)}_r + i\,F^{(n+1)}_i$$

R = radius of curvature of circular path of integration.

$$\therefore\ F^{(n)}_r = -R\left[\int F^{(n+1)}_r \sin\theta\, d\theta + \int F^{(n+1)}_i \cos\theta\, d\theta\right]$$

$$F^{(n)}_i = R\left[\int F^{(n+1)}_r \cos\theta\, d\theta - \int F^{(n+1)}_i \sin\theta\, d\theta\right]$$

The rules for differentiation along the circular path can also be stated as follows:-

$$F^{(n+1)}_r = -\frac{1}{R}\left(\frac{\partial F^{(n)}_r}{\partial\theta}\sin\theta - \frac{\partial F^{(n)}_i}{\partial\theta}\cos\theta\right)$$

$$F^{(n+1)}_i = \frac{1}{R}\left(\frac{\partial F^{(n)}_r}{\partial\theta}\cos\theta - \frac{\partial F^{(n)}_i}{\partial\theta}\sin\theta\right)$$

The procedure than to be followed, in solving the eigenvalue problem on the digital computer would be as follows.

(1) Solve equations (A) along the path of integration and store the solution in the machine.

(2) Using the solution obtained from step (1), solve

equations (B) and (C) simultaneously for various values of c and α that give real values of R. (The procedure is similar to that described in Part V.)

BIOGRAPHICAL NOTE

Martin Lessen was born on September 6, 1920, in New York City. He attended Townsend Harris High School in New York and in February, 1940, obtained the degree of B.M.E. from the College of the City of New York. After a year in the aircraft industry, he joined the staff of Diesel Design Section of the New York Navy Yard and, in time, was put in charge of design calculations. During his employment at the Navy Yard, he did graduate work in the evening at New York University from where he obtained the degree of M.M.E. in June, 1942. He later did further evening graduate study at Columbia University. In November, 1945, he left the Navy Yard to pursue his present course of study at the Institute.