Shock formation in solutions to the 2D compressible Euler equations in the presence of non-zero vorticity

SHOCK FORMATION IN SOLUTIONS TO THE $2D$ COMPRESSIBLE EULER EQUATIONS IN THE PRESENCE OF NON-ZERO VORTICITY

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Abstract. We study the Cauchy problem for the compressible Euler equations in two spatial dimensions under any physical barotropic equation of state except that of a Chaplygin gas. We prove that the well-known phenomenon of shock formation in simple plane wave solutions, starting from smooth initial conditions, is stable under perturbations that break the plane symmetry. Moreover, we provide a sharp asymptotic description of the singularity formation. The new feature of our work is that the perturbed solutions are allowed to have small but non-zero vorticity, even at the location of the shock. Thus, our results provide the first constructive description of the vorticity near a singularity formed from compression: relative to a system of geometric coordinates adapted to the acoustic characteristics, the vorticity remains many times differentiable, all the way up to the shock. In addition, relative to the Cartesian coordinates, the vorticity remains bounded, and the specific vorticity remains uniformly Lipschitz, up to the shock.

To control the vorticity, we rely on a coalition of new geometric and analytic insights that complement the ones used by Christodoulou in his groundbreaking, sharp proof of shock formation in vorticity-free regions. In particular, we rely on a new formulation of the compressible Euler equations (derived in a companion article) exhibiting remarkable structures. To derive estimates, we construct an eikonal function adapted to the acoustic characteristics (which correspond to sound wave propagation) and a related set of geometric coordinates and differential operators. Thanks to the remarkable structure of the equations, the same set of coordinates and differential operators can be used to analyze the vorticity, whose characteristics are transversal to the acoustic characteristics. In particular, our work provides the first constructive description of shock formation without symmetry assumptions in a system with multiple speeds.

Keywords: characteristics; eikonal equation; eikonal function; null condition; null hypersurface; null structure; singularity formation; vectorfield method; wave breaking

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1. Introduction

In two spatial dimensions, the isentropic compressible Euler equations are evolution equations for the velocity $v : \mathbb{R} \times \Sigma \to \mathbb{R}^2$ and the density $\rho : \mathbb{R} \times \Sigma \to [0, \infty)$, where Σ is the (two-dimensional) space manifold, which we assume throughout to be $\Sigma = \mathbb{R} \times \mathbb{T}$. Here and throughout, $\mathbb{T} := [0, 1)$ (with the endpoints identified) denotes the standard one-dimensional torus. We now fix a constant $\bar{\rho} > 0$ corresponding to a constant background density.[1] Under a barotropic[2] equation of state and in terms of the logarithmic density $\uprho := \ln\left(\frac{\rho}{\bar{\rho}}\right)$, the equations take[3] the following form relative to the usual Cartesian coordinates,[4] $(i = 1, 2)$:

$$B\uprho = -\partial_a v^a, \tag{1.0.1a}$$

$$Bv^i = -c_s^2 \delta^{ia} \partial_a \uprho, \tag{1.0.1b}$$

where $B = \partial_t + v^a \partial_a$ is the material derivative vectorfield (see (3.3.9)) and $c_s = c_s(\uprho)$ is the speed of sound, which depends on the equation of state (see (3.3.3)). Throughout this paper, we assume the normalization condition[5]

$$c_s(0) = 1, \tag{1.0.2}$$

which simplifies some aspects of the analysis and presentation.

[1]In this paper, we will study solutions with density close to $\bar{\rho}$.

[2]A barotropic equation of state is one in which the pressure p can be expressed as a function of the density ρ alone.

[3]Throughout, if V is a vectorfield and f is a function, then $Vf := V^\alpha \partial_\alpha f$ denotes the derivative of f in the direction V. Lower case Latin indices correspond to the Cartesian spatial coordinates and lower case Greek indices correspond to the Cartesian spacetime coordinates. We also use Einstein's summation convention.

[4]Throughout, $\{x^\alpha\}$ are the usual Cartesian coordinates with corresponding partial derivative vectorfields $\partial_\alpha := \frac{\partial}{\partial x^\alpha}$. We also set $t := x^0$ and $\partial_t := \partial_0$.

[5]As we explain in Sect. 3, this can always be achieved by a change of variables.

As has been known since the foundational work of Riemann [27] in one spatial dimension, initially smooth solutions to the compressible Euler equations can form shock singularities in finite time, even though the solutions enjoy a conserved energy.[6] We recall that a shock singularity is such that the velocity and density remain bounded while some first partial derivative of these quantities with respect to the Cartesian coordinates blows up in finite time. This phenomenon is also known in the literature as wave breaking. Our main result is a proof of finite-time shock formation for solutions generated by an open set of regular Sobolev-class initial data in two spatial dimensions verifying suitable relative smallness assumptions. The solutions that we study here are perturbations of simple plane wave solutions that are close, as measured by suitable Sobolev norms, to constant state solutions; see Subsect. 1.2 for further discussion. The main new feature of our work is that the vorticity of the perturbed solutions,[7] defined to be $\omega := \partial_1 v^2 - \partial_2 v^1$, is allowed to be non-zero in the region where the shock forms. Actually, in our analysis, it is more convenient to work with the *specific vorticity* $\boldsymbol{\omega}$, defined by

$$\boldsymbol{\omega} := \omega / \exp(\boldsymbol{\rho}),$$

since it satisfies a simpler evolution equation and better estimates. As we describe in more detail later in this section, *our proof applies in particular to data such that the solution's vorticity is provably non-zero at the location of the first shock singularity.* Therefore, to close the proof, we in particular have to control the vorticity (and, as it turns out, many of its derivatives) in a past neighborhood of the first singularity. To this end, we rely on a new formulation of the compressible Euler equations (see Prop. 3.1), which we describe below in detail.

We now provide a rough summary of our main results. We plan to extend our results to the case of three spatial dimensions in forthcoming work [24]. As we briefly describe below, the case of three spatial dimensions requires substantial new technical innovations compared to the case of two spatial dimensions. The new innovations are tied to the need to derive, in three spatial dimensions, elliptic estimates to control the vorticity at the top order. In contrast, elliptic estimates are not necessary in two spatial dimensions; see [25] for a more substantial overview of this issue.

Theorem 1.1 (Rough version). *For any physical equation of state except that of the Chaplygin gas,*[8] *there exists an <u>open</u> set of regular initial data, with elements close to the data of a subset of simple plane wave solutions, that leads to <u>stable</u> finite-time shock formation. The specific vorticity, which is <u>provably non-vanishing at the shock</u> for some of our solutions, remains uniformly Lipschitz relative to the Cartesian coordinates, all the way up to the shock. Moreover, the dynamics are "well-described" by the irrotational Euler equations.*

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[6]More precisely, solutions to the compressible Euler equations (1.0.1a)-(1.0.1b) enjoy the conserved energy

$$\int_{\Sigma_t} \rho e + \frac{1}{2} \rho \delta_{ab} v^a v^b \, d^2 x, \tag{1.0.3}$$

where e, the specific internal energy, is given by $e = h - \frac{p}{\rho}$, where p is the pressure and the enthalpy h is as in Subsect. 2.3. However, the energy (1.0.3) is far too weak to prevent the formation of singularities. For this reason, it plays no role in our analysis.

[7]Plane symmetric solutions have vanishing vorticity.

[8] The equation of state of a Chaplygin gas is $p = p(\rho) = C_0 - \frac{C_1}{\rho}$, where $C_0 \in \mathbb{R}$ and $C_1 > 0$; see (3.3.4).

Remark 1.1 (Assumption on the spatial manifold). Our assumption that $\Sigma = \mathbb{R} \times \mathbb{T}$ is mainly for technical convenience and is not of fundamental importance. For instance, the case $\Sigma = \mathbb{R}^2$ could be treated with a similar approach, though the set of initial data to which our methods apply might be quite different (see also the discussion at the end of Subsect. 1.2).

Remark 1.2 (Maximal classical development). Our main results provide information about the solution only up to the constant-time hypersurface of first blowup. However, thanks to the sharp estimates of Theorem 16.1, our results could in principle be extended to give a detailed description of a portion of the maximal classical development[9] of the data corresponding to times up to approximately twice the time[10] of first blowup, including the shape of the boundary and the behavior of the solution along it. More precisely, the estimates that we prove are similar to the ones used by Christodoulou in his work [6, Ch. 15], in which he revealed the structure of a large irrotational portion of the maximal classical development of solutions to the relativistic Euler equations; see also [9] for a similar picture of an irrotational portion of the maximal classical development for solutions to the non-relativistic compressible Euler equations. However, for the sake of brevity, we have chosen not to carry out those arguments. We clarify that in obtaining their sharp picture of the boundary of the maximal classical development, the authors of [6, 9] relied on technical non-degeneracy assumptions on the behavior of the solution at the boundary; we would have to make similar non-degeneracy assumptions if we were to study the boundary of the maximal classical development in regions with vorticity.

We will give a precise version of Theorem 1.1 in Theorem 16.1. The estimates in Theorem 16.1 give a precise sense in which the dynamics are "well-described" by the irrotational Euler equations. In particular, our proof shows that[11] some of the quantities $\{\partial_j v^i\}_{i,j=1,2,3}$ blow up, while $\{v^i\}_{i=1,2}$, $\ln\left(\dfrac{\rho}{\bar{\rho}}\right)$, and ω remain uniformly bounded, all the way up to the shock. Another important part of our proof is that the specific vorticity and vorticity are more regular than the velocity estimates would suggest, both in terms of Cartesian coordinates and geometric[12] coordinates. In particular, to close our estimates, we must show that relative to the geometric coordinates, the specific vorticity and vorticity are exactly as differentiable as the velocity and density, which represents a gain of one derivative compared to viewing vorticity as a first derivative of velocity.

Remark 1.3 (Assumption on spatial dimensions and the regularity of the vorticity). In our proof, we rely on the assumption of two spatial dimensions to control the specific vorticity and the top-order derivatives of the eikonal function (see Subsubsect. 2.1.1

9 Roughly, the maximal classical development is the largest possible classical solution that is uniquely determined by the data; see, for example, [28, 32] for further discussion.

10 Roughly, we could propagate our bootstrap assumptions for this amount of time.

11 We note here that in the irrotational case, Christodoulou–Miao have already proved [9] that the quantities $\{\partial_j v^i\}_{i,j=1,2,3}$ blow up, while $\{v^i\}_{i=1,2}$ and $\ln\left(\frac{\rho}{\bar{\rho}}\right)$ remain uniformly bounded, all the way up to the shock.

12 In order to capture the geometry of shock formation, we define geometric coordinates similar to the ones used by Christodoulou in [6]; see Subsect. 2.1.

for its definition). In particular, in two spatial dimensions, the specific vorticity equation is homogeneous (see (3.3.11c)), which allows for a relatively straightforward proof that the specific vorticity gains regularity. In three spatial dimensions, the specific vorticity equation contains an additional "vorticity stretching" term, which introduces significant technical complications into the analysis. As we overviewed in our companion article [25], the vorticity stretching term can be controlled using additional elliptic estimates,[13] and a similar gain in regularity for the specific vorticity with respect to the geometric coordinates can be achieved; we will treat this in detail in a future work. Nevertheless, since the case of two spatial dimensions already requires substantial new ideas, of interest in themselves, we have chosen to treat it separately here.

We also note that the gain in regularity for the vorticity is familiar to the community of researchers who have proved well-posedness results for the compressible Euler equations in the presence of a physical vacuum boundary; see, for example, [11–13, 16, 17]. However, in those works, the proof of the gain in regularity relied on the special properties of Lagrangian coordinate partial derivative vectorfields. In the study of shock formation, Lagrangian coordinates are entirely inadequate for measuring regularity since they are not adapted to the acoustic characteristics (which we describe in detail later on), whose intersection corresponds to the singularity. For this reason, in three spatial dimensions, we need to rely on a different approach, tied to our new formulation of the equations (see below and Prop. 3.1), which allows us to realize the gain in regularity relative to vectorfields adapted to the acoustic characteristics. In fact, if regularity were the only consideration, then our approach for gaining a derivative in the vorticity could be implemented with *any* sufficiently smooth spanning set of vectorfields, not just the geometric ones (described in Subsect. 1.2) that we use to study shock formation.

The aforementioned results [6, 9] on shock formation for compressible fluids, though foundational, crucially relied on the assumption that the fluid is irrotational, at least in a neighborhood near the shock.[14] In the irrotational case, the dynamics are completely determined by a fluid potential Φ and the Euler equations can be written as a single quasilinear scalar wave equation; this is a big simplification compared to the structure of the compressible Euler equations with vorticity. Moreover, we note that the assumption of irrotationality is very restrictive from a physical point of view. In particular, irrotational data constitute only a very small (infinite co-dimension with empty interior!) subset of all initial data. It is therefore of interest to prove, at the very least, that previous shock formation results still hold under perturbations with small vorticity. As we explain below, substantial new ideas are needed to accommodate the presence of even small amounts of vorticity near the singularity. In this context, let us note that accommodating vorticity is particularly relevant

[13]We also note that one encounters other new technical difficulties in three spatial dimensions compared to the case of two spatial dimensions. In particular, in three spatial dimensions, one must also derive elliptic estimates, distinct from those mentioned above for the specific vorticity, to control the top-order derivatives of the eikonal function. However, an approach to implementing the elliptic estimates for the eikonal function in three spatial dimensions has been well understood since the Christodoulou–Klainerman proof [7] of the stability of Minkowski spacetime, and, in the context of shock formation, since Christodoulou's work [6].

[14]A vorticity-free region near the shock can be achieved, for instance, in the small data dispersive setting by exploiting the fact that the characteristic speed for the vorticity is slower than the sound speed. See also the discussions in Subsect. 1.4.

when one is interested in extending the solution (in a weak sense) after the shock is formed. The reason is that even if the fluid is initially irrotational, vorticity may be generated after a shock has formed [6].

Moreover, in the larger context of the study of singularity formation for evolution partial differential equations, our theorem appears to be the first shock formation result in more than one spatial dimension that involves a system of quasilinear wave equations coupled to another quasilinear evolution equation *with a different characteristic speed*. More precisely, in the presence of vorticity, the Euler equations exhibit the following two kinds of characteristics: acoustic characteristics (corresponding to the propagation of sound waves) and the integral curves of the material derivative vectorfield (corresponding to the transporting of vorticity). We hope that the techniques introduced here will be relevant to other problems featuring multiple characteristic speeds.

As we have alluded to above, the starting point of our proof is a new formulation of the compressible Euler equations as a coupled system of covariant wave and transport equations. The new formulation, which we derived in the companion article [25], is a consequence of (1.0.1a)-(1.0.1b), obtained by differentiating the equations with suitable operators and observing remarkable cancellations. One key advantage of the new formulation, stated below as Prop. 3.1, is that the inhomogeneous terms exhibit surprisingly good null structures that are preserved under commutations with well-constructed geometric vectorfields, adapted to the acoustic characteristics. The good null structure, which we referred to as the "strong null condition" in [25], signifies the complete nonlinear absence of certain quadratic and higher-order interactions that we would not be able to control near the shock. The new formulation also allows for the aforementioned *gain of regularity in the specific vorticity*, both in terms of Cartesian and geometric coordinates, which is central to closing the proof. We will further discuss this in Subsect. 1.3.

1.0.1. *Organization of Sect. 1.* We have organized the remainder of Sect. 1 as follows: In Subsect. 1.1, we describe the setup of the problem, noting in particular that we will only study the solution in the causal future of a portion of the initial data. In Subsect. 1.2, we describe the solution regime that we study and the size parameters that we use in the analysis. In Subsect. 1.3, we describe some new ideas in the proof of our main theorem (although we will postpone a more detailed discussion of the main ideas to Sect. 2). Finally, in Subsect. 1.4, we close the introduction with a discussion on relevant previous works.

1.1. **Setup of the problem.** Instead of studying the solution in the entire spacetime $\mathbb{R}\times\Sigma$, we study only the future portion of the solution that is completely determined by the portion of the data lying in the subset $\Sigma_0^{U_0} \subset \Sigma_0$ of thickness U_0 (as measured by the eikonal function u, described below) and on the curved null hyperplane portion $\mathcal{P}_0^t$; see Figure 1.

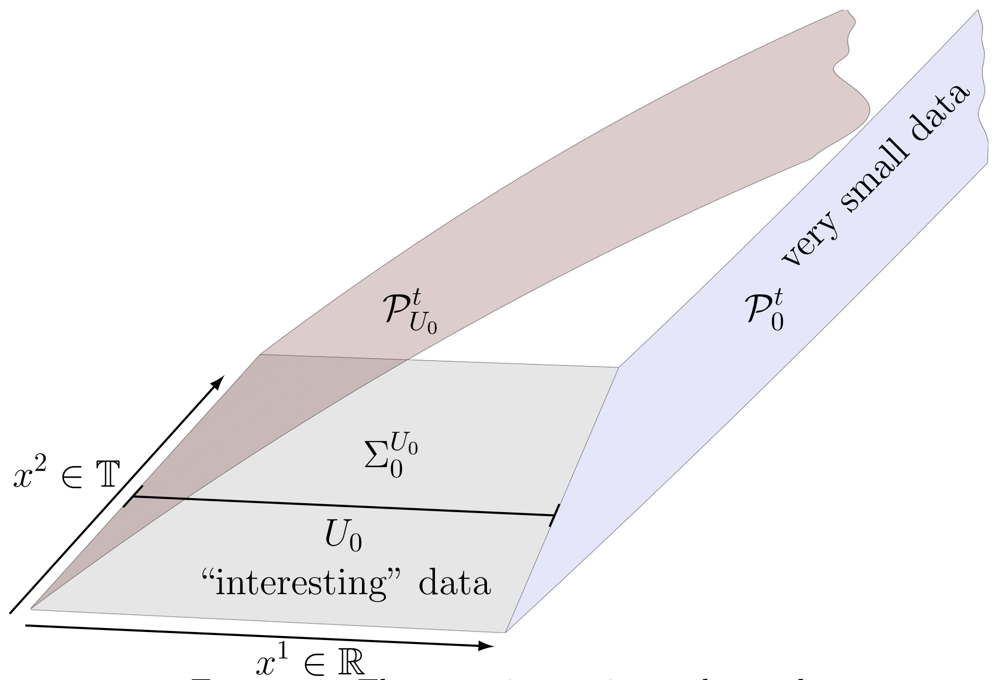

FIGURE 1. The spacetime region under study.

The set Σ_t is the level set of constant Cartesian time. Moreover, here and throughout, $0 \leq t < 2\mathring{\delta}_*^{-1}$, where $\mathring{\delta}_* > 0$ is a data-dependent parameter (see Def. 8.1) connected to the expected time of first shock formation, and

$$0 < U_0 \leq 1 \tag{1.1.1}$$

is a parameter, fixed until Theorem 16.1 (our main theorem). The data that we treat are such that $\mathring{\delta}_*$ is large *relative* to other parameters that control various seminorms of the data. That is, $\mathring{\delta}_* > 0$ can be arbitrary, but another parameter must be appropriately small; see Subsect. 1.2 for further discussion. We assume that the "interesting, relatively large" (in a sense that we explain below) portion of the data lies in $\Sigma_0^{U_0}$. Moreover, we assume that the data are "very small" on $\mathcal{P}_0^{2\mathring{\delta}_*^{-1}}$, though the vorticity is allowed to be non-zero everywhere there.

More precisely, in our main theorem, we consider perturbations of simple outgoing[15] plane wave solutions. We focus our attention on perturbations of the subset of these plane wave solutions that have data supported in Σ_0^1 and that satisfy the relative size condition mentioned above. Domain of dependence considerations imply that such plane wave solutions completely vanish along $\mathcal{P}_0^{2\mathring{\delta}_*^{-1}}$. Thus, "very small" (not necessarily symmetric) perturbations of their data on Σ_0, *which we now allow to have large spatial support*, *induce*[16] "very small" data along $\mathcal{P}_0^{2\mathring{\delta}_*^{-1}}$ such that the data on $\Sigma_0^{U_0} \cup \mathcal{P}_0^{2\mathring{\delta}_*^{-1}}$ is a perturbation of that of the simple plane wave solution. Moreover, it can be easily arranged that the vorticity is non-vanishing

[15]Here and throughout, outgoing simply means right-moving as is indicated in Figure 1.

[16]Notice that while in principle one can attempt to directly prescribe data on the null hypersurface $\mathcal{P}_0^{2\mathring{\delta}_*^{-1}}$, in practice, this involves solving a rather complicated system of *constraint* equations, which can be difficult to implement.

on $\Sigma_0^{U_0} \cup \mathcal{P}_0^{2\mathring{\delta}_*^{-1}}$; since the specific vorticity is transported by the material derivative vectorfield (see the one-dimensional curves in Figure 3 for a depiction of the integral curves of the material derivative vectorfield), if the vorticity is everywhere non-zero along $\Sigma_0^{U_0} \cup \mathcal{P}_0^{2\mathring{\delta}_*^{-1}}$, then it will be non-zero at the location of the first shock.

To summarize, the dynamic region of interest[17] lies in between $\Sigma_0^{U_0}$ and the curved null hyperplane portions $\mathcal{P}_{U_0}^t$ and $\mathcal{P}_0^t$, where $0 \leq t < 2\mathring{\delta}_*^{-1}$. We rigorously define these sets in Def. 3.9, but let us say a few words about them here and, at the same time, about some other important spacetime subsets depicted in Figure 2. The definitions of these sets refer to an eikonal function u, whose level sets are acoustic characteristics; we postpone our extensive discussion of u until later. For now, we simply note that the level sets of u are denoted by $\mathcal{P}_u$ or, when they are truncated at time t, by $\mathcal{P}_u^t$. We refer to the $\mathcal{P}_u$ and $\mathcal{P}_u^t$ as "null hypersurfaces," "null hyperplanes," "characteristics," or "acoustic characteristics." We use the notation $\mathcal{M}_{t,u}$ to denote the open-at-the-top region trapped in between Σ_0, Σ_t, $\mathcal{P}_0^t$, and $\mathcal{P}_u^t$. We refer to the portion of Σ_t trapped in between $\mathcal{P}_0^t$ and $\mathcal{P}_u^t$ as Σ_t^u. The trace of the level sets of u along Σ_0 are chosen (see condition (3.6.2)) to be straight lines, which we denote by $\ell_{0,u}$. For $t > 0$, the trace of the level sets of u along Σ_t are (typically) curves[18] $\ell_{t,u}$. We restrict our attention to spacetime regions with $0 \leq u \leq U_0$.

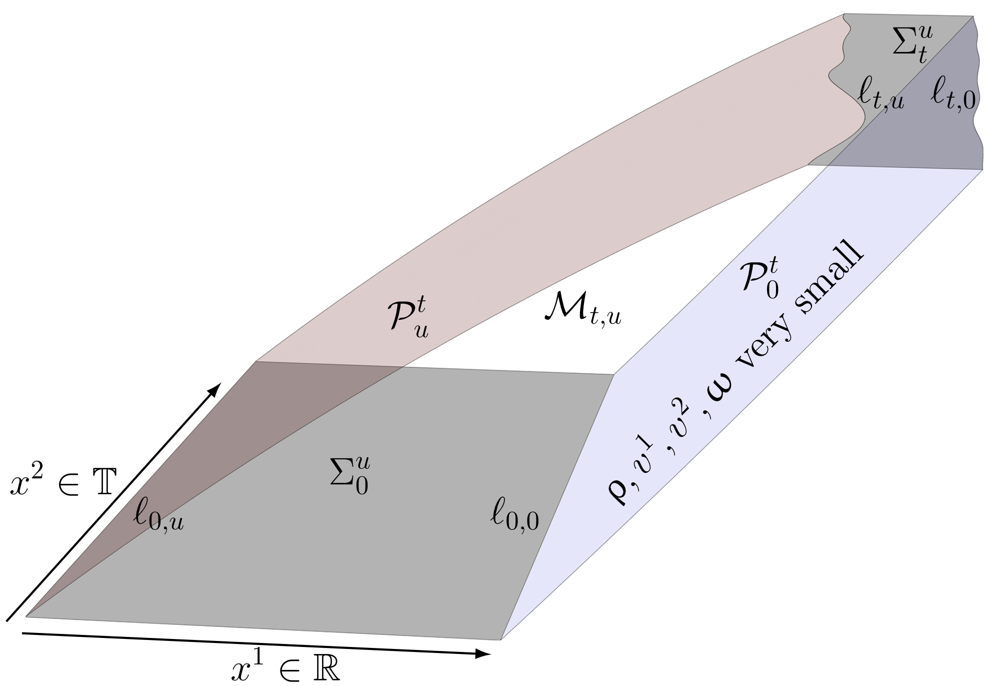

FIGURE 2. The spacetime region and various subsets.

1.2. **Further description of the data and the solution regime.** We now provide more details about the data and solutions that we study. They are close to plane symmetric solutions with data supported in the x^1 interval $[0, 1]$ such that one Riemann invariant $(\mathcal{R}_- := v^1 - \int_{\widetilde{\rho}=0}^{\rho} c_s(\widetilde{\rho})\, d\widetilde{\rho})$ completely vanishes, while the other one $(\mathcal{R}_+ := v^1 + \int_{\widetilde{\rho}=0}^{\rho} c_s(\widetilde{\rho})\, d\widetilde{\rho})$

[17] Notice that by domain of dependence considerations, the solution in this region indeed depends only on the initial data on $\Sigma_0^{U_0} \cup \mathcal{P}_0^{2\mathring{\delta}_*^{-1}}$.

[18] More precisely, the $\ell_{t,u}$ are diffeomorphic to the torus $\mathbb{T}$.

is initially small but with *relatively large*[19] spatial derivatives. Using the approach taken by Riemann in his famous work [27] (in which he invented Riemann invariants), one may show that $\partial_1 \mathcal{R}_+$ experiences a Riccati-type blowup along a characteristic curve while $\mathcal{R}_+$ remains bounded; we stress that these phenomena occur while $\mathcal{R}_-$ remains identically zero. Such a solution is known as a *simple plane wave* and arguably represents the simplest kind of symmetric shock-forming solution that one can study in a perturbative sense. It is for this reason that our main results apply to neighborhoods of a subclass of simple plane waves.

To further describe the class of data and solutions under study, we need to introduce the geometric vectorfields L, Y and $\breve{X}$, which are depicted in Figure 3 below, and also the parameters $\mathring{\delta}_*$, $\mathring{\delta}$, and $\mathring{\epsilon}$ describing the sizes of different quantities. At this point, let us just note that the vectorfields L and Y are chosen to be tangential to the acoustic characteristics, while $\breve{X}$ is transversal to them. Moreover, we use L, $\breve{X}$, and Y to commute the equations and obtain estimates for the solution's derivatives. We refer the readers to Subsect. 3.7 for rigorous definitions of these vectorfields and proofs of their basic properties. The parameters $\mathring{\delta}_*$ and $\mathring{\delta}$ are not necessarily small, but we require $\mathring{\epsilon}$ to be *relatively small* in a sense explained in Subsect. 8.6. With the help of these parameters, we can now further describe the data and solutions under study (where it may be seen that some of the conditions below are redundant):

- (Nearly plane symmetric perturbations of constant states) The initial data and solution are $\mathring{\epsilon}$ close in L^∞ to the constant state $(\rho, v^1, v^2) \equiv (0, 0, 0)$. By a "plane symmetric" solution, we mean one such that, relative to the standard Cartesian coordinates, we have $\rho = \rho(t, x^1)$, $v^1 = v^1(t, x^1)$, and $v^2 \equiv 0$. The factor $\mathbb{T}$ in the Cauchy hypersurface $\Sigma_0 \simeq \mathbb{R} \times \mathbb{T}$ corresponds to the direction of symmetry for the simple plane symmetric waves that we are perturbing. Thus, "nearly plane symmetric" means, roughly, small dependence in the $\mathbb{T}$ direction.
- (Finite-time shock formation occurs) $\mathring{\delta}_*$ is defined so that $\mathring{\delta}_*^{-1}$ is the expected blowup time, up to $\mathcal{O}(\mathring{\epsilon})$ error. $\mathring{\delta}_*$ is tied to the size of the (signed part of) $\breve{X} v^1|_{t=0}$; see (8.1.1) for the precise definition.
- (Boundedness of the transversal derivatives) $0 < \mathring{\delta} < \infty$ bounds the initial size of the $\breve{X}$ (and $\breve{X}\breve{X}$ and $\breve{X}\breve{X}\breve{X}$, etc.) derivatives of (ρ, v^1, v^2). In addition, we make smallness assumptions on the derivatives of $\rho - v^1$ and v^2, consistent with the behavior of the simple plane symmetric waves that we are perturbing; see the next item.
- (Nearly simple outgoing) Initially,[20] the L and Y derivatives of (ρ, v^1, v^2) are $\mathring{\epsilon}$-small at all derivative levels in appropriate norms. The same holds true for higher-order derivatives in terms of L, Y and $\breve{X}$, where at least one of the derivatives is L or Y. Moreover, we assume that the first-order $\breve{X}$ derivatives of $\rho - v^1$ and all directional derivatives of v^2 are of size $\mathring{\epsilon}$. Roughly, these conditions correspond to a solution whose dynamics are well-described by an outgoing (that is, moving in the direction of increasing x^1) and nearly simple.

[19] $\partial_1 \mathcal{R}_+$ is allowed to be initially small in an absolute sense, as long as $\mathcal{R}_+$ is restricted to be even smaller.

[20] Notice that only the initial data for (ρ, v^1, v^2) can be prescribed and that we cannot prescribe their time derivatives. Nevertheless, for perturbations of appropriate simple plane wave solutions, the desired smallness can be achieved.

- (Smallness of the Riemann invariant $\mathcal{R}_-$) Initially, $\mathcal{R}_- := v^1 - \int_{\widetilde{\uprho}=0}^{\uprho} c_s(\widetilde{\uprho})\, d\widetilde{\uprho}$ and *all of its directional derivatives* up to top-order are initially $\mathring{\upepsilon}$ small in appropriate norms. This represents a perturbation of the complete vanishing of $\mathcal{R}_-$, which, in view of the above discussion, therefore corresponds to a perturbation of a simple outgoing plane symmetric wave.
- (Small vorticity) This means that $\partial_1 v^2 - \partial_2 v^1$ and all of its derivatives up to top-order, in all directions, are initially $\mathring{\upepsilon}$ small in appropriate norms.
- (Near-acoustic regime) In this regime, the compressible Euler equations are well-approximated by transport equations along appropriately scaled null[21] generators L of the acoustic characteristics $\mathcal{P}_u$. This corresponds to a flow that is dominated by sound wave propagation.

Of course, one of our main tasks in our proof is showing that the smallness conditions stated above, at first only assumed for the initial data, are propagated by the flow. Actually, as we explain below in great detail, one of the main difficulties in the proof is that near the top-order, the smallness can only be understood in terms of some singular norms. By this, we mean that the best estimates we are able to prove allow for the possibility that the high-order energies might blow up.

The solution regime described above is not the only one for which we could prove a shock formation result. However, it is perhaps the simplest one allowing for non-zero vorticity. In particular, because the solutions are nearly plane symmetric, there is no wave dispersion and hence no decay. Therefore, powers of t and r do not play a role in our analysis. We expect that a similar shock formation result could be proved for small, nearly radially symmetric quickly decaying[22] data on $\mathbb{R}^2$. In this case, the analysis would involve factors of t and/or r, which would capture the dispersive decay (that one expects to occur until close to the shock). Moreover, one would have to assume that the vorticity is initially very small, so that it and its derivatives are not able to become large by the time that the shock forms.

1.3. **New ideas for the proof.** To prove our main theorem, we rely on the full strength of the technology developed in the works of Christodoulou [6] and Speck–Holzegel–Luk–Wong [30]. In particular, the same size parameters $\mathring{\updelta}_*$, $\mathring{\updelta}$, and $\mathring{\upepsilon}$, which are featured in the proof (and discussed in Subsect. 1.1), are also present in [30]. We therefore postpone a detailed discussion of our proof until Sect. 2, where we review the works [6, 30]. Here, we will simply highlight a few key new high-level ideas (see Subsect. 2.4 for a discussion of more technical new ideas):

(1) In Prop. 3.1, we reformulate the equations as a system of coupled wave and transport equations with remarkable geometric features, including the good null structures mentioned above.
(2) We prove that the transport part of the system "interacts well" with the wave part of the system. More precisely, one can commute geometric vectorfields adapted to the acoustic characteristics $\mathcal{P}_u$ through an appropriately weighted version of the material

[21]By null, we mean relative to the acoustical metric of Def. 3.3.

[22]By decaying, we mean in the Euclidean radial coordinate r, towards the data of a non-vacuum constant state.

derivative vectorfield, which is the principal part of the transport equation for the specific vorticity.

(3) We show that the specific vorticity is uniformly Lipschitz with respect to Cartesian coordinates up to the formation of the first shock, which is a much stronger estimate than what follows from simply viewing the specific vorticity as first derivatives of v^i divided by ρ.

(4) We prove that the specific vorticity is "one derivative better" with respect to geometrically defined vectorfields than one naively expects, thus avoiding an apparent loss of derivatives in the new formulation of the equations.

Our reformulation of the compressible Euler equations was derived in [25] in the case of three spatial dimensions but can be easily modified so as to apply in two spatial dimensions. We present the two-space-dimensional version in Prop. 3.1 below. In two spatial dimensions, the new formulation can be modeled by the following wave-transport system in the scalar unknowns Ψ (which models v^i and $\mathsf{\rho}$) and w (which models the specific vorticity, defined above as $\boldsymbol{\omega} = \omega / \exp(\mathsf{\rho})$):

$$\square_{g(\Psi)} \Psi = \partial w, \tag{1.3.1}$$

$$\partial_t w = 0. \tag{1.3.2}$$

In (1.3.1), $g = g(\Psi)$ is a Lorentzian metric whose Cartesian components $g_{\alpha\beta}$ are assumed to be explicit smooth functions of Ψ, $\square_{g(\Psi)}$ is the covariant wave operator[23] of $g(\Psi)$, and ∂w schematically denotes first-order Cartesian coordinate partial derivatives of w. In our study of the compressible Euler equations, g is the acoustical metric (see Def. 3.3) corresponding to the propagation of sound waves. In Cartesian coordinates, the expression $\square_{g(\Psi)}\Psi$ contains (quasilinear) principal terms of the schematic form $\mathrm{f}(\Psi)\partial^2\Psi$ and semilinear terms of the form $\mathrm{f}(\Psi)(\partial\Psi)^2$. The precise structures of both the quasilinear and the semilinear terms are important for our analysis. Equation (1.3.2) models the transporting of specific vorticity. In writing down (1.3.1)-(1.3.2), we have omitted the quadratic inhomogeneous terms from Prop. 3.1, all of which have a good null structure and remain negligible, all the way up the shock. The presence of this null structure, which is available thanks to the special form of the equations stated in Prop. 3.1, is fundamental for our proof; see Remark 3.2 for further discussion.

We also note the following aesthetically appealing feature of the formulation: the principal parts of the system are a wave operator and a transport operator. Thus, the two kinds of propagation phenomena present in the compressible Euler equations, namely the propagation of sound waves and the transporting of vorticity, become manifest. This stands in contrast to the usual first-order formulation (1.0.1a)-(1.0.1b), where the presence of the two kinds of propagation phenomena are not easily visible at the level of the equations.

Previous shock formation results, which we review in Sects. 1.4 and 2, apply to quasilinear wave equations. In contrast, in the model problem (1.3.1) and (1.3.2), we need to handle an extra transport equation and also additional inhomogeneous terms in the wave equation. In previous works on shock formation in quasilinear wave equations, starting from [2–4, 6], a crucial insight was to use geometric vectorfields that are adapted to the characteristics $\mathcal{P}_u$ and that, in directions transversal to the characteristics, are appropriately degenerate

[23] Relative to arbitrary coordinates, $\square_g f = \frac{1}{\sqrt{\mathrm{det} g}}\partial_\alpha\left(\sqrt{\mathrm{det} g}(g^{-1})^{\alpha\beta}\partial_\beta f\right)$.

(with respect to the Cartesian coordinate vectorfields) near the shock. Morally, this is equivalent to deriving estimates relative to a system of geometric coordinates adapted to the characteristics. To accommodate the term ∂w on RHS (1.3.1), it is therefore important when dealing with the coupled system to ensure that the derivatives of the specific vorticity with respect to the *same geometric vectorfields* can be controlled. To achieve this, we rely on the fact that the transport operator is a *first-order* differential operator and therefore, upon multiplying by a degeneration factor μ (explained below in great detail), that *commuting the transport equation with the geometric vectorfields generates only controllable error terms.*

Next, we note that RHS (1.3.1) involves a Cartesian coordinate partial derivative of w, which is therefore singular with respect to the geometric vectorfields.[24] However, the following crucial geometric fact is available in our formulation of the compressible Euler equations: the transport equation has a *strictly smaller speed compared to* the characteristic wave speed corresponding to the operator $\square_g$. For this reason, in the actual problem under study, we can use the transport equation to express the transversal (to the acoustic characteristics of g) derivatives of w in terms of the non-degenerate tangential derivatives of w. This can be used to show, among other things, that w is in fact uniformly Lipschitz up to the shock. The difference in the characteristic speeds for the transport operator and the wave operator is also important in that it leads to the availability of non-degenerate energies for w along the acoustic characteristics corresponding to g; see the last term on RHS (2.4.2).

Finally, we discuss the basic regularity of the solution variables, highlighting the role of the source term ∂w on the right-hand side of the wave equation (1.3.1). In the case of the compressible Euler equations, vorticity can be viewed as the first derivatives of the velocity and hence, in the context of the regularity of solutions to the model problem, one might be tempted to think of ∂w as corresponding to the *second* derivatives of Ψ. However, this perspective is insufficient from the point of view of regularity since energy estimates for the wave equation (without commutation) yield control of only one derivative of Ψ. Hence, this perspective leads to an apparent loss of a derivative. However, since (1.3.2) is a homogeneous transport equation, one expects to gain a derivative – this is indeed obvious[25] if one takes Cartesian coordinate partial derivatives of equation (1.3.2). What is less obvious is that in fact, the loss of derivatives can also be avoided if one differentiates the transport equation with the geometric vectorfields which, as it turns out, depend on Ψ. We note that while it is indeed possible to carry out commutations the transport equation with geometric derivatives, one encounters some singular terms tied to the degenerate top-order behavior of Ψ and the acoustic geometry, which we will discuss in detail in Sect. 2.

1.4. **History of the problem.** The study of the formation of shocks for the compressible Euler equations has a long history which traces back to the aforementioned foundational work of Riemann [27]. He introduced the *Riemann invariants* for the Euler equations in one spatial dimension and showed that shocks often form in finite time. In the one-dimensional case, the theory, at least in the small BV regime, is fairly complete. In particular, it is known

[24] This singularity is actually unimportant just from the point of view of the lower-order energy estimates. This is, however, of crucial importance at the top-order; see discussions in Sect. 2.

[25] From this point of view, the model system is oversimplified in that one can control an arbitrarily large number of Cartesian coordinate partial derivatives of w. In the actual system, since the transport operator depends also on the Cartesian components (v^1, v^2), only one derivative can be gained.

that there exist unique global weak solutions in the BV class. This theory in particular incorporates formation and interaction of shocks. We refer the readers to [5, 14] for surveys on the one-dimensional case.

Let us mention that the theory of finite-time blowup for solutions to hyperbolic systems in one spatial dimension has been developed way beyond the theory of the compressible Euler equations. For example, for general 2×2 genuinely nonlinear hyperbolic systems, finite-time blowup has been proven by Lax [22]. For $n \times n$ genuinely nonlinear hyperbolic systems, even though Riemann invariants are not available, John [18] has obtained a shock formation result in which the waves are simple by the time a shock forms.

In two or three spatial dimensions (without symmetry assumptions), the problem becomes considerably harder. The first general breakdown result for the compressible Euler equations in three spatial dimensions was achieved by Sideris [29] for a polytropic gas[26] with adiabatic index $\gamma > 1$. In particular, he exhibited an open set of small and regular initial data for which the corresponding solutions cease to be C^1 in finite time. However, his methods did not provide any information on the nature of the breakdown.

In a different direction, Alinhac studied the two-dimensional compressible isentropic Euler equations in *radial symmetry* [1]. He showed that a large class of small radially symmetric data (with potentially non-vanishing vorticity[27]) lead to a finite time blow up. While this result only applies to radial initial data, it gives a precise estimate on the blow up time (at least as the size of the data tends to 0).

Alinhac later achieved [2–4] important breakthroughs regarding shock formation. His works, which addressed solutions to a large class of quasilinear wave equations, were the first instances of proofs of shock formation for solutions to quasilinear equations in more than one spatial dimension that did not rely on any symmetry assumptions. In particular, his work yielded a precise description of the singularity and tied its formation to the intersection of the characteristics. While he did not explicitly study the compressible Euler equations, his works provided all of the main insights needed to extend the result to the *irrotational* Euler equations. More precisely, for all quasilinear wave equations $(g^{-1})^{\alpha\beta}(\partial\Phi)\partial_\alpha\partial_\beta\Phi = 0$ that fail to satisfy the null condition, Alinhac exhibited a set of initial data leading to finite-time shock formation. As we will discuss in Subsect. 2.3, under the irrotationality assumption, the compressible Euler equations can be written as a quasilinear wave equation in the above form. Moreover, the null condition is violated whenever the equation of state is not that of a Chaplygin gas. The data in Alinhac's works were small and satisfied a *non-degeneracy condition*. For this class of data, he gave precise estimates of the solution up to the first singular time. In his proof, he recognized the importance of deriving estimates relative to a geometric coordinate system tied to an eikonal function, which captures the geometry of shock formation. However, Alinhac's approach to deriving energy estimates was based on a Nash–Moser iteration scheme featuring a free boundary, and the iteration scheme relied in a fundamental way on his non-degeneracy condition on the initial data.

[26]That is, the equation of state is given by $p(\rho) = k\rho^\gamma$ for constants k and γ with $k > 0$. Actually, Sideris allowed for the presence of non-constant entropy; his breakdown result holds for the equations of state $p(\rho) = k\rho^\gamma \exp(s)$, where s, the specific entropy, verifies the evolution equation $Bs = 0$.

[27]However, since the initial vorticity is required to be compactly supported and the speed of the vorticity is much slower than the sound speed, the vorticity in Alinhac's solutions vanishes in a neighborhood to the past of the first singularity.

In a monumental work in 2007, Christodoulou [6] studied shock formation for all[28] wave equations of irrotational relativistic fluid mechanics.[29] Christodoulou proved that a large class of small initial data give rise to shock formation and he gave a precise description of a portion of the boundary of the maximal classical development of the data. Compared to the work of Alinhac, Christodoulou introduced a fully geometric framework such that the breakdown of the solution is completely described in terms of the vanishing of the inverse foliation density μ (see definition (2.1.3)) of the acoustic characteristics. As a consequence, his work applied to an open neighborhood of solutions whose data are small and compactly supported perturbations of the non-vacuum constant states. In particular, for data that are small as measured by a high-order Sobolev norm, he showed that (at least outside the causal future of a compact set) shocks are the only possible singularities. Moreover, he exhibited an open condition on the data that guarantees that a shock will form in finite time.[30]

The geometric framework introduced in [6] has proven to be useful for studying shock formation in other settings. Most relevant to our current work is the aforementioned work of Christodoulou–Miao [9], which used the geometric insights of [6] to study shock formation for small and compactly supported perturbations of non-vacuum constant state solutions to the non-relativistic compressible Euler equations. In particular,[31] the results of [9] provided a precise picture of the singularity formation exhibited by Sideris in [29].

While the shock formation result of [6] was proved in *irrotational* regions of spacetime, the result also applies to initial data with non-vanishing vorticity *which satisfies appropriate conditions on its (compact) support*. This is because for such initial data, using that the vorticity and sound travel with different speeds, one can show that in the complement of the causal future of an appropriate compact set, the vorticity vanishes. In particular, in the small-data regime, the vorticity travels with small speed and hence completely vanishes in the acoustic wave zone, where the shock forms. A similar result could be proved for the non-relativistic compressible Euler equations using the techniques of [9], even though such a result was not stated there. However, we stress that the approach of [6, 9] is not sufficient, in itself, for controlling solutions with non-vanishing at the first shock singularity; for this, one seems to need all of the new structural features afforded by Prop. 3.1.

The seminal work of Christodoulou also inspired some recent developments on shock formation for quasilinear wave equations in more than one spatial dimension. See [15, 26, 31] for a sample of such results. The work [31] in particular generalized the results in [6] to a

[28]Actually, there is one exceptional equation of state such that the null condition is satisfied, in which case the corresponding wave equation admits small data global solutions [23]; see [6] for further discussion. The exceptional equation of state corresponds to the equation of state of a Chaplygin gas in the non-relativistic case, for which our main shock formation results do not apply.

[29]Roughly speaking, these equations form a subclass of the equations studied by Alinhac and enjoy special additional properties, such as an Euler-Lagrange structure and invariance under the Poincaré group. However, as is shown in [31], the insights introduced in [6] can be applied to a much larger class of quasilinear wave equations.

[30]Given Christodoulou's result that shocks are the only possible singularities, there remains a possibility of some non-trivial global solutions arising from small data.

[31]Note that Sideris' proof of blowup by contradiction [29] applies only to adiabatic equations of state with index bigger than one, while the work [9] allows for an arbitrary barotropic equation of state (except that of the Chaplygin gas).

much larger class of quasilinear wave equations. We refer the readers also to [8, 10] for some recent developments in *symmetry-reduced* problems motivated by [6].

1.5. **Outline of the paper.** In Sect. 2, we describe the main ideas behind our proof. Although we need many new ideas to treat the vorticity, we extensively rely on the framework developed by Christodoulou [6] in his proof of shock formation in irrotational regions and on the methods of Speck–Holzegel–Luk–Wong, who proved [30] shock formation for perturbations of simple outgoing plane waves for general classes of quasilinear wave equations. Hence, we review the relevant aspects of those works in detail. Readers who are familiar with those works might prefer to skip to Subsect. 2.4, where we overview the main new ideas needed to handle the presence of vorticity at the shock.

Starting in Sect. 3, we give detailed proofs. Specifically, in Sects. 3-7, we construct all of the geometric quantities that we need to study the solution. In Sect. 8 we describe our assumptions on the data and formulate suitable L^∞-type bootstrap assumptions. In Sects. 9-11 and 14, we use the bootstrap assumptions to derive L^∞ and pointwise estimates for the solution. In Sect. 12, we construct the L^2-type quantities that we later bound with energy estimates. In Sect. 13, we provide a geometric Sobolev embedding theorem, which we will use to recover the L^∞ bootstrap assumptions from energy estimates. Sect. 15 is the most important part of the paper. There we use the previous estimates to derive a priori energy estimates for the L^2-type quantities mentioned above. In Sect. 16, we prove our main shock formation theorem, including recovering the bootstrap assumptions and showing that the shock forms. The theorem is relatively easy to prove given the estimates from the prior sections.

2. Ideas of the proof

In this section, we describe the ideas of the proof of our main theorem. While the main novelty in this paper is that we allow for non-vanishing vorticity all the way up to shock formation, in order to describe our proof, we nonetheless have to recall some of the main points in the work of Christodoulou [6] and the work of Speck–Holzegel–Luk–Wong [30]. In particular, in the present work, we will work with a solution regime similar to that in [30].

We have organized Sect. 2 as follows: In Subsect. 2.1, we review the work [6], emphasizing the geometric insights that are relevant to our present work. In Subsect. 2.2, we review the work [30]. In Subsect. 2.3, we discuss how the work [30] can be applied to the compressible Euler equations and how [30] is related to our present work. Finally, in Subsect. 2.4, we discuss the main new ingredients that we use to prove our main theorem, which requires controlling the interaction between sound waves and vorticity up to the first singularity caused by compression.

2.1. **Review of Christodoulou's work.** We begin with a review of the main ideas in [6]. However, in this subsection, we will not restrict ourselves to discussing the small-data regime in $1+3$ dimensions as in [6]. Instead, we focus on *general* principles regarding the geometric structure of shock formation for quasilinear wave equations, which can be applied to settings beyond the original work [6], for example in different solution regimes and for more general equations; see [9, 15, 26, 31] and also a more thorough discussion in the survey article [15] in the case of small compactly supported data on $\mathbb{R}^3$. In particular, in this subsection, we will suppress discussion of the precise estimates that are *specific* to each problem.[32]

In this subsection, we will consider $(1 + 2)$ dimensional[33] covariant (see Footnote 23) quasilinear wave equations of the form[34]

$$\square_{g(\Psi)}\Psi = 0 \tag{2.1.1}$$

for real-valued scalar functions Ψ with regular initial data. By assumption, the metric $g(\Psi)$ is a Lorentzian metric that we will refer to as the acoustic metric since, in the context of the Euler equations, the wave equations correspond to the propagation of sound waves. As in Subsect. 1.3, we assume that the Cartesian components $g_{\alpha\beta}$ are explicit smooth functions of Ψ. We require the nonlinearity in (2.1.1) to obey certain conditions so that a shock can form for appropriate initial conditions; see Footnote 41.

2.1.1. *Identification of the blowup-mechanism.* In the work [6], Christodoulou studied the formation of shocks by introducing a geometric framework tied to an eikonal function u, which is a solution to the eikonal equation (a hyperbolic PDE)

$$(g^{-1})^{\alpha\beta}(\Psi)\partial_\alpha u \partial_\beta u = 0, \qquad \partial_t u > 0, \tag{2.1.2}$$

supplemented with appropriate initial conditions. The level sets of u are null hypersurfaces (also known as characteristics) relative to $g(\Psi)$, which we denoted above by $\mathcal{P}_u$. The characteristics provide a foliation of the spacetime that is essential for understanding the shock. The most important quantity in the study of shock formation is the *inverse foliation density* $\mu > 0$, defined as

$$\mu := -\frac{1}{(g^{-1})^{\alpha\beta}(\Psi)\partial_\alpha t \partial_\beta u}, \tag{2.1.3}$$

where t is the Cartesian time coordinate. $1/\mu$ is a measure, relative to the constant time slices Σ_t, of how "densely packed" the characteristics are. That is, $\mu = 0$ corresponds to infinite density, the intersection of characteristics, and the formation of a shock. One of the key features of Christodoulou's work [6] is his proof that for small data, the only possible blowup in a certain solution regime is the formation of shocks. That is, he showed that the regularity of the solution is completely determined by μ and that, for a class of data, one has

[32] In particular, when studying shock formation in a particular solution regime, it is important to track the "smallness" in the problem. This plays a crucial role in [6], which specifically considers the small-data regime in $1+3$ dimensions for the irrotational relativistic Euler equations. We will completely suppress this discussion in this subsection, but in later subsections, we will emphasize the importance of the role of certain kinds of smallness present in the solution regime that we consider in the present paper.

[33] The work [6] was carried out in $(1 + 3)$ dimensions. However, since the rest of our present paper is in $(1 + 2)$ dimensions, we will discuss the ideas of [6] as adapted to that case instead. Notice that the $(1+2)$ dimensional case already requires almost all of the new ideas introduced in [6], with the exception of top-order elliptic estimates for the eikonal function.

[34] In [6], the equations were in fact a subclass of equations, derivable from a Lagrangian, which take the form $(g^{-1})^{\alpha\beta}(\partial\Phi)\partial_\alpha\partial_\beta\Phi = 0$. In particular, the metric depends on the first derivative of the unknown. However, by differentiation, the equation can be transformed into a system of scalar equations of type (2.1.1) that can be studied using essentially the same techniques needed for proving shock formation in solutions to (2.1.1); see the survey article [15] for further discussion. For this reason, we focus here on equation (2.1.1).

very good control on how $\upmu \to 0$. In particular, he proved the following facts for solutions generated by an open set[35] of data.

- The solution remains regular at time t if $\upmu_\star(t) > 0$, where $\upmu_\star(t) := \inf_{\Sigma_t} \upmu$.
- Given appropriate initial conditions (consistent with the assumptions for the solution regime), $\upmu_\star \to 0$ in finite time.
- It can be justified[36] that $\upmu_\star$ approaches 0 *linearly*, a fact which turns out to be crucial for deriving energy estimates.

The blowup-mechanism described above already suggests that the estimates proven for the solutions must take into account appropriate weights of $\upmu$ and $\upmu_\star$. The precise estimates, however, requires further geometric inputs, which we will explain in the next subsubsection.

2.1.2. *The geometric coordinates and the geometric vectorfields.* A second key feature of Christodoulou's proof, which was also found in Alinhac's works [2–4], is that relative to a *geometric coordinate system* (t, u, ϑ), the solution and its low-order partial derivatives remain bounded. That is, *relative to the geometric coordinates, one does not see the shock "singularity."* This suggests the main paradigm for approaching the problem: to the extent possible, prove "long-time-existence-type" estimates for the solution relative to the geometric coordinates and then recover the formation of the shock singularity as a degeneration between the geometric coordinates and the Cartesian ones. Above, t is the Cartesian time coordinate, u is the eikonal function, and ϑ is a geometrically defined coordinate that satisfies the transport equation $(g^{-1})^{\alpha\beta}(\Psi)\partial_\alpha u \partial_\beta \vartheta = 0$; we will downplay the role of ϑ here since it is better to avoid the use of coordinates in most of the analysis.

It turns out that deriving the regularity of the solution relative to the geometric coordinates is equivalent to proving that appropriately $\upmu$-rescaled derivatives of various quantities remain bounded. That is, one may insert factors of $\upmu$ into various estimates in such a way that the vanishing of $\upmu$ exactly compensates for the singularity. One might say that many quantities featured in the problem "blow up like $1/\upmu$." More specifically, the tangential (to the characteristics) derivatives of Ψ remain bounded *without any factor of* $\upmu$ while for the transversal derivative X (see the next paragraph for further discussions), the $\breve{X} := \upmu X$ derivatives of Ψ remain bounded. Furthermore, it was shown that $|\breve{X}\Psi|$ is bounded from below, strictly away from 0, when $\upmu$ becomes 0. Hence, at those points, $X\Psi$ blows up and the solution cannot be extended classically.

To prove that the above picture regarding shock formation holds, Christodoulou introduced an extensive geometric setup, tied to the eikonal function, which we now adapt to the context of the present article: the case of two space dimensions for solutions with approximate plane symmetry. In addition to the geometric coordinates described above, he also introduced geometric vectorfields L, Y and $\breve{X}$ adapted to the characteristics. L is defined to be tangential to the null generators of the $\mathcal{P}_u$, normalized such that $Lt = 1$. Specifically, we have $L^\alpha = -\upmu(g^{-1})^{\alpha\beta}(\Psi)\partial_\beta u$ and moreover, $L = \dfrac{\partial}{\partial t}$ relative to the geometric coordinates. Let $\ell_{t,u}$

[35]By open, we mean relative to a high-order Sobolev topology.

[36]This justification of course relies on a full bootstrap argument, for which the bounds for $\upmu_\star$ have to be obtained simultaneously with all the other estimates.

be the intersections[37] $\Sigma_t \cap \mathcal{P}_u$. Then Y is the g-orthogonal projection of[38] ∂_2 to $\ell_{t,u}$. The vectorfield Y is a replacement[39] for the geometric coordinate partial derivative vectorfield $\frac{\partial}{\partial \vartheta}$. Finally, define $\breve{X}$ to be tangential to Σ_t and g-orthogonal to $\ell_{t,u}$, normalized such that $\breve{X} u = 1$. That is, $\breve{X} = \frac{\partial}{\partial u}$ plus a small error vectorfield that is tangent to $\ell_{t,u}$. Importantly, $\breve{X}$ becomes degenerate (with respect to the Cartesian coordinate vectorfields) as $\mu \to 0$. That is, the Cartesian components $\breve{X}^\alpha$ vanish precisely at the points where $\mu = 0$. On the other hand, the vectorfield $X = \mu^{-1}\breve{X}$ remains non-degenerate, all the way up to the shock. See Figure 3 for a depiction of these vectorfields.

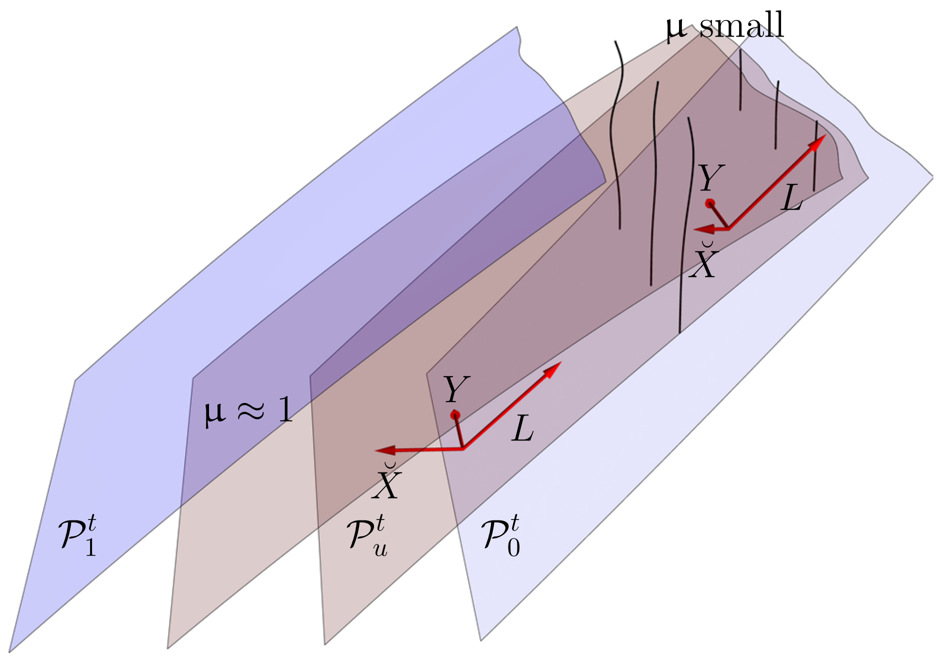

FIGURE 3. The dynamic vectorfield frame at two distinct points in $\mathcal{P}_u^t$ and the integral curves of B

Once these geometric vectorfields have been defined, the problem can be reduced to the following steps:

(1) Prove that Ψ and its lower-order derivatives with respect to the geometric vectorfields L, Y and $\breve{X}$ are appropriately bounded, with estimates that are independent of how small μ is.

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[37] Note that ϑ as defined above is a local coordinate on $\ell_{t,u}$.

[38] Recall that ∂_2 is a Cartesian coordinate partial derivative vectorfield.

[39] It turns out that Y has better regularity properties than $\frac{\partial}{\partial \vartheta}$, which are essential for closing the energy estimates.

(2) Prove that the higher-order derivatives of Ψ with respect to the geometric vectorfields are not too singular[40] in terms of $\mu_\star$. Moreover, show that these not-too-singular estimates can be used to derive the non-singular estimates for the lower-order derivatives of Ψ (as described in point (1) above).

(3) Justify the transport equation[41] (where G_{LL} depends on g and Ψ (cf. Def. 3.20) and the extra terms are small error terms by Step (1))

$$L\mu = \frac{1}{2} G_{LL} \breve{X}\Psi + \dots$$

and prove that $G_{LL}\breve{X}\Psi$ can be precisely controlled in terms of its initial value. Hence, under appropriate negativity assumptions on $G_{LL}\breve{X}\Psi|_{t=0}$, one can guarantee that μ approaches 0 within the time for which the solution is controlled.

2.1.3. *Degenerate energy estimates and the coercive spacetime bulk term.* In carrying out Step (1) of Subsubsect. 2.1.2, the most crucial estimates are of course L^2-based energy estimates. It turns out that in order to handle the shock, one needs to incorporate degenerate weights in the energies. To obtain suitable degenerate energy estimates, we apply the vectorfield multiplier method with the help of the energy-momentum tensor (see (4.2.1)) and the vectorfield multiplier $T = (1 + 2\mu)L + 2\breve{X}$. T has the property that it becomes null and tangential to the characteristics $\mathcal{P}_u$ as μ vanishes. Moreover, the degeneration is chosen precisely so that the energy controls the following quantities Σ_t hypersurfaces (truncated at eikonal function value u):

$$\int_{\Sigma_t^u} \Big((\breve{X}\Psi)^2 + \mu\left((L\Psi)^2 + (Y\Psi)^2\right) \Big)\, d\vartheta\, du. \tag{2.1.4}$$

In particular, only the estimate for $\breve{X}\Psi$ is non-degenerate, by which we mean the energy becomes very weak in $L\Psi$ and $Y\Psi$ along Σ_t when μ is small. On the other hand, the energy identities also yield control over the following quantities on the characteristics $\mathcal{P}_u^t$ (truncated at time t):

$$\int_{\mathcal{P}_u^t} \left((L\Psi)^2 + \mu(Y\Psi)^2 \right) d\vartheta\, dt, \tag{2.1.5}$$

i.e., one obtains a non-degenerate control for $L\Psi$ if one considers the energy flux on constant-u hypersurfaces. Notice that in both (2.1.4) and (2.1.5), the control for $Y\Psi$ is degenerate.

Naively, one might expect that in deriving energy estimates, one encounters terms that are not controllable by the energy itself. This is because proving degenerate energy estimates corresponds to putting μ weights in the "standard" energy estimates, and the weights are differentiated during integration by parts. If the weights were small with large derivatives of an unfavorable sign, then this would lead to potentially insurmountable obstacles to closing the estimates. However, it turns out that by obtaining detailed information about the way that μ behaves along the integral curves of L, one can suitably control the geometric

[40] As it turns out, the scheme in [6] does not show that the high-order derivatives of Ψ with respect to the geometric vectorfields are bounded. Of course, as we will explain in great detail below, the possible blowup of the solution's high-order derivatives is the source of many difficulties in the problem.

[41] In order to guarantee that shock forms, we need $G_{LL} \neq 0$. This can be viewed as a condition on the Cartesian components $g_{\alpha\beta}$, viewed as a function of Ψ.

derivatives of the $\upmu$ weights in the energy. Moreover, as was first observed by Christodoulou in his work [6], one of the spacetime terms in the energy identities in fact has a *good sign* and is bounded below by

$$\int_{\mathcal{M}_{t,u}} [L\upmu]_{-}(Y\Psi)^2 \, d\vartheta \, du \, dt. \tag{2.1.6}$$

It can be proven that when $\upmu$ is sufficiently small, then the negative part $[L\upmu]_{-}$ is *bounded below* and the integrated term above is non-degenerate and coercive. That is, one can quantify the following heuristic statement for the solution regime under consideration: the only way that $\upmu$ can become small is for $L\upmu$ to be sufficiently negative. It is this crucial observation that allows $Y\Psi$ to be controlled without degeneration.

One must also obtain similar energy estimates for the higher-order derivatives of Ψ. In order to prove estimates consistent with the expected shock formation picture, only the geometric vectorfields can be used as commutators to derive higher-order estimates; the Cartesian coordinate partial derivative vectorfields would generate uncontrollable error terms if they were used to commute the wave equation since they are generally transversal to the $\mathcal{P}_u$ and are not $\upmu$-weighted. The main technical difficulty that one encounters is that some of the commutator error terms are exceptionally difficulty to control. The reason is that the commutator terms depend on the derivatives of the vectorfields and thus, in view of their connection to the characteristics, on the derivatives of the eikonal function. As we describe in the next subsubsection, it turns out that one must work hard to avoid losing derivatives in the most difficult of these terms and, crucially, that avoiding the derivative loss comes with a price: it introduces a dangerous factor of $1/\upmu$ into the energy identities, which leads to energy estimates that are allowed to blow up in terms of powers of $\upmu_\star^{-1}$ as the shock forms.

2.1.4. *Top-order estimates for the eikonal function.* While the use of tensorfields adapted the characteristics, especially geometric commutation vectorfields, is necessary to prove shock formation, a naive implementation of this framework leads to a loss of derivatives that threatens to obstruct the closure of the energy estimates. The difficulty is that the commutator of[42] $\upmu\Box_g$ and the geometric vectorfields generates error terms that depend on the third derivatives of the eikonal function u, which, as is suggested by the eikonal equation (2.1.2), can be controlled only by obtaining control over three derivatives of Ψ. On the other hand, after one commutation of the wave equation $\upmu\Box_g\Psi = 0$ with the geometric vectorfields, only two derivatives of Ψ can be estimated. Nevertheless, as is known since the works[43] [7, 20], one can exploit the fact that the Cartesian components of the metric $g(\Psi)$ also satisfy a wave equation[44] to gain a derivative for certain special combinations of third derivatives of u and second derivatives of Ψ. The gain is tensorial in nature, and it is a happy fact that the vectorfields L, $\breve{X}$, and Y generate only commutation error terms featuring those special combinations.

[42]It turns out that in our proof, we must commute the weighted operator $\upmu\Box_g$ in order to avoid generating uncontrollable error terms.

[43]The work [7] exploited this gain of a derivative in the specific case of the Einstein vacuum equations, for which this structure is more easily seen. Nevertheless, the ideas in [7] already serve as a blueprint for gaining the derivative in the context of more general quasilinear wave equations.

[44]This claim is a simple consequence of the chain rule applied to the component functions $g_{\alpha\beta}(\Psi)$.

One example (in fact, the most important example in the problem) of a controllable error term depending on the eikonal function is the null mean curvature ${\mathrm{tr}}_{\not g}\chi$ of the characteristics $\mathcal{P}_u$. It turns out that after commuting the wave equation one time with geometric vectorfields, one encounters the first derivatives of ${\mathrm{tr}}_{\not g}\chi$, which, as we alluded to above, we must carefully treat to avoid losing a derivative. We now explain how to avoid this derivative loss. For convenience, instead of addressing this difficulty at the level of one commutation of the wave equation, we consider the analogous difficulty at the level of zero commutations. That is, we explain how to control the undifferentiated quantity ${\mathrm{tr}}_{\not g}\chi$ in terms of one derivative of Ψ (which is the allowed regularity for Ψ without commuting). The source of the difficulty is that ${\mathrm{tr}}_{\not g}\chi$ depends on two derivatives of the eikonal function u. This seems to be incompatible with the available regularity of Ψ since a general second derivative of u has to be estimated by two derivatives of the metric (and hence two derivatives of Ψ). However, ${\mathrm{tr}}_{\not g}\chi$ is a special combination of two derivatives of the eikonal function u and one can "gain" in derivatives with the following procedure. First, one derives the following transport equation for ${\mathrm{tr}}_{\not g}\chi$ (well-known in general relativity as the Raychaudhuri equation):

$$\mu L {\mathrm{tr}}_{\not g}\chi = (L\mu){\mathrm{tr}}_{\not g}\chi - \mu({\mathrm{tr}}_{\not g}\chi)^2 - \mu \mathrm{Ric}_{LL}, \tag{2.1.7}$$

where Ric_{LL} is the LL-component of the spacetime Ricci curvature of g. The second main observation is that Ric_{LL} is equal to a sum of terms controllable by only one derivative of Ψ and a term which can be written as $L\mathfrak{X}$, where $\mathfrak{X}$ can be expressed in terms of at most one derivative of Ψ. That this can be achieved crucially depends on the wave equation for Ψ. More precisely, using the wave equation $\mu\square_g\Psi = 0$, one can replace the second derivative term[45] $\mu\not\Delta\Psi$, which appears in the expression of $\mu\mathrm{Ric}_{LL}$, with $L(\mu L\Psi + 2\breve{X}\Psi)$ (and lower-order terms). As a consequence, instead of directly studying ${\mathrm{tr}}_{\not g}\chi$, we can instead study the *modified quantity* $\mu{\mathrm{tr}}_{\not g}\chi + \mathfrak{X}$. The key point is that the right hand side of the transport equation $L(\mu{\mathrm{tr}}_{\not g}\chi + \mathfrak{X})$ now depends on at most one derivative of Ψ. In total, this procedure allows us to control ${\mathrm{tr}}_{\not g}\chi$ using estimates for one derivative of Ψ only, which is better than what one would naively expect.

On the other hand, as the shock is approached, this procedure of gaining derivatives is coupled with the difficulty of[46] $\mu_\star \to 0$. This is because the modified quantity is $\mu{\mathrm{tr}}_{\not g}\chi + \mathfrak{X}$, where $\mathfrak{X}$ is merely bounded. Hence, to recover estimates for (the higher-order derivatives of) ${\mathrm{tr}}_{\not g}\chi$ from estimates for (the higher-order derivatives of) the modified quantity, one faces the critical difficulty of a discrepancy factor of $1/\mu$; this discrepancy is central to most of the difficulties that one faces in closing the problem.

We now illustrate how this difficulty enters into the top-order energy estimates for Ψ by keeping one of the most significant terms.[47] This leads to an estimate of the following form for the top-order energy $\mathbb{E}_{top}$ on Σ_t^u:

$$\mathbb{E}_{top}(t,u) \leq \mathrm{Data} + C_{fix}\int_{t'=0}^{t'=t}\left(\sup_{\Sigma_{t'}^u}\left|\frac{L\mu}{\mu}\right|\right)\mathbb{E}_{top}(t',u)\,dt + \ldots, \tag{2.1.8}$$

[45] Here, $\not\Delta$ denotes the Laplacian with respect to the Riemannian metric induced by $g(\Psi)$ on $\ell_{t,u} = \Sigma_t \cap \mathcal{P}_u$.

[46] Recall that $\mu_\star$ is the minimum of μ on a constant-t hypersurface.

[47] Notice that there are other terms which are of the same strength (from the point of view of the singularity) as the term that is shown.

where[48] $C_{fix} > 0$ is a *fixed* constant independent of how many derivatives we choose to be the "top level". To proceed, one needs very precise estimates for $\upmu$ and $L\upmu$ and to show that $\upmu_\star$ tends to 0 *linearly*. This implies,[49] via a difficult analog of Gronwall's inequality that relies on the sharp information for $\upmu_\star$, the estimate[50]

$$\mathbb{E}_{top}(t,u) \leq \text{Data} \times \upmu_\star^{-c_{fix}}(t,u), \tag{2.1.9}$$

for some *universal constant* $c_{fix} > 0$ *that is independent of the structure of the nonlinearities.*

2.1.5. *Energy hierarchy and the descent scheme.* In the previous subsubsection, we saw that in order not to lose derivatives, the top level energy estimates must degenerate in terms of $\upmu_\star$. To finish the argument, Christodoulou introduced a *descent scheme* in which he showed that for every order below the top-order, the degeneration can be improved by a fixed amount. In particular, at some sufficiently low-order of derivatives, the energy can be shown to be bounded. This then also yields, by a geometric Sobolev embedding estimate, the necessary low-order L^∞ estimates that allow the argument to be closed.

Let us describe[51] the relevant numerology in the adaption of [6] to the present paper. One proves estimates of the type

$$\sqrt{\mathbb{E}_{15+K}}(t,u) \leq C\mathring{\upepsilon}\upmu_\star^{-(K+.9)}(t,u), \qquad (0 \leq K \leq 5), \tag{2.1.10a}$$

$$\sqrt{\mathbb{E}_{N}}(t,u) \leq C\mathring{\upepsilon}, \qquad (1 \leq N \leq 14), \tag{2.1.10b}$$

where $\mathbb{E}_N$(t,u) denotes the energy on Σ_t^u after N commutations. Here, $\mathbb{E}_{20}$ is the top-order energy, and it can be controlled using the approach described in the previous subsubsection. On the other hand, when controlling $\mathbb{E}_{19}$, one can control the highest order (that is, 19^{th}) derivative of $\mathrm{tr}_{\not{g}}\chi$ appearing in the energy estimates by $\mathbb{E}_{20}$ instead of $\mathbb{E}_{19}$. Put differently, below-top-order, one can simply allow the loss of a derivative and avoid using the modified version of $\mathrm{tr}_{\not{g}}\chi$. In this way, one avoids introducing an explicit singular factor of $1/\upmu$ into the below-top-order energy estimates, at the expense of introducing a coupling to the energy at one higher level. A key point, which was exploited by Christodoulou in controlling the energies at all derivative levels, is that $\upmu_\star \to 0$ at worst linearly (with precise estimates). This in particular allows one to show that every integration in t reduces the strength of the singularity by a power of $\upmu_\star$. These "descent estimates," though exceptionally technical to implement, are nothing other than a "quasilinear version" of the estimate $\int_{s=0}^t s^{-b}\,ds \lesssim t^{1-b}$ (for $b > 1$), where $s = 0$ represents the "vanishing" of $\upmu_\star$.

To be more concrete, let us consider the most difficult inhomogeneous term on the right-hand side of the equation $\upmu\square_{g(\Psi)}Y^{19}\Psi = \cdots$, which modulo bounded factors is the term

[48] In the proof of our main theorem, C_{fix} will in fact be an explicit numerical constant.

[49] We also note that in closing the top-order energy estimates, one must perform some crucially important integrations-by-parts in time that lead to singular boundary terms that must be controlled. We will suppress this technical difficulty here in order to keep the discussion short, instead referring readers to Subsect. 15.9 for details regarding this estimate.

[50] To obtain some heuristic understanding of why (2.1.9) follows from (2.1.8), one may replace $\upmu$ and $\upmu_\star$ with $1-t$ and $L\upmu$ with -1. Then the standard Gronwall inequality yields (2.1.9) with $c_{fix} = C_{fix}$.

[51] We do not directly describe the numerology of [6] here as it is slightly different from that of the present paper and doing so might create some confusion. On the other hand, the main ideas can be traced back to [6].

$(LY^{19}\Psi)(Y^{19}\mathrm{tr}_{\not g}\chi)$. Since the control for $LY^{19}\Psi$ on a constant t-hypersurface in $\mathbb{E}_{19}^{\frac{1}{2}}$ is $\mu^{\frac{1}{2}}$ degenerate (recall (2.1.4)), these considerations lead to the estimate

$$\mathbb{E}_{19}(t,u) \leq \mathrm{Data} + \int_0^t \mu_\star^{-\frac{1}{2}} \mathbb{E}_{19}^{\frac{1}{2}}(t',u) \|Y^{19}\mathrm{tr}_{\not g}\chi\|_{L^2(\Sigma_{t'}^u)} dt' + \ldots$$

To estimate $Y^{19}\mathrm{tr}_{\not g}\chi$, one can again use (2.1.7), but this time directly controlling Ric_{LL}. Since $\mathbb{E}_{20}^{\frac{1}{2}}$ again has a $\mu^{\frac{1}{2}}$ degeneration for the Y derivative, integrating (2.1.7) yields

$$\|Y^{19}\mathrm{tr}_{\not g}\chi\|_{L^2(\Sigma_{t'}^u)} \leq \mathrm{Data} + \int_{t''=0}^{t''=t'} \mu_\star^{-\frac{1}{2}} \mathbb{E}_{20}^{\frac{1}{2}}(t'',u)\, dt'' + \ldots \tag{2.1.11}$$

Substituting this back into the estimate for $\mathbb{E}_{19}$ gives

$$\mathbb{E}_{19}(t,u) \leq \mathrm{Data} + \int_0^t \mu_\star^{-\frac{1}{2}} \mathbb{E}_{19}^{\frac{1}{2}}(t',u) \int_{t''=0}^{t''=t'} \mu_\star^{-\frac{1}{2}} \mathbb{E}_{20}^{\frac{1}{2}}(t'',u)\, dt'' dt' + \ldots \tag{2.1.12}$$

Now since $\mu_\star$ tends to 0 at worst linearly, we have the following estimate, which we alluded to above: for every $b > 1$, we have

$$\int_{t'=0}^{t'=t} \mu_\star^{-b}(t')\, dt \leq C\mu_\star^{-b+1}(t). \tag{2.1.13}$$

Therefore, (2.1.12) is indeed consistent[52] with the reduced blowup-rate for $\mathbb{E}_{19}$ compared to $\mathbb{E}_{20}$, as stated in (2.1.10a). One can continue the descent and show that the blowup-rates for the energies continues to improve as the number of derivatives is reduced, until one actually obtains boundedness of the lower-order energies.

As we can see from the above scheme, the number of derivatives that is needed for this descent scheme depends on the strength of the top-order singularity, represented by the constant c_{fix} in (2.1.9). In turn, c_{fix} depends on the constant in C_{fix} in (2.1.8).

2.1.6. *Formation of shocks.* Once one closes all the estimates, the formation of shocks follows easily. Indeed, with the estimates at hand, it is easy to conclude the solution remains regular relative to both[53] geometric and Cartesian coordinates as long as $\mu > 0$ and that $\mu \to 0$ corresponds indeed to a shock. Moreover, the non-degenerate low-level energy estimates imply, via Sobolev embedding, non-degenerate low-level L^∞ estimates that lead to

$$L\mu = \frac{1}{2} G_{LL} \breve{X}\Psi + \ldots,$$

where $\ldots$ denotes small error terms and also that $G_{LL}\breve{X}\Psi$ is essentially transported along the integral curves of L. Therefore with appropriate negativity assumptions on $G_{LL}\breve{X}\Psi|_{t=0}$, it is easy to prove that μ goes to 0 in finite time.

[52]To actually close the estimates, one needs to derive a Gronwall estimate for a coupled system featuring $\mathbb{E}_{19}$ and $\mathbb{E}_{20}$. We refer the readers to Subsect. 15.16 for the relevant details in the context of the present article.

[53]Indeed, the change of variables map from geometric to Cartesian coordinates is a diffeomorphism when $\mu > 0$.

2.2. **Review of the stability of shock formation for nearly simple outgoing plane symmetric solutions to quasilinear wave equations.** Together with Holzegel and Wong, we proved [30] stable shock formation for "nearly simple outgoing plane symmetric" solutions to[54] a class of quasilinear wave equations in two spatial dimensions. In this paper, we study a similar regime of nearly simple outgoing plane symmetric solutions. More precisely, we extend the results of [30] to the compressible Euler's equations *without the irrotationality assumption.*[55]

We now describe the case of *exact* simple outgoing plane symmetric solutions. We first recall that $(1+1)$-dimensional quasilinear wave equations can be greatly simplified using the conformal invariance of $\square_g$ and the fact that $(1+1)$-dimensional Lorentzian manifolds are (locally) conformally flat. Indeed, defining appropriate null functions u and w satisfying the eikonal equation

$$(g^{-1})^{\alpha\beta}\partial_\alpha u \partial_\beta u = (g^{-1})^{\alpha\beta}\partial_\alpha w \partial_\beta w = 0,$$

it is easy to show the quasilinear wave equation $\square_{g(\Psi)}\Psi = 0$ is equivalent to

$$\partial_w \partial_u \Psi = 0.$$

Note that the equation is still quasilinear as u and w depend on Ψ. We say that a solution Ψ is a *simple outgoing*[56] if $\partial_w \Psi = 0$. When expressed in terms of Ψ alone, $\partial_w \Psi = 0$ can be viewed as a Burger's-type equation.

In [30], the authors studied $(1+2)$-dimensional quasilinear wave equations. The equations admit plane symmetric solutions which do not depend on the Cartesian spatial coordinate x^2. Analogous to the $(1+1)$-dimensional case, they can be written as

$$\partial_w \partial_u \Psi = \mathcal{N},$$

where $\mathcal{N} = \mathcal{N}(\Psi, \partial\Psi)\partial_u \Psi \cdot \partial_w \Psi$ is a null form relative to g, and a solution is said to be *simple outgoing* if $\partial_w \Psi = 0$. It is not difficult to see that under a condition of genuine nonlinearity, there exist simple outgoing solutions for which shocks form in finite time. The main result of [30] is that a subclass of such shock-forming solutions to equation (2.1.1) is stable under *non-symmetric* perturbations. Proving this result requires, in addition to the ideas of [6] described in the previous subsection, a method to propagate smallness parameters relevant to this solution regime.[57]

In order to achieve this, the authors introduced the parameters[58] $\mathring{\delta}_*$, $\mathring{\delta}$ and $\mathring{\epsilon}$ to describe the relative sizes of the derivatives of Ψ. Here, $\mathring{\delta}_*$ and $\mathring{\delta}$ are not necessarily small: $\mathring{\delta}$ describes the size of the *transversal* (to the characteristics) derivatives of the data of Ψ and $\mathring{\delta}_*^{-1}$ is the

[54]In fact, similar methods could be used to show that the solutions are stable under non-symmetric perturbations in *three* spatial dimensions; see the discussion in [30].

[55]See Subsect. 2.3 for discussion on the relation between the class of equations discussed here and the compressible Euler equations.

[56]We use the term "outgoing" to mean that the solution travels towards the "right." That is, we have used the convention that initially $\frac{\partial w}{\partial x} > 0$. Of course, the restriction to outgoing waves is merely for notational convenience, as the analysis remains identical if we instead consider "incoming" solutions.

[57]In particular, this is in contrast to [6], where dispersion was crucially used to propagate smallness. The lack of dispersion in [30] requires the introduction of the $\mathring{\delta}_*$-$\mathring{\epsilon}$ size hierarchy of the initial data (to be described below), but it turns out that to propagate that smallness is slightly less involved than that in [6].

[58]These parameters are closely related to those introduced in Subsect. 1.2. See Subsect. 2.3 for further discussion.

"expected the blow up time", which depends on the first transversal derivative of Ψ at time 0 (compare with Definition 8.1). On the other hand, $\mathring{\epsilon}$, which, roughly speaking, describes the size of the initial L^∞ norm of Ψ as well as its initial outgoing derivatives and its derivatives in the direction ∂_2, is required to be small compared to $\mathring{\delta}_*$ and $\mathring{\delta}^{-1}$. In order to more precisely describe the smallness of the initial data, we will again use the geometric vectorfields L, Y and $\breve{X}$ described in the previous subsection.[59] In [30], the geometric derivatives of Ψ at time 0 are required to be $\mathring{\epsilon}$-small whenever *at least one* of the differentiations is in the direction of L or Y. It is straightforward to show that this initial smallness follows whenever the initial data are $\mathring{\epsilon}$-perturbations of simple outgoing plane symmetric solutions.

In order to control the solution, we do not need to explicitly subtract the simple outgoing plane symmetric solution from the full nonlinear solution. Instead, we show that the solution remains nearly plane symmetric and nearly simple outgoing up to the time of first shock formation in the sense that

> For the derivatives of Ψ with respect to the geometric vectorfields, if *at least one* of the vectorfields is L or Y, then the quantity in an appropriate norm is $\mathcal{O}(\mathring{\epsilon})$ small, all the way up to the shock.[60]

In other words, the smallness of the $\mathcal{P}_u^t$-tangential derivatives of Ψ, which is originally assumed for the initial data, is propagated by the flow. Notice that in this process, not only do we use the geometric vectorfields L, Y and $\breve{X}$ to capture the formation of shocks, we also use them to track the smallness in the problem. The following geometric and analytic properties are crucial in order to achieve this:

(1) (Commutation properties of the geometric vectorfields) We stress that we have crucially used the property that even if Ψ is $\breve{X}$-differentiated, as long it is also hit with one or more L or Y derivatives (for instance for the quantities $L\breve{X}\Psi$, $\breve{X}LL\Psi$, etc), then the quantity is still $\mathcal{O}(\mathring{\epsilon})$ small. That this holds of course relies on good commutation properties of the geometric vectorfields, in particular that the commutator of any two of $\{L, \breve{X}, Y\}$ is *tangential to the characteristics* $\mathcal{P}_u$ (in fact, the commutators may be seen to be tangent to $\ell_{t,u} = \Sigma_t \cap \mathcal{P}_u$)!

(2) (Null structure of the nonlinear terms when decomposed with respect to geometric vectorfields) The wave equation (2.1.1) is equivalent to (see Proposition 3.16)

$$- L(\upmu L\Psi + 2\breve{X}\Psi) + \upmu \slashed{\Delta}\Psi = \mathcal{N}, \tag{2.2.1}$$

where $\mathcal{N}$ denotes nonlinear terms with *at most one factor transversal* to $\mathcal{P}_u$, that is, with at most one factor equal to $\breve{X}\Psi$. Consequently, under appropriate bootstrap assumptions, the term $\mathcal{N}$ can be shown to be $\mathcal{O}(\mathring{\epsilon})$-small.[61] Notice that this smallness partly comes from the geometry associated to the problem. For instance, one of the terms in $\mathcal{N}$ is $\mathrm{tr}_{\slashed{g}}\chi\breve{X}\Psi$ (where as before, $\mathrm{tr}_{\slashed{g}}\chi$ is the null mean curvature of the $\mathcal{P}_u$).

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[59]Here, one could think of L as an analogue of ∂_w in the exact plane symmetric case and Y as in the direction ∂_2.

[60]We recall from the previous subsection that some higher-order norms are allowed to blow up. Therefore, the $\mathcal{O}(\mathring{\epsilon})$ "smallness" of the higher-order energies must be carefully interpreted as smallness relative to singular norms.

[61]The implicit constants are allowed to depend on $\mathring{\delta}$.

It was shown that $\mathrm{tr}_{\not g}\chi$ is $\mathcal{O}(\mathring{\varepsilon})$-small, which is a consequence of approximate plane symmetry the solution.[62]

(3) (Commutation properties between the geometric vectorfields and $\mu\square_g$))In order to prove size $\mathcal{O}(\mathring{\varepsilon})$ estimates for the (higher-order) L and Y derivatives of Ψ, we rely on the fact that the commutators terms[63] $[\mu\square_g, L]\Psi$ and $[\mu\square_g, Y]\Psi$ are $\mathcal{O}(\mathring{\varepsilon})$-small. Moreover, we also use the fact that $[\mu\square_g, L]\Psi$ and $[\mu\square_g, Y]\Psi$ do not generate $\breve{X}\breve{X}\Psi$ terms.[64] These can be viewed as a consequence of the commutation properties described in the first point above.

Using the above properties, we carry out our estimates as follows:

(1) (Higher-order energy estimates) For the energy estimates, we only use L and Y as commutators. We also only carry out the energy estimates after at least one commutation. Notice that this is sufficient from the point of view of regularity since $[\mu\square_g, L]$ and $[\mu\square_g, Y]$ do not generate $\breve{X}\breve{X}$ terms! Moreover, the energy corresponding to commuting the wave equation with one or more factors of L or Y is initially $O(\mathring{\varepsilon}^2)$-small, which is convenient[65] for deriving estimates.

(2) (Estimates for the eikonal function) The estimates for the eikonal function u up to the highest order are intimately tied to the energy estimates. For example, we show that $\mathrm{tr}_{\not g}\chi$ (and its higher-order L and Y derivatives) inherits the $\mathcal{O}(\mathring{\varepsilon})$ smallness from the energy estimates, as is expected since the solution is nearly outgoing simple plane symmetric.[66] Note that in order to implement steps (1) and (2), it is important that one can close the energy estimates and the estimates for the eikonal function (which, as we described above, are highly coupled!) by commuting only with L and Y.

(3) (Lower-order estimates for $\breve{X}\Psi$, $\breve{X}L\Psi$, $\breve{X}\breve{X}\Psi$, $\breve{X}\breve{X}\breve{X}\Psi$, etc.) Since we only derive energy estimates after commuting with at least one factor of L or Y, our energies cannot be directly combined with Sobolev embedding to yield pointwise control of $\breve{X}\Psi$. To obtain pointwise control of $\breve{X}\Psi$, we use the wave equation in the form (2.2.1) as a transport equation in the unknown $\breve{X}\Psi$. The pointwise estimates for $\breve{X}L\Psi$, $\breve{X}\breve{X}\Psi$, $\breve{X}\breve{X}\breve{X}\ Psi$, etc. are obtained in a similar manner, after commuting the wave equation. Notice that some of these terms, for instance $\breve{X}\Psi$, are of relatively large size $\mathring{\delta}$, but this size can be propagated since the error terms in (2.2.1) are all of smaller size $\mathcal{O}(\mathring{\varepsilon})$. In other words, in the nonlinear error terms, we never encounter, say, quadratic terms of size $\mathring{\delta}^2$.

62Notice also that this smallness is tied to our foliation of spacetime by the nearly flat characteristics $\mathcal{P}_u$. One might say that we made an "educated" guess about how to construct a foliation that allows us to propagate the smallness.

63The factor of μ generates important cancellations.

64Dimensional considerations imply that these terms, if present, would be multiplied by an uncontrollable factor of $1/\mu$. Furthermore, the absence of these terms in the commutators is also useful for the higher-order energy estimates; see point (1) below.

65The energy of the non-commuted equation is lower-bounded by the square of the L^2 norm of $\breve{X}\Psi$, which can be of a relatively large size $\mathring{\delta}^2$.

66Note that for exact outgoing simple plane symmetric solutions, we have $\mathrm{tr}_{\not g}\chi = 0$.

(4) (Sharp control of $\upmu$) Using the smallness above, we show that $\upmu$ satisfies the transport equation

$$L\upmu = \frac{1}{2} G_{LL} \breve{X}\Psi + \mathcal{O}(\mathring{\epsilon}),$$

where G_{LL} is a function of Ψ depending on the Cartesian components $g_{\alpha\beta}$. This equation, together with a precise estimate for $G_{LL}\breve{X}\Psi$, are crucial for obtaining sharp control[67] of $\upmu$ and $L\upmu$ and for showing that a shock indeed forms in finite time.

2.3. **Nearly simple outgoing plane symmetric solutions to irrotational Euler equations.** As was already discussed in the paper [30] on quasilinear wave equations of the type $\square_{g(\Psi)}\Psi = 0$, with very few modifications, the same methods can be used to prove shock formation in solutions to quasilinear equations of the form

$$(g^{-1})^{\alpha\beta}(\partial\Phi)\partial_\alpha\partial_\beta\Phi = 0. \tag{2.3.1}$$

This can be seen by considering the vector $\vec{\Psi} := (\Psi_\nu)_{\nu=0,1,2} := (\partial_\nu\Phi)_{\nu=0,1,2}$, differentiating (2.3.1) and deriving the system of equations

$$\square_{g(\vec{\Psi})}\Psi_\nu = \mathcal{Q}(\partial\vec{\Psi}, \partial\vec{\Psi}_\nu), \tag{2.3.2}$$

where $\square_{g(\vec{\Psi})}$ is to be understood as acting on scalar functions and the inhomogeneous terms $\mathcal{Q}$ are quadratic *null forms* (relative to g). Both the vectorial nature of the unknown and the additional nonlinear terms pose almost no additional challenge and (2.3.2) can be treated with essentially the same methods as the scalar equation (2.1.1). This in particular crucially relies on the structure of the null forms, which have only a negligible influence on the solution, all the way up to the shock.

To handle the equation (2.3.1), Speck–Holzegel–Luk–Wong [30] considered initial data such that each of the components Ψ_ν (which are viewed as scalar functions) obey the $\mathring{\epsilon}$-$\mathring{\delta}$ size estimates as described in Subsect. 2.2. Notice that, in view of the identity $\partial_\alpha\Psi_\beta = \partial_\beta\Psi_\alpha$, these assumptions imply smallness estimates for the $\breve{X}$ derivative of certain combinations of the Ψ_ν. We note that the result in [30] can be applied to the irrotational compressible Euler equations.[68] More precisely, by introducing a potential function Φ for the flow, we obtain an equation of the form (2.3.1).

Let us clarify the connection between the Riemann invariants (see Subsect. 1.2), our assumption that we are studying perturbations of simple plane waves, and the size assumptions on the potential Φ described in the previous two paragraphs. We first note that in the non-relativistic case, we have, relative to the Cartesian spatial coordinates,

$$\partial_i\Phi = -v^i.$$

Next, we note that in one spatial dimension, we have (see [9])

$$\partial_t\Phi - \frac{1}{2}(\partial_1\Phi)^2 = h, \tag{2.3.3}$$

[67]Let us recall from the previous subsection that we crucially need to show that $\upmu \to 0$ at worst linearly and also to prove that $[L\upmu]_-$ is bounded from below whenever $\upmu$ is sufficiently small.

[68]There is an explicit justification of this fact in [30] for the *relativistic* Euler equations. It can easily be seen that this also applies to the non-relativistic case.

where h is the enthalpy, defined such that $dh = \frac{c_s^2}{\rho}\,d\rho = c_s^2\,d\uprho$. The assumption (stated in the previous paragraph) that $L\Psi_1$ should initially be of small size $\mathcal{O}(\mathring{\upepsilon})$ (where $\Psi_1 := \partial_1\Phi$) can be stated as $L_{(plane)}\Psi_1 = \mathcal{O}(\mathring{\upepsilon})$ where, in one spatial dimension, $L_{(plane)} = L = \partial_t + (v + c_s)\partial_1$ is the outgoing null vectorfield with $L_{(plane)}t = 1$. This smallness assumption implies, via (2.3.3) and the formula $\mathcal{R}_- = v^1 - \int_{\widetilde{\uprho}=0}^{\uprho} c_s(\widetilde{\uprho})\,d\widetilde{\uprho}$, that, at time 0, we have $-\partial_1\mathcal{R}_- = \partial_1\left(-v_1 + \int_{\widetilde{\uprho}=0}^{\uprho} c_s(\widetilde{\uprho})\,d\widetilde{\uprho}\right) = \partial_1^2\Phi + c_s\partial_1\uprho = \partial_1^2\Phi + c_s^{-1}\partial_1 h = \partial_1^2\Phi + c_s^{-1}\partial_t\partial_1\Phi - c_s^{-1}(\partial_1\Phi)\partial_1^2\Phi = c_s^{-1}L_{(plane)}\Psi_1 = \mathcal{O}(\mathring{\upepsilon})$. Therefore, the smallness assumption implies that the derivatives of the Riemann invariant $\mathcal{R}_-$ are initially small, which is a perturbation of the simple plane wave case $\mathcal{R}_- \equiv 0$ described in Subsect. 1.2.

Finally, recalling the discussions of the initial data in Subsect. 1.2, we see that the data considered in this paper are a generalization of that considered in [30] such that in this paper, the *vorticity is not required to vanish identically.*

2.4. **New ideas in the case of non-vanishing vorticity.** We are now ready to discuss the main new ideas in the present paper, which are needed to handle the interaction between the vorticity and the sound waves. We recall that we described our assumptions on the initial data in Subsect. 1.2. In this paper, we also need all of the ideas as described in Subsects. 2.1 and 2.2, although for the sake of brevity we will often not repeat them. In particular, we mostly suppress in this subsection the issue of propagating L^∞-type smallness at the low derivative levels; most of the ideas in that regard are similar to those discussed in Subsect. 2.2. Instead, we focus on the crucially important issue of closing the energy estimates.

As we mentioned earlier, the starting point of our proof is the following reformulation of the compressible Euler equations, valid in two spatial dimensions:

$$\upmu\Box_g v^i = -[ia](\exp\uprho)c_s^2(\upmu\partial_a\upomega) + 2[ia](\exp\uprho)\upomega(\upmu B v^a) + \upmu\mathscr{Q}^i, \qquad (2.4.1a)$$

$$\upmu\Box_g\uprho = \upmu\mathscr{Q}, \qquad (2.4.1b)$$

$$\upmu B\upomega = 0. \qquad (2.4.1c)$$

Here, $B = \partial_t + v^a\partial_a$ (as before), g is the acoustical metric depending on v^i and $\uprho$ (see Definition 3.3 for the precise definition), $\upomega = (\partial_1 v^2 - \partial_2 v^1)/\exp(\uprho)$ is the specific vorticity, and $\mathscr{Q}$ and $\mathscr{Q}^i$ are *null forms relative to* g (which we sometimes refer to as g-null forms); see Proposition 3.1 for precise definitions. It turns out that the g-null form structure is crucially important. In contrast to a g-null form, a typical quadratic term could severely distort the dynamics near the shock and could in principle prevent it from forming; see Remark 3.2 for further comments. Thus, under this new formulation, we need to consider a coupled system of three quasilinear wave equations and one transport equation.[69]

Remark 2.1 (**Avoiding vacuum regions**)**.** In this article, we show that the solutions under study have densities that are from bounded from below, strictly away from 0. We

[69]Notice that this system is in principle over-determined, but its local-in-time well-posedness follows from that of the Euler equations (1.0.1a)-(1.0.1b).

therefore avoid the difficult problem of studying the dynamics of a fluid containing vacuum regions and hence there is no difficulty in dividing by the density to form the specific vorticity.

Our approach is to treat the wave part of the system using the ideas from [30] and to handle the additional terms involving the specific vorticity within the same geometric framework. In particular, we prove estimates for the geometric vectorfield derivatives of the specific vorticity. The following are the main tasks that we must accomplish:

- Make sure that we can control all terms at a consistent level of derivatives (that is, without derivative loss). As we have mentioned, as part of this scheme, we must show that the specific vorticity has the same differentiability as the velocity and density, representing a gain of one derivative.
- Understand the expected blowup-rate of the L^2 norms of all quantities at all orders in terms of powers of $\mu_\star^{-1}$; see Subsubsect. 2.4.7 for a summary of the blowup-rates. As before, we must distinguish between the top-order and the below-top-order energy estimates. An notable feature of the present work is that the top-order derivatives of the specific vorticity are allowed to blow up at a worse rate than any of the terms that arise in the irrotational case; see (2.4.13b). However, in the coupling to the wave equation, the top-derivatives of the specific vorticity appear as a source term multiplied by a *critically important factor of* μ (see the term $\mu\partial_a\omega$ on RHS (2.4.1a)). This factor of μ turns out to be enough to compensate for the especially singular behavior of the top-order derivatives of ω. We also note that we must ensure the viability of the energy descent scheme (see Subsubsect. 2.1.5) so that, in particular, we can obtain non-degenerate energy and L^∞ estimates at the low derivative levels.

Remark 2.2 (**An alternate approach to controlling the top-order derivatives of** ω)**.** Although we do not use it in the present article, there is alternate approach to controlling the top-order derivatives of ω. Specifically, one could differentiate the transport equation $B\omega = 0$ with the spatial Cartesian coordinate partial derivative vectorfields ∂_i to obtain the evolution equation $B\partial_i\omega = -(\partial_i v^a)\partial_a\omega$. One could then think of the quantities $\partial_i\omega$ as new variables that need to be controlled, in addition to ω. Although this approach would involve some additional analysis compared to analysis carried out here, the advantage would be that we could close the transport equation energy estimates by commuting the transport equations only up to 20 times with geometric vectorfields, as opposed to the approach of the present article, which relies on commuting the equation $B\omega = 0$ up to 21 times. In carrying out this alternate strategy, one would avoid generating error terms in the transport equations that depend on the top-order derivatives of the eikonal function. In particular, this would allow us to avoid the most singular terms and to therefore derive less degenerate estimates for ω at the top-order compared to the estimates that we obtain in this article. In our forthcoming work [24] on shock formation with vorticity in three spatial dimensions, it turns out that we are forced to employ a closely related strategy and, as we mentioned earlier, to complement it with elliptic estimates. The reason is that in three spatial dimensions, the evolution equation verified by ω is no longer homogeneous, but rather $B\omega^i = \omega^a\partial_a v^i$ (recall that in three spatial dimensions, ω is a Σ_t-tangent vectorfield). Thus, the simplified approach of the present article, which is based on commuting the *homogeneous* transport equation for ω up to top-order with geometric derivative vectorfields, would result in the loss of a derivative in three spatial dimensions (coming from the term generated when all derivatives fall on the factor $\partial_a v^i$ in the product $\omega^a \partial_a v^i$).

- Make sure that the expected blowup-rates are consistent in the sense that the coupling does not spoil the expectation. By coupling, we mean coupling between the "wave variables" v^i and ρ, the specific vorticity ω, and the acoustic geometry (that is, the eikonal function), which enters into the analysis in particular through the term $\mathrm{tr}_{\not g}\chi$.
- Go beyond "consistency" by actually closing the energy estimates. For this, it is important to exploit various kinds of smallness in the problem (in addition to those that are already present in the irrotational case). For instance, in the energy estimates for the specific vorticity, the wave variables and the acoustic geometry enter with an extra smallness constant $\mathring{\epsilon}^2$ (see (2.4.5) and (2.4.7)) so that the latter variables couple only weakly[70] to the specific vorticity. This allows the energy estimates for the specific vorticity to be closed semi-independently with the help of appropriate bootstrap assumptions for the behavior of the wave variables and the acoustic geometry. Another useful but more subtle source of smallness is tied to the fact that we are treating perturbations of simple outgoing (that is, right-moving) plane waves. For example, $\breve{X}(v^1 - \rho)$ is $\mathcal{O}(\mathring{\epsilon})$ small even though $\breve{X}v^1$ and $\breve{X}\rho$ are not. This smallness allows us to exploit effective decoupling between different solution variables, which turns out to be important for minimizing the size of certain key coefficients and therefore minimizing[71] the number of derivatives needed to close the problem; see the discussion in Subsubsect. 2.4.6.

In order to close the estimates, we will commute the wave equations with up to 20 geometric vectorfields and the transport equation with 21 geometric vectorfields;[72] see, however, Remark 2.2. As we described above, since ω is at the level of one derivative of v^i, this represents a gain of one derivative for ω. Define[73]

$$\mathbb{W}_N$$

to be the energy norm for v^i and ρ corresponding to N commutations of the wave equations with geometric vectorfields, where we require[74] $N \geq 1$ and allow at most one of them to be $\breve{X}$. Moreover, the case of a single pure $\breve{X}$ commutation is excluded. Notice that for technical reasons,[75] we have slightly modified the approach to commuting the wave equations taken in [30]. In particular, unlike in [30], we now commute with up to one $\breve{X}$ (recall that in Subsect. 2.2, only L and Y were used as commutators). We also note the energy includes a

[70] This is a big difference from the case of three spatial dimensions, where the coupling is much stronger. We will discuss this issue in our forthcoming work [24] in the three spatial dimensional case.

[71] The size of the coefficients is tied to the blowup-rate of the top-order energies which is in turn tied to the number of derivatives needed to close; see, for example, the "6" on RHS (2.4.10).

[72] With additional effort, we could slightly reduce the number of derivatives that we need to close.

[73] In the proof, we will denote the boundary energy norms by $\mathbb{Q}_N$ and the bulk spacetime norm by $\mathbb{K}_N$. In this subsection, in order to simplify the exposition, we will not make this distinction.

[74] Note that $N = 1$ corresponds to controlling *two* derivatives of v^i and ρ.

[75] In commuting the specific vorticity equation, we encounter a new term that forces us to commute the wave equations with one copy of $\breve{X}$, namely the terms $\breve{X}Y^{N-1}\mathrm{tr}_{\not g}\chi$ on RHSs (14.2.3a)-(14.2.3b). We will downplay this issue here.

term on Σ_t^u (cf. (2.1.4)), a term on the characteristics $\mathcal{P}_u^t$ (cf. (2.1.5)) and a spacetime bulk term (cf. (2.1.6)). Our choice of the structure of the strings of commutation vectorfields ensures that the energy is $\mathcal{O}(\mathring{\epsilon})$-small.[76]

To control the specific vorticity ω, we will use an energy norm $\mathbb{V}$, which we define below (see (2.4.2)). Since ω satisfies a *homogeneous* transport equation, the main challenge is to control the commutators of the weighted transport operator μB and the geometric commutation fields, which are adapted to the acoustic characteristics. In order to minimize the number of $\breve{X}$ commutators needed to control the "wave variables" (v^1, v^2, ρ) (which satisfy the wave equations) and $\mathrm{tr}_{\not g}\chi$, we commute the transport equation only with the $\mathcal{P}_u$-tangent vectorfields L and Y. Because the material derivative vectorfield is transversal[77] to the acoustic characteristics $\mathcal{P}_u$, this is sufficient for obtaining estimates for all directional derivatives of ω and closing the argument.

In the next few subsubsections, we will discuss the various energy estimates needed to control the specific vorticity, the eikonal function and the wave variables. Let us already note at this point that the main difficulty comes in the *top-order* derivative, where the singular behaviors of the wave variables, the specific vorticity, and the geometry of the null hypersurfaces are all coupled.

2.4.1. *Lower-order energy estimates for the specific vorticity.* The energy norm that we use to control the specific vorticity at the lowest order is

$$\mathbb{V}(t,u) := \int_{\Sigma_t^u} \mu\omega^2 \, d\vartheta \, du + \int_{\mathcal{P}_u^t} \omega^2 \, d\vartheta \, dt. \tag{2.4.2}$$

In other words, the energy for ω on a constant-t hypersurface Σ_t^u is "degenerate" in μ, while that for ω on a constant-u hypersurface $\mathcal{P}_u^t$ is "non-degenerate".[78]

In deriving energy estimates, we also control the derivatives of ω with respect to L and Y and use the notation $\mathbb{V}_N(t,u)$ to denote the corresponding energy norm after N commutations. In particular, for N sufficiently large, the non-degenerate control of $\mathbb{V}_0, \cdots, \mathbb{V}_N$ on the acoustic characteristics $\mathcal{P}_u^t$, when combined with Sobolev embedding, gives rise to[79] pointwise control of ω and its lower-order L and Y derivatives.

While ω itself satisfies a homogeneous transport equation (see (2.4.1c)), to control its derivatives, we need to bound the commutator terms and derive estimates for solutions to inhomogeneous transport equations. For the general inhomogeneous equation $\mu B\omega = \mathfrak{F}$, we

[76] Recall that the derivatives of (v^1, v^2, ρ) are small if at least one of the geometric vectorfields is L or Y.

[77] The transversality follows from a simple geometric fact: in all solution regimes, B is a g-timelike vectorfield (that is, $g(B,B) < 0$); thus, B cannot be tangent to any g-null hypersurface. In fact, we have $B = L + X$ and hence the X component of B is bounded below all the way up to the shock. This then allows us to use equation (2.4.1c) to algebraically express $X\omega = -L\omega$. Similarly, higher $\breve{X}$ derivatives of $X\omega$ can be expressed in terms of derivatives tangential to the g-null hypersurfaces.

[78] The energies for $V(t,u)$ are in fact very natural. If one changes variables and expresses the forms $\mu \, d\vartheta \, du$ and $d\vartheta \, dt$ relative to the Cartesian coordinates, then one sees that, up to $\mathcal{O}(1)$ multiplicative factors, these forms agree with the usual forms induced on the corresponding hypersurfaces by the Euclidean metric on $\mathbb{R}^{1+2}$.

[79] Let us note that pointwise control can alternatively be derived directly using the transport equation itself.

have the following estimate (see Proposition 4.4):

$$\int_{\Sigma_t^u} \mu\omega^2 \, d\vartheta \, du + \int_{\mathcal{P}_u^t} \omega^2 \, d\vartheta \, dt \lesssim \text{Data} + \int_{\mathcal{M}_{t,u}} \mathfrak{F}^2 \, d\vartheta \, du \, dt. \tag{2.4.3}$$

Both of the commutators $[\mu B, L]$ and $[\mu B, Y]$ generate controllable error terms that are regular with respect to μ; this of course is the main reason to commute the geometric vectorfields with μB (instead of, say, B). The following equation exhibits a typical difficult inhomogeneous term that we have to control after N commutations:

$$\mu B Y^N \omega = (Y\omega) \breve{X} Y^{N-2} \mathrm{tr}_{\not{g}} \chi + \dots, \tag{2.4.4}$$

where $\dots$ denotes terms that are easier to handle. Therefore, except for the top-order case $N = 21$, one can control the term $(Y\omega)\breve{X}Y^{N-2}\mathrm{tr}_{\not{g}}\chi$ by first showing[80] that $Y\omega$ is $\mathcal{O}(\mathring{\epsilon})$-small in L^∞ and that $\breve{X}Y^{N-2}\mathrm{tr}_{\not{g}}\chi$ can be controlled by an analogue of (2.1.11). Using (2.4.3), this roughly yields the following inequality for $N < 21$:

$$\mathbb{V}_N(t, u) \lesssim \mathring{\epsilon}^2 + \mathring{\epsilon}^2 \int_{t'=0}^{t} \left(\int_{s=0}^{t'} \mu_\star^{-\frac{1}{2}}(s, u) \mathbb{W}_N^{\frac{1}{2}}(s, u) \, ds \right)^2 dt' + \dots. \tag{2.4.5}$$

As we will see, the coupling with $\mathbb{W}_N$ in equation (2.4.5) is quite weak. More precisely, due to the small factor $\mathring{\epsilon}^2$ and the large number of time integrations on RHS (2.4.5), the influence of RHS (2.4.5) on $\mathbb{V}_N$ is easy to control.

On the other hand, for $N = 21$, one does not have the luxury of using the $\mathbb{W}_{21}$ norm on the right hand side, since $\mathbb{W}_{20}$ is top-order. Hence, as we will later see, at the top-order, we have to take a different approach to controlling certain terms in the top-order inhomogeneous transport equation, an approach which avoids relying on $\mathbb{W}_{21}$. The different approach forces us to confront the most singular terms in the 21-times-commuted transport equation: $\breve{X}Y^{19}\mathrm{tr}_{\not{g}}\chi$ and $Y^{20}\mathrm{tr}_{\not{g}}\chi$. Specifically, we have to account for the singular behavior of the L^2 norms of $\breve{X}Y^{19}\mathrm{tr}_{\not{g}}\chi$ and $Y^{20}\mathrm{tr}_{\not{g}}\chi$ in terms of powers of $\mu_\star^{-1}$. Note that, as we described in Subsubsect. 2.1.4, the singular behavior of the top-order derivatives of $\mathrm{tr}_{\not{g}}\chi$, which is tied to the necessity of using modified quantities to avoid derivative loss, is already present in the irrotational case as the primary source of degeneracy.

2.4.2. *Top-order estimates for the eikonal function.* Before we discuss the top-order estimates for ω, it makes sense to first consider the top-order derivatives of the eikonal function (in particular the top-order derivatives of the mean curvature $\mathrm{tr}_{\not{g}}\chi$ of $\mathcal{P}_u$), as they are the main source terms in the vorticity estimates (see equation (2.4.4)). In our setting, we again need to use modified quantities as described in Subsubsect. 2.1.4 in order to obtain sufficient top-order estimates for $\mathrm{tr}_{\not{g}}\chi$. However, since the "gain of a derivative" that one achieves with modified quantities relies on the wave equations satisfied by the Cartesian metric components,[81] which feature source terms depending on the specific vorticity, this procedure is now *coupled* with the estimates for the specific vorticity. As a consequence, at the top-order, *the specific vorticity is directly coupled to the evolution of* $\mathrm{tr}_{\not{g}}\chi$. Indeed, we recall from the

[80]Actually, the smallness of $Y\omega$ is one of our bootstrap assumptions.

[81]By (3.3.10a), the Cartesian metric components depend on v^1, v^2 and ρ.

discussion[82] in Subsubsect. 2.1.4 that in order to use (2.1.7) to gain a derivative for $\mathrm{tr}_{\not g}\chi$, we need to use the wave equation to exchange $\mu \not\Delta v^i$ with an exact L-derivative. Since the wave equation features the inhomogeneous terms $\mu\partial\omega$ and $\mu\omega\partial\rho$, these terms will couple into the estimates for $\mathrm{tr}_{\not g}\chi$. In the next paragraph, we describe the effect of this coupling.[83]

Let us focus on the term $\mu\partial\omega$ in equation (2.4.1a) since the second term $\mu\omega\partial\rho$, though it gives rise to some singular estimates, is easier to handle. A crucial observation, which we already made in Footnote 77, is that by algebraically using the transport equation for ω, one can express $\mu\partial\omega$ as linear combinations of $\mu L\omega$ and $\mu Y\omega$. As a consequence, the $\mathbb{V}_N$ norms suffice[84] to control these terms and we obtain the following bound for the top-order[85] derivatives of $\mathrm{tr}_{\not g}\chi$:

$$\|\mu \breve{X} Y^{19}\mathrm{tr}_{\not g}\chi\|_{L^2(\Sigma_t^u)},\ \|\mu Y^{20}\mathrm{tr}_{\not g}\chi\|_{L^2(\Sigma_t^u)} \lesssim \mathbb{W}_{20}^{\frac{1}{2}} + \int_{t'=0}^{t} \mathbb{V}_{21}^{\frac{1}{2}}(t',u)\,dt' + \dots, \tag{2.4.6}$$

where ... are similar or less singular terms. We stress that the time integral term on RHS (2.4.6) is exactly the term that accounts for the influence of the top-order derivatives of the specific vorticity on the acoustic geometry.

2.4.3. *Top-order energy estimates for the specific vorticity.* We now return to the discussions for the estimates for ω, but this time at the top-order derivative. According to (2.4.3) and (2.4.4), at the top-order, we need to bound[86] $\breve{X}Y^{19}\mathrm{tr}_{\not g}\chi$ and $Y^{20}\mathrm{tr}_{\not g}\chi$ in a suitable spacetime L^2 norm. With the help of the estimate (2.4.6) for $\breve{X}Y^{19}\mathrm{tr}_{\not g}\chi$ and $Y^{20}\mathrm{tr}_{\not g}\chi$, we can obtain the following top-order estimate (see Prop. 15.4 for the details):

$$\mathbb{V}_{21}(t,u) \lesssim \mathring{\epsilon}^2 \left(\underbrace{\int_{t'=0}^{t} \mu_\star^{-2}\mathbb{W}_{20}(t',u)\,dt'}_{VT_1} + \underbrace{\int_{t'=0}^{t} \mu_\star^{-2}(t',u) \left(\int_{s=0}^{t'} \mathbb{V}_{21}^{\frac{1}{2}}(s,u)\,ds \right)^2 dt'}_{VT_2} \right) + \dots \tag{2.4.7}$$

Notice that VT_2 features the singular factor $\mu_\star^{-2}$ and a total of three time integrations. Since $\mu_\star \to 0$ at worst linearly (as we described in Subsubsect. 2.1.1), as long as we are willing to settle for proving a sufficiently singular bound,[87] the term VT_2 can be treated with a Gronwall-type argument. On the other hand, as we will later see, the term VT_1

[82]Let us note that while in Subsubsect. 2.1.4 we were dealing with a scalar equation, the system case can be dealt with similarly. We discuss here only the estimates involving v^i (as it is slightly harder) and suppress those involving the wave equation for ρ.

[83]Note that at the same time, equation (2.4.4) shows that the top-order derivatives of $\mathrm{tr}_{\not g}\chi$ couple into the top-order transport equation for ω. However, we will postpone the discussion of the effect of the top-order derivatives of $\mathrm{tr}_{\not g}\chi$ on the top-order derivatives of ω until Subsubsect. 2.4.3.

[84]Recall that by definition, the norms $\mathbb{V}_N$ control only the L and Y derivatives of ω.

[85]The top-order derivatives of $\mathrm{tr}_{\not g}\chi$ involving at least one L differentiation are much easier to control since one can directly bound it by estimating the RHS of (2.1.7) and hence does not need to use the modified quantities of Subsect. 2.1.4 to handle them.

[86]There is in fact a similar term which features $Y^{20}\mathrm{tr}_{\not g}\chi$ that we have suppressed in (2.4.4). It is as difficult as the term featuring $\breve{X}Y^{19}\mathrm{tr}_{\not g}\chi$, although in view of (2.4.6), it can be estimated in a similar manner.

[87]That is, as long as we are proving that $\mathbb{V}_{21}(t,u)$ is bounded from above by some negative powers of $\mu_\star$.

determines the blowup-rate of $\mathbb{V}_{21}$ in terms of negative powers of $\mu_\star$. We now recall a crucial feature of the estimate (2.4.7) mentioned earlier, namely that the coupling to $\breve{X}Y^{19}\mathrm{tr}_{\not g}\chi$ and $Y^{20}\mathrm{tr}_{\not g}\chi$ is weak in that there is a small factor $\mathring{\varepsilon}^2$ on RHS (2.4.7). For this reason, one can actually derive sufficient estimates for $\mathbb{V}_{21}(t,u)$ using only bootstrap assumptions for the energy norms, based on having a good guess for the blowup-rates of all quantities. One is aided in this endeavor by the fact, justified later on, that $\mathbb{W}_{20}$ blows up at the same rate as in the irrotational case. This basic fact allows one to control the specific vorticity in a relatively straightforward fashion.

2.4.4. *Lower-order energy estimates for the wave variables.* To close the argument, we need to derive estimates for the "wave variables" v^1, v^2, and ρ, that is, for solutions to the wave equations (2.4.1a)-(2.4.1b), and in particular to estimate the vorticity terms arising on the right hand side of equation (2.4.1a). When we are bounding the below-top-order derivatives of the wave variables, we do not need to rely on modified quantities to control the eikonal function. For this reason, the below-top-order estimates are relatively easy to derive, as we now describe. For this discussion, we suppress most of the terms that do not involve ω except for one that is analogous to the term on RHS (2.1.12); we denote this analogous term by WL_1 below in (2.4.8). The inhomogeneous terms not involving ω can be bounded by using the same arguments as in the irrotational case, so we do not discuss them in detail. By equation (2.4.1a), the terms in the equation $\mu\square_g v^i = \cdots$ involving ω can be expressed in the form[88] $\mu L\omega$, $\mu Y\omega$ or $\omega(\mu L v^i + \breve{X}v^i)$. Since the terms $\mu L\omega$ and $\mu Y\omega$ contain factors of μ, when estimating $\mathbb{W}_N$, these terms can be controlled by the degenerate energy on constant-t hypersurfaces (that is, the analog of the first term on RHS (2.4.2)). On the other hand, since there are no extra factors of μ in the product $\omega\breve{X}v^i$, the ω factor cannot be bounded by the degenerate energy. Instead, we control it using the non-degenerate flux on $\mathcal{P}_u^t$ (that is, the analog of the second term on RHS (2.4.2)). Since the factor ω is not top-order, when estimating $\mathbb{W}_N$, one needs only to use $\mathbb{V}_N$ to control its up-to-order N derivatives. In total, we roughly obtain the following estimate for $N < 20$:

$$\begin{aligned}\mathbb{W}_N(t,u) \lesssim &\mathrm{Data} + \underbrace{\int_0^t \mu_\star^{-\frac{1}{2}} \mathbb{W}_N^{\frac{1}{2}}(t',u) \int_{t''=0}^{t''=t'} \mu_\star^{-\frac{1}{2}} \mathbb{W}_{N+1}^{\frac{1}{2}}(t'',u)\, dt''dt'}_{WL_1} \\ &+ \underbrace{\int_{t'=0}^t \mathbb{V}_{N+1}(t',u)\, dt'}_{WL_2} + \underbrace{\int_{u'=0}^u \mathbb{V}_N(t,u')\, du'}_{WL_3} + \ldots .\end{aligned} \tag{2.4.8}$$

The key point here is that the WL_2 term has a time integration and thus one can gain[89] a power of $\mu_\star$. Such gain cannot be achieved in WL_3, but on the other hand, the term only features the lower-order $\mathbb{V}_N$ norm and no singular factor of $\mu_\star^{-1}$.

2.4.5. *Top-order estimates for the wave variables.* In deriving estimates for the top derivative norm $\mathbb{W}_{20}$, we again encounter terms that are analogous to the terms WL_2 and WL_3 from

[88]Here, we have used the observation discussed in Footnote 77, namely that by using the transport equation for ω, the term $\breve{X}\omega$ can expressed as $-\mu L\omega$.

[89]Let us recall again that $\mu_\star$ goes to 0 at worst linearly and that (2.1.13) holds.

(2.4.8), which are respectively denoted by WT_1 and WT_2 below in (2.4.9). There are also additional terms involving the top derivatives of $\mathrm{tr}_{\not g}\chi$, which cannot be treated like the term WL_1 from (2.4.8). These additional terms are in fact precisely the ones described in Subsubsect. 2.1.4, which need to be bounded with the help of modified quantities. We stated an estimate for them in (2.4.6). To proceed, we use the estimate (2.4.6), but this time carefully tracking the precise numerical coefficient of the $\mathbb{W}_{20}^{\frac{1}{2}}$ term. These give rise to WT_{main} and WT_3 in (2.4.9) below. In total, we obtain the following estimate (see Prop. 15.3 for the details):

$$\begin{aligned} \mathbb{W}_{20}(t,u) \leq & \mathrm{Data} + \underbrace{C_{fix} \int_{t'=0}^{t} \left(\sup_{\Sigma_{t'}^u} \left| \frac{L\mu}{\mu} \right| \right) \mathbb{W}_{20}(t',u)\, dt'}_{WT_{main}} \\ & + \underbrace{C \int_{t'=0}^{t} \mathbb{V}_{21}(t',u)\, dt'}_{WT_1} + \underbrace{C \int_{u'=0}^{u} \mathbb{V}_{20}(t,u')\, du'}_{WT_2} \\ & + \underbrace{C \int_{t'=0}^{t} \frac{\mathbb{W}_{20}^{\frac{1}{2}}(t',u)}{\mu_\star(t',u)} \int_{s=0}^{t'} \mathbb{V}_{21}^{\frac{1}{2}}(s,u)\, ds\, dt'}_{WT_3} + \ldots . \end{aligned} \tag{2.4.9}$$

Notice that the term WT_{main} is analogous to the term in (2.1.8) and is the main term driving the blowup-rate of $\mathbb{W}_{20}(t,u)$. As we described in Sects. 2.1.4 and 2.1.5, the constant C_{fix} in WT_{main} is intimately tied to the number of derivatives needed to close the proof.

2.4.6. *Independent bounds for the "good" components.* The next ingredient of the proof is to derive independent estimates for v^2 and $v^1 - \rho$. This is crucial for obtaining a good estimate for the constant C_{fix} in (2.4.9). More precisely, we show that v^2 and $v^1 - \rho$ obey better bounds than either v^1 or ρ and that all geometric derivatives of v^2 and $v^1 - \rho$, *including their* $\breve{X}$ *derivatives*, are small. This is of course tied to the assumption that the solution is nearly simple outgoing plane symmetric. Indeed, for plane symmetric solutions, we have $v^2 = 0$. Moreover, for the simple outgoing plane symmetric solutions described in Subsect. 2.3, we have $0 = \mathcal{R}_- = v^1 - \int_{\rho'=0}^{\rho} c_s(\rho')\, d\rho'$. Hence, it follows that under our assumed normalization condition $c_s(\rho = 0) = 1$ from (1.0.2) and the L^∞-smallness conditions for ρ and v^1, all geometric derivatives of $v^1 - \rho$ are small for the perturbations of simple outgoing plane symmetric solutions under study.

In our proof, we take advantage of this smallness as follows. First, we explicitly prove that $\breve{X}(\rho - v^1)$ and $\breve{X} v^2$ are $\mathcal{O}(\mathring{\epsilon})$-small in the L^∞ sense; in the next paragraph, it will become clear why this is important. Next, we derive independent estimates for the top-order energy norms of v^2 and $v^1 - \rho$. Let us momentarily[90] denote the top-order energy norms of v^2 and $v^1 - \rho$ by $\mathbb{W}_{20}^{(Partial)}$ in order to distinguish it from the energy norm $\mathbb{W}_{20}$ (which controls all three of v^1, v^2 and ρ).

[90]In the proof of the main theorem, we define $\mathbb{Q}_{20}^{(Partial)}$ and $\mathbb{K}_{20}^{(Partial)}$ in an analogous manner to respectively denote the boundary terms and the bulk terms in the energy norms.

Roughly speaking, if we just track the WT_{main} term in (2.4.9), then we obtain the following system of energy inequalities:

$$\begin{aligned}\mathbb{W}_{20}(t,u) \leq &(6+C\mathring{\epsilon})\int_{t'=0}^{t}\left(\sup_{\Sigma_{t'}^u}\left|\frac{L\mu}{\mu}\right|\right)\mathbb{W}_{20}(t',u)\,dt' \\ &+ C_*\int_{t'=0}^{t}\left(\sup_{\Sigma_{t'}^u}\left|\frac{L\mu}{\mu}\right|\right)\sqrt{\mathbb{W}_{20}}(t',u)\sqrt{\mathbb{W}_{20}^{(Partial)}}(t',u)\,dt' + \dots, \qquad (2.4.10)\\ \mathbb{W}_{20}^{(Partial)}(t,u) \leq &C\mathring{\epsilon}\int_{t'=0}^{t}\left(\sup_{\Sigma_{t'}^u}\left|\frac{L\mu}{\mu}\right|\right)\mathbb{W}_{20}(t',u)\,dt' + \dots,\end{aligned}$$

where C_* is a possibly large constant that, unlike C_{fix}, depends on the equation of state. A crucial feature of the above system is that the main term in the inequality for $\mathbb{W}_{20}^{(Partial)}(t,u)$ is multiplied by a small factor $\mathring{\epsilon}$, which is available thanks to the L^∞ estimates mentioned in the previous paragraph. This small factor limits the contribution of the main term to the blowup-rate for $\mathbb{W}_{20}^{(Partial)}(t,u)$, which in turn allows us to obtain semi-independent control of $\mathbb{W}_{20}^{(Partial)}$ and thus show that the product $C_*\int_{t'=0}^{t}\left(\sup_{\Sigma_{t'}^u}\left|\frac{L\mu}{\mu}\right|\right)\sqrt{\mathbb{W}_{20}}(t',u)\sqrt{\mathbb{W}_{20}^{(Partial)}}(t',u)\,dt'$ does not significantly influence the blowup-rate of $\mathbb{W}_{20}(t,u)$. In total, this structure allows us to show that the constant C_{fix} in (2.4.9) is essentially 6. This fact, together with similar estimates for a few other related terms that we have suppressed, determines the total number of derivatives that we need in the argument. We clarify that if we did not split the energies into $\mathbb{W}_{20}$ and $\mathbb{W}_{20}^{(Partial)}$, then the constant C_* could in principle increase the blowup-rate of $\mathbb{W}_{20}$, which would in turn increase number of derivatives we need to close the problem. Thanks to the splitting, we are able to close the estimates by differentiating v and ρ up to[91] 22 times.

2.4.7. *Putting everything together.* We now combine the estimates discussed in the previous subsubsections and show, at least heuristically, that they can close. The detailed proof is based on a lengthy Gronwall argument that is located in Subsects. 15.15 and 15.16. As is already clear from the discussions above, the estimates for the lower-order derivatives and for the top derivatives are rather different and the most difficult terms are found in the estimates for the top-order energies.

We first consider the lower-order estimates, where the blowup-rates are determined by (2.4.5) and (2.4.8). Recall the discussions of the descent scheme in Subsubsect. 2.1.5: *For every order of descent, one gains two powers of $\mu_\star$ until one shows that the energy is bounded.* Moreover, we recall that the descent scheme is based mainly on the fact that each time integration reduces the power of the singularity by one:

$$\int_{s=0}^{t}\mu_\star^{-b}(s,u)\,ds \lesssim \mu_\star^{1-b}(t,u), \quad \text{for } b>1. \qquad (2.4.11)$$

[91] Note that we take up to 21 derivatives of ω, which corresponds to up to 22 derivatives of v and 21 derivatives of ρ.

For this reason, inequality (2.4.5) suggests that when $\mathbb{W}_N$ is sufficiently singular, one can prove that $\mathbb{V}_N$ is less singular than $\mathbb{W}_N$ by a factor of $\upmu_\star^2$. This suggests proving the following estimates:[92]

Below-top-order energy hierarchy

$$
\begin{aligned}
\sqrt{\mathbb{W}_{15+K}}(t,u) &\leq C\mathring{\upepsilon}\upmu_\star^{-(K+.9)}(t,u), && (0 \leq K \leq 4),\\
\sqrt{\mathbb{W}_N}(t,u) &\leq C\mathring{\upepsilon}, && (0 \leq N \leq 14)\\
\sqrt{\mathbb{V}_{16+K}}(t,u) &\leq C\mathring{\upepsilon}\upmu_\star^{-(K+.9)}(t,u), && (0 \leq K \leq 4),\\
\sqrt{\mathbb{V}_N}(t,u) &\leq C\mathring{\upepsilon} && (0 \leq N \leq 15).
\end{aligned}
$$

Notice that the above hierarchy is consistent in the following sense: when one substitutes the hierarchy estimates for $\mathbb{V}_{N+1}$ and $\mathbb{V}_N$ into the terms WL_2 and WL_3 on RHS (2.4.8) and uses (2.4.11), one finds that these terms contribute to the blowup-rate for the term $\mathbb{W}_N$ on the LHS in a manner that is compatible with the estimates for $\mathbb{W}_N$ stated in the hierarchy. In fact, there is even extra room in these estimates.

Finally, we consider the top-order estimates, which are determined by (2.4.7) and (2.4.9), modulo the discussion surrounding equation (2.4.10). As in Christodoulou's work [6], the top-order blowup-rate is determined mainly by the term WT_{main} on RHS (2.4.9). As we mentioned in the previous subsubsection, the blowup-rate of the top-order energies depends[93] on the constant C_{fix}, which can be precisely estimated, independent of the equation of state;[94] see also Footnote 50.

To see how the top-order estimates for $\mathbb{W}_{20}$ and $\mathbb{V}_{21}$ couple, we combine (2.4.7) and (2.4.9) to see that $\mathbb{W}_{20}$ is better than $\mathbb{V}_{21}$ by a single factor of $\upmu_\star$. Notice that such an estimate is *borderline* in the sense that if either the term VT_1 in (2.4.7) or the term WT_1 in (2.4.9) involved a slightly worse power of $\upmu_\star^{-1}$, then the estimates could not close. We are therefore led to prove the following estimate (see Subsect. 15.16 for the precise details concerning the blowup-rates in (2.4.13a)-(2.4.13b)):

Top-order energy estimates

$$\sqrt{\mathbb{W}_{20}}(t,u) \leq C\mathring{\upepsilon}\upmu_\star^{-5.9}(t,u), \tag{2.4.13a}$$

$$\sqrt{\mathbb{V}_{21}}(t,u) \leq C\mathring{\upepsilon}\upmu_\star^{-6.4}(t,u). \tag{2.4.13b}$$

We clarify that (2.4.13a) should be viewed as the main estimate determining the blowup-rates of not only for v^1, v^2, and $\uprho$, but also $\mathrm{tr}_{\not g}\upchi$ (in view of the discussion of Subsubsect. 2.1.4,

[92]Here, we emphasize the relative singularity between different norms. The precise absolute strength of the singularity depends on the estimates at the top level, which we will discuss immediately below.

[93]Let us emphasize again that this is a slight simplification, as the strength of the singularity in fact depends on the constant in front of *all* of the singular terms, only one of which is written in (2.4.9).

[94]In particular, for *all* equations of state other than that of the Chaplygin gas, the estimates can be closed with a total of 22 derivatives of v^i and 21 derivatives of $\uprho$. On the other hand, the relative smallness that is required for $\mathring{\upepsilon}$ *does* depend on the equation of state.

where we described how to control $\mathrm{tr}_{g\!\!\!/}\chi$ at the top-order by using modified quantities). More precisely, one can trace through the above logic to discover that the blowup-rate $\mu_\star^{-5.9}(t,u)$ on RHS (2.4.13a), if taken as given, controls the blowup-rates of all other energy quantities. We also note that (2.4.13a)-(2.4.13b) are consistent for the terms VT_2 and WT_3, neither of which are borderline. This concludes our discussion of the main ideas of the proof.

3. Geometric setup

In this section, we construct most of the geometric objects that we use to the shock formation and exhibit their basic properties. We postpone our construction of energies and the corresponding integration measures until Sect. 4. We postpone our construction of modified quantities, which are needed for top-order energy estimates, until Sect. 7.

3.1. **Notational conventions and shorthand notation.** We start by summarizing some of our notational conventions; the precise definitions of some of the concepts referred to here are provided later in the article.

- Lowercase Greek spacetime indices α, β, etc. correspond to the Cartesian spacetime coordinates defined in Sect. 3.3 and vary over $0,1,2$. Lowercase Latin spatial indices a,b, etc. correspond to the Cartesian spatial coordinates and vary over $1,2$. All lowercase Greek indices are lowered and raised with the spacetime metric g and its inverse g^{-1}, and *not with the Minkowski metric.*
- We sometimes use $\cdot$ to denote the natural contraction between two tensors (and thus raising or lowering indices with a metric is not needed). For example, if ξ is a spacetime one-form and V is a spacetime vectorfield, then $\xi \cdot V := \xi_\alpha V^\alpha$.
- If ξ is an $\ell_{t,u}$-tangent one-form (as defined in Sect. 3.8), then $\xi^\#$ denotes its $g\!\!\!/$-dual vectorfield, where $g\!\!\!/$ is the Riemannian metric induced on $\ell_{t,u}$ by g. Similarly, if ξ is a symmetric type $\binom{0}{2}$ $\ell_{t,u}$-tangent tensor, then $\xi^\#$ denotes the type $\binom{1}{1}$ $\ell_{t,u}$-tangent tensor formed by raising one index with $g\!\!\!/^{-1}$ and $\xi^{\#\#}$ denotes the type $\binom{2}{0}$ $\ell_{t,u}$-tangent tensor formed by raising both indices with $g\!\!\!/^{-1}$.
- Unless otherwise indicated, all quantities in our estimates that are not explicitly under an integral are viewed as functions of the geometric coordinates (t,u,ϑ) of Def. 3.11. Unless otherwise indicated, quantities under integrals have the functional dependence established below in Def. 4.1.
- If Q_1 and Q_2 are two operators, then $[Q_1,Q_2] = Q_1Q_2 - Q_2Q_1$ denotes their commutator.
- $A \lesssim B$ means that there exists $C > 0$ such that $A \leq CB$.
- $A \approx B$ means that $A \lesssim B$ and $B \lesssim A$.
- $A = \mathcal{O}(B)$ means that $|A| \lesssim |B|$.
- Constants such as C and c are free to vary from line to line. **Explicit and implicit constants are allowed to depend in an increasing, continuous fashion on the data-size parameters $\mathring{\delta}$ and $\mathring{\delta}_*^{-1}$ from Sect. 8.1. However, the constants can be chosen to be independent of the parameters $\mathring{\epsilon}$ and ε whenever $\mathring{\epsilon}$ and ε are sufficiently small relative to $\mathring{\delta}^{-1}$ and $\mathring{\delta}_*$.**
- $\lfloor\cdot\rfloor$ and $\lceil\cdot\rceil$ respectively denote the floor and ceiling functions.

3.2. **A caveat on citations.** We often cite [30] for equations and identities We now point out some minor discrepancies between the work [30] and the present work; we will not explicitly comment on them again, even though they occur throughout our work. Some of the concepts referred to here are defined later in the article.

- In citing [30], we sometimes adjust the formulas involving Cartesian metric components to take into account the explicit form of $g_{\alpha\beta}$ and $(g^{-1})^{\alpha\beta}$ stated in Def. 3.3.
- In [30], the metric components $g_{\alpha\beta}$ were functions of a scalar-valued function Ψ, as opposed to the array $\vec{\Psi}$ (defined in Def. 3.4). For this reason, we must make minor adjustments to many of the formulas from [30] to account for the fact that in the present article, $\vec{\Psi}$ is an array. In all cases, our minor adjustments can easily be verified by examining the corresponding proof in [30].
- In [30], the array $\underline{\gamma}$ (see definition (3.19.1)) was defined to contain the entry $\mu - 1$ rather than μ. However, that difference is not important and in particular does not affect the validity of any of the schematic formulas that we cite from [30].

3.3. **Formulation of the equations.** We now formulate the evolution equations, the main result being Prop. 3.1. As we mentioned at the beginning, we assume that the space manifold, on which the equations are posed, is

$$\Sigma := \mathbb{R} \times \mathbb{T}, \tag{3.3.1}$$

where $\mathbb{R}$ corresponds to time and Σ to space. We fix a standard Cartesian coordinate system $\{x^\alpha\}_{\alpha=0,1,2}$ on $\mathbb{R}\times\Sigma$, where $x^0 \in \mathbb{R}$ is the time coordinate and $(x^1, x^2) \in \mathbb{R}\times\mathbb{T}$ are the spatial coordinates. The coordinate x^2 corresponds to perturbations away from plane symmetry. We denote the corresponding Cartesian coordinate partial derivative vectorfields by $\partial_\alpha := \dfrac{\partial}{\partial x^\alpha}$. The coordinate x^2 is only locally defined even though ∂_2 can be extended to a globally defined vectorfield on $\mathbb{T}$. We often use the alternate notation $t = x^0$ and $\partial_t = \partial_0$.

The compressible Euler equations are evolution equations for the velocity $v : \mathbb{R}\times\Sigma \to \mathbb{R}^2$ and the density $\rho : \mathbb{R}\times\Sigma \to [0,\infty)$. To close the system, we assume a barotropic equation of state

$$p = p(\rho), \tag{3.3.2}$$

where p is the pressure. To the equation of state, we associate the quantity c_s, known as the *speed of sound*

$$c_s := \sqrt{\frac{dp}{d\rho}}. \tag{3.3.3}$$

Physical equations of state are such that

- $c_s \geq 0$.
- $c_s > 0$ when $\rho > 0$.

We study solutions with $\rho > 0$, which, under the above assumptions, ensures the hyperbolicity of the system. In particular, we avoid the study of fluid-vacuum boundaries, which is accompanied by technical difficulties tied to the degeneracy of the hyperbolicity along the boundary.

Our shock formation results apply to all equations of state except for those corresponding to a Chaplygin gas, which are of the form

$$p = C_0 - \frac{C_1}{\rho} \tag{3.3.4}$$

for constants $C_0 \in \mathbb{R}$ and $C_1 > 0$.

3.3.1. *Vorticity, modified variables, the speed of sound and its derivative with respect to* $\mathsf{\rho}$. In two spatial dimensions, the vorticity ω is the scalar-valued function

$$\omega := \partial_1 v^2 - \partial_2 v^1. \tag{3.3.5}$$

Although ω is an auxiliary variable, it plays a fundamental role in our analysis.

Rather than directly studying the density and the vorticity, we find it convenient to instead study the logarithmic density and the specific vorticity.

Definition 3.1 (Modified variables). We define the *logarithmic density* $\mathsf{\rho}$ and the *specific vorticity* $\boldsymbol{\omega}$ as follows:

$$\mathsf{\rho} := \ln \rho, \qquad \boldsymbol{\omega} := \frac{\omega}{\rho} = \frac{\omega}{\exp \mathsf{\rho}}. \tag{3.3.6}$$

From now on, we view c_s, defined in (3.3.3), as a function of $\mathsf{\rho}$:

$$c_s = c_s(\mathsf{\rho}). \tag{3.3.7}$$

Moreover, we set

$$c_s' = c_s'(\mathsf{\rho}) := \frac{d}{d\mathsf{\rho}} c_s(\mathsf{\rho}). \tag{3.3.8}$$

3.3.2. *Geometric tensorfields associated to the flow.* To derive our main results, we rely on a geometric formulation of the Euler equations, derived in the companion article [25], which exhibit remarkable structures. Before stating the equations, we define some tensorfields that lie at the heart of our analysis.

We start by defining the material derivative vectorfield, which transports the specific vorticity.

Definition 3.2 (Material derivative vectorfield). The *material derivative vectorfield* B is defined as follows relative to the Cartesian coordinates:

$$B := \partial_t + v^a \partial_a. \tag{3.3.9}$$

Next, we define the acoustical metric g. It is the Lorentzian spacetime metric corresponding to the propagation of sounds waves.

Definition 3.3 (The acoustical metric and its inverse). We define the *acoustical metric* g and the *inverse acoustical metric* g^{-1} relative to the Cartesian coordinates as follows:

$$g := -dt \otimes dt + c_s^{-2} \sum_{a=1}^{2} (dx^a - v^a dt) \otimes (dx^a - v^a dt), \tag{3.3.10a}$$

$$g^{-1} := -B \otimes B + c_s^2 \sum_{a=1}^{2} \partial_a \otimes \partial_a. \tag{3.3.10b}$$

Remark 3.1. It is straightforward to verify that g^{-1} is the matrix inverse of g, that is, we have $(g^{-1})^{\mu\alpha} g_{\alpha\nu} = \delta^{\mu}_{\nu}$, where δ^{μ}_{ν} is the standard Kronecker delta.

3.3.3. *Statement of the geometric form of the equations.* We now state the form of the equations that we use to analyze solutions. The equations were essentially derived in [25], up to the following three remarks: **i)** We have multiplied the equations by a weight $\upmu > 0$ that we explain in great detail below. The reason is that the $\upmu$-weighted equations have better commutation properties with various differential operators compared to the unweighted equations. **ii)** In [25], the equations were derived in three space dimensions, in which case the specific vorticity is a vectorfield. In that context, the analog of equation (3.3.11c) is a vector equation for the Cartesian components ω^i, $(i = 1, 2, 3)$. The equation features the non-zero "vorticity stretching" source term $\sum_{a=1}^{3} \omega^a \partial_a v^i$. In $2D$, this source term vanishes, as we now explain. We may view the $2D$ Euler equations as a special case of the $3D$ Euler equations in which $v^3 \equiv 0$, $\partial_3 v^i \equiv 0$, and the vectorfield ω is proportional to $(\partial_2 v^3 - \partial_3 v^2)\partial_3$. It follows that $\sum_{a=1}^{3} \omega^a \partial_a v^i \equiv 0$ in $2D$, that is, the vorticity stretching term vanishes. Hence, in the remainder of the article, we view ω to be the scalar-valued function in (3.3.6). **iii)** In [25], an additional term $-c_s^{-1} c_s' (g^{-1})^{\alpha\beta} \partial_\alpha \rho \partial_\beta v^i$ appeared on the analog of RHS (3.3.12a) and the coefficient of the first product of the analog of RHS (3.3.12b) was -3 instead of -2. The reason for the discrepancy is that relative to Cartesian coordinates, $\det g = c_s^{-6}$ in three space dimensions while $\det g = c_s^{-4}$ in the present case of two space dimensions; in view of this fact and Footnote 23, we see that the form of $\square_g$ relative to the Cartesian coordinates depends on the number of spatial dimensions. In turn, this affects the coefficients of the semilinear terms present on the RHS of the wave equations. However, this is a minor point that has no substantial bearing on the analysis; the products under discussion are null forms and thus have only a negligible effect on the dynamics; see Remark 3.2.

Proposition 3.1 (The geometric wave-transport formulation of the compressible Euler equations). *Let* $\square_g$ *denote the covariant wave operator of the acoustic metric* g *defined by* (3.3.10a). *In two spatial dimensions, classical solutions to the compressible Euler equations* (1.0.1a)-(1.0.1b) *verify the following equations, where the Cartesian components* v^i, $(i = 1, 2)$, *are viewed as scalar-valued functions under covariant differentiation:*[95]

$$\upmu \square_g v^i = -[ia](\exp \rho) c_s^2 (\upmu \partial_a \omega) + 2[ia](\exp \rho)\omega(\upmu B v^a) + \upmu \mathscr{Q}^i, \tag{3.3.11a}$$

$$\upmu \square_g \rho = \upmu \mathscr{Q}, \tag{3.3.11b}$$

$$\upmu B \omega = 0. \tag{3.3.11c}$$

In (3.3.11a)-(3.3.11c), $\mathscr{Q}^i$ *and* $\mathscr{Q}$ *are the* ***null forms relative to g****, defined by*

$$\mathscr{Q}^i := -(g^{-1})^{\alpha\beta} \partial_\alpha \rho \partial_\beta v^i, \tag{3.3.12a}$$

$$\mathscr{Q} := -2 c_s^{-1} c_s' (g^{-1})^{\alpha\beta} \partial_\alpha \rho \partial_\beta \rho + 2\left\{ \partial_1 v^1 \partial_2 v^2 - \partial_2 v^1 \partial_1 v^2 \right\}. \tag{3.3.12b}$$

Remark 3.2 (The importance of the null forms relative to g**).** For the proof of our main theorem, it is critically important that $\mathscr{Q}^i$ and $\mathscr{Q}$ are null forms relative to g. The reason is that, due to their special structure, $\upmu \mathscr{Q}^i$ and $\upmu \mathscr{Q}$ remain uniformly small, all the way up to the shock. Thus, they do not interfere with the singularity formation mechanisms. In contrast, a general quadratic term $\upmu(\partial v, \partial \rho) \cdot (\partial v, \partial \rho)$ could become large near the expected

[95] Here, we use the square bracket $[\cdot]$ to denote the anti-symmetrization of the indices.

singularity and dominate the dynamics; had such a term been present in the equations, it would have completely obstructed our approach.

3.4. **Constant state background solutions and the array of solution variables.** We will study perturbations of the following constant state background solution to the system (3.3.11a)-(3.3.11c):

$$(\rho, v^1, v^2, \omega) \equiv (0,0,0,0). \tag{3.4.1}$$

The solution (3.4.1) corresponds to a motionless fluid of constant density $\bar{\rho}$, where $\bar{\rho} > 0$ is a constant. Note that a more general constant state $(\rho, v^1, v^2, \omega) \equiv (0, \bar{v}^1, \bar{v}^2, 0)$, the $\bar{v}^i$ are constants, may be brought into the form (3.4.1) via a Galilean transformation.[96] Let

$$\bar{c_s} := c_s(\rho = 0) \tag{3.4.2}$$

denote the speed of sound (3.3.3) evaluated at the background solution (3.4.1). Without loss of generality, we assume[97] that

$$\bar{c_s} = 1. \tag{3.4.3}$$

The advantage of the assumption (3.4.3) is that it simplifies many of our formulas.

Many of our estimates will apply uniformly to the "wave variables" ρ, v^1, and v^2. For this reason, we collect them into an array.

Definition 3.4 (**The array $\vec{\Psi}$ of wave variables**).

$$\vec{\Psi} = (\Psi_0, \Psi_1, \Psi_2) := (\rho, v^1, v^2). \tag{3.4.4}$$

Since we are studying perturbations of the solution (3.4.1), we may think of $\vec{\Psi}$ as small. However, for the solutions under study, some of the derivatives of $\vec{\Psi}$ are relatively large.

Remark 3.3 (**$\vec{\Psi}$ is not a tensor**). Throughout, we view $\vec{\Psi}$ to be an array of scalar-valued functions; we will not attribute any tensorial structure to the labeling index of $\Psi_\imath$ besides simple contractions, denoted by $\diamond$, corresponding to the chain rule; see Def. 3.5.

3.5. **The metric components and their derivatives with respect to the solution.** Throughout the paper, we often view the Cartesian metric component functions $g_{\alpha\beta}$ (see (3.3.10a)) to be (explicitly known) functions of $\vec{\Psi}$: $g_{\alpha\beta} = g_{\alpha\beta}(\vec{\Psi})$. From the expression (3.3.10a) and the assumption (3.4.3), it follows that we can decompose

$$g_{\alpha\beta}(\vec{\Psi}) = m_{\alpha\beta} + g_{\alpha\beta}^{(Small)}(\vec{\Psi}), \qquad (\alpha, \beta = 0,1,2), \tag{3.5.1}$$

---

[96]By this, we mean the change of coordinates $\widetilde{t} := t$ and $\widetilde{x}^i := x^i - \bar{v}^i t$, which implies that $\frac{\partial}{\partial \widetilde{t}} = \partial_t + \bar{v}^a \partial_a$ and $\frac{\partial}{\partial \widetilde{x}^i} = \partial_i$. Note that the expression $x^2 - \bar{v}^2 t$ should be interpreted as the translation of the point $x^2 \in \mathbb{T}$ by the flow of of the vectorfield $-\bar{v}^2 \partial_2$ for t units of time.

[97]We can always ensure the condition (3.4.3) by making the following changes of variables:

$$\widetilde{v}^i = \frac{v^i}{\bar{c_s}}, \qquad \widetilde{t} = \bar{c_s} t, \qquad \widetilde{g} = \bar{c_s}^2 g, \qquad \widetilde{c_s} = \frac{c_s}{\bar{c_s}}.$$

These changes of variables leave the expressions (3.3.10a)-(3.3.10b) and the Euler equations (1.0.1a)-(1.0.1b) invariant and are such that the desired normalization $\widetilde{c_s}(\rho = 0) = 1$ holds.

where

$$m_{\alpha\beta} = \mathrm{diag}(-1,1,1) \tag{3.5.2}$$

is the standard Minkowski metric and $g_{\alpha\beta}^{(Small)}(\vec{\Psi})$ is a smooth function of $\vec{\Psi}$ with

$$g_{\alpha\beta}^{(Small)}(\vec{0}) = 0. \tag{3.5.3}$$

Specifically, we have the formula

$$g^{(Small)} = \sum_{a=1}^{2} c_s^{-2}(v^a)^2 dt \otimes dt - c_s^{-2}\sum_{a=1}^{2} v^a dt \otimes dx^a - c_s^{-2}\sum_{i=1}^{2} v^a dx^a \otimes dt + \left\{c_s^{-2} - 1\right\}\sum_{a=1}^{2} dx^a \otimes dx^a. \tag{3.5.4}$$

The following quantities arise in many of the equations that we study.

Definition 3.5 (**Derivatives of $g_{\alpha\beta}$ with respect to $\vec{\Psi}$**)**.** For $\alpha,\beta = 0,1,2$ and $\imath,\jmath = 0,1,2$, we define

$$G_{\alpha\beta}^{\imath}(\vec{\Psi}) := \frac{\partial}{\partial \Psi_{\imath}} g_{\alpha\beta}(\vec{\Psi}), \tag{3.5.5a}$$

$$\vec{G}_{\alpha\beta} = \vec{G}_{\alpha\beta}(\vec{\Psi}) := \left(G_{\alpha\beta}^{0}(\vec{\Psi}), G_{\alpha\beta}^{1}(\vec{\Psi}), G_{\alpha\beta}^{2}(\vec{\Psi})\right), \tag{3.5.5b}$$

$$H_{\alpha\beta}^{\imath\jmath}(\vec{\Psi}) := \frac{\partial}{\partial \Psi_{\imath}}\frac{\partial}{\partial \Psi_{\jmath}} g_{\alpha\beta}(\vec{\Psi}), \tag{3.5.5c}$$

$$\vec{H}_{\alpha\beta} = \vec{H}_{\alpha\beta}(\vec{\Psi}) := \left(H_{\alpha\beta}^{00}(\vec{\Psi}), H_{\alpha\beta}^{01}(\vec{\Psi}), \cdots, H_{\alpha\beta}^{22}(\vec{\Psi})\right). \tag{3.5.5d}$$

For $\imath = 0,1,2$, we think of the $\{G_{\alpha\beta}^{\imath}\}_{\alpha,\beta=0,1,2}$, as the Cartesian components of a spacetime tensorfield. Similarly, we think of $\{\vec{G}_{\alpha\beta}\}_{\alpha,\beta=0,1,2}$ as the Cartesian components of an array-valued spacetime tensorfield. Similar remarks apply to $H_{\alpha\beta}^{\imath\jmath}$ and $\vec{H}_{\alpha\beta}$.

The following operators naturally arise in our analysis of solutions.

Definition 3.6 (**Operators involving the array $\vec{\Psi}$**)**.** Let U, V, U_1, U_2, V_1, V_2 be vectorfields. We define

$$V\vec{\Psi} := (V\Psi_0, V\Psi_1, V\Psi_2), \tag{3.5.6a}$$

$$\vec{G}_{U_1U_2} \diamond V\vec{\Psi} := \sum_{\imath=0}^{2} G_{\alpha\beta}^{\imath} U_1^{\alpha} U_2^{\beta} V\Psi_{\imath}, \tag{3.5.6b}$$

$$\vec{H}_{U_1U_2} \diamond\diamond (V_1\vec{\Psi})V_2\vec{\Psi} := \sum_{\imath,\jmath=0}^{2} H_{\alpha\beta}^{\imath\jmath} U_1^{\alpha} U_2^{\beta} (V_1\Psi_{\imath}) V_2\Psi_{\jmath}. \tag{3.5.6c}$$

We use similar notation with other differential operators in place of vectorfield differentiation. For example, $\vec{G}_{U_1U_2} \diamond \not{\Delta}\vec{\Psi} := \sum_{\imath=0}^{2} G_{\alpha\beta}^{\imath} U_1^{\alpha} U_2^{\beta} \not{\Delta}\Psi_{\imath}$.

3.6. **The eikonal function and related constructions.** To track the solution all the way to the shock, we construct a new set of geometric coordinates, one of which is the eikonal function.

Definition 3.7 (Eikonal function). The eikonal function u solves the eikonal equation initial value problem

$$(g^{-1})^{\alpha\beta}\partial_\alpha u \partial_\beta u = 0, \qquad \partial_t u > 0, \tag{3.6.1}$$

$$u|_{\Sigma_0} = 1 - x^1, \tag{3.6.2}$$

where Σ_0 is the hypersurface of constant Cartesian time 0.

Using u, we can construct many geometric quantities that can be used to derive sharp information about the solution. We start by defining the most important quantity in the study of shock formation: the *inverse foliation density*.

Definition 3.8 (Inverse foliation density). We define the inverse foliation density μ as follows:

$$\mu := \frac{-1}{(g^{-1})^{\alpha\beta}(\vec{\Psi})\partial_\alpha t \partial_\beta u} > 0, \tag{3.6.3}$$

where t is the Cartesian time coordinate. We note that the identity (3.7.14) below implies that $\mu = \frac{1}{Bu}$, where B is the material derivative vectorfield defined in (3.3.9).

The quantity $1/\mu$ measures the density of the level sets of u relative to the constant-time hypersurfaces Σ_t. When μ becomes 0, the density becomes infinite and the level sets of u intersect. For the initial data under consideration, μ starts out near unity. It turns out that the formation of the shock, the blowup of the eikonal function's first Cartesian coordinate partial derivatives, and the blowup of the first derivatives of v and ρ with respect to the Cartesian coordinate partial derivatives are all simultaneously tied to the vanishing of μ. We also note that the vanishing of μ is equivalent to the blowup of Bu.

We now define the spacetime subsets on which we analyze solutions. They are depicted in Fig. 2 on pg. 10.

Definition 3.9 (Subsets of spacetime). For $1 \leq t'$ and $0 \leq u \leq U_0$, we define the following subsets of spacetime:

$$\Sigma_{t'} := \{(t, x^1, x^2) \in \mathbb{R} \times \mathbb{R} \times \mathbb{T} \mid t = t'\}, \tag{3.6.4a}$$

$$\Sigma_{t'}^{u'} := \{(t, x^1, x^2) \in \mathbb{R} \times \mathbb{R} \times \mathbb{T} \mid t = t',\ 0 \leq u(t, x^1, x^2) \leq u'\}, \tag{3.6.4b}$$

$$\mathcal{P}_{u'}^{t'} := \{(t, x^1, x^2) \in \mathbb{R} \times \mathbb{R} \times \mathbb{T} \mid 1 \leq t \leq t',\ u(t, x^1, x^2) = u'\}, \tag{3.6.4c}$$

$$\ell_{t',u'} := \mathcal{P}_{u'}^{t'} \cap \Sigma_{t'}^{u'} = \{(t, x^1, x^2) \in \mathbb{R} \times \mathbb{R} \times \mathbb{T} \mid t = t',\ u(t, x^1, x^2) = u'\}, \tag{3.6.4d}$$

$$\mathcal{M}_{t',u'} := \cup_{u \in [0,u']} \mathcal{P}_u^{t'} \cap \{(t, x^1, x^2) \in \mathbb{R} \times \mathbb{R} \times \mathbb{T} \mid 1 \leq t < t'\}. \tag{3.6.4e}$$

We refer to the Σ_t and Σ_t^u as "constant time slices," the $\mathcal{P}_u^t$ as "null hyperplanes," and the $\ell_{t,u}$ as "curves" or "tori." We sometimes use the notation $\mathcal{P}_u$ in place of $\mathcal{P}_u^t$ when we are not concerned with the truncation time t. Note that $\mathcal{M}_{t,u}$ is "open at the top" by construction.

We now construct a local coordinate function on the tori $\ell_{t,u}$.

Definition 3.10 (**Geometric torus coordinate**)**.** We define the geometric torus coordinate ϑ to be the solution to the following transport equation:

$$(g^{-1})^{\alpha\beta}\partial_\alpha u \partial_\beta \vartheta = 0, \tag{3.6.5}$$

$$\vartheta|_{\Sigma_0} = x^2. \tag{3.6.6}$$

Definition 3.11 (**Geometric coordinates and partial derivatives**)**.** We refer to (t, u, ϑ) as the geometric coordinates, where t is the Cartesian time coordinate. We denote the corresponding geometric coordinate partial derivative vectorfields by

$$\left\{ \frac{\partial}{\partial t}, \frac{\partial}{\partial u}, \Theta := \frac{\partial}{\partial \vartheta} \right\}. \tag{3.6.7}$$

Remark 3.4. Θ is globally defined even though ϑ is only locally defined along $\ell_{t,u}$.

Definition 3.12. We define $\Upsilon : [0,T) \times [0,U_0] \times \mathbb{T} \to \mathcal{M}_{T,U_0}$, $\Upsilon(t,u,\vartheta) := (t, x^1, x^2)$, to be the change of variables map from geometric to Cartesian coordinates.

Remark 3.5 (C^1**-equivalent differential structures until shock formation**)**.** We often identify spacetime regions of the form $\mathcal{M}_{t,U_0}$ (see (3.6.4e)) with the region $[0,t) \times [0,U_0] \times \mathbb{T}$ corresponding to the geometric coordinates. This identification is justified by the fact that during the classical lifespan of the solutions under consideration, the differential structure on $\mathcal{M}_{t,U_0}$ corresponding to the geometric coordinates is C^1-equivalent to the differential structure on $\mathcal{M}_{t,U_0}$ corresponding to the Cartesian coordinates. The reason is that Υ is C^1 with a C^1 inverse until a shock forms; this fact was proved in [30, Theorem 15.1] and is revisited in limited form in the proof of Theorem 16.1. In contrast, at points where $\upmu$ vanishes, the partial derivatives of $\uprho$ and v^1 with respect to the Cartesian coordinates blow up, the inverse map Υ^{-1} becomes singular, and the equivalence of the differential structures breaks down.

3.7. **Important vectorfields, the rescaled frame, and the unit frame.** In this section, we define some vectorfields that we use in our analysis and exhibit their basic properties.

We start by defining the (negative) gradient vectorfield associated to the eikonal function:

$$L_{(Geo)}^\nu := -(g^{-1})^{\nu\alpha}\partial_\alpha u. \tag{3.7.1}$$

It is easy to see that $L_{(Geo)}$ is future-directed[98] with

$$g(L_{(Geo)}, L_{(Geo)}) := g_{\alpha\beta} L_{(Geo)}^\alpha L_{(Geo)}^\beta = 0, \tag{3.7.2}$$

that is, $L_{(Geo)}$ is g-null. Moreover, we can differentiate the eikonal equation with $\mathscr{D}^\nu := (g^{-1})^{\nu\alpha}\mathscr{D}_\alpha$ and use the torsion-free property of the connection $\mathscr{D}$ to deduce that $0 = (g^{-1})^{\alpha\beta}\mathscr{D}_\alpha u \mathscr{D}_\beta \mathscr{D}^\nu u = -\mathscr{D}^\alpha u \mathscr{D}_\alpha L_{(Geo)}^\nu = L_{(Geo)}^\alpha \mathscr{D}_\alpha L_{(Geo)}^\nu$. That is, $L_{(Geo)}$ is geodesic:

$$\mathscr{D}_{L_{(Geo)}} L_{(Geo)} = 0. \tag{3.7.3}$$

In addition, since $L_{(Geo)}$ is proportional to the metric dual of the one-form du, which is co-normal to the level sets $\mathcal{P}_u$ of the eikonal function, it follows that $L_{(Geo)}$ is g-orthogonal to $\mathcal{P}_u$. Hence, the $\mathcal{P}_u$ have null normals. Such hypersurfaces are known as *null hypersurfaces*

[98]Here and throughout, a vectorfield V is "future-directed" if its Cartesian component V^0 is positive.

or *characteristics*. Our analysis will show that the Cartesian components of $L_{(Geo)}$ blow up when the shock forms.

In our analysis, we work with a rescaled version of $L_{(Geo)}$ that we denote by L. Our proof reveals that the Cartesian components of L remain near those of $L_{(Flat)} := \partial_t + \partial_1$ all the way up to the shock.

Definition 3.13 (Rescaled null vectorfield). We define the rescaled null (see (3.7.2)) vectorfield L as follows:

$$L := \upmu L_{(Geo)}. \tag{3.7.4}$$

Note that L is g-null since $L_{(Geo)}$ is. We also note that by (3.6.5), we have

$$L\vartheta = 0. \tag{3.7.5}$$

We now define the vectorfields $\breve{X}$ and X, which are transversal to the characteristics $\mathcal{P}_u$. It is critically important for our work that $\breve{X}$ is rescaled by a factor of $\upmu$.

Definition 3.14 (X and $\breve{X}$). We define X to be the unique vectorfield that is Σ_t-tangent, g-orthogonal to the $\ell_{t,u}$, and normalized by

$$g(L, X) = -1. \tag{3.7.6}$$

We define

$$\breve{X} := \upmu X. \tag{3.7.7}$$

We use the following two vectorfield frames in our analysis.

Definition 3.15 (Two frames). We define, respectively, the rescaled frame and the non-rescaled frame as follows:

$$\{L, \breve{X}, \Theta\}, \qquad \text{Rescaled frame,} \tag{3.7.8a}$$

$$\{L, X, \Theta\}, \qquad \text{Non-rescaled frame.} \tag{3.7.8b}$$

In the next lemma, we exhibit the basic properties of some of the vectorfields that we have defined.

Lemma 3.2 (Basic properties of X, $\breve{X}$, L, and B). *The following identities hold:*

$$Lu = 0, \qquad Lt = L^0 = 1, \tag{3.7.9a}$$

$$\breve{X}u = 1, \qquad \breve{X}t = \breve{X}^0 = 0, \tag{3.7.9b}$$

$$g(X, X) = 1, \qquad g(\breve{X}, \breve{X}) = \upmu^2, \tag{3.7.10a}$$

$$g(L, X) = -1, \qquad g(L, \breve{X}) = -\upmu. \tag{3.7.10b}$$

Moreover, relative to the geometric coordinates, we have

$$L = \frac{\partial}{\partial t}. \tag{3.7.11}$$

In addition, there exists an $\ell_{t,u}$-tangent vectorfield $\Xi = \xi\Theta$ (where ξ is a scalar-valued function) such that

$$\breve{X} = \frac{\partial}{\partial u} - \Xi = \frac{\partial}{\partial u} - \xi\Theta. \tag{3.7.12}$$

The material derivative vectorfield B defined in (3.3.9) *is future-directed, g-orthogonal to Σ_t and is normalized by*

$$g(B,B) = -1. \tag{3.7.13}$$

In addition, relative to Cartesian coordinates, we have (for $\nu = 0,1,2$):

$$B^\nu = -(g^{-1})^{0\nu}. \tag{3.7.14}$$

Moreover, we have

$$B = L + X. \tag{3.7.15}$$

Finally, the following identities hold relative to the Cartesian coordinates (for $\nu = 0,1,2$):

$$X_\nu = -L_\nu - \delta_\nu^0, \qquad X^\nu = -L^\nu - (g^{-1})^{0\nu}, \tag{3.7.16}$$

where δ_ν^0 is the standard Kronecker delta.

Proof. The identity (3.7.14) follows trivially from (3.3.10b). The remaining statements in the lemmas were proved in [30, Lemma 2.1], where the vectorfield B was denoted by N. ☐

3.8. **Projection tensorfields, $\vec{G}_{(Frame)}$, and projected Lie derivatives.** Many of our constructions involve projections onto Σ_t and $\ell_{t,u}$.

Definition 3.16 (Projection tensorfields). We define the Σ_t projection tensorfield $\underline{\Pi}$ and the $\ell_{t,u}$ projection tensorfield $\not{\Pi}$ relative to Cartesian coordinates as follows:

$$\underline{\Pi}_\nu{}^\mu := \delta_\nu{}^\mu - B_\nu B^\mu = \delta_\nu{}^\mu + \delta_\nu{}^0 L^\mu + \delta_\nu{}^0 X^\mu, \tag{3.8.1a}$$

$$\not{\Pi}_\nu{}^\mu := \delta_\nu{}^\mu + X_\nu L^\mu + L_\nu(L^\mu + X^\mu) = \delta_\nu{}^\mu - \delta_\nu{}^0 L^\mu + L_\nu X^\mu. \tag{3.8.1b}$$

In (3.8.1a)-(3.8.1b), $\delta_\nu{}^\mu$ is the standard Kronecker delta.

Definition 3.17 (Projections of tensorfields). Given any spacetime tensorfield ξ, we define its Σ_t projection $\underline{\Pi}\xi$ and its $\ell_{t,u}$ projection $\not{\Pi}\xi$ as follows:

$$(\underline{\Pi}\xi)^{\mu_1\cdots\mu_m}_{\nu_1\cdots\nu_n} := \underline{\Pi}^{\mu_1}_{\widetilde{\mu}_1}\cdots\underline{\Pi}^{\mu_m}_{\widetilde{\mu}_m}\underline{\Pi}^{\widetilde{\nu}_1}_{\nu_1}\cdots\underline{\Pi}^{\widetilde{\nu}_n}_{\nu_n}\xi^{\widetilde{\mu}_1\cdots\widetilde{\mu}_m}_{\widetilde{\nu}_1\cdots\widetilde{\nu}_n}, \tag{3.8.2a}$$

$$(\not{\Pi}\xi)^{\mu_1\cdots\mu_m}_{\nu_1\cdots\nu_n} := \not{\Pi}^{\mu_1}_{\widetilde{\mu}_1}\cdots\not{\Pi}^{\mu_m}_{\widetilde{\mu}_m}\not{\Pi}^{\widetilde{\nu}_1}_{\nu_1}\cdots\not{\Pi}^{\widetilde{\nu}_n}_{\nu_n}\xi^{\widetilde{\mu}_1\cdots\widetilde{\mu}_m}_{\widetilde{\nu}_1\cdots\widetilde{\nu}_n}. \tag{3.8.2b}$$

We say that a spacetime tensorfield ξ is Σ_t-tangent (respectively $\ell_{t,u}$-tangent) if $\underline{\Pi}\xi = \xi$ (respectively if $\not{\Pi}\xi = \xi$). Alternatively, we say that ξ is a Σ_t tensor (respectively $\ell_{t,u}$ tensor).

Definition 3.18 ($\ell_{t,u}$ projection notation). If ξ is a spacetime tensor, then we define

$$\not{\xi} := \not{\Pi}\xi. \tag{3.8.3}$$

If ξ is a symmetric type $\binom{0}{2}$ spacetime tensor and V is a spacetime vectorfield, then we define

$$\not{\xi}_V := \not{\Pi}(\xi_V), \tag{3.8.4}$$

where ξ_V is the spacetime one-form with Cartesian components $\xi_{\alpha\nu}V^\alpha$, $(\nu = 0,1,2)$.

Throughout, $\mathcal{L}_V \xi$ denotes the Lie derivative of the tensorfield ξ with respect to the vectorfield V. We often use the Lie bracket notation $[V, W] := \mathcal{L}_V W$ when V and W are vectorfields.

In our analysis, we will apply the Leibniz rule for Lie derivatives to contractions of tensor products of $\ell_{t,u}$-tensorfields. Due in part to the special properties (such as (3.13.5)) of the vectorfields that we use to differentiate, the non-$\ell_{t,u}$ components of the differentiated factor in the products typically cancel. This motivates the following definition.

Definition 3.19 (**$\ell_{t,u}$ and Σ_t-projected Lie derivatives**)**.** Given a tensorfield ξ and a vectorfield V, we define the Σ_t-projected Lie derivative $\underline{\mathcal{L}}_V \xi$ of ξ and the $\ell_{t,u}$-projected Lie derivative $\not{\mathcal{L}}_V \xi$ of ξ as follows:

$$\underline{\mathcal{L}}_V \xi := \underline{\Pi} \mathcal{L}_V \xi, \qquad \not{\mathcal{L}}_V \xi := \not{\Pi} \mathcal{L}_V \xi. \tag{3.8.5}$$

Definition 3.20 (**Components of $\vec{G}$ and $\vec{H}$ relative to the non-rescaled frame**)**.** We define

$$\vec{G}_{(Frame)} := \left\{ \vec{G}_{LL}, \vec{G}_{LX}, \vec{\not{G}}_L, \vec{\not{G}}_X, \vec{\not{G}} \right\}, \qquad \vec{H}_{(Frame)} := \left\{ \vec{H}_{LL}, \vec{H}_{LX}, \vec{\not{H}}_L, \vec{\not{H}}_X, \vec{\not{H}} \right\} \tag{3.8.6}$$

to be the arrays of components of the tensorfield arrays (3.5.5b) and (3.5.5d) relative the non-rescaled frame (3.7.8b).

We adopt the convention that when we differentiate $\vec{G}_{(Frame)}$ or $\vec{H}_{(Frame)}$, we by definition form a new array consisting of the differentiated components. For example,

$$\not{\mathcal{L}}_L \vec{G}_{(Frame)} := \left\{ L(\vec{G}_{LL}), L(\vec{G}_{LX}), \not{\mathcal{L}}_L(\vec{\not{G}}_L), \not{\mathcal{L}}_L(\vec{\not{G}}_X), \not{\mathcal{L}}_L \vec{\not{G}} \right\}, \tag{3.8.7}$$

where $L(\vec{G}_{LL}) := \left\{ L(G^0_{LL}), L(G^1_{LL}), L(G^2_{LL}) \right\}$, $\not{\mathcal{L}}_L(\vec{\not{G}}_X) := \left\{ \not{\mathcal{L}}_L(\not{G}^0_X), \not{\mathcal{L}}_L(\not{G}^1_X), \not{\mathcal{L}}_L(\not{G}^2_X) \right\}$, etc.

3.9. **First and second fundamental forms and covariant differential operators.**

Definition 3.21 (**First fundamental forms**)**.** We define the first fundamental form $\underline{g}$ of Σ_t and the first fundamental form $\not{g}$ of $\ell_{t,u}$ as follows:

$$\underline{g} := \underline{\Pi} g, \qquad \not{g} := \not{\Pi} g. \tag{3.9.1}$$

We define the inverse first fundamental forms by raising the indices with g^{-1}:

$$(\underline{g}^{-1})^{\mu\nu} := (g^{-1})^{\mu\alpha}(g^{-1})^{\nu\beta}\underline{g}_{\alpha\beta}, \qquad (\not{g}^{-1})^{\mu\nu} := (g^{-1})^{\mu\alpha}(g^{-1})^{\nu\beta}\not{g}_{\alpha\beta}. \tag{3.9.2}$$

Note that $\underline{g}$ is the Riemannian metric on Σ_t induced by g and that $\not{g}$ is the Riemannian metric on $\ell_{t,u}$ induced by g. Moreover, simple calculations yield $(\underline{g}^{-1})^{\mu\alpha}\underline{g}_{\alpha\nu} = \underline{\Pi}_\nu{}^\mu$ and $(\not{g}^{-1})^{\mu\alpha}\not{g}_{\alpha\nu} = \not{\Pi}_\nu{}^\mu$.

Remark 3.6. Because the $\ell_{t,u}$ are one-dimensional manifolds, it follows that symmetric type $\binom{0}{2}$ $\ell_{t,u}$-tangent tensorfields ξ satisfy $\xi = (\mathrm{tr}_{\not{g}}\xi)\not{g}$, where $\mathrm{tr}_{\not{g}}\xi := \not{g}^{-1} \cdot \xi$. This simple fact simplifies some of our formulas compared to the case of higher space dimensions. In the remainder of the article, we often use this fact without explicitly mentioning it.

Definition 3.22 (**Differential operators associated to the metrics**)**.** We use the following notation for various differential operators associated to the spacetime metric g and the Riemannian metric $\not{g}$ induced on $\ell_{t,u}$.

- $\mathscr{D}$ denotes the Levi-Civita connection of the acoustical metric g.
- $\nabla\!\!\!/$ denotes the Levi-Civita connection of $g\!\!\!/$.
- If ξ is an $\ell_{t,u}$-tangent one-form, then $\mathrm{di}\!\!\!/\mathrm{v}\,\xi$ is the scalar-valued function $\mathrm{di}\!\!\!/\mathrm{v}\,\xi := g\!\!\!/^{-1} \cdot \nabla\!\!\!/\xi$.
- Similarly, if V is an $\ell_{t,u}$-tangent vectorfield, then $\mathrm{di}\!\!\!/\mathrm{v}\,V := g\!\!\!/^{-1} \cdot \nabla\!\!\!/V_\flat$, where $V_\flat$ is the one-form $g\!\!\!/$-dual to V.
- If ξ is a symmetric type $\binom{0}{2}$ $\ell_{t,u}$-tangent tensorfield, then $\mathrm{di}\!\!\!/\mathrm{v}\,\xi$ is the $\ell_{t,u}$-tangent one-form $\mathrm{di}\!\!\!/\mathrm{v}\,\xi := g\!\!\!/^{-1} \cdot \nabla\!\!\!/\xi$, where the two contraction indices in $\nabla\!\!\!/\xi$ correspond to the operator $\nabla\!\!\!/$ and the first index of ξ.
- $\Delta\!\!\!/ := g\!\!\!/^{-1} \cdot \nabla\!\!\!/^2$ denotes the covariant Laplacian corresponding to $g\!\!\!/$.

Definition 3.23 (Geometric torus differential). If f is a scalar-valued function on $\ell_{t,u}$, then $d\!\!\!/ f := \nabla\!\!\!/ f = \Pi\!\!\!/ \mathscr{D} f$, where $\mathscr{D} f$ is the gradient one-form associated to f.

Def. 3.23 allows us to avoid potentially confusing notation such as $\nabla\!\!\!/ L^i$ by instead writing $d\!\!\!/ L^i$; the latter notation signifies to view L^i as a scalar function under differentiation.

Definition 3.24 (Second fundamental forms). We define the second fundamental form k of Σ_t, by

$$k := \frac{1}{2}\underline{\mathcal{L}}_B \underline{g}. \tag{3.9.3}$$

We define the null second fundamental form χ of $\ell_{t,u}$ by

$$\chi := \frac{1}{2}\mathcal{L}\!\!\!/_L g\!\!\!/. \tag{3.9.4}$$

As was shown in [30, Subsection 2.6], we have the following alternate expressions:

$$k = \frac{1}{2}\underline{\mathcal{L}}_B g, \qquad \chi = \frac{1}{2}\mathcal{L}\!\!\!/_L g. \tag{3.9.5}$$

Lemma 3.3. [30, Lemma 2.3; **Alternate expressions for the second fundamental forms**] *We have the following identities:*

$$\chi_{\Theta\Theta} = g(\mathscr{D}_\Theta L, \Theta), \qquad k\!\!\!/_{X\Theta} = g(\mathscr{D}_\Theta L, X). \tag{3.9.6}$$

Lemma 3.4. [30, Lemma 2.13; **Decompositions of some $\ell_{t,u}$ tensorfields into μ^{-1}-singular and μ^{-1}-regular pieces**] *Let ζ be the $\ell_{t,u}$-tangent one-form defined by (see* (3.9.6)*)*

$$\zeta_\Theta := k\!\!\!/_{X\Theta} = g(\mathscr{D}_\Theta L, X) = \mu^{-1} g(\mathscr{D}_\Theta L, \breve{X}). \tag{3.9.7}$$

Then we can decompose the frame components of the $\ell_{t,u}$-tangent tensorfields $k\!\!\!/$ and ζ into μ^{-1}-singular and μ^{-1}-regular pieces as follows:

$$\zeta = \mu^{-1}\zeta^{(Trans-\vec{\Psi})} + \zeta^{(Tan-\vec{\Psi})}, \tag{3.9.8a}$$

$$k\!\!\!/ = \mu^{-1}k\!\!\!/^{(Trans-\vec{\Psi})} + k\!\!\!/^{(Tan-\vec{\Psi})}, \tag{3.9.8b}$$

where

$$\zeta^{(Trans-\vec{\Psi})} := -\frac{1}{2}\vec{\mathcal{G}}_L \diamond \breve{X}\vec{\Psi}, \tag{3.9.9a}$$

$$\not{k}^{(Trans-\vec{\Psi})} := \frac{1}{2}\vec{\mathcal{G}} \diamond \breve{X}\vec{\Psi}, \tag{3.9.9b}$$

$$\zeta^{(Tan-\vec{\Psi})} := \frac{1}{2}\vec{\mathcal{G}}_X \diamond L\vec{\Psi} - \frac{1}{2}\vec{G}_{LX} \diamond \not{d}\vec{\Psi} - \frac{1}{2}\vec{G}_{XX} \diamond \not{d}\vec{\Psi}, \tag{3.9.9c}$$

$$\not{k}^{(Tan-\vec{\Psi})} := \frac{1}{2}\vec{\mathcal{G}} \diamond L\vec{\Psi} - \frac{1}{2}\vec{\mathcal{G}}_L \overset{\diamond}{\otimes} \not{d}\vec{\Psi} - \frac{1}{2}\not{d}\vec{\Psi} \overset{\diamond}{\otimes} \vec{\mathcal{G}}_L - \frac{1}{2}\vec{\mathcal{G}}_X \overset{\diamond}{\otimes} \not{d}\vec{\Psi} - \frac{1}{2}\not{d}\vec{\Psi} \overset{\diamond}{\otimes} \vec{\mathcal{G}}_X. \tag{3.9.9d}$$

3.10. **Pointwise norms.** We always measure the magnitude of $\ell_{t,u}$ tensors using the Riemannian metric $\not{g}$, as is captured by the following definition.

Definition 3.25 (Pointwise norms). If $\xi_{\nu_1\cdots\nu_n}^{\mu_1\cdots\mu_m}$ is a type $\binom{m}{n}$ $\ell_{t,u}$ tensor, then we define the norm $|\xi| \geq 0$ by

$$|\xi|^2 := \not{g}_{\mu_1\widetilde{\mu}_1}\cdots\not{g}_{\mu_m\widetilde{\mu}_m}(\not{g}^{-1})^{\nu_1\widetilde{\nu}_1}\cdots(\not{g}^{-1})^{\nu_n\widetilde{\nu}_n}\xi_{\nu_1\cdots\nu_n}^{\mu_1\cdots\mu_m}\xi_{\widetilde{\nu}_1\cdots\widetilde{\nu}_n}^{\widetilde{\mu}_1\cdots\widetilde{\mu}_m}. \tag{3.10.1}$$

In (3.10.1), $\not{g}$ is the Riemannian metric on $\ell_{t,u}$ induced by g, as given by Def. 3.21.

3.11. **Expressions for the metrics.**

Lemma 3.5. [30, Lemma 2.4; **Expressions for g and g^{-1} in terms of the non-rescaled frame**] *We have the following identities:*

$$g_{\mu\nu} = -L_\mu L_\nu - (L_\mu X_\nu + X_\mu L_\nu) + \not{g}_{\mu\nu}, \tag{3.11.1a}$$

$$(g^{-1})^{\mu\nu} = -L^\mu L^\nu - (L^\mu X^\nu + X^\mu L^\nu) + (\not{g}^{-1})^{\mu\nu}. \tag{3.11.1b}$$

The following scalar-valued function captures the $\ell_{t,u}$ part of g.

Definition 3.26 (The metric component υ). We define the function $\upsilon > 0$ by

$$\upsilon^2 := g(\Theta,\Theta) = \not{g}(\Theta,\Theta). \tag{3.11.2}$$

It follows that relative to the geometric coordinates, we have $\not{g}^{-1} = \upsilon^{-2}\Theta \otimes \Theta$.

Lemma 3.6. [30, Corollary 2.6; **The geometric volume form factors of g and $\underline{g}$**] *The following identity is verified by the acoustcial metric g:*

$$|\det g| = \mu^2\upsilon^2, \tag{3.11.3}$$

where the determinant on the LHS is taken relative to the geometric coordinates (t,u,ϑ).

Furthermore, the following identity is verified by the first fundamental form $\underline{g}$ of $\Sigma_t^{U_0}$:

$$\det\underline{g}|_{\Sigma_t^{U_0}} = \mu^2\upsilon^2, \tag{3.11.4}$$

where the determinant on the LHS is taken relative to the geometric coordinates (u,ϑ) induced on $\Sigma_t^{U_0}$.

3.12. **Commutation vectorfields.** To derive estimates for the solution's higher-order derivatives, we commute the equations with the elements of $\{L, \breve{X}, Y\}$, where Y is the $\ell_{t,u}$-tangent vectorfield given in the next definition. We use Y rather than Θ because commuting Θ through $\square_g$ seems to produce error terms that are uncontrollable because they lose a derivative.

Definition 3.27 (**The vectorfields $Y_{(Flat)}$ and Y**)**.** We define the Cartesian components of the Σ_t-tangent vectorfields $Y_{(Flat)}$ and Y as follows ($i = 1, 2$):

$$Y^i_{(Flat)} := \delta^i_2, \tag{3.12.1}$$

$$Y^i := \not{\Pi}_a{}^i Y^a_{(Flat)} = \not{\Pi}_2{}^i, \tag{3.12.2}$$

where $\not{\Pi}$ is the $\ell_{t,u}$ projection tensorfield defined in (3.8.1b).

When commuting the equations, we use elements of the commutation sets $\mathscr{Z}$ and $\mathscr{P}$.

Definition 3.28 (**Commutation vectorfields**)**.** We define the commutation set $\mathscr{Z}$ as follows:

$$\mathscr{Z} := \{L, \breve{X}, Y\}, \tag{3.12.3}$$

where L, $\breve{X}$, and Y are respectively defined by (3.7.4), (3.7.7), and (3.12.2).

We define the $\mathcal{P}_u$-tangent commutation set $\mathscr{P}$ as follows:

$$\mathscr{P} := \{L, Y\}. \tag{3.12.4}$$

The Cartesian spatial components of L, X, and Y deviate from their flat values by a small amount that we denote by $L^i_{(Small)}$, $X^i_{(Small)}$, and $Y^i_{(Small)}$.

Definition 3.29 (**Perturbed part of various vectorfields**)**.** For $i = 1, 2$, we define the following scalar-valued functions:

$$L^i_{(Small)} := L^i - \delta^i_1, \qquad X^i_{(Small)} := X^i + \delta^i_1, \qquad Y^i_{(Small)} := Y^i - \delta^i_2. \tag{3.12.5}$$

Lemma 3.7 (**Identity connecting $L^i_{(Small)}$, $X^i_{(Small)}$, and v^i**)**.** *The following identity holds:*

$$X^i_{(Small)} = -L^i_{(Small)} + v^i. \tag{3.12.6}$$

Proof. The identity (3.12.6) follows from (3.5.1), (3.5.3), (3.7.16), (3.12.5), and the fact that $(g^{-1})^{0i} = -v^i$. □

In the next lemma, we characterize the discrepancy between $Y_{(Flat)}$ and Y.

Lemma 3.8. [30, Lemma 2.8; **Decomposition of** $Y_{(Flat)}$] *We can decompose $Y_{(Flat)}$ into an $\ell_{t,u}$-tangent vectorfield and a vectorfield parallel to X as follows: since Y is $\ell_{t,u}$-tangent, there exists a scalar-valued function y such that*

$$Y^i_{(Flat)} = Y^i + yX^i, \tag{3.12.7a}$$

$$Y^i_{(Small)} = -yX^i. \tag{3.12.7b}$$

Moreover, we have

$$y = g(Y_{(Flat)}, X) = g_{ab} Y^a_{(Flat)} X^b = g_{2a} X^a = -c_s^{-2} X^2_{(Small)}. \tag{3.12.8}$$

3.13. **Deformation tensors and basic vectorfield commutator properties.** In this section, we recall the standard definition of the deformation of a vectorfield V. We then provide some simple commutator lemmas.

Definition 3.30 (Deformation tensor of a vectorfield V). If V is a spacetime vectorfield, then its deformation tensor ${}^{(V)}\pi$ (relative to g) is the symmetric type $\binom{0}{2}$ tensorfield

$$ {}^{(V)}\pi_{\alpha\beta} := \mathcal{L}_V g_{\alpha\beta} = \mathscr{D}_\alpha V_\beta + \mathscr{D}_\beta V_\alpha, \tag{3.13.1} $$

where the second equality follows from the torsion-free property of $\mathscr{D}$.

Lemma 3.9 (Basic vectorfield commutator properties). *The vectorfields $[L, \breve{X}]$, $[L, Y]$, and $[\breve{X}, Y]$ are $\ell_{t,u}$-tangent, and the following identities hold:*

$$ [L, \breve{X}] = {}^{(\breve{X})}\not\pi_L^{\#}, \qquad [L, Y] = {}^{(Y)}\not\pi_L^{\#}, \qquad [\breve{X}, Y] = {}^{(Y)}\not\pi_{\breve{X}}^{\#}. \tag{3.13.2} $$

In addition, we have

$$ [\mu B, L] = -(L\mu)L + {}^{(L)}\not\pi_{\breve{X}}^{\#}, \tag{3.13.3a} $$

$$ [\mu B, Y] = -(Y\mu)L + \mu {}^{(Y)}\not\pi_L^{\#} + {}^{(Y)}\not\pi_{\breve{X}}^{\#}. \tag{3.13.3b} $$

Furthermore, if $Z \in \mathscr{Z}$, then

$$ \not\mathcal{L}_Z \not g = {}^{(Z)}\not\pi, \qquad \not\mathcal{L}_Z \not g^{-1} = -{}^{(Z)}\not\pi^{\#\#}. \tag{3.13.4} $$

Finally, if V is an $\ell_{t,u}$-tangent vectorfield, then

$$ [L, V] \text{ and } [\breve{X}, V] \text{ are } \ell_{t,u} - \text{tangent}. \tag{3.13.5} $$

Proof. All aspects of the proposition except for (3.13.3a)-(3.13.3b) were derived in [30, Lemma 2.9]. The identities (3.13.3a)-(3.13.3b) are straightforward consequences of the decomposition (3.7.15) and the identities in (3.13.2). ☐

Lemma 3.10. [30, Lemma 2.10; L, $\breve{X}$, Y **commute with** $\not d$] *If $V \in \{L, \breve{X}, Y\}$ and f is a scalar-valued function, then*

$$ \not\mathcal{L}_V \not d f = \not d V f. \tag{3.13.6} $$

3.14. **Transport equations for the eikonal function quantities.** We now provide transport equations verified by the scalar-valued functions μ and $L^i_{(Small)}$. These are the main equations we use to estimate the eikonal function quantities below-top-order. For top-order estimates, we use the modified quantities of Sect. 7.

Lemma 3.11. [30, Lemma 2.12; **The transport equations verified by μ and L^i**] *The inverse foliation density μ defined in (3.6.3) verifies the following transport equation:*

$$ L\mu = \frac{1}{2}\vec{G}_{LL} \diamond \breve{X}\vec{\Psi} - \frac{1}{2}\mu\vec{G}_{LL} \diamond L\vec{\Psi} - \mu\vec{G}_{LX} \diamond L\vec{\Psi}. \tag{3.14.1} $$

The scalar-valued Cartesian component functions $L^i_{(Small)}$, $(i = 1, 2)$, defined in (3.12.5), verify the following transport equation:

$$ \begin{aligned} LL^i_{(Small)} = & -\frac{1}{2}\vec{G}_{LL} \diamond (L\vec{\Psi})L^i + \frac{1}{2}\vec{G}_{LL} \diamond (L\vec{\Psi})v^i \\ & - \vec{\not G}_L^{\#} \diamond (L\vec{\Psi}) \cdot (\not d x^i) + \frac{1}{2}\vec{G}_{LL} \diamond (\not d^{\#}\vec{\Psi}) \cdot \not d x^i. \end{aligned} \tag{3.14.2} $$

3.15. **Calculations connected to the failure of the null condition.** Many of our most important estimates are tied to the coefficients $\vec{G}_{LL}$. In the next lemma, we derive expressions for them. Then, in the subsequent lemma, we derive an expression for the product $\frac{1}{2}\vec{G}_{LL} \diamond \breve{X}\vec{\Psi}$. This presence of this product is tied to the failure of Klainerman's null condition [19] and thus one expects that the product must be non-zero for shocks to form; this is explained in the survey article [15] in a slightly different context.

Lemma 3.12 (**Formula for** G_{LL}^{ι})**.** *Let* $G_{\alpha\beta}^{\iota}$ *be as in Def. 3.5. Then for* $i = 1, 2$, *we have*

$$G_{LL}^0 = -2c_s^{-1}c_s', \tag{3.15.1a}$$

$$G_{LL}^i = 2c_s^{-2}(v^i - L^i) = 2c_s^{-2}X^i. \tag{3.15.1b}$$

Proof. We first prove (3.15.1b). From the formula (3.3.10a), Defs. 3.5 and 3.6, the fact that $L^0 = 1$, the identity (3.7.16), and the fact that $L^i + X^i = v^i$ (see (3.3.9) and (3.7.15)), we compute the desired identity as follows:

$$\begin{aligned} G_{LL}^i = \left(\frac{\partial}{\partial v^i} g_{\alpha\beta}\right) L^\alpha L^\beta \left(\frac{\partial}{\partial v^i} g_{\alpha\beta}\right) L^\alpha L^\beta &= 2c_s^{-2} v^i (L^0)^2 - 2c_s^{-2} L^0 L^i \\ &= 2c_s^{-1}(v^i - L^i) = 2c_s^{-2}X^i. \end{aligned} \tag{3.15.2}$$

We now prove (3.15.1a). Since $g_{\alpha\beta}L^\alpha L^\beta = 0$, it suffices to prove $\left(\frac{\partial}{\partial \rho}(c_s^2 g_{\alpha\beta})\right) L^\alpha L^\beta = -2c_s c_s'$. Since, among the components $\{c_s^2 g_{\alpha\beta}\}_{\alpha,\beta=0,1,2}$, only $c_s^2 g_{00}$ depends on ρ (see (3.3.10a)), the desired identity is a simple consequence of the fact that $L^0 = 1$.

□

Lemma 3.13 (**Formula for** $\frac{1}{2}\vec{G}_{LL} \diamond \breve{X}\vec{\Psi}$)**.** *For solutions to the compressible Euler equations* (1.0.1a)-(1.0.1b) *the following identity holds for the first product on RHS* (3.14.1)*:*

$$\frac{1}{2}\vec{G}_{LL} \diamond \breve{X}\vec{\Psi} = c_s^{-2}\left\{c_s^{-1}c_s' + 1\right\}\delta_{ab}X^a\breve{X}v^b + \mu c_s^{-3}c_s'\delta_{ab}Lv^aX^b. \tag{3.15.3}$$

Proof. Using Lemma 3.12, we deduce $\frac{1}{2}\vec{G}_{LL} \diamond \breve{X}\vec{\Psi} = -c_s^{-1}c_s'\breve{X}\rho + c_s^{-2}\delta_{ab}X^a\breve{X}v^b$. Contracting (1.0.1b) against $\delta_{ij}\breve{X}^j$ and inserting the resulting identity into the first product in the previous expression, we can rewrite it as $c_s^{-3}c_s'\delta_{ab}Bv^a\breve{X}^b + c_s^{-2}\delta_{ab}X^a\breve{X}v^b$. Using (3.7.15) to substitute $L + X$ for B in the previous expression and recalling that $\breve{X} = \mu X$, we conclude (3.15.3). □

Note that for the equation of state $p = C_0 - C_1\exp(-\rho)$ of a Chaplygin gas, we have $c_s^{-1}c_s' + 1 = 0$. For such a gas, the product $\frac{1}{2}\vec{G}_{LL} \diamond \breve{X}\vec{\Psi}$ vanishes and our main shock formation results do not apply. In fact, even in the plane symmetric case, it is not known whether shocks form in Chaplygin gas. In that case, only a very different type of singularity (where in particular the density itself blows up) is known to form [21]. Moreover, in the case of the Chaplygin gas without vorticity, the wave equations (3.3.11a)-(3.3.11b) verify Klainerman's null condition. While it is not directly related to the regime we study, we point out that in that case small-data global existence is known[99] [23] when the data are given on the Cauchy hypersurface $\mathbb{R}^2$.

[99]Note that the equation for the irrotational Chaplygin gas is equivalent to that of a Minkowskian minimal surface equation, which is treated in [23].

3.16. **Deformation tensor calculations.** In the next lemma, we provide explicit expressions for the frame components of the deformation tensors of the commutation vectorfields.

Lemma 3.14. [30, Lemma 2.18; **The frame components of** $^{(Z)}\pi$] *The following identities are verified by the deformation tensors (see Def. 3.30) of the elements of* $\mathscr{Z}$ *(see* (3.12.3)*):*

$$^{(\breve{X})}\pi_{LL} = 0, \qquad ^{(\breve{X})}\pi_{\breve{X}X} = 2\breve{X}\upmu, \qquad ^{(\breve{X})}\pi_{L\breve{X}} = -\breve{X}\upmu, \tag{3.16.1a}$$

$$^{(\breve{X})}\not{\pi}_{L} = -\not{d}\upmu - 2\zeta^{(Trans-\vec{\Psi})} - 2\upmu\zeta^{(Tan-\vec{\Psi})}, \qquad ^{(\breve{X})}\not{\pi}_{\breve{X}} = 0, \tag{3.16.1b}$$

$$^{(\breve{X})}\not{\pi} = -2\upmu \mathrm{tr}_{\not{g}}\chi \not{g} + 2\not{k}^{(Trans-\vec{\Psi})} + 2\upmu\not{k}^{(Tan-\vec{\Psi})}, \tag{3.16.1c}$$

$$^{(L)}\pi_{LL} = 0, \qquad ^{(L)}\pi_{\breve{X}X} = 2L\upmu, \qquad ^{(L)}\pi_{L\breve{X}} = -L\upmu, \tag{3.16.2a}$$

$$^{(L)}\not{\pi}_{L} = 0, \qquad ^{(L)}\not{\pi}_{\breve{X}} = \not{d}\upmu + 2\zeta^{(Trans-\vec{\Psi})} + 2\upmu\zeta^{(Tan-\vec{\Psi})}, \tag{3.16.2b}$$

$$^{(L)}\not{\pi} = 2\mathrm{tr}_{\not{g}}\chi\not{g}, \tag{3.16.2c}$$

$$^{(Y)}\pi_{LL} = 0, \qquad ^{(Y)}\pi_{\breve{X}X} = 2Y\upmu, \qquad ^{(Y)}\pi_{L\breve{X}} = -Y\upmu, \tag{3.16.3a}$$

$$\begin{aligned} ^{(Y)}\not{\pi}_{L} = & -\mathrm{tr}_{\not{g}}\chi Y_\flat + \frac{1}{2}(\vec{\not{G}}\cdot Y)\diamond L\vec{\Psi} + y\vec{\not{G}}_X \diamond L\vec{\Psi} \\ & + \frac{1}{2}(\vec{\not{G}}_L\cdot Y)\diamond \not{d}\vec{\Psi} - y\vec{G}_{LX}\diamond\not{d}\vec{\Psi} - \frac{1}{2}y\vec{G}_{XX}\diamond\not{d}\vec{\Psi}, \end{aligned} \tag{3.16.3b}$$

$$\begin{aligned} ^{(Y)}\not{\pi}_{\breve{X}} = & \ \upmu\mathrm{tr}_{\not{g}}\chi Y_\flat + y\not{d}\upmu + y\vec{\not{G}}_X \diamond \breve{X}\vec{\Psi} - \frac{1}{2}\upmu y\vec{G}_{XX}\diamond\not{d}\vec{\Psi} \\ & - \frac{1}{2}\upmu(\vec{\not{G}}\cdot Y)\diamond L\vec{\Psi} + \upmu(\vec{\not{G}}_L\cdot Y)\diamond\not{d}\vec{\Psi} + \upmu(\vec{\not{G}}_X\cdot Y)\diamond\not{d}\vec{\Psi}, \end{aligned} \tag{3.16.3c}$$

$$\begin{aligned} ^{(Y)}\not{\pi} = & \ 2y\mathrm{tr}_{\not{g}}\chi\not{g} + \frac{1}{2}(\vec{\not{G}}\cdot Y)\overset{\diamond}{\otimes}\not{d}\vec{\Psi} + \frac{1}{2}\not{d}\vec{\Psi}\overset{\diamond}{\otimes}(\vec{\not{G}}\cdot Y) - y\vec{\not{G}}\diamond L\vec{\Psi} \\ & + y\vec{\not{G}}_L\overset{\diamond}{\otimes}\not{d}\vec{\Psi} + y\not{d}\vec{\Psi}\overset{\diamond}{\otimes}\vec{\not{G}}_L + y\vec{\not{G}}_X\overset{\diamond}{\otimes}\not{d}\vec{\Psi} + y\not{d}\vec{\Psi}\overset{\diamond}{\otimes}\vec{\not{G}}_X. \end{aligned} \tag{3.16.3d}$$

The scalar-valued function y *from above is as in Lemma 3.8, while the* $\ell_{t,u}$*-tangent tensorfields* χ, $\zeta^{(Trans-\vec{\Psi})}$, $\not{k}^{(Trans-\vec{\Psi})}$, $\zeta^{(Tan-\vec{\Psi})}$, *and* $\not{k}^{(Tan-\vec{\Psi})}$ *from above are as in* (3.9.4)*,* (3.9.9a)*,* (3.9.9b)*,* (3.9.9c)*, and* (3.9.9c)*. In* (3.16.3d)*,* $\vec{\not{G}}_L \overset{\diamond}{\otimes} \not{d}\vec{\Psi} := \sum_{\imath=0}^{2} \not{G}_L^{\imath}\otimes\not{d}\vec{\Psi}_\imath$*, and similarly for the other terms involving* $\overset{\diamond}{\otimes}$.

3.17. **Useful expressions for the null second fundamental form.** The next lemma provides explicit formulas for χ, $\mathrm{tr}_{\not{g}}\chi$, and $L\ln\upsilon$.

Lemma 3.15. [30, Lemma 2.15; **Identities involving** χ] *Let* χ *be the* $\ell_{t,u}$ *tensorfield defined in* (3.9.4) *and let* υ *be the metric component from Def. 3.26. We have the following identities:*

$$\chi = g_{ab}(\not{d}L^a) \otimes \not{d}x^b + \frac{1}{2}\vec{\not{G}} \diamond L\vec{\Psi}, \tag{3.17.1a}$$

$$\mathrm{tr}_{\not{g}}\chi = g_{ab}\not{g}^{-1} \cdot \left\{(\not{d}L^a) \otimes \not{d}x^b\right\} + \frac{1}{2}\not{g}^{-1} \cdot \vec{\not{G}} \diamond L\vec{\Psi}, \tag{3.17.1b}$$

$$L \ln \upsilon = \mathrm{tr}_{\not{g}}\chi. \tag{3.17.1c}$$

3.18. **Decomposition of differential operators.** We start by decomposing $\mu\Box_{g(\vec{\Psi})}$ relative to the rescaled frame. The factor of μ is important for our decompositions.

Proposition 3.16 (**Frame decomposition of** $\mu\Box_{g(\vec{\Psi})}f$)**.** *Let f be a scalar-valued function. Then $\mu\Box_{g(\vec{\Psi})}f$ can be expressed in either of the following two forms:*

$$\mu\Box_{g(\vec{\Psi})}f = -L(\mu Lf + 2\breve{X}f) + \mu\not{\Delta}f - \mathrm{tr}_{\not{g}}\chi\breve{X}f - \mu\mathrm{tr}_{\not{g}}\not{k}\, Lf - 2\mu\zeta^{\#} \cdot \not{d}f, \tag{3.18.1a}$$

$$= -(\mu L + 2\breve{X})(Lf) + \mu\not{\Delta}f - \mathrm{tr}_{\not{g}}\chi\breve{X}f - (L\mu)Lf + 2\mu\zeta^{\#} \cdot \not{d}f + 2(\not{d}^{\#}\mu) \cdot \not{d}f, \tag{3.18.1b}$$

where the $\ell_{t,u}$-tangent tensorfields χ, ζ, and $\not{k}$ can be expressed via (3.17.1a), (3.9.8a), *and* (3.9.8b)*.*

Lemma 3.17 (**Expression for ∂_ν in terms of geometric vectorfields**)**.** *We can express the Cartesian coordinate partial derivative vectorfields in terms of L, X, and Y as follows, $(i = 1, 2)$:*

$$\partial_t = L - (g_{\alpha 0}L^\alpha)X + \left(\frac{g_{a0}Y^a}{g_{cd}Y^cY^d}\right)Y, \tag{3.18.2a}$$

$$\partial_i = (g_{ai}X^a)X + \left(\frac{g_{ai}Y^a}{g_{cd}Y^cY^d}\right)Y. \tag{3.18.2b}$$

Proof. We expand $\partial_i = \alpha_i X + \beta_i Y$ for scalar-valued functions α_i and β_i. Taking the g-inner product of each side with respect to X, we obtain $\alpha_i = g(X, \partial_i) = g_{ab}X^a\delta_i^b = g_{ai}X^a$. Similarly, we take the inner product with respect to Y to deduce $\beta_i g_{cd}Y^cY^d = g_{ai}Y^a$. Using these identities to substitute for α_i and β_i, we conclude (3.18.2b). A similar argument yields (3.18.2a), though in this case we must use an expansion of the form $\partial_t = \alpha L + \beta X + \gamma Y$; we omit the details. □

With the help of Lemma 3.17, we can now express the products on RHS (3.3.11a) involving $\partial_a\omega$ in terms of $\mathcal{P}_u$**-tangent** geometric derivatives of ω.

Corollary 3.18 (**Decomposition of the vorticity derivatives in equation** (3.3.11a))**.** *We have the following identity for the vorticity derivative-involving product on RHS* (3.3.11a)*:*

$$-[ia](\exp\rho)c_s^2(\mu\partial_a\omega) = [ia]\mu(\exp\rho)c_s^2(g_{ab}X^b)L\omega - [ia]\mu(\exp\rho)c_s^2\left(\frac{g_{ab}Y^b}{g_{cd}Y^cY^d}\right)Y\omega. \tag{3.18.3}$$

Proof. We first use the formula (3.18.2b) to express the factor $\partial_a\omega$ on LHS (3.18.3) in terms of $X\omega$ and $Y\omega$. We then use (3.3.11c) and (3.7.15) to replace $X\omega$ with $-L\omega$. □

3.19. **Arrays of fundamental unknowns and schematic notation.** In Lemma 3.19, we show that many scalar-valued functions and tensorfields that we have introduced depend on just a handful of more fundamental functions and tensorfields. This simplifies various aspects of our analysis. We start by introducing some convenient shorthand notation.

Definition 3.31 (Shorthand notation for the unknowns). We define the following arrays $\upgamma$ and $\underline{\upgamma}$ of scalar-valued functions:

$$\upgamma := \left(\vec{\Psi}, L^1_{(Small)}, L^2_{(Small)}\right), \qquad \underline{\upgamma} := \left(\vec{\Psi}, \upmu, L^1_{(Small)}, L^2_{(Small)}\right). \tag{3.19.1}$$

Remark 3.7 (Schematic functional dependence). Throughout, $\mathrm{f}(\xi_{(1)}, \xi_{(2)}, \cdots, \xi_{(m)})$ schematically denotes an expression (often tensorial and involving contractions) that depends smoothly on the $\ell_{t,u}$-tangent tensorfields $\xi_{(1)}, \xi_{(2)}, \cdots, \xi_{(m)}$. In general, we have $\mathrm{f}(0) \neq 0$. We sometimes use the notation $\vec{x} := (x^1, x^2)$ and $\not{d}\vec{x} := (\not{d}x^1, \not{d}x^2)$ in our schematic depictions.

Lemma 3.19 (Schematic structure of various tensorfields). *We have the following schematic relations for scalar-valued functions:*

$$g_{\alpha\beta}, (g^{-1})^{\alpha\beta}, (\underline{g}^{-1})^{ab}, \not{g}_{\alpha\beta}, (\not{g}^{-1})^{\alpha\beta}, G^{\imath}_{\alpha\beta}, H^{\imath\jmath}_{\alpha\beta}, \not{\Pi}_{\beta}^{\ \alpha}, L^\alpha, X^\alpha, Y^\alpha, c_s = \mathrm{f}(\upgamma), \tag{3.19.2a}$$

$$G^{\imath}_{LL}, G^{\imath}_{LX}, G^{\imath}_{XX}, H^{\imath\jmath}_{LL}, H^{\imath\jmath}_{LX}, H^{\imath\jmath}_{XX} = \mathrm{f}(\upgamma), \tag{3.19.2b}$$

$$g^{(Small)}_{\alpha\beta}, \not{\Pi}_{\beta}^{\ \alpha} - \delta_{\beta 2}\delta^{\alpha 2}, Y^\alpha_{(Small)}, X^\alpha_{(Small)}, y = \mathrm{f}(\upgamma)\upgamma, \tag{3.19.2c}$$

$$\breve{X}^\alpha = \mathrm{f}(\underline{\upgamma}). \tag{3.19.2d}$$

Moreover, we have the following schematic relations for $\ell_{t,u}$-tangent tensorfields:

$$\not{g}, \vec{\not{G}}_L, \vec{\not{G}}_X, \vec{\not{G}}, \not{H}^{\imath\jmath}_L, \not{H}^{\imath\jmath}_X, \not{H}^{\imath\jmath} = \mathrm{f}(\upgamma, \not{d}\vec{x}), \tag{3.19.3a}$$

$$Y = \mathrm{f}(\upgamma, \not{g}^{-1}, \not{d}\vec{x}), \tag{3.19.3b}$$

$$\upzeta^{(Tan-\vec{\Psi})}, \not{k}^{(Tan-\vec{\Psi})} = \mathrm{f}(\upgamma, \not{d}\vec{x})P\vec{\Psi}, \tag{3.19.3c}$$

$$\not{k}^{(Trans-\vec{\Psi})} = \mathrm{f}(\upgamma, \not{d}\vec{x})\breve{X}\vec{\Psi}, \tag{3.19.3d}$$

$$\upzeta^{(Trans-\vec{\Psi})} = \mathrm{f}(\breve{X}\vec{\Psi}, \not{d}\vec{x})\upgamma, \tag{3.19.3e}$$

$$\upchi = \mathrm{f}(\upgamma, \not{d}\vec{x})P\upgamma, \tag{3.19.3f}$$

$$\mathrm{tr}_{\not{g}}\upchi = \mathrm{f}(\upgamma, \not{g}^{-1}, \not{d}\vec{x})P\upgamma. \tag{3.19.3g}$$

Finally, the null forms $\mathscr{Q}^i$ and $\mathscr{Q}$ defined by (3.3.12a) *and* (3.3.12b)*, upon being multiplied by $\upmu$, have the following schematic structure:*

$$\upmu\mathscr{Q}^i, \ \upmu\mathscr{Q} = \mathrm{f}(\underline{\upgamma}, \breve{X}\vec{\Psi}, P\vec{\Psi})P\vec{\Psi}. \tag{3.19.4}$$

Proof. Except for (3.19.4), the desired relations were proved as [30, Lemma 2.19]. We now prove (3.19.4). The desired result for the term $c_s^{-1}c_s'(g^{-1})^{\alpha\beta}\partial_\alpha\uprho\partial_\beta\uprho$ on RHS (3.3.12b) is a simple consequence of the identity $g^{-1} = -L \otimes L - L \otimes X - X \otimes L + \frac{1}{g_{ab}Y^aY^b} Y \otimes Y$, which is easy to verify by contracting each side against the g-duals of the elements of $\{L, X, Y\}$. We now consider the quadratic term $\partial_1 v^1 \partial_2 v^2 - \partial_2 v^1 \partial_1 v^2$ on RHS (3.3.12b). Similar remarks apply to the quadratic term $-(g^{-1})^{\alpha\beta}\partial_\alpha\uprho\partial_\beta v^i$ on RHS (3.3.12a). We use (3.18.2b) to write the Cartesian coordinate partial derivatives in the previous expression in terms of X and

Y derivatives. In view of the antisymmetry of the expression in v^1 and v^2, we see that the terms proportional to $(Xv^1)Xv^2$ cancel (although it is not important for the main results of this paper, we note that the terms proportional to $(Yv^1)Yv^2$ also cancel). Multiplying by μ, we conclude that the quadratic term under consideration is of the form RHS (3.19.4). $\square$

3.20. **Geometric decompositions involving Y.** In this section, we express various $\ell_{t,u}$ tensorfields and operators in terms of Y. This allows for a simplified approach to deriving various formulas and estimates.

Lemma 3.20 (Formula for $\not{g}^{-1}$ in terms of Y). *Let $\not{g}^{-1}$ be the inverse of the first fundamental form $\not{g}$ from Def. 3.21 and let Y be the $\ell_{t,u}$-tangent vectorfield from Def. 3.27. We have the following identity:*

$$\not{g}^{-1} = \frac{1}{g(Y,Y)} Y \otimes Y. \tag{3.20.1}$$

Proof. Since the $\ell_{t,u}$ are one-dimensional, $\not{g}^{-1}$ must be a multiple of $Y \otimes Y$. Contracting (3.20.1) against $Y_\flat \otimes Y_\flat$, we easily obtain that the correct proportionality factor is $\frac{1}{g(Y,Y)}$. $\square$

Lemma 3.21 (ξ in terms of $\mathrm{tr}_{\not{g}}\xi$). *Let Y be the $\ell_{t,u}$-tangent vectorfield from Def. 3.27. We have the following identity, valid for symmetric type $\binom{0}{2}$ $\ell_{t,u}$ tensorfields ξ:*

$$\xi = \frac{1}{g(Y,Y)} \mathrm{tr}_{\not{g}}\xi Y_\flat \otimes Y_\flat. \tag{3.20.2}$$

Proof. Since $\ell_{t,u}$ is one-dimensional, we have $\xi = AY_\flat \otimes Y_\flat$ for some scalar-valued function A. Taking the $\not{g}$-trace of this equation, we find that $A = \frac{1}{g(Y,Y)} \mathrm{tr}_{\not{g}}\xi$ as desired. $\square$

4. Area and Volume Forms and Energy-Null Flux Identities

In this section, we first define geometric area and volume forms and corresponding integrals. Using these, we construct the energies and null fluxes that we use to control the solution and its derivatives in L^2. We then exhibit the basic coercive properties of the energies and null fluxes and provide the fundamental energy identities that we use to derive a priori estimates. There are two identities: one for wave equations, which we use to control $\vec{\Psi}$ (see Prop. 4.2), and one for transport equations, which we use to control ω (see Prop. 4.4).

4.1. **Area and volume forms and geometric integrals.** We define our geometric integrals in terms of length, area, and volume forms that remain non-degenerate throughout the evolution, all the way up to the shock.

Definition 4.1 (Non-degenerate forms and related integrals). We define the length form $d\lambda_{\not{g}}$ on $\ell_{t,u}$, the area form $d\underline{\varpi}$ on Σ_t^u, the area form $d\overline{\varpi}$ on $\mathcal{P}_u^t$, and the volume form $d\varpi$ on $\mathcal{M}_{t,u}$ as follows (relative to the geometric coordinates):

$$\begin{aligned} d\lambda_{\not{g}} &= d\lambda_{\not{g}}(t,u,\vartheta) := \upsilon(t,u,\vartheta)\, d\vartheta, & d\underline{\varpi} &= d\underline{\varpi}(t,u',\vartheta) := d\lambda_{\not{g}}(t,u',\vartheta)du', \\ d\overline{\varpi} &= d\overline{\varpi}(t',u,\vartheta) := d\lambda_{\not{g}}(t',u,\vartheta)dt', & d\varpi &= d\varpi(t',u',\vartheta') := d\lambda_{\not{g}}(t',u',\vartheta')du'dt', \end{aligned} \tag{4.1.1}$$

where υ is the scalar-valued function from Def. 3.26.

If f is a scalar-valued function, then we define

$$\int_{\ell_{t,u}} f\, d\lambda_{\not g} := \int_{\vartheta\in\mathbb{T}} f(t,u,\vartheta)\, \upsilon(t,u,\vartheta) d\vartheta, \tag{4.1.2a}$$

$$\int_{\Sigma_t^u} f\, d\underline{\varpi} := \int_{u'=0}^{u} \int_{\vartheta\in\mathbb{T}} f(t,u',\vartheta)\, \upsilon(t,u',\vartheta) d\vartheta du', \tag{4.1.2b}$$

$$\int_{\mathcal{P}_u^t} f\, d\overline{\varpi} := \int_{t'=0}^{t} \int_{\vartheta\in\mathbb{T}} f(t',u,\vartheta)\, \upsilon(t',u,\vartheta) d\vartheta dt', \tag{4.1.2c}$$

$$\int_{\mathcal{M}_{t,u}} f\, d\varpi := \int_{t'=0}^{t} \int_{u'=0}^{u} \int_{\vartheta\in\mathbb{T}} f(t',u',\vartheta)\, \upsilon(t',u',\vartheta) d\vartheta du' dt'. \tag{4.1.2d}$$

Remark 4.1. The canonical forms associated to $\underline{g}$ and g are respectively $\mu d\underline{\varpi}$ and $\mu d\varpi$.

4.2. **Basic ingredients and the definitions of the energies and null fluxes.** We construct our fundamental energies and null fluxes for scalar-valued functions Ψ with the help of the *energy-momentum tensor*

$$Q_{\mu\nu} = Q_{\mu\nu}[\Psi] := \mathscr{D}_\mu \Psi \mathscr{D}_\nu \Psi - \frac{1}{2} g_{\mu\nu} (g^{-1})^{\alpha\beta} \mathscr{D}_\alpha \Psi \mathscr{D}_\beta \Psi. \tag{4.2.1}$$

We construct our energies and null fluxes for $\vec{\Psi}$ by contracting the following multiplier vectorfield T against Q.

Definition 4.2 (The timelike multiplier vectorfield T). We define

$$T := (1+2\mu)L + 2\breve{X}. \tag{4.2.2}$$

Note that $g(T,T) = -4\mu(1+\mu) < 0$. This property leads to coercive energy identities.

Definition 4.3 (Energies and null fluxes). In terms of the non-degenerate forms of Def. 4.1.1, we define the energy functional $\mathbb{E}^{(Wave)}[\cdot]$ and null flux functional $\mathbb{F}^{(Wave)}[\cdot]$ as follows:

$$\mathbb{E}^{(Wave)}[\Psi](t,u) := \int_{\Sigma_t^u} \mu Q_{BT}[\Psi]\, d\underline{\varpi}, \qquad \mathbb{F}^{(Wave)}[\Psi](t,u) := \int_{\mathcal{P}_u^t} Q_{LT}[\Psi]\, d\overline{\varpi}, \tag{4.2.3}$$

where B and T are the vectorfields defined in (3.3.9) and (4.2.2).

We define the energy functional $\mathbb{E}^{(Vort)}[\cdot]$ and null flux functional $\mathbb{F}^{(Vort)}[\cdot]$ as follows:

$$\mathbb{E}^{(Vort)}[\omega](t,u) := \int_{\Sigma_t^u} \mu\omega^2\, d\underline{\varpi}, \qquad \mathbb{F}^{(Vort)}[\omega](t,u) := \int_{\mathcal{P}_u^t} \omega^2\, d\overline{\varpi}. \tag{4.2.4}$$

Lemma 4.1. [30, Lemma 3.4; **Coerciveness of the energy and null flux**] *The energy $\mathbb{E}^{(Wave)}[\Psi]$ and null flux $\mathbb{F}^{(Wave)}[\Psi]$ from Def. 4.3 enjoy the following coerciveness properties:*

$$\mathbb{E}^{(Wave)}[\Psi](t,u) = \int_{\Sigma_t^u} \frac{1}{2}(1+2\mu)\mu(L\Psi)^2 + 2\mu(L\Psi)\breve{X}\Psi + 2(\breve{X}\Psi)^2 + \frac{1}{2}(1+2\mu)\mu|\not d\Psi|^2\, d\underline{\varpi}, \tag{4.2.5a}$$

$$\mathbb{F}^{(Wave)}[\Psi](t,u) = \int_{\mathcal{P}_u^t} (1+\mu)(L\Psi)^2 + \mu|\not d\Psi|^2\, d\overline{\varpi}. \tag{4.2.5b}$$

4.3. **The main energy-null flux identities for wave and transport equations.** We now provide the fundamental energy-null flux identity for solutions to $\upmu\square_{g(\vec{\Psi})}\Psi = \mathfrak{F}$.

Remark 4.2 (**Picture of the regions of integration for the energy identities**)**.** See Fig. 2 on pg. 10 for a picture of the regions of integration. Note that the (unlabeled) front and back boundaries in that figure should be identified.

Proposition 4.2. [30, Proposition 3.5; **Fundamental energy-null flux identity for the wave equation**] *For scalar-valued functions* Ψ *that solve the covariant wave equation*

$$\upmu\square_{g(\vec{\Psi})}\Psi = \mathfrak{F},$$

the following identity involving the energy and null flux from Def. 4.3 holds for $t \geq 1$ *and* $u \in [0, U_0]$*:*

$$\begin{aligned}\mathbb{E}^{(Wave)}[\Psi](t,u) + \mathbb{F}^{(Wave)}[\Psi](t,u) = \mathbb{E}^{(Wave)}[\Psi](0,u) + \mathbb{F}^{(Wave)}[\Psi](t,0) \\ - \int_{\mathcal{M}_{t,u}} \left\{(1+2\upmu)(L\Psi) + 2\breve{X}\Psi\right\}\mathfrak{F}\, d\varpi \\ - \frac{1}{2}\int_{\mathcal{M}_{t,u}} \upmu Q^{\alpha\beta}[\Psi]^{(T)}\pi_{\alpha\beta}\, d\varpi.\end{aligned} \tag{4.3.1}$$

Furthermore, with $f_+ := \max\{f,0\}$ *and* $f_- := \max\{-f,0\}$*, we have*

$$^{(T)}\mathfrak{P}[\Psi] := -\frac{1}{2}\upmu Q^{\alpha\beta}[\Psi]^{(T)}\pi_{\alpha\beta} = -\frac{1}{2}\upmu|\not{d}\Psi|^2\frac{[L\upmu]_-}{\upmu} + \sum_{i=1}^{5}{}^{(T)}\mathfrak{P}_{(i)}[\Psi], \tag{4.3.2}$$

where

$$^{(T)}\mathfrak{P}_{(1)}[\Psi] := (L\Psi)^2\left\{-\frac{1}{2}L\upmu + \breve{X}\upmu - \frac{1}{2}\upmu\mathrm{tr}_{\not{g}}\upchi - \mathrm{tr}_{\not{g}}\not{k}^{(Trans-\vec{\Psi})} - \upmu\mathrm{tr}_{\not{g}}\not{k}^{(Tan-\vec{\Psi})}\right\}, \tag{4.3.3a}$$

$$^{(T)}\mathfrak{P}_{(2)}[\Psi] := -(L\Psi)(\breve{X}\Psi)\left\{\mathrm{tr}_{\not{g}}\upchi + 2\mathrm{tr}_{\not{g}}\not{k}^{(Trans-\vec{\Psi})} + 2\upmu\mathrm{tr}_{\not{g}}\not{k}^{(Tan-\vec{\Psi})}\right\}, \tag{4.3.3b}$$

$$^{(T)}\mathfrak{P}_{(3)}[\Psi] := \upmu|\not{d}\Psi|^2\left\{\frac{1}{2}\frac{[L\upmu]_+}{\upmu} + \frac{\breve{X}\upmu}{\upmu} + 2L\upmu - \frac{1}{2}\mathrm{tr}_{\not{g}}\upchi - \mathrm{tr}_{\not{g}}\not{k}^{(Trans-\vec{\Psi})} - \upmu\mathrm{tr}_{\not{g}}\not{k}^{(Tan-\vec{\Psi})}\right\}, \tag{4.3.3c}$$

$$^{(T)}\mathfrak{P}_{(4)}[\Psi] := (L\Psi)(\not{d}^{\#}\Psi)\cdot\left\{(1-2\upmu)\not{d}\upmu + 2\upzeta^{(Trans-\vec{\Psi})} + 2\upmu\upzeta^{(Tan-\vec{\Psi})}\right\}, \tag{4.3.3d}$$

$$^{(T)}\mathfrak{P}_{(5)}[\Psi] := -2(\breve{X}\Psi)(\not{d}^{\#}\Psi)\cdot\left\{\not{d}\upmu + 2\upzeta^{(Trans-\vec{\Psi})} + 2\upmu\upzeta^{(Tan-\vec{\Psi})}\right\}. \tag{4.3.3e}$$

The tensorfields $\upchi$*,* $\upzeta^{(Trans-\vec{\Psi})}$*,* $\not{k}^{(Trans-\vec{\Psi})}$*,* $\upzeta^{(Tan-\vec{\Psi})}$*, and* $\not{k}^{(Tan-\vec{\Psi})}$ *from above are as in* (3.9.4)*,* (3.9.9a)*,* (3.9.9b)*,* (3.9.9c)*, and* (3.9.9d)*.*

In the next proposition, we provide the fundamental energy-null flux identity for solutions to the transport equation $\upmu B\upomega = \mathfrak{F}$. The proof relies on the following divergence identity.

Lemma 4.3. [30, Lemma 4.3; **Spacetime divergence in terms of derivatives of frame components**] *Let* $\mathscr{J}$ *be a spacetime vectorfield. Let* $\upmu\mathscr{J} = -\upmu\mathscr{J}_L L - \mathscr{J}_{\breve{X}}L - \mathscr{J}_L\breve{X} + \upmu\not{\mathscr{J}}$

be its decomposition relative to the rescaled frame, where $\mathscr{J}_L = \mathscr{J}^\alpha L_\alpha$, $\mathscr{J}_{\breve{X}} = \mathscr{J}^\alpha \breve{X}_\alpha$, *and* $\not{\mathscr{J}} = \not{\Pi}\mathscr{J}$. *Then*

$$\mu \mathscr{D}_\alpha \mathscr{J}^\alpha = -L(\mu \mathscr{J}_L) - L(\mathscr{J}_{\breve{X}}) - \breve{X}(\mathscr{J}_L) + \not{\mathrm{div}}(\mu \not{\mathscr{J}}) - \mu \mathrm{tr}_{\not{g}} \not{k} \, \mathscr{J}_L - \mathrm{tr}_{\not{g}} \chi \, \mathscr{J}_{\breve{X}}, \qquad (4.3.4)$$

where where the $\ell_{t,u}$*-tangent tensorfields* $\not{k}$ *and* χ *can be expressed via* (3.9.8b) *and* (3.17.1a).

Proposition 4.4 (**Energy-null flux identity for the specific vorticity**). *For scalar-valued functions* ω *that solve the transport equation*

$$\mu B \omega = \mathfrak{F}, \qquad (4.3.5)$$

the following identity involving the energy and null flux from Def. 4.3 holds for $t \geq 1$ *and* $u \in [0, U_0]$:

$$\begin{aligned} \mathbb{E}^{(Vort)}[\omega](t,u) + \mathbb{F}^{(Vort)}[\omega](t,u) &= \mathbb{E}^{(Vort)}[\omega](0,u) + \mathbb{F}^{(Vort)}[\omega](t,0) \\ &\quad + 2 \int_{\mathcal{M}_{t,u}} \omega \mathfrak{F} \, d\varpi \\ &\quad + \int_{\mathcal{M}_{t,u}} \left\{ L\mu + \mu \mathrm{tr}_{\not{g}} \not{k} \right\} \omega^2 \, d\varpi. \end{aligned} \qquad (4.3.6)$$

Proof. We define the vectorfield $J := \omega^2 B = \omega^2 L + \omega^2 X$ and note that $J_L = -\omega^2$, $J_X = J_\Theta = 0$. Thus, using Lemma 4.3 and equation (4.3.5), we compute that

$$\mu \mathscr{D}_\alpha J^\alpha = (L\mu)\omega^2 + \mu \mathrm{tr}_{\not{g}} \not{k} \, \omega^2 + 2\omega \mathfrak{F}. \qquad (4.3.7)$$

Next, using the identities $L = \frac{\partial}{\partial t}$ and $\breve{X} = \frac{\partial}{\partial u} - \Xi$ (see (3.7.12)) and the relations $J_L = -\omega^2$, $J_X = J_\Theta = 0$ obtained above, we obtain the following decomposition from straightforward computations: $J = J^t \frac{\partial}{\partial t} + J^u \frac{\partial}{\partial u} + J^\Theta \Theta$, where $J^t = \omega^2$, $J^u = \mu^{-1}\omega^2$. Next, we note the following formula, which is the standard identity for the divergence of a vectorfield expressed relative to a coordinate frame (here the geometric coordinates) and the formula (3.11.3), which implies that $|\mathrm{det} g|^{1/2} = \mu \upsilon$ (where the determinant is taken relative to the geometric coordinates): $\mu \upsilon \mathscr{D}_\alpha J^\alpha = \frac{\partial}{\partial t}\left(\mu \upsilon J^t\right) + \frac{\partial}{\partial u}\left(\mu \upsilon J^u\right) + \frac{\partial}{\partial \vartheta}\left(\mu \upsilon J^\Theta\right)$. Integrating this identity over $\mathcal{M}_{t,u}$ with respect to $dt' \, du' \, d\vartheta$ and referring to Def. 4.1, we obtain

$$\int_{\mathcal{M}_{t,u}} \mu \mathscr{D}_\alpha J^\alpha \, d\varpi = \int_{t'=0}^{t} \int_{u'=0}^{u} \int_{\vartheta \in \mathbb{T}} \frac{\partial}{\partial t}\left(\mu \upsilon \omega^2\right) + \frac{\partial}{\partial u}\left(\upsilon \omega^2\right) + \frac{\partial}{\partial \vartheta}\left(\mu \upsilon J^\Theta\right) \, dt' \, du' \, d\vartheta. \qquad (4.3.8)$$

The desired identity (4.3.6) now follows from (4.3.7), (4.3.8), definition (4.2.4), Fubini's theorem, and the fact that the integral of the last term $\frac{\partial}{\partial \vartheta}\left(\mu \upsilon J^\Theta\right)$ over $\mathbb{T}$ vanishes. □

4.4. **Additional integration by parts identities.** In this section, we provide, for future use, some integration by parts identities. We highlight here the identity (4.4.2), which plays a critical role in our top-order energy estimates; see equation (15.14.4) and just below it.

Lemma 4.5. [30, Lemma 3.6; **Identities connected to integration by parts**] *The following identities hold for scalar-valued functions f:*

$$\frac{\partial}{\partial t}\int_{\ell_{t,u}} f\, d\lambda_{\not g} = \int_{\ell_{t,u}} \left\{ Lf + \mathrm{tr}_{\not g}\chi f \right\} d\lambda_{\not g}, \tag{4.4.1a}$$

$$\frac{\partial}{\partial u}\int_{\ell_{t,u}} f\, d\lambda_{\not g} = \int_{\ell_{t,u}} \left\{ \breve{X} f + \frac{1}{2}\mathrm{tr}_{\not g}{}^{(\breve{X})}\not\pi f \right\} d\lambda_{\not g}. \tag{4.4.1b}$$

In addition, the following integration by parts identity holds for scalar-valued functions Ψ and η (see Sect. 6.2 regarding the vectorfield operator notation):

$$\begin{aligned}
&\int_{\mathcal{M}_{t,u}} (1+2\mu)(\breve{X}\Psi)(L\mathscr{Z}_*^{N;\leq 1}\Psi)Y\eta\, d\varpi \qquad (4.4.2)\\
&= \int_{\mathcal{M}_{t,u}} (1+2\mu)(\breve{X}\Psi)(Y\mathscr{Z}_*^{N;\leq 1}\Psi)L\eta\, d\varpi \\
&\quad - \int_{\Sigma_t^u} (1+2\mu)(\breve{X}\Psi)(Y\mathscr{Z}_*^{N;\leq 1}\Psi)\eta\, d\varpi + \int_{\Sigma_0^u} (1+2\mu)(\breve{X}\Psi)(Y\mathscr{Z}_*^{N;\leq 1}\Psi)\eta\, d\varpi \\
&\quad + \int_{\mathcal{M}_{t,u}} \mathrm{Error}_1[\mathscr{Z}_*^{N;\leq 1}\Psi;\eta]\, d\varpi + \int_{\Sigma_t^u} \mathrm{Error}_2[\mathscr{Z}_*^{N;\leq 1}\Psi;\eta]\, d\varpi - \int_{\Sigma_0^u} \mathrm{Error}_2[\mathscr{Z}_*^{N;\leq 1}\Psi;\eta]\, d\varpi,
\end{aligned}$$

where

$$\begin{aligned}
\mathrm{Error}_1[\mathscr{Z}_*^{N;\leq 1}\Psi;\eta] &:= 2(L\mu)(\breve{X}\Psi)(Y\mathscr{Z}_*^{N;\leq 1}\Psi)\eta + (1+2\mu)(L\breve{X}\Psi)(Y\mathscr{Z}_*^{N;\leq 1}\Psi)\eta \qquad (4.4.3a)\\
&\quad + (1+2\mu)(\breve{X}\Psi)({}^{(Y)}\not\pi_L^{\#}\cdot \not d\mathscr{Z}_*^{N;\leq 1}\Psi)\eta + (1+2\mu)(\breve{X}\Psi)\mathrm{tr}_{\not g}\chi(Y\mathscr{Z}_*^{N;\leq 1}\Psi)\eta \\
&\quad + 2(Y\mu)(\breve{X}\Psi)(\mathscr{Z}_*^{N;\leq 1}\Psi)L\eta + (1+2\mu)(Y\breve{X}\Psi)(\mathscr{Z}_*^{N;\leq 1}\Psi)L\eta \\
&\quad + \frac{1}{2}(1+2\mu)(\breve{X}\Psi)\mathrm{tr}_{\not g}{}^{(Y)}\not\pi(\mathscr{Z}_*^{N;\leq 1}\Psi)L\eta \\
&\quad + 2(LY\mu)(\breve{X}\Psi)(\mathscr{Z}_*^{N;\leq 1}\Psi)\eta + 2(Y\mu)(L\breve{X}\Psi)(\mathscr{Z}_*^{N;\leq 1}\Psi)\eta \\
&\quad + 2(Y\mu)(\breve{X}\Psi)\mathrm{tr}_{\not g}\chi(\mathscr{Z}_*^{N;\leq 1}\Psi)\eta + (L\mu)(\breve{X}\Psi)\mathrm{tr}_{\not g}{}^{(Y)}\not\pi(\mathscr{Z}_*^{N;\leq 1}\Psi)\eta \\
&\quad + (1+2\mu)(LY\breve{X}\Psi)(\mathscr{Z}_*^{N;\leq 1}\Psi)\eta + (1+2\mu)(\breve{X}\Psi)(Y\mathrm{tr}_{\not g}\chi)(\mathscr{Z}_*^{N;\leq 1}\Psi)\eta \\
&\quad + (1+2\mu)(Y\breve{X}\Psi)\mathrm{tr}_{\not g}\chi(\mathscr{Z}_*^{N;\leq 1}\Psi)\eta + \frac{1}{2}(1+2\mu)(L\breve{X}\Psi)\mathrm{tr}_{\not g}{}^{(Y)}\not\pi(\mathscr{Z}_*^{N;\leq 1}\Psi)\eta \\
&\quad + (1+2\mu)(\breve{X}\Psi)(\not{\mathrm{div}}\,{}^{(Y)}\not\pi_L^{\#})(\mathscr{Z}_*^{N;\leq 1}\Psi)\eta + \frac{1}{2}(1+2\mu)(\breve{X}\Psi)\mathrm{tr}_{\not g}\chi\,\mathrm{tr}_{\not g}{}^{(Y)}\not\pi(\mathscr{Z}_*^{N;\leq 1}\Psi)\eta, \\
\mathrm{Error}_2[\mathscr{Z}_*^{N;\leq 1}\Psi;\eta] &:= -2(Y\mu)(\breve{X}\Psi)(\mathscr{Z}_*^{N;\leq 1}\Psi)\eta - (1+2\mu)(Y\breve{X}\Psi)(\mathscr{Z}_*^{N;\leq 1}\Psi)\eta \qquad (4.4.3b)\\
&\quad - \frac{1}{2}(1+2\mu)(\breve{X}\Psi)\mathrm{tr}_{\not g}{}^{(Y)}\not\pi(\mathscr{Z}_*^{N;\leq 1}\Psi)\eta.
\end{aligned}$$

5. The commutator of the covariant wave operator and a vectorfield

In this section, we provide expressions for the commutators $[\mu\Box_{g(\vec{\Psi})}, Z]$ for vectorfields Z belonging to the commutation set $\mathscr{Z}$ defined in (3.12.3). The following lemma provides a first decomposition. In Prop. 5.2, we further decompose the main term from the lemma.

Lemma 5.1. [30, Lemma 4.2; **Vectorfield-covariant wave operator commutator identity**] *For $Z \in \mathscr{Z}$ (see Def. 3.12.3), we have the following commutation identity for scalar functions Ψ, where $\mathrm{tr}_g{}^{(Z)}\pi := (g^{-1})^{\alpha\beta(Z)}\pi_{\alpha\beta}$:*

$$\upmu\square_{g(\Psi)}(Z\Psi) = \upmu\mathscr{D}_\alpha\left\{{}^{(Z)}\pi^{\alpha\beta}\mathscr{D}_\beta\Psi - \frac{1}{2}\mathrm{tr}_g{}^{(Z)}\pi\mathscr{D}^\alpha\Psi\right\} + Z(\upmu\square_{g(\Psi)}\Psi) + \frac{1}{2}\mathrm{tr}_{\not g}{}^{(Z)}\not\pi(\upmu\square_{g(\Psi)}\Psi). \tag{5.0.1}$$

We now decompose the first term on RHS (5.0.1) relative to the rescaled frame.

Proposition 5.2. [30, Proposition 4.4; **Frame decomposition of the divergence of the main inhomogeneous term in the commuted wave equation**] *For vectorfields $Z \in \mathscr{Z}$, we have the following identity for the first term on RHS* (5.0.1)*:*

$$\begin{aligned}\upmu\mathscr{D}_\alpha\left\{{}^{(Z)}\pi^{\alpha\beta}\mathscr{D}_\beta\Psi - \frac{1}{2}\mathrm{tr}_g{}^{(Z)}\pi\mathscr{D}^\alpha\Psi\right\} &= \mathscr{K}_{(\pi-Danger)}^{(Z)}[\Psi] \\ &\quad + \mathscr{K}_{(\pi-Cancel-1)}^{(Z)}[\Psi] + \mathscr{K}_{(\pi-Cancel-2)}^{(Z)}[\Psi] \\ &\quad + \mathscr{K}_{(\pi-Less\ \ Dangerous)}^{(Z)}[\Psi] + \mathscr{K}_{(\pi-Good)}^{(Z)}[\Psi] \\ &\quad + \mathscr{K}_{(\Psi)}^{(Z)}[\Psi] + \mathscr{K}_{(Low)}^{(Z)}[\Psi],\end{aligned} \tag{5.0.2}$$

where

$$\mathscr{K}_{(\pi-Danger)}^{(Z)}[\Psi] := -(\mathrm{d}\!\!/\mathrm{iv}\,{}^{(Z)}\not\pi_L^{\#})\breve{X}\Psi, \tag{5.0.3a}$$

$$\mathscr{K}_{(\pi-Cancel-1)}^{(Z)}[\Psi] := \left\{\frac{1}{2}\breve{X}\mathrm{tr}_{\not g}{}^{(Z)}\not\pi - \mathrm{d}\!\!/\mathrm{iv}\,{}^{(Z)}\not\pi_{\breve{X}}^{\#} - \upmu\mathrm{d}\!\!/\mathrm{iv}\,{}^{(Z)}\not\pi_L^{\#}\right\}L\Psi, \tag{5.0.3b}$$

$$\mathscr{K}_{(\pi-Cancel-2)}^{(Z)}[\Psi] := \left\{-\not{\mathcal{L}}_{\breve{X}}{}^{(Z)}\not\pi_L^{\#} + \not d^{\#(Z)}\pi_{L\breve{X}}\right\}\cdot\not d\Psi, \tag{5.0.3c}$$

$$\mathscr{K}_{(\pi-Less\ \ Dangerous)}^{(Z)}[\Psi] := \frac{1}{2}\upmu(\not d^{\#}\mathrm{tr}_{\not g}{}^{(Z)}\not\pi)\cdot\not d\Psi, \tag{5.0.3d}$$

$$\begin{aligned}\mathscr{K}_{(\pi-Good)}^{(Z)}[\Psi] &:= \frac{1}{2}\upmu(L\mathrm{tr}_{\not g}{}^{(Z)}\not\pi)L\Psi + (L^{(Z)}\pi_{L\breve{X}})L\Psi + (L^{(Z)}\pi_{\breve{X}X})L\Psi \\ &\quad + \frac{1}{2}(L\mathrm{tr}_{\not g}{}^{(Z)}\not\pi)\breve{X}\Psi - \upmu(\not{\mathcal{L}}_L{}^{(Z)}\not\pi_L^{\#})\cdot\not d\Psi - (\not{\mathcal{L}}_L{}^{(Z)}\not\pi_{\breve{X}}^{\#})\cdot\not d\Psi,\end{aligned} \tag{5.0.3e}$$

$$\begin{aligned}\mathscr{K}_{(\Psi)}^{(Z)}[\Psi] &:= \left\{\frac{1}{2}\upmu\mathrm{tr}_{\not g}{}^{(Z)}\not\pi + {}^{(Z)}\pi_{L\breve{X}} + {}^{(Z)}\pi_{\breve{X}X}\right\}LL\Psi \\ &\quad + \mathrm{tr}_{\not g}{}^{(Z)}\not\pi L\breve{X}\Psi \\ &\quad - 2\upmu^{(Z)}\not\pi_L^{\#}\cdot\not dL\Psi - 2^{(Z)}\not\pi_{\breve{X}}^{\#}\cdot\not dL\Psi - 2^{(Z)}\not\pi_L^{\#}\cdot\not d\breve{X}\Psi \\ &\quad + {}^{(Z)}\pi_{L\breve{X}}\not\Delta\Psi + \frac{1}{2}\upmu\mathrm{tr}_{\not g}{}^{(Z)}\not\pi\not\Delta\Psi,\end{aligned} \tag{5.0.4}$$

and

$$\mathscr{K}_{(Low)}^{(Z)}[\Psi] := \left\{ \frac{1}{2}(L\upmu)\mathrm{tr}_{\not g}{}^{(Z)}\not\pi + \frac{1}{2}\upmu\mathrm{tr}_{\not g}\not k\,\mathrm{tr}_{\not g}{}^{(Z)}\not\pi + \mathrm{tr}_{\not g}\upchi^{(Z)}\pi_{L\breve{X}} + \mathrm{tr}_{\not g}\upchi^{(Z)}\pi_{\breve{X}X} - {}^{(Z)}\not\pi_L^{\#}\cdot\not d\upmu \right\} L\Psi \tag{5.0.5}$$

$$+ \frac{1}{2}\mathrm{tr}_{\not g}\upchi\mathrm{tr}_{\not g}{}^{(Z)}\not\pi\breve{X}\Psi$$

$$+ \left\{ -(L\upmu)^{(Z)}\not\pi_L^{\#} - \upmu\mathrm{tr}_{\not g}\not k^{(Z)}\not\pi_L^{\#} - \mathrm{tr}_{\not g}\upchi^{(Z)}\not\pi_{\breve{X}}^{\#} + \mathrm{tr}_{\not g}{}^{(Z)}\not\pi\not d^{\#}\upmu + \mathrm{tr}_{\not g}\upchi\upmu\zeta^{\#} \right\}\cdot\not d\Psi.$$

In the above expressions, the $\ell_{t,u}$-tangent tensorfields $\upchi$, ζ, and $\not k$, are as in (3.17.1a), (3.9.8a), *and* (3.9.8b).

6. Norms and Strings of Commutation Vectorfields

In this section, we define various norms and seminorms. We also introduce schematic notation that succinctly captures the most important properties of strings of commutation vectorfields.

6.1. **Norms.** We now define some norms that we use in our analysis. We recall that we defined the pointwise norm of $\ell_{t,u}$-tensors (relative to $\not g$) in Subsect. 3.10.

6.1.1. *Lebesgue norms.*

Definition 6.1 (**L^2 and L^∞ norms**)**.** In terms of the **non-degenerate** forms of Def. 4.1, we define the following norms for $\ell_{t,u}$-tangent tensorfields:

$$\|\xi\|_{L^2(\ell_{t,u})}^2 := \int_{\ell_{t,u}} |\xi|^2\, d\lambda_{\not g}, \qquad \|\xi\|_{L^2(\Sigma_t^u)}^2 := \int_{\Sigma_t^u} |\xi|^2\, d\underline{\varpi}, \tag{6.1.1a}$$

$$\|\xi\|_{L^2(\mathcal{P}_u^t)}^2 := \int_{\mathcal{P}_u^t} |\xi|^2\, d\overline{\varpi},$$

$$\|\xi\|_{L^\infty(\ell_{t,u})} := \mathrm{ess\ sup}_{\vartheta\in\mathbb{T}}|\xi|(t,u,\vartheta), \qquad \|\xi\|_{L^\infty(\Sigma_t^u)} := \mathrm{ess\ sup}_{(u',\vartheta)\in[0,u]\times\mathbb{T}}|\xi|(t,u',\vartheta), \tag{6.1.1b}$$

$$\|\xi\|_{L^\infty(\mathcal{P}_u^t)} := \mathrm{ess\ sup}_{(t',\vartheta)\in[0,t]\times\mathbb{T}}|\xi|(t',u,\vartheta).$$

Remark 6.1 (**Subset norms**)**.** In our analysis below, we occasionally use norms $\|\cdot\|_{L^2(\Omega)}$ and $\|\cdot\|_{L^\infty(\Omega)}$, where Ω is a subset of Σ_t^u. These norms are defined by replacing Σ_t^u with Ω in (6.1.1a) and (6.1.1b).

6.1.2. *Norms of arrays.* We define the norms of the arrays $\vec{G}_{(Frame)}$ and $\vec{H}_{(Frame)}$ from Def. 3.5 to be the sums of the norms of their $\imath,\jmath$-indexed entries. For example,

$$\left|\vec{G}_{(Frame)}\right| := \left|\vec{G}_{LL}\right| + \left|\vec{G}_{LX}\right| + \left|\vec{\not G}_L\right| + \left|\vec{\not G}_X\right| + \left|\vec{\not G}\right|, \tag{6.1.2}$$

where $\left|\vec{G}_{LL}\right| := \sum_{\imath=0}^2 |G_{LL}^\imath|$, $\left|\vec{\not G}_X\right| := \sum_{\imath=0}^2 |\not G_X^\imath|$, etc. We similarly define $\left\|\vec{G}_{(Frame)}\right\|_{L^\infty(\Sigma_t^u)}$ and $\left\|\vec{H}_{(Frame)}\right\|_{L^\infty(\Sigma_t^u)}$, and similarly for other norms.

6.2. **Strings of commutation vectorfields and vectorfield seminorms.** The following shorthand notation captures the important structural features of various differential operators corresponding to repeated differentiation with respect to the commutation vectorfields. The notation allows us to schematically depict identities and estimates.

Definition 6.2 (**Strings of commutation vectorfields and vectorfield seminorms**).

- $\mathscr{Z}^{N;M} f$ denotes an arbitrary string of N commutation vectorfields in $\mathscr{Z}$ (see (3.12.3)) applied to f, where the string contains *precisely* M factors of the $\mathcal{P}_t^u$-transversal vectorfield $\breve{X}$. We also set $\mathscr{Z}^{0;0} f := f$. Similarly, we write $\mathscr{Z}^{N;\leq M} f$ when the string is allowed to contain $\leq M$ factors of $\breve{X}$.
- $\mathscr{P}^N f$ denotes an arbitrary string of N commutation vectorfields in $\mathscr{P}$ (see (3.12.4)) applied to f.
- For $N \geq 1$, $\mathscr{Z}_*^{N;M} f$ denotes an arbitrary string of N commutation vectorfields in $\mathscr{Z}$ applied to f, where the string contains *at least* one $\mathcal{P}_u$-tangent factor and *precisely* M factors of $\breve{X}$. We also set $\mathscr{Z}_*^{0;0} f := f$. Similarly, we write $\mathscr{Z}_*^{N;\leq M} f$ when the string is allowed to contain $\leq M$ factors of $\breve{X}$.
- For $N \geq 1$, $\mathscr{Z}_{**}^{N;M} f$ denotes an arbitrary string of N commutation vectorfields in $\mathscr{Z}$ applied to f, where the string contains *at least* two factors of L or *at least* one factor of Y and *precisely* M factors of $\breve{X}$. Similarly, we write $\mathscr{Z}_{**}^{N;\leq M} f$ when the string is allowed to contain $\leq M$ factors of $\breve{X}$.
- For $\ell_{t,u}$-tangent tensorfields ξ, we similarly define strings of $\ell_{t,u}$-projected Lie derivatives such as $\not{\mathcal{L}}_{\mathscr{Z}}^{N;M} \xi$.

We also define pointwise seminorms constructed out of sums of strings of vectorfields:

- $\left|\mathscr{Z}^{N;M} f\right|$ simply denotes the magnitude of one of the $\mathscr{Z}^{N;M} f$ as defined above (there is no summation). Similarly, $\left|\mathscr{Z}^{N;\leq M} f\right|$ denotes the magnitude of one of the $\mathscr{Z}^{N;\leq M} f$ as defined above.
- $\left|\mathscr{Z}^{\leq N;M} f\right|$ is the *sum* over all terms of the form $\left|\mathscr{Z}^{N';M} f\right|$ with $N' \leq N$.
- $\left|\mathscr{Z}^{\leq N;\leq M} f\right|$ is the sum over all terms of the form $\left|\mathscr{Z}^{N';M'} f\right|$ with $N' \leq N$ and $M' \leq M$.
- $\left|\mathscr{Z}^{[1,N];M} f\right|$ is the sum over all terms of the form $\left|\mathscr{Z}^{N';M} f\right|$ with $1 \leq N' \leq N$.
- $\left|\mathscr{Z}^{[1,N];\leq M} f\right|$ is the sum over all terms of the form $\left|\mathscr{Z}^{N';M'} f\right|$ with $1 \leq N' \leq N$ and $M' \leq M$.
- Quantities such as $\left|\mathscr{P}^N f\right|$, $\left|\mathscr{Z}_*^{N;M} f\right|$, $\left|\mathscr{Z}_{**}^{N;M} f\right|$, and $\left|\mathscr{Z}_{**}^{N;\leq M} f\right|$ are defined analogously (without summation).
- Sums such as $\left|\mathscr{P}^{\leq N} f\right|$, $\left|\mathscr{P}^{[1,N]} f\right|$, $\left|\mathscr{Z}_*^{[1,N];M} f\right|$, $\left|\mathscr{Z}_*^{[1,N];\leq M} f\right|$, $\left|\mathscr{Z}_{**}^{[1,N];M} f\right|$, $\left|\mathscr{Z}_{**}^{[1,N];\leq M} f\right|$, $\left|Y^{\leq N} f\right|$, and $\left|\breve{X}^{[1,N]} f\right|$ are defined analogously. For example, $\left|\breve{X}^{[1,N]} f\right| = |\breve{X} f| + |\breve{X}\breve{X} f| + \cdots + \left|\overbrace{\breve{X}\breve{X}\cdots\breve{X}}^{N \text{ copies}} f\right|$.

Remark 6.2 (**Operators decorated with $*$ or $**$**). The purpose of the symbols $*$ and $**$ in Def. 6.2 is to highlight the presence of special structures in vectorfield operators, which helps us track smallness in the estimates. That is, in our analysis, we typically display operators decorated with a $*$ and $**$ when they lead to quantities that are initially[100] of small size $\mathcal{O}(\mathring{\upepsilon})$, where $\mathring{\upepsilon}$ is the data-size parameter defined in Sect. 8. We note here that the quantities $\mathscr{Z}_*^{N;M}\upgamma$ and $\mathscr{Z}_{**}^{N;M}\underline{\upgamma}$ are always initially small, while $\mathscr{Z}_*^{N;M}\underline{\upgamma}$ may not be. The reason that $\mathscr{Z}_*^{N;M}\underline{\upgamma}$ may not be small is: for the solutions under consideration, $L\upmu$ and its $\breve{X}$ derivatives are large quantities. We also note that the notation $*$ and $**$ is not important[101] for treating the specific vorticity variable $\upomega$ because our initial conditions are such that all directional derivatives of the specific vorticity are initially small.

7. Modified quantities

In this section, we define the modified quantities that allow us to avoid losing a derivative at the top-order. We also define the partially modified quantities that allow us to avoid some top-order error integrals with magnitudes that are too large for us to control. We then provide transport-type evolution equations for these quantities.

7.1. Curvature tensors and the key Ricci component identity.

We use use curvature tensors of g to help us organize the calculations in this section.

Definition 7.1 (**Curvature tensors of g**). The Riemann curvature tensor $\mathscr{R}_{\alpha\beta\kappa\lambda}$ of the spacetime metric g is the type $\binom{0}{4}$ spacetime tensorfield defined by

$$g(\mathscr{D}^2_{UV}W - \mathscr{D}^2_{VU}W, Z) = -\mathscr{R}(U, V, W, Z), \tag{7.1.1}$$

where U, V, W, and Z are arbitrary spacetime vectors. In (7.1.1), $\mathscr{D}^2_{UV}W := U^\alpha V^\beta \mathscr{D}_\alpha \mathscr{D}_\beta W$.

The Ricci curvature tensor $\mathrm{Ric}_{\alpha\beta}$ of g is the following type $\binom{0}{2}$ tensorfield:

$$\mathrm{Ric}_{\alpha\beta} := (g^{-1})^{\kappa\lambda}\mathscr{R}_{\alpha\kappa\beta\lambda}. \tag{7.1.2}$$

The next lemma lies at the heart of the construction of the modified quantities.

Lemma 7.1 (**The key identity verified by $\upmu \mathrm{Ric}_{LL}$**). *Assume that the entries of $\vec{\Psi} = (\uprho, v^1, v^2)$ verify the geometric wave equation system* (3.3.11a)-(3.3.11b). *Then the following identity holds for the Ricci curvature component* $\mathrm{Ric}_{LL} := \mathrm{Ric}_{\alpha\beta}L^\alpha L^\beta$*:*

$$\upmu \mathrm{Ric}_{LL} = L\left\{-\vec{G}_{LL} \diamond \breve{X}\vec{\Psi} - \frac{1}{2}\upmu \mathrm{tr}_{g\!\!/}\vec{G\!\!/} \diamond L\vec{\Psi} - \frac{1}{2}\upmu\vec{G}_{LL} \diamond L\vec{\Psi} + \upmu \vec{G\!\!/}_L^{\#} \diamond \cdot d\!\!/\vec{\Psi}\right\} + \mathfrak{A}, \tag{7.1.3}$$

where $\mathfrak{A}$ has the following schematic structure:

$$\mathfrak{A} = \mathrm{f}(\underline{\upgamma}, g\!\!/^{-1}, d\!\!/\vec{x}, \breve{X}\vec{\Psi}, P\vec{\Psi})P\vec{\Psi} + \upmu \mathrm{f}(\upgamma)P\upomega + \upomega \mathrm{f}(\upgamma)\breve{X}\vec{\Psi} + \upmu\upomega \mathrm{f}(\upgamma)P\vec{\Psi}. \tag{7.1.4}$$

Furthermore, without assuming that equations (3.3.11a)-(3.3.11b) *hold, we have*

$$\mathrm{Ric}_{LL} = \frac{(L\upmu)}{\upmu}\mathrm{tr}_{g\!\!/}\upchi + L\left\{-\frac{1}{2}\mathrm{tr}_{g\!\!/}\vec{G\!\!/} \diamond L\vec{\Psi} - \frac{1}{2}\vec{G}_{LL} \diamond L\vec{\Psi} + \vec{G\!\!/}_L^{\#} \diamond \cdot d\!\!/\vec{\Psi}\right\} - \frac{1}{2}\vec{G}_{LL} \diamond \Delta\!\!\!/\vec{\Psi} + \mathfrak{B}, \tag{7.1.5}$$

[100] At the high derivative levels, the "initially small" quantities are allowed to blow up like $\mathring{\upepsilon}(\min_{\Sigma_t} \upmu)^{-P}$ for some power P as the shock forms.

[101] We use it nonetheless for consistency.

where $\mathfrak{B}$ *has the following schematic structure:*

$$\mathfrak{B} = \mathrm{f}(\gamma, \not{g}^{-1}, \not{d}\vec{x})(P\vec{\Psi})P\gamma. \tag{7.1.6}$$

Sketch of proof. The identities (7.1.3) and (7.1.5) were essentially proved in [30, Lemma 6.1] using calculations along the lines of those in [6, Chapter 8]. The only new feature in the present work is that RHS (7.1.3) depends on the inhomogeneous terms on the right-hand sides of the wave equations (3.3.11a)-(3.3.11b), which were absent in the previous works. The inhomogeneous terms appear because at the key point in the proof, one uses (3.18.1a) and the wave equations (3.3.11a)-(3.3.11b) to express

$$\begin{aligned} -\frac{1}{2}\mu\vec{G}_{LL} \diamond \not{\Delta}\vec{\Psi} = & -\frac{1}{2}L\left\{\vec{G}_{LL} \diamond (\mu L\vec{\Psi} + 2\breve{X}\vec{\Psi})\right\} - \frac{1}{2}\mathrm{tr}_{\not{g}}\chi\vec{G}_{LL} \diamond \breve{X}\vec{\Psi} \\ & + \mathrm{Inhom} \\ & + \begin{pmatrix} \vec{G}^2_{(Frame)}\not{g}^{-1} \\ \vec{H}_{(Frame)} \end{pmatrix} \begin{pmatrix} \mu L\vec{\Psi} \\ \breve{X}\vec{\Psi} \\ \mu\not{d}\vec{\Psi} \end{pmatrix} \begin{pmatrix} L\vec{\Psi} \\ \not{d}\vec{\Psi} \end{pmatrix}, \end{aligned} \tag{7.1.7}$$

where the last line of RHS (7.1.7) is schematically depicted and term Inhom on RHS (7.1.7) denotes the inhomogeneous terms on RHSs (3.3.11a)-(3.3.11b). The first term on RHS (7.1.7) is incorporated into the perfect L derivative term on the first line of RHS (7.1.3). It is straightforward to see that the term Inhom is of the form of RHS (7.1.4): we use (3.19.4) to decompose the null forms on RHSs (3.3.11a)-(3.3.11b), Cor. 3.18 to decompose the product on RHS (3.3.11a) depending on the first Cartesian coordinate partial derivatives of ω, (3.7.15) to decompose the material derivative vectorfield on RHS (3.3.11a), and Lemma 3.19. In a detailed proof (see [30, Lemma 6.1]), one would find that the term $-\frac{1}{2}\mathrm{tr}_{\not{g}}\chi\vec{G}_{LL} \diamond \breve{X}\vec{\Psi}$ on RHS (7.1.7) is canceled by another term and hence does not appear on RHS (7.1.3). This completes our proof sketch of the lemma. □

7.2. The definitions of the modified quantities and their transport equations.

Definition 7.2 (**Modified versions of the derivatives of** $\mathrm{tr}_{\not{g}}\chi$)**.** Let $\mathscr{Z}_*^{N;\leq 1}$ be an N^{th} order commutation vectorfield operator (see Sect. 6.2 regarding the notation). We define the fully modified function ${}^{(\mathscr{Z}_*^{N;\leq 1})}\mathscr{X}$ as follows:

$$ {}^{(\mathscr{Z}_*^{N;\leq 1})}\mathscr{X} := \mu\mathscr{Z}_*^{N;\leq 1}\mathrm{tr}_{\not{g}}\chi + \mathscr{Z}_*^{N;\leq 1}\mathfrak{X}, \tag{7.2.1a}$$

$$\mathfrak{X} := -\vec{G}_{LL} \diamond \breve{X}\vec{\Psi} - \frac{1}{2}\mu\mathrm{tr}_{\not{g}}\vec{\not{G}} \diamond L\vec{\Psi} - \frac{1}{2}\mu\vec{G}_{LL} \diamond L\vec{\Psi} + \mu\vec{\not{G}}_L^{\#} \diamond \cdot\not{d}\vec{\Psi}. \tag{7.2.1b}$$

We define the partially modified function ${}^{(\mathscr{Z}_*^{N;\leq 1})}\widetilde{\mathscr{X}}$ as follows:

$$ {}^{(\mathscr{Z}_*^{N;\leq 1})}\widetilde{\mathscr{X}} := \mathscr{Z}_*^{N;\leq 1}\mathrm{tr}_{\not{g}}\chi + {}^{(\mathscr{Z}_*^{N;\leq 1})}\widetilde{\mathfrak{X}}, \tag{7.2.2a}$$

$$ {}^{(\mathscr{Z}_*^{N;\leq 1})}\widetilde{\mathfrak{X}} := -\frac{1}{2}\mathrm{tr}_{\not{g}}\vec{\not{G}} \diamond L\mathscr{Z}_*^{N;\leq 1}\vec{\Psi} - \frac{1}{2}\vec{G}_{LL} \diamond L\mathscr{Z}_*^{N;\leq 1}\vec{\Psi} + \vec{\not{G}}_L^{\#} \diamond \cdot\not{d}\mathscr{Z}_*^{N;\leq 1}\vec{\Psi}. \tag{7.2.2b}$$

We also define the following "0^{th}-order" version of (7.2.2b):

$$\widetilde{\mathfrak{X}} := -\frac{1}{2}\mathrm{tr}_{\not{g}}\vec{\not{G}} \diamond L\vec{\Psi} - \frac{1}{2}\vec{G}_{LL} \diamond L\vec{\Psi} + \vec{\not{G}}_L^{\#} \diamond \cdot\not{d}\vec{\Psi}. \tag{7.2.3}$$

Proposition 7.2. [30, Proposition 6.2; **The transport equation for the fully modified version of** $\mathscr{Z}_*^{N;\leq 1}\mathrm{tr}_{\not g}\chi$] *Assume that the entries of* $\vec{\Psi} = (\rho, v^1, v^2)$ *verify the geometric wave equation system* (3.3.11a)-(3.3.11b). *Let* $\mathscr{Z}_*^{N;\leq 1}$ *be an* N^{th} *order commutation vectorfield operator (see Sect. 6.2 regarding the notation) and let* ${}^{(\mathscr{Z}_*^{N;\leq 1})}\mathscr{X}$ *and* $\mathfrak{X}$ *be the corresponding quantities defined in* (7.2.1a) *and* (7.2.1b). *Then the fully modified quantity* ${}^{(\mathscr{Z}_*^{N;\leq 1})}\mathscr{X}$ *verifies the following transport equation:*

$$L^{(\mathscr{Z}_*^{N;\leq 1})}\mathscr{X} - \left(2\frac{L\mu}{\mu}\right){}^{(\mathscr{Z}_*^{N;\leq 1})}\mathscr{X} = \mu[L, \mathscr{Z}_*^{N;\leq 1}]\mathrm{tr}_{\not g}\chi - 2\mu\mathrm{tr}_{\not g}\chi\mathscr{Z}_*^{N;\leq 1}\mathrm{tr}_{\not g}\chi - \left(2\frac{L\mu}{\mu}\right)\mathscr{Z}_*^{N;\leq 1}\mathfrak{X} \tag{7.2.4}$$
$$+ [L, \mathscr{Z}_*^{N;\leq 1}]\mathfrak{X} + [\mu, \mathscr{Z}_*^{N;\leq 1}]L\mathrm{tr}_{\not g}\chi + [\mathscr{Z}_*^{N;\leq 1}, L\mu]\mathrm{tr}_{\not g}\chi$$
$$- \left\{\mathscr{Z}_*^{N;\leq 1}\left(\mu(\mathrm{tr}_{\not g}\chi)^2\right) - 2\mu\mathrm{tr}_{\not g}\chi\mathscr{Z}_*^{N;\leq 1}\mathrm{tr}_{\not g}\chi\right\} - \mathscr{Z}_*^{N;\leq 1}\mathfrak{A},$$

where the term $\mathfrak{A}$ *on the last line of RHS* (7.2.4) *is the one appearing in* (7.1.3)-(7.1.4).

Proposition 7.3. [30, Proposition 6.3; **The transport equation for the partially modified version of** $\mathscr{Z}_*^{N-1;\leq 1}\mathrm{tr}_{\not g}\chi$] *Let* $\mathscr{Z}_*^{N-1;\leq 1}$ *be an* $(N-1)^{st}$ *order commutation vectorfield operator (see Sect. 6.2 regarding the notation) and let* ${}^{(\mathscr{Z}_*^{N-1;\leq 1})}\widetilde{\mathscr{X}}$ *be the corresponding partially modified quantity defined in* (7.2.2a). *Then* ${}^{(\mathscr{Z}_*^{N-1;\leq 1})}\widetilde{\mathscr{X}}$ *verifies the following transport equation:*

$$L^{(\mathscr{Z}_*^{N-1;\leq 1})}\widetilde{\mathscr{X}} = \frac{1}{2}\vec{G}_{LL} \diamond \not\Delta\mathscr{Z}_*^{N-1;\leq 1}\vec{\Psi} + {}^{(\mathscr{Z}_*^{N-1;\leq 1})}\mathfrak{B}, \tag{7.2.5}$$

where the inhomogeneous term ${}^{(\mathscr{Z}_*^{N-1;\leq 1})}\mathfrak{B}$ *is given by*

$${}^{(\mathscr{Z}_*^{N-1;\leq 1})}\mathfrak{B} = -\mathscr{Z}_*^{N-1;\leq 1}\mathfrak{B} - \mathscr{Z}_*^{N-1;\leq 1}(\mathrm{tr}_{\not g}\chi)^2 \tag{7.2.6}$$
$$+ \frac{1}{2}[\mathscr{Z}_*^{N-1;\leq 1}, \vec{G}_{LL}] \diamond \not\Delta\vec{\Psi} + \frac{1}{2}\vec{G}_{LL}[\mathscr{Z}_*^{N-1;\leq 1}, \not\Delta] \diamond \vec{\Psi} + [L, \mathscr{Z}_*^{N-1;\leq 1}]\mathrm{tr}_{\not g}\chi$$
$$+ [L, \mathscr{Z}_*^{N-1;\leq 1}]\widetilde{\mathfrak{X}} + L\left\{{}^{(\mathscr{Z}_*^{N-1;\leq 1})}\widetilde{\mathfrak{X}} - \mathscr{Z}_*^{N-1;\leq 1}\widetilde{\mathfrak{X}}\right\},$$

$\mathfrak{B}$ *is defined in* (7.1.6), ${}^{(\mathscr{Z}_*^{N-1;\leq 1})}\widetilde{\mathfrak{X}}$ *is defined in* (7.2.2b), *and* $\widetilde{\mathfrak{X}}$ *is defined in* (7.2.3).

7.3. **Some identities connected to curvature.** We now show that $\breve{X}\mathrm{tr}_{\not g}\chi$ and $\not\Delta\mu$ are equal up to simple error terms. This fact allows for a simplified approach to various estimates appearing later in the paper.

Lemma 7.4. [30, Lemma 11.4; **Connection between** $\breve{X}\mathrm{tr}_{\not g}\chi$ **and** $\not\Delta\mu$] $\breve{X}\mathrm{tr}_{\not g}\chi$ *can be expressed as follows, where the term* $\not\Delta\mu$ *on RHS* (7.3.1) *and* $\mathrm{f}(\cdots)$ *is schematic:*

$$\breve{X}\mathrm{tr}_{\not g}\chi = \not\Delta\mu + \mathrm{f}(\gamma, \not g^{-1}, \not d\vec{x})P\breve{X}\vec{\Psi} + \mathrm{f}(\underline{\gamma}, \not g^{-1}, \not d\vec{x})PP\vec{\Psi} \tag{7.3.1}$$
$$+ \mathrm{f}(\underline{\gamma}, \breve{X}\vec{\Psi}, P\underline{\gamma}, \not g^{-1}, \not dx)P\gamma.$$

Discussion of proof. Lemma 7.4 was essentially proved as [30, Lemma 11.4] and is based on an analysis of the Riemann curvature component $\mathrm{tr}_{\not g}\mathscr{R}_{\breve{X}\cdot L\cdot}$. We remark that in the identity provided by [30, Lemma 11.4], one finds a term proportional to $\not\nabla^2\vec{x}$. However, using (9.1.1b) with $f = x^i$ and Lemma 3.19, we can write $\not\nabla^2\vec{x} = \mathrm{f}(\gamma, \not d\vec{x})P\gamma$, and thus the corresponding error terms are part of the last term on RHS (7.3.1). This completes our discussion of the lemma. We remark that similar calculations are presented in [6, Chapter 4]. □

8. Assumptions on the initial state of the solution and bootstrap assumptions

In this section, we introduce our Sobolev norm assumptions on the data for $\vec{\Psi}$, ω, and the eikonal function quantities. We also state the bootstrap assumptions that we use in analyzing solutions. By data, we mean the state of the solution along Σ_0^1 and a large potion of the outgoing null hypersurface $\mathcal{P}_0$. Our assumptions involve several size parameters, and in Sect. 8.6, we describe our assumptions on their relative sizes. In Subsubsect. 8.7, we show that there exists an open set of nearly plane symmetric data verifying the size assumptions.

8.1. **Assumptions on the initial state of the fluid variables.**

8.1.1. *The quantity that controls the blowup-time.* We start by introducing the data-dependent number $\mathring{\delta}_*$, which is of crucial importance. Our main theorem shows that if $\mathring{\epsilon}$ (defined just below) is sufficiently small, then the time of first shock formation is $(1+\mathcal{O}(\mathring{\epsilon}))\mathring{\delta}_*^{-1}$

Definition 8.1 (**The quantity that controls the blowup-time**)**.** We define

$$\mathring{\delta}_* := \frac{1}{2}\sup_{\Sigma_0^1}\left[\sum_{\imath=0}^{1} G_{LL}^{\imath}\breve{X}v^1\right]_- . \tag{8.1.1}$$

Remark 8.1 (**Significance of $\mathring{\delta}_*$**)**.** Equation (3.14.1) and the estimates of Props. 9.12 and 10.1 can be used to show that there exist $u_* \in [0,U_0]$ and $\vartheta_* \in \mathbb{T}$ such that for $t \geq 1$, we have $L\mu(t,u_*,\vartheta_*) = -\mathring{\delta}_* + \mathrm{Error}$, where (under suitable assumptions on the data) Error is small compared to $\mathring{\delta}_*$. That is, the maximal shrinking rate of μ along the integral curves of L is determined by $\mathring{\delta}_*$. It is for this reason that $\mathring{\delta}_*^{-1}$ is connected to the time of shock formation.

8.1.2. *Size assumptions for the fluid variables.* We make the following size assumptions along Σ_0^1 and $\mathcal{P}_0^{2\mathring{\delta}_*^{-1}}$ (see Sect. 6.2 regarding the vectorfield operator notation).

<u>L^2 **assumptions along** Σ_0^1</u>.

$$\left\|\mathscr{Z}_*^{\leq 21;\leq 2}\vec{\Psi}\right\|_{L^2(\Sigma_0^1)}, \begin{pmatrix} \left\|\breve{X}(\rho - v^1)\right\|_{L^2(\Sigma_0^1)} \\ \left\|\breve{X}^{[0,2]}v^2\right\|_{L^2(\Sigma_0^1)} \end{pmatrix}, \left\|\mathscr{P}^{\leq 21}\omega\right\|_{L^2(\Sigma_0^1)} \leq \mathring{\epsilon}. \tag{8.1.2}$$

L^∞ **assumptions along** Σ_0^1.

$$\left\|\mathscr{Z}_*^{\leq 13;\leq 1}\vec{\Psi}\right\|_{L^\infty(\Sigma_0^1)}, \left\|\mathscr{Z}_*^{\leq 12;\leq 2}\vec{\Psi}\right\|_{L^\infty(\Sigma_0^1)}, \tag{8.1.3a}$$
$$\begin{pmatrix} \left\|\breve{X}(\uprho - v^1)\right\|_{L^\infty(\Sigma_0^1)} \\ \left\|\breve{X}^{[1,3]}v^2\right\|_{L^\infty(\Sigma_0^1)} \end{pmatrix}, \left\|L\breve{X}\breve{X}\breve{X}\vec{\Psi}\right\|_{L^\infty(\Sigma_0^1)},$$
$$\left\|\mathscr{P}^{\leq 13}\upomega\right\|_{L^\infty(\Sigma_0^1)} \leq \mathring{\upepsilon},$$
$$\begin{pmatrix} \left\|\breve{X}^{[1,3]}\uprho\right\|_{L^\infty(\Sigma_0^1)} \\ \left\|\breve{X}^{[1,3]}v^1\right\|_{L^\infty(\Sigma_0^1)} \end{pmatrix} \leq \mathring{\updelta}. \tag{8.1.3b}$$

L^2 **assumptions along** $\mathcal{P}_0^{2\mathring{\updelta}_*^{-1}}$.

$$\left\|\mathscr{Z}^{\leq 21;\leq 1}\vec{\Psi}\right\|_{L^2\left(\mathcal{P}_0^{2\mathring{\updelta}_*^{-1}}\right)}, \left\|\mathscr{P}^{\leq 21}\upomega\right\|_{L^2\left(\mathcal{P}_0^{2\mathring{\updelta}_*^{-1}}\right)} \leq \mathring{\upepsilon}. \tag{8.1.4}$$

L^∞ **assumptions along** $\mathcal{P}_0^{2\mathring{\updelta}_*^{-1}}$.

$$\left\|\mathscr{Z}^{\leq 19;\leq 1}\vec{\Psi}\right\|_{L^\infty\left(\mathcal{P}_0^{2\mathring{\updelta}_*^{-1}}\right)}, \left\|\mathscr{P}^{\leq 19}\upomega\right\|_{L^\infty\left(\mathcal{P}_0^{2\mathring{\updelta}_*^{-1}}\right)} \leq \mathring{\upepsilon}. \tag{8.1.5}$$

L^2 **assumptions along** $\ell_{1,u}$.

$$\left\|\mathscr{Z}_*^{\leq 20;\leq 1}\vec{\Psi}\right\|_{L^2(\ell_{1,u})}, \begin{pmatrix} \left\|\breve{X}(\uprho - v^1)\right\|_{L^2(\ell_{1,u})} \\ \left\|\breve{X}v^2\right\|_{L^2(\ell_{1,u})} \end{pmatrix}, \left\|\mathscr{P}^{\leq 20}\upomega\right\|_{L^2(\ell_{1,u})} \leq \mathring{\upepsilon}. \tag{8.1.6}$$

L^2 **assumptions along** $\ell_{t,0}$.

$$\left\|\mathscr{Z}^{\leq 20;\leq 1}\vec{\Psi}\right\|_{L^2(\ell_{t,0})}, \left\|\mathscr{P}^{\leq 20}\upomega\right\|_{L^2(\ell_{t,0})} \leq \mathring{\upepsilon}. \tag{8.1.7}$$

Remark 8.2 (**A concise summary of the effect of the size assumptions**)**.** The assumptions (8.1.2)-(8.1.7) will allow us to prove that among $\vec{\Psi}$, $\upomega$ and their relevant derivatives, the only relatively large (in all relevant norms) quantities in our analysis are $\breve{X}^{[1,3]}v^1$ and $\breve{X}^{[1,3]}\uprho$ along Σ_t^u. Moreover, even $\breve{X}(\uprho - v^1)$ is small along Σ_t^u, and $\breve{X}v^1$ and $\breve{X}\uprho$ are small along $\mathcal{P}_0^{2\mathring{\updelta}_*^{-1}}$. This division into small and large quantities is fundamental for our analysis.

To prove our main theorem, we make assumptions on the relative sizes of the above parameters; see Sect. 8.6.

8.2. **Assumptions on the initial conditions of the eikonal function quantities.** We now state our size assumptions for the initial conditions of the eikonal function quantities (see Sect. 6.2 regarding the vectorfield operator notation).

L^2 assumptions along Σ_0^1. We assume that there exist (implicit) constants, depending on $\mathring{\delta}$, such that

$$\left\| \mathscr{Z}_*^{\leq 21;\leq 2} L^i_{(Small)} \right\|_{L^2(\Sigma_0^1)} \lesssim \mathring{\epsilon}, \tag{8.2.1a}$$

$$\left\| \breve{X}^{[1,3]} L^i_{(Small)} \right\|_{L^2(\Sigma_0^1)} \lesssim 1, \tag{8.2.1b}$$

$$\left\| \mathscr{Z}_{**}^{[1,21];\leq 1} \mu \right\|_{L^2(\Sigma_0^1)} \lesssim \mathring{\epsilon}, \tag{8.2.2a}$$

$$\left\| L\breve{X}^{[0,2]} \mu \right\|_{L^2(\Sigma_0^1)}, \left\| \breve{X}^{[0,2]} L\mu \right\|_{L^2(\Sigma_0^1)}, \left\| \breve{X} L \breve{X} \mu \right\|_{L^2(\Sigma_0^1)}, \left\| \breve{X}^{[0,2]} \mu \right\|_{L^2(\Sigma_0^1)} \lesssim 1. \tag{8.2.2b}$$

L^∞ assumptions along Σ_0^1.

$$\left\| \mathscr{Z}_*^{\leq 11;2} L^i_{(Small)} \right\|_{L^\infty(\Sigma_0^1)} \lesssim \mathring{\epsilon}, \tag{8.2.3}$$

$$\left\| \breve{X}^{[0,2]} L^i_{(Small)} \right\|_{L^\infty(\Sigma_0^1)} \lesssim 1, \tag{8.2.4}$$

$$\|\mu - 1\|_{L^\infty(\Sigma_0^1)}, \left\| \mathscr{Z}_{**}^{[1,11];\leq 1} \mu \right\|_{L^\infty(\Sigma_0^1)} \lesssim \mathring{\epsilon}, \tag{8.2.5a}$$

$$\left\| L\breve{X}^{[0,2]} \mu \right\|_{L^\infty(\Sigma_0^1)}, \left\| \breve{X}^{[0,2]} L\mu \right\|_{L^\infty(\Sigma_0^1)}, \left\| \breve{X} L \breve{X} \mu \right\|_{L^\infty(\Sigma_0^1)}, \left\| \breve{X}^{[0,2]} \mu \right\|_{L^\infty(\Sigma_0^1)} \lesssim 1, \tag{8.2.5b}$$

$$\left\| \mathscr{Z}^{\leq 18;\leq 3} (\Theta^i - \delta_2^i) \right\|_{L^\infty(\Sigma_0^1)} \lesssim \mathring{\epsilon}, \tag{8.2.6}$$

$$\left\| \mathscr{Z}^{\leq 18;\leq 2} \Xi^i \right\|_{L^\infty(\Sigma_0^1)} \lesssim \mathring{\epsilon}. \tag{8.2.7}$$

L^∞ assumptions along $\mathcal{P}_0^{2\mathring{\delta}_*^{-1}}$.

$$\|\mu - 1\|_{L^\infty(\mathcal{P}_0^{2\mathring{\delta}_*^{-1}})} \lesssim \mathring{\epsilon}. \tag{8.2.8}$$

8.3. **$T_{(Boot)}$, the positivity of μ, and the diffeomorphism property of Υ.** We now state some basic bootstrap assumptions. We start by fixing a real number $T_{(Boot)}$ with

$$0 < T_{(Boot)} \leq 2\mathring{\delta}_*^{-1}. \tag{8.3.1}$$

We assume that on the spacetime domain $\mathcal{M}_{T_{(Boot)},U_0}$ (see (3.6.4e)), we have

$$\mu > 0. \tag{$\mathbf{BA}\mu > 0$}$$

Inequality ($\mathbf{BA}\mu > 0$) implies that no shocks are present in $\mathcal{M}_{T_{(Boot)},U_0}$.

We also assume that

The change of variables map Υ from Def. 3.12 is a C^1 diffeomorphism from $[0, T_{(Boot)}) \times [0, U_0] \times \mathbb{T}$ onto its image. (8.3.2)

8.4. **Fundamental L^{∞} bootstrap assumptions.** Our fundamental bootstrap assumptions for $\vec{\Psi}$ and ω are that the following inequalities hold on $\mathcal{M}_{T_{(Boot)},U_0}$ (see Sect. 6.2 regarding the vectorfield operator notation):

$$\left\| \mathscr{Z}_*^{\leq 13;\leq 1} \vec{\Psi} \right\|_{L^{\infty}(\Sigma_t^u)}, \left\| \mathscr{Z}^{\leq 13;\leq 1} (\rho - v^1) \right\|_{L^{\infty}(\Sigma_t^u)} \leq \varepsilon, \qquad (\mathbf{BA}\vec{\Psi})$$

$$\left\| \mathscr{P}^{\leq 13} \omega \right\|_{L^{\infty}(\Sigma_t^u)} \leq \varepsilon, \qquad (\mathbf{BA}\omega)$$

where ε is a small positive bootstrap parameter whose smallness we describe in Sect. 8.6.

8.5. **Auxiliary L^{∞} bootstrap assumptions.** In deriving pointwise estimates, we find it convenient to make the following auxiliary bootstrap assumptions. In Prop. 9.12, we will derive strict improvements of these assumptions.

Auxiliary bootstrap assumptions for small quantities. We assume that the following inequalities hold on $\mathcal{M}_{T_{(Boot)},U_0}$:

$$\left\| \mathscr{Z}_*^{\leq 12;\leq 2} \vec{\Psi} \right\|_{L^{\infty}(\Sigma_t^u)} \leq \varepsilon^{1/2}, \qquad (\mathbf{AUX}\vec{\Psi}\mathbf{SMALL})$$

$$\left\| \mathscr{Z}_*^{\leq 11;\leq 2} L^i_{(Small)} \right\|_{L^{\infty}(\Sigma_t^u)} \leq \varepsilon^{1/2}, \qquad (\mathbf{AUX}L_{(Small)}\mathbf{SMALL})$$

$$\left\| \mathscr{Z}_{**}^{[1,11];1} \mu \right\|_{L^{\infty}(\Sigma_t^u)} \leq \varepsilon^{1/2}, \qquad (\mathbf{AUX}\mu\mathbf{SMALL})$$

$$\left\| \not{\mathcal{L}}_{\mathscr{Z}}^{\leq 11;\leq 1} \chi \right\|_{L^{\infty}(\Sigma_t^u)}, \left\| \not{\mathcal{L}}_{\mathscr{Z}}^{\leq 10;\leq 2} \chi \right\|_{L^{\infty}(\Sigma_t^u)} \leq \varepsilon^{1/2}. \qquad (\mathbf{AUX}\chi)$$

Auxiliary bootstrap assumptions for quantities that are allowed to be large. We assume that the following inequalities hold on $\mathcal{M}_{T_{(Boot)},U_0}$ for $M = 1, 2$:

$$\left\| \breve{X}^M \rho \right\|_{L^{\infty}(\Sigma_t^u)} \leq \left\| \breve{X}^M \rho \right\|_{L^{\infty}(\Sigma_0^u)} + \varepsilon^{1/2}, \qquad (\mathbf{AUX}\rho\mathbf{LARGE})$$

$$\left\| \breve{X}^M v^1 \right\|_{L^{\infty}(\Sigma_t^u)} \leq \left\| \breve{X}^M v^1 \right\|_{L^{\infty}(\Sigma_0^u)} + \varepsilon^{1/2}. \qquad (\mathbf{AUX}v^1\mathbf{LARGE})$$

We assume that the following inequalities hold on $\mathcal{M}_{T_{(Boot)},U_0}$ for $M = 0, 1$:

$$\left\| L \breve{X}^M \mu \right\|_{L^{\infty}(\Sigma_t^u)} \leq \frac{1}{2} \left\| \breve{X}^M \left\{ \vec{G}_{LL} \diamond \breve{X} \vec{\Psi} \right\} \right\|_{L^{\infty}(\Sigma_0^u)} + \varepsilon^{1/2}, \qquad (\mathbf{AUX}L\mu)$$

$$\left\| \breve{X}^M \mu \right\|_{L^{\infty}(\Sigma_t^u)} \leq \left\| \breve{X}^M \mu \right\|_{L^{\infty}(\Sigma_0^u)} + 2 \mathring{\delta}_*^{-1} \left\| \breve{X}^M \left\{ \vec{G}_{LL} \diamond \breve{X} \vec{\Psi} \right\} \right\|_{L^{\infty}(\Sigma_0^u)} + \varepsilon^{1/2}. \qquad (\mathbf{AUX}\mu)$$

We assume that the following inequalities hold on $\mathcal{M}_{T_{(Boot)},U_0}$ for $M = 1, 2$:

$$\left\| \breve{X}^M L^i_{(Small)} \right\|_{L^{\infty}(\Sigma_t^u)} \leq \left\| \breve{X}^M L^i_{(Small)} \right\|_{L^{\infty}(\Sigma_0^u)} + \varepsilon^{1/2}. \qquad (\mathbf{AUX}L_{(Small)}\mathbf{LARGE})$$

8.6. **Smallness assumptions.** For the remainder of the article, when we say that "A is small relative to B," we mean that there exists a continuous increasing function $f : [0,\infty) \to (0,\infty)$ such that $A \leq f(B)$. In principle, the functions f could always be chosen to be polynomials with positive coefficients or exponential functions.[102] However, to avoid lengthening the paper, we typically do not specify the form of f.

Throughout the rest of the paper, we make the following relative smallness assumptions. We continually adjust the required smallness in order to close our estimates.

- ε is small relative to $\mathring{\delta}^{-1}$, where $\mathring{\delta}$ is the data-size parameter from (8.1.3b).
- ε is small relative to the data-size parameter $\mathring{\delta}_*$ from (8.1.1).

The first assumption will allow us to control error terms that, roughly speaking, are of size $\varepsilon\mathring{\delta}^k$ for some integer $k \geq 0$. The second assumption is relevant because the expected blowup-time is approximately $\mathring{\delta}_*^{-1}$, and the assumption will allow us to show that various error products featuring a small factor ε remain small for $t < 2\mathring{\delta}_*^{-1}$, which is plenty of time for us to show that a shock forms.

Finally, we assume that

$$\varepsilon^{3/2} \leq \mathring{\epsilon} \leq \varepsilon, \tag{8.6.1}$$

where $\mathring{\epsilon}$ is the data smallness parameter from Sects. 8.1.2 and 8.2.

Remark 8.3 (**Relationship between ε and $\mathring{\epsilon}$ in the proof of our main theorem**)**.** In the proof of our main theorem, we will set $\varepsilon = C'\mathring{\epsilon}$, where $C' > 1$ is chosen to be sufficiently large and $\mathring{\epsilon}$ is assumed to be sufficiently small. This is compatible with (8.6.1).

8.7. **The existence of initial data verifying the size assumptions.** In this section, we show that there exists an open set of data verifying the size the assumptions of Subsects. 8.1,8.2, and 8.6. By "open," we mean open relative to the Sobolev topologies corresponding to the size assumptions stated in those subsections. By Cauchy stability,[103] it is enough to exhibit smooth plane symmetric data that are compactly supported in Σ_0^1 (which can be identified here with the unit x^1 interval $[0,1]$) and that verify the size assumptions. By a plane symmetric solution, we mean that $\rho = \rho(t,x^1)$, $v^1 = v^1(t,x^1)$, and $v^2 \equiv 0$. The data that we exhibit launch simple plane symmetric solutions. By "simple," we mean that one Riemann invariant completely vanishes.

Remark 8.4 (**Strictly non-zero vorticity along Σ_0^1 and $\mathcal{P}_0^{2\mathring{\delta}_*^{-1}}$**)**.** Once we have exhibited the plane symmetric data described above, it is easy to perturb it so that the vorticity is everywhere non-zero along Σ_0^1 and $\mathcal{P}_0^{2\mathring{\delta}_*^{-1}}$. One can simply leave the data for ρ and v^1 along Σ_0 unchanged and set $v^2|_{\Sigma_0} = f(x^1)$, where $\lambda > 0$ is small and f is smooth with $f' > 0$ in an interval I of length $|I| \gg 2\mathring{\delta}_*^{-1}$ containing the origin. Then ω will be small but non-zero on $I \times \mathbb{T} \subset \Sigma_0$. Hence, using the transport equation (3.3.11c), it is easy to show[104] (under suitable smallness assumptions) that the corresponding solution "induces" data for ω along $\mathcal{P}_0^{2\mathring{\delta}_*^{-1}}$ such that ω is everywhere non-zero along $\mathcal{P}_0^{2\mathring{\delta}_*^{-1}}$.

[102] The exponential functions appear, for example, in our energy estimates, during our Gronwall argument; see the proof of Prop. 15.1 given in Sect. 15.16.

[103] Here we mean continuous dependence of the solution on the data.

[104] In the solution regime under consideration, equation (3.3.11c) reads $\partial_t\omega =$ Quadratically small error terms.

We remind the reader (see (3.6.2)) that the initial condition for the eikonal function is $u|_{\Sigma_0} := 1 - x^1$. Our discussion relies on the following simple lemma.

Lemma 8.1. [30, Lemma 7.2; **Algebraic identities along** Σ_0] *Consider a solution launched by data given along* Σ_0*. Assume that the datum for the eikonal function is* $u|_{\Sigma_0} := 1 - x^1$*. Then the following identities hold along* Σ_0 *(for* $i = 1, 2$*):*

$$\upmu = \frac{1}{c_s}, \qquad L^i_{(Small)} = (c_s - 1)\delta^{i1} + v^i, \qquad \Xi^i = 0, \qquad \Theta^i = \delta^i_2, \tag{8.7.1}$$

where Ξ *is the* $\ell_{t,u}$*-tangent vectorfield from* (3.7.12)*.*

We now turn to the construction of plane symmetric initial data that lead to the desired size assumptions. Our approach is based on Riemann's method of Riemann invariants [27]. The results that we present here are standard. Hence, for brevity, we do not provide detailed proofs. In plane symmetry, in terms of the Riemann invariants

$$\mathcal{R}_\pm = v^1 \pm F(\uprho), \tag{8.7.2}$$

the compressible Euler equations (1.0.1a)-(1.0.1b) are equivalent to the system

$$\breve{\underline{L}}\mathcal{R}_- = 0, \qquad\qquad L\mathcal{R}_+ = 0, \tag{8.7.3}$$

where

$$\breve{\underline{L}} := \upmu\underline{L}, \tag{8.7.4}$$

$$L = \partial_t + (v^1 + c_s)\partial_1, \qquad\qquad \underline{L} = \partial_t + (v^1 - c_s)\partial_1, \tag{8.7.5}$$

and L coincides with the vectorfield defined in Def. 3.13. The function F in (8.7.3) solves the following initial value problem in $\uprho$:

$$\frac{d}{d\uprho}F(\uprho) = c_s(\uprho), \qquad\qquad F(\uprho = 0) = 0, \tag{8.7.6}$$

where $F(\uprho = 0) = 0$ is just a convenient normalization condition. It is straightforward to show that

$$\underline{L} = L + 2X, \tag{8.7.7}$$

$$X = -c_s\partial_1, \tag{8.7.8}$$

where X coincides with the vectorfield defined in Def. 3.14. Then by (8.7.1), we have

$$\breve{X}|_{\Sigma_0} = -\partial_1. \tag{8.7.9}$$

Hence, by (3.7.7), we have

$$\breve{\underline{L}} = \upmu L + 2\upmu X = \upmu L + 2\breve{X}. \tag{8.7.10}$$

The desired initial data can be constructed by simply taking smooth data $(\mathcal{R}_-|_{\Sigma_0^1}, \mathcal{R}_+|_{\Sigma_0^1})$ for the system (8.7.3) that are supported in Σ_0^1 such that $\mathcal{R}_-|_{\Sigma_0^1} \equiv 0$, $\sum_{M=1}^3 \|\breve{X}^M\mathcal{R}_+\|_{L^\infty(\Sigma_0^1)} \leq \mathring{\updelta}'$, and $\|\mathcal{R}_+\|_{L^\infty(\Sigma_0^1)} \leq \mathring{\upepsilon}'$, where $\mathring{\upepsilon}'$ and $\mathring{\updelta}'$ verify the same relative size assumptions as the parameters $\mathring{\upepsilon}$ and $\mathring{\updelta}$ described in Subsect. 8.6. As we now outline, this leads to the desired size assumptions stated in Subsects. 8.1,8.2, and 8.6, where the smallness of $\mathring{\upepsilon}$ is induced by the smallness of $\mathring{\upepsilon}'$ and the relative largeness of $\mathring{\updelta}$ is tied to the relative largeness of $\mathring{\updelta}'$.

We first note that the support assumption on the data implies that the solution completely vanishes along $\mathcal{P}_0$, consistent with the data assumptions made in Subsects. 8.1 and 8.2. We next note that the first evolution equation in (8.7.3) implies that $\mathcal{R}_- \equiv 0$. One can derive estimates for the mixed derivatives of $\mathcal{R}_+|_{\Sigma_0^1}$ with respect to L and $\breve{X}$ by commuting the second evolution equation in (8.7.3). In view of the simple commutation relation $[L, \breve{X}] = 0$, valid in plane symmetry, we obtain that $L^{M_1}\breve{X}^{M_2}\mathcal{R}_+ = 0$ if $M_1 \geq 1$, from which it easily follows (see equation (3.14.1)) that if $M_1 \geq 1$, then $LL^{M_1}\breve{X}^{M_2}\upmu = 0$.

From these facts, one can show that all of the data assumptions stated in Subsects. 8.1 and 8.2 are verified if $\mathring{\upepsilon}'$ is sufficiently small. We do not give a the full proof here because it is straightforward but tedious; instead, we prove four representative estimates. First, using (3.4.3), (8.7.2), Taylor expansions, and the fact that $\mathcal{R}_- \equiv 0$, we obtain $\breve{X}(v^1 - \uprho) = \breve{X}\mathcal{R}_- + (1 - c_s)\breve{X}\uprho = \mathcal{O}(\mathcal{R}_+)\breve{X}\mathcal{R}_+$. Hence, using the above estimates, we obtain $\|\breve{X}(v^1 - \uprho)\|_{L^\infty(\Sigma_0^1)} \lesssim \mathring{\upepsilon}'$, which is consistent with the smallness assumption (8.1.2) for the first entry of the second term on the LHS. As a second example, we note that with the help of (8.7.1), we have $\upmu|_{\Sigma_0^1} = 1 + \mathcal{O}(\mathcal{R}_+)$. Hence, using the above estimates, we obtain $\|\upmu - 1\|_{L^\infty(\Sigma_0^1)} \lesssim \mathring{\upepsilon}'$ (consistent with the smallness assumption (8.2.5a) for the first term on the LHS) and $\|\breve{X}^M\upmu\|_{L^\infty(\Sigma_0^1)} \lesssim \mathring{\updelta}' \lesssim 1$ for $M = 1, 2$, consistent with the assumptions stated in (8.2.5b) for the last term on the LHS. As a third example, we note that with the help of (8.7.1), it is easy to show that[105] $\Xi^i \equiv 0$ and $\Theta^i \equiv \delta_2^i$ in the maximal development of the data, which in particular is consistent with (8.2.6)-(8.2.7). As a last example, we note that $\upomega \equiv 0$ in plane symmetry, which is consistent with the smallness assumptions (8.1.2), (8.1.3a), and (8.1.7) for $\upomega$.

9. Preliminary pointwise estimates

In this section, we derive preliminary pointwise estimates for the simplest error terms that appear in the commuted equations. Our arguments rely on the data-size assumptions and bootstrap assumptions stated in Sect. 8 and are tedious to carry out but not too difficult.

In the remainder of the article, we schematically express many equations and inequalities by stating them in terms of the arrays $\upgamma$ and $\underline{\upgamma}$ from Def. 3.31. We also remind the reader that we often use the abbreviations introduced Sect. 6.2 to schematically indicate the structure of various differential operators.

9.1. **Differential operator comparison estimates.**

In this section, we provide quantitative comparison estimates relating various differential operators on $\ell_{t,u}$.

We start by providing a simple lemma in which we express $\not\Delta f$ and $\not\nabla^2 f$ in terms of derivatives with respect to the vectorfield Y.

[105] Here we are viewing plane symmetric solutions to be solutions on $\mathbb{R} \times \mathbb{R} \times \mathbb{T}$.

Lemma 9.1 (**$\Delta\!\!\!/$ and $\nabla\!\!\!/^2$ in terms of Y derivatives**). *Let Y be the $\ell_{t,u}$-tangent vectorfield defined in Def. 3.27. We have the following differential operator identities, valid for scalar-valued functions f defined on $\ell_{t,u}$:*

$$\Delta\!\!\!/ f = \frac{1}{g(Y,Y)} YYf - \frac{1}{g(Y,Y)} \{Y \ln g(Y,Y)\} Yf, \tag{9.1.1a}$$

$$\nabla\!\!\!/^2 f = \frac{1}{\{g(Y,Y)\}^2} (YYf) Y_\flat \otimes Y_\flat - \frac{1}{\{g(Y,Y)\}^2} \{Y \ln g(Y,Y)\} Yf. \tag{9.1.1b}$$

Proof. Using (3.20.1), we obtain

$$\Delta\!\!\!/ f = \frac{1}{g(Y,Y)} YYf - \frac{1}{g(Y,Y)} (\nabla\!\!\!/_Y Y) \cdot d\!\!\!/ f. \tag{9.1.2}$$

Since $\nabla\!\!\!/_Y Y$ is $\ell_{t,u}$-tangent, it must be a scalar-valued function multiple, denoted by M, of Y: $MY = \nabla\!\!\!/_Y Y$. Taking the inner product of this identity with Y, we obtain $Mg(Y,Y) = g\!\!\!/(\nabla\!\!\!/_Y Y, Y) = \frac{1}{2}\nabla\!\!\!/_Y \{g\!\!\!/(Y,Y)\} = \frac{1}{2} Y \{g(Y,Y)\}$. Solving for M and substituting into (9.1.2), we conclude (9.1.1a).

(9.1.1b) then follows from (9.1.1a) and the identity (3.20.2) with $\xi = \nabla\!\!\!/^2 f$.

□

The next lemma shows that the pointwise norms of $\ell_{t,u}$ tensors are controlled by contractions against Y.

Lemma 9.2 (**The norm of $\ell_{t,u}$-tangent tensors can be measured via Y contractions**). *Let $\xi_{\alpha_1 \cdots \alpha_n}$ be a type $\binom{0}{n}$ $\ell_{t,u}$-tangent tensor with $n \geq 1$ and let Y be the $\ell_{t,u}$-tangent vectorfield defined in Def. 3.27. Under the data-size and bootstrap assumptions of Sects. 8.1-8.5 and the smallness assumptions of Sect. 8.6, we have*

$$|\xi| = \left\{1 + \mathcal{O}(\varepsilon^{1/2})\right\} |\xi_{YY \cdots Y}|. \tag{9.1.3}$$

The same result holds if $|\xi_{YY \cdots Y}|$ is replaced with $|\xi_{Y\cdot}|$, $|\xi_{YY\cdot}|$, etc., where $\xi_{Y\cdot}$ is the type $\binom{0}{n-1}$ tensor with components $Y^{\alpha_1}\xi_{\alpha_1\alpha_2\cdots\alpha_n}$, and similarly for $\xi_{YY\cdot}$, etc.

Proof. (9.1.3) is easy to derive relative to Cartesian coordinates by using the decomposition $(g\!\!\!/^{-1})^{ij} = \frac{1}{|Y|^2} Y^i Y^j$ (see (3.20.1)) and the estimate $|Y| = 1 + \mathcal{O}(\varepsilon^{1/2})$, which follows from the identity $|Y|^2 = g_{ab} Y^a Y^b = (\delta_{ab} + g_{ab}^{(Small)})(\delta_2^a + Y_{(Small)}^a)(\delta_2^b + Y_{(Small)}^b)$, the schematic relations $g_{ab}^{(Small)}, Y_{(Small)}^a = \mathrm{f}(\gamma)\gamma$ (see Lemma 3.19), and the bootstrap assumptions. □

We now establish some comparison estimates for various differential operators on $\ell_{t,u}$.

Lemma 9.3 (**Controlling $\nabla\!\!\!/$ derivatives in terms of Y derivatives**). *Let f be a scalar-valued function on $\ell_{t,u}$. Under the data-size and bootstrap assumptions of Sects. 8.1-8.5 and the smallness assumptions of Sect. 8.6, the following comparison estimates hold on $\mathcal{M}_{T_{(Boot)},U_0}$:*

$$|d\!\!\!/ f| \leq (1 + C\varepsilon^{1/2}) |Yf|, \qquad |\nabla\!\!\!/^2 f| \leq (1 + C\varepsilon^{1/2}) |d\!\!\!/(Yf)| + C\varepsilon |d\!\!\!/ f|. \tag{9.1.4}$$

Proof. The first inequality in (9.1.4) follows directly from Lemma 9.2. To prove the second, we first use Lemma 9.2, the identity $\slashed{\nabla}^2_{YY} f = Y \cdot \slashed{d}(Yf) - \slashed{\nabla}_Y Y \cdot \slashed{d} f$, and the estimate $|Y| = 1 + \mathcal{O}(\varepsilon^{1/2})$ noted in the proof of Lemma 9.2 to deduce

$$|\slashed{\nabla}^2 f| \leq (1 + C\varepsilon^{1/2})|\slashed{\nabla}^2_{YY} f| \leq (1 + C\varepsilon^{1/2})|\slashed{d}(Yf)| + |\slashed{\nabla}_Y Y||\slashed{d} f|. \tag{9.1.5}$$

Next, we use Lemma 9.2 and the identity ${}^{(Y)}\slashed{\pi}_{YY} = \slashed{\nabla}_Y(\slashed{g}(Y, Y)) = Y(g_{ab} Y^a Y^b)$ to deduce that

$$|\slashed{\nabla}_Y Y| \lesssim |g(\slashed{\nabla}_Y Y, Y)| \lesssim \left|{}^{(Y)}\slashed{\pi}_{YY}\right| \lesssim \left|Y(g_{ab} Y^a Y^b)\right|. \tag{9.1.6}$$

Since Lemma 3.19 implies that $g_{ab} Y^a Y^b = \mathrm{f}(\gamma)$ with f smooth, the bootstrap assumptions yield that RHS (9.1.6) $\lesssim |Y\gamma| \lesssim \varepsilon^{1/2}$. The desired estimate for $|\slashed{\nabla}^2 f|$ now follows from this estimate, (9.1.5), and (9.1.6). □

Lemma 9.4 (Controlling $\slashed{\mathcal{L}}_V$ and $\slashed{\nabla}$ derivatives in terms of $\slashed{\mathcal{L}}_Y$ derivatives). *Let $\xi_{\alpha_1 \cdots \alpha_n}$ be a type $\binom{0}{n}$ $\ell_{t,u}$-tangent tensor with $n \geq 1$ and let V be an $\ell_{t,u}$-tangent vectorfield. Under the data-size and bootstrap assumptions of Sects. 8.1-8.5 and the smallness assumptions of Sect. 8.6, the following comparison estimates hold on $\mathcal{M}_{T_{(Boot)}, U_0}$:*

$$\begin{aligned} |\slashed{\mathcal{L}}_V \xi| &\lesssim |V|\, |\slashed{\mathcal{L}}_Y \xi| + |\xi|\, |\slashed{\mathcal{L}}_Y V| + |Y\gamma|\, |\xi||V| \\ &\lesssim |V|\, |\slashed{\mathcal{L}}_Y \xi| + |\xi|\, |\slashed{\mathcal{L}}_Y V| + \varepsilon^{1/2}|\xi||V|, \end{aligned} \tag{9.1.7}$$

$$\begin{aligned} |\slashed{\nabla} \xi| &\lesssim |\slashed{\mathcal{L}}_Y \xi| + |Y\gamma|\, |\xi| \\ &\lesssim |\slashed{\mathcal{L}}_Y \xi| + \varepsilon^{1/2}|\xi|. \end{aligned} \tag{9.1.8}$$

Proof. To prove (9.1.7), we use the schematic Lie derivative identity $\slashed{\mathcal{L}}_V \xi = \slashed{\nabla}_V \xi + \sum \xi \cdot \slashed{\nabla} V$ and Lemma 9.2 to deduce

$$|\slashed{\mathcal{L}}_V \xi| \lesssim |V||\slashed{\nabla}_Y \xi| + |\xi||\slashed{\nabla}_Y V|. \tag{9.1.9}$$

Next, we note that the torsion-free property of $\slashed{\nabla}$ implies that $\slashed{\nabla}_Y V = \slashed{\mathcal{L}}_Y V + \slashed{\nabla}_V Y$. Hence, using Lemma 9.2, (9.1.6), and the estimate $|\slashed{\nabla}_Y Y| \lesssim |Y\gamma| \lesssim \varepsilon^{1/2}$ shown in the proof of Lemma 9.3, we find that

$$\begin{aligned} |\slashed{\nabla}_Y V| &\lesssim |\slashed{\mathcal{L}}_Y V| + |V||\slashed{\nabla} Y| \lesssim |\slashed{\mathcal{L}}_Y V| + |V||\slashed{\nabla}_Y Y| \lesssim |\slashed{\mathcal{L}}_Y V| + |Y\gamma||V| \\ &\lesssim |\slashed{\mathcal{L}}_Y V| + \varepsilon^{1/2}|V|. \end{aligned} \tag{9.1.10}$$

Similarly, we have

$$|\slashed{\nabla}_Y \xi| \lesssim |\slashed{\mathcal{L}}_Y \xi| + \varepsilon^{1/2}|\xi|. \tag{9.1.11}$$

The desired estimate (9.1.7) now follows from (9.1.9), (9.1.10), and (9.1.11).

The estimate (9.1.8) follows from applying Lemma 9.2 to $\slashed{\nabla}\xi$ and using (9.1.11). □

9.2. **Basic facts and estimates that we use silently.** For the reader's convenience, we present here some basic facts and estimates that we silently use throughout the rest of the paper when deriving estimates.

(1) All quantities that we estimate can be controlled in terms of $\underline{\gamma} = \{\vec{\Psi}, \upmu, L^1_{(Small)}, L^2_{(Small)}\}$ and the specific vorticity ω.

(2) We typically use the Leibniz rule for the operators $\not{\mathcal{L}}_Z$ and $\nabla\!\!\!/$ when deriving pointwise estimates for the $\not{\mathcal{L}}_Z$ and $\nabla\!\!\!/$ derivatives of tensor products of the schematic form $\prod_{i=1}^m v_i$, where the v_i are scalar functions or $\ell_{t,u}$-tangent tensors. Our derivative counts are such that all v_i except at most one are uniformly bounded in L^∞ on $\mathcal{M}_{T_{(Boot)},U_0}$. Thus, our pointwise estimates often explicitly feature (on the right-hand sides) only one factor with many derivatives on it, multiplied by a constant that uniformly bounds the other factors. In some estimates, the right-hand sides also gain a smallness factor, such as $\varepsilon^{1/2}$, generated by the remaining $v_i's$.

(3) The operators $\not{\mathcal{L}}_{\mathscr{Z}}^N$ commute through $d\!\!/$, as shown by Lemma 3.10.

(4) As differential operators acting on scalar functions, we have $Y = (1+\mathcal{O}(\gamma))\,d\!\!/ = (1+\mathcal{O}(\varepsilon^{1/2}))d\!\!/$, a fact which follows from Lemma 9.1.4, (9.4.2a), and the bootstrap assumptions. Hence, for scalar functions f, we sometimes schematically depict $d\!\!/ f$ as $(1+\mathcal{O}(\gamma))\,Pf$ or $(1+\mathcal{O}(\gamma))\,\mathscr{Z}_{**}^{1;0}f$, or alternatively as Pf or $\mathscr{Z}_{**}^{1;0}f$ when the factor $1+\mathcal{O}(\gamma)$ is not important. Similarly, by Lemma 9.3 we can depict $\Delta\!\!\!/ f$ by $\mathrm{f}(\mathscr{P}^{\leq 1}\gamma, g\!\!\!/^{-1})\mathscr{Z}_{**}^{[0,2];0}f$ (or $\mathscr{Z}_{**}^{[0,2];0}f$ when the factor $\mathrm{f}(\mathscr{P}^{\leq 1}\gamma, g\!\!\!/^{-1})$ is not important). Similarly, by Lemma 9.4, for type $\binom{0}{n}$ $\ell_{t,u}$-tangent tensorfields ξ, we can depict $\nabla\!\!\!/\xi$ by $\mathrm{f}(\mathscr{P}^{\leq 1}\gamma, g\!\!\!/^{-1})\not{\mathcal{L}}_{\mathscr{P}}^{\leq 1}\xi$ (or $\not{\mathcal{L}}_{\mathscr{P}}^{\leq 1}\xi$ when the factor $\mathrm{f}(\mathscr{P}^{\leq 1}\gamma, g\!\!\!/^{-1})$ is not important).

(5) We remind the reader that all constants are allowed to depend on the data-size parameters $\mathring{\delta}$ and $\mathring{\delta}_*^{-1}$.

9.3. **Pointwise estimates for the Cartesian coordinates and the Cartesian components of some vectorfields.**

Lemma 9.5 (Pointwise estimates for x^i and the Cartesian components of several vectorfields). *Assume that*[106] $1 \leq N \leq 20$, $0 \leq M \leq \min\{N,2\}$, *and* $V \in \{L, X, Y\}$. *Let* $x^i = x^i(t,u,\vartheta)$ *denote the Cartesian spatial coordinate function and let* $\mathring{x}^i = \mathring{x}^i(u,\vartheta) := x^i(0,u,\vartheta)$. *Under the data-size and bootstrap assumptions of Sects. 8.1-8.5 and the smallness assumptions of Sect. 8.6, the following pointwise estimates hold on* $\mathcal{M}_{T_{(Boot)},U_0}$, *for* $i=1,2$ *(see Sect. 6.2 regarding the vectorfield operator notation):*

$$\left|V^i\right| \lesssim 1 + |\gamma|\,, \tag{9.3.1a}$$

$$\left|\mathscr{Z}^{[1,N];M} V^i\right| \lesssim \left|\mathscr{Z}^{[1,N];\leq M}\gamma\right|, \tag{9.3.1b}$$

$$\left|\mathscr{Z}_*^{[1,N];M} V^i\right| \lesssim \left|\mathscr{Z}_*^{[1,N];\leq M}\gamma\right|. \tag{9.3.1c}$$

[106]Throughout, we use the convention that terms in our formulas and estimates involving operators that do not make sense are absent. $\mathscr{Z}_*^{1;1}$ is an example of such an operator.

Similarly, if $1 \leq N \leq 20$ *and* $0 \leq M \leq \min\{N, 1\}$, *then*

$$\left| \breve{X}^i \right| \lesssim 1 + |\underline{\gamma}|, \tag{9.3.2a}$$

$$\left| \mathscr{Z}^{[1,N];M} \breve{X}^i \right| \lesssim \left| \mathscr{Z}^{[1,N];\leq M} \underline{\gamma} \right|, \tag{9.3.2b}$$

$$\left| \mathscr{Z}_*^{[1,N];M} \breve{X}^i \right| \lesssim \left| \mathscr{Z}_*^{[1,N];\leq M} \underline{\gamma} \right|, \tag{9.3.2c}$$

$$\left| \mathscr{Z}_{**}^{[1,N];M} \breve{X}^i \right| \lesssim \left| \begin{pmatrix} \mathscr{Z}_{**}^{[1,N];\leq M} \underline{\gamma} \\ \mathscr{Z}_*^{[1,N];\leq M} \gamma \end{pmatrix} \right|. \tag{9.3.2d}$$

Moreover, if $1 \leq N \leq 20$ *and* $0 \leq M \leq \min\{N, 2\}$, *then*

$$\left| x^i - \mathring{x}^i \right| \lesssim 1, \tag{9.3.3a}$$

$$\left| \not{d} x^i \right| \lesssim 1 + |\gamma|, \tag{9.3.3b}$$

$$\left| \not{d} \mathscr{Z}^{[1,N];M} x^i \right| \lesssim \left| \begin{pmatrix} \mathscr{Z}_{**}^{[1,N];\leq (M-1)_+} \underline{\gamma} \\ \mathscr{Z}_*^{[1,N];\leq M} \gamma \end{pmatrix} \right|. \tag{9.3.3c}$$

Finally, if $0 \leq N \leq 20$ *and* $0 \leq M \leq \min\{N, 2\}$, *then*

$$\left| \mathscr{Z}^{N;M} Y^i_{(Small)} \right| \lesssim \left| \mathscr{Z}^{\leq N;\leq M} \gamma \right|, \tag{9.3.4a}$$

$$\left| \mathscr{Z}_*^{N;M} Y^i_{(Small)} \right| \lesssim \left| \mathscr{Z}_*^{\leq N;\leq M} \gamma \right|. \tag{9.3.4b}$$

In the case $i = 2$ *at fixed* u, ϑ, *LHS* (9.3.3a) *is to be interpreted as the Euclidean distance traveled by the point* x^2 *in the flat universal covering space* $\mathbb{R}$ *of* $\mathbb{T}$ *along the corresponding integral curve of* L *over the time interval* $[0, t]$.

Proof. See Sect. 9.2 for some comments on the analysis. Lemma 3.19 implies that for $V \in \{L, X, Y\}$, the component $V^i = V x^i$ verifies $V^i = \mathrm{f}(\gamma)$ with f smooth. Similarly, $Y^i_{(Small)}$ verifies $Y^i_{(Small)} = \mathrm{f}(\gamma)\gamma$ with f smooth and $\breve{X} x^i = \breve{X}^i$ verifies $\breve{X}^i = \mu \mathrm{f}(\gamma)$ with f smooth. The estimates of the lemma therefore follow easily from the bootstrap assumptions, except for the estimates (9.3.3a)-(9.3.3c). To obtain (9.3.3a), we first argue as above to deduce $|Lx^i| = |L^i| = |\mathrm{f}(\gamma)| \lesssim 1$. Since $L = \frac{\partial}{\partial t}$, we may integrate along the integral curves of L starting from time 1 to deduce, via the fundamental theorem of calculus, that

$$x^i(t, u, \vartheta) = x^i(0, u, \vartheta) + \int_{s=0}^{t} L x^i(s, u, \vartheta) \, ds. \tag{9.3.5}$$

Taking the absolute value of (12.3.11) and using the estimate $|Lx^i| \lesssim 1$ to bound the time integral by $\lesssim t \lesssim 1$, we conclude (9.3.3a). To derive (9.3.3b), we use (9.1.4) with $f = x^i$ to deduce $|\not{d} x^i| \lesssim |Y x^i| = |Y^i| = |\mathrm{f}(\gamma)| \lesssim 1 + |\gamma|$ as desired. The proof of (9.3.3c) is similar, but we also use Lemma 3.10 to commute vectorfields under $\not{d}$. ☐

9.4. **Pointwise estimates for various $\ell_{t,u}$-tensorfields.**

Lemma 9.6 (Crude pointwise estimates for the Lie derivatives of $\not{g}$ and $\not{g}^{-1}$). *Assume that* $1 \leq N \leq 20$ *and* $0 \leq M \leq \min\{N, 2\}$. *Under the data-size and bootstrap assumptions of Sects. 8.1-8.5 and the smallness assumptions of Sect. 8.6, the following pointwise* *estimates hold on* $\mathcal{M}_{T_{(Boot)},U_0}$ *(see Sect. 6.2 regarding the vectorfield operator notation):*

$$\left|\not{\mathcal{L}}_{\mathscr{Z}}^{N;M}\not{g}\right|, \left|\not{\mathcal{L}}_{\mathscr{Z}}^{N;M}\not{g}^{-1}\right| \lesssim \left(\begin{array}{c} \left|\mathscr{Z}_{**}^{[1,N];\leq(M-1)_+}\underline{\gamma}\right| \\ \left|\mathscr{Z}^{[1,N];\leq M}\gamma\right| \end{array} \right), \tag{9.4.1a}$$

$$\left|\not{\mathcal{L}}_{\mathscr{Z}_*}^{N;M}\not{g}\right|, \left|\not{\mathcal{L}}_{\mathscr{Z}_*}^{N;M}\not{g}^{-1}\right| \lesssim \left(\begin{array}{c} \left|\mathscr{Z}_{**}^{[1,N];\leq(M-1)_+}\underline{\gamma}\right| \\ \left|\mathscr{Z}_*^{[1,N];\leq M}\gamma\right| \end{array} \right). \tag{9.4.1b}$$

Moreover, if $0 \leq N \leq 20$ *and* $0 \leq M \leq \min\{N, 2\}$*, then*

$$\left|\not{\mathcal{L}}_{\mathscr{Z}}^{N;M}\chi\right|, \left|\mathscr{Z}^{N;M}\mathrm{tr}_{\not{g}}\chi\right| \lesssim \left(\begin{array}{c} \left|\mathscr{Z}_{**}^{[1,N+1];\leq(M-1)_+}\underline{\gamma}\right| \\ \left|\mathscr{Z}_*^{[1,N+1];\leq M}\gamma\right| \end{array} \right). \tag{9.4.1c}$$

Proof. See Sect. 9.2 for some comments on the analysis. By Lemma 3.19, we have $\not{g} = \mathrm{f}(\gamma, \not{d}\vec{x})$. The desired estimates for $\not{\mathcal{L}}_{\mathscr{Z}}^{N;M}\not{g}$ and $\not{\mathcal{L}}_{\mathscr{Z}_*}^{N;M}\not{g}$ thus follow from Lemma 9.5 and the bootstrap assumptions. The desired estimates for $\not{\mathcal{L}}_{\mathscr{Z}}^{N;M}\not{g}^{-1}$ and $\not{\mathcal{L}}_{\mathscr{Z}_*}^{N;M}\not{g}^{-1}$ then follow from repeated use of the second identity in (3.13.4) and the estimates for $\not{\mathcal{L}}_{\mathscr{Z}}^{N;M}\not{g}$ and $\not{\mathcal{L}}_{\mathscr{Z}_*}^{N;M}\not{g}$. The estimates for $\not{\mathcal{L}}_{\mathscr{Z}}^{N;M}\chi$ and $\mathscr{Z}^{N;M}\mathrm{tr}_{\not{g}}\chi$ follow from the estimates for $\not{\mathcal{L}}_{\mathscr{Z}}^{N+1;M}\not{g}$ and $\not{\mathcal{L}}_{\mathscr{Z}}^{N+1;M}\not{g}^{-1}$ since $\chi \sim \not{\mathcal{L}}_P\not{g}$ (see (3.9.4)) and $\mathrm{tr}_{\not{g}}\chi \sim \not{g}^{-1} \cdot \not{\mathcal{L}}_P\not{g}$. □

Lemma 9.7 (Pointwise estimates for the Lie derivatives of Y and some deformation tensor components). *Assume that* $1 \leq N \leq 20$ *and* $0 \leq M \leq \min\{N, 2\}$*. Under the data-size and bootstrap assumptions of Sects. 8.1-8.5 and the smallness assumptions of Sect. 8.6, the following pointwise estimates hold on* $\mathcal{M}_{T_{(Boot)},U_0}$ *(see Sect. 6.2 regarding the vectorfield operator notation):*

$$||Y| - 1| \leq C\,|\gamma|, \tag{9.4.2a}$$

$$\left|\not{\mathcal{L}}_{\mathscr{Z}}^{N;M}Y\right| \leq C\left|\left(\begin{array}{c} \mathscr{Z}_{**}^{[1,N];\leq(M-1)_+}\underline{\gamma} \\ \mathscr{Z}^{[1,N+1];\leq M}\gamma \end{array} \right)\right|, \tag{9.4.2b}$$

$$\left|\not{\mathcal{L}}_{\mathscr{Z}_*}^{N;M}Y\right| \leq C\left|\left(\begin{array}{c} \mathscr{Z}_{**}^{[1,N-1];\leq(M-1)_+}\underline{\gamma} \\ \mathscr{Z}_*^{[1,N];\leq M}\gamma \end{array} \right)\right|. \tag{9.4.2c}$$

Similarly, if $0 \leq N \leq 20$ *and* $0 \leq M \leq \min\{N, 2\}$*, then we have*

$$\left|\not{\mathcal{L}}_{\mathscr{Z}}^{N;M}{}^{(Y)}\not{\pi}_L^{\#}\right| \lesssim \left|\left(\begin{array}{c} \mathscr{Z}_{**}^{[1,N];\leq(M-1)_+}\underline{\gamma} \\ \mathscr{Z}_*^{[1,N+1];\leq M}\gamma \end{array} \right)\right|. \tag{9.4.3a}$$

In addition, if $0 \leq N \leq 20$ *and* $0 \leq M \leq \min\{N, 1\}$*, then we have*

$$\left|\not{\mathcal{L}}_{\mathscr{Z}}^{N;M}{}^{(\breve{X})}\not{\pi}_L^{\#}\right|, \left|\not{\mathcal{L}}_{\mathscr{Z}}^{N;M}{}^{(Y)}\not{\pi}_{\breve{X}}^{\#}\right| \lesssim \left|\mathscr{Z}_*^{[1,N+1];\leq M+1}\vec{\Psi}\right| + \left|\left(\begin{array}{c} \mathscr{Z}_{**}^{[1,N+1];\leq M}\underline{\gamma} \\ \mathscr{Z}^{\leq N+1;\leq M}\gamma \end{array} \right)\right|, \tag{9.4.4a}$$

and if $1 \leq N \leq 20$ *and* $0 \leq M \leq \min\{N-1, 1\}$*, then*

$$\left|\not{\mathcal{L}}_{\mathscr{Z}_*}^{N;M}{}^{(\breve{X})}\not{\pi}_L^{\#}\right|, \left|\not{\mathcal{L}}_{\mathscr{Z}_*}^{N;M}{}^{(Y)}\not{\pi}_{\breve{X}}^{\#}\right| \lesssim \left|\mathscr{Z}_*^{[1,N+1];\leq M+1}\vec{\Psi}\right| + \left|\left(\begin{array}{c} \mathscr{Z}_{**}^{[1,N+1];\leq M}\underline{\gamma} \\ \mathscr{Z}_*^{[1,N+1];\leq M}\underline{\gamma} \end{array} \right)\right|. \tag{9.4.4b}$$

Moreover, if $0 \leq N \leq 20$ *and* $0 \leq M \leq \min\{N, 2\}$*, then we have*

$$\left| \not{\mathcal{L}}_{\mathscr{Z}}^{N;M} {}^{(L)}\not{\pi} \right|, \left| \not{\mathcal{L}}_{\mathscr{Z}}^{N;M} {}^{(Y)}\not{\pi} \right| \lesssim \left| \begin{pmatrix} \mathscr{Z}_{**}^{[1,N];\leq (M-1)_+} \underline{\gamma} \\ \mathscr{Z}_{*}^{[1,N+1];\leq M} \gamma \end{pmatrix} \right|. \tag{9.4.5}$$

In addition, if $0 \leq N \leq 20$ *and* $0 \leq M \leq \min\{N, 1\}$*, then we have*

$$\left| \not{\mathcal{L}}_{\mathscr{Z}}^{N;M} {}^{(\breve{X})}\not{\pi} \right| \lesssim \left| \mathscr{Z}^{[1,N+1];\leq M+1} \vec{\Psi} \right| + \left| \begin{pmatrix} \mathscr{Z}_{**}^{[1,N];\leq (M-1)_+} \underline{\gamma} \\ \mathscr{Z}_{*}^{[1,N+1];\leq M} \gamma \end{pmatrix} \right|, \tag{9.4.6a}$$

and if $1 \leq N \leq 20$ *and* $0 \leq M \leq \min\{N-1, 1\}$*, then we have*

$$\left| \not{\mathcal{L}}_{\mathscr{Z}_*}^{N;M} {}^{(\breve{X})}\not{\pi} \right| \lesssim \left| \mathscr{Z}_{*}^{[1,N+1];\leq M+1} \vec{\Psi} \right| + \left| \begin{pmatrix} \mathscr{Z}_{**}^{[1,N];\leq (M-1)_+} \underline{\gamma} \\ \mathscr{Z}_{*}^{[1,N+1];\leq M} \gamma \end{pmatrix} \right|. \tag{9.4.6b}$$

Proof. See Sect. 9.2 for some comments on the analysis. To prove (9.4.3a), we first note that by Lemma 3.19 and (3.16.3b), we have ${}^{(Y)}\not{\pi}_L^{\#} = \mathrm{f}(\gamma, \not{g}^{-1}, \not{d}\vec{x}) P\gamma$. We now apply $\not{\mathcal{L}}_{\mathscr{Z}}^{N;M}$ to the previous relation. We bound the derivatives of $\not{g}^{-1}$ and $\not{d}x$ with Lemmas 9.5 and 9.6. Also using the bootstrap assumptions, we conclude the desired result.

Since Lemma 3.19 implies that $Y = \mathrm{f}(\gamma, \not{g}^{-1}, \not{d}\vec{x})$, similar reasoning yields (9.4.2b)-(9.4.2c).

Inequality (9.4.2a) follows from the slightly more precise arguments already given in the proof of Lemma 9.2.

The proofs of (9.4.4a)-(9.4.4b) for ${}^{(Y)}\not{\pi}_{\breve{X}}^{\#}$ are similar and are based on the observation that by Lemma 3.19 and (3.16.3c), we have

$${}^{(Y)}\not{\pi}_{\breve{X}}^{\#} = \mathrm{f}(\underline{\gamma}, \not{g}^{-1}, \not{d}\vec{x}) P\gamma + \mathrm{f}(\gamma, \not{g}^{-1}, \not{d}\vec{x}, \breve{X}\vec{\Psi})\gamma + \mathrm{f}(\gamma, \not{g}^{-1}) \not{d}\mu.$$

The proofs of (9.4.4a)-(9.4.4b) for ${}^{(\breve{X})}\not{\pi}_L^{\#}$ are similar and are based on the observation that by Lemma 3.19, (3.9.8a), and (3.16.1b), we have

$${}^{(\breve{X})}\not{\pi}_L^{\#} = \mathrm{f}(\underline{\gamma}, \not{g}^{-1}, \not{d}\vec{x}) P\vec{\Psi} + \mathrm{f}(\gamma, \not{g}^{-1}, \not{d}\vec{x}, \breve{X}\vec{\Psi})\gamma + \not{g}^{-1} \not{d}\mu.$$

The proof of (9.4.5) is similar and is based on the fact that by Lemma 3.19, (3.16.2c), and (3.16.3d), we have ${}^{(L)}\not{\pi}, {}^{(Y)}\not{\pi} = \mathrm{f}(\gamma, \not{g}^{-1}, \not{d}\vec{x}) P\gamma$.

The proofs of (9.4.6a)-(9.4.6b) are similar and are based on the fact that by Lemma 3.19 and (3.16.1c), we have ${}^{(\breve{X})}\not{\pi} = \mathrm{f}(\underline{\gamma}, \not{g}^{-1}, \not{d}\vec{x}) P\gamma + \mathrm{f}(\gamma, \not{d}\vec{x}) \breve{X}\vec{\Psi}$. □

9.5. **Multi-indices and commutator estimates.** In this section, we establish some commutator estimates.

We start by defining some sets of multi-indices corresponding to repeated differentiation with respect to the commutation vectorfields.

Definition 9.1 (Sets of multi-indices). We define

$$\mathcal{I}^{N;M} \tag{9.5.1}$$

to be the set of $\mathscr{Z}$ multi-indices $\vec{I}$ with the following properties:

- $|\vec{I}| = N$.
- $\mathscr{Z}^{\vec{I}}$ contains at least one factor belonging to $\mathscr{P} = \{L, Y\}$.
- $\mathscr{Z}^{\vec{I}}$ contains precisely M factors of $\breve{X}$.

We define

$$\mathcal{I}^{N;\leq M}, \tag{9.5.2}$$

in the same way, except the last condition above is replaced with the following one:

- $\mathscr{Z}^{\vec{I}}$ contains no more than M factors of $\breve{X}$.

We now provide two preliminary lemmas from [30].

Lemma 9.8. [30, Lemma 5.1; **Preliminary identities for commuting $Z \in \mathscr{Z}$ with $\nabla\!\!\!/$**] *For each $\mathscr{Z}$-multi-index $\vec{I}$ and integer $n \geq 1$, the following commutator identity, correct up to constant factors, holds for all type $\binom{0}{n}$ $\ell_{t,u}$-tangent tensorfields ξ:*

$$[\nabla\!\!\!/, \mathcal{L}\!\!\!/_{\mathscr{Z}}^{\vec{I}}]\xi = \sum_{M=1}^{|\vec{I}|} \sum_{\substack{\vec{I}_1+\cdots+\vec{I}_{M+1}=\vec{I} \\ |\vec{I}_a|\geq 1 \text{ for } 1\leq a\leq M}} (g\!\!\!/^{-1})^M \underbrace{(\mathcal{L}\!\!\!/_{\mathscr{Z}}^{\vec{I}_1} g\!\!\!/) \cdots (\mathcal{L}\!\!\!/_{\mathscr{Z}}^{\vec{I}_{M-1}} g\!\!\!/)}_{\textit{absent when } M=1} (\nabla\!\!\!/ \mathcal{L}\!\!\!/_{\mathscr{Z}}^{\vec{I}_M} g\!\!\!/)(\mathcal{L}\!\!\!/_{\mathscr{Z}}^{\vec{I}_{M+1}} \xi). \tag{9.5.3}$$

Moreover, with $\mathrm{d}\!\!\!/\mathrm{iv}$ denoting the torus divergence operator from Def. 3.22, for each $\mathscr{Z}$-multi-index $\vec{I}$, the following commutator identity, correct up to constant factors, holds for all symmetric type $\binom{0}{2}$ $\ell_{t,u}$-tangent tensorfields ξ:

$$[\mathrm{d}\!\!\!/\mathrm{iv}, \mathcal{L}\!\!\!/_{\mathscr{Z}}^{\vec{I}}]\xi = \sum_{i_1+i_2=1} \sum_{M=1}^{|\vec{I}|} \sum_{\substack{\vec{I}_1+\cdots+\vec{I}_{M+1}=\vec{I} \\ |\vec{I}_a|\geq 1 \text{ for } 1\leq a\leq M}} (g\!\!\!/^{-1})^{M+1} \underbrace{(\mathcal{L}\!\!\!/_{\mathscr{Z}}^{\vec{I}_1} g\!\!\!/) \cdots (\mathcal{L}\!\!\!/_{\mathscr{Z}}^{\vec{I}_{M-1}} g\!\!\!/)}_{\textit{absent when } i_1=M=1} (\nabla\!\!\!/^{i_1} \mathcal{L}\!\!\!/_{\mathscr{Z}}^{\vec{I}_M} g\!\!\!/)(\nabla\!\!\!/^{i_2} \mathcal{L}\!\!\!/_{\mathscr{Z}}^{\vec{I}_{M+1}} \xi). \tag{9.5.4}$$

Finally, for each $\mathscr{Z}$-multi-index $\vec{I}$ and each commutation vectorfield $Z \in \mathscr{Z}$, the following commutator identity, correct up to constant factors, holds for all scalar-valued functions f:

$$[\nabla\!\!\!/^2, \mathcal{L}\!\!\!/_{\mathscr{Z}}^{\vec{I}}]f = \sum_{M=1}^{|\vec{I}|} \sum_{\substack{\vec{I}_1+\cdots+\vec{I}_{M+1}=\vec{I} \\ |\vec{I}_a|\geq 1 \text{ for } 1\leq a\leq M}} (g\!\!\!/^{-1})^M \underbrace{(\mathcal{L}\!\!\!/_{\mathscr{Z}}^{\vec{I}_1} g\!\!\!/) \cdots (\mathcal{L}\!\!\!/_{\mathscr{Z}}^{\vec{I}_{M-1}} g\!\!\!/)}_{\textit{absent when } M=1} (\nabla\!\!\!/ \mathcal{L}\!\!\!/_{\mathscr{Z}}^{\vec{I}_M} g\!\!\!/)(d\!\!\!/ \mathscr{Z}^{\vec{I}_{M+1}} f), \tag{9.5.5a}$$

$$[\Delta\!\!\!/, \mathscr{Z}^{\vec{I}}]f = \sum_{i_1+i_2=1} \sum_{M=1}^{|\vec{I}|} \sum_{\substack{\vec{I}_1+\cdots+\vec{I}_{M+1}=\vec{I} \\ |\vec{I}_a|\geq 1 \text{ for } 1\leq a\leq M}} (g\!\!\!/^{-1})^{M+1} \underbrace{(\mathcal{L}\!\!\!/_{\mathscr{Z}}^{\vec{I}_1} g\!\!\!/) \cdots (\mathcal{L}\!\!\!/_{\mathscr{Z}}^{\vec{I}_{M-1}} g\!\!\!/)}_{\textit{absent when } i_1=M=1} (\nabla\!\!\!/^{i_1} \mathcal{L}\!\!\!/_{\mathscr{Z}}^{\vec{I}_M} g\!\!\!/)(\nabla\!\!\!/^{i_2+1} \mathscr{Z}^{\vec{I}_{M+1}} f). \tag{9.5.5b}$$

In (9.5.3)-(9.5.5b)*, we have omitted all tensorial contractions to condense the presentation.*

Lemma 9.9. [30, Lemma 5.2; **Preliminary Lie derivative commutation identities**] *Let $\vec{I} = (\iota_1, \iota_2, \cdots, \iota_N)$ be an N^{th}-order $\mathscr{Z}$ multi-index, let f be a scalar-valued function, and let*

ξ be a type $\binom{m}{n}$ $\ell_{t,u}$-tangent tensorfield with $m+n \geq 1$. Let $i_1, i_2, \cdots, i_N$ be any permutation of $1, 2, \cdots, N$ and let $\vec{I}' = (\iota_{i_1}, \iota_{i_2}, \cdots, \iota_{i_N})$. Then, up to omitted constant factors, we have

$$\left\{ \mathscr{Z}^{\vec{I}} - \mathscr{Z}^{\vec{I}'} \right\} f = \sum_{\substack{\vec{I}_1 + \vec{I}_2 + \iota_{k_1} + \iota_{k_2} = \vec{I} \\ Z_{(\iota_{k_1})} \in \{L, \breve{X}\},\ Z_{(\iota_{k_2})} \in \{\breve{X}, Y\},\ Z_{(\iota_{k_1})} \neq Z_{(\iota_{k_2})}}} \not{\mathcal{L}}_{\mathscr{Z}}^{\vec{I}_1} ({}^{(Z_{(\iota_{k_2})})} \not{\pi}^{\#}_{Z_{(\iota_{k_1})}}) \cdot \not{d} \mathscr{Z}^{\vec{I}_2} f, \tag{9.5.6a}$$

$$\left\{ \not{\mathcal{L}}_{\mathscr{Z}}^{\vec{I}} - \not{\mathcal{L}}_{\mathscr{Z}}^{\vec{I}'} \right\} \xi = \sum_{\substack{\vec{I}_1 + \vec{I}_2 + \iota_{k_1} + \iota_{k_2} = \vec{I} \\ Z_{(\iota_{k_1})} \in \{L, \breve{X}\},\ Z_{(\iota_{k_2})} \in \{\breve{X}, Y\},\ Z_{(\iota_{k_1})} \neq Z_{(\iota_{k_2})}}} \not{\mathcal{L}}_{\not{\mathcal{L}}_{\mathscr{Z}}^{\vec{I}_1} ({}^{(Z_{(\iota_{k_2})})} \not{\pi}^{\#}_{Z_{(\iota_{k_1})}})} \not{\mathcal{L}}_{\mathscr{Z}}^{\vec{I}_2} \xi. \tag{9.5.6b}$$

In (9.5.6a)-(9.5.6b), $\vec{I}_1 + \vec{I}_2 + \iota_{k_1} + \iota_{k_2} = \vec{I}$ means that $\vec{I}_1 = (\iota_{k_3}, \iota_{k_4}, \cdots, \iota_{k_m})$, and $\vec{I}_2 = (\iota_{k_{m+1}}, \iota_{k_{m+2}}, \cdots, \iota_{k_N})$, where $k_1, k_2, \cdots, k_N$ is a permutation of $1, 2, \cdots, N$. In particular, $|\vec{I}_1| + |\vec{I}_2| = N - 2$.

We now provide the main estimates of this section.

Lemma 9.10 (**Commutator estimates**). *Assume that $1 \leq N \leq 20$ and $0 \leq M \leq \min\{2, N\}$. Let $\vec{I}$ be a multi-index belonging to the set $\mathcal{I}^{N+1;M}$ from Def. 9.1 and let $\vec{I}'$ be any permutation of $\vec{I}$. Let f be a scalar-valued function. Under the data-size and bootstrap assumptions of Sects. 8.1-8.5 and the smallness assumptions of Sect. 8.6, the following commutator estimates hold on $\mathcal{M}_{T_{(Boot)}, U_0}$ (see Sect. 6.2 regarding the vectorfield operator notation):*

$$\begin{aligned} \left| \mathscr{Z}^{\vec{I}} f - \mathscr{Z}^{\vec{I}'} f \right| \lesssim & \left| \begin{pmatrix} \mathscr{Z}_{**}^{[1,\lceil N/2 \rceil];\leq (M-1)_+} \underline{\gamma} \\ \mathscr{Z}_{*}^{[1,\lceil N/2 \rceil];\leq (M-1)_+} \gamma \end{pmatrix} \right| \left| \mathscr{Z}_{**}^{[1,N];\leq M} f \right| + \underbrace{\left| \mathscr{Z}_{**}^{[1,N];\leq (M-1)_+} f \right|}_{\text{Absent if } M=0} \\ & + \left| \mathscr{Z}_{**}^{[1,\lfloor N/2 \rfloor];\leq M} f \right| \left| \begin{pmatrix} \mathscr{Z}_{**}^{[1,N];\leq (M-1)_+} \underline{\gamma} \\ \mathscr{Z}_{*}^{[1,N];\leq M} \gamma \end{pmatrix} \right|, \end{aligned} \tag{9.5.7a}$$

$$\left| \mathscr{Z}^{\vec{I}} f - \mathscr{Z}^{\vec{I}'} f \right| \lesssim \left| \mathscr{Z}_{**}^{[1,N];\leq M} f \right| + \left| \mathscr{Z}_{**}^{[1,\lfloor N/2 \rfloor];\leq M} f \right| \left| \begin{pmatrix} \mathscr{Z}_{**}^{[1,N];\leq (M-1)_+} \underline{\gamma} \\ \mathscr{Z}_{*}^{[1,N];\leq M} \gamma \end{pmatrix} \right|, \tag{9.5.7b}$$

where $(M-1)_+ := \max\{0, M-1\}$.

Moreover, if $1 \leq N \leq 19$ and $0 \leq M \leq \min\{2, N\}$, then the following commutator estimates hold:

$$\left| [\not{\nabla}^2, \not{\mathcal{L}}_{\mathscr{Z}}^{N;M}] f \right| \lesssim \left| \mathscr{Z}_{**}^{[1,N];\leq M} f \right| + \left| \mathscr{Z}_{**}^{[1,\lceil N/2 \rceil];\leq M} f \right| \left| \begin{pmatrix} \mathscr{Z}_{**}^{[1,N+1];\leq (M-1)_+} \underline{\gamma} \\ \mathscr{Z}_{*}^{[1,N+1];\leq M} \gamma \end{pmatrix} \right|, \tag{9.5.8a}$$

$$\left| [\not{\Delta}, \mathscr{Z}^{N;M}] f \right| \lesssim \left| \mathscr{Z}_{**}^{[1,N+1];\leq M} f \right| + \left| \mathscr{Z}_{**}^{[1,\lceil N/2 \rceil];\leq M} f \right| \left| \begin{pmatrix} \mathscr{Z}_{**}^{[1,N+1];\leq (M-1)_+} \underline{\gamma} \\ \mathscr{Z}_{*}^{[1,N+1];\leq M} \gamma \end{pmatrix} \right|. \tag{9.5.8b}$$

Finally, if ξ is an $\ell_{t,u}$-tangent one-form or a type $\binom{0}{2}$ $\ell_{t,u}$-tangent tensorfield, $1 \leq N \leq 19$, $0 \leq M \leq \min\{2, N\}$, *and* $\vec{I} \in \mathcal{I}^{N+1;M}$, *then the following commutator estimates hold:*

$$\left| \not{\mathcal{L}}_{\mathscr{Z}}^{\vec{I}} \xi - \not{\mathcal{L}}_{\mathscr{Z}}^{\vec{I}'} \xi \right| \lesssim \left| \not{\mathcal{L}}_{\mathscr{Z}_*}^{[1,N];\leq M} \xi \right| + \left| \not{\mathcal{L}}_{\mathscr{Z}}^{\leq \lceil N/2 \rceil;\leq M} \xi \right| \left| \begin{pmatrix} \mathscr{Z}_{**}^{[1,N+1];\leq (M-1)_+} \underline{\gamma} \\ \mathscr{Z}_*^{[1,N+1];\leq M} \gamma \end{pmatrix} \right|, \tag{9.5.9a}$$

$$\left| [\not{\nabla}, \not{\mathcal{L}}_{\mathscr{Z}}^{N;M}] \xi \right| \lesssim \left| \not{\mathcal{L}}_{\mathscr{Z}_*}^{[1,N-1];\leq M} \xi \right| + \left| \not{\mathcal{L}}_{\mathscr{Z}}^{\leq \lceil N/2 \rceil;\leq M} \xi \right| \left| \begin{pmatrix} \mathscr{Z}_{**}^{[1,N+1];\leq (M-1)_+} \underline{\gamma} \\ \mathscr{Z}_*^{[1,N+1];\leq M} \gamma \end{pmatrix} \right|, \tag{9.5.9b}$$

$$\left| [\not{\mathrm{div}}, \not{\mathcal{L}}_{\mathscr{Z}}^{N;\leq M}] \xi \right| \lesssim \left| \not{\mathcal{L}}_{\mathscr{Z}_*}^{[1,N];\leq M} \xi \right| + \left| \not{\mathcal{L}}_{\mathscr{Z}}^{\leq \lceil N/2 \rceil;\leq M} \xi \right| \left| \begin{pmatrix} \mathscr{Z}_{**}^{[1,N+1];\leq (M-1)_+} \underline{\gamma} \\ \mathscr{Z}_*^{[1,N+1];\leq M} \gamma \end{pmatrix} \right|. \tag{9.5.9c}$$

Proof. See Sect. 9.2 for some comments on the analysis.

Proof of (9.5.7a): We consider the commutation formula (9.5.6a). We will bound the products $\not{\mathcal{L}}_{\mathscr{Z}}^{\vec{I}_1} {}^{(Z_{(\iota_{k_2})})}\not{\pi}^{\#}_{Z_{(\iota_{k_1})}} \cdot \not{d}\mathscr{Z}^{\vec{I}_2} f$ on RHS (9.5.6a) on a case by case basis. Let M' be the number of factors of $\breve{X}$ in $\mathscr{Z}^{\vec{I}_2}$. Note that $M' \leq M$ in view of the summation constraint $\vec{I}_1 + \vec{I}_2 + \iota_{k_1} + \iota_{k_2} = \vec{I}$ on RHS (9.5.6a).

Case i): $M' = M$ and $|\vec{I}_2| \in [\lfloor N/2 \rfloor, N-1]$. Clearly we have $\left| \not{d}\mathscr{Z}^{\vec{I}_2} f \right| \lesssim \left| \mathscr{Z}_{**}^{[1,N];\leq M} f \right|$. To bound the remaining factor $\not{\mathcal{L}}_{\mathscr{Z}}^{\vec{I}_1} {}^{(Z_{(\iota_{k_2})})}\not{\pi}^{\#}_{Z_{(\iota_{k_1})}}$, where $|\vec{I}_1| \in [0, \lfloor (N-1)/2 \rfloor]$, we note that since $M' = M$, it must be that $\not{\mathcal{L}}_{\mathscr{Z}}^{\vec{I}_1}$ comprises only $\mathcal{P}_u^t$-tangent vectorfield factors and that $(Z_{(\iota_{k_1})}, Z_{(\iota_{k_2})}) = (L, Y)$. Hence, with the help of (9.4.3a), we see that the remaining factor under consideration is bounded in magnitude by $\lesssim \left| \not{\mathcal{L}}_{\mathscr{P}}^{\leq \lfloor (N-1)/2 \rfloor} {}^{(Y)}\not{\pi}^{\#}_L \right| \lesssim \left| \begin{pmatrix} \mathscr{Z}_{**}^{[1,\lceil N/2 \rceil];0} \underline{\gamma} \\ \mathscr{Z}_*^{[1,\lceil N/2 \rceil];0} \gamma \end{pmatrix} \right|$. In particular, the product $\not{\mathcal{L}}_{\mathscr{Z}}^{\vec{I}_1} {}^{(Z_{(\iota_{k_2})})}\not{\pi}^{\#}_{Z_{(\iota_{k_1})}} \cdot \not{d}\mathscr{Z}^{\vec{I}_2} f$ under consideration is bounded in magnitude by $\lesssim$ the first product on RHS (9.5.7a).

Case ii): $M' = M$ and $|\vec{I}_2| \in [0, \lfloor N/2 \rfloor - 1]$. Clearly we have $\left| \not{d}\mathscr{Z}^{\vec{I}_2} f \right| \lesssim \left| \mathscr{Z}_{**}^{[1,\lfloor N/2 \rfloor];\leq M} f \right|$. To bound the remaining factor $\not{\mathcal{L}}_{\mathscr{Z}}^{\vec{I}_1} {}^{(Z_{(\iota_{k_2})})}\not{\pi}^{\#}_{Z_{(\iota_{k_1})}}$, where $|\vec{I}_1| \in [\lceil N/2 \rceil, N-1]$, we note that since $M' = M$, it must be that $\not{\mathcal{L}}_{\mathscr{Z}}^{\vec{I}_1}$ comprises only $\mathcal{P}_u^t$-tangent vectorfield factors and $(Z_{(\iota_{k_1})}, Z_{(\iota_{k_2})}) = (L, Y)$. Hence, with the help of (9.4.3a), we see that the remaining factor under consideration is bounded in magnitude by $\lesssim \left| \not{\mathcal{L}}_{\mathscr{P}}^{\leq N-1} {}^{(Y)}\not{\pi}^{\#}_L \right| \lesssim \left| \begin{pmatrix} \mathscr{Z}_{**}^{[1,N];0} \underline{\gamma} \\ \mathscr{Z}_*^{[1,N+1];0} \gamma \end{pmatrix} \right|$. In particular, the product $\not{\mathcal{L}}_{\mathscr{Z}}^{\vec{I}_1} {}^{(Z_{(\iota_{k_2})})}\not{\pi}^{\#}_{Z_{(\iota_{k_1})}} \cdot \not{d}\mathscr{Z}^{\vec{I}_2} f$ under consideration is bounded in magnitude by $\lesssim$ the last product on RHS (9.5.7a).

We note that we have now proved inequality (9.5.7a) in the case $M = 0$, and it holds without the second term on the RHS (as is indicated in (9.5.7a)).

Case iii): $1 \leq M \leq 2$, $M' \leq M - 1$ and $|\vec{I}_2| \in [\lfloor N/2 \rfloor, N-1]$. Clearly we have $\left| \not{d}\mathscr{Z}^{\vec{I}_2} f \right| \lesssim \left| \mathscr{Z}_{**}^{[1,N];\leq M-1} f \right|$. Since $|\vec{I}_1| \in [0, \lfloor (N-1)/2 \rfloor]$, we may bound the remaining factor

$\not{\mathcal{L}}_{\mathscr{Z}}^{\vec{I}_1 (Z_{(\iota_{k_2})})} \not{\pi}_{Z_{(\iota_{k_1})}}^{\#}$ in the norm $\| \cdot \|_{L^\infty(\Sigma_t^u)}$ by $\lesssim 1$ with the help of the pointwise estimates of Lemma 9.7 and the bootstrap assumptions. We note that in view of the summation constraint $\vec{I}_1 + \vec{I}_2 + \iota_{k_1} + \iota_{k_2} = \vec{I}$ and the assumption $M \leq 2$, we do not encounter terms of the form $\not{\mathcal{L}}_{\mathscr{Z}}^{|\vec{I}_1|;2(L)} \not{\pi}_{\breve{X}}^{\#}$ or $\not{\mathcal{L}}_{\mathscr{Z}}^{|\vec{I}_1|;2(Y)} \not{\pi}_{\breve{X}}^{\#}$, which would contain factors involving derivatives of $\breve{X}\breve{X}\upmu$ or $\breve{X}\breve{X}\breve{X}\vec{\Psi}$, which we are not able to control in L^∞ based on the current bootstrap assumptions.[107] In total, we find that the product $\not{\mathcal{L}}_{\mathscr{Z}}^{\vec{I}_1 (Z_{(\iota_{k_2})})} \not{\pi}_{Z_{(\iota_{k_1})}}^{\#} \cdot \not{d} \mathscr{Z}^{\vec{I}_2} f$ under consideration is bounded in magnitude by $\lesssim$ the second term on RHS (9.5.7a).

Case iv): $1 \leq M \leq 2$, $M' \leq M - 1$, $|\vec{I}_2| \in [0, \lfloor N/2 \rfloor - 1]$, and $1 \leq N \leq 4$. Clearly we have $\left| \not{d} \mathscr{Z}^{\vec{I}_2} f \right| \lesssim \left| \mathscr{Z}_{**}^{[1,\lfloor N/2 \rfloor];\leq M-1} f \right|$. Since $1 \leq N \leq 4$, we may bound the remaining factor $\not{\mathcal{L}}_{\mathscr{Z}}^{\vec{I}_1 (Z_{(\iota_{k_2})})} \not{\pi}_{Z_{(\iota_{k_1})}}^{\#}$ in the norm $\| \cdot \|_{L^\infty(\Sigma_t^u)}$ by $\lesssim 1$, as in Case iii). It follows that the product $\not{\mathcal{L}}_{\mathscr{Z}}^{\vec{I}_1 (Z_{(\iota_{k_2})})} \not{\pi}_{Z_{(\iota_{k_1})}}^{\#} \cdot \not{d} \mathscr{Z}^{\vec{I}_2} f$ under consideration is bounded in magnitude by $\lesssim$ the second term on RHS (9.5.7a).

Case v): $1 \leq M \leq 2$, $M' \leq M-1$, $|\vec{I}_2| \in [0, \lfloor N/2 \rfloor - 1]$, and $5 \leq N \leq 20$. Clearly we have $\left| \not{d} \mathscr{Z}^{\vec{I}_2} f \right| \lesssim \left| \mathscr{Z}_{**}^{[1,\lfloor N/2 \rfloor];\leq M-1} f \right|$. We now bound the remaining factor $\not{\mathcal{L}}_{\mathscr{Z}}^{\vec{I}_1 (Z_{(\iota_{k_2})})} \not{\pi}_{Z_{(\iota_{k_1})}}^{\#}$, starting with the sub-case in which either $Z_{(\iota_{k_1})} = \breve{X}$ or $Z_{(\iota_{k_2})} = \breve{X}$. In view of the summation constraint $\vec{I}_1 + \vec{I}_2 + \iota_{k_1} + \iota_{k_2} = \vec{I}$, we see that it suffices to bound $\left| \not{\mathcal{L}}_{\mathscr{Z}}^{|\vec{I}_1|;\leq M-1(Y)} \not{\pi}_{\breve{X}}^{\#} \right|$, $\left| \not{\mathcal{L}}_{\mathscr{Z}}^{|\vec{I}_1|;\leq M-1(\breve{X})} \not{\pi}_{L}^{\#} \right|$. Since $N \geq 5$, and $|\vec{I}_1| \in [\lceil N/2 \rceil, N-1]$, we have $|\vec{I}_1| \geq 3$. Thus, since $M \leq 2$, at least 2 vectorfield factors in the operator $\not{\mathcal{L}}_{\mathscr{Z}}^{|\vec{I}_1|;\leq M-1}$ must be $\mathcal{P}_u^t$-tangent. That is, $\not{\mathcal{L}}_{\mathscr{Z}}^{|\vec{I}_1|;\leq M-1} = \not{\mathcal{L}}_{\mathscr{Z}_*}^{|\vec{I}_1|;\leq M-1}$. We may therefore use (9.4.4b) with $M - 1$ in the role of M to deduce that $\left| \not{\mathcal{L}}_{\mathscr{Z}_*}^{|\vec{I}_1|;\leq M-1(Y)} \not{\pi}_{\breve{X}}^{\#} \right|, \left| \not{\mathcal{L}}_{\mathscr{Z}_*}^{|\vec{I}_1|;\leq M-1(\breve{X})} \not{\pi}_{L}^{\#} \right| \lesssim \left| \begin{pmatrix} \mathscr{Z}_{**}^{[1,N];\leq M-1} \underline{\upgamma} \\ \mathscr{Z}_{*}^{[1,N];\leq M} \upgamma \end{pmatrix} \right|$. It follows that the product $\not{\mathcal{L}}_{\mathscr{Z}}^{\vec{I}_1 (Z_{(\iota_{k_2})})} \not{\pi}_{Z_{(\iota_{k_1})}}^{\#} \cdot \not{d} \mathscr{Z}^{\vec{I}_2} f$ under consideration is bounded in magnitude by $\lesssim$ the last product on RHS (9.5.7a) as desired. Finally, we address the remaining sub-case in which $(Z_{(\iota_{k_1})}, Z_{(\iota_{k_2})}) = (L, Y)$. Thus, using (9.4.3a), we see that the factor $\not{\mathcal{L}}_{\mathscr{Z}}^{\vec{I}_1 (Z_{(\iota_{k_2})})} \not{\pi}_{Z_{(\iota_{k_1})}}^{\#}$ is bounded in magnitude by $\lesssim \left| \not{\mathcal{L}}_{\mathscr{Z}_*}^{\leq N-1;\leq M(Y)} \not{\pi}_{L}^{\#} \right| \lesssim \left| \begin{pmatrix} \mathscr{Z}_{**}^{[1,N-1];\leq M-1} \underline{\upgamma} \\ \mathscr{Z}_{*}^{[1,N];\leq M} \upgamma \end{pmatrix} \right|$. It follows that the product $\not{\mathcal{L}}_{\mathscr{Z}}^{\vec{I}_1 (Z_{(\iota_{k_2})})} \not{\pi}_{Z_{(\iota_{k_1})}}^{\#} \cdot \not{d} \mathscr{Z}^{\vec{I}_2} f$ under consideration is bounded in magnitude by $\lesssim$ the last product on RHS (9.5.7a) as desired. We have thus proved (9.5.7a).

Proof of (9.5.7b): The estimate (9.5.7b) is a simplified version of (9.5.7a) that follows as a simple consequence of (9.5.7a) and the bootstrap assumptions.

Proof of (9.5.8a) **and** (9.5.8b): The proofs of these estimates are similar to the proof of (9.5.7a) and are based on the commutation identities (9.5.5a)-(9.5.5b), the estimates (9.4.1a) and (9.4.1b), and Lemma 9.3; we omit the details.

[107] See, however, Sect. 10.

Proof of (9.5.9a)**,** (9.5.9b) **and** (9.5.9c): The proofs of these estimates are similar to the proof of (9.5.7a) and are based on the commutation identities (9.5.3)-(9.5.4) and (9.5.6b), the estimates (9.4.1a) and (9.4.1b), and Lemma 9.4. We omit the details, noting only that the right-hand side of (9.5.9a) involves one more derivative of $\underline{\upgamma}$ and $\upgamma$ compared to the estimates (9.5.7a)-(9.5.7b); the reason is that we use the estimate (9.1.7) when bounding the terms on RHS (9.5.6b), which leads to the presence of one additional derivative on ${}^{(Z_{(\iota_{k_2})})}\not{\pi}^{\#}_{Z_{(\iota_{k_1})}}$. □

Corollary 9.11. *Assume that* $1 \leq N \leq 20$ *and* $0 \leq M \leq \min\{2, N\}$*. Let* $\Psi \in \{\uprho, v^1, v^2\}$*. Under the data-size and bootstrap assumptions of Sects. 8.1-8.5 and the smallness assumptions of Sect. 8.6, the following estimates hold on* $\mathcal{M}_{T_{(Boot)},U_0}$ *(see Sect. 6.2 regarding the vectorfield operator notation):*

$$\left|\mathscr{Z}^{N-1;M}\not{\Delta}\Psi\right| \lesssim \left|\mathscr{Z}_{**}^{[1,N+1];\leq M}\Psi\right| + \left|\begin{pmatrix} \mathscr{Z}_{**}^{[1,N];\leq (M-1)_+}\underline{\upgamma} \\ \mathscr{Z}_{*}^{[1,N];\leq M}\upgamma \end{pmatrix}\right|. \tag{9.5.10}$$

Proof. See Sect. 9.2 for some comments on the analysis. We decompose $\not{\Delta}\Psi = \not{\Delta}\mathscr{Z}^{N-1;M}\Psi + [\mathscr{Z}^{N-1;M}, \not{\Delta}]\Psi$. The first term in the decomposition is bounded in magnitude by $\lesssim$ the first term on RHS (9.5.10). To obtain $\left|[\mathscr{Z}^{N-1;M}, \not{\Delta}]\Psi \lesssim \text{RHS (9.5.10)}\right|$, we use the commutator estimate (9.5.8b) with $f = \Psi$ and $N-1$ in the role of N and the bootstrap assumptions. □

9.6. **Transport inequalities and improvements of the auxiliary bootstrap assumptions.** In the next proposition, we use the previous estimates to derive transport inequalities for the eikonal function quantities and improvements of the auxiliary bootstrap assumptions. The transport inequalities form the starting point for our derivation of L^2 estimates for the below-top-order derivatives of the eikonal function quantities (see Sect. 15.4). In proving the proposition, we must propagate the smallness of the $\mathring{\upepsilon}$-sized quantities even though some terms in their evolution equations involve the relatively large $\mathring{\updelta}$-sized quantities.

Proposition 9.12 (**Transport inequalities and improvements of the auxiliary bootstrap assumptions**)**.** *Under the data-size and bootstrap assumptions of Sects. 8.1-8.5 and the smallness assumptions of Sect. 8.6, the following estimates hold on* $\mathcal{M}_{T_{(Boot)},U_0}$ *(see Sect. 6.2 regarding the vectorfield operator notation).*

Transport inequalities for the eikonal function quantities*.*

• ***Transport inequalities for*** $\upmu$*. The following pointwise estimate holds:*

$$|L\upmu| \lesssim \left|\mathscr{Z}^{\leq 1}\vec{\Psi}\right|. \tag{9.6.1a}$$

Moreover, for $1 \leq N \leq 20$ *and* $0 \leq M \leq \min\{1, N-1\}$ *the following pointwise estimates hold:*

$$\left|L\mathscr{Z}_{*}^{N;M}\upmu\right|, \left|\mathscr{Z}_{*}^{N;M}L\upmu\right| \lesssim \left|\mathscr{Z}_{*}^{[1,N+1];\leq M+1}\vec{\Psi}\right| + \left|\begin{pmatrix} \mathscr{Z}_{**}^{[1,N];\leq M}\upgamma \\ \mathscr{Z}_{*}^{[1,N];\leq M}\underline{\upgamma} \end{pmatrix}\right|. \tag{9.6.1b}$$

• ***Transport inequalities for*** $L^i_{(Small)}$ *and* $\mathrm{tr}_{\not g}\chi$. *For* $0 \leq N \leq 20$ *and* $0 \leq M \leq \min\{2, N\}$, *the following pointwise estimates hold:*

$$\left|\begin{pmatrix} L\mathscr{Z}^{N;M} L^i_{(Small)} \\ L\mathscr{Z}^{N-1;M}\mathrm{tr}_{\not g}\chi \end{pmatrix}\right|, \left|\begin{pmatrix} \mathscr{Z}^{N;M} L L^i_{(Small)} \\ \mathscr{Z}^{N-1;M} L\mathrm{tr}_{\not g}\chi \end{pmatrix}\right| \lesssim \left|\mathscr{Z}_*^{[1,N+1];\leq M}\vec{\Psi}\right| + \underbrace{\left|\begin{pmatrix} \mathscr{Z}_{**}^{[1,N];\leq (M-1)_+}\underline{\gamma} \\ \mathscr{Z}_*^{[1,N];\leq M}\gamma \end{pmatrix}\right|}_{\text{Absent when } N=0}. \tag{9.6.2}$$

<u>L^∞ ***estimates for*** $\vec{\Psi}$ ***and the eikonal function quantities***.</u>

•L^∞ ***estimates for*** $\vec{\Psi}$. *The following estimates hold for* $M = 1, 2$:

$$\left\|\breve{X}^M v^2\right\|_{L^\infty(\Sigma_t^u)} \leq C\varepsilon, \tag{9.6.3a}$$

$$\left\|\breve{X}^M \rho\right\|_{L^\infty(\Sigma_t^u)} \leq \left\|\breve{X}^M \rho\right\|_{L^\infty(\Sigma_0^u)} + C\varepsilon, \tag{9.6.3b}$$

$$\left\|\breve{X}^M v^1\right\|_{L^\infty(\Sigma_t^u)} \leq \left\|\breve{X}^M v^1\right\|_{L^\infty(\Sigma_0^u)} + C\varepsilon, \tag{9.6.3c}$$

$$\left\|\mathscr{Z}_*^{\leq 12;\leq 2}\vec{\Psi}\right\|_{L^\infty(\Sigma_t^u)} \leq C\varepsilon. \tag{9.6.3d}$$

Moreover,

$$\left\|\breve{X}(\rho - v^1)\right\|_{L^\infty(\Sigma_t^u)} \leq C\varepsilon. \tag{9.6.4}$$

•L^∞ ***estimates for*** μ. *The following estimates hold for* $M = 0, 1$:

$$\left\|\breve{X}^M \mu\right\|_{L^\infty(\Sigma_t^u)} \leq \left\|\breve{X}^M \mu\right\|_{L^\infty(\Sigma_0^u)} + \mathring{\delta}_*^{-1}\left\|\breve{X}^M\left\{\vec{G}_{LL}\diamond\breve{X}\vec{\Psi}\right\}\right\|_{L^\infty(\Sigma_0^u)} + C\varepsilon, \tag{9.6.5a}$$

$$\left\|L\breve{X}^M \mu\right\|_{L^\infty(\Sigma_t^u)} = \frac{1}{2}\left\|\breve{X}^M\left\{\vec{G}_{LL}\diamond\breve{X}\vec{\Psi}\right\}\right\|_{L^\infty(\Sigma_0^u)} + \mathcal{O}(\varepsilon), \tag{9.6.5b}$$

$$\left\|\mathscr{Z}_{**}^{[1,11];\leq 1}\mu\right\|_{L^\infty(\Sigma_t^u)} \leq C\varepsilon. \tag{9.6.5c}$$

Moreover, we have

$$\|\mu - 1\|_{L^\infty\left(\mathcal{P}_0^{T_{(Boot)}}\right)} \leq C\varepsilon. \tag{9.6.6}$$

•L^∞ ***estimates for*** $L^i_{(Small)}$ ***and*** χ. *The following estimates hold for* $M = 1, 2$:

$$\left\|\mathscr{Z}_*^{\leq 11;\leq 2} L^i_{(Small)}\right\|_{L^\infty(\Sigma_t^u)} \leq C\varepsilon, \tag{9.6.7a}$$

$$\left\|\breve{X}^M L^i_{(Small)}\right\|_{L^\infty(\Sigma_t^u)} \leq \left\|\breve{X}^M L^i_{(Small)}\right\|_{L^\infty(\Sigma_0^u)} + C\varepsilon, \tag{9.6.7b}$$

$$\left\|\not{\mathcal{L}}_{\mathscr{Z}}^{\leq 10;\leq 2}\chi\right\|_{L^\infty(\Sigma_t^u)}, \left\|\not{\mathcal{L}}_{\mathscr{Z}}^{\leq 10;\leq 2}\chi^\#\right\|_{L^\infty(\Sigma_t^u)}, \left\|\mathscr{Z}^{\leq 10;\leq 2}\mathrm{tr}_{\not g}\chi\right\|_{L^\infty(\Sigma_t^u)} \leq C\varepsilon. \tag{9.6.8}$$

<u>L^∞ ***estimates for*** ω.</u> *The following estimates hold:*

$$\left\|\mathscr{Z}^{\leq 12;\leq 2}\omega\right\|_{L^\infty(\Sigma_t^u)} \leq C\varepsilon. \tag{9.6.9}$$

Proof. See Sect. 9.2 for some comments on the analysis. Throughout the proof, we use the phrase "conditions on the data" to refer to the assumptions stated in Sects. 8.1 and 8.2. Also, we often silently use inequality (8.6.1).

Proof of (9.6.9): The estimate (9.6.9) for $\left\|\mathscr{P}^{\leq 12}\upomega\right\|_{L^\infty(\Sigma_t^u)}$ is a direct consequence of the bootstrap assumptions. To prove (9.6.9) for $\left\|\mathscr{Z}^{\leq 12;1}\upomega\right\|_{L^\infty(\Sigma_t^u)}$, we first note that by (3.7.15), equation (3.3.11c) is equivalent to $\breve{X}\upomega = -\upmu L\upomega$. Applying $\mathscr{P}^{\leq 11}$ to this equation and using the bootstrap assumptions, we deduce that $\left\|\mathscr{P}^{\leq 11}\breve{X}\upomega\right\|_{L^\infty(\Sigma_t^u)} \lesssim \varepsilon$. Next, for $2 \leq K \leq 12$, we repeatedly use the commutator estimate (9.5.7b) and the bootstrap assumptions to deduce

$$\left|\mathscr{Z}^{\leq K;1}\upomega\right| \lesssim \left|\mathscr{P}^{[1,K-1]}\breve{X}\upomega\right| + \left|\mathscr{P}^{\leq K-1}\upomega\right|. \tag{9.6.10}$$

We have already shown that the first term on RHS (9.6.10) is $\lesssim \varepsilon$, while the bootstrap assumptions imply that the second term is $\lesssim \varepsilon$. In total, we have shown that we can permute the vectorfield factors in $\mathscr{P}^{\leq 11}\breve{X}\upomega$ up to $\mathcal{O}(\varepsilon)$ errors, which yields the desired bound $\left\|\mathscr{Z}^{\leq 12;1}\upomega\right\|_{L^\infty(\Sigma_t^u)} \lesssim \varepsilon$. To prove (9.6.9) for $\left\|\mathscr{Z}^{\leq 12;2}\upomega\right\|_{L^\infty(\Sigma_t^u)}$, we first apply $\mathscr{P}^{\leq 10}\breve{X}$ to the equation $\breve{X}\upomega = -\upmu L\upomega$ and use the already proven bound $\left\|\mathscr{Z}^{\leq 12;\leq 1}\upomega\right\|_{L^\infty(\Sigma_t^u)} \lesssim \varepsilon$, and the bootstrap assumptions to deduce $\left\|\mathscr{P}^{\leq 10}\breve{X}\breve{X}\upomega\right\|_{L^\infty(\Sigma_t^u)} \lesssim \varepsilon$. Using this bound, the estimate $\left\|\mathscr{Z}^{\leq 12;\leq 1}\upomega\right\|_{L^\infty(\Sigma_t^u)} \lesssim \varepsilon$, the commutator estimate (9.5.7b) with $M = 2$, and the bootstrap assumptions, we may use an argument similar to the one given just below (9.6.10) in order to permute the vectorfield factors in $\mathscr{P}^{\leq 10}\breve{X}\breve{X}\upomega$ up to $\mathcal{O}(\varepsilon)$ errors. In total, we have shown that $\left\|\mathscr{Z}^{\leq 12;2}\upomega\right\|_{L^\infty(\Sigma_t^u)} \lesssim \varepsilon$, which completes the proof of (9.6.9).

Proof of (9.6.1a) **and** (9.6.1b): The estimate (9.6.1a) is a simple consequence of the evolution equation $L\upmu = \mathrm{f}(\underline{\upgamma})P\vec{\Psi} + \mathrm{f}(\upgamma)\breve{X}\vec{\Psi}$ (see equation (3.14.1) and Lemma 3.19) and the bootstrap assumptions.

We now prove (9.6.1b). We show only how to obtain the estimates for $\left|L\mathscr{Z}_*^{N;M}\upmu\right|$ since the estimates for $\left|\mathscr{Z}_*^{N;M}L\upmu\right|$ are simpler because they do not involve commutations. To proceed, for $1 \leq N \leq 20$ and $\min\{1, N-1\}$, we commute the evolution equation from the previous paragraph with $\mathscr{Z}_*^{N;M}$ to deduce the schematic identity

$$L\mathscr{Z}_*^{N;M}\upmu = [L, \mathscr{Z}_*^{N;M}]\upmu + \mathscr{Z}_*^{N;M}\left\{\mathrm{f}(\upgamma)\breve{X}\Psi + \mathrm{f}(\underline{\upgamma})P\vec{\Psi}\right\}. \tag{9.6.11}$$

To bound the magnitude of the second term on RHS (9.6.11) by $\lesssim$ RHS (9.6.1b), we use the bootstrap assumptions. To derive $\left|[L, \mathscr{Z}_*^{N;M}]\upmu\right| \lesssim$ RHS (9.6.1b), we use the commutator estimate (9.5.7b) with $f = \upmu$ and the bootstrap assumptions. We have thus proved (9.6.1b).

Proof of (9.6.2) **for** $L\mathscr{Z}^{N;M}L^i_{(Small)}$ **and** $\mathscr{Z}^{N;M}LL^i_{(Small)}$: We first write the evolution equation (3.14.2) in the schematic form $LL^i_{(Small)} = \mathrm{f}(\upgamma, \not{g}^{-1}, \not{d}\vec{x})P\vec{\Psi}$. For $0 \leq N \leq 20$ and $0 \leq M \leq \min\{2, N\}$, we commute this evolution equation with $\mathscr{Z}^{N;M}$ to obtain

$$L\mathscr{Z}^{N;M}L^i_{(Small)} = [L, \mathscr{Z}^{N;M}]L^i_{(Small)} + \mathscr{Z}^{N;M}\left\{\mathrm{f}(\upgamma, \not{g}^{-1}, \not{d}\vec{x})P\vec{\Psi}\right\}. \tag{9.6.12}$$

To bound the magnitude of the second term on RHS (9.6.12) by $\lesssim$ RHS (9.6.2), we use the estimates (9.3.3b)-(9.3.3c) and (9.4.1a)-(9.4.1b) and the bootstrap assumptions. To deduce $\left|[L, \mathscr{Z}^{N;M}]L^i_{(Small)}\right| \lesssim$ RHS (9.6.2) we use the commutator estimate (9.5.7b) with $f = L^i_{(Small)}$ and the bootstrap assumptions.

Proof of (9.6.2) **for** $L\mathscr{Z}^{N-1;M}\mathrm{tr}_{\not{g}}\chi$ **and** $\mathscr{Z}^{N-1;M}L\mathrm{tr}_{\not{g}}\chi$: We first apply L to equation (3.17.1b) and use the schematic identity $\not{\mathcal{L}}_L\not{g}^{-1} = (\not{g}^{-1})^{-2}\chi = \mathrm{f}(\gamma, \not{g}^{-1}, \not{d}\vec{x})P\gamma$ (see (3.13.4), (3.16.2c), and Lemma 3.19) to deduce that $L\mathrm{tr}_{\not{g}}\chi = \mathrm{f}(\gamma, \not{g}^{-1}, \not{d}\vec{x})PL\gamma + l.o.t.$, where $l.o.t. := \left\{\mathrm{f}(\mathscr{P}^{\leq 1}\gamma, \not{\mathcal{L}}^{\leq 1}_{\mathscr{P}}\not{g}^{-1}, \not{d}\mathscr{P}^{\leq 1}\vec{x})\right\} P\gamma$. Applying $\mathscr{Z}^{N-1;M}$ to this identity and using Lemmas 9.5 and 9.6 and the bootstrap assumptions, we find that $\left|\mathscr{Z}^{N-1;M}L\mathrm{tr}_{\not{g}}\chi\right| \lesssim \sum_{i=1}^2 \left|\mathscr{Z}^{N+1;M}_* L^i_{(Small)}\right| +$ RHS (9.6.2), where the operator $\mathscr{Z}^{N+1;M}_*$ acting on $L^i_{(Small)}$ contains a factor of L. By arguing as in our proof of the bound for the commutator term on RHS (9.6.12), we may commute the factor of L to the front (so that L acts last), thereby obtaining that $\sum_{i=1}^2 \left|\mathscr{Z}^{N+1;M}_* L^i_{(Small)}\right| \lesssim \left|L\mathscr{Z}^{N;M}L^i_{(Small)}\right| +$ RHS (9.6.2). Moreover, we already showed in the previous paragraph that $\left|L\mathscr{Z}^{N;M}L^i_{(Small)}\right| \lesssim$ RHS (9.6.2), which completes our proof of the estimate (9.6.2) for $\left|\mathscr{Z}^{N-1;M}L\mathrm{tr}_{\not{g}}\chi\right|$. To obtain the same estimate for $\left|L\mathscr{Z}^{N-1;M}\mathrm{tr}_{\not{g}}\chi\right|$, we use the commutator estimate (9.5.7b) with $f = \mathrm{tr}_{\not{g}}\chi$, (9.4.1c), and the bootstrap assumptions to deduce that $\left|L\mathscr{Z}^{N-1;M}\mathrm{tr}_{\not{g}}\chi\right| \lesssim \left|\mathscr{Z}^{N-1;M}L\mathrm{tr}_{\not{g}}\chi\right| + \left|\begin{pmatrix} \mathscr{Z}^{[1,N];\leq(M-1)_+}_{**}\underline{\gamma} \\ \mathscr{Z}^{[1,N];\leq M}_{*}\gamma \end{pmatrix}\right|$. The desired bound (9.6.2) for $\left|L\mathscr{Z}^{N-1;M}\mathrm{tr}_{\not{g}}\chi\right|$ now follows from this estimate and the one we established just above for $\left|\mathscr{Z}^{N-1;M}L\mathrm{tr}_{\not{g}}\chi\right|$.

Proof of an intermediate estimate: As an intermediate step, we now show that

$$\left|\begin{pmatrix} \mathscr{Z}^{[1,11];0}_{**}\mu \\ \mathscr{Z}^{\leq 11;\leq 1}_{*}(L^1_{(Small)}, L^2_{(Small)}) \end{pmatrix}\right|(t,u,\vartheta) \lesssim \varepsilon. \tag{9.6.13}$$

We first recall that $L = \frac{\partial}{\partial t}$. Hence, we can use (9.6.1b)-(9.6.2) and the bootstrap assumptions and integrate along the integral curves of L as in (9.3.5) to deduce

$$\begin{aligned} &\left|\begin{pmatrix} \mathscr{Z}^{[1,11];\leq 0}_{**}\mu \\ \mathscr{Z}^{\leq 11;\leq 1}_{*}(L^1_{(Small)}, L^2_{(Small)}) \end{pmatrix}\right|(t,u,\vartheta) \\ &\leq C\left|\begin{pmatrix} \mathscr{Z}^{[1,11];0}_{**}\mu \\ \mathscr{Z}^{\leq 11;\leq 1}_{*}(L^1_{(Small)}, L^2_{(Small)}) \end{pmatrix}\right|(0,u,\vartheta) \\ &\quad + C\int_{s=0}^{t}\left|\begin{pmatrix} \mathscr{Z}^{[1,11];0}_{**}\mu \\ \mathscr{Z}^{\leq 11;\leq 1}_{*}(L^1_{(Small)}, L^2_{(Small)}) \end{pmatrix}\right|(s,u,\vartheta)\,ds + C\varepsilon, \end{aligned} \tag{9.6.14}$$

where the $C\varepsilon$ term on RHS (9.6.14) comes from the terms $\left|\mathscr{Z}^{[1,12];\leq 1}_{*}\vec{\Psi}\right|$ (which are $\lesssim \varepsilon$ in view of the bootstrap assumptions) on RHS (9.6.1b) and RHS (9.6.2). The conditions on the data imply that the first term on RHS (9.6.14) is $\leq C\varepsilon$. Hence, from Gronwall's inequality, we conclude $\left|\begin{pmatrix} \mathscr{Z}_{**}^{[1,11];0}\upmu \\ \mathscr{Z}_{*}^{\leq 11;\leq 1}(L_{(Small)}^1, L_{(Small)}^2) \end{pmatrix}\right|(t,u,\vartheta) \lesssim \varepsilon \exp(C\mathring{\updelta}_*^{-1}) \lesssim \varepsilon$, which yields (9.6.13).

Proof of (9.6.3d): Using (3.18.1a), (3.7.15), Cor. 3.18, Lemma 3.19, the identity (9.1.1a), and the schematic relation $L\upmu = \mathrm{f}(\underline{\gamma})P\Psi + \mathrm{f}(\gamma)\breve{X}\Psi$ (which follows from Lemmas 3.14.1 and 3.19), we write the wave equations (3.3.11a)-(3.3.11b) verified by $\Psi \in \{\uprho, v^1, v^2\}$ in the following schematic form:

$$L\breve{X}\Psi = \mathrm{f}(\underline{\gamma}, \not{g}^{-1}, \not{d}\vec{x}, \mathscr{Z}^{\leq 1}\vec{\Psi})\mathscr{P}^{\leq 2}\Psi + \mathrm{f}(\underline{\gamma}, \not{g}^{-1}, \not{d}\vec{x}, \mathscr{Z}^{\leq 1}\vec{\Psi})P\gamma \\ + \mathrm{f}(\underline{\gamma}, \mathscr{Z}^{\leq 1}\vec{\Psi})\mathscr{P}^{\leq 1}\upomega. \tag{9.6.15}$$

We now show that

$$\left|L\mathscr{Z}_*^{[1,11];1}\breve{X}\Psi\right| \lesssim \left|\mathscr{Z}_*^{[1,11];1}\breve{X}\Psi\right| + \varepsilon. \tag{9.6.16}$$

To derive (9.6.16), we first apply $\mathscr{Z}_*^{[1,11];1}$ to (9.6.15). Using the bootstrap assumptions and the already proven estimates (9.6.9) and (9.6.13) (to bound the derivatives of the terms $L_{(Small)}^1$ and $L_{(Small)}^2$ found in the factor $P\gamma$ on RHS (9.6.15)), we deduce the estimate $\left\|\mathscr{Z}_*^{[1,11];1}\text{RHS (9.6.15)}\right\|_{L^\infty(\Sigma_t^u)} \lesssim \varepsilon$. To finish the proof of (9.6.16), we consider the term $\mathscr{Z}_*^{[1,11];1}L\breve{X}\Psi$ obtained by applying $\mathscr{Z}_*^{[1,11];1}$ to LHS (9.6.15). We use the commutator estimate (9.5.7b) with $f = \breve{X}\Psi$, $1 \leq N \leq 11$, and $M = 1$ and the bootstrap assumptions to arbitrarily permute the vectorfield factors in $\mathscr{Z}_*^{[1,11];1}L$ up to error terms that are $\lesssim \left|\mathscr{Z}_{**}^{[1,11];\leq 1}\breve{X}\Psi\right| \lesssim \left|\mathscr{Z}_{**}^{[1,11];1}\breve{X}\Psi\right| + \varepsilon$, where we bounded the last factor on RHS (9.5.7b) as follows: $\left|\begin{pmatrix} \mathscr{Z}_{**}^{[1,11];0}\gamma \\ \mathscr{Z}_{*}^{[1,11];\leq 1}\underline{\gamma} \end{pmatrix}\right| \lesssim 1$. We have therefore proved (9.6.16). We now integrate inequality (9.6.16) along the integral curves of L as in (9.3.5), use the conditions on the data, and apply Gronwall's inequality to deduce

$$\left|\mathscr{Z}_*^{[1,11];1}\breve{X}\Psi\right| \lesssim \varepsilon. \tag{9.6.17}$$

Using (9.6.17), the commutator estimate (9.5.7b) with $M = 2$, and the bootstrap assumptions (including $\left\|\mathscr{Z}_*^{[1,12];\leq 1}\Psi\right\|_{L^\infty(\Sigma_t^u)} \lesssim \varepsilon$), we use a commutator argument similar to the one surrounding equation (9.6.10), which allows us to arbitrarily permute the vectorfield factors in the operator $\mathscr{Z}_*^{[1,11];1}\breve{X}$ on LHS (9.6.17) up to $\mathcal{O}(\varepsilon)$ errors. In total, we have obtained the desired bound $\left\|\mathscr{Z}_*^{[1,12];\leq 2}\Psi\right\|_{L^\infty(\Sigma_t^u)} \lesssim \varepsilon$, which completes the proof of (9.6.3d).

Proof of (9.6.5c)**,** (9.6.7a)**, and** (9.6.8): We first prove (9.6.5c) and (9.6.7a). Much like in our proof of (9.6.13), we may use (9.6.1b)-(9.6.2) and the bootstrap assumptions and integrate along the integral curves of L to deduce

$$\left|\begin{pmatrix} \mathscr{Z}_{**}^{[1,11];\leq 1}\upmu \\ \mathscr{Z}_{*}^{\leq 11;\leq 2}(L_{(Small)}^1, L_{(Small)}^2) \end{pmatrix}\right|(t,u,\vartheta) \tag{9.6.18}$$
$$\leq C\left|\begin{pmatrix} \mathscr{Z}_{**}^{[1,11];\leq 1}\upmu \\ \mathscr{Z}_{*}^{\leq 11;\leq 2}(L_{(Small)}^1, L_{(Small)}^2) \end{pmatrix}\right|(0,u,\vartheta)$$
$$+ C\int_{s=0}^{t}\left|\begin{pmatrix} \mathscr{Z}_{**}^{[1,11];\leq 1}\upmu \\ \mathscr{Z}_{*}^{\leq 11;\leq 2}(L_{(Small)}^1, L_{(Small)}^2) \end{pmatrix}\right|(s,u,\vartheta)\,ds + C\varepsilon,$$

where the $C\varepsilon$ term on RHS (9.6.18) comes from the terms $\left|\mathscr{Z}_{*}^{[1,12];\leq 2}\vec{\Psi}\right|$ (which are $\lesssim \varepsilon$ in view of the already proven estimate (9.6.3d)) on RHS (9.6.1b) and RHS (9.6.2). The conditions on the data imply that the first term on RHS (9.6.18) is $\leq C\varepsilon$. Hence, from Gronwall's inequality, we conclude $\left|\begin{pmatrix} \mathscr{Z}_{**}^{[1,11];\leq 1}\upmu \\ \mathscr{Z}_{*}^{\leq 11;\leq 2}(L_{(Small)}^1, L_{(Small)}^2) \end{pmatrix}\right|(t,u,\vartheta) \lesssim \varepsilon \exp(C\mathring{\updelta}_*^{-1}) \lesssim \varepsilon$, which yields (9.6.5c) and (9.6.7a). The estimate (9.6.8) then follows as a consequence of inequality (9.4.1c) and the estimates (9.6.3d), (9.6.5c), and (9.6.7a).

Proof of (9.6.6): The estimate (9.6.6) is a trivial consequence of (8.2.8) and (8.6.1).

Proof of (9.6.3a)**,** (9.6.3b)**,** (9.6.3c)**,** (9.6.4)**,** (9.6.5a)**,** (9.6.5b)**, and** (9.6.7b): We first prove (9.6.3a), (9.6.3b), (9.6.3c), and (9.6.4). We note that a special case of (9.6.3d) is the estimate $\left\|L\breve{X}^M\vec{\Psi}\right\|_{L^\infty(\Sigma_t^u)} \lesssim \varepsilon$, valid for $M = 1, 2$. Hence, we can integrate along the integral curves of L as in (9.3.5) and use this estimate to obtain $\left|\breve{X}^M\vec{\Psi}\right|(t,u,\vartheta) = \left|\breve{X}^M\vec{\Psi}\right|(0,u,\vartheta) + \mathcal{O}(\varepsilon)$ and $\left|\breve{X}(\uprho - v^1)\right|(t,u,\vartheta) = \left|\breve{X}(\uprho - v^1)\right|(0,u,\vartheta) + \mathcal{O}(\varepsilon)$. Using the conditions on the data, we arrive at the desired four estimates (note that our assumptions on the data and (8.6.1) imply the smallness bounds $\left\|\breve{X}(\uprho - v^1)\right\|_{L^\infty(\Sigma_0^1)}$, $\left\|\breve{X}^{[0,2]}v^2\right\|_{L^\infty(\Sigma_0^1)} \lesssim \varepsilon$ and the non-small bounds $\left\|\breve{X}^{[0,2]}\uprho\right\|_{L^\infty(\Sigma_0^1)} \lesssim 1$ and $\left\|\breve{X}^{[0,2]}v^1\right\|_{L^\infty(\Sigma_0^1)} \lesssim 1$).

Inequality (9.6.7b) follows in a similar fashion from the already proven estimate (9.6.7a).

We now prove (9.6.5b). From the evolution equation (3.14.1), Lemma 3.19, the commutator estimate (9.5.7b) with $f = \upmu$, the estimates (9.6.3d) (9.6.7a), and (9.6.5c) and the bootstrap assumptions, we see that for $M = 0, 1$, we have

$$L\breve{X}^M\upmu = \frac{1}{2}\breve{X}^M\left\{\vec{G}_{LL}\diamond\breve{X}\vec{\Psi}\right\} + \breve{X}^M\left\{\mathrm{f}(\underline{\gamma})P\vec{\Psi}\right\} + [L,\breve{X}^M]\upmu \tag{9.6.19}$$
$$= \frac{1}{2}\breve{X}^M\left\{\vec{G}_{LL}\diamond\breve{X}\vec{\Psi}\right\} + \mathcal{O}(\varepsilon).$$

Moreover, from the schematic identity (3.19.2b), the estimates (9.6.3d) and (9.6.7a) and the bootstrap assumptions, we deduce $L\breve{X}^M\left\{\vec{G}_{LL}\diamond\breve{X}\vec{\Psi}\right\} = \mathcal{O}(\varepsilon)$. Integrating this estimate along the integral curves of L as in (9.3.5), we find that $\left\|\breve{X}^M\left\{\vec{G}_{LL}\diamond\breve{X}\vec{\Psi}\right\}\right\|_{L^\infty(\Sigma_t^u)} = \left\|\breve{X}^M\left\{\vec{G}_{LL}\diamond\breve{X}\vec{\Psi}\right\}\right\|_{L^\infty(\Sigma_0^u)} + \mathcal{O}(\varepsilon)$. From this estimate and (9.6.19), we conclude (9.6.5b).

Finally, we prove (9.6.5a). We first integrate along the integral curves of L as in (9.3.5) to deduce the following inequality, valid for $M = 0,1$: $\left|\breve{X}^M \upmu\right|(t,u,\vartheta) \leq \left|\breve{X}^M \upmu\right|(0,u,\vartheta) + \int_{s=0}^t \left|L\breve{X}^M \upmu\right|(s,u,\vartheta)\,ds$. We now use (9.6.5b) to bound the time integral in the previous inequality by $\leq \mathring{\updelta}_*^{-1} \left\| \breve{X}^M \left\{ \vec{G}_{LL} \diamond \breve{X}\vec{\Psi} \right\} \right\|_{L^\infty(\Sigma_0^u)} + C\varepsilon$, where we have used the assumption $t \leq 2\mathring{\updelta}_*^{-1}$. The desired bound (9.6.5a) now readily follows from these estimates. □

10. L^∞ Estimates Involving Higher Transversal Derivatives

In Sect. 11, we derive sharp pointwise estimates for $\upmu$ and some of its derivative, estimates which play a crucial role in the energy estimates. The proofs of some of the estimates of Sect. 11 rely on the bound $\left\| \breve{X}\breve{X}\upmu \right\|_{L^\infty(\Sigma_t^u)} \lesssim 1$. In this section, we derive this bound and some related ones, some of which are needed to prove it.

10.1. **Auxiliary bootstrap assumptions.** We will use auxiliary bootstrap assumptions to simplify the analysis. In Prop. 10.1, we derive strict improvements of the assumptions.

Our auxiliary bootstrap assumptions are that following inequalities hold on $\mathcal{M}_{T_{(Boot)},U_0}$, where ε is the small positive bootstrap parameter from Sect. 8.4.

Auxiliary bootstrap assumptions involving three transversal derivatives of $\vec{\Psi}$. For $\Psi \in \{\uprho, v^1, v^2\}$, we have

$$\left\| \breve{X}\breve{X}\breve{X}\Psi \right\|_{L^\infty(\Sigma_t^u)} \leq \left\| \breve{X}\breve{X}\breve{X}\Psi \right\|_{L^\infty(\Sigma_0^u)} + \varepsilon^{1/2}. \qquad (\mathbf{AUX}\breve{X}\breve{X}\breve{X}\vec{\Psi})$$

Auxiliary bootstrap assumptions involving two transversal derivatives of $\upmu$.

$$\left\| L\breve{X}\breve{X}\upmu \right\|_{L^\infty(\Sigma_t^u)} \leq \frac{1}{2} \left\| \breve{X}\breve{X} \left\{ \vec{G}_{LL} \diamond \breve{X}\vec{\Psi} \right\} \right\|_{L^\infty(\Sigma_0^u)} + \varepsilon^{1/2}, \qquad (\mathbf{AUX}L\breve{X}\breve{X}\upmu)$$

$$\left\| \breve{X}\breve{X}\upmu \right\|_{L^\infty(\Sigma_t^u)} \leq \left\| \breve{X}\breve{X}\upmu \right\|_{L^\infty(\Sigma_0^u)} + 2\mathring{\updelta}_*^{-1} \left\| \breve{X}\breve{X} \left\{ \vec{G}_{LL} \diamond \breve{X}\vec{\Psi} \right\} \right\|_{L^\infty(\Sigma_0^u)} + \varepsilon^{1/2}. \quad (\mathbf{AUX}\breve{X}\breve{X}\upmu)$$

10.2. **The main estimates involving higher-order transversal derivatives.** In the next proposition, we provide the main estimates of Sect. 10. The proposition yields, in particular, strict improvements of the bootstrap assumptions of Sect. 10.1.

Proposition 10.1 (L^∞ **estimates involving higher-order transversal derivatives).** *Under the data-size and bootstrap assumptions of Sects. 8.1-8.4 and Sect. 10.1 and the smallness assumptions of Sect. 8.6, the following estimates hold on $\mathcal{M}_{T_{(Boot)},U_0}$.*

L^∞ estimates involving three transversal derivatives of $\vec{\Psi}$.

$$\left\| L\breve{X}\breve{X}\breve{X}\vec{\Psi} \right\|_{L^\infty(\Sigma_t^u)} \leq C\varepsilon. \qquad (10.2.1)$$

Moreover, for $\Psi \in \{\uprho, v^1, v^2\}$, we have

$$\left\| \breve{X}\breve{X}\breve{X}\Psi \right\|_{L^\infty(\Sigma_t^u)} \leq \left\| \breve{X}\breve{X}\breve{X}\Psi \right\|_{L^\infty(\Sigma_0^u)} + C\varepsilon. \qquad (10.2.2a)$$

L^∞ ***estimates involving two transversal derivatives of*** $\upmu$.

$$\left\| L\breve{X}\breve{X}\upmu \right\|_{L^\infty(\Sigma_t^u)} \leq \frac{1}{2}\left\| \breve{X}\breve{X}\left\{ \vec{G}_{LL} \diamond \breve{X}\vec{\Psi} \right\}\right\|_{L^\infty(\Sigma_0^u)} + C\varepsilon, \tag{10.2.3a}$$

$$\left\| \breve{X}\breve{X}\upmu \right\|_{L^\infty(\Sigma_t^u)} \leq \left\| \breve{X}\breve{X}\upmu \right\|_{L^\infty(\Sigma_0^u)} + \mathring{\updelta}_*^{-1}\left\| \breve{X}\breve{X}\left\{ \vec{G}_{LL} \diamond \breve{X}\vec{\Psi} \right\}\right\|_{L^\infty(\Sigma_0^u)} + C\varepsilon. \tag{10.2.3b}$$

Sharp pointwise estimates involving the critical factor $\vec{G}_{LL}$. *Moreover, if* $0 \leq M \leq 2$ *and* $0 \leq s \leq t < T_{(Boot)}$*, then we have the following estimates:*

$$\left| \breve{X}^M \vec{G}_{LL}(t,u,\vartheta) - \breve{X}^M \vec{G}_{LL}(s,u,\vartheta) \right| \leq C\varepsilon(t-s), \tag{10.2.4}$$

$$\left| \left\{ \breve{X}^M \left(\vec{G}_{LL} \diamond \breve{X}\vec{\Psi} \right) \right\}(t,u,\vartheta) - \left\{ \breve{X}^M \left(\vec{G}_{LL} \diamond \breve{X}\vec{\Psi} \right) \right\}(s,u,\vartheta) \right| \leq C\varepsilon(t-s). \tag{10.2.5}$$

Furthermore, we have

$$\left\| G_{LL}^2 \right\|_{L^\infty(\Sigma_t^u)} \leq C\varepsilon. \tag{10.2.6}$$

Finally, with $\bar{c_s}' := \frac{d}{d\uprho} c_s(\uprho = 0)$*, we have*

$$L\upmu(t,u,\vartheta) = -(\bar{c_s}' + 1)\breve{X}v^1(t,u,\vartheta) + \mathcal{O}(\varepsilon). \tag{10.2.7}$$

Proof of Prop. 10.1. See Sect. 9.2 for some comments on the analysis. We must derive the estimates in a viable order. Throughout this proof, we use the data-size assumptions of Sects. 8.1 and 8.2 and the assumption (8.6.1) without explicitly mentioning them each time. We refer to these as "conditions on the data."

Proof of (10.2.1)-(10.2.2a): By (9.6.15), for $\Psi \in \{\uprho, v^1, v^2\}$, we have

$$\begin{aligned} L\breve{X}\Psi = {} & \mathrm{f}(\underline{\upgamma}, \not{g}^{-1}, \not{d}\vec{x}, P\Psi, \breve{X}\Psi)PP\Psi + \mathrm{f}(\underline{\upgamma}, \not{g}^{-1}, \not{d}\vec{x}, P\Psi, \breve{X}\Psi)P\upgamma \\ & + \mathrm{f}(\underline{\upgamma}, \mathscr{Z}^{\leq 1}\vec{\Psi})\mathscr{P}^{\leq 1}\upomega. \end{aligned} \tag{10.2.8}$$

Commuting (10.2.8) with $\breve{X}\breve{X}$ and using Lemmas 9.5 and 9.6, the L^∞ estimates of Prop. 9.12, and the auxiliary bootstrap assumptions of Sect. 10.1, we find that

$$\begin{aligned} \left| L\breve{X}\breve{X}\breve{X}\Psi \right| & \leq \left| L\breve{X}\breve{X}\breve{X}\Psi - \breve{X}\breve{X}L\breve{X}\Psi \right| + \left| \breve{X}\breve{X}L\breve{X}\Psi \right| \\ & \lesssim \left| L\breve{X}\breve{X}\breve{X}\Psi - \breve{X}\breve{X}L\breve{X}\Psi \right| + \varepsilon. \end{aligned} \tag{10.2.9}$$

Using in addition the commutator estimate (9.5.7b) with $f = \breve{X}\Psi$, we obtain the bound $\left| L\breve{X}\breve{X}\breve{X}\Psi - \breve{X}\breve{X}L\breve{X}\Psi \right| \lesssim \varepsilon$ as well. We have therefore proved (10.2.1). The estimates (10.2.2a) then follow from integrating along the integral curves of L as in (9.3.5) and using the estimate (10.2.1) and the conditions on the data.

Proof of (10.2.4)-(10.2.5): It suffices to prove that for $M = 0,1,2$ and $\imath = 0,1,2$, we have

$$\left| L\breve{X}^M G_{LL}^\imath \right|, \left| L\breve{X}^M \breve{X}\Psi_\imath \right| \lesssim \varepsilon, \tag{10.2.10}$$

$$\left| \breve{X}^M G_{LL}^\imath \right|, \left| \breve{X}^M \breve{X}\Psi_\imath \right| \lesssim 1; \tag{10.2.11}$$

once we have shown (10.2.10)-(10.2.11), we can obtain the desired estimates by integrating along the integral curves of L from time s to t (in analogy with (9.3.5)) and using (10.2.10)-(10.2.11). The estimate for the second term on LHS (10.2.10) follows from (9.6.3d) and (10.2.1). To prove the desired bound (10.2.11) for $\left|L\breve{X}^M G_{LL}^{\imath}\right|$, we first use Lemma 3.19 to deduce that $\vec{G}_{LL} = \mathrm{f}(\gamma)$. Differentiating this identity with $L\breve{X}^M$ and using the L^∞ estimates of Prop. 9.12, we obtain the desired bound. The estimate (10.2.11) is a simple consequence of the relation $\vec{G}_{LL} = \mathrm{f}(\gamma)$, the L^∞ estimates of Prop. 9.12, and the estimates (10.2.2a).

Proof of (10.2.6): From (3.15.1b), the fact that $X^2 = X^2_{(Small)}$ (see Def. 3.29), and Lemma 3.19, we see that $G_{LL}^2 = \mathrm{f}(\gamma)\gamma$. The desired estimate (10.2.6) now follows as a simple consequence of the L^∞ estimates of Prop. 9.12.

Proof of (10.2.7): From the evolution equation (3.14.1), (3.12.5), the identities (3.15.1a)-(3.15.1b), Lemma 3.19, and the assumption $\bar{c}_s = 1$, we deduce that

$$\begin{aligned} L\mu &= \frac{1}{2}\sum_{\imath=0}^{1} G_{LL}^{\imath}\breve{X}v^1 + \frac{1}{2}G_{LL}^0\breve{X}(\rho - v^1) + \frac{1}{2}G_{LL}^2\breve{X}v^2 + \mathrm{f}(\underline{\gamma})P\vec{\Psi} \\ &= -(\bar{c}_s' + 1)\breve{X}v^1 + \mathrm{f}(\gamma)\breve{X}(\rho - v^1) + \mathrm{f}(\underline{\gamma}, \breve{X}\vec{\Psi})\gamma + \mathrm{f}(\underline{\gamma})P\vec{\Psi}. \end{aligned} \tag{10.2.12}$$

The desired estimate (10.2.7) now follows from (10.2.12) and the L^∞ estimates of Prop. 9.12 (see especially (9.6.4)).

Proof of (10.2.3a)**-**(10.2.3b): With the help of the L^∞ estimates of Prop. 9.12 and the bootstrap assumptions, we can use the same argument that we used to prove (9.6.19) in order to conclude that (9.6.19) also holds with $M = 2$. The remainder of the proof of (10.2.3a)-(10.2.3b) now proceeds as in the proof of (9.6.5a)-(9.6.5b) (which is given just below (9.6.19)), thanks to the availability of the already proven estimates (10.2.4)-(10.2.5) in the case $M = 2$.

□

11. Sharp Estimates for μ

In this section, we derive sharp pointwise estimates for μ and some of its derivatives. These estimates provide much more information than the crude estimates we obtained in Sects. 9 and 10. The sharp estimates play an essential role in our derivation a priori energy estimates (see Sect. 15). The reason is that in order to obtain the energy estimates, we must know exactly how μ vanishes[108] and how certain ratios with μ in the denominator behave. This is the main information that we derive in this section.

Many results derived in this section are based on a posteriori estimates. By this, we mean estimates for quantities at times $0 \leq s \leq t$ that depend on the behavior of other quantities at the "late time" t, where $t < T_{(Boot)}$. For this reason, some of our analysis involves functions $q = q(s, u, \vartheta; t)$, which we view to be functions of the geometric coordinates (s, u, ϑ) that depend on the "late time parameter" t. When we state and derive estimates for such quantities, s is the "moving" time variable verifying $0 \leq s \leq t$.

[108]It vanishes linearly; see (11.2.5a). This fact is of fundamental importance for our a priori energy estimates.

11.1. **Definitions and preliminary ingredients in the analysis.**

Definition 11.1 (**Auxiliary quantities used to analyze** $\upmu$)**.** We define the following quantities, where $0 \leq s \leq t$ for those quantities that depend on both s and t :

$$M(s,u,\vartheta;t) := \int_{s'=s}^{s'=t} \{L\upmu(t,u,\vartheta) - L\upmu(s',u,\vartheta)\}\, ds', \tag{11.1.1a}$$

$$\mathring{\upmu}(u,\vartheta) := \upmu(s=0,u,\vartheta), \tag{11.1.1b}$$

$$\widetilde{M}(s,u,\vartheta;t) := \frac{M(s,u,\vartheta;t)}{\mathring{\upmu}(u,\vartheta) - M(0,u,\vartheta;t)}, \tag{11.1.1c}$$

$$\upmu_{(Approx)}(s,u,\vartheta;t) := 1 + \frac{L\upmu(t,u,\vartheta)}{\mathring{\upmu}(u,\vartheta) - M(0,u,\vartheta;t)} s + \widetilde{M}(s,u,\vartheta;t). \tag{11.1.1d}$$

The following quantity $\upmu_\star$ captures the worst-case smallness of $\upmu$ along Σ_t^u. Our high-order energies are allowed to blow up like a positive power of $\upmu_\star^{-1}$.

Definition 11.2 (**Definition of** $\upmu_\star$)**.** We define

$$\upmu_\star(t,u) := \min\{1, \min_{\Sigma_t^u} \upmu\}. \tag{11.1.2}$$

The next lemma provides basic pointwise estimates for the auxiliary quantities.

Lemma 11.1 (**First estimates for the auxiliary quantities**)**.** *Under the data-size and bootstrap assumptions of Sects. 8.1-8.4 and the smallness assumptions of Sect. 8.6, the following estimates hold for* $(t,u,\vartheta) \in [0, T_{(Boot)}) \times [0, U_0] \times \mathbb{T}$ *and* $0 \leq s \leq t$*:*

$$\mathring{\upmu}(u,\vartheta) = 1 + \mathcal{O}(\varepsilon), \tag{11.1.3}$$

$$\mathring{\upmu}(u,\vartheta) = 1 + M(0,u,\vartheta;t) + \mathcal{O}(\varepsilon). \tag{11.1.4}$$

In addition, the following pointwise estimates hold:

$$|L\upmu(t,u,\vartheta) - L\upmu(s,u,\vartheta)| \lesssim \varepsilon(t-s), \tag{11.1.5}$$

$$|M(s,u,\vartheta;t)|, |\widetilde{M}(s,u,\vartheta;t)| \lesssim \varepsilon(t-s)^2, \tag{11.1.6}$$

$$\upmu(s,u,\vartheta) = (1 + \mathcal{O}(\varepsilon))\upmu_{(Approx)}(s,u,\vartheta;t). \tag{11.1.7}$$

Proof. (11.1.3) follows from (8.2.5a) and (8.6.1). The estimate (11.1.5) follows from the mean value theorem and the estimate $|LL\upmu| \lesssim \varepsilon$, which is a special case of (9.6.5c). The estimate (11.1.4) and the estimate (11.1.6) for M then follow from definition (11.1.1a) and the estimates (11.1.3) and (11.1.5). The estimate (11.1.6) for $\widetilde{M}$ follows from definition (11.1.1c), the estimate (11.1.6) for M, and (11.1.4). To prove (11.1.7), we first note the following identity, which is a straightforward consequence of Def. 11.1:

$$\upmu(s,u,\vartheta) = \{\mathring{\upmu}(u,\vartheta) - M(0,u,\vartheta;t)\}\, \upmu_{(Approx)}(s,u,\vartheta;t). \tag{11.1.8}$$

The desired estimate (11.1.7) now follows from (11.1.8) and (11.1.4).

□

To derive some of the estimates of this section, it is convenient to partition various subsets of spacetime into regions where $L\upmu < 0$ (and hence $\upmu$ is decaying) and regions where $L\upmu \geq 0$ (and hence $\upmu$ is not decaying). This motivates the sets given in the next definition.

Definition 11.3 (**Regions of distinct $\upmu$ behavior**)**.** For each $t \in [0, T_{(Boot)})$, $s \in [0,t]$, and $u \in [0, U_0]$, we partition

$$[0,u] \times \mathbb{T} = {}^{(+)}\mathcal{V}_t^u \cup {}^{(-)}\mathcal{V}_t^u, \tag{11.1.9a}$$

$$\Sigma_s^u = {}^{(+)}\Sigma_{s;t}^u \cup {}^{(-)}\Sigma_{s;t}^u, \tag{11.1.9b}$$

where

$${}^{(+)}\mathcal{V}_t^u := \left\{ (u',\vartheta) \in [0,u] \times \mathbb{T} \mid \frac{L\upmu(t,u',\vartheta)}{\mathring{\upmu}(u',\vartheta) - M(0,u',\vartheta;t)} \geq 0 \right\}, \tag{11.1.10a}$$

$${}^{(-)}\mathcal{V}_t^u := \left\{ (u',\vartheta) \in [0,u] \times \mathbb{T} \mid \frac{L\upmu(t,u',\vartheta)}{\mathring{\upmu}(u',\vartheta) - M(0,u',\vartheta;t)} < 0 \right\}, \tag{11.1.10b}$$

$${}^{(+)}\Sigma_{s;t}^u := \left\{ (s,u',\vartheta) \in \Sigma_s^u \mid (u',\vartheta) \in {}^{(+)}\mathcal{V}_t^u \right\}, \tag{11.1.10c}$$

$${}^{(-)}\Sigma_{s;t}^u := \left\{ (s,u',\vartheta) \in \Sigma_s^u \mid (u',\vartheta) \in {}^{(-)}\mathcal{V}_t^u \right\}. \tag{11.1.10d}$$

Remark 11.1 (**The role of the denominators in** (11.1.10a)-(11.1.10b))**.** Note that (11.1.4) implies that the denominator $\mathring{\upmu}(u',\vartheta) - M(0,u',\vartheta;t)$ in (11.1.10a)-(11.1.10b) remains strictly positive all the way up to the shock. We include the denominator in the definitions (11.1.10a)-(11.1.10b) because it helps to clarify the connection between the parameter $\upkappa$ defined in (11.2.4) and the sets ${}^{(+)}\mathcal{V}_t^u$ and ${}^{(-)}\mathcal{V}_t^u$.

11.2. **Sharp pointwise estimates for $\upmu$ and its derivatives.** In the next proposition, we provide the sharp pointwise estimates for $\upmu$ that we use to close our energy estimates.

Proposition 11.2 (**Sharp pointwise estimates for $\upmu$, $L\upmu$, and $\breve{X}\upmu$**)**.** *Under the data-size and bootstrap assumptions of Sects. 8.1-8.4 and the smallness assumptions of Sect. 8.6, the following estimates hold for $(t,u,\vartheta) \in [0, T_{(Boot)}) \times [0, U_0] \times \mathbb{T}$ and $0 \leq s \leq t$.*

Upper bound for $\frac{[L\upmu]_+}{\upmu}$.

$$\left\| \frac{[L\upmu]_+}{\upmu} \right\|_{L^\infty(\Sigma_s^u)} \leq C. \tag{11.2.1}$$

Small $\upmu$ implies $L\upmu$ is negative.

$$\upmu(s,u,\vartheta) \leq \frac{1}{4} \implies L\upmu(s,u,\vartheta) \leq -\frac{1}{4}\mathring{\updelta}_*, \tag{11.2.2}$$

where $\mathring{\updelta}_* > 0$ is defined in (8.1.1).

Upper bound for $\frac{[\breve{X}\upmu]_+}{\upmu}$.

$$\left\| \frac{[\breve{X}\upmu]_+}{\upmu} \right\|_{L^\infty(\Sigma_s^u)} \leq \frac{C}{\sqrt{T_{(Boot)} - s}}. \tag{11.2.3}$$

Sharp spatially uniform estimates. *Consider a time interval* $s \in [0,t]$ *and define the* $(t,u$*-dependent) constant* $\upkappa$ *by*

$$\upkappa := \sup_{(u',\vartheta)\in[0,u]\times\mathbb{T}} \frac{[L\upmu]_-(t,u',\vartheta)}{\mathring{\upmu}(u',\vartheta) - M(0,u',\vartheta;t)}, \tag{11.2.4}$$

and note that $\upkappa \geq 0$ *in view of the estimate* (11.1.4). *Then*

$$\upmu_\star(s,u) = \{1+\mathcal{O}(\varepsilon)\}\{1-\upkappa s\}, \tag{11.2.5a}$$

$$\|[L\upmu]_-\|_{L^\infty(\Sigma_s^u)} = \begin{cases} \left\{1+\mathcal{O}(\varepsilon^{1/2})\right\}\upkappa, & \text{if } \upkappa \geq \sqrt{\varepsilon}, \\ \mathcal{O}(\varepsilon^{1/2}), & \text{if } \upkappa \leq \sqrt{\varepsilon}. \end{cases} \tag{11.2.5b}$$

Furthermore, we have

$$\upkappa \leq \{1+\mathcal{O}(\varepsilon)\}\,\mathring{\updelta}_*. \tag{11.2.6a}$$

Moreover, when $u=1$*, we have*

$$\upkappa = \{1+\mathcal{O}(\varepsilon)\}\,\mathring{\updelta}_*. \tag{11.2.6b}$$

Sharp estimates when $(u',\vartheta) \in {}^{(+)}\mathcal{V}_t^u$. *We recall that the set* ${}^{(+)}\mathcal{V}_t^u$ *is defined in* (11.1.10a). *If* $0 \leq s_1 \leq s_2 \leq t$*, then the following estimate holds:*

$$\sup_{(u',\vartheta)\in{}^{(+)}\mathcal{V}_t^u} \frac{\upmu(s_2,u',\vartheta)}{\upmu(s_1,u',\vartheta)} \leq C. \tag{11.2.7}$$

In addition, if $s \in [0,t]$ *and* ${}^{(+)}\Sigma_{s;t}^u$ *is as defined in* (11.1.10c)*, then*

$$\inf_{{}^{(+)}\Sigma_{s;t}^u} \upmu \geq 1 - C\varepsilon. \tag{11.2.8}$$

In addition, if $s \in [0,t]$ *and* ${}^{(+)}\Sigma_{s;t}^u$ *is as defined in* (11.1.10c)*, then*

$$\left\| \frac{[L\upmu]_-}{\upmu} \right\|_{L^\infty({}^{(+)}\Sigma_{s;t}^u)} \leq C\varepsilon. \tag{11.2.9}$$

Sharp estimates when $(u',\vartheta) \in {}^{(-)}\mathcal{V}_t^u$. *Assume that the set* ${}^{(-)}\mathcal{V}_t^u$ *defined in* (11.1.10b) *is non-empty, and consider a time interval* $s \in [0,t]$*. Let* $\upkappa \geq 0$ *be as in* (11.2.4)*. Then the following estimate holds:*

$$\sup_{\substack{0\leq s_1\leq s_2\leq t \\ (u',\vartheta)\in{}^{(-)}\mathcal{V}_t^u}} \frac{\upmu(s_2,u',\vartheta)}{\upmu(s_1,u',\vartheta)} \leq 1 + C\varepsilon. \tag{11.2.10}$$

Furthermore, if $s \in [0,t]$ *and* ${}^{(-)}\Sigma_{s;t}^u$ *is as defined in* (11.1.10d)*, then*

$$\|[L\upmu]_+\|_{L^\infty({}^{(-)}\Sigma_{s;t}^u)} \leq C\varepsilon. \tag{11.2.11}$$

Finally, there exists a constant $C>0$ *such that if* $0 \leq s \leq t$*, then*

$$\|[L\upmu]_-\|_{L^\infty({}^{(-)}\Sigma_{s;t}^u)} \leq \begin{cases} \left\{1+C\varepsilon^{1/2}\right\}\upkappa, & \text{if } \upkappa \geq \sqrt{\varepsilon}, \\ C\varepsilon^{1/2}, & \text{if } \upkappa \leq \sqrt{\varepsilon}. \end{cases} \tag{11.2.12}$$

Approximate time-monotonicity of $\upmu_\star^{-1}(s,u)$. *There exists a constant $C>0$ such that if $0 \leq s_1 \leq s_2 \leq t$, then*

$$\upmu_\star^{-1}(s_1,u) \leq (1+C\varepsilon)\upmu_\star^{-1}(s_2,u). \tag{11.2.13}$$

Proof. See Sect. 9.2 for some comments on the analysis.

Proof of (11.2.1): Clearly it suffices to show that $\frac{[L\upmu(t,u,\vartheta)]_+}{\upmu(t,u,\vartheta)} \leq C$ for $0 \leq t < T_{(Boot)}$, $u \in [0,U_0]$, and $\vartheta \in \mathbb{T}$. We may assume that $L\upmu(t,u,\vartheta) > 0$ since otherwise (11.2.1) is trivial. Then by (11.1.5), for $0 \leq s' \leq s \leq t < T_{(Boot)} \leq 2\mathring{\delta}_*^{-1}$, we have that $L\upmu(s',u,\vartheta) \geq L\upmu(s,u,\vartheta) - C\varepsilon(s-s') \geq -C\varepsilon$. Integrating this estimate with respect to s' starting from $s'=0$ and using (11.1.3), we find that $\upmu(s,u,\vartheta) \geq 1 - C\varepsilon s \geq 1 - C\varepsilon$ and thus $1/\upmu(s,u,\vartheta) \leq 1 + C\varepsilon$. Also using the bound $|L\upmu(s,u,\vartheta)| \leq C$ proved in (9.6.5b), we conclude the desired estimate.

Proof of (11.2.2): By (11.1.5), for $0 \leq s \leq t < T_{(Boot)} \leq 2\mathring{\delta}_*^{-1}$, we have that $L\upmu(s,u,\vartheta) = L\upmu(0,u,\vartheta) + \mathcal{O}(\varepsilon)$. Integrating this estimate with respect to s starting from $s=0$ and using (11.1.3), we find that $\upmu(s,u,\vartheta) = 1 + \mathcal{O}(\varepsilon) + sL\upmu(0,u,\vartheta)$. It follows that whenever $\upmu(s,u,\vartheta) < 1/4$, we have $L\upmu(0,u,\vartheta) < -\frac{1}{2}\mathring{\delta}_*(3/4 + \mathcal{O}(\varepsilon)) = -\frac{3}{8}\mathring{\delta}_* + \mathcal{O}(\varepsilon)$. Again using (11.1.5) to deduce that $L\upmu(s,u,\vartheta) = L\upmu(0,u,\vartheta) + \mathcal{O}(\varepsilon)$, we arrive at the desired estimate (11.2.2).

Proof of (11.2.6a) **and** (11.2.6b): We prove only (11.2.6a) since (11.2.6b) follows from nearly identical arguments. From the first line of (10.2.12), (11.1.4), (11.1.5), and the L^∞ estimates of Prop. 9.12, we have

$$\frac{L\upmu(t,u,\vartheta)}{\mathring{\upmu}(u,\vartheta) - M(0,u,\vartheta;t)} = L\upmu(0,u,\vartheta) + \mathcal{O}(\varepsilon) = \frac{1}{2}\sum_{\imath=0}^{1}[G_{LL}^{\imath}\breve{X}v^1](0,u,\vartheta) + \mathcal{O}(\varepsilon).$$

From this estimate and definitions (8.1.1) and (11.2.4), we conclude that $\upkappa \leq \mathring{\delta}_* + \mathcal{O}(\varepsilon) = (1+\mathcal{O}(\varepsilon))\mathring{\delta}_*$, which yields the desired bound (11.2.6a).

Proof of (11.2.5a) **and** (11.2.13): We first prove (11.2.5a). We start by establishing the following preliminary estimate for the crucial quantity $\upkappa = \upkappa(t,u)$ (see (11.2.4)):

$$t\upkappa < 1. \tag{11.2.14}$$

To proceed, we use (11.1.1d), (11.1.8), (11.1.4), and (11.1.6) to deduce that the following estimate holds for $(s,u',\vartheta) \in [0,t]\times[0,u]\times\mathbb{T}$:

$$\upmu(s,u',\vartheta) = (1+\mathcal{O}(\varepsilon))\left\{1 + \frac{L\upmu(t,u',\vartheta)}{\mathring{\upmu}(u',\vartheta) - M(0,u',\vartheta;t)}s + \mathcal{O}(\varepsilon)(t-s)^2\right\}. \tag{11.2.15}$$

Setting $s=t$ in equation (11.2.15), taking the min of both sides over $(u',\vartheta) \in [0,u]\times\mathbb{T}$, and appealing to definitions (11.1.2) and (11.2.4), we deduce that $\upmu_\star(t,u) = (1+\mathcal{O}(\varepsilon))(1-\upkappa t)$. Since $\upmu_\star(t,u) > 0$ by ($\mathbf{BA}\upmu > 0$), we conclude (11.2.14).

Having established the preliminary estimate, we now take the min of both sides of (11.2.15) over $(u',\vartheta) \in [0,u]\times\mathbb{T}$ and appeal to definitions (11.1.2) and (11.2.4) to obtain:

$$\min_{(u',\vartheta)\in[0,u]\times\mathbb{T}} \upmu(s,u',\vartheta) = (1+\mathcal{O}(\varepsilon))\left\{1 - \upkappa s + \mathcal{O}(\varepsilon)(t-s)^2\right\}. \tag{11.2.16}$$

We will show that the terms in braces on RHS (11.2.16) verify

$$1-\upkappa s+\mathcal{O}(\varepsilon)(t-s)^2=(1+\mathrm{f}(s,u;t))\left\{1-\upkappa s\right\}, \tag{11.2.17}$$

where

$$\mathrm{f}(s,u;t)=\mathcal{O}(\varepsilon). \tag{11.2.18}$$

The desired estimate (11.2.5a) then follows easily from (11.2.16)-(11.2.18) and definition (11.1.2). To prove (11.2.18), we first use (11.2.17) to solve for $\mathrm{f}(s,u;t)$:

$$\mathrm{f}(s,u;t)=\frac{\mathcal{O}(\varepsilon)(t-s)^2}{1-\upkappa s}=\frac{\mathcal{O}(\varepsilon)(t-s)^2}{1-\upkappa t+\upkappa(t-s)}. \tag{11.2.19}$$

We start by considering the case $\upkappa\leq(1/4)\mathring{\delta}_*$. Since $0\leq s\leq t<T_{(Boot)}\leq 2\mathring{\delta}_*^{-1}$, the denominator in the middle expression in (11.2.19) is $\geq 1/2$, and the desired estimate (11.2.18) follows easily whenever ε is sufficiently small. In remaining case, we have $\upkappa>(1/4)\mathring{\delta}_*$. Using (11.2.14), we deduce that RHS (11.2.19) $\leq\frac{1}{\upkappa}\mathcal{O}(\varepsilon)(t-s)\leq C\varepsilon\mathring{\delta}_*^{-2}\lesssim\varepsilon$ as desired.

Inequality (11.2.13) then follows as a simple consequence of (11.2.5a).

Proof of (11.2.5b) **and** (11.2.12): To prove (11.2.5b), we first use (11.1.5) to deduce that for $0\leq s\leq t<T_{(Boot)}\leq 2\mathring{\delta}_*^{-1}$ and $(u',\vartheta)\in[0,u]\times\mathbb{T}$, we have $L\upmu(s,u',\vartheta)=L\upmu(t,u',\vartheta)+\mathcal{O}(\varepsilon)$. Appealing to definition (11.2.4) and using the estimate (11.1.4), we find that $\|[L\upmu]_-\|_{L^\infty(\Sigma_s^u)}=\upkappa+\mathcal{O}(\varepsilon)$. If $\sqrt{\varepsilon}\leq\upkappa$, we see that as long as ε is sufficiently small, we have the desired bound $\upkappa+\mathcal{O}(\varepsilon)=(1+\mathcal{O}(\varepsilon^{1/2}))\upkappa$. On the other hand, if $\upkappa\leq\sqrt{\varepsilon}$, then similar reasoning yields that $\|[L\upmu]_-\|_{L^\infty(\Sigma_s^u)}=\upkappa+\mathcal{O}(\varepsilon)=\mathcal{O}(\sqrt{\varepsilon})$ as desired. We have thus proved (11.2.5b).

The estimate (11.2.12) can be proved via a similar argument and we omit the details.

Proof of (11.2.3): We fix times s and t with $0\leq s\leq t<T_{(Boot)}$ and a point $p\in\Sigma_s^u$ with geometric coordinates $(s,\widetilde{u},\widetilde{\vartheta})$. Let $\iota:[0,u]\to\Sigma_s^u$ be the integral curve of $\breve{X}$ that passes through p and that is parametrized by the values u' of the eikonal function. We set

$$F(u'):=\upmu\circ\iota(u'),\qquad \dot{F}(u'):=\frac{d}{du'}F(u')=(\breve{X}\upmu)\circ\iota(u').$$

We must bound $\frac{[\breve{X}\upmu]_+}{\upmu}|_p=\frac{[\dot{F}(\widetilde{u})]_+}{F(\widetilde{u})}$. We may assume that $\dot{F}(\widetilde{u})>0$ since otherwise the desired estimate is trivial. We now set

$$H:=\sup_{\mathcal{M}_{T_{(Boot)},U_0}}\breve{X}\breve{X}\upmu.$$

If $F(\widetilde{u})>\frac{1}{2}$, then the desired estimate is a simple consequence of (9.6.5a) with $M=1$. We may therefore also assume that $F(\widetilde{u})\leq\frac{1}{2}$. Then in view of the estimate $\|\upmu-1\|_{L^\infty\left(\mathcal{P}_0^{T_{(Boot)}}\right)}\lesssim\varepsilon$ along $\mathcal{P}_0^{T_{(Boot)}}$ (see (9.6.6)), we deduce that there exists a $u''\in[0,\widetilde{u}]$ such that $\dot{F}(u'')<0$. Considering also the assumption $\dot{F}(\widetilde{u})>0$, we see that $H>0$. Moreover, by (10.2.3b), we have $H \leq C$. Furthermore, by continuity, there exists a smallest $u_* \in [0, \widetilde{u}]$ such that $\dot{F}(u') \geq 0$ for $u' \in [u_*, \widetilde{u}]$. We also set

$$\upmu_{(Min)}(s, u') := \min_{(u'', \vartheta) \in [0, u'] \times \mathbb{T}} \upmu(s, u'', \vartheta). \tag{11.2.20}$$

The two main steps in the proof are showing that

$$\frac{[\breve{X}\upmu(s, \widetilde{u}, \widetilde{\vartheta})]_+}{\upmu(s, \widetilde{u}, \widetilde{\vartheta})} \leq H^{1/2} \frac{1}{\sqrt{\upmu_{(Min)}(s, \widetilde{u})}} \tag{11.2.21}$$

and that for $0 \leq s \leq t < T_{(Boot)}$, we have

$$\upmu_{(Min)}(s, u) \geq \max\left\{(1 - C\varepsilon)\upkappa(t - s), (1 - C\varepsilon)(1 - \upkappa s)\right\}, \tag{11.2.22}$$

where $\upkappa = \upkappa(t, u)$ is defined in (11.2.4). Once we have obtained (11.2.21)-(11.2.22) (see below), we split the remainder of the proof (which is relatively easy) into the two cases $\upkappa \leq \frac{1}{4}\mathring{\updelta}_*$ and $\upkappa > \frac{1}{4}\mathring{\updelta}_*$. In the first case $\upkappa \leq \frac{1}{4}\mathring{\updelta}_*$, we have $1 - \upkappa s \geq 1 - \frac{1}{4}\mathring{\updelta}_* T_{(Boot)} \geq \frac{1}{2}$, and the desired bound $\frac{[\breve{X}\upmu(s, \widetilde{u}, \widetilde{\vartheta})]_+}{\upmu(s, \widetilde{u}, \widetilde{\vartheta})} \leq C \leq \frac{C}{T_{(Boot)}^{1/2}} \leq \frac{C}{\sqrt{T_{(Boot)} - s}} \leq$ RHS (11.2.3) follows easily from (11.2.21) and the second term in the min on RHS (11.2.22). In the remaining case $\upkappa > \frac{1}{4}\mathring{\updelta}_*$, we have $\frac{1}{\upkappa} \leq C$, and using the first term in the min on RHS (11.2.22), we deduce that $\frac{[\breve{X}\upmu(s, \widetilde{u}, \widetilde{\vartheta})]_+}{\upmu(s, \widetilde{u}, \widetilde{\vartheta})} \leq \frac{C}{\sqrt{t - s}}$. Since this estimate holds for all $t < T_{(Boot)}$ with a uniform constant C, we conclude (11.2.3) in this case.

We now prove (11.2.21). To this end, we will show that

$$\frac{[\breve{X}\upmu(s, \widetilde{u}, \widetilde{\vartheta})]_+}{\upmu(s, \widetilde{u}, \widetilde{\vartheta})} \leq 2H^{1/2} \frac{\sqrt{\upmu(s, \widetilde{u}, \widetilde{\vartheta}) - \upmu_{(Min)}(s, \widetilde{u})}}{\upmu(s, \widetilde{u}, \widetilde{\vartheta})}. \tag{11.2.23}$$

Then viewing RHS (11.2.21) as a function of the real variable $\upmu(s, \widetilde{u}, \widetilde{\vartheta})$ (with all other parameters fixed) on the domain $[\upmu_{(Min)}(s, \widetilde{u}), \infty)$, we carry out a simple calculus exercise to find that RHS (11.2.21) $\leq H^{1/2} \frac{1}{\sqrt{\upmu_{(Min)}(s, \widetilde{u})}}$, which yields (11.2.21). We now prove (11.2.23). For any $u' \in [u_*, \widetilde{u}]$, we use the mean value theorem to obtain

$$\dot{F}(\widetilde{u}) - \dot{F}(u') \leq H(\widetilde{u} - u'), \qquad F(\widetilde{u}) - F(u') \geq \min_{u'' \in [u', \widetilde{u}]} \dot{F}(u'')(\widetilde{u} - u'). \tag{11.2.24}$$

Setting $u_1 := \widetilde{u} - \frac{1}{2}\frac{\dot{F}(\widetilde{u})}{H}$, we find from the first estimate in (11.2.24) that for $u' \in [u_1, \widetilde{u}]$, we have $\dot{F}(u') \geq \frac{1}{2}\dot{F}(\widetilde{u})$. Using also the second estimate in (11.2.24), we find that for $u' \in [u_1, \widetilde{u}]$, we have $F(\widetilde{u}) - F(u_1) \geq \frac{1}{2}\dot{F}(\widetilde{u})(\widetilde{u} - u_1) = \frac{1}{4}\frac{\dot{F}^2(\widetilde{u})}{H}$. Noting that the definition of $\upmu_{(Min)}$

implies that $F(u_1) \geq \upmu_{(Min)}(s,\widetilde{u})$, we deduce that

$$\upmu(s,\widetilde{u},\widetilde{\vartheta}) - \upmu_{(Min)}(s,\widetilde{u}) \geq \frac{1}{4}\frac{[\breve{X}\upmu(s,\widetilde{u},\widetilde{\vartheta})]_+^2}{H}. \tag{11.2.25}$$

Taking the square root of (11.2.25), rearranging, and dividing by $\upmu(s,\widetilde{u},\widetilde{\vartheta})$, we conclude the desired estimate (11.2.23).

It remains for us to prove (11.2.22). Reasoning as in the proof of (11.2.15)-(11.2.18) and using (11.2.14), we find that for $0 \leq s \leq t < T_{(Boot)}$ and $u' \in [0,u]$, we have $\upmu_{(Min)}(s,u') \geq (1-C\varepsilon)\{1-\upkappa s\} \geq (1-C\varepsilon)\upkappa(t-s)$. From these two inequalities, we conclude (11.2.22).

Proof of (11.2.10): A straightforward modification of the proof of (11.2.5a), based on equation (11.2.15) and on replacing $\upkappa$ in (11.2.16)-(11.2.17) with $L\upmu(t,u',\vartheta)$ (without taking the min on the LHS of the analog of (11.2.16)), yields that for $0 \leq s \leq t < T_{(Boot)}$ and $(u',\vartheta) \in {}^{(-)}\mathcal{V}_t^u$, we have $\upmu(s,u',\vartheta) = \{1+\mathcal{O}(\varepsilon)\}\{1-|L\upmu(t,u',\vartheta)|\,s\}$. The estimate (11.2.10) then follows as a simple consequence.

Proof of (11.2.7), (11.2.8), **and** (11.2.9): By (11.1.5), if $(u',\vartheta) \in {}^{(+)}\mathcal{V}_t^u$ and $0 \leq s \leq t < T_{(Boot)}$, then $[L\upmu]_-(s,u,\vartheta) \leq C\varepsilon$ and $L\upmu(s,u,\vartheta) \geq -C\varepsilon$. Integrating the latter estimate with respect to s from 0 to t and using (11.1.3), we find that $\upmu(s,u',\vartheta) \geq 1 - C\varepsilon s \geq 1 - C\varepsilon$. Moreover, from (9.6.5a) with $M=0$, we have the crude bound $\upmu(s,u',\vartheta) \leq C$. The desired bounds (11.2.7), (11.2.8), and (11.2.9) now readily follow from these estimates.

Proof of (11.2.11): By (11.1.5), if $(u',\vartheta) \in {}^{(-)}\mathcal{V}_t^u$ and $0 \leq s \leq t < T_{(Boot)}$, then $[L\upmu]_+(s,u',\vartheta) = [L\upmu]_+(t,u',\vartheta) + \mathcal{O}(\varepsilon) = \mathcal{O}(\varepsilon)$. The desired bound (11.2.11) thus follows.

□

11.3. **Sharp time-integral estimates involving $\upmu$.** Some error integrals appearing in our top-order energy identities contain a dangerous factor of $1/\upmu$. This forces us, in our Gronwall argument for a priori energy estimates, to derive estimates for time integrals involving various powers of $\upmu_\star^{-1}$. In Prop. 11.3, we derive estimates for these time integrals. The estimates of Prop. 11.3 directly influence the blowup-exponents P featured in our high-order energy estimates, which are allowed to blow up like $\upmu_\star^{-P}$. In particular, when controlling the size of the blowup-exponents, we will use the fact that the estimates (11.3.1) and (11.3.2) have coefficient factors $1+C\sqrt{\varepsilon}$ on the right-hand sides; larger coefficient factors would lead to larger blowup-exponents.

Proposition 11.3 (**Fundamental estimates for time integrals involving $\upmu^{-1}$**). *Let $\upmu_\star(t,u)$ be as defined in* (11.1.2). *Let*

$$b > 1$$

be a real number. Under the data-size and bootstrap assumptions of Sects. 8.1-8.4 and the smallness assumptions of Sect. 8.6, the following estimates hold for $(t,u) \in [0,T_{(Boot)}] \times [0,U_0]$.

Estimates relevant for borderline top-order spacetime integrals*. There exists a constant $C>0$ such that if $b\sqrt{\varepsilon} \leq 1$, then*

$$\int_{s=0}^{t} \frac{\|[L\upmu]_-\|_{L^\infty(\Sigma_s^u)}}{\upmu_\star^b(s,u)}\,ds \leq \frac{1+C\sqrt{\varepsilon}}{b-1}\upmu_\star^{1-b}(t,u). \tag{11.3.1}$$

Estimates relevant for borderline top-order hypersurface integrals. *There exists a constant $C > 0$ such that*

$$\|L\upmu\|_{L^\infty({}^{(-)}\Sigma_{t;t}^u)} \int_{s=0}^t \frac{1}{\upmu_\star^b(s,u)}\,ds \le \frac{1+C\sqrt{\varepsilon}}{b-1}\upmu_\star^{1-b}(t,u). \tag{11.3.2}$$

Estimates relevant for less dangerous top-order spacetime integrals. *There exists a constant $C > 0$ such that if $b\sqrt{\varepsilon} \le 1$, then*

$$\int_{s=0}^t \frac{1}{\upmu_\star^b(s,u)}\,ds \le C\left\{2^b + \frac{1}{b-1}\right\}\upmu_\star^{1-b}(t,u). \tag{11.3.3}$$

Estimates for integrals that lead to only $\ln\upmu_\star^{-1}$ degeneracy. *There exists a constant $C > 0$ such that*

$$\int_{s=0}^t \frac{\|[L\upmu]_-\|_{L^\infty(\Sigma_s^u)}}{\upmu_\star(s,u)}\,ds \le (1+C\sqrt{\varepsilon})\ln\upmu_\star^{-1}(t,u) + C\sqrt{\varepsilon}. \tag{11.3.4}$$

In addition, there exists a constant $C > 0$ such that

$$\int_{s=0}^t \frac{1}{\upmu_\star(s,u)}\,ds \le C\left\{\ln\upmu_\star^{-1}(t,u) + 1\right\}. \tag{11.3.5}$$

Estimates for integrals that break the $\upmu_\star^{-1}$ degeneracy. *There exists a constant $C > 0$ such that*

$$\int_{s=0}^t \frac{1}{\upmu_\star^{9/10}(s,u)}\,ds \le C. \tag{11.3.6}$$

Proof. **Proof of** (11.3.1), (11.3.2), and (11.3.4): To prove (11.3.1), we first consider the case $\upkappa \ge \sqrt{\varepsilon}$ in (11.2.5b). Using (11.2.5a) and (11.2.5b), we deduce that

$$\begin{aligned}\int_{s=0}^t \frac{\|[L\upmu]_-\|_{L^\infty(\Sigma_s^u)}}{\upmu_\star^b(s,u)}\,ds &= (1+\mathcal{O}(\varepsilon^{1/2}))\int_{s=0}^t \frac{\upkappa}{(1-\upkappa s)^b}\,ds \\ &\le \frac{1+\mathcal{O}(\varepsilon^{1/2})}{b-1}\frac{1}{(1-\upkappa t)^{b-1}} = \frac{1+\mathcal{O}(\varepsilon^{1/2})}{b-1}\upmu_\star^{1-b}(t,u)\end{aligned} \tag{11.3.7}$$

as desired. We now consider the remaining case $\upkappa \le \sqrt{\varepsilon}$ in (11.2.5b). Using (11.2.5a) and (11.2.5b) and the fact that $0 \le s \le t < T_{(Boot)} \le 2\mathring{\updelta}_*^{-1}$, we see that for ε sufficiently small relative to $\mathring{\updelta}_*$, we have

$$\begin{aligned}\int_{s=0}^t \frac{\|[L\upmu]_-\|_{L^\infty(\Sigma_s^u)}}{\upmu_\star^b(s,u)}\,ds &\le C\varepsilon^{1/2}\int_{s=0}^t \frac{1}{(1-\upkappa s)^b}\,ds \\ &\le C\varepsilon^{1/2}\frac{1}{(1-\upkappa t)^{b-1}} \le \frac{1}{b-1}\upmu_\star^{1-b}(t,u)\end{aligned} \tag{11.3.8}$$

as desired. We have thus proved (11.3.1).

Inequality (11.3.4) can be proved using similar arguments and we omit the details.

Inequality (11.3.2) can be proved using similar arguments with the help of the estimate (11.2.12) and we omit the details.

Proof of (11.3.3), (11.3.5), and (11.3.6): To prove (11.3.3), we first use (11.2.5a) to deduce

$$\int_{s=0}^{t} \frac{1}{\upmu_{\star}^{b}(s,u)}\,ds \leq C \int_{s=0}^{t} \frac{1}{(1-\upkappa s)^{b}}\,ds, \tag{11.3.9}$$

where $\upkappa = \upkappa(t,u)$ is defined in (11.2.4). We first assume that $\upkappa \leq \frac{1}{4}\mathring{\updelta}_*$. Then since $0 \leq t < T_{(Boot)} < 2\mathring{\updelta}_*^{-1}$, we see from (11.2.5a) that $\upmu_{\star}(s,u) \geq \frac{1}{4}$ for $0 \leq s \leq t$ and that RHS (11.3.9) $\leq C2^b t \leq C2^b \mathring{\updelta}_*^{-1} \leq C2^b \leq C2^b \upmu_{\star}^{1-b}(t,u)$ as desired. In the remaining case, we have $\upkappa > \frac{1}{4}\mathring{\updelta}_*$, and we can use (11.2.5a) and the estimate $\frac{1}{\upkappa} \leq C$ to bound RHS (11.3.9) by

$$\leq \frac{C}{\upkappa}\frac{1}{b-1}\frac{1}{(1-\upkappa t)^{b-1}} \leq \frac{C}{b-1}\upmu_{\star}^{1-b}(t,u) \tag{11.3.10}$$

as desired.

Inequalities (11.3.5) and (11.3.6) can be proved in a similar fashion. We omit the details, aside from remarking that the last step of the proof of (11.3.6) relies on the trivial estimate $(1-\upkappa t)^{1/10} \leq 1$.

□

12. The fundamental L^2-controlling quantities

In this section, we define the "fundamental L^2-controlling quantities" that we use to control $\vec{\Psi}$, $\upomega$, and their derivatives in L^2. We also exhibit their coerciveness properties. These are the quantities that, in Sect. 15, we will estimate using a Gronwall-type inequality.

12.1. Definitions of the fundamental L^2-controlling quantities.

Definition 12.1 (The main coercive quantities used for controlling the solution and its derivatives in L^2). In terms of the energy-null flux quantities of Def. 4.3 and the multi-index set $\mathcal{I}^{N;\leq 1}$ of Def. 9.5.1, we define

$$\mathbb{Q}_N(t,u) := \max_{\substack{\vec{I}\in\mathcal{I}^{N;\leq 1}\\ \Psi\in\{\uprho-v^1,v^1,v^2\}}} \sup_{(t',u')\in[0,t]\times[0,u]} \left\{\mathbb{E}^{(Wave)}[\mathscr{Z}^{\vec{I}}\Psi](t',u') + \mathbb{F}^{(Wave)}[\mathscr{Z}^{\vec{I}}\Psi](t',u')\right\}, \tag{12.1.1a}$$

$$\mathbb{Q}_N^{(Partial)}(t,u) := \max_{\substack{\vec{I}\in\mathcal{I}^{N;\leq 1}\\ \Psi\in\{\uprho-v^1,v^2\}}} \sup_{(t',u')\in[0,t]\times[0,u]} \left\{\mathbb{E}^{(Wave)}[\mathscr{Z}^{\vec{I}}\Psi](t',u') + \mathbb{F}^{(Wave)}[\mathscr{Z}^{\vec{I}}\Psi](t',u')\right\}, \tag{12.1.1b}$$

$$\mathbb{V}_N(t,u) := \max_{|\vec{I}|=N} \sup_{(t',u')\in[0,t]\times[0,u]} \left\{\mathbb{E}^{(Vort)}[\mathscr{P}^{\vec{I}}\upomega](t',u') + \mathbb{F}^{(Vort)}[\mathscr{P}^{\vec{I}}\upomega](t',u')\right\}, \tag{12.1.1c}$$

$$\mathbb{Q}_{[1,N]}(t,u) := \max_{1\leq M\leq N} \mathbb{Q}_M(t,u), \tag{12.1.1d}$$

$$\mathbb{V}_{\leq N}(t,u) := \max_{0\leq M\leq N} \mathbb{V}_M(t,u). \tag{12.1.1e}$$

Remark 12.1 (**Carefully note what is controlled by $\mathbb{Q}_N$ and $\mathbb{Q}_N^{(Partial)}$**). Note that $\mathbb{Q}_N$ directly controls the derivatives of $\rho - v^1$, v^1, and v^2, while control of derivatives of ρ in terms of $\mathbb{Q}_N$ (and thus for all three entries of the array $\vec{\Psi}$) can be achieved via the triangle inequality. Similarly, $\mathbb{Q}_N^{(Partial)}$ directly controls the derivatives of $\rho - v^1$ and v^2. The quantity $\mathbb{Q}_N^{(Partial)}$ might seem to be unnecessary, but it plays an important role in our energy estimates. The reason is that the most degenerate error terms in the energy estimates for $\mathbb{Q}_N^{(Partial)}$ are multiplied by a small factor ε; see (15.2.2). This is important because $\mathbb{Q}_N^{(Partial)}$ appears as a large coefficient (denoted by C_*) source term in the energy estimates for $\mathbb{Q}_N$; see (15.2.1a). The smallness of ε compensates for the largeness of C_* and allows us to close our energy estimates; see also the discussion in Subsubsect. 2.4.6. Similar remarks apply to the integrals of Def. 12.2.

The following spacetime integrals are indispensable for controlling the geometric torus derivatives of $\vec{\Psi}$. Their key property is that they are strong in regions where μ is small; see Lemma 12.4. Recall that these spacetime integrals are featured in the basic energy identity for solutions to the wave equation; see (4.3.2).

Definition 12.2 (**Key coercive spacetime integrals**). Let $\mathcal{I}^{N;\leq 1}$ be the multi-index set from Def. 9.1. We associate the following integrals to Ψ, where $[L\mu]_- = |L\mu|$ when $L\mu < 0$ and $[L\mu]_- = 0$ when $L\mu \geq 0$:

$$\mathbb{K}[\Psi](t,u) := \frac{1}{2}\int_{\mathcal{M}_{t,u}} [L\mu]_- |\not{d}\Psi|^2 \, d\varpi, \tag{12.1.2a}$$

$$\mathbb{K}_N(t,u) := \max_{\substack{\vec{I}\in\mathcal{I}^{N;\leq 1} \\ \Psi\in\{\rho - v^1, v^1, v^2\}}} \mathbb{K}[\mathscr{Z}^{\vec{I}}\Psi](t,u), \tag{12.1.2b}$$

$$\mathbb{K}_{[1,N]}(t,u) := \max_{1\leq M\leq N} \mathbb{K}_M(t,u), \tag{12.1.2c}$$

$$\mathbb{K}_N^{(Partial)}(t,u) := \max_{\substack{\vec{I}\in\mathcal{I}^{N;\leq 1} \\ \Psi\in\{\rho - v^1, v^2\}}} \mathbb{K}[\mathscr{Z}^{\vec{I}}\Psi](t,u), \tag{12.1.2d}$$

$$\mathbb{K}_{[1,N]}^{(Partial)}(t,u) := \max_{1\leq M\leq N} \mathbb{K}_M^{(Partial)}(t,u). \tag{12.1.2e}$$

12.2. **Comparison of area forms and estimates for the L^2- norm of time integrals.** We now provide some preliminary lemmas that we will use in our L^2 analysis.

Lemma 12.1 (**Pointwise estimates for υ**). *Let υ be the metric component from Def. 3.26. Under the data-size and bootstrap assumptions of Sects. 8.1-8.4 and the smallness assumptions of Sect. 8.6, the following estimate holds on $\mathcal{M}_{T_{(Boot)},U_0}$:*

$$\upsilon = 1 + \mathcal{O}(\varepsilon). \tag{12.2.1}$$

Proof. See Sect. 9.2 for some comments on the analysis. Using the identity (3.17.1c) and the estimate (9.6.8), we deduce that $L \ln \upsilon = \mathcal{O}(\varepsilon)$. Integrating this estimate along the integral curves of L as in (9.3.5), we find that $\ln \upsilon(t,u,\vartheta) = \ln \upsilon(0,u,\vartheta) + \mathcal{O}(\varepsilon)$. From this estimate and the small-data bound $\upsilon(0,u,\vartheta) = 1 + \mathcal{O}(\varepsilon)$, which we derive just below, we conclude the desired bound (12.2.1). To derive the small-data bound, we use (3.5.1)-(3.5.3), the small-data bound (8.2.6), and the bootstrap assumptions ($\mathbf{BA}\vec{\Psi}$) to obtain $\upsilon^2|_{t=0} = g(\Theta, \Theta)|_{t=0} = g_{22}|_{t=0} + \mathcal{O}(\mathring{\epsilon}) = 1 + \mathcal{O}(\varepsilon)$, which implies the desired bound. □

Lemma 12.2 (**Comparison of the forms** $d\lambda_{\not g}$ **and** $d\vartheta$). *Let $p = p(\vartheta)$ be a non-negative function of ϑ. Then the following estimates hold for $(t, u) \in [0, T_{(Boot)}) \times [0, U_0]$:*

$$(1 - C\varepsilon) \int_{\vartheta \in \mathbb{T}} p(\vartheta) d\vartheta \leq \int_{\ell_{t,u}} p(\vartheta) d\lambda_{\not g(t,u,\vartheta)} \leq (1 + C\varepsilon) \int_{\vartheta \in \mathbb{T}} p(\vartheta) d\vartheta, \tag{12.2.2}$$

where $d\vartheta$ denotes the standard integration measure on $\mathbb{T}$.

Furthermore, let $p = p(u', \vartheta)$ be a non-negative function of $(u', \vartheta) \in [0, u] \times \mathbb{T}$ that ***does not depend on*** *t. Then for $s, t \in [0, T_{(Boot)})$ and $u \in [0, U_0]$, we have:*

$$(1 - C\varepsilon) \int_{\Sigma_s^u} p \, d\underline{\varpi} \leq \int_{\Sigma_t^u} p \, d\underline{\varpi} \leq (1 + C\varepsilon) \int_{\Sigma_s^u} p \, d\underline{\varpi}. \tag{12.2.3}$$

Proof. See Sect. 9.2 for some comments on the analysis. From (4.1.1) and inequality (12.2.1), we deduce that $d\lambda_{\not g} = (1 + \mathcal{O}(\varepsilon)) \, d\vartheta$, which yields (12.2.2). (12.2.3) then follows from (12.2.2) and the fact that $d\underline{\varpi} = d\lambda_{\not g(t,u,\vartheta)} du'$ along Σ_t^u. □

Lemma 12.3 (**Estimate for the norm** $\|\cdot\|_{L^2(\Sigma_t^u)}$ **of time-integrated functions**). *Let f be a scalar-valued function on $\mathcal{M}_{T_{(Boot)}, U_0}$ and let*

$$F(t, u, \vartheta) := \int_{t'=0}^{t} f(t', u, \vartheta) \, dt'. \tag{12.2.4}$$

Under the data-size and bootstrap assumptions of Sects. 8.1-8.4 and the smallness assumptions of Sect. 8.6, the following estimate holds for $(t, u) \in [0, T_{(Boot)}) \times [0, U_0]$:

$$\|F\|_{L^2(\Sigma_t^u)} \leq (1 + C\varepsilon) \int_{t'=0}^{t} \|f\|_{L^2(\Sigma_{t'}^u)} \, dt'. \tag{12.2.5}$$

Proof. Recall that $\|F\|_{L^2(\Sigma_t^u)} := \left\{ \int_{u'=0}^{u} \int_{\ell_{t,u'}} F^2(t, u', \vartheta) \, d\lambda_{\not g} \, du' \right\}^{1/2}$. Using the estimate (12.2.2), we may replace $d\lambda_{\not g}$ in the previous formula with the standard integration measure $d\vartheta$ up to an overall multiplicative error factor of $1 + \mathcal{O}(\varepsilon)$. The desired estimate (12.2.5) follows from this estimate and from applying Minkowski's inequality for integrals to equation (12.2.4). □

12.3. **The coerciveness of the fundamental square-integral controlling quantities.** In this section, we quantify the coercive nature of the fundamental L^2-controlling quantities.

We start by quantifying the coercive nature of the spacetime integrals of Def. 12.2.

Lemma 12.4 (**Strength of the coercive spacetime integral**). *Let $\mathbf{1}_{\{\mu \leq 1/4\}}$ denote the characteristic function of the spacetime subset $\{(t, u, \vartheta) \in [1, \infty) \times [0, 1] \times \mathbb{T} \mid \mu(t, u, \vartheta) \leq 1/4\}$. Under the data-size and bootstrap assumptions of Sects. 8.1-8.4 and the smallness assumptions of Sect. 8.6, the following lower bound holds for $(t, u) \in [0, T_{(Boot)}) \times [0, U_0]$:*

$$\mathbb{K}[\Psi](t, u) \geq \frac{1}{8} \mathring{\delta}_* \int_{\mathcal{M}_{t,u}} \mathbf{1}_{\{\mu \leq 1/4\}} |\not d\Psi|^2 \, d\varpi. \tag{12.3.1}$$

Proof. Inequality (12.3.1) follows from definition (12.1.2a) and the estimate (11.2.2). □

We now quantify the coercivity of the fundamental L^2-controlling quantities from Def. 12.1.

Lemma 12.5 (**The coercivity of the fundamental controlling quantities**). *Assume that $1 \leq N \leq 20$ and $0 \leq M \leq \min\{N-1, 1\}$. Under the assumptions of Lemma 12.4, the following lower bounds hold for $(t,u) \in [0, T_{(Boot)}) \times [0, U_0]$:*

$$\mathbb{Q}_N(t,u) \geq \max_{\Psi \in \{\rho - v^1, v^1, v^2\}} \Big\{ \frac{1}{2} \left\| \sqrt{\mu} L \mathscr{Z}_*^{N;M} \Psi \right\|_{L^2(\Sigma_t^u)}^2, \left\| \breve{X} \mathscr{Z}_*^{N;M} \Psi \right\|_{L^2(\Sigma_t^u)}^2, \frac{1}{2} \left\| \sqrt{\mu} {d\mkern-9mu /} \mathscr{Z}_*^{N;M} \Psi \right\|_{L^2(\Sigma_t^u)}^2, \tag{12.3.2a}$$
$$\left\| L \mathscr{Z}_*^{N;M} \Psi \right\|_{L^2(\mathcal{P}_u^t)}^2, \left\| \sqrt{\mu} {d\mkern-9mu /} \mathscr{Z}_*^{N;M} \Psi \right\|_{L^2(\mathcal{P}_u^t)}^2 \Big\},$$

$$\mathbb{Q}_N(t,u) \geq \max \Big\{ \frac{1}{8} \left\| \sqrt{\mu} L \mathscr{Z}_*^{N;M} \rho \right\|_{L^2(\Sigma_t^u)}^2, \frac{1}{4} \left\| \breve{X} \mathscr{Z}_*^{N;M} \rho \right\|_{L^2(\Sigma_t^u)}^2, \frac{1}{8} \left\| \sqrt{\mu} {d\mkern-9mu /} \mathscr{Z}_*^{N;M} \rho \right\|_{L^2(\Sigma_t^u)}^2, \tag{12.3.2b}$$
$$\frac{1}{4} \left\| L \mathscr{Z}_*^{N;M} \rho \right\|_{L^2(\mathcal{P}_u^t)}^2, \frac{1}{4} \left\| \sqrt{\mu} {d\mkern-9mu /} \mathscr{Z}_*^{N;M} \rho \right\|_{L^2(\mathcal{P}_u^t)}^2 \Big\},$$

$$\mathbb{Q}_N^{(Partial)}(t,u) \geq \max_{\Psi \in \{\rho - v^1, v^2\}} \Big\{ \frac{1}{2} \left\| \sqrt{\mu} L \mathscr{Z}_*^{N;M} \Psi \right\|_{L^2(\Sigma_t^u)}^2, \left\| \breve{X} \mathscr{Z}_*^{N;M} \Psi \right\|_{L^2(\Sigma_t^u)}^2, \frac{1}{2} \left\| \sqrt{\mu} {d\mkern-9mu /} \mathscr{Z}_*^{N;M} \Psi \right\|_{L^2(\Sigma_t^u)}^2, \tag{12.3.3}$$
$$\left\| L \mathscr{Z}_*^{N;M} \Psi \right\|_{L^2(\mathcal{P}_u^t)}^2, \left\| \sqrt{\mu} {d\mkern-9mu /} \mathscr{Z}_*^{N;M} \Psi \right\|_{L^2(\mathcal{P}_u^t)}^2 \Big\}.$$

In addition, if $N \leq 21$, then

$$\mathbb{V}_N(t,u) \geq \max \Big\{ \left\| \sqrt{\mu} \mathscr{P}^N \omega \right\|_{L^2(\Sigma_t^u)}^2, \left\| \mathscr{P}^N \omega \right\|_{L^2(\mathcal{P}_u^t)}^2 \Big\}. \tag{12.3.4}$$

Moreover, if $1 \leq N \leq 20$ and $0 \leq M \leq 1$, then

$$\left\| \vec{\Psi} \right\|_{L^2(\Sigma_t^u)}^2 \leq C \mathring{\epsilon}^2 + C \mathbb{Q}_1(t,u), \tag{12.3.5a}$$

$$\left\| \mathscr{Z}_*^{N;M} \vec{\Psi} \right\|_{L^2(\Sigma_t^u)}^2, \left\| \mathscr{Z}_*^{N;M} \vec{\Psi} \right\|_{L^2(\ell_{t,u})}^2 \leq C \mathring{\epsilon}^2 + C \mathbb{Q}_N(t,u), \tag{12.3.5b}$$

$$\left\| \breve{X} \vec{\Psi} \right\|_{L^2(\Sigma_t^u)}^2 \leq C \left\| \breve{X} \vec{\Psi} \right\|_{L^2(\Sigma_0^u)}^2 + C \mathring{\epsilon}^2 + C \mathbb{Q}_1(t,u), \tag{12.3.5c}$$

$$\left\| \breve{X} \breve{X} \vec{\Psi} \right\|_{L^2(\Sigma_t^u)}^2 \leq C \left\| \breve{X} \breve{X} \vec{\Psi} \right\|_{L^2(\Sigma_0^u)}^2 + C \mathring{\epsilon}^2 + C \mathbb{Q}_2(t,u). \tag{12.3.5d}$$

Finally, if $N \leq 20$, then

$$\left\| \mathscr{P}^{\leq N} \omega \right\|_{L^2(\ell_{t,u})}^2 \leq C \mathring{\epsilon}^2 + C \mathbb{V}_{\leq N+1}(t,u). \tag{12.3.6}$$

Remark 12.2. The constants 1 and $\frac{1}{2}$ in front of the term $\left\| \breve{X} \mathscr{Z}_*^{N;M} \Psi \right\|_{L^2(\Sigma_t^u)}^2$ and the term $\frac{1}{2} \left\| \sqrt{\mu} {d\mkern-9mu /} \mathscr{Z}_*^{N;M} \Psi \right\|_{L^2(\Sigma_t^u)}^2$ on RHS (12.3.2a) influence the blowup-rate of our top-order energy estimates. In turn, this affects the number of derivatives that we need to close our estimates.

Proof of Lemma 12.5. The estimates stated in (12.3.4) follow easily from Defs. 12.1 and 4.3.

The estimates stated in (12.3.2a) follow easily from Lemma 4.1, Def. 12.1, and Young's inequality.

The estimates stated in (12.3.3) follow easily from (12.3.2a) and the triangle inequality.

We now prove (12.3.5b). We first note that the estimates for $\left\| \mathscr{Z}_*^{N;M} \vec{\Psi} \right\|_{L^2(\Sigma_t^u)}^2$ follow easily from integrating the estimates for $\left\| \mathscr{Z}_*^{N;M} \vec{\Psi} \right\|_{L^2(\ell_{t,u})}^2$ with respect to u. Hence, it suffices to prove the estimates for $\left\| \mathscr{Z}_*^{N;M} \right\|_{L^2(\ell_{t,u})}^2$. Our proof is based on the identity (4.4.1b). To proceed, we first use (3.13.4), the estimate (9.4.1a) and the L^∞ estimates of Prop. 9.12 to bound the factor $(1/2)\mathrm{tr}_{\not{g}}{}^{(\breve{X})}\not{\pi}$ in (4.4.1b) as follows: $(1/2)\left|\mathrm{tr}_{\not{g}}{}^{(\breve{X})}\not{\pi}\right| \lesssim |\not{\mathcal{L}}_{\breve{X}} \not{g}| \lesssim 1$. Using this estimate, the identity (4.4.1b) with $f = (\mathscr{Z}_*^{N;M}\Psi)^2$, and Young's inequality, we deduce that

$$\begin{aligned} \left\| \mathscr{Z}_*^{N;M} \Psi \right\|_{L^2(\ell_{t,u})}^2 \leq \left\| \mathscr{Z}_*^{N;M} \Psi \right\|_{L^2(\ell_{t,0})}^2 + \int_{u'=0}^{u} \left\| \breve{X} \mathscr{Z}_*^{N;M} \Psi \right\|_{L^2(\ell_{t,u'})}^2 du' \\ + C \int_{u'=0}^{u} \left\| \mathscr{Z}_*^{N;M} \Psi \right\|_{L^2(\ell_{t,u'})}^2 du'. \end{aligned} \tag{12.3.7}$$

From (12.3.7), the smallness assumption (8.1.7), and Gronwall's inequality, we deduce

$$\begin{aligned} \left\| \mathscr{Z}_*^{N;M} \right\|_{L^2(\ell_{t,u})}^2 &\leq e^{cu} \mathring{\epsilon}^2 + e^{cu} \int_{u'=0}^{u} \left\| \breve{X} \mathscr{Z}_*^{N;M} \Psi \right\|_{L^2(\ell_{t,u'})}^2 du' \\ &= e^{cu} \mathring{\epsilon}^2 + C e^{cu} \left\| \breve{X} \mathscr{Z}_*^{N;M} \Psi \right\|_{L^2(\Sigma_t^u)}^2 \leq C \mathring{\epsilon}^2 + C \left\| \breve{X} \mathscr{Z}_*^{N;M} \Psi \right\|_{L^2(\Sigma_t^u)}^2. \end{aligned} \tag{12.3.8}$$

The desired bound for $\left\| \mathscr{Z}_*^{N;M} \Psi \right\|_{L^2(\ell_{t,u})}^2$ now follows from (12.3.8) and the already proven estimate (12.3.2a) for $\left\| \breve{X} \mathscr{Z}_*^{N;M} \Psi \right\|_{L^2(\Sigma_t^u)}^2$.

We now prove (12.3.6). We first use (4.4.1a) with $f = (\mathscr{P}^N \omega)^2$, the estimate (9.6.8), and Young's inequality to deduce

$$\frac{\partial}{\partial t} \left\| \mathscr{P}^N \omega \right\|_{L^2(\ell_{t,u})}^2 \leq \left\| L \mathscr{P}^N \omega \right\|_{L^2(\ell_{t,u})}^2 + C \left\| \mathscr{P}^N \omega \right\|_{L^2(\ell_{t,u})}^2. \tag{12.3.9}$$

Integrating (12.3.9) from the initial time 0 to time t, using Gronwall's inequality, and using the small data assumption (8.1.6), we obtain the desired bound (12.3.6) as follows:

$$\left\| \mathscr{P}^N \omega \right\|_{L^2(\ell_{t,u})}^2 \lesssim \left\| \mathscr{P}^N \omega \right\|_{L^2(\ell_{0,u})}^2 + \int_{s=0}^{t} \left\| L \mathscr{P}^N \omega \right\|_{L^2(\ell_{s,u})}^2 ds \lesssim \mathring{\epsilon}^2 + \mathbb{V}_{N+1}(t,u). \tag{12.3.10}$$

To prove (12.3.5a), we first use the fundamental theorem of calculus to express

$$\vec{\Psi}(t,u,\vartheta) = \vec{\Psi}(0,u,\vartheta) + \int_{t'=0}^{t} L \vec{\Psi}(t',0,\vartheta)\, dt'. \tag{12.3.11}$$

From (12.3.11), (12.2.5), and the estimate $\left\| L\vec{\Psi} \right\|_{L^2(\Sigma_t^u)} \lesssim \sqrt{\mathbb{Q}_1}(t,u) + \mathring{\upepsilon}$ (which is a particular case of the already proven bound (12.3.5b)), we deduce

$$\begin{aligned} \|\vec{\Psi}\|_{L^2(\Sigma_t^u)} &\lesssim \|\vec{\Psi}(0,\cdot)\|_{L^2(\Sigma_t^u)} + \int_{t'=0}^{t} \left\| L\vec{\Psi} \right\|_{L^2(\Sigma_{t'}^u)} \, dt' \qquad (12.3.12) \\ &\lesssim \|\vec{\Psi}(0,\cdot)\|_{L^2(\Sigma_t^u)} + \int_{t'=0}^{t} \sqrt{\mathbb{Q}_1}(t',u) \, dt' + \mathring{\upepsilon}. \end{aligned}$$

From (12.2.3) with $s = 0$ and the smallness assumption (8.1.2), we deduce that $\|\vec{\Psi}(0,\cdot)\|_{L^2(\Sigma_t^u)} \lesssim \|\vec{\Psi}\|_{L^2(\Sigma_0^u)} \lesssim \mathring{\upepsilon}$. Moreover, using the fact that $\mathbb{Q}_1$ is increasing in its arguments, we deduce that $\int_{t'=0}^{t} \sqrt{\mathbb{Q}_1}(t',u) \, dt' \lesssim \mathbb{Q}_1^{1/2}(t,u)$. Inserting the above estimates into RHS (12.3.12), we arrive at the desired bound (12.3.5a).

The remaining estimates (12.3.5c) and (12.3.5d) follow from arguments similar to the ones we used to prove (12.3.5a) and we omit those details. □

13. Sobolev embedding

Our main goal in this section is to prove Cor. 13.2, which is the Sobolev embedding result that we will use to improve the fundamental bootstrap assumptions ($\mathbf{BA}\vec{\Psi}$)-($\mathbf{BA}\upomega$).

Lemma 13.1 (Sobolev embedding along $\ell_{t,u}$). *Under the data-size and bootstrap assumptions of Sects. 8.1-8.4 and the smallness assumptions of Sect. 8.6, the following estimate holds for scalar-valued functions f defined on $\ell_{t,u}$ for $(t,u) \in [0, T_{(Boot)}) \times [0, U_0]$:*

$$\|f\|_{L^\infty(\ell_{t,u})} \leq C \left\| Y^{\leq 1} f \right\|_{L^2(\ell_{t,u})}. \qquad (13.0.13)$$

Proof. Standard Sobolev embedding yields that $\|f\|_{L^\infty(\mathbb{T})} \leq C \left\| \Theta^{\leq 1} f \right\|_{L^2(\mathbb{T})}$, where the integration measure defining $\|\cdot\|_{L^2(\mathbb{T})}$ is $d\vartheta$. From Def. 3.26, (9.4.2a), Lemma 12.1, and the L^∞ estimates of Prop. 9.12, we find that $|\Theta^{\leq 1} f| = (1 + \mathcal{O}(\varepsilon))|Y^{\leq 1} f|$. From these estimates and Lemma 12.2, we conclude the desired estimate (13.0.13).

□

Corollary 13.2 (L^∞ bounds for $\vec{\Psi}$ and $\upomega$ in terms of the fundamental controlling quantities). *Under the assumptions of Lemma 13.1, the following estimates hold for $(t,u) \in [0, T_{(Boot)}) \times [0, U_0]$:*

$$\left\| \mathscr{Z}_*^{\leq 13; \leq 1} \vec{\Psi} \right\|_{L^\infty(\Sigma_t^u)} \lesssim \mathbb{Q}_{[1,14]}^{1/2}(t,u) + \mathring{\upepsilon}, \qquad (13.0.14a)$$

$$\left\| \mathscr{P}^{\leq 13} \upomega \right\|_{L^\infty(\Sigma_t^u)} \lesssim \mathbb{V}_{\leq 15}^{1/2}(t,u) + \mathring{\upepsilon}. \qquad (13.0.14b)$$

Proof. See Sect. 9.2 for some comments on the analysis. The bound $\left\| \mathscr{Z}_*^{[1,13]; \leq 1} \vec{\Psi} \right\|_{L^\infty(\Sigma_t^u)} \lesssim \mathbb{Q}_{[1,14]}^{1/2}(t,u) + \mathring{\upepsilon}$ follows from (12.3.5b) and Lemma 13.1. This estimate implies in particular the pointwise bound $\left| L\vec{\Psi} \right| \lesssim \mathbb{Q}_{[1,14]}^{1/2}(t,u) + \mathring{\upepsilon}$. Integrating along the integral curves of L as in

(9.3.5) and using this bound and the small-data assumption $\left\| \vec{\Psi} \right\|_{L^\infty(\Sigma_0^u)} \leq \mathring{\upepsilon}$ (see (8.1.3a)), we deduce that $\left\| \vec{\Psi} \right\|_{L^\infty(\Sigma_t^u)} \leq C \mathbb{Q}_{[1,14]}^{1/2}(t,u) + C\mathring{\upepsilon}$. We have thus proved (13.0.14a).

The estimate (13.0.14b) is a direct consequence of Lemma 13.1 and inequality (12.3.6). □

14. Pointwise estimates for the error integrands

In order to derive a priori estimates for the fundamental L^2-controlling quantities of Defs. 12.1 and 12.2, we must first obtain pointwise estimates for the error terms in the energy identities corresponding to the wave equations verified by $\mathscr{Z}_*^{[1,20];\leq 1}(\uprho - v^1)$, $\mathscr{Z}_*^{[1,20];\leq 1} v^1$, $\mathscr{Z}_*^{[1,20];\leq 1} v^2$, and the transport equations verified by $\mathscr{P}^{\leq 21}\upomega$. By "energy identities," we mean the ones provided by Props. 4.2 and 4.4. In this section, we derive these pointwise estimates. The error terms consist of the following three types, ordered in increasing difficulty: **i)** Error terms generated by differentiating the inhomogeneous terms on RHSs (3.3.11a)-(3.3.11b); **ii)** Error terms corresponding to the last integral on RHS (4.3.6); **ii')** Error terms corresponding to the deformation tensor of the multiplier vectorfields, which correspond to the last integral on RHS (4.3.1); and **iii)** Error terms generated by the commutator terms of the form $[\upmu\square_g, \mathscr{Z}_*^{N;\leq 1}]\Psi$ and $[\upmu B, \mathscr{P}^N]\upomega$ and their derivatives up to top-order.

We prove the two main propositions in Sects. 14.9 and 14.10. The rest of Sect. 14 consists of preliminary estimates.

14.1. Harmless terms.

We start by defining error terms of type $Harmless_{(Wave)}^{\leq N}$, which appear in the energy estimates for the wave variables, and of type $Harmless_{(Vort)}^{\leq N}$, which appear in the energy estimates for the specific vorticity. These terms have a negligible effect on the dynamics, even near the shock. Most error terms that we encounter are of these types.

Definition 14.1 (Harmless terms). $Harmless_{(Wave)}^{\leq N}$ and $Harmless_{(Vort)}^{\leq N}$ denote any terms such that under the data-size and bootstrap assumptions of Sects. 8.1-8.4 and the smallness assumptions of Sect. 8.6, the following bound holds on $\mathcal{M}_{T_{(Boot)},U_0}$, where $1 \leq N \leq 20$ in (14.1.1a) and $N \leq 21$ in (14.1.1b) (see Sect. 6.2 regarding the vectorfield operator notation):

$$\left| Harmless_{(Wave)}^{\leq N} \right| \lesssim \left| \mathscr{Z}_*^{[1,N+1];\leq 2} \vec{\Psi} \right| + \left| \mathscr{Z}_*^{[1,N];\leq 2} \upgamma \right| + \left| \mathscr{Z}_{**}^{[1,N];\leq 1} \underline{\upgamma} \right| + \left| \mathscr{Z}^{\leq N;\leq 1} \upomega \right|, \quad (14.1.1a)$$

$$\left| Harmless_{(Vort)}^{\leq N} \right| \lesssim \varepsilon \left| \mathscr{Z}_*^{[1,N];\leq 2} \vec{\Psi} \right| + \varepsilon \left| \mathscr{Z}_*^{[1,N-1];\leq 2} \upgamma \right| + \varepsilon \left| \mathscr{Z}_{**}^{[1,N-1];\leq 1} \underline{\upgamma} \right| + \left| \mathscr{P}^{\leq N} \upomega \right|. \quad (14.1.1b)$$

By definition, the first three terms on RHS (14.1.1b) are absent when $N = 0$ and the second and third terms on RHS (14.1.1b) are absent when $N = 1$.

Remark 14.1 (The role of the factors ε). The smallness factors ε on RHS (14.1.1b) allow us to derive (see Lemma 15.21), via a bootstrap argument, energy estimates for $\upomega$ without having to simultaneously derive energy estimates for $\vec{\Psi}$. This allows us to obtain energy estimates for $\upomega$ in a much simpler way compared to the energy estimates for $\vec{\Psi}$.

14.2. **Identification of the difficult error terms in the commuted equations.** In the next proposition, which we prove in Sect. 14.9, we identify the main error terms in the inhomogeneous wave equations verified by the higher-order versions of $\uprho - v^1$, v^1, and v^2. The main terms will require careful treatment in the energy estimates, while the terms denoted by $Harmless_{(Wave)}^{\leq N}$ will be easy to control.

Proposition 14.1 (**Identification of the key difficult error term factors in the commuted wave equations**). *Assume that $1 \leq N \leq 20$ and recall that y is the scalar-valued appearing in Lemma 3.8. Under the data-size and bootstrap assumptions of Sects. 8.1-8.4 and the smallness assumptions of Sect. 8.6, the following pointwise estimates hold for $i = 1, 2$ on $\mathcal{M}_{T_{(Boot)}, U_0}$ (see Sect. 6.2 regarding the vectorfield operator notation):*

$$
\begin{aligned}
\upmu \square_g (Y^{N-1} L v^i) &= (\rlap{/}d^{\#} v^i) \cdot (\upmu \rlap{/}d Y^{N-1} \mathrm{tr}_{\rlap{/}g} \upchi) \\
&\quad + [ia] \upmu (\exp \uprho) c_s^2 (g_{ab} X^b) Y^{N-1} L L \upomega \\
&\quad - [ia] \upmu (\exp \uprho) c_s^2 \left(\frac{g_{ab} Y^b}{g_{cd} Y^c Y^d} \right) Y^{N-1} L Y \upomega \\
&\quad + Harmless_{(Wave)}^{\leq N},
\end{aligned}
\tag{14.2.1a}
$$

$$
\begin{aligned}
\upmu \square_g (Y^N v^i) &= (\breve{X} v^i) Y^N \mathrm{tr}_{\rlap{/}g} \upchi + y (\rlap{/}d^{\#} v^i) \cdot (\upmu \rlap{/}d Y^{N-1} \mathrm{tr}_{\rlap{/}g} \upchi) \\
&\quad + [ia] \upmu (\exp \uprho) c_s^2 (g_{ab} X^b) Y^N L \upomega \\
&\quad - [ia] \upmu (\exp \uprho) c_s^2 \left(\frac{g_{ab} Y^b}{g_{cd} Y^c Y^d} \right) Y^{N+1} \upomega \\
&\quad + Harmless_{(Wave)}^{\leq N}.
\end{aligned}
\tag{14.2.1b}
$$

Moreover, if $2 \leq N \leq 20$, then

$$
\begin{aligned}
\upmu \square_g (Y^{N-1} \breve{X} v^i) &= (\breve{X} v^i) Y^{N-1} \breve{X} \mathrm{tr}_{\rlap{/}g} \upchi - (\upmu \rlap{/}d^{\#} v^i) \cdot (\upmu \rlap{/}d Y^{N-1} \mathrm{tr}_{\rlap{/}g} \upchi) \\
&\quad - [ia] \upmu^2 (\exp \uprho) c_s^2 (g_{ab} X^b) Y^{N-1} L L \upomega \\
&\quad + [ia] \upmu^2 (\exp \uprho) c_s^2 \left(\frac{g_{ab} Y^b}{g_{cd} Y^c Y^d} \right) Y^N L \upomega \\
&\quad + Harmless_{(Wave)}^{\leq N}.
\end{aligned}
\tag{14.2.1c}
$$

Moreover, if $2 \leq N \leq 20$ *and* $\mathscr{Z}^{N-1;1}$ *contains exactly one factor of* $\breve{X}$ *with all other factors equal to* Y*, then we have*

$$\begin{aligned}\upmu\square_g(\mathscr{Z}^{N-1;1}Lv^i) &= (\not{d}^{\#}v^i)\cdot(\upmu\not{d}Y^{N-2}\breve{X}\mathrm{tr}_{\not{g}}\chi) \\ &\quad - [ia]\upmu^2(\exp\rho)c_s^2(g_{ab}X^b)Y^{N-2}LLL\omega \\ &\quad + [ia]\upmu^2(\exp\rho)c_s^2\left(\frac{g_{ab}Y^b}{g_{cd}Y^cY^d}\right)Y^{N-1}LL\omega \\ &\quad + Harmless^{\leq N}_{(Wave)},\end{aligned} \tag{14.2.1d}$$

$$\begin{aligned}\upmu\square_g(\mathscr{Z}^{N-1;1}Yv^i) &= (\breve{X}v^i)Y^{N-1}\breve{X}\mathrm{tr}_{\not{g}}\chi + y(\not{d}^{\#}v^i)\cdot(\upmu\not{d}Y^{N-2}\breve{X}\mathrm{tr}_{\not{g}}\chi) \\ &\quad - [ia]\upmu^2(\exp\rho)c_s^2(g_{ab}X^b)Y^{N-1}LL\omega \\ &\quad + [ia]\upmu^2(\exp\rho)c_s^2\left(\frac{g_{ab}Y^b}{g_{cd}Y^cY^d}\right)Y^{N}L\omega \\ &\quad + Harmless^{\leq N}_{(Wave)}.\end{aligned} \tag{14.2.1e}$$

Furthermore, if $1 \leq N \leq 20$ *and* $\mathscr{P}^{N-1}$ *contains a factor of* L*, then*

$$\begin{aligned}\upmu\square_g(\mathscr{P}^{N-1}Lv^i) &= +[ia]\upmu(\exp\rho)c_s^2(g_{ab}X^b)\mathscr{P}^{N-2}\mathscr{P}^{N-1}LL\omega \\ &\quad - [ia]\upmu(\exp\rho)c_s^2\left(\frac{g_{ab}Y^b}{g_{cd}Y^cY^d}\right)\mathscr{P}^{N-2}\mathscr{P}^{N-1}LY\omega \\ &\quad + Harmless^{\leq N}_{(Wave)}.\end{aligned} \tag{14.2.2a}$$

Likewise, if $1 \leq N \leq 20$ *and* $\mathscr{Z}^{N-1;1}$ *contains one or more factors of* L*, then*

$$\begin{aligned}\upmu\square_g(\mathscr{Z}^{N-1;1}Lv^i) &= -[ia]\upmu^2(\exp\rho)c_s^2(g_{ab}X^b)\mathscr{P}^{N-2}LLL\omega \\ &\quad + [ia]\upmu^2(\exp\rho)c_s^2\left(\frac{g_{ab}Y^b}{g_{cd}Y^cY^d}\right)\mathscr{P}^{N-2}LYL\omega \\ &\quad + Harmless^{\leq N}_{(Wave)}.\end{aligned} \tag{14.2.2b}$$

Similarly, if $1 \leq N \leq 20$ *and* $\mathscr{P}^{N-1}$ *contains one or more factors of* L*, then*

$$\begin{aligned}\upmu\square_g(\mathscr{P}^{N-1}\breve{X}v^i) &= -[ia]\upmu^2(\exp\rho)c_s^2(g_{ab}X^b)\mathscr{P}^{N-1}LL\omega \\ &\quad + [ia]\upmu^2(\exp\rho)c_s^2\left(\frac{g_{ab}Y^b}{g_{cd}Y^cY^d}\right)\mathscr{P}^{N-1}YL\omega \\ &\quad + Harmless^{\leq N}_{(Wave)}.\end{aligned} \tag{14.2.2c}$$

In addition, if $1 \leq N \leq 20$ *and* $\mathscr{Z}_*^{N-1;1}$ *contains one or more factors of* L*, then*

$$\begin{aligned}\upmu\square_g(\mathscr{Z}_*^{N-1;1}Yv^i) &= -[ia]\upmu^2(\exp\rho)c_s^2(g_{ab}X^b)\mathscr{P}^{N-2}YLL\omega \\ &\quad + [ia]\upmu^2(\exp\rho)c_s^2\left(\frac{g_{ab}Y^b}{g_{cd}Y^cY^d}\right)\mathscr{P}^{N-2}YYL\omega \\ &\quad + Harmless^{\leq N}_{(Wave)}.\end{aligned} \tag{14.2.2d}$$

Finally, $\rho - v^1$ *verifies similar estimates according to the following prescription:*

$$\begin{aligned}&\textit{in (14.2.1a)-(14.2.2d) with } i = 1,\\ &\textit{we may replace the explicit factors of } v^1 \textit{ on the LHSs and RHSs with } \rho - v^1\\ &\textit{as long as we change the sign of all explicit } \omega\textit{-containing products on the RHSs.}\end{aligned} \tag{14.2.2e}$$

The next proposition, which we prove in Sect. 14.10, is an analog of Prop. 14.1 for ω. Specifically, the proposition identifies the main error terms in the inhomogeneous transport equations verified by the higher-order versions of ω.

Proposition 14.2 (**Identification of the key difficult error term factors in the commuted transport equation**)**.** *Assume that* $1 \leq N \leq 20$*. Under the data-size and bootstrap assumptions of Sects. 8.1-8.4 and the smallness assumptions of Sect. 8.6, the following pointwise estimates hold on* $\mathcal{M}_{T_{(Boot)},U_0}$ *(see Sect. 6.2 regarding the vectorfield operator notation):*

$$\mu B Y^N L\omega = (Y\omega) Y^{N-1}\breve{X}\mathrm{tr}_{\not g}\chi + Harmless^{\leq N+1}_{(Vort)}, \tag{14.2.3a}$$

$$\mu B Y^{N+1}\omega = -g(Y,Y)(L\omega)Y^{N-1}\breve{X}\mathrm{tr}_{\not g}\chi + y(Y\omega)Y^{N-1}\breve{X}\mathrm{tr}_{\not g}\chi + Harmless^{\leq N+1}_{(Vort)}, \tag{14.2.3b}$$

where y *is the scalar-valued function from* (3.12.8).

Furthermore, if $1 \leq N \leq 20$ *and* $\mathscr{P}^{N+1}$ *is any* $(N+1)^{st}$ *order* $\mathcal{P}_u$*-tangent operator except for* $Y^N L$ *or* Y^{N+1}*, then*

$$\mu B \mathscr{P}^{N+1}\omega = Harmless^{\leq N+1}_{(Vort)}. \tag{14.2.4}$$

Finally, if $P \in \mathscr{P}$*, then*

$$\mu B P\omega = Harmless^{\leq 1}_{(Vort)}. \tag{14.2.5}$$

14.3. **Technical estimates involving the eikonal function quantities.** In this section, we provide two technical lemmas that will allow us to reduce the analysis of the top-order derivatives of μ to those of $\mathrm{tr}_{\not g}\chi$. This is mainly for convenience.

We start with a lemma in which we obtain higher-order analogs of Lemma 7.4.

Lemma 14.3 (**Estimate connecting** $\mathscr{Z}^{\vec{I}}\mathrm{tr}_{\not g}\chi$ **to** $\not\Delta\mathscr{P}^{\vec{J}}\mu$)**.** *Assume that* $1 \leq N \leq 20$*. Let* $\vec{I} \in \mathcal{I}^{N;1}$*, where the multi-index set* $\mathcal{I}^{N;1}$ *is defined in Def. 9.1. Let* $\vec{J}$ *be any multi-index formed by deleting the one entry in* $\vec{I}$ *corresponding to the single* $\breve{X}$ *differentiation and by possibly permuting the remaining entries (and thus* $|\vec{J}| = N-1$ *and the corresponding operator is* $\mathscr{P}^{\vec{J}}$*). Under the data-size and bootstrap assumptions of Sects. 8.1-8.4 and the smallness assumptions of Sect. 8.6, the following estimate holds on* $\mathcal{M}_{T_{(Boot)},U_0}$ *(see Sect. 6.2 regarding the vectorfield operator notation):*

$$\left|\mathscr{Z}^{\vec{I}}\mathrm{tr}_{\not g}\chi - \not\Delta\mathscr{P}^{\vec{J}}\mu\right| \lesssim \left|\mathscr{Z}^{[1,N+1];\leq 1}_{*}\Psi\right| + \left|\begin{pmatrix} \mathscr{Z}^{[1,N];0}_{**}\gamma \\ \mathscr{Z}^{[1,N];\leq 1}_{*}\gamma \end{pmatrix}\right|. \tag{14.3.1}$$

Proof. See Sect. 9.2 for some comments on the analysis. First, using the commutator estimate (9.5.7b) with $f = \mathrm{tr}_{\not g}\chi$, N in the role of $N+1$, and $M = 1$, the estimate (9.4.1c), and the L^∞ estimates of Prop. 9.12, we write $\mathscr{Z}^{\vec{I}}\mathrm{tr}_{\not g}\chi = \mathscr{P}^{\vec{J}}\breve{X}\mathrm{tr}_{\not g}\chi$ plus error terms with magnitudes $\lesssim$ RHS (14.3.1). Next, we apply $\mathscr{P}^{\vec{J}}$ to (7.3.1). Using the estimates (9.3.3b)-(9.3.3c) and

(9.4.1b) and the L^∞ estimates of Prop. 9.12, we write $\mathscr{P}^{\vec{J}}\breve{X}\mathrm{tr}_{\not g}\chi = \mathscr{P}^{\vec{J}}\not\Delta\upmu$ plus error terms with magnitudes $\lesssim$ RHS (14.3.1). Finally, we use the commutator estimate (9.5.8b) with $f = \upmu$, $N-1$ in the role of N, and $M = 0$ and the L^∞ estimates of Prop. 9.12 to write $\mathscr{P}^{\vec{J}}\not\Delta\upmu = \not\Delta\mathscr{P}^{\vec{J}}\upmu$ plus error terms with magnitudes $\lesssim$ RHS (14.3.1). Combining the above estimates, we conclude (14.3.1). □

Lemma 14.4 (**Connecting derivatives of $\upmu$ to derivatives of $\mathrm{tr}_{\not g}\chi$ up to error terms**). *Assume that $1 \leq N \leq 20$. Under the data-size and bootstrap assumptions of Sects. 8.1-8.4 and the smallness assumptions of Sect. 8.6, the following estimates hold on $\mathcal{M}_{T_{(Boot)},U_0}$ (see Sect. 6.2 regarding the vectorfield operator notation):*

$$\left|Y^{N+1}\upmu - g(Y,Y)Y^{N-1}\breve{X}\mathrm{tr}_{\not g}\chi\right|, \left|\not d^{\#}Y^N\upmu - (Y^{N-1}\breve{X}\mathrm{tr}_{\not g}\chi)Y\right| \tag{14.3.2}$$
$$\lesssim \left|\mathscr{Z}_{*}^{[1,N+1];\leq 1}\vec{\Psi}\right| + \left|\begin{pmatrix} \mathscr{Z}_{**}^{[1,N];0}\underline{\gamma} \\ \mathscr{Z}_{*}^{[1,N];\leq 1}\gamma \end{pmatrix}\right|.$$

Proof. See Sect. 9.2 for some comments on the analysis. We start by proving (14.3.2) for the first term on the LHS. Using (9.1.1a) with $f = Y^{N-1}\upmu$, we obtain

$$Y^{N+1}\upmu = g(Y,Y)\not\Delta Y^{N-1}\upmu + \{Y\ln g(Y,Y)\}\,Y^N\upmu. \tag{14.3.3}$$

Using (14.3.1) and $|Y| = 1 + \mathcal{O}(\varepsilon)$ (which is a simple consequence of (9.4.2a) and the L^∞ estimates of Prop. 9.12), we deduce that $g(Y,Y)\not\Delta Y^{N-1}\upmu = g(Y,Y)Y^{N-1}\breve{X}\mathrm{tr}_{\not g}\chi$ plus error terms that are bounded in magnitude by $\lesssim$ RHS (14.3.2). Next, we use Lemma 3.19 and the L^∞ estimates of Prop. 9.12 to deduce that $|Y\ln g(Y,Y)| = |Y\mathrm{f}(\gamma)| \lesssim \varepsilon$. It follows that the last product on RHS (14.3.3) is bounded in magnitude by $\lesssim |\mathscr{Z}_{**}^{[1,N];0}\underline{\gamma}|$. We have thus obtained the desired estimate.

We now prove (14.3.2) for $\left|\not d^{\#}Y^N\upmu - (Y^{N-1}\breve{X}\mathrm{tr}_{\not g}\chi)Y\right|$. By (3.20.1), we have $\not d^{\#}Y^N\upmu = \frac{1}{g(Y,Y)}(Y^{N+1}\upmu)Y$. The desired estimate is therefore a simple consequence of the estimates we obtained above for $\left|Y^{N+1}\upmu - g(Y,Y)Y^{N-1}\breve{X}\mathrm{tr}_{\not g}\chi\right|$ and the estimate $|Y| = 1 + \mathcal{O}(\varepsilon)$ noted above. □

14.4. **Pointwise estimates for the deformation tensors of the commutation vectorfields.** In the next lemma, we identify the main terms in various derivatives of the frame components of the deformation tensors of the commutation vectorfields defined in (3.12.3). The main terms are located on the left-hand sides of the estimates stated in the lemma, while the right-hand sides of the estimates contain simple error terms that will be easy to bound in the energy estimates. The main terms involve top-order derivatives of the eikonal function quantities and are difficult to control in the energy estimates.

Lemma 14.5 (**Identification of the important terms in ${}^{(L)}\pi$, ${}^{(\breve{X})}\pi$, and ${}^{(Y)}\pi$**). *Assume that $1 \leq N \leq 20$. Under the data-size and bootstrap assumptions of Sects. 8.1-8.4 and the smallness assumptions of Sect. 8.6, the following estimates hold on $\mathcal{M}_{T_{(Boot)},U_0}$ (see Sect. 6.2 regarding the vectorfield operator notation).*

Important terms in the derivatives of $^{(L)}\pi$. *For* $M = 0, 1$, *we have*

$$\left| \mathscr{Z}^{N-1;M} \breve{X} \mathrm{tr}_{\not g} {}^{(L)}\not\pi - 2 \not\Delta \mathscr{Z}^{N-1;M} \upmu \right| \lesssim \left| \mathscr{Z}_*^{[1,N+1];\leq 2} \Psi \right| + \left| \begin{pmatrix} \mathscr{Z}_{**}^{[1,N];\leq 1} \underline{\gamma} \\ \mathscr{Z}_*^{[1,N];\leq 2} \gamma \end{pmatrix} \right|, \tag{14.4.1a}$$

$$\begin{gathered} \left| \not{\mathcal{L}}_{\mathscr{Z}}^{N-1;M} \not d^{\#(L)} \pi_{L\breve{X}} \right|, \left| \mathscr{Z}^{N-1;M} \not{\mathrm{div}}^{(L)} \not\pi_{\breve{X}}^{\#} - \not\Delta \mathscr{Z}^{N-1;M} \upmu \right|, \left| \not{\mathcal{L}}_{\mathscr{Z}}^{N-1;M} \not d^{\#} \mathrm{tr}_{\not g} {}^{(L)}\not\pi - 2 \not d^{\#} \mathscr{Z}^{N-1;M} \mathrm{tr}_{\not g} \upchi \right| \\ \lesssim \left| \mathscr{Z}_*^{[1,N+1];\leq 2} \Psi \right| + \left| \begin{pmatrix} \mathscr{Z}_{**}^{[1,N];\leq 1} \underline{\gamma} \\ \mathscr{Z}_*^{[1,N];\leq 2} \gamma \end{pmatrix} \right|. \end{gathered} \tag{14.4.1b}$$

Important terms in the derivatives of $^{(\breve{X})}\pi$. *We have*

$$\begin{gathered} \left| \not{\mathcal{L}}_{\mathscr{P}}^{N-1} \not{\mathcal{L}}_{\breve{X}} {}^{(\breve{X})}\not\pi_L^{\#} + \not d^{\#} \mathscr{P}^{N-1} \breve{X} \upmu \right|, \left| \mathscr{P}^{N-1} \breve{X} \mathrm{tr}_{\not g} {}^{(\breve{X})}\not\pi + 2 \upmu \mathscr{P}^{N-1} \breve{X} \mathrm{tr}_{\not g} \upchi \right| \\ \lesssim \left| \mathscr{Z}_*^{[1,N+1];\leq 2} \Psi \right| + \left| \begin{pmatrix} \mathscr{Z}_{**}^{[1,N];\leq 1} \underline{\gamma} \\ \mathscr{Z}_*^{[1,N];\leq 2} \gamma \end{pmatrix} \right| + 1, \end{gathered} \tag{14.4.2a}$$

$$\begin{gathered} \left| \mathscr{P}^{N-1} \not{\mathrm{div}}^{(\breve{X})} \not\pi_L^{\#} + \mathscr{P}^{N-1} \breve{X} \mathrm{tr}_{\not g} \upchi \right|, \\ \left| \not{\mathcal{L}}_{\mathscr{P}}^{N-1} \not d^{\#(\breve{X})} \pi_{L\breve{X}} + \not d^{\#} \mathscr{P}^{N-1} \breve{X} \upmu \right|, \left| \not{\mathcal{L}}_{\mathscr{P}}^{N-1} \not d^{\#} \mathrm{tr}_{\not g} {}^{(\breve{X})}\not\pi + 2 \upmu \not d^{\#} \mathscr{P}^{N-1} \mathrm{tr}_{\not g} \upchi \right| \\ \lesssim \left| \mathscr{Z}_*^{[1,N+1];\leq 2} \Psi \right| + \left| \begin{pmatrix} \mathscr{Z}_{**}^{[1,N];\leq 1} \underline{\gamma} \\ \mathscr{Z}_*^{[1,N];\leq 2} \gamma \end{pmatrix} \right|. \end{gathered} \tag{14.4.2b}$$

Important terms in the derivatives of $^{(Y)}\pi$. *For* $M = 0, 1$, *we have*

$$\begin{gathered} \left| \not{\mathcal{L}}_{\mathscr{Z}}^{N-1;M} \not{\mathcal{L}}_{\breve{X}} {}^{(Y)}\not\pi_L^{\#} + (\not\Delta \mathscr{Z}^{N-1;M} \upmu) Y \right|, \left| \mathscr{Z}^{N-1;M} \breve{X} \mathrm{tr}_{\not g} {}^{(Y)}\not\pi - 2y \not\Delta \mathscr{Z}^{N-1;M} \upmu \right| \\ \lesssim \left| \mathscr{Z}_*^{[1,N+1];\leq 2} \Psi \right| + \left| \begin{pmatrix} \mathscr{Z}_{**}^{[1,N];\leq 1} \underline{\gamma} \\ \mathscr{Z}_*^{[1,N];\leq 2} \gamma \end{pmatrix} \right|, \end{gathered} \tag{14.4.3a}$$

$$\begin{gathered} \left| \mathscr{Z}^{N-1;M} \not{\mathrm{div}}^{(Y)} \not\pi_L^{\#} + Y \mathscr{Z}^{N-1;M} \mathrm{tr}_{\not g} \upchi \right|, \left| \mathscr{Z}^{N-1;M} \not{\mathrm{div}}^{(Y)} \not\pi_{\breve{X}}^{\#} - \left\{ \upmu Y \mathscr{Z}^{N-1;M} \mathrm{tr}_{\not g} \upchi + y \not\Delta \mathscr{Z}^{N-1;M} \upmu \right\} \right|, \\ \left| \not{\mathcal{L}}_{\mathscr{Z}}^{N-1;M} \not d^{\#(Y)} \pi_{L\breve{X}} + (\not\Delta \mathscr{Z}^{N-1;M} \upmu) Y \right|, \left| \not{\mathcal{L}}_{\mathscr{Z}}^{N-1;M} \not d^{\#} \mathrm{tr}_{\not g} {}^{(Y)}\not\pi - 2y \not d^{\#} \mathscr{Z}^{N-1;M} \mathrm{tr}_{\not g} \upchi \right| \\ \lesssim \left| \mathscr{Z}_*^{[1,N+1];\leq 2} \Psi \right| + \left| \begin{pmatrix} \mathscr{Z}_{**}^{[1,N];\leq 1} \underline{\gamma} \\ \mathscr{Z}_*^{[1,N];\leq 2} \gamma \end{pmatrix} \right|. \end{gathered} \tag{14.4.3b}$$

Moreover, we have

$$\left|\upmu \not{\mathcal{L}}_Y^{N(Y)}\not{\pi}_L^{\#} + \upmu(\not{\mathcal{L}}_Y^N \mathrm{tr}_{\not{g}}\upchi)Y\right|, \left|\not{\mathcal{L}}_Y^{N(Y)}\not{\pi}_{\breve{X}}^{\#} - \upmu(\not{\mathcal{L}}_Y^N \mathrm{tr}_{\not{g}}\upchi)Y - y\not{d}^{\#}Y^N\upmu\right| \tag{14.4.4}$$
$$\lesssim \left|\mathscr{P}^{[1,N+1]}\vec{\Psi}\right| + \left|\begin{pmatrix} \mathscr{Z}_{**}^{[1,N];0}\underline{\upgamma} \\ \mathscr{P}^{[1,N]}\upgamma \end{pmatrix}\right|,$$

$$\left|\not{\mathcal{L}}_Y^{N(L)}\not{\pi}_{\breve{X}}^{\#} - \not{d}^{\#}Y^N\upmu\right| \lesssim \left|\mathscr{Z}_{*}^{[1,N+1];\leq 1}\vec{\Psi}\right| + \left|\begin{pmatrix} \mathscr{Z}_{**}^{[1,N];0}\underline{\upgamma} \\ \mathscr{P}^{[1,N]}\upgamma \end{pmatrix}\right|. \tag{14.4.5}$$

Above, within a given inequality, the symbol $\mathscr{Z}^{N-1;M}$ on the LHS always denotes the same order $N-1$ vectorfield operator, and similarly for the symbol $\mathscr{P}^{N-1}$.

Proof. See Sect. 9.2 for some comments on the analysis. The main point of the proof is to identity the products featuring the top-order derivatives of the eikonal function quantities, which we place on the LHS of the estimates. More precisely, we aim to identify the products containing a factor with $N+1$ derivatives on $\upmu$ or N derivatives on $\mathrm{tr}_{\not{g}}\upchi$, *with none of the derivatives being in the L direction*; all other terms are error terms that can be shown to be bounded in magnitude by $\lesssim$ the RHSs of the inequalities (including top-order derivatives of $\upmu$ or $\mathrm{tr}_{\not{g}}\upchi$ involving an L derivative, which we bound with the estimates (9.6.1b) and (9.6.2)).

We start by proving (14.4.2a) for the first term $\not{\mathcal{L}}_{\mathscr{P}}^{N-1}\not{\mathcal{L}}_{\breve{X}}{}^{(\breve{X})}\not{\pi}_L^{\#} + \not{d}^{\#}\breve{X}\mathscr{P}^{N-1}\upmu$ on the LHS. We apply $\not{\mathcal{L}}_{\mathscr{P}}^{N-1}\not{\mathcal{L}}_{\breve{X}}$ to the $\not{g}$-dual of the equation (3.16.1b) for ${}^{(\breve{X})}\not{\pi}_L$. By Lemma 3.19, the $\not{g}$-dual of the terms $-2\upzeta^{(Trans-\vec{\Psi})} - 2\upmu\upzeta^{(Tan-\vec{\Psi})}$ is of the form $\mathrm{f}(\not{g}^{-1}, \not{d}\vec{x}, \breve{X}\vec{\Psi})\upgamma + \mathrm{f}(\underline{\upgamma}, \not{g}^{-1}, \not{d}\vec{x})P\vec{\Psi}$. Hence, using the estimates (9.3.3b)-(9.3.3c) and (9.4.1a) and the L^∞ estimates of Prop. 9.12, we find that the $\not{\mathcal{L}}_{\mathscr{P}}^{N-1}\not{\mathcal{L}}_{\breve{X}}$ derivative of these terms are bounded in magnitude by $\lesssim$ RHS (14.4.2a) as desired. It remains for us to consider the terms $-\not{\mathcal{L}}_{\mathscr{P}}^{N-1}\not{\mathcal{L}}_{\breve{X}}\not{d}^{\#}\upmu = -\not{\mathcal{L}}_{\mathscr{P}}^{N-1}\not{\mathcal{L}}_{\breve{X}}(\not{g}^{-1}\cdot\not{d}\upmu)$ generated by the first term on the RHS of the equation (3.16.1b) for ${}^{(\breve{X})}\not{\pi}_L$. We first use (9.4.1a) and the L^∞ estimates of Prop. 9.12 to deduce that all terms in the Leibniz expansion of $-\not{\mathcal{L}}_{\mathscr{P}}^{N-1}\not{\mathcal{L}}_{\breve{X}}(\not{g}^{-1}\cdot\not{d}\upmu)$ are bounded in magnitude by $\lesssim$ RHS (14.4.2a) except for the top-order-in-$\upmu$ term $-\not{g}^{-1}\cdot\not{\mathcal{L}}_{\mathscr{P}}^{N-1}\not{\mathcal{L}}_{\breve{X}}\not{d}\upmu$. To handle this term, we use the commutator estimate (9.5.9a) with $\xi = \not{d}\upmu$, $N-1$ in the role of N, and $M=1$ and the L^∞ estimates of Prop. 9.12 to express $-\not{g}^{-1}\cdot\not{\mathcal{L}}_{\mathscr{P}}^{N-1}\not{\mathcal{L}}_{\breve{X}}\not{d}\upmu = -\not{d}^{\#}\breve{X}\mathscr{P}^{N-1}\upmu$ plus error terms that are in magnitude $\lesssim$ RHS (14.4.2a). We then bring the top-order term $\not{d}^{\#}\breve{X}\mathscr{P}^{N-1}\upmu$ over to the left, as is indicated on LHS (14.4.2a), which completes the proof of the desired estimate.

The proof of (14.4.2a) for the second term $\mathscr{P}^{N-1}\breve{X}\mathrm{tr}_{\not{g}}{}^{(\breve{X})}\not{\pi} + 2\upmu\mathscr{P}^{N-1}\breve{X}\mathrm{tr}_{\not{g}}\upchi$ on the LHS is based on the formula (3.16.1c) but is otherwise similar. We omit the details, noting only that the top-order eikonal function term occurs when all derivatives fall on the $\mathrm{tr}_{\not{g}}\upchi$ factor in the first product on RHS (3.16.1c), that we use the estimate (9.4.1c) to bound the below-top-order derivatives of $\mathrm{tr}_{\not{g}}\upchi$, and that we use the commutator estimate (9.5.7b) with $f = \mathrm{tr}_{\not{g}}\upchi$ to commute derivatives on $\mathrm{tr}_{\not{g}}\upchi$ (rather than the commutator estimate (9.5.9a) used above).

All three estimates in (14.4.2b) can be proved using essentially the same ideas with the help of the formulas (3.16.1a), (3.16.1b), and (3.16.1c). More precisely, in the case of the first term on LHS (14.4.2b), we use two new ingredients **i)** the commutator estimate (9.5.9b) with $\xi = {}^{(X)}\not{\pi}_L$ (to commute the operator $\mathscr{P}^{N-1}$ through the operator $\not{\mathrm{div}}$ in the term

$\mathscr{P}^{N-1}\mathrm{d}\!\!/\mathrm{iv}\,^{(\check{X})}\not\pi_L^{\#} = \mathscr{P}^{N-1}\left\{\not g^{-1}\cdot\not\nabla^{(\check{X})}\not\pi_L\right\}$) and **ii)** we use Lemma 14.3 to replace, up to error terms bounded in magnitude by $\lesssim$ RHS (14.4.2b), the top-order eikonal function quantity term $-\not\Delta\mathscr{P}^{N-1}\upmu$ (generated by the first term on RHS (3.16.1b)) with the term $-\mathscr{P}^{N-1}\check{X}\mathrm{tr}_{\not g}\chi$ (which we then bring over to LHS (14.4.2b)).

The estimates (14.4.1a)-(14.4.1b) can be proved with the help of (3.16.2a)-(3.16.2c) and are based on the same ideas plus one new ingredient: to bound the top derivatives of the quantities $L\upmu$ in equation (3.16.2a), we use the estimate (9.6.1b) (and the resulting terms are bounded in magnitude by $\lesssim$ RHS (14.4.1b)) as desired; we omit the remaining details.

The proofs of (14.4.3a)-(14.4.3b) are based on (3.16.3a)-(3.16.3d) and require no new ingredients beyond the ones we used above; we therefore omit the details. The same remarks apply to the estimates (14.4.4)-(14.4.5) (except to obtain (14.4.5), we rely on (3.16.2b)). □

The next lemma complements Lemma 14.5 by providing bounds for the derivatives of the deformation tensor frame components when an L differentiation is involved or when the number of derivatives is below-top-order. In contrast with Lemma 14.5, no difficult terms appear in the estimates.

Lemma 14.6 (Pointwise estimates for the negligible derivatives of $^{(L)}\pi$ and $^{(Y)}\pi$). *Assume that $N = 20$ and let $P \in \mathscr{P} = \{L, Y\}$. Under the data-size and bootstrap assumptions of Sects. 8.1-8.4 and the smallness assumptions of Sect. 8.6, the following estimates hold on $\mathcal{M}_{T_{(Boot)},U_0}$ (see Sect. 6.2 regarding the vectorfield operator notation).*

First, if $\mathscr{Z}^{N;\leq 1}$ contains a factor of L, then

$$\begin{aligned}
&\left|\mathscr{Z}^{N;\leq 1}\mathrm{tr}_{\not g}\,^{(P)}\not\pi\right|, \left|\mathscr{Z}^{N;\leq 1(P)}\pi_{L\check{X}}\right|, \left|\mathscr{Z}^{N;\leq 1(P)}\pi_{\check{X}X}\right|, \\
&\left|\not\mathcal{L}_{\mathscr{Z}}^{N;\leq 1(P)}\not\pi_L^{\#}\right|, \left|\not\mathcal{L}_{\mathscr{Z}}^{N;\leq 1(P)}\not\pi_{\check{X}}^{\#}\right| \\
&\lesssim \left|\mathscr{Z}_*^{[1,N+1];\leq 2}\Psi\right| + \left|\begin{pmatrix} \mathscr{Z}_{**}^{[1,N];\leq 1}\upgamma \\ \mathscr{Z}_*^{[1,N];\leq 1}\underline{\upgamma} \end{pmatrix}\right|.
\end{aligned} \tag{14.4.6}$$

In addition, if $\mathscr{P}^N$ contains a factor of L, then

$$\left|\mathscr{P}^N\mathrm{tr}_{\not g}\,^{(\check{X})}\not\pi\right|, \left|\not\mathcal{L}_{\mathscr{P}}^{N}\,^{(\check{X})}\not\pi_L^{\#}\right| \lesssim \left|\mathscr{Z}_*^{[1,N+1];\leq 2}\Psi\right| + \left|\begin{pmatrix} \mathscr{Z}_{**}^{[1,N];\leq 1}\upgamma \\ \mathscr{Z}_*^{[1,N];\leq 1}\underline{\upgamma} \end{pmatrix}\right|, \tag{14.4.7a}$$

$$\left|\mathscr{P}^{N(\check{X})}\pi_{L\check{X}}\right|, \left|\mathscr{P}^{N(\check{X})}\pi_{\check{X}X}\right| \lesssim \left|\mathscr{Z}_*^{[1,N+1];\leq 2}\Psi\right| + \left|\begin{pmatrix} \mathscr{Z}_{**}^{[1,N];\leq 1}\upgamma \\ \mathscr{Z}_*^{[1,N];\leq 1}\underline{\upgamma} \end{pmatrix}\right|. \tag{14.4.7b}$$

Moreover, for $1 \leq N \leq 20$, the following below-top-order estimates hold, where the operator $\mathscr{Z}_*^{N-1;\leq 1}$ does not necessarily contain any factor of L:

$$\begin{aligned}
&\left|\mathscr{Z}^{N-1;\leq 1}\mathrm{tr}_{\not g}\,^{(P)}\not\pi\right|, \left|\mathscr{Z}^{N-1;\leq 1(P)}\pi_{L\check{X}}\right|, \left|\mathscr{Z}^{N-1;\leq 1(P)}\pi_{\check{X}X}\right|, \left|\not\mathcal{L}_{\mathscr{Z}}^{N-1;\leq 1(P)}\not\pi_L^{\#}\right|, \left|\not\mathcal{L}_{\mathscr{Z}}^{N-1;\leq 1(P)}\not\pi_{\check{X}}^{\#}\right| \qquad (14.4.8a) \\
&\lesssim \left|\mathscr{Z}_*^{[1,N];\leq 2}\Psi\right| + \left|\begin{pmatrix} \mathscr{Z}_{**}^{[1,N];\leq 1}\upgamma \\ \mathscr{Z}_*^{[1,N];\leq 1}\underline{\upgamma} \end{pmatrix}\right| + 1.
\end{aligned}$$

In addition, for $1 \leq N \leq 20$, we have the following below-top-order estimates:

$$\left|\mathscr{P}^{N-1}\mathrm{tr}_{\not g}{}^{(\breve{X})}\not\pi\right|, \left|\mathscr{P}^{N-1}L^{(\breve{X})}\pi_{L\breve{X}}\right|, \left|\mathscr{P}^{N-1}L^{(\breve{X})}\pi_{\breve{X}X}\right|, \left|\not\mathcal{L}_{\mathscr{P}}^{N-1(\breve{X})}\not\pi_L^{\#}\right|, \left|\not\mathcal{L}_{\mathscr{P}}^{N-1(\breve{X})}\not\pi_{\breve{X}}^{\#}\right| \tag{14.4.8b}$$
$$\lesssim \left|\mathscr{Z}_*^{[1,N];\leq 2}\Psi\right| + \left|\begin{pmatrix} \mathscr{Z}_{**}^{[1,N];\leq 1}\underline{\gamma} \\ \mathscr{Z}_*^{[1,N];\leq 1}\gamma \end{pmatrix}\right| + 1.$$

Proof. See Sect. 9.2 for some comments on the analysis. We first prove (14.4.6). From Prop. 3.14, equation (3.14.1), and Lemma 3.19, we see that the deformation tensor components $\mathrm{tr}_{\not g}{}^{(P)}\not\pi$, ${}^{(P)}\pi_{L\breve{X}}$, $\cdots$, ${}^{(P)}\not\pi_{\breve{X}}^{\#}$ on LHS (14.4.6) are schematically of the form

$$\mathrm{f}(\underline{\gamma}, \not g^{-1}, \not d\vec{x})P\vec{\Psi} + \mathrm{f}(\gamma, \not g^{-1}, \not d\vec{x})\breve{X}\vec{\Psi} + \mathrm{f}(\gamma, \not g^{-1}, \not d\vec{x})\mathrm{tr}_{\not g}\chi + \mathrm{f}(\gamma, \not g^{-1}, \not d\vec{x})\not d\mu.$$

We now apply $\not\mathcal{L}_{\mathscr{Z}_*}^{\leq N;\leq 1}$. Recall that $\mathscr{Z}_*^{N;\leq 1}$ contains a factor of L by assumption. Let $\mathscr{Z}_*^{N-1;\leq 1}$ denote the factors obtained by removing the factor of L. If all derivatives fall on $\mathrm{tr}_{\not g}\chi$, then we the commutator estimate (9.5.7a) with $f = \mathrm{tr}_{\not g}\chi$, the estimate (9.4.1c), and the L^∞ estimates of Prop. 9.12, to commute the factor of L in $\mathscr{Z}_*^{N;\leq 1}$ so that it hits $\mathrm{tr}_{\not g}\chi$ first, which implies that $\left|\mathscr{Z}_*^{N;\leq 1}\mathrm{tr}_{\not g}\chi\right| \lesssim \left|\mathscr{Z}_*^{N-1;\leq 1}L\mathrm{tr}_{\not g}\chi\right|$ plus error terms that are bounded by $\lesssim$ RHS (14.4.6). Moreover, using the pointwise estimate (9.6.2), we obtain that $\left|\mathscr{Z}_*^{N-1;\leq 1}L\mathrm{tr}_{\not g}\chi\right| \lesssim$ RHS (14.4.6). Moreover, we bound the remaining factors multiplying $\mathscr{Z}^{N;\leq 1}\mathrm{tr}_{\not g}\chi$ in magnitude by $\lesssim 1$ via Lemmas 9.5 and 9.6 and the L^∞ estimates of Prop. 9.12. In total, we find that the product under consideration is $\lesssim$ RHS (14.4.6) as desired. Similarly, if all derivatives fall on $\not d\mu$, we use the commutator estimate (9.5.7a) with $f = \mu$ and the L^∞ estimates of Prop. 9.12, we can commute the factor of L so that it hits μ first, which implies that $\left|\not d\mathscr{Z}_*^{N;\leq 1}\mu\right| \lesssim \left|\mathscr{Z}_*^{N+1;\leq 1}\mu\right| \lesssim \left|\mathscr{Z}_*^{N;\leq 1}L\mu\right|$ plus error terms that are bounded by $\lesssim$ RHS (14.4.6). Moreover, we bound the remaining factors multiplying $\not d\mathscr{Z}^{N;\leq 1}\mu$ in magnitude by $\lesssim 1$ via Lemmas 9.5 and 9.6 and the L^∞ estimates of Prop. 9.12. In total, we find that the product under consideration is $\lesssim$ RHS (14.4.6) as desired. If most (but not all) derivatives fall on $\mathrm{tr}_{\not g}\chi$ or $\not d\mu$, then we bound all terms using the above arguments. If most derivatives fall on $P\vec{\Psi}$, $\breve{X}\vec{\Psi}$, $\underline{\gamma}$, γ, $\not g^{-1}$, or $\not d\vec{x}$, then we bound these factors by $\lesssim$ RHS (14.4.6) with the help of Lemmas 9.5 and 9.6. Moreover, we bound the remaining factors (which multiply the factor with many derivatives on it) in magnitude by $\lesssim 1$ via Lemmas 9.5 and 9.6 and the L^∞ estimates of Prop. 9.12. We have therefore proved (14.4.6).

To prove (14.4.7a)-(14.4.7b), we first use Prop. 3.14, equation (3.14.1), and Lemma 3.19 to deduce that the deformation tensor components $\mathrm{tr}_{\not g}{}^{(\breve{X})}\not\pi$, ${}^{(\breve{X})}\pi_{L\breve{X}}$, ${}^{(\breve{X})}\pi_{\breve{X}X}$, ${}^{(\breve{X})}\not\pi_L^{\#}$, and ${}^{(\breve{X})}\not\pi_{\breve{X}}^{\#}$ are schematically of the form

$$\mathrm{f}(\gamma, \not g^{-1}, \not d\vec{x}, \breve{X}\vec{\Psi}) + \mathrm{f}(\underline{\gamma}, \not g^{-1}, \not d\vec{x})P\vec{\Psi} + \mathrm{f}(\underline{\gamma}, \not g^{-1}, \not d\vec{x})\mathrm{tr}_{\not g}\chi + \breve{X}\mu + \not g^{-1}\not d\mu.$$

From this schematic formula and the assumption that $\mathscr{P}^N$ contains a factor of L, we see that all terms can be bounded by using the arguments given in the previous paragraph.

The estimates (14.4.8a) and (14.4.8b) can be proved using similar but simpler arguments and we therefore omit the details. This completes our proof of the lemma.

□

14.5. **Pointwise estimates involving the fully modified quantities.** Our main goal in this section is to prove Prop. 14.9, in which we obtain pointwise estimates for the most difficult product that appears in our energy estimates: $(\breve{X}v^1)\mathscr{Z}_*^{N;\leq 1}\mathrm{tr}_{\not{g}}\upchi$. As a preliminary step, we prove Lemma 14.8, in which we use the transport equation (7.2.4) to derive pointwise estimates for the fully modified quantities ${}^{(\mathscr{Z}_*^{N;\leq 1})}\mathscr{X}$ From Def. 7.2.

Before proving Lemma 14.8, we first provide a lemma in which we derive pointwise estimates for the source term $\mathfrak{X}$ appearing in the transport equation (7.2.4) satisfied by ${}^{(\mathscr{Z}_*^{N;\leq 1})}\mathscr{X}$. At the same time, for later use, we derive pointwise estimates for the terms $\widetilde{\mathfrak{X}}$ and ${}^{(\mathscr{Z}_*^{N-1;\leq 1})}\mathfrak{B}$ appearing in the transport equation (7.2.5) verified by the partially modified quantity ${}^{(\mathscr{Z}_*^{N-1;\leq 1})}\widetilde{\mathscr{X}}$.

Lemma 14.7 (Pointwise estimates for $\mathscr{Z}_*^{N;\leq 1}\mathfrak{X}$, $\mathscr{Z}_*^{N-1;\leq 1}\widetilde{\mathfrak{X}}$, and ${}^{(\mathscr{Z}_*^{N-1;\leq 1})}\mathfrak{B}$). *Assume that $1 \leq N \leq 20$. Let $\mathfrak{X}$ be the quantity defined in* (7.2.1b)*, let $\widetilde{\mathfrak{X}}$ be the quantity defined in* (7.2.3)*, let ${}^{(\mathscr{Z}_*^{N-1;\leq 1})}\widetilde{\mathfrak{X}}$ be the quantity defined in* (7.2.2b)*, and let ${}^{(\mathscr{Z}_*^{N-1;\leq 1})}\mathfrak{B}$ be the quantity defined in* (7.2.6)*. Under the data-size and bootstrap assumptions of Sects. 8.1-8.4 and the smallness assumptions of Sect. 8.6, the following pointwise estimates hold on $\mathcal{M}_{T_{(Boot)},U_0}$:*

$$\left|\mathscr{Z}_*^{N;\leq 1}\mathfrak{X} + \vec{G}_{LL}\diamond\breve{X}\mathscr{Z}_*^{N;\leq 1}\vec{\Psi}\right| \lesssim \upmu\left|\mathscr{Z}_*^{[1,N+1];\leq 1}\vec{\Psi}\right| + \left|\mathscr{Z}_*^{[1,N];\leq 2}\vec{\Psi}\right| + \left|\mathscr{Z}_*^{[1,N];\leq 1}\upgamma\right| + \left|\mathscr{Z}_{**}^{[1,N];\leq 1}\underline{\upgamma}\right|, \tag{14.5.1a}$$

$$\left|\mathscr{Z}_*^{N;\leq 1}\mathfrak{X}\right| \lesssim \left|\mathscr{Z}_*^{[1,N+1];\leq 2}\vec{\Psi}\right| + \left|\mathscr{Z}_*^{[1,N];\leq 1}\upgamma\right| + \left|\mathscr{Z}_{**}^{[1,N];\leq 1}\underline{\upgamma}\right|, \tag{14.5.1b}$$

$$\left|\mathscr{Z}_*^{N;\leq 1}\widetilde{\mathfrak{X}}\right| \lesssim \left|\mathscr{Z}_*^{[1,N+1];\leq 1}\vec{\Psi}\right| + \left|\mathscr{Z}_*^{[1,N];\leq 1}\upgamma\right| + \left|\mathscr{Z}_{**}^{[1,N];0}\underline{\upgamma}\right|, \tag{14.5.1c}$$

$$\left|L^{(\mathscr{Z}_*^{N-1;\leq 1})}\widetilde{\mathfrak{X}}\right|, \left|Y^{(\mathscr{Z}_*^{N-1;\leq 1})}\widetilde{\mathfrak{X}}\right| \lesssim \left|\mathscr{Z}_*^{[1,N+1];\leq 1}\vec{\Psi}\right|, \tag{14.5.1d}$$

$$\left|{}^{(\mathscr{Z}_*^{N-1;\leq 1})}\mathfrak{B}\right| \lesssim \left|\mathscr{Z}_*^{[1,N];\leq 1}\upgamma\right| + \left|\mathscr{Z}_{**}^{[1,N];0}\underline{\upgamma}\right|. \tag{14.5.1e}$$

Proof. See Sect. 9.2 for some comments on the analysis. Throughout this proof, we silently use the L^∞ estimates of Prop. 9.12.

To prove (14.5.1a), we first use (7.2.1b) and Lemma 3.19 to deduce $\mathfrak{X} = -\vec{G}_{LL}\diamond\breve{X}\vec{\Psi} + \upmu\mathrm{f}(\upgamma, \not{g}^{-1}, \not{d}\vec{x})P\vec{\Psi}$. We now apply $\mathscr{Z}_*^{N;\leq 1}$ to this identity and bring the top-order term $\vec{G}_{LL}\diamond\breve{X}\mathscr{Z}_*^{N;\leq 1}\vec{\Psi}$ over to the left (as indicated on LHS (14.5.1a)), which leaves the commutator terms $[\vec{G}_{LL}, \mathscr{Z}_*^{N;\leq 1}]\diamond\breve{X}\vec{\Psi}$ and $\vec{G}_{LL}\diamond[\breve{X}, \mathscr{Z}_*^{N;\leq 1}]\vec{\Psi}$ on the RHS. To bound the term $\left|\mathscr{Z}_*^{N;\leq 1}\left\{\upmu\mathrm{f}(\upgamma, \not{g}^{-1}, \not{d}\vec{x})P\vec{\Psi}\right\}\right|$ by $\leq$ RHS (14.5.1a), we use Lemmas 9.5 and 9.6. Note that we have paid special attention to terms in which all derivatives $\mathscr{Z}_*^{N;\leq 1}$ fall on $P\vec{\Psi}$; these terms are bounded by the first term on RHS (14.5.1a). To bound $\left|[\vec{G}_{LL}, \mathscr{Z}_*^{N;\leq 1}]\diamond\breve{X}\vec{\Psi}\right|$ by $\leq$ RHS (14.5.1a), we use the fact that $\vec{G}_{LL} = \mathrm{f}(\upgamma)$ (see Lemma 3.19). To bound $\left|\vec{G}_{LL}\diamond[\breve{X}, \mathscr{Z}_*^{N;\leq 1}]\vec{\Psi}\right|$ by $\leq$ RHS (14.5.1a), we also use the commutator estimate (9.5.7b) with $f = \vec{\Psi}$ and $M \leq 1$. Combining the above estimates, we arrive at (14.5.1a). The proof of (14.5.1b) is similar but simpler and we omit the details. The same is true for the proof of (14.5.1c) since by Lemma 3.19, we have $\widetilde{\mathfrak{X}} = \mathrm{f}(\upgamma, \not{g}^{-1}, \not{d}\vec{x})P\vec{\Psi}$.

We now prove (14.5.1d). We give the proof only for the second term on the LHS since the proof for the first one is similar. To proceed, we use (7.2.2b) and Lemma 3.19 to deduce that

$Y^{(\mathscr{Z}_*^{N-1;\leq 1})}\widetilde{\mathfrak{X}} = Y\left\{\mathrm{f}(\gamma, \not{g}^{-1}, \not{d}\vec{x})\mathscr{Z}_*^{N;\leq 1}\vec{\Psi}\right\}$. The estimate (14.5.1d) now follows easily from the previous expression and Lemmas 9.5 and 9.6.

We now prove (14.5.1e). We bound the term $\mathscr{Z}_*^{N-1;\leq 1}(\mathrm{tr}_{\not{g}}\chi)^2$ from RHS (7.2.6) by $\leq$ RHS (14.5.1e) with the help of inequality (9.4.1c). We bound the term $[\mathscr{Z}_*^{N-1;\leq 1}, \vec{G}_{LL}] \diamond \not{\Delta}\vec{\Psi}$ with the help of the aforementioned relation $\vec{G}_{LL} = \mathrm{f}(\gamma)$ and Cor. 9.11. To bound $\vec{G}_{LL} \diamond [\mathscr{Z}_*^{N-1;\leq 1}, \not{\Delta}]\vec{\Psi}$, we also use the commutator estimate (9.5.8b) with $f = \vec{\Psi}$. We bound the term $[L, \mathscr{Z}_*^{N-1;\leq 1}]\mathrm{tr}_{\not{g}}\chi$ with the help of the commutator estimate (9.5.7a) with $\mathrm{tr}_{\not{g}}\chi$ in the role of f and inequality (9.4.1c). We bound $[L, \mathscr{Z}_*^{N-1;\leq 1}]\widetilde{\mathfrak{X}}$ with the help of the commutator estimate (9.5.7a) with $f = \widetilde{\mathfrak{X}}$ and (14.5.1c). To bound $L\left\{^{(\mathscr{Z}_*^{N-1;\leq 1})}\widetilde{\mathfrak{X}} - \mathscr{Z}_*^{N-1;\leq 1}\widetilde{\mathfrak{X}}\right\}$, we first note that (7.2.2b), (7.2.3), and the Leibniz rule imply that the magnitude of this term is

$$\lesssim \sum_{\substack{N_1+N_2\leq N \\ N_1\geq 1}} \sum_{M_1+M_2\leq 1} \left|\not{\mathcal{L}}_{\mathscr{Z}}^{N_1;M_1}\vec{G}_{(Frame)}^{\#}\right|\left|\mathscr{Z}^{N_2+1;M_2}\vec{\Psi}\right| + \left|\vec{G}_{(Frame)}^{\#}\right|\left|[L, L\mathscr{Z}_*^{N-1;\leq 1}]\vec{\Psi}\right|.$$

Since Lemma 3.19 implies that $\vec{G}_{(Frame)}^{\#} = \mathrm{f}(\gamma, \not{g}^{-1}, \not{d}\vec{x})$, the desired bound for the sum follows from Lemmas 9.5 and 9.6. To bound the term $\left|[L, L\mathscr{Z}_*^{N-1;\leq 1}]\vec{\Psi}\right|$, we also use the commutator estimate (9.5.7b) with $f = \vec{\Psi}$ and $M \leq 1$. We have therefore proved (14.5.1e), which completes the proof of the lemma.

□

With the help of the previous lemma, we now derive pointwise estimates for the fully modified quantities $^{(\mathscr{Z}_*^{N;\leq 1})}\mathscr{X}$.

Lemma 14.8 (Estimates for solutions to the transport equation verified by $^{(\mathscr{Z}_*^{N;\leq 1})}\mathscr{X}$). *Assume that $N = 20$ and let $^{(\mathscr{Z}_*^{N;\leq 1})}\mathscr{X}$ and $\mathfrak{X}$ be as in Prop. 7.2. Assume first that $\mathscr{Z}_*^{N;\leq 1} = Y^N$. Under the data-size and bootstrap assumptions of Sects. 8.1-8.4 and the smallness assumptions of Sect. 8.6, the following pointwise estimate holds on $\mathcal{M}_{T_{(Boot)},U_0}$:*

$$\begin{aligned}\left|^{(Y^N)}\mathscr{X}\right|(t,u,\vartheta) &\leq C\left|^{(Y^N)}\mathscr{X}\right|(0,u,\vartheta) \\ &\quad + 2(1+C\varepsilon)\int_{s=0}^t \frac{[L\upmu(s,u,\vartheta)]_-}{\upmu(s,u,\vartheta)}\left|Y^N\mathfrak{X}\right|(s,u,\vartheta)\,ds \\ &\quad + C\int_{s=0}^t \left\{\left|\mathscr{Z}_*^{[1,N+1];\leq 2}\vec{\Psi}\right| + \left|\begin{pmatrix}\mathscr{Z}_{**}^{[1,N];\leq 1}\gamma \\ \mathscr{Z}_*^{[1,N];\leq 1}\underline{\gamma}\end{pmatrix}\right|\right\}(s,u,\vartheta)\,ds \\ &\quad + C\int_{s=0}^t \left\{\upmu\left|\mathscr{Z}^{N+1;\leq 1}\omega\right|\right\}(s,u,\vartheta)\,ds + C\int_{s=0}^t \left|\mathscr{Z}^{\leq N;\leq 1}\omega\right|(s,u,\vartheta)\,ds.\end{aligned} \tag{14.5.2}$$

Assume now that $\mathscr{Z}_*^{N;\leq 1} = Y^{N-1}\breve{X}$. Then $\left|^{(Y^{N-1}\breve{X})}\mathscr{X}\right|(t,u,\vartheta)$ verifies inequality (14.5.2), but with the term $\left|^{(Y^N)}\mathscr{X}\right|(0,u,\vartheta)$ on the RHS replaced by $\left|^{(Y^{N-1}\breve{X})}\mathscr{X}\right|(0,u,\vartheta) + \left|^{(Y^N)}\mathscr{X}\right|(0,u,\vartheta)$, *with the term* $\left|Y^N \mathfrak{X}\right|(s,u,\vartheta)$ *replaced by* $\left|Y^{N-1}\breve{X}\mathfrak{X}\right|(s,u,\vartheta)$*, and with the following additional double time integral present on the RHS:*

$$C\int_{s=0}^{t}\int_{s'=0}^{s}\frac{1}{\upmu_\star(s',u)}\left\{\left|\mathscr{Z}_*^{[1,N+1];\leq 2}\Psi\right| + \left|\left(\begin{array}{c}\mathscr{Z}_{**}^{[1,N];\leq 1}\underline{\gamma}\\ \mathscr{Z}_*^{[1,N];\leq 1}\gamma\end{array}\right)\right|\right\}(s',u,\vartheta)\,ds'\,ds. \qquad (14.5.3)$$

Proof. See Sect. 9.2 for some comments on the analysis. We first prove (14.5.2). We set $\mathscr{Z}_*^{N;\leq 1} = Y^N$ in (7.2.4) and view both sides of the equation as functions of (s,u,ϑ). Noting that $L = \frac{\partial}{\partial s}$ in the present context, we define the integrating factor

$$\iota(s,u,\vartheta) := \exp\left(\int_{t'=0}^{s} -2\frac{L\upmu(t',u,\vartheta)}{\upmu(t',u,\vartheta)}\,dt'\right) = \frac{\upmu^2(0,u,\vartheta)}{\upmu^2(s,u,\vartheta)} \qquad (14.5.4)$$

corresponding to the coefficient of ${}^{(Y^N)}\mathscr{X}$ on LHS (7.2.4). We then rewrite (7.2.4) as $L\left(\iota^{(Y^N)}\mathscr{X}\right) = \iota \times$ RHS (7.2.4) and integrate the resulting equation with respect to s from time 0 to time t. From Def. 11.3 and the estimates (11.2.7) and (11.2.10), we deduce

$$\sup_{0\leq s'\leq t}\frac{\upmu(t,u,\vartheta)}{\upmu(s',u,\vartheta)} \leq C. \qquad (14.5.5)$$

From (14.5.4) and (14.5.5), it is straightforward to see that the desired bound (14.5.2) follows once we establish the following bounds for the terms generated by the terms on RHS (7.2.4):

$$\begin{aligned}
&\left|\upmu[L,Y^N]\mathrm{tr}_{\not g}\chi\right|(s,u,\vartheta) && (14.5.6)\\
&\leq C\varepsilon\left|{}^{(Y^N)}\mathscr{X}\right|(s,u,\vartheta)\\
&\quad + C\varepsilon\left|\mathscr{Z}_*^{[1,N+1];\leq 2}\vec{\Psi}\right|(s,u,\vartheta) + C\varepsilon\left|\mathscr{Z}_*^{[1,N];\leq 1}\gamma\right|(s,u,\vartheta) + C\varepsilon\left|\mathscr{Z}_{**}^{[1,N];\leq 1}\underline{\gamma}\right|(s,u,\vartheta),\\
&2\left|\upmu\mathrm{tr}_{\not g}\chi Y^N\mathrm{tr}_{\not g}\chi\right|(s,u,\vartheta) && (14.5.7)\\
&\leq C\varepsilon\left|{}^{(Y^N)}\mathscr{X}\right|(s,u,\vartheta)\\
&\quad + C\varepsilon\left|\mathscr{Z}_*^{[1,N+1];\leq 2}\vec{\Psi}\right|(s,u,\vartheta) + C\varepsilon\left|\mathscr{Z}_*^{[1,N];\leq 1}\gamma\right|(s,u,\vartheta) + C\varepsilon\left|\mathscr{Z}_{**}^{[1,N];\leq 1}\underline{\gamma}\right|(s,u,\vartheta),\\
&2\left(\frac{\upmu(t,u,\vartheta)}{\upmu(s,u,\vartheta)}\right)^2\left|\frac{L\upmu(s,u,\vartheta)}{\upmu(s,u,\vartheta)}\right|\left|Y^N\mathfrak{X}\right|(s,u,\vartheta) && (14.5.8)\\
&\leq 2(1+C\varepsilon)\left|\frac{[L\upmu]_-(s,u,\vartheta)}{\upmu(s,u,\vartheta)}\right|\left|Y^N\mathfrak{X}\right|(s,u,\vartheta)\\
&\quad + C\left|\mathscr{Z}_*^{[1,N+1];\leq 2}\vec{\Psi}\right|(s,u,\vartheta) + C\left|\mathscr{Z}_*^{[1,N];\leq 1}\gamma\right|(s,u,\vartheta) + C\left|\mathscr{Z}_{**}^{[1,N];\leq 1}\underline{\gamma}\right|(s,u,\vartheta),\\
&\text{all remaining terms on RHS (7.2.4) are in magnitude} && (14.5.9)\\
&\leq C\left|\mathscr{Z}_*^{[1,N+1];\leq 2}\vec{\Psi}\right|(s,u,\vartheta) + C\left|\mathscr{Z}_*^{[1,N];\leq 1}\gamma\right|(s,u,\vartheta) + C\left|\mathscr{Z}_{**}^{[1,N];\leq 1}\underline{\gamma}\right|(s,u,\vartheta)\\
&\quad + \upmu\left|\mathscr{Z}^{N+1;\leq 1}\omega\right|(s,u,\vartheta) + \left|\mathscr{Z}^{[1,N];\leq 1}\omega\right|(s,u,\vartheta).
\end{aligned}$$

We note that in deriving (14.5.2), the product $C\varepsilon\iota\left|{}^{(Y^N)}\mathscr{X}\right|$ arising from the first term on RHS (14.5.7) needs to be treated with Gronwall's inequality. However, due to the small factor ε, this product has only the negligible effect of contributing to the factors $C\varepsilon$ on RHS (14.5.2). We remark that some of the estimates (14.5.6)-(14.5.9) are non-optimal in the sense of the number of $\breve{X}$ derivatives allowed on the right-hand sides. However, later in the proof, when we are analyzing ${}^{(Y^{N-1}\breve{X})}\mathscr{X}$, the same number of $\breve{X}$ derivatives appear on the right-hand sides of the analogous estimates, and they are optimal.

We now prove (14.5.6). We first use the commutator estimate (9.5.7a) with $M = 0$ and $f = \mathrm{tr}_{\not g}\chi$, the L^∞ estimates of Prop. 9.12, (9.6.2) with $M = 0$, and (9.4.1c) to deduce that $\left|\mu[L, Y^N]\mathrm{tr}_{\not g}\chi\right| \lesssim \varepsilon\left|\mu Y^N \mathrm{tr}_{\not g}\chi\right|$ plus error terms with magnitude $\leq$ the sum of the last three terms on RHS (14.5.6). We then use definition (7.2.1a) and the estimate (14.5.1b) to deduce that $\varepsilon\left|\mu Y^N \mathrm{tr}_{\not g}\chi\right| = \varepsilon\left|{}^{(Y^N)}\mathscr{X}\right|$ plus error terms with magnitude $\leq$ the sum of the last three terms on RHS (14.5.6), which yields the desired bound. Inequality (14.5.7) can be proved using similar arguments (without the help of a commutator estimate).

We now prove (14.5.8). We first note the simple inequality $\left|\frac{L\mu(s,u,\vartheta)}{\mu(s,u,\vartheta)}\right| \leq \left|\frac{[L\mu]_-(s,u,\vartheta)}{\mu(s,u,\vartheta)}\right| + \left|\frac{[L\mu]_+(s,u,\vartheta)}{\mu(s,u,\vartheta)}\right|$. To bound terms on LHS (14.5.8) arising from the factor $\left|\frac{[L\mu]_+(s,u,\vartheta)}{\mu(s,u,\vartheta)}\right|$ by $\leq$ the terms on the last line of RHS (14.5.8), we use (11.2.1), (14.5.5), and the estimate (14.5.1b). To bound terms on LHS (14.5.8) arising from the factor $\left|\frac{[L\mu]_-(s,u,\vartheta)}{\mu(s,u,\vartheta)}\right|$, we consider the partitions from Def. 11.3. When $(u,\vartheta) \in {}^{(+)}\mathcal{V}_t^u$, we use the bounds (11.2.9) and (14.5.5) to deduce that $\left(\frac{\mu(t,u,\vartheta)}{\mu(s,u,\vartheta)}\right)^2\left|\frac{[L\mu]_-(s,u,\vartheta)}{\mu(s,u,\vartheta)}\right| \leq C\varepsilon$. Combining this bound with (14.5.1b), we easily conclude that the terms of interest are $\leq$ the terms on the last line of RHS (14.5.8). Finally, when $(u,\vartheta) \in {}^{(-)}\mathcal{V}_t^u$, we use (11.2.10) to deduce that

$$2\left(\frac{\mu(t,u,\vartheta)}{\mu(s,u,\vartheta)}\right)^2\left|\frac{[L\mu]_-(s,u,\vartheta)}{\mu(s,u,\vartheta)}\right| \leq 2(1+C\varepsilon)\left|\frac{[L\mu]_-(s,u,\vartheta)}{\mu(s,u,\vartheta)}\right|.$$

Thus, we conclude that the terms under consideration are $\leq$ the terms on the first line of RHS (14.5.8), which completes the proof of (14.5.8).

We now prove (14.5.9), starting with the estimate for the term $[L, Y^N]\mathfrak{X}$ on RHS (7.2.4). Using the commutator estimate (9.5.7a) with $M = 0$ and $f = \mathfrak{X}$, the L^∞ estimates of Prop. 9.12, (9.6.2) with $M = 0$, and the estimate (14.5.1b), we deduce that $\left|[L, Y^N]\mathfrak{X}\right| \leq$ RHS (14.5.9) as desired. We next bound bound the term $[\mu, Y^N]L\mathrm{tr}_{\not g}\chi$ on RHS (7.2.4). Using the L^∞ estimates of Prop. 9.12 and the estimate (9.6.2) with $M = 0$, we deduce that $\left|[\mu, Y^N]L\mathrm{tr}_{\not g}\chi\right| \leq$ RHS (14.5.9) as desired. We next bound the term $[Y^N, L\mu]\mathrm{tr}_{\not g}\chi$ on RHS (7.2.4). Using the estimate (9.4.1c), the L^∞ estimates of Prop. 9.12, and the estimate (9.6.1b) with $M = 0$, we deduce that $\left|[Y^N, L\mu]\mathrm{tr}_{\not g}\chi\right| \leq$ RHS (14.5.9) as desired. We now bound the term $Y^N\left(\mu(\mathrm{tr}_{\not g}\chi)^2\right) - 2\mu\mathrm{tr}_{\not g}\chi Y^N\mathrm{tr}_{\not g}\chi$ on RHS (7.2.4). By the Leibniz rule, we see that the magnitude of this term is $\lesssim \sum_{\substack{N_1+N_2+N_3\leq N\\ N_2,N_3\leq N-1}}\left|Y^{N_1}\mu\right|\left|Y^{N_2}\mathrm{tr}_{\not g}\chi\right|\left|Y^{N_3}\mathrm{tr}_{\not g}\chi\right|$. Hence, from the estimate (9.4.1c) and the L^∞ estimates of Prop. 9.12, we deduce that all products in the sum are $\leq$ RHS (14.5.9) as desired. Finally, to bound the term $Y^N\mathfrak{A}$ on RHS (7.2.4), we apply Y^N to both sides of (7.1.4). We bound the products of interest in magnitude by $\leq$ RHS (14.5.9) with the help of the estimates (9.3.3b)-(9.3.3c), (9.4.1a), and the L^∞ estimates of Prop. 9.12. This completes the proof of (14.5.9) and finishes the proof of (14.5.2).

We now derive the desired bound for $\left|{}^{(Y^{N-1}\breve{X})}\mathscr{X}\right|(t,u,\vartheta)$. We first note that by using essentially the same arguments used in the proof of (14.5.2), we can show that (14.5.6)-(14.5.9) hold with the operator Y^N on the LHS replaced by $Y^{N-1}\breve{X}$, but with the following changes: (14.5.6)-(14.5.7) are replaced with

$$\begin{aligned}
&\left|\upmu[L,Y^{N-1}\breve{X}]\mathrm{tr}_{\not g}\upchi\right|(s,u,\vartheta) && (14.5.10)\\
&\leq C\varepsilon\left|{}^{(Y^{N-1}\breve{X})}\mathscr{X}\right|(s,u,\vartheta)+\left|{}^{(Y^{N})}\mathscr{X}\right|(s,u,\vartheta)\\
&\quad+C\varepsilon\left|\mathscr{Z}_*^{[1,N+1];\leq 2}\vec{\Psi}\right|(s,u,\vartheta)+C\varepsilon\left|\mathscr{Z}_*^{[1,N];\leq 1}\upgamma\right|(s,u,\vartheta)+C\varepsilon\left|\mathscr{Z}_{**}^{[1,N];\leq 1}\underline{\upgamma}\right|(s,u,\vartheta),
\end{aligned}$$

$$\begin{aligned}
&2\left|\upmu\mathrm{tr}_{\not g}\upchi Y^{N-1}\breve{X}\mathrm{tr}_{\not g}\upchi\right|(s,u,\vartheta) && (14.5.11)\\
&\leq C\varepsilon\left|{}^{(Y^{N-1}\breve{X})}\mathscr{X}\right|(s,u,\vartheta)+C\varepsilon\left|{}^{(Y^{N})}\mathscr{X}\right|(s,u,\vartheta)\\
&\quad+C\varepsilon\left|\mathscr{Z}_*^{[1,N+1];\leq 2}\vec{\Psi}\right|(s,u,\vartheta)+C\varepsilon\left|\mathscr{Z}_*^{[1,N];\leq 1}\upgamma\right|(s,u,\vartheta)+C\varepsilon\left|\mathscr{Z}_{**}^{[1,N];\leq 1}\underline{\upgamma}\right|(s,u,\vartheta).
\end{aligned}$$

The new features are that the second term on RHS (14.5.10) does not contain a small factor ε and that both RHS (14.5.10) and RHS (14.5.11) depend on ${}^{(Y^N)}\mathscr{X}$ (that is, the estimate for ${}^{(Y^{N-1}\breve{X})}\mathscr{X}$ does not decouple from the one for ${}^{(Y^N)}\mathscr{X}$). To obtain (14.5.10), we use the commutator estimate (9.5.7a) with $f=\mathrm{tr}_{\not g}\upchi$ as before, but now with $M=1$. Also using the L^∞ estimates of Prop. 9.12, (9.6.2) with $M=1$, and (9.4.1c) we deduce that $\left|\upmu[L,Y^{N-1}\breve{X}]\mathrm{tr}_{\not g}\upchi\right| \lesssim \varepsilon\left|\upmu Y^{N-1}\breve{X}\mathrm{tr}_{\not g}\upchi\right|+\left|\upmu Y^N\mathrm{tr}_{\not g}\upchi\right|$ plus error terms that are bounded in magnitude by $\lesssim$ RHS (14.5.10). We then use definition (7.2.1a) and the estimate (14.5.1b) to deduce that $\varepsilon\left|\upmu Y^{N-1}\breve{X}\mathrm{tr}_{\not g}\upchi\right| = \varepsilon\left|{}^{(Y^{N-1}\breve{X})}\mathscr{X}\right|$ plus error terms with magnitude $\leq$ the sum of the last three terms on RHS (14.5.10) and $\left|\upmu Y^N\mathrm{tr}_{\not g}\upchi\right| = \left|{}^{(Y^N)}\mathscr{X}\right|$ plus error terms with magnitude $\leq$ the sum of the last three terms on RHS (14.5.10). We have thus proved (14.5.10). Inequality (14.5.11) can be proved using similar arguments (without the help of a commutator estimate).

We now recall that we can rewrite (7.2.4) (with $Y^{N-1}\breve{X}$ in the role of $\mathscr{Z}_*^{\leq N;\leq 1}$ in that equation) in the form $L\left(\iota^{(Y^{N-1}\breve{X})}\mathscr{X}\right)=\iota\times$RHS (7.2.4) and integrate the resulting equation with respect to s from the initial time 0 to time t. With the help of of the estimates obtained in the previous paragraph, we can obtain a pointwise estimate for $\iota^{(Y^{N-1}\breve{X})}\mathscr{X}(t,u,\vartheta)$, much as in the case of $\iota^{(Y^N)}\mathscr{X}(t,u,\vartheta)$, where we use Gronwall's inequality to handle the first terms $C\varepsilon\left|{}^{(Y^{N-1}\breve{X})}\mathscr{X}\right|$ on RHS (14.5.10) and RHS (14.5.11). The new step compared to the argument for ${}^{(Y^N)}\mathscr{X}$ is that we insert the already proven bound (14.5.2) in order to handle the terms $\left| {}^{(Y^N)}\mathscr{X} \right| (s,u,\vartheta)$ on RHS (14.5.10) and RHS (14.5.11). In view of the fact that we are using Gronwall's inequality, we see that the bound (14.5.2) leads to the presence (on the RHS of the pointwise estimate $\left| {}^{(Y^{N-1}\breve{X})}\mathscr{X} \right| (t,u,\vartheta) \leq \cdots$) of additional integrals of the form

$$\int_{s=0}^{t} \frac{\upmu^2(t,u,\vartheta)}{\upmu^2(s,u,\vartheta)} \times \text{RHS } (14.5.2)(s,u,\vartheta)\, ds, \tag{14.5.12}$$

which we did not encounter in our proof of (14.5.2). To handle these additional integrals, we first bound the factor $\dfrac{\upmu^2(t,u,\vartheta)}{\upmu^2(s,u,\vartheta)}$ in (14.5.12) by $\leq C$ with the help of (14.5.5). Next, we use the L^∞ estimates of Prop. 9.12 and the estimate (14.5.1b) to bound the first integrand on RHS (14.5.2), evaluated at (s',u,ϑ), as follows:

$$\frac{[L\upmu(s',u,\vartheta)]_-}{\upmu(s',u,\vartheta)} \left| Y^N \mathfrak{X} \right| (s',u,\vartheta) \lesssim \frac{1}{\upmu_\star(s',u)} \left\{ \left| \mathscr{Z}_*^{[1,N+1];\leq 2}\Psi \right| + \left| \begin{pmatrix} \mathscr{Z}_{**}^{[1,N];\leq 1}\upgamma \\ \mathscr{Z}_{*}^{[1,N];\leq 1}\underline{\upgamma} \end{pmatrix} \right| \right\} (s',u,\vartheta). \tag{14.5.13}$$

Inserting the estimate (14.5.13) into the first integrand on RHS (14.5.2) (with s in the role of t on RHS (14.5.2) and s' in the role of t), we generate the double time integral stated in (14.5.3). The remaining two time integrals on RHS (14.5.13) also generate double time integrals, but they are less singular in that they do not involve the factor of $\dfrac{1}{\upmu_\star}$ present on RHS (14.5.13). Hence, these double time integrals are $\lesssim$ the single time integrals on RHS (14.5.2), in view of the following simple bound, which holds for non-negative scalar-valued functions f:

$$\begin{aligned} \int_{s=0}^{t}\int_{s'=0}^{s} f(s',u,\vartheta)\, ds'\, ds &\leq \int_{s=0}^{t}\int_{s'=0}^{t} f(s',u,\vartheta)\, ds'\, ds \\ &\leq t\int_{s'=0}^{t} f(s',u,\vartheta)\, ds' \leq C\int_{s=0}^{t} f(s,u,\vartheta)\, ds. \end{aligned} \tag{14.5.14}$$

Similarly, with the help of (14.5.5), we bound the time integral generated by the initial data term $\left| {}^{(Y^N)}\mathscr{X} \right| (0,u,\vartheta)$ on RHS (14.5.2) by $\leq C \int_{s'=0}^{t} \left| {}^{(Y^N)}\mathscr{X} \right| (0,u,\vartheta)\, ds' \leq C \left| {}^{(Y^N)}\mathscr{X} \right| (0,u,\vartheta)$. We have thus obtained the desired bound for $\left| {}^{(Y^{N-1}\breve{X})}\mathscr{X} \right| (t,u,\vartheta)$, which completes the proof of the lemma.

□

Armed with Lemma 14.8, we now derive the main result of this section.

Remark 14.2 (**Boxed constants affect high-order energy blowup-rates**). The "boxed constants" such as the $\boxed{2}$ and $\boxed{4}$ appearing on the RHS of inequality (14.5.15) are important because they affect the blowup-rates (that is, the powers of $\upmu_\star^{-1}$) featured on the right-hand sides of high-order energy estimates. Similar remarks apply to the boxed constants appearing on RHSs (15.2.1a), (15.7.1), (15.7.4), (15.9.1a), and (15.9.1b).

Proposition 14.9 (The key pointwise estimates for $(\breve{X}v^1)\mathscr{Z}_*^{N;\leq 1}\mathrm{tr}_{\not g}\upchi$). *Assume that $N=20$ and let $\mathscr{Z}_*^{N;\leq 1}\in\{Y^N, Y^{N-1}\breve{X}\}$. There exists a constant $C_*>0$ such that under the data-size and bootstrap assumptions of Sects. 8.1-8.4 and the smallness assumptions of Sect. 8.6, the following pointwise estimate holds on $\mathcal{M}_{T_{(Boot)},U_0}$:*

$$
\begin{aligned}
\left|(\breve{X}v^1)\mathscr{Z}_*^{N;\leq 1}\mathrm{tr}_{\not g}\upchi\right| &\leq \boxed{2}\left\|\frac{[L\upmu]_-}{\upmu}\right\|_{L^\infty(\Sigma_t^u)}\left|\breve{X}\mathscr{Z}_*^{N;\leq 1}v^1\right| \\
&\quad + C_*\frac{1}{\upmu_\star(t,u)}\left|\breve{X}\mathscr{Z}_*^{N;\leq 1}(\uprho-v^1)\right| \\
&\quad + \boxed{4}(1+C\varepsilon)\frac{\|[L\upmu]_-\|_{L^\infty(\Sigma_t^u)}}{\upmu_\star(t,u)}\int_{t'=0}^t\frac{\|[L\upmu]_-\|_{L^\infty(\Sigma_{t'}^u)}}{\upmu_\star(t',u)}\left|\breve{X}\mathscr{Z}_*^{N;\leq 1}v^1\right|(t',u,\vartheta)\,dt' \\
&\quad + C_*\frac{1}{\upmu_\star(t,u)}\int_{t'=0}^t\frac{1}{\upmu_\star(t',u)}\left|\breve{X}\mathscr{Z}_*^{N;\leq 1}(\uprho-v^1)\right|(t',u,\vartheta)\,dt' \\
&\quad + \mathrm{Error},
\end{aligned}
\tag{14.5.15}
$$

where

$$
\begin{aligned}
|\mathrm{Error}|(t,u,\vartheta) &\lesssim \frac{1}{\upmu_\star(t,u)}\left\{\left|{}^{(Y^N)}\mathscr{X}\right| + \left|{}^{(Y^{N-1}\breve{X})}\mathscr{X}\right|\right\}(0,u,\vartheta) \\
&\quad + \varepsilon\frac{1}{\upmu_\star}\left|\mathscr{Z}_*^{[1,N+1];\leq 2}\vec{\Psi}\right|(t,u,\vartheta) + \left|\mathscr{Z}_*^{[1,N+1];\leq 2}\vec{\Psi}\right|(t,u,\vartheta) \\
&\quad + \frac{1}{\upmu_\star(t,u)}\left|\mathscr{Z}_*^{[1,N];\leq 2}\vec{\Psi}\right|(t,u,\vartheta) + \frac{1}{\upmu_\star(t,u)}\left|\begin{pmatrix}\mathscr{Z}_{**}^{[1,N];\leq 1}\upgamma \\ \mathscr{Z}_*^{[1,N];\leq 1}\underline{\upgamma}\end{pmatrix}\right|(t,u,\vartheta) \\
&\quad + \frac{1}{\upmu_\star(t,u)}\int_{t'=0}^t\int_{s=0}^{t'}\frac{1}{\upmu_\star(s,u)}\left\{\left|\mathscr{Z}_*^{[1,N+1];\leq 2}\vec{\Psi}\right| + \left|\begin{pmatrix}\mathscr{Z}_{**}^{[1,N];\leq 1}\upgamma \\ \mathscr{Z}_*^{[1,N];\leq 1}\underline{\upgamma}\end{pmatrix}\right|\right\}(s,u,\vartheta)\,ds\,dt' \\
&\quad + \varepsilon\frac{1}{\upmu_\star(t,u)}\int_{t'=0}^t\frac{1}{\upmu_\star(t',u)}\left|\mathscr{Z}_*^{[1,N+1];\leq 2}\vec{\Psi}\right|(t',u,\vartheta)\,dt' \\
&\quad + \frac{1}{\upmu_\star(t,u)}\int_{t'=0}^t\left|\mathscr{Z}_*^{[1,N+1];\leq 2}\vec{\Psi}\right|(t',u,\vartheta)\,dt' \\
&\quad + \frac{1}{\upmu_\star(t,u)}\int_{t'=0}^t\frac{1}{\upmu_\star(t',u)}\left\{\left|\mathscr{Z}_*^{[1,N];\leq 2}\vec{\Psi}\right| + \left|\begin{pmatrix}\mathscr{Z}_{**}^{[1,N];\leq 1}\upgamma \\ \mathscr{Z}_*^{[1,N];\leq 1}\underline{\upgamma}\end{pmatrix}\right|\right\}(t',u,\vartheta)\,dt' \\
&\quad + \frac{1}{\upmu_\star(t,u)}\int_{t'=0}^t\upmu(t',u,\vartheta)\left|\mathscr{Z}_*^{N+1;\leq 1}\upomega\right|(t',u,\vartheta)\,dt' \\
&\quad + \frac{1}{\upmu_\star(t,u)}\int_{t'=0}^t\left|\mathscr{Z}^{\leq N;\leq 1}\upomega\right|(t',u,\vartheta)\,dt'.
\end{aligned}
\tag{14.5.16}
$$

Furthermore, we have the following less precise pointwise estimate:

$$
\begin{aligned}
&\left|\upmu\mathscr{Z}_*^{N;\leq 1}\mathrm{tr}_{\not g}\upchi\right| \\
&\lesssim \left\{\left|{}^{(Y^N)}\mathscr{X}\right| + \left|{}^{(Y^{N-1}\breve{X})}\mathscr{X}\right|\right\}(0,u,\vartheta)
\end{aligned}
\tag{14.5.17}
$$

$$
\begin{aligned}
&+ \upmu \left| \mathscr{P}^{N+1} \vec{\Psi} \right| + \left| \breve{X} \mathscr{Z}_*^{N;\leq 1} \vec{\Psi} \right| \\
&+ \left| \mathscr{Z}_*^{[1,N];\leq 2} \vec{\Psi} \right| + \left| \begin{pmatrix} \mathscr{Z}_{**}^{[1,N];\leq 1} \underline{\upgamma} \\ \mathscr{Z}_*^{[1,N];\leq 1} \upgamma \end{pmatrix} \right| \\
&+ \int_{t'=0}^t \frac{1}{\upmu_\star(t',u)} \left| \breve{X} \mathscr{Z}_*^{N;\leq 1} \vec{\Psi} \right| (t',u,\vartheta)\, dt' + \int_{t'=0}^t \left| \mathscr{Z}_*^{N+1;2} \vec{\Psi} \right| (t',u,\vartheta)\, dt' \\
&+ \int_{t'=0}^t \int_{s=0}^{t'} \frac{1}{\upmu_\star(s,u)} \left\{ \left| \mathscr{Z}_*^{[1,N+1];\leq 2} \vec{\Psi} \right| + \left| \begin{pmatrix} \mathscr{Z}_{**}^{[1,N];\leq 1} \underline{\upgamma} \\ \mathscr{Z}_*^{[1,N];\leq 1} \upgamma \end{pmatrix} \right| \right\} (s,u,\vartheta)\, ds\, dt' \\
&+ \int_{t'=0}^t \frac{1}{\upmu_\star(t',u)} \left\{ \left| \mathscr{Z}_*^{[1,N];\leq 2} \vec{\Psi} \right| + \left| \begin{pmatrix} \mathscr{Z}_{**}^{[1,N];\leq 1} \underline{\upgamma} \\ \mathscr{Z}_*^{[1,N];\leq 1} \upgamma \end{pmatrix} \right| \right\} (t',u,\vartheta)\, dt' \\
&+ \int_{t'=0}^t \upmu(t',u,\vartheta) \left| \mathscr{Z}_*^{N+1;\leq 1} \upomega \right| (t',u,\vartheta)\, dt' \\
&+ \int_{t'=0}^t \left| \mathscr{Z}^{\leq N;\leq 1} \upomega \right| (t',u,\vartheta)\, dt'.
\end{aligned}
$$

Proof. See Sect. 9.2 for some comments on the analysis. Throughout this proof, Error denotes any term verifying the estimate (14.5.16). We first prove (14.5.15) in the case $\mathscr{Z}_*^{N;\leq 1} = Y^N$. Using (7.2.1a)-(7.2.1b), the estimate (14.5.1a), and the simple bound $\|\breve{X} v^1\|_{L^\infty} \lesssim 1$ (see (9.6.3c)), we decompose

$$
(\breve{X} v^1) Y^N \mathrm{tr}_{\not g} \upchi = \frac{1}{\upmu} (\breve{X} v^1)^{(Y^N)} \mathscr{X} + \frac{1}{\upmu} (\breve{X} v^1) \vec{G}_{LL} \diamond Y^N \breve{X} \vec{\Psi} + \text{Error}. \tag{14.5.18}
$$

Next, we use the commutator estimate (9.5.7b) with $f = \vec{\Psi}$, the schematic identity (3.19.2b), and the L^∞ estimates of Prop. 9.12 to obtain

$$
\frac{1}{\upmu} (\breve{X} v^1) \vec{G}_{LL} \diamond Y^N \breve{X} \vec{\Psi} = \frac{1}{\upmu} (\breve{X} v^1) \vec{G}_{LL} \diamond \breve{X} Y^N \vec{\Psi} + \text{Error}. \tag{14.5.19}
$$

Recalling the definition (3.4.4) of $\vec{\Psi}$ and that $\vec{G}_{LL} \diamond \breve{X} \vec{\Psi} = \sum_{\imath=0}^2 G_{LL}^\imath \breve{X} \Psi_\imath$, and using the transport equation (3.14.1), we compute that

$$
\begin{aligned}
&\frac{1}{\upmu} (\breve{X} v^1) \vec{G}_{LL} \diamond \breve{X} Y^N \vec{\Psi} \qquad (14.5.20) \\
&= 2 \left(\frac{L \upmu}{\upmu} \right) \breve{X} Y^N v^1 \\
&\quad + \frac{1}{\upmu} (\breve{X} v^1) G_{LL}^0 \breve{X} Y^N (\uprho - v^1) \\
&\quad - \frac{1}{\upmu} (\breve{X} v^2) G_{LL}^2 \breve{X} Y^N v^1 + \frac{1}{\upmu} (\breve{X} v^1) G_{LL}^2 \breve{X} Y^N v^2 + \frac{1}{\upmu} \left\{ \breve{X} (v^1 - \uprho) \right\} G_{LL}^0 \breve{X} Y^N v^1 \\
&\quad + \left(\vec{G}_{LL} \diamond L \vec{\Psi} \right) \breve{X} Y^N v^1 + 2 \left(\vec{G}_{LX} \diamond L \vec{\Psi} \right) \breve{X} Y^N v^1.
\end{aligned}
$$

Using the schematic identity (3.19.2b) and the L^∞ estimates of Prop. 9.12, we deduce that the product on the second line of RHS (14.5.20) is bounded in magnitude by the second term $C_* \cdots$ on RHS (14.5.15). Using the L^∞ estimates of Prop. 9.12 (in particular (9.6.3a) and (9.6.4)) and the estimate (10.2.6), we deduce that the terms on the last two lines of RHS (14.5.20) are bounded in magnitude by $\lesssim$ the terms on the second line of RHS (14.5.16) and are therefore Error. To bound the product $2\left(\frac{L\upmu}{\upmu}\right)\breve{X}Y^N v^1$, we first split $L\upmu = [L\upmu]_+ - [L\upmu]_-$. From (11.2.1), we find that the product corresponding to $[L\upmu]_+$ is Error, while the product corresponding to $[L\upmu]_-$ is clearly bounded in magnitude by the first term on RHS (14.5.15).

It remains for us to bound the first product $\frac{1}{\upmu}(\breve{X}v^1)^{(Y^N)}\mathscr{X}$ on RHS (14.5.18). Our argument is based on equation (14.5.2). To proceed, we multiply both sides of (14.5.2) by $\frac{1}{\upmu}(\breve{X}v^1)$. We first bound the product generated by the first time integral $2(1+C\varepsilon)\cdots$ on RHS (14.5.2). Using (14.5.1a) and the simple bounds $|[L\upmu]_-|(s,u,\vartheta) \lesssim |L\upmu|(s,u,\vartheta) \lesssim 1$ (that is, (9.6.5b)) and $\left|\frac{1}{\upmu}(\breve{X}v^1)\right|(t,u) \lesssim \frac{1}{\upmu_\star(t,u)}$ (see (9.6.3c)), we express the product under consideration as

$$-2(1+C\varepsilon)\frac{1}{\upmu}(\breve{X}v^1)\int_{s=0}^{t}\frac{[L\upmu]_-(s,u,\vartheta)}{\upmu(s,u,\vartheta)}\left|\vec{G}_{LL}\diamond\breve{X}Y^N\vec{\Psi}\right|(s,u,\vartheta)\,ds + \text{Error}. \tag{14.5.21}$$

Next, we algebraically decompose the second factor in the integrand in (14.5.21) as

$$\vec{G}_{LL}\diamond\breve{X}Y^N\vec{\Psi} = \sum_{\imath=0}^{1} G_{LL}^{\imath}\breve{X}Y^N v^1 + G_{LL}^{2}\breve{X}Y^N v^2 + G_{LL}^{0}\breve{X}Y^N(\uprho - v^1). \tag{14.5.22}$$

Using the schematic identity (3.19.2b) and the L^∞ estimates of Prop. 9.12, we bound the magnitude of the time integral corresponding to the product $G_{LL}^0\breve{X}Y^N(\uprho - v^1)$ in (14.5.22) by $\leq$ the fourth term $C_*\frac{1}{\upmu_\star(t,u)}\int_{t'=0}^{t}\frac{1}{\upmu_\star(t',u)}\left|\breve{X}\mathscr{Z}_*^{N;\leq 1}(\uprho - v^1)\right|(t',u,\vartheta)\,dt'$ on RHS (14.5.15). Next, using (10.2.6), we bound the magnitude of the time integral corresponding to the product $G_{LL}^2\breve{X}Y^N v^2$ in (14.5.22) by the time-integral-involving product on RHS (14.5.16) featuring the small coefficient ε (and thus the time integral under consideration is of the form Error). We now bound the remaining time integral

$$2(1+C\varepsilon)\frac{1}{\upmu}(\breve{X}v^1)\int_{s=0}^{t}\frac{[L\upmu(s,u,\vartheta)]_-}{\upmu(s,u,\vartheta)}\left|\sum_{\imath=0}^{1} G_{LL}^{\imath}(s,u,\vartheta)\breve{X}Y^N v^1(s,u,\vartheta)\right|\,ds,$$

which is generated by the sum in (14.5.22). We first algebraically decompose

$$\begin{aligned}&\sum_{\imath=0}^{1} G_{LL}^{\imath}(s,u,\vartheta)\breve{X}Y^N v^1(s,u,\vartheta) \qquad (14.5.23)\\ &= \sum_{\imath=0}^{1} G_{LL}^{\imath}(t,u,\vartheta)\breve{X}Y^N v^1(s,u,\vartheta) + \left\{\sum_{\imath=0}^{1} G_{LL}^{\imath}(s,u,\vartheta) - \sum_{\imath=0}^{1} G_{LL}^{\imath}(t,u,\vartheta)\right\}\breve{X}Y^N v^1(s,u,\vartheta).\end{aligned}$$

Using the estimate (10.2.4) and the bounds $|[L\upmu]_-| \lesssim 1$ and $\left|\frac{1}{\upmu}(\breve{X}v^1)\right|(t,u) \lesssim \frac{1}{\upmu_\star(t,u)}$ noted above, we bound the magnitude of the time integral featuring the integrand factor $\sum_{\imath=0}^1 G_{LL}^\imath(s,u,\vartheta) - \sum_{\imath=0}^1 G_{LL}^\imath(t,u,\vartheta)$ by $\lesssim$ the time-integral-involving product on RHS (14.5.16) featuring the small coefficient ε (and thus the time integral under consideration is of the form Error). The remaining time integral that we must estimate contains the integrand factor $\sum_{\imath=0}^1 G_{LL}^\imath(t,u,\vartheta)$, which we may pull out of the ds integral. That is, we must bound

$$2(1+C\varepsilon)\left|\frac{1}{\upmu}(\breve{X}v^1)\sum_{\imath=0}^1 G_{LL}^\imath\right|(t,u,\vartheta)\int_{s=0}^t \frac{[L\upmu]_-(s,u,\vartheta)}{\upmu(s,u,\vartheta)}\left|\breve{X}Y^N v^1\right|(s,u,\vartheta)\,ds. \tag{14.5.24}$$

Next, using the transport equation (3.14.1), we algebraically decompose the factor outside of the integral in (14.5.24) as follows:

$$\begin{aligned}\frac{1}{\upmu}(\breve{X}v^1)\sum_{\imath=0}^1 G_{LL}^\imath &= 2\frac{L\upmu}{\upmu} \\ &\quad - \frac{1}{\upmu}G_{LL}^0\breve{X}(\uprho - v^1) - \frac{1}{\upmu}G_{LL}^2\breve{X}v^2 + \vec{G}_{LL}\diamond L\vec{\Psi} + 2\vec{G}_{LX}\diamond L\vec{\Psi}.\end{aligned} \tag{14.5.25}$$

Substituting the decomposition (14.5.25) into (14.5.24) and using the same arguments given in the lines just below (14.5.20), we bound the term (14.5.24) by

$$\leq 4(1+C\varepsilon)\left|\frac{[L\upmu]_-}{\upmu}(\breve{X}v^1)\right|(t,u,\vartheta)\int_{s=0}^t \frac{[L\upmu]_-(s,u,\vartheta)}{\upmu(s,u,\vartheta)}\left|\breve{X}Y^N v^1\right|(s,u,\vartheta)\,ds \tag{14.5.26}$$

plus a term that is $\leq$ the time-integral-involving product on RHS (14.5.16) featuring the small coefficient ε (and thus is of the form Error). Finally, we note that RHS (14.5.26) is $\leq$ the third term $\boxed{4}(1+C\varepsilon)\cdots$ on RHS (14.5.15) as desired. We have thus proved (14.5.15) in the case $\mathscr{Z}_*^{N;\leq 1} = Y^N$.

We now prove (14.5.15) in the remaining case $\mathscr{Z}_*^{N;\leq 1} = Y^{N-1}\breve{X}$. The proof is nearly identical to the case $\mathscr{Z}_*^{N;\leq 1} = Y^N$. The only difference is the presence of some additional error terms Error, which appear on RHS (14.5.16). The additional error terms, namely the second term on the first line of RHS (14.5.16) and the double time integral on RHS (14.5.16), are generated in view of equation (14.5.3) and the remarks located just above it.

The proof of (14.5.17) is based on a subset of the above arguments and is much simpler. We therefore omit the details, noting only that the main simplification is that we do not have to rely on the algebraic decompositions (14.5.22) and (14.5.25); we can instead crudely bound the terms on LHS (14.5.22) and (14.5.25).

□

14.6. **Pointwise estimates for the partially modified quantities.** In this section, we derive pointwise estimates for the partially modified quantities from Def. 7.2. We also derive pointwise estimates for their L derivative.

Lemma 14.10 (**Pointwise estimates for the partially modified quantities and their L derivative**). *Assume that $N = 20$ and let $\mathscr{Z}_*^{N-1;\leq 1} \in \{Y^{N-1}, Y^{N-2}\breve{X}\}$. Let* ${}^{(\mathscr{Z}_*^{N-1;\leq 1})}\widetilde{\mathscr{X}}$

be the partially modified quantity defined by (7.2.2a). *There exist constants*[109] $C > 0$ *and* $C_* > 0$ *such that under the data-size and bootstrap assumptions of Sects. 8.1-8.4 and the smallness assumptions of Sect. 8.6, the following pointwise estimate holds on* $\mathcal{M}_{T_{(Boot)},U_0}$*:*

$$\left|L^{(\mathscr{Z}_*^{N-1;\leq 1})}\widetilde{\mathscr{X}}\right| \leq \frac{1}{2}\left|\sum_{\imath=0}^{1} G_{LL}^{\imath}\right| \left|\not{\Delta}\mathscr{Z}_*^{N-1;\leq 1}v^1\right| + C_*\left|\not{d}\mathscr{Z}_*^{N;\leq 1}(\uprho - v^1)\right| \tag{14.6.1a}$$
$$+ C\varepsilon\left|\mathscr{Z}_*^{[1,N+1];\leq 1}\right| + C\left|\begin{pmatrix} \mathscr{Z}_{**}^{[1,N];0}\upgamma \\ \mathscr{Z}_*^{[1,N];\leq 1}\underline{\upgamma}\end{pmatrix}\right|,$$

$$\left|{}^{(\mathscr{Z}_*^{N-1;\leq 1})}\widetilde{\mathscr{X}}\right|(t,u,\vartheta) \leq \left|{}^{(\mathscr{Z}_*^{N-1;\leq 1})}\widetilde{\mathscr{X}}\right|(0,u,\vartheta) \tag{14.6.1b}$$
$$+ \frac{1}{2}\left|\sum_{\imath=0}^{1} G_{LL}^{\imath}\right|(t,u,\vartheta)\int_{t'=0}^{t}\left|\not{\Delta}\mathscr{Z}_*^{N-1;\leq 1}v^1\right|(t',u,\vartheta)\,dt'$$
$$+ C_*\int_{t'=0}^{t}\left|\not{d}\mathscr{Z}_*^{N;\leq 1}(\uprho - v^1)\right|(t',u,\vartheta)\,dt$$
$$+ C\varepsilon\int_{t'=0}^{t}\left|\mathscr{Z}_*^{[1,N+1];\leq 1}\vec{\Psi}\right|(t',u,\vartheta)\,dt'$$
$$+ C\int_{t'=0}^{t}\left|\begin{pmatrix} \mathscr{Z}_{**}^{[1,N];0}\upgamma \\ \mathscr{Z}_*^{[1,N];\leq 1}\underline{\upgamma}\end{pmatrix}\right|(t',u,\vartheta)\,dt'.$$

Proof. See Sect. 9.2 for some comments on the analysis. We first prove (14.6.1a) with the help of equation (7.2.5). We begin by algebraically decomposing the first product on RHS (7.2.5) as follows:

$$\frac{1}{2}\vec{G}_{LL}\diamond\not{\Delta}\mathscr{Z}_*^{N-1;\leq 1}\vec{\Psi} = \frac{1}{2}\sum_{\imath=0}^{1} G_{LL}^{\imath}\not{\Delta}\mathscr{Z}_*^{N-1;\leq 1}v^1 \tag{14.6.2}$$
$$+ \frac{1}{2}G_{LL}^{2}\not{\Delta}\mathscr{Z}_*^{N-1;\leq 1}v^2 + \frac{1}{2}G_{LL}^{0}\not{\Delta}\mathscr{Z}_*^{N-1;\leq 1}(\uprho - v^1).$$

Clearly the first sum on RHS (14.6.1a) arises from the first sum on RHS (14.6.2). Next, using (3.19.2b), the L^∞ estimates of Prop. 9.12, and (10.2.6), we find that $\|G_{LL}^0\|_{L^\infty(\Sigma_t^u)} \lesssim 1$ and $\|G_{LL}^2\|_{L^\infty(\Sigma_t^u)} \lesssim \varepsilon$. Hence, we can bound the terms on the last line of RHS (14.6.2) by the second and third terms on RHS (14.6.1a). Finally, to bound the terms on RHS (7.2.6), we simply quote (14.5.1e). We have thus proved (14.6.1a).

To derive (14.6.1b), we integrate (14.6.1a) along the integral curves of L as in (9.3.5). The only subtle point is that we bound the time integral of the first sum on RHS (14.6.1a) as

[109]For the purpose of the remainder of the proof, there is no need to distinguish between the constants C and C_*. Here, we just use C_* to denote the (large) constants which would in principle have caused the top-order energy to blow up with a worse rate if it were not for the fact that we have carefully distinguished between the energies $\mathbb{Q}_N$ and $\mathbb{Q}_N^{(Partial)}$ (and $\mathbb{K}_N$ and $\mathbb{K}_N^{(Partial)}$); see Remark 12.1. Similar remarks apply to later appearances of C_*.

follows by using (10.2.4) with $M=0$ and $s=t'$:

$$\begin{aligned}
&\int_{t'=0}^{t}\left\{\left|\sum_{\imath=0}^{1} G_{LL}^{\imath}\right|\left|\not\Delta \mathscr{Z}_{*}^{N-1;\leq 1} v^1\right|\right\}(t',u,\vartheta)\,dt' \qquad (14.6.3)\\
&\leq \left|\sum_{\imath=0}^{1} G_{LL}^{\imath}\right|(t,u,\vartheta)\int_{t'=0}^{t}\left|\not\Delta \mathscr{Z}_{*}^{N-1;\leq 1} v^1\right|(t',u,\vartheta)\,dt' \\
&\quad + C\varepsilon \int_{t'=0}^{t}\left|\mathscr{Z}_{*}^{[1,N+1];\leq 1} v^1\right|(t',u,\vartheta)\,dt'.
\end{aligned}$$

Note that the last term on RHS (14.6.3) is bounded by the next-to-last term on RHS (14.6.1b). We have thus proved (14.6.1b). □

14.7. **Pointwise estimates for the inhomogeneous terms in the wave equations.** In this section, we derive pointwise estimates for the derivatives of the inhomogeneous terms in the geometric wave equations (3.3.11a)-(3.3.11b).

We start with a lemma in which we decompose the derivatives of the ω-involving inhomogeneous terms on RHS (3.3.11a) into the main terms and error terms. By "main terms," we mean those products that involve the top-order derivatives of ω.

Lemma 14.11 (**Identification of the important wave equation inhomogeneous terms involving the top-order derivatives of the vorticity**). *Assume that* $1 \leq N \leq 20$. *Under the data-size and bootstrap assumptions of Sects. 8.1-8.4 and the smallness assumptions of Sect. 8.6, the following pointwise estimate holds on* $\mathcal{M}_{T_{(Boot)},U_0}$ *(see Sect. 6.2 regarding the vectorfield operator notation):*

$$\mathscr{P}^N\left\{[ia]\mu(\exp\rho)c_s^2 g_{ab}X^b L\omega\right\} = [ia]\mu(\exp\rho)c_s^2 g_{ab}X^b \mathscr{P}^N L\omega + \text{Error}, \qquad (14.7.1a)$$

$$\mathscr{P}^N\left\{[ia]\mu(\exp\rho)c_s^2\left(\frac{g_{ab}Y^b}{g_{cd}Y^cY^d}\right)Y\omega\right\} = [ia]\mu(\exp\rho)c_s^2\left(\frac{g_{ab}Y^b}{g_{cd}Y^cY^d}\right)\mathscr{P}^N Y\omega + \text{Error}, \qquad (14.7.1b)$$

where

$$|\text{Error}| \lesssim \varepsilon\left|\mathscr{Z}_{**}^{[1,N];\leq 1}\underline{\gamma}\right| + \left|\mathscr{P}^{\leq N}\omega\right|. \qquad (14.7.2)$$

Moreover, let $\mathscr{Z}_*^{N;1}$ *be a* N^{th} *order vectorfield operator containing exactly one factor of* $\breve{X}$*, and let* $\mathscr{P}^{N-1}$ *denote the remaining non-*$\breve{X}$ *factors. Then we have the following estimates:*

$$\mathscr{Z}_*^{N;1}\left\{[ia]\mu(\exp\rho)c_s^2 g_{ab}X^b L\omega\right\} = -[ia]\mu^2(\exp\rho)c_s^2 g_{ab}X^b \mathscr{P}^{N-1}LL\omega + \text{Error}, \qquad (14.7.3a)$$

$$\mathscr{Z}_*^{N;1}\left\{[ia]\mu(\exp\rho)c_s^2\left(\frac{g_{ab}Y^b}{g_{cd}Y^cY^d}\right)Y\omega\right\} = -[ia]\mu^2(\exp\rho)c_s^2\left(\frac{g_{ab}Y^b}{g_{cd}Y^cY^d}\right)\mathscr{P}^{N-1}YL\omega + \text{Error}, \qquad (14.7.3b)$$

where Error *satisfies* (14.7.2).

Proof. See Sect. 9.2 for some comments on the analysis. The estimates (14.7.1a)-(14.7.2) are a straightforward consequence of Lemma 3.19, (which implies that $\upmu(\exp\uprho)c_s^2 g_{ab}X^b = \mathrm{f}(\underline{\upgamma})$ and $\upmu(\exp\uprho)c_s^2\left(\frac{g_{ab}Y^b}{g_{cd}Y^cY^d}\right) = \mathrm{f}(\underline{\upgamma})$) and the L^∞ estimates of Prop. 9.12.

Similar remarks apply to the estimates (14.7.3a)-(14.7.3b). However, when bounding derivatives of $\upomega$ that contain the factor of $\breve{X}$, we first use the commutator estimate (9.5.7b) with $f = \upomega$ and the L^∞ estimates of Prop. 9.12 to commute the factor of $\breve{X}$ so that it hits $\upomega$ first; the commutator terms are of the form Error, where Error verifies (14.7.2). We then use the transport equation (3.3.11c) and the identity (3.7.15) to algebraically replace $\breve{X}\upomega$ with $-\upmu L\upomega$; this replacement is the origin of the extra factor of $\upmu$ on RHSs (14.7.3a)-(14.7.3b) compared to (14.7.1a)-(14.7.1b). Finally, we explicitly place the products containing the top-order (that is, order $N+1$) derivatives of $\upomega$ on RHSs (14.7.3a)-(14.7.3b) and again use the L^∞ estimates of Prop. 9.12 to conclude that the remaining products are of the form Error, where Error verifies (14.7.2). This completes the proof of the lemma. □

We now derive estimates for the derivatives of the null forms on RHSs (3.3.11a)-(3.3.11b).

Lemma 14.12 (**Estimates for the null forms**). *Assume that* $1 \leq N \leq 20$ *and let* $\mathscr{Q}^i$ *and* $\mathscr{Q}$ *be the null forms defined by* (3.3.12a) *and* (3.3.12b). *Under the data-size and bootstrap assumptions of Sects. 8.1-8.4 and the smallness assumptions of Sect. 8.6, the following estimates hold on* $\mathcal{M}_{T_{(Boot)},U_0}$ *(see Sect. 6.2 regarding the vectorfield operator notation):*

$$\left|\mathscr{Z}^{N;\leq 1}(\upmu\mathscr{Q}^i)\right|, \left|\mathscr{Z}^{N;\leq 1}(\upmu\mathscr{Q})\right| \lesssim \left|\mathscr{Z}_*^{[1,N+1];\leq 2}\vec{\Psi}\right| + \varepsilon\left|\mathscr{Z}_{**}^{[1,N];\leq 1}\underline{\upgamma}\right|. \tag{14.7.4}$$

Proof. See Sect. 9.2 for some comments on the analysis. The estimate (14.7.4) is a straightforward consequence of (3.19.4) and the L^∞ estimates of Prop. 9.12. □

14.8. **Pointwise estimates for the error terms generated by the multiplier vectorfield.** In this section, we derive pointwise estimates for the wave equation energy estimate error terms generated by the deformation tensor of the multiplier vectorfield T. That is, we obtain pointwise bounds for the terms ${}^{(T)}\mathfrak{P}_{(i)}[\Psi]$ (see (4.3.2)) corresponding the integrand $\upmu Q^{\alpha\beta}[\Psi]^{(T)}\pi_{\alpha\beta}$ on RHS (4.3.1).

Lemma 14.13 (**Pointwise bounds for the error terms generated by the deformation tensor of** T). *Let* Ψ *be a function*[110] *and consider the multiplier vectorfield error terms* ${}^{(T)}\mathfrak{P}_{(1)}[\Psi], \cdots, {}^{(T)}\mathfrak{P}_{(5)}[\Psi]$ *defined in* (4.3.3a)-(4.3.3e). *Let* $\varsigma > 0$ *be a real number. Under the data-size and bootstrap assumptions of Sects. 8.1-8.4 and the smallness assumptions of Sect. 8.6, the following pointwise inequality holds on* $\mathcal{M}_{T_{(Boot)},U_0}$ *(without any absolute value taken on the left), where the implicit constants are independent of* ς*:*

$$\begin{aligned}\sum_{i=1}^{5} {}^{(T)}\mathfrak{P}_{(i)}[\Psi] &\lesssim (1+\varsigma^{-1})(L\Psi)^2 + (1+\varsigma^{-1})(\breve{X}\Psi)^2 + \upmu|\not{d}\Psi|^2 + \varsigma\mathring{\updelta}_*|\not{d}\Psi|^2 \\ &\quad + \frac{1}{\sqrt{T_{(Boot)}-t}}\upmu|\not{d}\Psi|^2.\end{aligned} \tag{14.8.1}$$

[110]We will eventually apply this estimate with the role of Ψ played by a derivative of an element of $\{\uprho - v^1, v^1, v^2\}$.

Proof. See Sect. 9.2 for some comments on the analysis. Only the term ${}^{(T)}\mathfrak{P}_{(3)}[\Psi]$ is difficult to treat. Specifically, using the schematic relations (3.19.3c), (3.19.3d), and (3.19.3e), the estimate (9.3.3b), and the L^∞ estimates of Props. 9.12 and 10.1, it is straightforward to verify that the terms in braces on RHS (4.3.3a), (4.3.3b), (4.3.3d), and (4.3.3e) are bounded in magnitude by $\lesssim 1$. It follows that for $i = 1, 2, 4, 5$, $\left|{}^{(T)}\mathfrak{P}_{(i)}[\Psi]\right|$ is $\lesssim$ the sum of the terms on the first line of RHS (14.8.1). The quantities ς and $\mathring{\delta}_*$ appear on RHS (14.8.1) because we use Young's inequality to bound ${}^{(T)}\mathfrak{P}_{(4)}[\Psi] \lesssim |L\Psi||d\!\!/\Psi| \leq \varsigma^{-1}\mathring{\delta}_*^{-1}(L\Psi)^2 + \varsigma\mathring{\delta}_*|d\!\!/\Psi|^2 \leq C\varsigma^{-1}(L\Psi)^2 + C\varsigma\mathring{\delta}_*|d\!\!/\Psi|^2$. Similar remarks apply to ${}^{(T)}\mathfrak{P}_{(5)}[\Psi]$.

To bound the difficult term ${}^{(T)}\mathfrak{P}_{(3)}[\Psi]$, we also use the estimates (11.2.1) and (11.2.3), which allow us to bound the first two terms in braces on RHS (4.3.3c). Note that since no absolute value is taken on LHS (14.8.1), we may replace the factor $(\breve{X}\upmu)/\upmu$ from RHS (4.3.3c) with the factor $[\breve{X}\upmu]_+/\upmu$, which is bounded by (11.2.3). This completes our proof of (14.8.1). □

14.9. **Proof of Prop. 14.1.** See Sect. 9.2 for some comments on the analysis. We must derive estimates for the elements $\Psi \in \{\uprho - v^1, v^1, v^2\}$. To condense the notation, we use the following notation for the term in braces on RHS (5.0.1):

$$ {}^{(Z)}\mathscr{J}^{\alpha}[\Psi] := {}^{(Z)}\pi^{\alpha\beta}\mathscr{D}_\beta\Psi - \frac{1}{2}\mathrm{tr}_g{}^{(Z)}\pi\mathscr{D}^\alpha\Psi. \tag{14.9.1} $$

Throughout we silently use the Definition 14.1 of $Harmless^{\leq N}_{(Wave)}$ terms. We prove the estimates (14.2.1e) and (14.2.2d) (corresponding to $\Psi = v^i$), whose proofs are closely related, in detail. Later in the proof, we indicate the minor changes needed to obtain (14.2.1a)-(14.2.1d) and (14.2.2a)-(14.2.2c). At the very end of the proof, we indicate the minor changes needed to obtain (14.2.2e), that is, the estimates in the case $\Psi = \uprho - v^1$. To proceed, we iterate (5.0.1), use the wave equation (3.3.11a), use the decomposition (3.18.3), use the estimates (14.7.3a)-(14.7.3b) and (14.7.4), and use the estimates

$$ \left\|\mathscr{Z}^{\leq 10;\leq 1}\mathrm{tr}_{g\!\!/}{}^{(L)}\pi\!\!\!/\right\|_{L^\infty(\Sigma_t^u)}, \left\|\mathscr{Z}^{\leq 10;\leq 1}\mathrm{tr}_{g\!\!/}{}^{(Y)}\pi\!\!\!/\right\|_{L^\infty(\Sigma_t^u)}, \left\|\mathscr{P}^{\leq 10}\mathrm{tr}_{g\!\!/}{}^{(\breve{X})}\pi\!\!\!/\right\|_{L^\infty(\Sigma_t^u)} \lesssim 1 \tag{14.9.2} $$

(which follow from (3.13.4), (9.4.1a), and the L^∞ estimates of Prop. 9.12) to deduce that

$$ \begin{aligned} \upmu\square_{g(\vec{\Psi})}(\mathscr{Z}^{N-1;1}Yv^i) = {} & \mathscr{Z}^{N-1;1}\left(\upmu\mathscr{D}_\alpha{}^{(Y)}\mathscr{J}^{\alpha}[v^i]\right) \\ & - [ia]\upmu^2(\exp\uprho)c_s^2(g_{ab}X^b)\mathscr{P}^{N-1}LL\upomega \\ & + [ia]\upmu^2(\exp\uprho)c_s^2\left(\frac{g_{ab}Y^b}{g_{cd}Y^cY^d}\right)\mathscr{P}^{N-1}YL\upomega \\ & + \mathrm{Error}, \end{aligned} \tag{14.9.3} $$

where

$$|\text{Error}| \lesssim \sum_{\substack{N_1+N_2+N_3\leq N-1\\ N_2\leq N-2}} \sum_{M_1+M_2+M_3\leq 1} \sum_{P_1,P_2\in\mathscr{P}} \left(1+\left|\mathscr{Z}^{N_1;M_1}\mathrm{tr}_{\not g}{}^{(P_1)}\not\pi\right|\right)\left|\mathscr{Z}^{N_2;M_2}\left(\mu\mathscr{D}_\alpha^{(P_2)}\mathscr{J}^\alpha[\mathscr{Z}^{N_3;M_3}v^i]\right)\right| \tag{14.9.4}$$

$$+\sum_{\substack{N_1+N_2+N_3\leq N-1\\ N_2\leq N-2}} \sum_{P\in\mathscr{P}} \left(1+\left|\mathscr{P}^{N_1}\mathrm{tr}_{\not g}{}^{(P)}\not\pi\right|\right)\left|\mathscr{P}^{N_2}\left(\mu\mathscr{D}_\alpha^{(\breve{X})}\mathscr{J}^\alpha[\mathscr{P}^{N_3}v^i]\right)\right|$$

$$+\sum_{\substack{N_1+N_2+N_3\leq N-1\\ N_2\leq N-2}} \sum_{P\in\mathscr{P}} \left(1+\left|\mathscr{P}^{N_1}\mathrm{tr}_{\not g}{}^{(\breve{X})}\not\pi\right|\right)\left|\mathscr{P}^{N_2}\left(\mu\mathscr{D}_\alpha^{(P)}\mathscr{J}^\alpha[\mathscr{P}^{N_3}v^i]\right)\right|$$

$$+\left|\mathscr{Z}_*^{[1,N+1];\leq 2}\vec{\Psi}\right| + \begin{pmatrix} \left|\mathscr{Z}_{**}^{[1,N];\leq 1}\underline{\gamma}\right| \\ \left|\mathscr{Z}_*^{[1,N];\leq 2}\gamma\right| \end{pmatrix} + \left|\mathscr{P}^{\leq N}\omega\right|.$$

Note that the terms on the last line of RHS (14.9.3) are $Harmless^{\leq N}_{(Wave)}$ as desired.

Remark 14.3. For the purpose of proving (14.2.1e) and (14.2.2d), the estimate (14.9.4) is non-optimal in the sense that some terms on RHS (14.9.4) could be deleted and the inequality would remain true. However, those terms later appear when we are deriving the other estimates of the proposition. For this reason, we find it convenient to already include them on RHS (14.9.4).

Most of our effort goes towards estimating the first term on RHS (14.9.3). Equivalently, we may analyze the $\mathscr{Z}^{N-1;1}$ derivatives of the seven terms on RHS (5.0.2) (with $\Psi = v^i$ and $Z = Y$ in (5.0.2)). We will show that if $\mathscr{Z}^{N-1;1}$ contains no factor of L, then

$$\mathscr{Z}^{N-1;1}\mathscr{K}^{(Y)}_{(\pi-Danger)}[v^i] = (Y^{N-1}\breve{X}\mathrm{tr}_{\not g}\chi)\breve{X}v^i + Harmless^{\leq N}_{(Wave)}, \tag{14.9.5}$$

$$\mathscr{Z}^{N-1;1}\mathscr{K}^{(Y)}_{(\pi-Less\ Dangerous)}[v^i] = \mu y(\not d^{\#}Y^{N-2}\breve{X}\mathrm{tr}_{\not g}\chi)\cdot\not d v^i + Harmless^{\leq N}_{(Wave)}, \tag{14.9.6}$$

and

$$\begin{aligned} &\mathscr{Z}^{N-1;1}\mathscr{K}^{(Y)}_{(\pi-Cancel-1)}[v^i],\ \mathscr{Z}^{N-1;1}\mathscr{K}^{(Y)}_{(\pi-Cancel-2)}[v^i],\ \mathscr{Z}^{N-1;1}\mathscr{K}^{(Y)}_{(\pi-Good)}[v^i], \\ &\mathscr{Z}^{N-1;1}\mathscr{K}^{(Y)}_{(\Psi)}[v^i],\ \mathscr{Z}^{N-1;1}\mathscr{K}^{(Y)}_{(Low)}[v^i] \\ &= Harmless^{\leq N}_{(Wave)}. \end{aligned} \tag{14.9.7}$$

At the same time, we will show that if $\mathscr{Z}^{N-1;1}$ contains one or more factors of L (and thus $\mathscr{Z}^{N-1;1} = \mathscr{Z}_*^{N-1;1}$), then

$$\begin{aligned} &\mathscr{Z}^{N-1;1}\mathscr{K}^{(Y)}_{(\pi-Danger)}[v^i],\ \mathscr{Z}^{N-1;1}\mathscr{K}^{(Y)}_{(\pi-Less\ Dangerous)}[v^i], \\ &\mathscr{Z}^{N-1;1}\mathscr{K}^{(Y)}_{(\pi-Cancel-1)}[v^i],\ \mathscr{Z}^{N-1;1}\mathscr{K}^{(Y)}_{(\pi-Cancel-2)}[v^i],\ \mathscr{Z}^{N-1;1}\mathscr{K}^{(Y)}_{(\pi-Good)}[v^i], \\ &\mathscr{Z}^{N-1;1}\mathscr{K}^{(Y)}_{(\Psi)}[v^i],\ \mathscr{Z}^{N-1;1}\mathscr{K}^{(Y)}_{(Low)}[v^i] \\ &= Harmless^{\leq N}_{(Wave)}. \end{aligned} \tag{14.9.8}$$

After establishing (14.9.7)-(14.9.8), we will show that

$$\text{RHS (14.9.4)} = Harmless^{\leq N}_{(Wave)}. \tag{14.9.9}$$

Then combining (14.9.3), (14.9.7), (14.9.8), and (14.9.9), we conclude the desired estimates (14.2.1e) and (14.2.2d).

We now return to our analysis of the first term on RHS (14.9.3). We will separately analyze the $\mathscr{Z}^{N-1;1}$ derivative of each of the seven terms on RHS (5.0.2) (with $Z = Y$ and $\Psi = v^i$ in (5.0.2)).

Analysis of $\mathscr{Z}^{N-1;1}\mathscr{K}^{(Y)}_{(\pi-Danger)}[v^1]$. We apply to $\mathscr{Z}^{N-1;1}$ to (5.0.3a) (with $\Psi = v^i$ and $Z = Y$). We first analyze the difficult product in which all derivatives fall on the factor $\operatorname{d\!/iv}{}^{(Y)}\not{\pi}^{\#}_{L}$:

$$-(\mathscr{Z}^{N-1;1}\operatorname{d\!/iv}{}^{(Y)}\not{\pi}^{\#}_{L})\breve{X}v^i. \tag{14.9.10}$$

Using the estimate (14.4.3b) for the first term on the LHS and the simple bound $\|\breve{X}v^i\|_{L^\infty(\Sigma^u_t)} \lesssim 1$ (see (9.6.3a) and (9.6.3c)), we deduce from (14.9.10) that

$$-(\mathscr{Z}^{N-1;1}\operatorname{d\!/iv}{}^{(Y)}\not{\pi}^{\#}_{L})\breve{X}v^i = (Y\mathscr{Z}^{N-1;1}\mathrm{tr}_{\not{g}}\chi)\breve{X}v^i + Harmless^{\leq N}_{(Wave)}. \tag{14.9.11}$$

We first consider the case in which $\mathscr{Z}^{N-1;1}$ contains no factor of L, which is relevant for proving (14.9.5). Then $\mathscr{Z}^{N-1;1}$ contains $N-2$ factors of Y and one factor of $\breve{X}$. Thus, using using the commutator estimate (9.5.7b) with $f = \mathrm{tr}_{\not{g}}\chi$, N in the role of $N+1$, and $M = 1$, the estimate (9.4.1c), and the L^∞ estimates of Prop. 9.12, we may commute the factor of $\breve{X}$ so that it hits $\mathrm{tr}_{\not{g}}\chi$ first, thereby obtaining $(Y\mathscr{Z}^{N-1;1}\mathrm{tr}_{\not{g}}\chi)\breve{X}v^i = (Y^{N-1}\breve{X}\mathrm{tr}_{\not{g}}\chi)\breve{X}v^i + Harmless^{\leq N}_{(Wave)}$. The remaining terms obtained from applying $\mathscr{Z}^{N-1;1}$ to (5.0.3a) generate products involving $\leq N-2$ derivatives of $\operatorname{d\!/iv}{}^{(Y)}\not{\pi}^{\#}_{L}$. We will show that these products are $Harmless^{\leq N}_{(Wave)}$, which completes the proof of (14.9.5). To proceed, we again use the L^∞ estimates of Prop. 9.12, the estimate (9.4.1c), and (14.4.3b) (with $\leq N-2$ in the role of $N-1$ in (14.4.3b)) to deduce that all of the products under consideration are $Harmless^{\leq N}_{(Wave)}$. We clarify that the estimate (14.4.3b) (for the first term on the LHS) generates a factor of $\mathrm{tr}_{\not{g}}\chi$ with $\leq N-1$ derivatives on it (located on LHS (14.4.3b)), which is in contrast to the factor from (14.9.11) with N derivatives. This factor is below-top-order in the sense that we may bound it with (9.4.1c) and hence the corresponding product of this factor and $\breve{X}v^i$ contributes only to the $Harmless^{\leq N}_{(Wave)}$ terms. We have thus proved (14.9.5).

We now consider the case in which $\mathscr{Z}^{N-1;1}$ contains a factor of L, which is relevant for the estimate (14.2.2d) Noting that the estimate (14.9.11) still holds, we use the same commutator argument given in the previous paragraph to obtain $(Y\mathscr{Z}^{N-1;1}\mathrm{tr}_{\not{g}}\chi)\breve{X}v^i = (\mathscr{Z}^{N-1;1}L\mathrm{tr}_{\not{g}}\chi)\breve{X}v^i + Harmless^{\leq N}_{(Wave)}$, where the operators $\mathscr{Z}^{N-1;1}$ on the LHS and RHS are not necessarily the same. Using (9.6.2) with $M = 1$ and the bound $\|\breve{X}v^i\|_{L^\infty(\Sigma^u_t)} \lesssim 1$ mentioned above, we deduce that $(\mathscr{Z}^{N-1;1}L\mathrm{tr}_{\not{g}}\chi) = Harmless^{\leq N}_{(Wave)}$. The remaining terms obtained from applying $\mathscr{Z}^{N-1;1}$ to (5.0.3a) are $Harmless^{\leq N}_{(Wave)}$ for the same reasons given in the previous paragraph. We have thus proved (14.9.8) for the term $\mathscr{Z}^{N-1;1}\mathscr{K}^{(Y)}_{(\pi-Danger)}[v^i]$.

Analysis of $\mathscr{Z}^{N-1;1}\mathscr{K}_{(\pi-Cancel-1)}^{(Y)}[v^i]$. We apply $\mathscr{Z}^{N-1;1}$ to (5.0.3b) (with $\Psi = v^i$ and $Z = Y$). We first analyze the difficult product in which all derivatives fall on the deformation tensor components:

$$\left\{\frac{1}{2}\mathscr{Z}^{N-1;1}\breve{X}\mathrm{tr}_{\not g}{}^{(Y)}\not\pi - \mathscr{Z}^{N-1;1}\mathrm{d\not{i}v}\,{}^{(Y)}\not\pi_{\breve{X}}^{\#} - \upmu\mathscr{Z}^{N-1;1}\mathrm{d\not{i}v}\,{}^{(Y)}\not\pi_{L}^{\#}\right\} Lv^i. \tag{14.9.12}$$

Using the second estimate in (14.4.3a) and the first and second estimates in (14.4.3b), we express the terms in brace in (14.9.12) as the sum of $Harmless_{(Wave)}^{\leq N}$ terms and terms involving the order N derivatives of $\upmu$ and $\mathrm{tr}_{\not g}\chi$, which *exactly cancel*. Also using the bound $\|Lv^i\|_{L^\infty(\Sigma_t^u)} \lesssim \varepsilon$ (see (9.6.3d)), we conclude that (14.9.12) $= Harmless_{(Wave)}^{\leq N}$. The remaining terms obtained from applying $\mathscr{Z}^{N-1;1}$ to (5.0.3b) can be shown to be $Harmless_{(Wave)}^{\leq N}$ by combining essentially the same argument with the schematic identity (3.19.2c) for y, the estimates (9.4.1c) and (9.4.2a), and the L^∞ estimates of Prop. 9.12. We have thus proved (14.9.7) and (14.9.8) for $\mathscr{Z}^{N-1;1}\mathscr{K}_{(\pi-Cancel-1)}^{(Y)}[v^i]$.

Analysis of $\mathscr{Z}^{N-1;1}\mathscr{K}_{(\pi-Cancel-2)}^{(Y)}[v^i]$. We apply $\mathscr{Z}^{N-1;1}$ to (5.0.3c) (with $\Psi = v^i$ and $Z = Y$). We first analyze the difficult product in which all derivatives fall on the deformation tensor components:

$$\left\{-\not{\mathcal{L}}_{\mathscr{Z}}^{N-1;1}\not{\mathcal{L}}_{\breve{X}}{}^{(Y)}\not\pi_{L}^{\#} + \not d^{\#}\mathscr{Z}^{N-1;1(Y)}\pi_{L\breve{X}}\right\}\cdot \not dv^i. \tag{14.9.13}$$

Using the first estimate in (14.4.3a) and the third estimate in (14.4.3b), we express the terms in braces in (14.9.13) as the sum of $Harmless_{(Wave)}^{\leq N}$ terms and terms involving the order N derivatives of $\upmu$ and $\mathrm{tr}_{\not g}\chi$, which *exactly cancel*. Also using the bound $\|\not dv^i\|_{L^\infty(\Sigma_t^u)} \lesssim \varepsilon$ (see (9.6.3d)), we conclude that (14.9.13) $= Harmless_{(Wave)}^{\leq N}$. The remaining terms obtained from applying $\mathscr{Z}^{N-1;1}$ to (5.0.3c) can be shown to be $Harmless_{(Wave)}^{\leq N}$ by combining essentially the same argument with the estimate (9.4.2a) and the L^∞ estimates of Prop. 9.12. We have thus proved (14.9.7) and (14.9.8) for $\mathscr{Z}^{N-1;1}\mathscr{K}_{(\pi-Cancel-2)}^{(Y)}[v^i]$.

Analysis of $\mathscr{Z}^{N-1;1}\mathscr{K}_{(\pi-Less\ Dangerous)}^{(Y)}[v^i]$. We apply $\mathscr{Z}^{N-1;1}$ to (5.0.3d) (with $\Psi = v^i$ and $Z = Y$). We first analyze the difficult product in which all derivatives fall on the deformation tensor component:

$$\frac{1}{2}\upmu(\not{\mathcal{L}}_{\mathscr{Z}}^{N-1;1}\not d^{\#}\mathrm{tr}_{\not g}{}^{(Y)}\not\pi)\cdot \not dv^i. \tag{14.9.14}$$

Using the fourth estimate in (14.4.3b) and the simple bounds $\|\not dv^i\|_{L^\infty(\Sigma_t^u)} \lesssim \varepsilon$ (see (9.6.3d)), $\|\upmu\|_{L^\infty(\Sigma_t^u)} \lesssim 1$ (see (9.6.5a)), and $\|y\|_{L^\infty(\Sigma_t^u)} \lesssim \varepsilon$ (which follows from (3.19.2c), (9.6.3d), and (9.6.7a)), we deduce that

$$\frac{1}{2}\upmu(\not{\mathcal{L}}_{\mathscr{Z}}^{N-1;1}\not d^{\#}\mathrm{tr}_{\not g}{}^{(Y)}\not\pi)\cdot \not dv^i = \upmu y(\not d^{\#}\mathscr{Z}_*^{N-1;M}\mathrm{tr}_{\not g}\chi)\cdot \not dv^i + Harmless_{(Wave)}^{\leq N}. \tag{14.9.15}$$

We first consider the case in which $\mathscr{Z}^{N-1;1}$ contains no factor of L, which is relevant for proving (14.9.6). Then $\mathscr{Z}^{N-1;1}$ contains $N-2$ factors of Y and one factor of $\breve{X}$. We write $\not d^{\#}\mathscr{Z}^{N-1;M}\mathrm{tr}_{\not g}\chi = \not g^{-1}\cdot\not{\mathcal{L}}_{\mathscr{Z}}^{N-1;M}\not d\mathrm{tr}_{\not g}\chi$, and use the commutator estimate (9.5.9a) with

$\xi = \not{d}\mathrm{tr}_{\not{g}}\chi$, $N-1$ in the role of N, and $M=1$, the estimate (9.4.1c), and the L^∞ estimates of Prop. 9.12 to commute the factor of $\breve{X}$ so that it hits $\mathrm{tr}_{\not{g}}\chi$ first, thereby obtaining $\upmu(\not{d}^{\#}\mathscr{Z}^{N-1;M}\mathrm{tr}_{\not{g}}\chi)\cdot\not{d}v^i = \upmu(\not{d}^{\#}Y^{N-2}\breve{X}\mathrm{tr}_{\not{g}}\chi)\cdot\not{d}v^i + Harmless^{\leq N}_{(Wave)}$. The remaining terms obtained from applying $\mathscr{Z}^{N-1;1}$ to (5.0.3d) generate products involving $\leq N-2$ derivatives of $\not{d}\mathrm{tr}_{\not{g}}{}^{(Y)}\not{\pi}$. We will show that these products are $Harmless^{\leq N}_{(Wave)}$. To proceed, we again use the L^∞ estimates of Prop. 9.12, the estimate (9.4.1c), the estimate $\|y\|_{L^\infty(\Sigma^u_t)} \lesssim \varepsilon$ mentioned above, and the fourth estimate in (14.4.3b) (with $\leq N-2$ in the role of $N-1$ in (14.4.3b)) to deduce that all of the products under consideration are $Harmless^{\leq N}_{(Wave)}$. We clarify that the estimate (14.4.3b) generates a factor of $\mathrm{tr}_{\not{g}}\chi$ with $\leq N-1$ derivatives on it (located on LHS (14.4.3b)), which is in contrast to the factor from (14.9.15) with N derivatives. This factor is below-top-order in the sense that we may bound it with (9.4.1c), and the corresponding product $\upmu y(\not{d}^{\#}\mathscr{Z}^{\leq N-2;M}\mathrm{tr}_{\not{g}}\chi)\cdot\not{d}v^i$ contributes only to the $Harmless^{\leq N}_{(Wave)}$ terms. The remaining terms obtained from applying $\mathscr{Z}^{N-1;1}$ to (5.0.3d) can be shown to be $Harmless^{\leq N}_{(Wave)}$ by using essentially the same argument and the L^∞ estimates of Prop. 9.12. We have thus proved (14.9.6).

We now consider the case in which $\mathscr{Z}^{N-1;1}$ contains a factor of L, which is relevant for proving (14.2.2d). Noting that the formula (14.9.15) still holds, we use the same commutator argument given in the previous paragraph to obtain $\frac{1}{2}\upmu(\not{\mathcal{L}}^{N-1;1}_{\mathscr{Z}}\not{d}^{\#}\mathrm{tr}_{\not{g}}{}^{(Y)}\not{\pi})\cdot\not{d}v^i = \upmu y(\not{d}^{\#}\mathscr{Z}^{N-1;M}L\mathrm{tr}_{\not{g}}\chi)\cdot\not{d}v^i + Harmless^{\leq N}_{(Wave)}$. Moreover, using the L^∞ estimates of Prop. 9.12, (3.19.2c), and(9.6.2), we deduce that $\left|\upmu y(\not{d}^{\#}\mathscr{Z}^{N-1;M}L\mathrm{tr}_{\not{g}}\chi)\cdot\not{d}v^i\right| \lesssim \left|\mathscr{Z}^{\leq N-1;\leq 1}L\mathrm{tr}_{\not{g}}\chi\right| = Harmless^{\leq N}_{(Wave)}$. We have thus proved (14.9.8) for the term $\mathscr{Z}^{N-1;1}\mathscr{K}^{(Y)}_{(\pi-Less\ Dangerous)}[v^i]$.

Analysis of $\mathscr{Z}^{N-1;1}\mathscr{K}^{(Y)}_{(\pi-Good)}[v^i]$. We apply $\mathscr{Z}^{N-1;1}$ to (5.0.3e) (with $\Psi = v^i$ and $Z = Y$). The main point is that all deformation tensor components on RHS (5.0.3e) are hit with an L derivative. We may therefore bound the products under consideration using (14.4.6)-(14.4.7b) and the L^∞ estimates of Prop. 9.12, thus concluding that all products are $Harmless^{\leq N}_{(Wave)}$. We have therefore proved (14.9.7) and (14.9.8) for $\mathscr{Z}^{N-1;1}\mathscr{K}^{(Y)}_{(\pi-Good)}[v^i]$.

Analysis of $\mathscr{Z}^{N-1;1}\mathscr{K}^{(Y)}_{(\Psi)}[v^i]$. The terms in $\mathscr{K}^{(Y)}_{(\Psi)}[v^i]$ (see (5.0.4), where $\Psi = v^i$ and $Z = Y$) are of the form $\mathrm{f}(\underline{\gamma})\pi PZv^i + \mathrm{f}(\underline{\gamma})\not{\Delta}v^i$ where $P \in \mathscr{P}$, $Z \in \mathscr{Z}$, and π is one of the following components of ${}^{(Y)}\pi$: $\pi \in \left\{\mathrm{tr}_{\not{g}}{}^{(Y)}\not{\pi}, {}^{(Y)}\pi_{L\breve{X}}, {}^{(Y)}\pi_{\breve{X}X}, {}^{(Y)}\not{\pi}^{\#}_{L}, {}^{(Y)}\not{\pi}^{\#}_{\breve{X}}\right\}$. We now apply $\mathscr{Z}^{N-1;1}$ to the expression $\mathrm{f}(\underline{\gamma})\pi PZv^i + \mathrm{f}(\underline{\gamma})\not{\Delta}v^i$ and use (9.5.10), (14.4.8a)-(14.4.8b), and the L^∞ estimates of Prop. 9.12, thereby concluding that all products under consideration are $Harmless^{\leq N}_{(Wave)}$ as desired. We have thus proved (14.9.7) and (14.9.8) for $\mathscr{Z}^{N-1;1}\mathscr{K}^{(Y)}_{(\Psi)}[v^i]$.

Analysis of $\mathscr{Z}^{N-1;1}\mathscr{K}^{(Y)}_{(Low)}[v^i]$. Using Lemma 3.19, we see that (see (5.0.5), where $\Psi = v^i$ and $Z = Y$) $\mathscr{K}^{(Y)}_{(Low)}[v^i] = \mathrm{f}(\mathscr{P}^{\leq 1}\underline{\gamma}, \not{g}^{-1}, \not{d}\vec{x}, \breve{X}\vec{\Psi})\pi P\gamma$ where π and P are as in the previous paragraph. Hence, we conclude that $\mathscr{Z}^{N-1;1}\mathscr{K}^{(Y)}_{(Low)}[v^i] = Harmless^{\leq N}$ by using the same arguments as in the previous paragraph together with Lemmas 9.5 and 9.6 (to bound the derivatives of $\not{g}^{-1}$ and $\not{d}\vec{x}$). We have thus proved (14.9.7) and (14.9.8) for $\mathscr{Z}^{N-1;1}\mathscr{K}^{(Y)}_{(Low)}[v^i]$.

We have thus established (14.9.7)-(14.9.8), which completes the analysis of the desired estimates for $\Psi \in \{v^1, v^2\}$, except for the error term bound (14.9.4) (which we derive below), in the case that the commutator operator is of the form $\mathscr{Z}^{N-1;1}Y$, where $\mathscr{Z}^{N-1;1}$ contains exactly one factor of $\breve{X}$. We must also establish similar estimates in the remaining cases, corresponding to the following operators on LHSs (14.2.1a)-(14.2.1d) and (14.2.2a)-(14.2.2c):

(1) $Y^{N-1}L$
(2) Y^N
(3) $Y^{N-1}\breve{X}$
(4) $\mathscr{Z}^{N-1;1}L$ (where $\mathscr{Z}^{N-1;1}$ contains exactly one factor of $\breve{X}$ with all other factors equal to Y)
(5) $\mathscr{P}^{N-1}L$ (where $\mathscr{P}^{N-1}$ contains one or more factors of L)
(6) $\mathscr{Z}^{N-1;1}L$ (where $\mathscr{Z}^{N-1;1}$ contains one or more factors of L)
(7) $\mathscr{P}^{N-1}\breve{X}$ (where $\mathscr{P}^{N-1}$ contains one or more factors of L)

In these remaining seven cases, we can obtain an analog of the estimate (14.9.3) by using the same arguments, which are based on Lemma 14.11 and the estimates (14.7.4) and (14.9.2). We note that the error term bound (14.9.4) remains correct as stated in all of these cases. We also note that the estimates of Lemma 14.11 yield the explicitly listed terms on RHSs (14.2.1a)-(14.2.1d) and (14.2.2a)-(14.2.2c) that depend on the order $N+1$ derivatives of ω. Moreover, in these remaining cases, we can use essentially the same arguments that we used in the case $\mathscr{Z}^{N-1;1}Y$ to establish pointwise estimates for the main term (that is, the analog of the first one on RHS (14.9.3)). That is, with the help of Lemmas 14.3 and 14.5, we can establish analogs of (14.9.7)-(14.9.8). The estimates are very similar in nature, the only difference being the details of the important terms generated by Lemma 14.5; the corresponding important products are precisely the ones on explicitly listed on RHSs (14.2.1a)-(14.2.1d) that depend on N derivatives of $\mathrm{tr}_{\not g}\chi$. More precisely, an argument similar to the one that we gave in the case $\mathscr{Z}^{N-1;1}Y$ yields that in the remaining seven cases stated above, the estimate (14.9.5) must respectively be replaced with

$$Y^{N-1}\mathscr{K}_{(\pi-Danger)}^{(L)}[v^i] = 0, \tag{14.9.16}$$

$$Y^{N-1}\mathscr{K}_{(\pi-Danger)}^{(Y)}[v^i] = (\breve{X}v^i)Y^N\mathrm{tr}_{\not g}\chi + Harmless_{(Wave)}^{\leq N}, \tag{14.9.17}$$

$$Y^{N-1}\mathscr{K}_{(\pi-Danger)}^{(\breve{X})}[v^i] = (\breve{X}v^i)Y^{N-1}\breve{X}\mathrm{tr}_{\not g}\chi + Harmless_{(Wave)}^{\leq N}, \tag{14.9.18}$$

$$\mathscr{Z}^{N-1;1}\mathscr{K}_{(\pi-Danger)}^{(L)}[v^i] = 0, \tag{14.9.19}$$

$$\mathscr{P}^{N-1}\mathscr{K}_{(\pi-Danger)}^{(L)}[v^i] = 0, \tag{14.9.20}$$

$$\mathscr{Z}^{N-1;1}\mathscr{K}_{(\pi-Danger)}^{(L)}[v^i] = 0, \tag{14.9.21}$$

$$\mathscr{P}^{N-1}\mathscr{K}_{(\pi-Danger)}^{(\breve{X})}[v^i] = Harmless_{(Wave)}^{\leq N}, \tag{14.9.22}$$

while the estimate (14.9.6) must respectively be replaced with

$$Y^{N-1}\mathscr{K}_{(\pi-Less\ Dangerous)}^{(L)}[v^i] = (\not{d}^{\#}v^i)\cdot(\upmu\not{d}Y^{N-1}\mathrm{tr}_{\not{g}}\chi) + Harmless_{(Wave)}^{\leq N}, \tag{14.9.23}$$

$$Y^{N-1}\mathscr{K}_{(\pi-Less\ Dangerous)}^{(Y)}[v^i] = y(\not{d}^{\#}v^i)\cdot(\upmu\not{d}Y^{N-1}\mathrm{tr}_{\not{g}}\chi) + Harmless_{(Wave)}^{\leq N}, \tag{14.9.24}$$

$$Y^{N-1}\mathscr{K}_{(\pi-Less\ Dangerous)}^{(\breve{X})}[v^i] = -(\upmu\not{d}^{\#}v^i)\cdot(\upmu\not{d}Y^{N-1}\mathrm{tr}_{\not{g}}\chi) + Harmless_{(Wave)}^{\leq N}, \tag{14.9.25}$$

$$\mathscr{Z}^{N-1;1}\mathscr{K}_{(\pi-Less\ Dangerous)}^{(L)}[v^i] = (\not{d}^{\#}v^i)\cdot(\upmu\not{d}Y^{N-2}\breve{X}\mathrm{tr}_{\not{g}}\chi) + Harmless_{(Wave)}^{\leq N}, \tag{14.9.26}$$

$$\mathscr{P}^{N-1;1}\mathscr{K}_{(\pi-Less\ Dangerous)}^{(L)}[v^i] = Harmless_{(Wave)}^{\leq N}, \tag{14.9.27}$$

$$\mathscr{Z}^{N-1;1}\mathscr{K}_{(\pi-Less\ Dangerous)}^{(L)}[v^i] = Harmless_{(Wave)}^{\leq N}, \tag{14.9.28}$$

$$\mathscr{P}^{N-1}\mathscr{K}_{(\pi-Less\ Dangerous)}^{(\breve{X})}[v^i] = Harmless_{(Wave)}^{\leq N} \tag{14.9.29}$$

(and all remaining terms are $Harmless_{(Wave)}^{\leq N}$, as in (14.9.7)).

Having treated the difficult main term in all cases, we now establish (14.9.9). We start by bounding the terms on RHS (14.9.4) involving ≤ 10 derivatives of the factors $\mathrm{tr}_{\not{g}}{}^{(L)}\not{\pi}$, $\mathrm{tr}_{\not{g}}{}^{(\breve{X})}\not{\pi}$, and $\mathrm{tr}_{\not{g}}{}^{(Y)}\not{\pi}$. Using (14.9.2), we see that it suffices to show that

$$\begin{aligned}
&\sum_{\substack{N_2+N_3\leq N-1\\ N_2\leq N-2}}\ \sum_{M_2+M_3\leq 1}\ \sum_{P_1,P_2\in\mathscr{P}} \left|\mathscr{Z}^{N_2;M_2}\left(\upmu\mathscr{D}_{\alpha}^{(P_2)}\mathscr{J}^{\alpha}[\mathscr{Z}^{N_3;M_3}v^i]\right)\right|, \\
&\sum_{\substack{N_2+N_3\leq N-1\\ N_2\leq N-2}}\ \sum_{P\in\mathscr{P}} \left|\mathscr{P}^{N_2}\left(\upmu\mathscr{D}_{\alpha}^{(\breve{X})}\mathscr{J}^{\alpha}[\mathscr{P}^{N_3}v^i]\right)\right|, \\
&\sum_{\substack{N_2+N_3\leq N-1\\ N_2\leq N-2}}\ \sum_{P\in\mathscr{P}} \left|\mathscr{P}^{N_2}\left(\upmu\mathscr{D}_{\alpha}^{(P)}\mathscr{J}^{\alpha}[\mathscr{P}^{N_3}v^i]\right)\right| \\
&= Harmless_{(Wave)}^{\leq N}.
\end{aligned} \tag{14.9.30}$$

It suffices to again decompose, with the help of (5.0.2), the terms $\mathscr{D}_{\alpha}^{(\cdot)}\mathscr{J}^{\alpha}$ in (14.9.30) and to show that all constituent parts, such as $\mathscr{Z}^{N_2;M_2}\mathscr{K}_{(\pi-Danger)}^{(Y)}[\mathscr{Z}^{N_3;M_3}v^i]$ and $\mathscr{P}^{N_2}\mathscr{K}_{(Low)}^{(\breve{X})}[\mathscr{P}^{N_3}v^i]$, are $Harmless_{(Wave)}^{\leq N}$. To this end, we repeat the proofs of the above estimates, including (14.9.16)-(14.9.29), but with N_2 (from LHS (14.9.30)) in place of $N-1$ and $\mathscr{Z}^{N_3;M_3}v^i$ or $\mathscr{P}^{N_3}v^i$ in place of the explicitly written v^i factors. The same arguments given above yield that all products $= Harmless_{(Wave)}^{\leq N_2+N_3+1} \leq Harmless_{(Wave)}^{\leq N}$, except for the ones corresponding to the explicitly written ones on RHSs (14.9.5)-(14.9.6), (14.9.17), (14.9.18), and (14.9.23)-(14.9.26). For example, the analog of the explicitly written term on RHS (14.9.5) is $(Y^{N_2}\breve{X}\mathrm{tr}_{\not{g}}\chi)\breve{X}\mathscr{Z}^{N_3;M_3}v^i$ while the analog of the explicitly written term on RHS (14.9.6) is $\upmu y(\not{d}^{\#}Y^{N_2}\breve{X}\mathrm{tr}_{\not{g}}\chi)\cdot\not{d}\mathscr{Z}^{N_3;M_3}v^i$. We now explain why these explicitly written products are $Harmless_{(Wave)}^{\leq N}$ too. The important point is that since $N_2 \leq N-2$ on LHS (14.9.30), the factors of $\mathrm{tr}_{\not{g}}\chi$ in these products are hit with no more than $N-1$ derivatives. We may therefore pointwise bound these factors using (9.4.1c). Given this observation, the fact that the products under consideration are $Harmless_{(Wave)}^{\leq \max\{N_3,N_2+2\}} \leq Harmless_{(Wave)}^{\leq N}$ follows from the same arguments given in our prior analysis of $\mathscr{Z}^{N-1;1}\mathscr{K}^{(Y)}_{(\pi-Danger)}[\Psi], \cdots, \mathscr{Z}^{N-1;1}\mathscr{K}^{(Y)}_{(Low)}[v^i]$. We have thus shown that the products on RHS (14.9.4) involving ≤ 10 derivatives of the factors $\mathrm{tr}_{\not g}{}^{(L)}\not\pi$, $\mathrm{tr}_{\not g}{}^{(\breve{X})}\not\pi$, and $\mathrm{tr}_{\not g}{}^{(Y)}\not\pi$ are $Harmless^{\leq N}_{(Wave)}$.

To complete the proof of (14.9.9), we must bound the terms on RHS (14.9.4) with $11 \leq N_1 \leq N-1 \leq 19$ (which implies that $N_2 + N_3 \leq 8$). The arguments given in the previous paragraph imply that the factors on RHS (14.9.4) corresponding to N_2, such as $\mathscr{Z}^{N_2;M_2}\left(\upmu\mathscr{D}^{(P_2)}_\alpha\mathscr{J}^\alpha[\mathscr{Z}^{N_3;M_3}\Psi]\right)$, are $Harmless^{\leq\max\{N_2+N_3+1,N_2+2\}}_{(Wave)} \leq Harmless^{\leq 10}_{(Wave)}$. In particular, the L^∞ estimates of Prop. 9.12 imply that $\|Harmless^{\leq 10}_{(Wave)}\|_{L^\infty(\Sigma_t^u)} \lesssim \varepsilon$. Moreover, from (3.13.4) and (9.4.1b), we find that the factors $\mathscr{Z}^{N_1;M_1}\mathrm{tr}_{\not g}{}^{(P_1)}\not\pi$, $\mathscr{P}^{N_1}\mathrm{tr}_{\not g}{}^{(P)}\not\pi$, and $\mathscr{P}^{N_1}\mathrm{tr}_{\not g}{}^{(\breve{X})}\not\pi$ on RHS (14.9.4) $= Harmless^{\leq N_1+1}_{(Wave)}$. Combining this bound with the estimate $\|Harmless^{\leq 10}_{(Wave)}\|_{L^\infty(\Sigma_t^u)} \lesssim \varepsilon$, we conclude that the products under consideration $= Harmless^{\leq N_1+1}_{(Wave)} \leq Harmless^{\leq N}_{(Wave)}$ as desired. We have thus proved (14.9.9), which completes the proof of Prop. 14.1 except for the estimates (14.2.2e) for the quantity $\uprho - v^1$.

To prove (14.2.2e), we first subtract (3.3.11a) with $i = 1$ from (3.3.11b) to obtain

$$\upmu\square_{g(\vec{\Psi})}(\uprho - v^1) = \upmu\mathscr{Q} - \upmu\mathscr{Q}^1 + [1a](\exp\uprho)c_s^2(\upmu\partial_a\upomega) - 2[1a](\exp\uprho)\upomega(\upmu B v^a). \tag{14.9.31}$$

That is, $\uprho - v^1$ solves the covariant wave equation with inhomogeneous terms equal to RHS (14.9.31). We now repeat the above proofs of the estimates for v^i, with $\uprho - v^1$ in the role of v^i and RHS (14.9.31) in the role of RHS (3.3.11a). Using nearly identical arguments, we obtain (14.2.2e). For clarity, we note that $\uprho - v^1 = \Psi_0 - \Psi_1$ (see Def. 3.4) and hence the derivatives of $\uprho - v^1$ can be controlled, via the triangle inequality, in terms of the derivatives of $\vec{\Psi}$. This completes the proof of Prop. 14.1.

□

14.10. **Proof of Prop. 14.2.** See Sect. 9.2 for some comments on the analysis. Throughout we silently use the Definition 14.1 of $Harmless^{\leq N}_{(Vort)}$ terms.

We first prove (14.2.3b). The main point is to identify the products that depend on the order $N+1$ derivatives of $\upmu$ or the order N derivatives of $\mathrm{tr}_{\not g}\upchi$; all other products will be shown to be $Harmless^{\leq N+1}_{(Vort)}$. To proceed, we apply Y^{N+1} to the transport equation (3.3.11c) and use (3.13.3b) to commute Y^{N+1} through $\upmu B$. Using also (9.4.3a) and (9.4.4b) with $M = 0$ and the L^∞ estimates of Prop. 9.12, we find that

$$\upmu B Y^{N+1}\upomega = -(Y^{N+1}\upmu)L\upomega + \left\{\upmu\not{\mathcal{L}}^{N}_{Y}{}^{(Y)}\not\pi^{\#}_{L} + \not{\mathcal{L}}^{N}_{Y}{}^{(Y)}\not\pi^{\#}_{\breve{X}}\right\}\cdot\not d\upomega + Harmless^{\leq N+1}_{(Vort)}. \tag{14.10.1}$$

Note that we have isolated all of the top-order derivatives of deformation tensors in the terms in braces on RHS (14.10.1). Moreover, we clarify that the smallness factors ε on RHS (14.1.1b) (with $N+1$ in the role of N) come from the estimate (9.6.9) for the low-order derivatives of $\upomega$, which appear as factors in quadratic terms that multiply high-order derivatives of deformation tensors. In addition, from (14.3.2), (14.4.4), and the simple bounds

$\|P\omega\|_{L^\infty(\Sigma_t^u)}$, $\|y\|_{L^\infty(\Sigma_t^u)} \lesssim \varepsilon$ (which follow from Lemma 3.19 and Prop. 9.12), we find that

$$(Y^{N+1}\mu)L\omega = g(Y,Y)(Y^{N-1}\breve{X}\mathrm{tr}_{\not g}\chi)L\omega + \mathrm{Error}, \tag{14.10.2}$$

$$\begin{aligned}\left\{\mu \not\mathcal{L}_Y^{N}\,{}^{(Y)}\not\pi_L^{\#} + \not\mathcal{L}_Y^{N}\,{}^{(Y)}\not\pi_{\breve{X}}^{\#}\right\}\cdot \not d\omega &= y(\not d^{\#} Y^N\mu)\cdot \not d\omega + Harmless_{(Vort)}^{\leq N+1} \\ &= y(Y^{N-1}\breve{X}\mathrm{tr}_{\not g}\chi)Y\omega + Harmless_{(Vort)}^{\leq N+1}.\end{aligned} \tag{14.10.3}$$

Combining (14.10.1) and (14.10.2)-(14.10.3), we arrive at the desired estimate (14.2.3b).

The proof of (14.2.3a) is similar but relies on (3.13.3a) in place of (3.13.3b) and (14.4.5) in place of (14.4.4); we omit the details, noting only that the term $-(Y^N L\mu)L\omega$, which is an analog of the first term on RHS (14.10.1), is $Harmless_{(Vort)}^{\leq N+1}$ in view of the bound $\|L\omega\|_{L^\infty(\Sigma_t^u)} \lesssim \varepsilon$ mentioned above and inequality (9.6.1b) with $M = 0$.

We now prove (14.2.5). Using the same arguments we used to derive (14.2.3a) and (14.2.3b), except now bounding the deformation tensor components from the formulas (3.13.3a) and (3.13.3b) in magnitude by $\lesssim 1$ via (14.4.8b) with $N = 1$ and the L^∞ estimates of Prop. 9.12, we deduce that the RHS of the equation $\mu BP\omega = \cdots$ is in magnitude $\lesssim |L\omega| + |\not d\omega| \lesssim \left|\mathscr{P}^{\leq 1}\omega\right|$. This implies (14.2.5).

We now prove (14.2.4), starting with the case $\mathscr{P}^{N+1} = \mathscr{P}^N Y$. Then the same arguments we used to prove (14.2.3b) yield

$$\mu B\mathscr{P}^N Y\omega = -(\mathscr{P}^N Y\mu)L\omega + \left\{\mu \not\mathcal{L}_{\mathscr{P}}^{N}\,{}^{(Y)}\not\pi_L^{\#} + \not\mathcal{L}_{\mathscr{P}}^{N}\,{}^{(Y)}\not\pi_{\breve{X}}^{\#}\right\}\cdot \not d\omega + Harmless_{(Vort)}^{\leq N+1}. \tag{14.10.4}$$

The key point is that by assumption, the operator $\mathscr{P}^N$ contains a factor of L. Hence, we may use the commutator estimate (9.5.7b) with $f = \mu$ and the L^∞ estimates of Prop. 9.12 to commute the factor of L in $\mathscr{P}^N Y\mu$ so that it hits μ first. Also using the pointwise estimate (9.6.1b), we find that $\mathscr{P}^N Y\mu = Harmless_{(Vort)}^{\leq N+1}$. To bound the terms $\not\mathcal{L}_{\mathscr{P}}^{N}\,{}^{(Y)}\not\pi_L^{\#}$ and $\not\mathcal{L}_{\mathscr{P}}^{N}\,{}^{(Y)}\not\pi_{\breve{X}}^{\#}$ in magnitude, we use the pointwise estimate (14.4.6). Combining these pointwise estimates with the L^∞ estimates of Prop. 9.12, we conclude that all terms on RHS (14.10.4) are of the form $Harmless_{(Vort)}^{\leq N+1}$. We have thus proved (14.2.4) in the case $\mathscr{P}^{N+1} = \mathscr{P}^N Y$.

To finish the proof of (14.2.4), it remains only for us to consider the case $\mathscr{P}^{N+1} = \mathscr{P}^N L$ where $\mathscr{P}^N$ contains a factor of L. Using the same arguments that we used to prove (14.2.3a), we derive the following analog of (14.10.4):

$$\mu B\mathscr{P}^N L\omega = -(\mathscr{P}^N L\mu)L\omega + \left\{\not\mathcal{L}_{\mathscr{P}}^{N}\,{}^{(L)}\not\pi_{\breve{X}}^{\#}\right\}\cdot \not d\omega + Harmless_{(Vort)}^{\leq N+1}, \tag{14.10.5}$$

where the operator $\mathscr{P}^N$ in (14.10.5) contains a factor of L. The remainder of the proof now proceeds as in the case $\mathscr{P}^{N+1} = \mathscr{P}^N Y$, but with the estimate (14.4.7a) in place of (14.4.6). This completes the proof of Prop. 14.2.

□

15. Energy estimates

In this section, we derive the most important estimates of the article: a priori energy estimates for the solution. The main result is Prop. 15.1 (see Sect. 15.1), which we prove in Sect. 15.16 via a lengthy Gronwall argument, after deriving many preliminary estimates. The main preliminary results of this section are Props. 15.3 and 15.4 (see Sect. 15.2), in which we derive, with the help of the pointwise estimates of Sect. 14 , integral inequalities for the fundamental L^2-controlling quantities defined in Sect. 12. To prove Props. 15.3 and 15.4, we must bound the error integrals on the right-hand sides of the energy-null flux identities of Props. 4.2 and 4.4 and their higher-order analogs. We divide the error integrals into various classes, which we bound in Sects. 15.4-15.12. We combine all of these estimates into proofs of Props. 15.3 and 15.4 in Sects. 15.14 and 15.13 respectively. Compared to the energy estimates in previous works, the new feature of the present work is that we derive a priori energy estimates for the specific vorticity and control various error integrals that depend on it. In particular, we control the influence that the specific vorticity has on the variables $\uprho$, v^1, and v^2, which solve covariant wave equations with vorticity-dependent source terms. We derive estimates for the specific vorticity itself in Sect. 15.15. To simplify our proofs, we use an energy bootstrap argument; see Sect. 15.3 for the bootstrap assumptions.

15.1. **Statement of the main a priori energy estimates.** We start by stating the proposition featuring our main a priori energy estimates. Its proof is located in Sect. 15.16.

Proposition 15.1 (**The main a priori energy estimates**). *Consider the L^2-controlling quantities $\{\mathbb{Q}_N(t,u)\}_{N=1,\cdots,20}$, $\{\mathbb{V}_N(t,u)\}_{N=0,\cdots,21}$, and $\{\mathbb{K}_N(t,u)\}_{N=1,\cdots,20}$ from Defs. 12.1 and 12.2. There exists a constant $C>0$ such that under the data-size and bootstrap assumptions of Sects. 8.1-8.4 and the smallness assumptions of Sect. 8.6, the following estimates hold for $(t,u)\in[0,T_{(Boot)})\times[0,U_0]$:*

$$\sqrt{\mathbb{Q}_{15+K}}(t,u)+\sqrt{\mathbb{K}_{15+K}}(t,u)\leq C\mathring{\upepsilon}\upmu_\star^{-(K+.9)}(t,u), \qquad (0\leq K\leq 5), \tag{15.1.1a}$$

$$\sqrt{\mathbb{Q}_{[1,14]}}(t,u)+\sqrt{\mathbb{K}_{[1,14]}}(t,u)\leq C\mathring{\upepsilon}, \tag{15.1.1b}$$

$$\sqrt{\mathbb{V}_{21}}(t,u)\leq C\mathring{\upepsilon}\upmu_\star^{-6.4}(t,u), \tag{15.1.1c}$$

$$\sqrt{\mathbb{V}_{16+K}}(t,u)\leq C\mathring{\upepsilon}\upmu_\star^{-(K+.9)}(t,u), \qquad (0\leq M\leq 4), \tag{15.1.1d}$$

$$\sqrt{\mathbb{V}_{\leq 15}}(t,u)\leq C\mathring{\upepsilon}. \tag{15.1.1e}$$

To initiate the proof of Prop. 15.1, we provide the following simple lemma, which shows that the fundamental L^2-controlling quantities are initially $\lesssim\mathring{\upepsilon}^2$.

Lemma 15.2 (**The fundamental controlling quantities are initially small**). *Under the data-size assumptions of Sects. 8.1 and 8.2, the following estimates hold for $t\in[0,2\mathring{\updelta}_*^{-1}]$ and $u\in[0,U_0]$:*

$$\mathbb{Q}_{[0,20]}(0,u),\ \mathbb{Q}_{[0,20]}(t,0)\leq C\mathring{\upepsilon}^2, \tag{15.1.2a}$$

$$\mathbb{V}_{\leq 21}(0,u),\ \mathbb{V}_{\leq 21}(t,0)\leq C\mathring{\upepsilon}^2. \tag{15.1.2b}$$

Proof. We first note that by (8.2.5a) and (8.2.8), we have $\upmu\approx 1$ along Σ_0^1 and along $\mathcal{P}_0^{2\mathring{\updelta}_*^{-1}}$. Using these estimates, Def. 12.1, and Lemma 4.1, we see that $\mathbb{Q}_{[0,20]}(0,u)\lesssim\sum_{\Psi\in\{\uprho-v^1,v^1,v^2\}}\left\|\mathscr{Z}_*^{[1,21];\leq 2}\Psi\right\|_{L^2(\Sigma_0^u)}^2$ and $\mathbb{V}_{\leq 21}(0,u)\lesssim\left\|\mathscr{P}^{\leq 21}\upomega\right\|_{L^2(\Sigma_0^u)}^2$. Similarly, for $0\leq t\leq 2\mathring{\updelta}_*^{-1}$, we have $\mathbb{Q}_{[0,20]}(t,0)\lesssim\sum_{\Psi\in\{\uprho-v^1,v^1,v^2\}}\left\|\mathscr{Z}_*^{[1,21];\leq 1}\Psi\right\|_{L^2(\mathcal{P}_0^t)}^2$ and $\mathbb{V}_{\leq 21}(t,0)\lesssim\left\|\mathscr{P}^{\leq 21}\upomega\right\|_{L^2(\mathcal{P}_0^t)}^2$. The estimates (15.1.2a)-(15.1.2b) follow from these estimates and the initial data assumptions (8.1.2) and (8.1.4). $\square$

15.2. **Statement of the integral inequalities that we use to derive a priori estimates.** We prove Prop. 15.1 using a lengthy Gronwall argument based on the sharp estimates for $\upmu$ derived in Sect. 11 and the energy inequalities provided by the next two proposition, Props. 15.3 and 15.4, which we prove in Sects. 15.14 and 15.13 respectively. See Remark 14.2 regarding the boxed constants on RHS (15.2.1a).

Proposition 15.3 (**Integral inequalities for the wave variable controlling quantities**). *Consider the L^2-controlling quantities $\{\mathbb{Q}_N(t,u)\}_{N=1,\cdots,20}$, $\{\mathbb{Q}_N^{(Partial)}(t,u)\}_{N=1,\cdots,20}$, $\{\mathbb{V}_N(t,u)\}_{N=0,\cdots,21}$, and $\{\mathbb{K}_N(t,u)\}_{N=1,\cdots,20}$ from Defs. 12.1 and 12.2. Assume that $N=20$ and $\varsigma>0$. There exist constants $C>0$ and $C_*>0$, independent of ς, such that under the data-size and bootstrap assumptions of Sects. 8.1-8.4 and the smallness assumptions of Sect. 8.6, the following estimates hold for $(t,u)\in[0,T_{(Boot)})\times[0,U_0]$:*

$$
\begin{aligned}
&\max\{\mathbb{Q}_N(t,u),\mathbb{K}_N(t,u)\} \qquad (15.2.1a)\\
&\leq C(1+\varsigma^{-1})\mathring{\epsilon}^2\upmu_\star^{-3/2}(t,u)\\
&\quad+\boxed{6}\int_{t'=0}^t\frac{\|[L\upmu]_-\|_{L^\infty(\Sigma_{t'}^u)}}{\upmu_\star(t',u)}\mathbb{Q}_N(t',u)\,dt'\\
&\quad+\boxed{8.1}\int_{t'=0}^t\frac{\|[L\upmu]_-\|_{L^\infty(\Sigma_{t'}^u)}}{\upmu_\star(t',u)}\sqrt{\mathbb{Q}_N}(t',u)\int_{s=0}^{t'}\frac{\|[L\upmu]_-\|_{L^\infty(\Sigma_s^u)}}{\upmu_\star(s,u)}\sqrt{\mathbb{Q}_N}(s,u)\,ds\,dt'\\
&\quad+\boxed{2}\frac{1}{\upmu_\star^{1/2}(t,u)}\sqrt{\mathbb{Q}_N}(t,u)\,\|L\upmu\|_{L^\infty({}^{(-)}\Sigma_{t;t}^u)}\int_{t'=0}^t\frac{1}{\upmu_\star^{1/2}(t',u)}\sqrt{\mathbb{Q}_N}(t',u)\,dt'\\
&\quad+C_*\int_{t'=0}^t\frac{1}{\upmu_\star(t',u)}\sqrt{\mathbb{Q}_N}(t',u)\sqrt{\mathbb{Q}_N^{(Partial)}}(t',u)\,dt'\\
&\quad+C_*\int_{t'=0}^t\frac{1}{\upmu_\star(t',u)}\sqrt{\mathbb{Q}_N}(t',u)\int_{s=0}^{t'}\frac{1}{\upmu_\star(s,u)}\sqrt{\mathbb{Q}_N^{(Partial)}}(s,u)\,ds\,dt'\\
&\quad+C_*\frac{1}{\upmu_\star^{1/2}(t,u)}\sqrt{\mathbb{Q}_N}(t,u)\int_{t'=0}^t\frac{1}{\upmu_\star^{1/2}(t',u)}\sqrt{\mathbb{Q}_N^{(Partial)}}(t',u)\,dt'\\
&\quad+C\varepsilon\int_{t'=0}^t\frac{1}{\upmu_\star(t',u)}\mathbb{Q}_N(t',u)\,dt'\\
&\quad+C\varepsilon\int_{t'=0}^t\frac{1}{\upmu_\star(t',u)}\sqrt{\mathbb{Q}_N}(t',u)\int_{s=0}^{t'}\frac{1}{\upmu_\star(s,u)}\sqrt{\mathbb{Q}_N}(s,u)\,ds\,dt'\\
&\quad+C\varepsilon\frac{1}{\upmu_\star^{1/2}(t,u)}\sqrt{\mathbb{Q}_N}(t,u)\int_{t'=0}^t\frac{1}{\upmu_\star^{1/2}(t',u)}\sqrt{\mathbb{Q}_N}(t',u)\,dt'\\
&\quad+C\sqrt{\mathbb{Q}_N}(t,u)\int_{t'=0}^t\frac{1}{\upmu_\star^{1/2}(t',u)}\sqrt{\mathbb{Q}_N}(t',u)\,dt'\\
&\quad+C\int_{t'=0}^t\frac{1}{\sqrt{T_{(Boot)}-t'}}\mathbb{Q}_N(t',u)\,dt'+C(1+\varsigma^{-1})\int_{t'=0}^t\frac{1}{\upmu_\star^{1/2}(t',u)}\mathbb{Q}_N(t',u)\,dt'\\
&\quad+C\int_{t'=0}^t\frac{1}{\upmu_\star(t',u)}\sqrt{\mathbb{Q}_N}(t',u)\int_{s=0}^{t'}\frac{1}{\upmu_\star^{1/2}(s,u)}\sqrt{\mathbb{Q}_N}(s,u)\,ds\,dt'
\end{aligned}
$$

$$
\begin{aligned}
&+ C \int_{t'=0}^{t} \frac{1}{\upmu_{\star}(t',u)} \sqrt{\mathbb{Q}_N}(t',u) \int_{s=0}^{t'} \frac{1}{\upmu_{\star}(s,u)} \int_{s'=0}^{s} \frac{1}{\upmu_{\star}^{1/2}(s',u)} \sqrt{\mathbb{Q}_N}(s',u)\, ds'\, ds\, dt' \\
&+ C(1+\varsigma^{-1}) \int_{u'=0}^{u} \mathbb{Q}_N(t,u')\, du' + C\varepsilon \mathbb{Q}_N(t,u) + C\varsigma \mathbb{Q}_N(t,u) + C\varsigma \mathbb{K}_N(t,u) \\
&+ C(1+\varsigma^{-1}) \int_{t'=0}^{t} \frac{1}{\upmu_{\star}^{5/2}(t',u)} \mathbb{Q}_{[1,N-1]}(t',u)\, dt' + C(1+\varsigma^{-1}) \int_{u'=0}^{u} \mathbb{Q}_{[1,N-1]}(t,u')\, du' \\
&+ C\varepsilon \mathbb{Q}_{[1,N-1]}(t,u) + C\varsigma \mathbb{Q}_{[1,N-1]}(t,u) + C\varsigma \mathbb{K}_{[1,N-1]}(t,u) \\
&+ C \int_{t'=0}^{t} \frac{1}{\upmu_{\star}^{3/2}(t',u)} \left\{ \int_{s=0}^{t'} \sqrt{\mathbb{V}_{N+1}}(s,u)\, ds \right\}^2 dt' \\
&+ C \int_{t'=0}^{t} \frac{1}{\upmu_{\star}^{3/2}(t',u)} \left\{ \int_{s=0}^{t'} \frac{1}{\upmu_{\star}^{1/2}(s,u)} \sqrt{\mathbb{V}_{\leq N}}(s,u)\, ds \right\}^2 dt' \\
&+ C \int_{t'=0}^{t} \mathbb{V}_{\leq N+1}(t',u)\, dt' + C \int_{u'=0}^{u} \mathbb{V}_{\leq N}(t,u')\, du',
\end{aligned}
$$

Moreover,[111]

inequality (15.2.1a) *holds with the LHS replaced with* (15.2.2)
$\max\left\{\mathbb{Q}_N^{(Partial)}(t,u), \mathbb{K}_N^{(Partial)}(t,u)\right\}$
and <u>*without*</u> *the six "large-coefficient" terms* $\boxed{6}\cdots, \boxed{2}\cdots, \boxed{8.1}\cdots, C_{*}\cdots$ *on the RHS.*

[111]Instead of the "large coefficient" terms, one only has corresponding "small coefficient" terms

$$
\begin{aligned}
&C\varepsilon \int_{t'=0}^{t} \frac{1}{\upmu_{\star}(t',u)} \sqrt{\mathbb{Q}_N}(t',u) \int_{s=0}^{t'} \frac{1}{\upmu_{\star}(s,u)} \sqrt{\mathbb{Q}_N}(s,u)\, ds\, dt' \\
&+ C\varepsilon \int_{t'=0}^{t} \frac{1}{\upmu_{\star}(t',u)} \mathbb{Q}_N(t',u)\, dt' \\
&+ C\varepsilon \frac{1}{\upmu_{\star}^{1/2}(t,u)} \sqrt{\mathbb{Q}_N}(t,u) \int_{t'=0}^{t} \frac{1}{\upmu_{\star}^{1/2}(t',u)} \sqrt{\mathbb{Q}_N}(t',u)\, dt',
\end{aligned}
$$

which are also featured on RHS (15.2.1a).

In addition if $2 \leq N \leq 20$ *and* $\varsigma > 0$, *then*

$$\begin{aligned}
& \max\left\{\mathbb{Q}_{[1,N-1]}(t,u), \mathbb{K}_{[1,N-1]}(t,u)\right\} \\
& \leq C\mathring{\epsilon}^2 \\
& \quad + C\int_{t'=0}^{t} \frac{1}{\mu_\star^{1/2}(t',u)} \sqrt{\mathbb{Q}_{[1,N-1]}}(t',u) \int_{s=0}^{t'} \frac{1}{\mu_\star^{1/2}(s,u)} \sqrt{\mathbb{Q}_N}(s,u)\, ds\, dt' \\
& \quad + C\int_{t'=0}^{t} \frac{1}{\sqrt{T_{(Boot)} - t'}} \mathbb{Q}_{[1,N-1]}(t',u)\, dt' \\
& \quad + C(1+\varsigma^{-1})\int_{t'=0}^{t} \frac{1}{\mu_\star^{1/2}(t',u)} \mathbb{Q}_{[1,N-1]}(t',u)\, dt' \\
& \quad + C(1+\varsigma^{-1})\int_{u'=0}^{u} \mathbb{Q}_{[1,N-1]}(t,u')\, du' \\
& \quad + C\varsigma\mathbb{K}_{[1,N-1]}(t,u) \\
& \quad + C\int_{t'=0}^{t} \mathbb{V}_{\leq N}(t',u)\, dt' + C\int_{u'=0}^{u} \mathbb{V}_{\leq N-1}(t,u')\, du'.
\end{aligned} \tag{15.2.3}$$

Proposition 15.4 (Integral inequalities for the specific vorticity-controlling quantities). *Consider the* L^2*-controlling quantities* $\{\mathbb{Q}_N(t,u)\}_{N=1,\cdots,20}$, $\{\mathbb{Q}_N^{(Partial)}(t,u)\}_{N=1,\cdots,20}$, $\{\mathbb{V}_N(t,u)\}_{N=0,\cdots,21}$, *and* $\{\mathbb{K}_N(t,u)\}_{N=1,\cdots,20}$ *from Defs. 12.1 and 12.2. Assume that* $N = 20$. *There exists a constant* $C > 0$ *such that under the data-size and bootstrap assumptions of Sects. 8.1-8.4 and the smallness assumptions of Sect. 8.6, the following estimates hold for* $(t,u) \in [0, T_{(Boot)}) \times [0, U_0]$:

$$\begin{aligned}
\mathbb{V}_{N+1}(t,u) & \leq C\mathring{\epsilon}^2 \frac{1}{\mu_\star^{3/2}(t,u)} + C\varepsilon^2 \frac{1}{\mu_\star(t,u)} \mathbb{Q}_N(t,u) \\
& \quad + C\varepsilon^2 \int_{t'=0}^{t} \frac{1}{\mu_\star^2(t',u)} \left\{ \int_{s=0}^{t} \frac{1}{\mu_\star(s,u)} \sqrt{\mathbb{Q}_N}(s,u)\, ds \right\}^2 dt' \\
& \quad + C\varepsilon^2 \frac{1}{\mu_\star^2(t,u)} \mathbb{Q}_{[1,N-1]}(t,u) + C\varepsilon^2 \mathbb{K}_{[1,N]}(t,u) \\
& \quad + C\varepsilon^2 \int_{t'=0}^{t} \frac{1}{\mu_\star^2(t',u)} \left\{ \int_{s=0}^{t} \sqrt{\mathbb{V}_{N+1}}(s,u)\, ds \right\}^2 dt' \\
& \quad + C\varepsilon^2 \int_{t'=0}^{t} \frac{1}{\mu_\star^2(t',u)} \left\{ \int_{s=0}^{t} \frac{1}{\mu_\star^{1/2}(s,u)} \sqrt{\mathbb{V}_{[1,N]}}(s,u)\, ds \right\}^2 dt' \\
& \quad + C\int_{u'=0}^{u} \mathbb{V}_{\leq N+1}(t,u')\, du'.
\end{aligned} \tag{15.2.4a}$$

Similarly, if $N \leq 20$, then

$$\begin{aligned} \mathbb{V}_{\leq N}(t,u) \leq C\mathring{\epsilon}^2 + C\varepsilon^2 \int_{t'=0}^{t} \left\{ \int_{s=0}^{t'} \frac{1}{\upmu_\star^{1/2}(s,u)} \sqrt{\mathbb{Q}_{[1,N]}}(s,u)\, ds \right\}^2 dt' \\ + \underbrace{C\varepsilon^2 \mathbb{Q}_{[1,N-1]}(t,u) + C\varepsilon^2 \mathbb{K}_{[1,N-1]}(t,u)}_{\text{Absent if } N=0} \\ + C \int_{u'=0}^{u} \mathbb{V}_{\leq N}(t,u')\, du'. \end{aligned} \tag{15.2.4b}$$

15.3. **Bootstrap assumptions for the fundamental L^2- controlling quantities of the wave variables.** To facilitate our proof of Prop. 15.1, it is convenient to make L^2-type bootstrap assumptions, which we state in this section. Specifically, let $\{\mathbb{Q}_N(t,u)\}_{N=1,\cdots,20}$, $\{\mathbb{V}_N(t,u)\}_{N=0,\cdots,21}$, and $\{\mathbb{K}_N(t,u)\}_{N=1,\cdots,20}$ be the L^2-controlling quantities from Defs. 12.1 and 12.2. We assume that the following inequalities hold for $(t,u) \in [0, T_{(Boot)}) \times [0, U_0]$, where ε is the small bootstrap parameter appearing in Sect. 8.4:

$$\sqrt{\mathbb{Q}_{15+M}}(t,u) + \sqrt{\mathbb{K}_{15+M}}(t,u) \leq \sqrt{\varepsilon}\upmu_\star^{-(M+.9)}(t,u), \qquad (0 \leq M \leq 5), \tag{15.3.1a}$$

$$\sqrt{\mathbb{Q}_{[1,14]}}(t,u) + \sqrt{\mathbb{K}_{[1,14]}}(t,u) \leq \sqrt{\varepsilon}, \tag{15.3.1b}$$

$$\sqrt{\mathbb{V}_{21}}(t,u) \leq \sqrt{\varepsilon}\upmu_\star^{-6.4}(t,u), \tag{15.3.1c}$$

$$\sqrt{\mathbb{V}_{16+M}}(t,u) \leq \sqrt{\varepsilon}\upmu_\star^{-(M+.9)}(t,u), \qquad (0 \leq M \leq 4), \tag{15.3.1d}$$

$$\sqrt{\mathbb{V}_{\leq 15}}(t,u) \leq \sqrt{\varepsilon}. \tag{15.3.1e}$$

15.4. **Preliminary L^2 estimates for the eikonal function quantities that do not require modified quantities.** In Lemma 15.6, we derive a priori estimates for the below-top-order derivatives of the eikonal function quantities and, in the case that at least one L-differentiation is involved, their top-order derivatives. These estimates are simple consequence of the transport inequalities derived in Prop. 9.12 and can be derived without using the modified quantities.

We start with a simple commutator lemma.

Lemma 15.5 (Simple commutator lemma). *Assume that $1 \leq N \leq 21$ and $0 \leq M \leq \min\{2, N-1\}$. Under the data-size and bootstrap assumptions of Sects. 8.1-8.4 and the smallness assumptions of Sect. 8.6, the following pointwise estimates hold on $\mathcal{M}_{T_{(Boot)},U_0}$:*

$$\left| \mathscr{Z}_*^{[1,N];\leq M} \vec{\Psi} \right| \lesssim \sum_{\widetilde{M}=0}^{M} \left| \breve{X}^{\widetilde{M}} \mathscr{P}^{[1,N-\widetilde{M}]} \vec{\Psi} \right| + \varepsilon \left| \begin{pmatrix} \mathscr{Z}_{**}^{[1,N-1];(M-1)_+} \underline{\upgamma} \\ \mathscr{Z}_{*}^{[1,N-1];\leq M} \upgamma \end{pmatrix} \right|, \tag{15.4.1}$$

where $(M-1)_+ = \max\{0, M-1\}$ and when $N=1$, only $\left|\mathscr{P}\vec{\Psi}\right|$ appears on RHS (15.4.1).

Proof. We repeatedly use (9.5.7a) with $f = \vec{\Psi}$ and the L^∞ estimates of Prop. 9.12 to commute the factors of $\breve{X}$ acting on $\vec{\Psi}$ on LHS (15.4.1) to the front (so they are the last to hit $\vec{\Psi}$). □

We now provide the main estimates of this section.

Lemma 15.6 (**L^2 bounds for the eikonal function quantities that do not require modified quantities**). *Assume that $1 \leq N \leq 20$. Under the data-size and bootstrap assumptions of Sects. 8.1-8.4 and the smallness assumptions of Sect. 8.6, the following L^2 estimates hold for $(t,u) \in [0, T_{(Boot)}) \times [0, U_0]$ (see Sect. 6.2 regarding the vectorfield operator notation):*

$$\left(\begin{array}{c} \left\| L \mathscr{Z}_*^{[1,N];\leq 1} \upmu \right\|_{L^2(\Sigma_t^u)} \\ \left\| L \mathscr{Z}_*^{\leq N;\leq 2} L^i_{(Small)} \right\|_{L^2(\Sigma_t^u)} \\ \left\| L \mathscr{Z}^{\leq N-1;\leq 2} \mathrm{tr}_{\not g} \chi \right\|_{L^2(\Sigma_t^u)} \end{array} \right) \lesssim \mathring{\upepsilon} + \frac{\sqrt{\mathbb{Q}_{[1,N]}}(t,u)}{\upmu_\star^{1/2}(t,u)}, \tag{15.4.2a}$$

$$\left(\begin{array}{c} \left\| \mathscr{Z}_{**}^{[1,N];\leq 1} \upmu \right\|_{L^2(\Sigma_t^u)} \\ \left\| \mathscr{Z}_*^{[1,N];\leq 2} L^i_{(Small)} \right\|_{L^2(\Sigma_t^u)} \\ \left\| \mathscr{Z}^{\leq N-1;\leq 2} \mathrm{tr}_{\not g} \chi \right\|_{L^2(\Sigma_t^u)} \end{array} \right) \lesssim \mathring{\upepsilon} + \int_{s=0}^t \frac{\sqrt{\mathbb{Q}_{[1,N]}}(s,u)}{\upmu_\star^{1/2}(s,u)} \, ds. \tag{15.4.2b}$$

Proof. See Sect. 9.2 for some comments on the analysis. We set $q_N(t) :=$ LHS (15.4.2b). From (9.6.1b), (9.6.2), Lemma 12.5, (15.4.1), (12.2.3), and Lemma 12.3, we deduce

$$q_N(t) \leq C q_N(0) + C \int_{s=0}^t q_N(s) \, ds + C \int_{s=0}^t \frac{\sqrt{\mathbb{Q}_{[1,N]}}(s,u)}{\upmu_\star^{1/2}(s,u)} \, ds. \tag{15.4.3}$$

Next, we note that $q_N(0) \lesssim \mathring{\upepsilon}$, an estimate that follows from the estimate (9.4.1c) for $\mathrm{tr}_{\not g} \chi$ and our data-size assumptions. We now apply Gronwall's inequality to (15.4.3) to conclude that $q_N(t) \lesssim$ RHS (15.4.2b) as desired. We have thus proved (15.4.2b).

To obtain the estimates (15.4.2a), we take the norm $\|\cdot\|_{L^2(\Sigma_t^u)}$ of the inequalities (9.6.1b), (9.6.2) and argue as above using the already proven estimates (15.4.2b). In these estimates, we encounter the integrals $\int_{s=0}^t \frac{\sqrt{\mathbb{Q}_{[1,N]}}(s,u)}{\upmu_\star^{1/2}(s,u)} \, ds$, which we (inefficiently) bound by $\lesssim \sqrt{\mathbb{Q}_{[1,N]}}(t,u) \leq \upmu_\star^{-1/2}(t,u) \sqrt{\mathbb{Q}_{[1,N]}}(t,u)$ with the help of inequality (11.3.6).

□

15.5. **Estimates for the easiest error integrals.** In this section, we derive estimates for the easiest error integrals that we encounter in our energy estimates for $\vec{\Psi}$ and $\upomega$. These error integrals do not contribute to the blowup featured in our high-order energy estimates.

We start with a lemma relevant for bounding the specific vorticity. Specifically, we bound the error integrals corresponding to the last integral on RHS (4.3.6).

Lemma 15.7 (**The simplest transport equation error integrals**). *Assume that $N \leq 21$. Under the data-size and bootstrap assumptions of Sects. 8.1-8.4 and the smallness assumptions of Sect. 8.6, the following integral estimates hold for $(t,u) \in [0, T_{(Boot)}) \times [0, U_0]$ (see Sect. 6.2 regarding the vectorfield operator notation):*

$$\left| \int_{\mathcal{M}_{t,u}} \left\{ L\upmu + \upmu \mathrm{tr}_{\not g} \not k \right\} (\mathscr{P}^N \upomega)^2 \, d\varpi \right| \lesssim \int_{u'=0}^u \mathbb{V}_N(t,u') \, du'. \tag{15.5.1}$$

Proof. From equation (3.9.8b), Lemma 3.19, Lemma 9.5, and the L^∞ estimates of Prop. 9.12, we deduce $\|L\upmu + \upmu \mathrm{tr}_{g\!\!/} k\!\!\!/ \|_{L^\infty(\Sigma_t^u)} \lesssim 1$. Hence, using Lemma 12.5, we conclude that

$$\text{LHS (15.5.1)} \lesssim \int_{\mathcal{M}_{t,u}} (\mathscr{P}^N \omega)^2 \, d\varpi = \int_{u'=0}^{u} \|\mathscr{P}^N \omega\|_{L^2(\mathcal{P}_{u'}^t)}^2 \, du' \lesssim \int_{u'=0}^{u} \mathbb{V}_N(t,u') \, du'. \quad (15.5.2)$$

□

The next lemma is a more complicated analog of Lemma 15.7 for the wave variables $\{\uprho - v^1, v^1, v^2\}$. We recall that in (4.3.2), we decomposed the integrand corresponding to the last term on the right-hand side of the wave equation energy identity (4.3.1). Moreover, we recall that one of the integrand pieces is coercive and is critically important for controlling geometric torus derivatives; we isolated it in Def. 12.2. In the next lemma, we bound the error integrals corresponding to the remaining terms in the decomposition (4.3.2).

Lemma 15.8 (Error integrals involving the deformation tensor of the multiplier vectorfield). *Let* $\Psi \in \{\uprho - v^1, v^1, v^2\}$. *Assume that* $1 \leq N \leq 20$ *and* $\varsigma > 0$. *Let* ${}^{(T)}\mathfrak{P}_{(i)}[\mathscr{Z}_*^{N;\leq 1}\Psi]$, $(i = 1, \cdots, 5)$, *be the quantities defined by* (4.3.3a)-(4.3.3e) *(with* $\mathscr{Z}_*^{N;\leq 1}\Psi$ *in the role of* Ψ*). Under the data-size and bootstrap assumptions of Sects. 8.1-8.4 and the smallness assumptions of Sect. 8.6, the following integral estimates hold for* $(t,u) \in [0, T_{(Boot)}) \times [0, U_0]$, *where the implicit constants are independent of* ς *(and without any absolute value taken on the left):*

$$\begin{aligned} \int_{\mathcal{M}_{t,u}} \sum_{i=1}^{5} {}^{(T)}\mathfrak{P}_{(i)}[\mathscr{Z}_*^{N;\leq 1}\Psi] \, d\varpi \lesssim & \int_{t'=0}^{t} \frac{1}{\sqrt{T_{(Boot)} - t'}} \mathbb{Q}_{[1,N]}(t',u) \, dt' \qquad (15.5.3) \\ & + (1 + \varsigma^{-1}) \int_{t'=0}^{t} \mathbb{Q}_{[1,N]}(t',u) \, dt' \\ & + (1 + \varsigma^{-1}) \int_{u'=0}^{u} \mathbb{Q}_{[1,N]}(t,u') \, du' + \varsigma \mathbb{K}_{[1,N]}(t,u). \end{aligned}$$

Proof. We integrate inequality (14.8.1) (with $\mathscr{Z}_*^{N;\leq 1}\Psi$ in the role of Ψ) over the domain $\mathcal{M}_{t,u}$ and use Lemmas 12.4 and 12.5. □

In the next lemma, we bound $\left\|\mathscr{Z}^{N;1}\omega\right\|_{L^2(\Sigma_{t,u})}$ in terms of the fundamental L^2-controlling quantities. These estimates play a role in bounding some of the inhomogeneous terms in the wave equations that depend on the derivatives of the vorticity up to top-order.

Lemma 15.9 (L^2 estimates involving one transversal derivative of the specific vorticity). *Assume that* $1 \leq N \leq 21$. *Under the data-size and bootstrap assumptions of Sects. 8.1-8.4 and the smallness assumptions of Sect. 8.6, the following* L^2 *estimates hold for* $(t,u) \in [0, T_{(Boot)}) \times [0, U_0]$:

$$\left\|\mathscr{Z}^{N;1}\omega\right\|_{L^2(\Sigma_{t,u})} \lesssim \mathring{\epsilon} + \sqrt{\mathbb{V}_{\leq N}}(t,u) + \varepsilon \int_{s=0}^{t} \frac{\sqrt{\mathbb{Q}_{[1,N-1]}}(t,u)}{\upmu_\star^{1/2}(s,u)} \, ds, \qquad (15.5.4)$$

where the last term on RHS (15.5.4) *is absent when* $N = 1$.

Proof. We use (9.5.7a) with $f = \upomega$ and the L^∞ estimates of Prop. 9.12 to commute the factors of $\breve{X}$ acting on $\upomega$ on LHS (15.5.4) so that it is the first to hit $\upomega$, thereby obtaining

$$\left|\mathscr{Z}^{N;1}\upomega\right| \lesssim \left|\mathscr{P}^{\leq N-1}\breve{X}\upomega\right| + \left|\mathscr{P}^{\leq N-1}\upomega\right| + \varepsilon\left|\mathscr{Z}_*^{[1,N-1];\leq 1}\upgamma\right| + \varepsilon\left|\mathscr{Z}_{**}^{[1,N-1]}\underline{\upgamma}\right|, \qquad (15.5.5)$$

where the last three terms on RHS (15.5.5) are absent when $N = 1$. Using Lemma 12.5, and the estimate (15.4.2b), we see that the norms $\|\cdot\|_{L^2(\Sigma_{t,u})}$ of the last two terms on RHS (15.5.5) are $\lesssim$ RHS (15.5.4). Next, we use the fundamental theorem of calculus to obtain the following pointwise estimate for the second term on RHS (15.5.5):

$$\left|\mathscr{P}^{\leq N-1}\upomega\right|(t,u,\vartheta) \lesssim \left|\mathscr{P}^{\leq N-1}\upomega\right|(0,u,\vartheta) + \int_{s=0}^{t}\left|L\mathscr{P}^{\leq N-1}\upomega\right|(s,u,\vartheta)\,ds. \qquad (15.5.6)$$

Using Lemma 12.3 and Lemma 12.5, we see that the norm $\|\cdot\|_{L^2(\Sigma_{t,u})}$ of the last term on RHS (15.5.6) is $\lesssim \mathring{\upepsilon} + \int_{s=0}^{t}\frac{\sqrt{\mathbb{V}_{[1,N]}}(s,u)}{\upmu_\star^{1/2}(s,u)}\,ds$. Manifestly, we have $\mathring{\upepsilon} \lesssim$ RHS (15.5.4). Moreover, using inequality (11.3.6) and the fact that $\mathbb{V}_{[1,N]}$ is increasing in its arguments, we see that the previous time integral is $\lesssim \sqrt{\mathbb{V}_{[1,N]}}(t,u)$, which is $\lesssim$ RHS (15.5.4) as desired. To bound the norm $\|\cdot\|_{L^2(\Sigma_{t,u})}$ of the first term on RHS (15.5.6), we use (12.2.3) with $s = 0$ and the small-data assumption (8.1.2) to obtain $\|\mathscr{P}^{\leq N-1}\upomega(1,\cdot)\|_{L^2(\Sigma_{t,u})} \lesssim \|\mathscr{P}^{\leq N-1}\upomega\|_{L^2(\Sigma_{1,u})} \leq \mathring{\upepsilon}$ as desired.

It remains for us to bound the norm $\|\cdot\|_{L^2(\Sigma_{t,u})}$ of the first term on RHS (15.5.5). Using equations (3.3.11c) and (3.7.15) to algebraically express $\breve{X}\upomega = -\upmu L\upomega$ and using the estimates (9.6.5a), (9.6.5b), (9.6.5c), and (9.6.9), we find that $\left|\mathscr{P}^{\leq N-1}\breve{X}\upomega\right| \lesssim \left|\sqrt{\upmu}\mathscr{P}^N\upomega\right| + \left|\mathscr{P}^{\leq N-1}\upomega\right| + \varepsilon\left|\mathscr{Z}_{**}^{[1,N-1]}\underline{\upgamma}\right|$, where the last two terms on the RHS are absent when $N = 1$. Lemma 12.5 immediately yields that $\left\|\sqrt{\upmu}\mathscr{P}^N\upomega\right\|_{L^2(\Sigma_{t,u})} \lesssim \sqrt{\mathbb{V}_{\leq N}}(t,u)$, while the arguments given below (15.5.6) imply that $\left\|\mathscr{P}^{\leq N-1}\upomega\right\|_{L^2(\Sigma_{t,u})} \lesssim \mathring{\upepsilon} + \sqrt{\mathbb{V}_{\leq N}}(t,u)$ as desired. Moreover, above we showed that $\varepsilon\left\|\mathscr{Z}_{**}^{[1,N-1]}\underline{\upgamma}\right\|_{L^2(\Sigma_{t,u})} \lesssim$ RHS (15.5.4). This completes the proof of (15.5.4). □

In the next lemma, we bound the error integrals corresponding to the $Harmless_{(Wave)}^{\leq N}$ and the $Harmless_{(Vort)}^{\leq N}$ terms.

Lemma 15.10 (**L^2 bounds for error integrals involving $Harmless_{(Wave)}^{\leq N}$ or $Harmless_{(Vort)}^{\leq N}$ terms**)**.** *Let $\Psi \in \{\uprho - v^1, v^1, v^2\}$. Assume that $1 \leq N \leq 20$ and $\varsigma > 0$. Recall that the terms $Harmless_{(Wave)}^{\leq N}$ and $Harmless_{(Vort)}^{\leq N}$ are defined in Def. 14.1. Under the data-size and bootstrap assumptions of Sects. 8.1-8.4 and the smallness assumptions of Sect. 8.6, the following* *integral estimates hold for* $(t,u) \in [0, T_{(Boot)}) \times [0, U_0]$, *where the implicit constants are independent of* ς*:*

$$\begin{aligned} &\int_{\mathcal{M}_{t,u}} \left| \begin{pmatrix} (1+\upmu) L \mathscr{Z}_*^{N;\leq 1} \Psi \\ \breve{X} \mathscr{Z}_*^{N;\leq 1} \Psi \end{pmatrix} \right| \left| Harmless_{(Wave)}^{\leq N} \right| \, d\varpi \\ &\quad \lesssim (1+\varsigma^{-1}) \int_{t'=0}^{t} \frac{\mathbb{Q}_{[1,N]}(t',u)}{\upmu_\star^{1/2}(t',u)} \, dt' + (1+\varsigma^{-1}) \int_{u'=0}^{u} \mathbb{Q}_{[1,N]}(t,u') \, du' \\ &\quad\quad + \varsigma \mathbb{K}_{[1,N]}(t,u) + \int_{t'=0}^{t} \mathbb{V}_{\leq N}(t',u) \, dt' + \int_{u'=0}^{u} \mathbb{V}_{\leq N}(t,u') \, du' + \mathring{\upepsilon}^2. \end{aligned} \tag{15.5.7}$$

Moreover, if $N \leq 21$, *then*

$$\begin{aligned} &\int_{\mathcal{M}_{t,u}} \left| \mathscr{P}^N \upomega \right| \left| Harmless_{(Vort)}^{\leq N} \right| \, d\varpi \\ &\quad \lesssim \underbrace{\varepsilon^2 \mathbb{Q}_{[1,N-1]}(t,u) + \varepsilon^2 \mathbb{K}_{[1,N-1]}(t,u)}_{\text{Absent if } N=0,1} + \int_{u'=0}^{u} \mathbb{V}_{\leq N}(t,u') \, du' + \varepsilon^2 \mathring{\upepsilon}^2. \end{aligned} \tag{15.5.8}$$

Proof. See Sect. 9.2 for some comments on the analysis. To prove (15.5.7) and (15.5.8) we must estimate the spacetime integrals of various quadratic terms. We derive the desired estimates for five representative quadratic terms: four in the case of (15.5.7) and one in the case of (15.5.8). The remaining terms can be bounded using similar or simpler arguments and we omit those details. As our first example, we bound the spacetime integral of $\left| L \mathscr{Z}_*^{N;\leq 1} \Psi \right| \left| Y^{N+1} \Psi \right|$ (note that $Y^{N+1}\Psi = Harmless_{(Wave)}^{\leq N}$). Using spacetime Cauchy-Schwarz, Lemmas 12.4 and 12.5, and simple estimates of the form $ab \lesssim a^2 + b^2$, and separately treating the regions $\{\upmu \geq 1/4\}$ and $\{\upmu < 1/4\}$ when bounding the integral of $\left| Y^{N+1}\Psi \right|^2$, we deduce the desired estimate as follows:

$$\begin{aligned} &\int_{\mathcal{M}_{t,u}} \left| L \mathscr{Z}_*^{N;\leq 1} \Psi \right| \left| Y^{N+1} \Psi \right| \, d\varpi \\ &\lesssim \left\{ \int_{\mathcal{M}_{t,u}} \left| L \mathscr{Z}_*^{N;\leq 1} \Psi \right|^2 \, d\varpi \right\}^{1/2} \left\{ \int_{\mathcal{M}_{t,u}} \left| Y^{N+1} \Psi \right|^2 \, d\varpi \right\}^{1/2} \\ &\lesssim (1+\varsigma^{-1}) \int_{u'=0}^{u} \int_{\mathcal{P}_{u'}^t} \left| L \mathscr{Z}_*^{N;\leq 1} \Psi \right|^2 \, d\overline{\varpi} \, du' \\ &\quad + \int_{u'=0}^{u} \int_{\mathcal{P}_{u'}^t} \upmu \left| d\!\!/ Y^N \Psi \right|^2 \, d\overline{\varpi} \, du' + \varsigma \mathring{\updelta}_* \int_{\mathcal{M}_{t,u}} \mathbf{1}_{\{\upmu < 1/4\}} \left| d\!\!/ Y^N \Psi \right|^2 \, d\varpi \\ &\lesssim (1+\varsigma^{-1}) \int_{u'=0}^{u} \mathbb{Q}_{[1,N]}(t,u') \, du' + \varsigma \mathbb{K}_{[1,N]}(t,u). \end{aligned} \tag{15.5.9}$$

As our second example, we bound the spacetime integral of $\left| L \mathscr{Z}_*^{N;\leq 1} \Psi \right| \left| \mathscr{Z}_{**}^{[1,N];\leq 1} \upmu \right|$. Using spacetime Cauchy-Schwarz, Lemmas 12.4 and 12.5, inequalities (11.3.6) and (15.4.2b), simple estimates of the form $ab \lesssim a^2 + b^2$, and the fact that $\mathbb{Q}_{[1,N]}$ is increasing in its arguments, we derive the desired estimate as follows:

$$\begin{aligned}
&\int_{\mathcal{M}_{t,u}} \left|L \mathscr{Z}_*^{N;\leq 1}\Psi\right| \left|\mathscr{Z}_{**}^{[1,N];\leq 1}\upmu\right| \, d\varpi \qquad (15.5.10)\\
&\lesssim \int_{u'=0}^{u} \int_{\mathcal{P}_{u'}^t} \left|L \mathscr{Z}_*^{N;\leq 1}\Psi\right|^2 \, d\overline{\varpi} \, du' + \int_{t'=0}^{t} \int_{\Sigma_{t'}^u} \left|\mathscr{Z}_{**}^{[1,N];\leq 1}\upmu\right|^2 \, d\underline{\varpi} \, dt' \\
&\lesssim \int_{u'=0}^{u} \mathbb{Q}_{[1,N]}(t,u') \, du' + \int_{t'=0}^{t} \left\{ \int_{s=0}^{t'} \frac{\sqrt{\mathbb{Q}_{[1,N]}}(s,u)}{\upmu_\star^{1/2}(s,u)} \, ds \right\}^2 + \mathring{\upepsilon}^2 \, dt' \\
&\lesssim \int_{u'=0}^{u} \mathbb{Q}_{[1,N]}(t,u') \, du' + \int_{t'=0}^{t} \mathbb{Q}_{[1,N]}(t',u) \, dt' + \mathring{\upepsilon}^2.
\end{aligned}$$

As our third example, we bound the spacetime integral of $\left|L \mathscr{Z}_*^{N;\leq 1}\Psi\right| \left|\mathscr{Z}_*^{N+1;2}\Psi\right|$. From Lemma 15.5 with $M = 2$ and $N+1$ in the role of N, we obtain $|\mathscr{Z}_*^{N+1;2}\Psi| \lesssim |\breve{X}\breve{X}\mathscr{P}^{[1,N-1]}\Psi| + |\breve{X}\mathscr{P}^{[1,N]}\Psi| + |\mathscr{P}^{[1,N+1]}\Psi| + \varepsilon|\mathscr{Z}_*^{[1,N];\leq 2}\upgamma| + \varepsilon|\mathscr{Z}_{**}^{[1,N];\leq 1}\underline{\upgamma}|$. Thus, we must bound the integral of the five corresponding products from the RHS of the previous inequality. To bound the integral of the first product, we argue as in the proof of (15.5.9) to deduce that

$$\begin{aligned}
&\int_{\mathcal{M}_{t,u}} \left|L \mathscr{Z}_*^{N;\leq 1}\Psi\right| \left|\breve{X}\breve{X}\mathscr{P}^{[1,N-1]}\Psi\right| \, d\varpi \qquad (15.5.11)\\
&\lesssim \int_{u'=0}^{u} \int_{\mathcal{P}_{u'}^t} \left|L \mathscr{Z}_*^{N;\leq 1}\Psi\right|^2 \, d\overline{\varpi} \, du' + \int_{t'=0}^{t} \int_{\Sigma_{t'}^u} \left|\breve{X}\breve{X}\mathscr{P}^{[1,N-1]}\Psi\right|^2 \, d\underline{\varpi} \, dt' \\
&\lesssim \int_{u'=0}^{u} \mathbb{Q}_{[1,N]}(t,u') \, du' + \int_{t'=0}^{t} \mathbb{Q}_{[1,N]}(t',u) \, dt',
\end{aligned}$$

which is $\lesssim$ RHS (15.5.7) as desired. The second product $\left|L \mathscr{Z}_*^{N;\leq 1}\Psi\right| \left|\breve{X}\mathscr{P}^{[1,N]}\Psi\right|$ can be bounded in the same way. Similar reasoning yields that the integral of the third product $|L \mathscr{Z}_*^{N;\leq 1}\Psi||\mathscr{P}^{[1,N+1]}\Psi|$ is $\lesssim$ RHS (15.5.9) plus RHS (15.5.10) as desired. We clarify that the factor $\mathring{\upepsilon}^2$ is generated by the square of RHS (12.3.5b), which is needed to bound $P\Psi$. Similar reasoning, together with inequality (15.4.2b), yields that the integral of the third product $\varepsilon|L \mathscr{Z}_*^{N;\leq 1}\Psi||\mathscr{Z}_*^{[1,N];\leq 2}\upgamma|$ and the integral of the fourth product $\varepsilon|L \mathscr{Z}_*^{N;\leq 1}\Psi||\mathscr{Z}_{**}^{[1,N];\leq 1}\underline{\upgamma}|$ are $\lesssim$ RHS (15.5.9) plus RHS (15.5.10) as desired. We clarify that we have used the fact that $\mathbb{Q}_{[1,N]}$ is increasing in its arguments and the estimate (11.3.6) to bound the time integrals on RHSs (15.4.2b) by $\lesssim \mathbb{Q}_{[1,N]}(t,u)$, as we did in passing to the last line of (15.5.10).

As our fourth example, we bound the integral of $\left|L\mathscr{Z}_*^{N;\leq 1}\Psi\right|\left|\mathscr{Z}^{N;1}\upomega\right|$. We argue as in the proof of (15.5.9) and use Lemmas 12.5 and 15.9 to deduce that

$$\begin{aligned}
&\int_{\mathcal{M}_{t,u}}\left|L\mathscr{Z}_*^{N;\leq 1}\Psi\right|\left|\mathscr{Z}^{N;1}\upomega\right|\,d\varpi && (15.5.12)\\
&\lesssim \int_{u'=0}^{u}\int_{\mathcal{P}_{u'}^t}\left|L\mathscr{Z}_*^{N;\leq 1}\Psi\right|^2\,d\overline{\varpi}\,du' + \int_{t'=0}^{t}\int_{\Sigma_{t'}^u}\left|\mathscr{Z}^{N;1}\upomega\right|^2\,d\underline{\varpi}\,dt'\\
&\lesssim \int_{u'=0}^{u}\mathbb{Q}_{[1,N]}(t,u')\,du' + \int_{t'=0}^{t}\left\{\int_{s=0}^{t'}\frac{\sqrt{\mathbb{Q}_{[1,N-1]}}(t,u)}{\upmu_\star^{1/2}(s,u)}\,ds\right\}^2\,dt'\\
&\quad + \int_{t'=0}^{t}\mathbb{V}_{\leq N}(t',u)\,dt' + \mathring{\upepsilon}^2.
\end{aligned}$$

Using inequality (11.3.6) and the fact that the $\mathbb{Q}_{[1,N-1]}$ is increasing in its arguments, we bound the double time integral on RHS (15.5.12) by $\lesssim \int_{t'=0}^{t}\mathbb{Q}_{[1,N-1]}(t',u)\,dt' \leq \int_{t'=0}^{t}\mathbb{Q}_{[1,N]}(t',u)\,dt'$. We conclude that RHS (15.5.12) is $\lesssim$ RHS (15.5.7) as desired. This completes our proof of the representative estimates from (15.5.7).

We now prove one representative estimate from (15.5.8). Specifically, we bound the integral of the product $\varepsilon|\mathscr{P}^{\leq N}\upomega||\mathscr{Z}_*^{N;\leq 2}\vec{\Psi}|$ in the cases $1\leq N\leq 21$. Using Lemma 15.5, we bound the last factor as follows:

$$\begin{aligned}
|\mathscr{Z}_*^{N;\leq 2}\vec{\Psi}| &\lesssim |\breve{X}\mathscr{Z}_*^{[1,N-1];\leq 1}\vec{\Psi}| + |L\mathscr{P}^{\leq N-1}\vec{\Psi}| + |{\not{d}}\mathscr{P}^{\leq N-1}\vec{\Psi}| && (15.5.13)\\
&\quad + \sum_{i=1}^{2}|\mathscr{Z}_*^{[1,N-1];\leq 2}L_{(Small)}^i| + |\mathscr{Z}_{**}^{[1,N-1];\leq 1}\upmu|,
\end{aligned}$$

where the first, fourth, and fifth terms on RHS (15.5.13) are absent when $N=1$. Thus, we must bound the integral of ε times the five corresponding products from the RHS of the previous inequality. To this end, we first use Young's inequality to obtain

$$\begin{aligned}
&\varepsilon\int_{\mathcal{M}_{t,u}}\left|\mathscr{P}^{\leq N}\upomega\right|\left|\mathscr{Z}_*^{N;\leq 2}\vec{\Psi}\right|\,d\varpi && (15.5.14)\\
&\lesssim \int_{u'=0}^{u}\left\|\mathscr{P}^{\leq N}\upomega\right\|_{L^2(\mathcal{P}_{u'}^t)}^2\,du' + \varepsilon^2\int_{t'=0}^{t}\left\|\breve{X}\mathscr{Z}_*^{[1,N-1];\leq 1}\vec{\Psi}\right\|_{L^2(\Sigma_{t'}^u)}^2\,dt'\\
&\quad + \varepsilon^2\int_{u'=0}^{u}\left\|L\mathscr{P}^{\leq N-1}\vec{\Psi}\right\|_{L^2(\mathcal{P}_{u'}^t)}^2\,du' + \varepsilon^2\int_{u'=0}^{u}\left\|{\not{d}}\mathscr{P}^{\leq N-1}\vec{\Psi}\right\|_{L^2(\mathcal{P}_{u'}^t)}^2\,du'\\
&\quad + \varepsilon^2\int_{t'=0}^{t}\sum_{i=1,2}\left\|\mathscr{Z}_*^{[1,N-1];\leq 2}L_{(Small)}^i\right\|_{L^2(\Sigma_{t'}^u)}^2\,dt' + \varepsilon^2\int_{t'=0}^{t}\left\|\mathscr{Z}_{**}^{[1,N-1];\leq 1}\upmu\right\|_{L^2(\Sigma_{t'}^u)}^2\,dt'.
\end{aligned}$$

By Lemma 12.5, the integrals on the first line of RHS (15.5.14) and the $L\mathscr{P}^{\leq N-1}\vec{\Psi}$ integral on the second line are $\lesssim$ RHS (15.5.8) as desired. Moreover, using Lemma 12.4, we find that the ${\not{d}}\mathscr{P}^{\leq N-1}\vec{\Psi}$ integral on RHS (15.5.14) is, for $N\geq 2$, $\lesssim$ the $\varepsilon^2\mathbb{K}_{[1,N-1]}(t,u)$ term on RHS (15.5.8) as desired. In the case $N=1$, to bound the ${\not{d}}\vec{\Psi}$ integral on RHS (15.5.14) by $\lesssim$ RHS (15.5.8), we again use Lemma 12.5. Finally, the argument given in our second example above implies that the terms on the last line of RHS (15.5.14) are $\lesssim$ RHS (15.5.8) as desired. This completes our proof of the representative estimate from (15.5.8).

□

15.6. **Estimates for wave equation error integrals involving top-order vorticity terms.** Recall that the geometric wave equation (3.3.11a) has inhomogeneous terms depending on ω. In the next lemma, we derive simple estimates for the corresponding error integrals that depend on the top-order derivatives of ω. The precise form of the terms that we bound corresponds to the explicit vorticity-involving terms on RHS (14.2.1a)-(14.2.2e).

Lemma 15.11 (**Estimates for wave equation integrals involving top-order vorticity terms**). *Assume that $\Psi \in \{\rho - v^1, v^1, v^2\}$ and that $1 \leq N \leq 20$. Under the data-size and bootstrap assumptions of Sects. 8.1-8.4 and the smallness assumptions of Sect. 8.6, the following integral estimates hold for $(t,u) \in [0, T_{(Boot)}) \times [0, U_0]$ (see Sect. 6.2 regarding the vectorfield operator notation):*

$$\int_{\mathcal{M}_{t,u}} \left| \begin{pmatrix} \breve{X} \mathscr{Z}_*^{N;\leq 1} \Psi \\ (1+\mu) L \mathscr{Z}_*^{N;\leq 1} \Psi \end{pmatrix} \right| \left| \begin{pmatrix} [ia] \mu (\exp \rho) c_s^2 (g_{ab} X^b) \mathscr{P}^{N+1} \omega \\ [ia] \mu (\exp \rho) c_s^2 \left(\frac{g_{ab} Y^b}{g_{cd} Y^c Y^d} \right) \mathscr{P}^{N+1} \omega \end{pmatrix} \right| d\varpi \tag{15.6.1}$$
$$\lesssim \int_{t'=0}^{t} \mathbb{Q}_N(t',u)\, dt' + \int_{t'=0}^{t} \mathbb{V}_{N+1}(t',u)\, dt'.$$

Proof. Using the schematic relations (3.19.2a), the L^∞ estimates of Prop. 9.12, Young's inequality, and Lemma 12.5, we bound LHS (15.6.1) as follows:

$$\begin{aligned}
&\int_{\mathcal{M}_{t,u}} \left| \begin{pmatrix} \breve{X} \mathscr{Z}_*^{N;\leq 1} \Psi \\ (1+\mu) L \mathscr{Z}_*^{N;\leq 1} \Psi \end{pmatrix} \right| \left| \begin{pmatrix} [ia] \mu (\exp \rho) c_s^2 (g_{ab} X^b) \mathscr{P}^{N+1} \omega \\ [ia] \mu (\exp \rho) c_s^2 \left(\frac{g_{ab} Y^b}{g_{cd} Y^c Y^d} \right) \mathscr{P}^{N+1} \omega \end{pmatrix} \right| d\varpi \\
&\lesssim \int_{\mathcal{M}_{t,u}} \left| \breve{X} \mathscr{Z}_*^{N;\leq 1} \Psi \right|^2 d\varpi + \int_{\mathcal{M}_{t,u}} \mu \left| L \mathscr{Z}_*^{N;\leq 1} \Psi \right|^2 d\varpi + \int_{\mathcal{M}_{t,u}} \mu \left| \mathscr{P}^{N+1} \omega \right|^2 d\varpi \\
&\lesssim \int_{t'=0}^{t} \left\| \breve{X} \mathscr{Z}_*^{N;\leq 1} \Psi \right\|_{L^2(\Sigma_{t'}^u)}^2 dt' + \int_{t'=0}^{t} \left\| \sqrt{\mu} L \mathscr{Z}_*^{N;\leq 1} \Psi \right\|_{L^2(\Sigma_{t'}^u)}^2 dt' \\
&\quad + \int_{t'=0}^{t} \left\| \sqrt{\mu} \mathscr{P}^{N+1} \omega \right\|_{L^2(\Sigma_{t'}^u)}^2 dt' \\
&\lesssim \int_{t'=0}^{t} \mathbb{Q}_N(t',u)\, dt' + \int_{t'=0}^{t} \mathbb{V}_{N+1}(t',u)\, dt'.
\end{aligned} \tag{15.6.2}$$

□

15.7. L^2 **bounds for the difficult top-order error integrals in terms of** $\mathbb{Q}_{[1,N]}$**.** In this section, we derive estimates for the difficult error integrals that we encounter in our energy estimates for $\vec{\Psi}$. These error integrals would cause derivative loss if they were not treated carefully and moreover, they make a substantial contribution to the blowup-rates featured in our high-order energy estimates. Our arguments here rely on the fully modified quantities defined in Sect. 7.

The main result is Lemma 15.13. We start with a preliminary lemma in which we estimate the most difficult product that appears in our wave equation energy estimates.

Lemma 15.12 (**L^2 bound for the most difficult product**)**.** *Assume that $N = 20$ and that $\mathscr{Z}_*^{N;\leq 1} \in \{Y^N, Y^{N-1}\breve{X}\}$. There exist constants $C > 0$ and $C_* > 0$ such that under the data-size and bootstrap assumptions of Sects. 8.1-8.4 and the smallness assumptions of Sect. 8.6, the following L^2 estimate holds for the difficult product $(\breve{X}v^1)\mathscr{Z}_*^{N;\leq 1}\mathrm{tr}_{\not g}\upchi$ from Prop. 14.9 whenever $(t,u) \in [0, T_{(Boot)}) \times [0, U_0]$:*

$$\begin{aligned}
\left\|(\breve{X}v^1)\mathscr{Z}_*^{N;\leq 1}\mathrm{tr}_{\not g}\upchi\right\|_{L^2(\Sigma_t^u)} &\leq \boxed{2}\frac{\|[L\upmu]_-\|_{L^\infty(\Sigma_t^u)}}{\upmu_\star(t,u)}\sqrt{\mathbb{Q}_N}(t,u) \\
&+ \boxed{4.05}\frac{\|[L\upmu]_-\|_{L^\infty(\Sigma_t^u)}}{\upmu_\star(t,u)}\int_{s=0}^t \frac{\|[L\upmu]_-\|_{L^\infty(\Sigma_s^u)}}{\upmu_\star(s,u)}\sqrt{\mathbb{Q}_N}(s,u)\,ds \\
&+ C_*\frac{1}{\upmu_\star(t,u)}\sqrt{\mathbb{Q}_N^{(Partial)}}(t,u) \\
&+ C_*\frac{1}{\upmu_\star(t,u)}\int_{s=0}^t \frac{1}{\upmu_\star(s,u)}\sqrt{\mathbb{Q}_N^{(Partial)}}(s,u)\,ds \\
&+ C\varepsilon\frac{1}{\upmu_\star(t,u)}\int_{s=0}^t \frac{1}{\upmu_\star(s,u)}\sqrt{\mathbb{Q}_N}(s,u)\,ds \\
&+ C\frac{1}{\upmu_\star(t,u)}\int_{s'=0}^t \frac{1}{\upmu_\star(s',u)}\int_{s=0}^{s'} \frac{1}{\upmu_\star^{1/2}(s,u)}\sqrt{\mathbb{Q}_N}(s,u)\,ds\,ds' \\
&+ C\frac{1}{\upmu_\star(t,u)}\int_{s=0}^t \frac{1}{\upmu_\star^{1/2}(s,u)}\sqrt{\mathbb{Q}_N}(s,u)\,ds \\
&+ C\frac{1}{\upmu_\star^{1/2}(t,u)}\sqrt{\mathbb{Q}_N}(t,u) + C\frac{1}{\upmu_\star^{3/2}(t,u)}\sqrt{\mathbb{Q}_{[1,N-1]}}(t,u) \\
&+ C\frac{1}{\upmu_\star(t,u)}\int_{s=0}^t \sqrt{\mathbb{V}_{N+1}}(s,u)\,ds \\
&+ C\frac{1}{\upmu_\star(t,u)}\int_{s=0}^t \frac{1}{\upmu_\star^{1/2}(s,u)}\sqrt{\mathbb{V}_{\leq N}}(s,u)\,ds \\
&+ C\frac{1}{\upmu_\star^{3/2}(t,u)}\mathring{\upepsilon}.
\end{aligned} \tag{15.7.1}$$

Moreover, we have the following less degenerate estimates:

$$\begin{aligned}
\begin{pmatrix} \left\|(\breve{X}v^2)\mathscr{Z}_*^{N;\leq 1}\mathrm{tr}_{\not g}\upchi\right\|_{L^2(\Sigma_t^u)} \\ \left\|\left\{\breve{X}(\uprho - v^1)\right\}\mathscr{Z}_*^{N;\leq 1}\mathrm{tr}_{\not g}\upchi\right\|_{L^2(\Sigma_t^u)} \end{pmatrix} &\lesssim \varepsilon\frac{1}{\upmu_\star(t,u)}\sqrt{\mathbb{Q}_N}(t,u) \\
&+ \varepsilon\frac{1}{\upmu_\star(t,u)}\int_{s=0}^t \frac{1}{\upmu_\star(s,u)}\sqrt{\mathbb{Q}_N}(s,u)\,ds \\
&+ \varepsilon\frac{1}{\upmu_\star(t,u)}\int_{s=0}^t \sqrt{\mathbb{V}_{N+1}}(s,u)\,ds
\end{aligned} \tag{15.7.2}$$

$$+\,\varepsilon\frac{1}{\upmu_\star(t,u)}\int_{s=0}^{t}\frac{1}{\upmu_\star^{1/2}(s,u)}\sqrt{\mathbb{V}_{\leq N}}(s,u)\,ds$$
$$+\,\varepsilon\frac{1}{\upmu_\star^{3/2}(t,u)}\sqrt{\mathbb{Q}_{[1,N-1]}}(t,u)+\mathring{\upepsilon}\frac{1}{\upmu_\star^{3/2}(t,u)}.$$

Furthermore, we have the following less precise estimate:

$$\left\|\upmu\mathscr{Z}_*^{N;\leq 1}\mathrm{tr}_{\not g}\chi\right\|_{L^2(\Sigma_t^u)}\lesssim\sqrt{\mathbb{Q}_{[1,N]}}(t,u)+\int_{s=0}^{t}\frac{1}{\upmu_\star(s,u)}\sqrt{\mathbb{Q}_{[1,N]}}(s,u)\,ds \tag{15.7.3}$$
$$+\int_{s=0}^{t}\sqrt{\mathbb{V}_{N+1}}(s,u)\,ds+\int_{s=0}^{t}\frac{1}{\upmu_\star^{1/2}(s,u)}\sqrt{\mathbb{V}_{\leq N}}(s,u)\,ds$$
$$+\,\mathring{\upepsilon}\left\{\ln\upmu_\star^{-1}(t,u)+1\right\}.$$

Proof. We first prove (15.7.1). We take the norm $\|\cdot\|_{L^2(\Sigma_t^u)}$ of both sides of inequality (14.5.15). Using Lemma 12.5, we see that the norm $\|\cdot\|_{L^2(\Sigma_t^u)}$ of the first term on RHS (14.5.15) is bounded by the first term on RHS (15.7.1). Similarly, using Lemma 12.5, we see that the norm $\|\cdot\|_{L^2(\Sigma_t^u)}$ of the second term on RHS (14.5.15) is bounded by the term $C_*\frac{1}{\upmu_\star(t,u)}\sqrt{\mathbb{Q}_N^{(Partial)}}(t,u)$ on RHS (15.7.1). Next we use Lemmas 12.3 and 12.5 to bound the norm $\|\cdot\|_{L^2(\Sigma_t^u)}$ of the third term on RHS (14.5.15) by the term $\boxed{4.05}\cdots$ on RHS (15.7.1). Similarly, using Lemmas 12.3 and 12.5 we see that the norm $\|\cdot\|_{L^2(\Sigma_t^u)}$ of the fourth term on RHS (14.5.15) is bounded by the $\sqrt{\mathbb{Q}_N^{(Partial)}}$-involving time integral term on RHS (15.7.1) (which is multiplied by C_*).

It remains for us to explain why the norm $\|\cdot\|_{L^2(\Sigma_t^u)}$ of the terms Error on RHS (14.5.15) are $\leq$ the sum of the terms on lines five to eleven of RHS (15.7.1). With the exception of the bound for the terms on the first line RHS (14.5.16), the desired bounds follow from the same estimates used above together with those of Lemmas 15.6 and 15.9, inequalities (11.3.3), (11.3.5), and (11.3.6), the fact that the $\mathbb{Q}_M$ are increasing in their arguments, and simple inequalities of the form $ab\lesssim a^2+b^2$. Finally, we must bound the norm $\|\cdot\|_{L^2(\Sigma_t^u)}$ of the terms on the first line of RHS (14.5.16). To this end, we first use (12.2.3) with $s=0$ to deduce $\left\|\left|^{(Y^N)}\mathscr{X}\right|(1,\cdot)+\left|^{(Y^{N-1}\breve{X})}\mathscr{X}\right|(1,\cdot)\right\|_{L^2(\Sigma_t^u)}\lesssim\left\|\left|^{(Y^N)}\mathscr{X}\right|+\left|^{(Y^{N-1}\breve{X})}\mathscr{X}\right|\right\|_{L^2(\Sigma_0^u)}$. Next, from definition (7.2.2a), the simple inequality $|\vec{G}_{(Frame)}|=|\mathrm{f}(\upgamma,\not d\vec{x})|\lesssim 1$ (which follows from Lemmas 3.19 and 9.5 and the L^∞ estimates of Prop. 9.12), the estimate (9.4.1c), and our assumptions on the data, we find that $\left\|\left|^{(Y^N)}\mathscr{X}\right|+\left|^{(Y^{N-1}\breve{X})}\mathscr{X}\right|\right\|_{L^2(\Sigma_0^u)}\lesssim\mathring{\upepsilon}$. It follows that the norm $\|\cdot\|_{L^2(\Sigma_t^u)}$ of the terms on the first line of RHS (14.5.16) is $\lesssim\mathring{\upepsilon}\frac{1}{\upmu_\star(t,u)}$ as desired. This completes the proof of (15.7.1).

The proof of (15.7.3) is based on inequality (14.5.17) and is similar but much simpler; we omit the details, noting only that inequality (11.3.5) leads to the presence of the factor $\ln\upmu_\star^{-1}(t,u)+1$.

The estimate (15.7.2) then follows from (15.7.3) and the estimates $\|\breve{X}v^2\|_{L^\infty(\Sigma_t^u)}\lesssim\varepsilon$ and $\|\breve{X}(\uprho-v^1)\|_{L^\infty(\Sigma_t^u)}\lesssim\varepsilon$ (that is, (9.6.3a) and (9.6.4)).

□

Armed with Lemma 15.12, we now derive the main result of this section.

Lemma 15.13 (**Bound for the most difficult error integrals**). *Assume that* $N = 20$. *There exist constants* $C > 0$ *and* $C_* > 0$ *such that under the data-size and bootstrap assumptions of Sects. 8.1-8.4 and the smallness assumptions of Sect. 8.6, the following integral inequalities hold for* $(t,u) \in [0, T_{(Boot)}) \times [0, U_0]$*:*

$$
\begin{aligned}
& 2 \left| \int_{\mathcal{M}_{t,u}} (\breve{X} \mathscr{Z}_*^{N;\leq 1} v^1)(\breve{X} v^1) Y^N \mathrm{tr}_{\not g}\chi \, d\varpi \right|, \, 2 \left| \int_{\mathcal{M}_{t,u}} (\breve{X} \mathscr{Z}_*^{N;\leq 1} v^1)(\breve{X} v^1) Y^{N-1} \breve{X} \mathrm{tr}_{\not g}\chi \, d\varpi \right| \qquad (15.7.4) \\
& \leq \boxed{4} \int_{t'=0}^{t} \frac{\| [L\mu]_- \|_{L^\infty(\Sigma_{t'}^u)}}{\mu_\star(t',u)} \mathbb{Q}_N(t',u) \, dt' \\
& \quad + \boxed{8.1} \int_{t'=0}^{t} \frac{\| [L\mu]_- \|_{L^\infty(\Sigma_{t'}^u)}}{\mu_\star(t',u)} \sqrt{\mathbb{Q}_N}(t',u) \int_{s=0}^{t'} \frac{\| [L\mu]_- \|_{L^\infty(\Sigma_s^u)}}{\mu_\star(s,u)} \sqrt{\mathbb{Q}_N}(s,u) \, ds \, dt' \\
& \quad + C_* \int_{t'=0}^{t} \frac{1}{\mu_\star(t',u)} \sqrt{\mathbb{Q}_N}(t',u) \sqrt{\mathbb{Q}_N^{(Partial)}}(t',u) \, dt' \\
& \quad + C_* \int_{t'=0}^{t} \frac{1}{\mu_\star(t',u)} \sqrt{\mathbb{Q}_N}(t',u) \int_{s=0}^{t'} \frac{1}{\mu_\star(s,u)} \sqrt{\mathbb{Q}_N^{(Partial)}}(s,u) \, ds \, dt' \\
& \quad + C\varepsilon \int_{t'=0}^{t} \frac{1}{\mu_\star(t',u)} \sqrt{\mathbb{Q}_N}(t',u) \int_{s=0}^{t'} \frac{1}{\mu_\star(s,u)} \sqrt{\mathbb{Q}_N}(s,u) \, ds \, dt' \\
& \quad + C \int_{t'=0}^{t} \frac{1}{\mu_\star(t',u)} \sqrt{\mathbb{Q}_N}(t',u) \int_{s'=0}^{t'} \frac{1}{\mu_\star(s',u)} \int_{s=0}^{s'} \frac{1}{\mu_\star^{1/2}(s,u)} \sqrt{\mathbb{Q}_N}(s,u) \, ds \, ds' \, dt' \\
& \quad + C \int_{t'=0}^{t} \frac{1}{\mu_\star(t',u)} \sqrt{\mathbb{Q}_N}(t',u) \int_{s=0}^{t'} \frac{1}{\mu_\star^{1/2}(s,u)} \sqrt{\mathbb{Q}_N}(s,u) \, ds \, dt' \\
& \quad + C \int_{t'=0}^{t} \frac{1}{\mu_\star^{1/2}(t,u)} \mathbb{Q}_N(t',u) \, dt' + C \int_{t'=0}^{t} \frac{1}{\mu_\star^{5/2}(t',u)} \mathbb{Q}_{[1,N-1]}(t',u) \, dt' \\
& \quad + C \int_{t'=0}^{t} \frac{1}{\mu_\star^{3/2}(t',u)} \left\{ \int_{s=0}^{t'} \sqrt{\mathbb{V}_{N+1}}(s,u) \, ds \right\}^2 dt' \\
& \quad + C \int_{t'=0}^{t} \frac{1}{\mu_\star^{3/2}(t',u)} \left\{ \int_{s=0}^{t'} \frac{1}{\mu_\star^{1/2}(s,u)} \sqrt{\mathbb{V}_{\leq N}}(s,u) \, ds \right\}^2 dt' \\
& \quad + C \mathring{\epsilon} \frac{1}{\mu_\star^{3/2}(t,u)}.
\end{aligned}
$$

Moreover, we have the following less degenerate estimates:

$$2\left|\int_{\mathcal{M}_{t,u}}(\breve{X}\mathscr{Z}_*^{N;\leq 1}v^2)(\breve{X}v^2)Y^N\mathrm{tr}_{\not g}\upchi\,d\varpi\right|,\ 2\left|\int_{\mathcal{M}_{t,u}}(\breve{X}\mathscr{Z}_*^{N;\leq 1}v^2)(\breve{X}v^2)Y^{N-1}\breve{X}\mathrm{tr}_{\not g}\upchi\,d\varpi\right|, \tag{15.7.5}$$

$$2\left|\int_{\mathcal{M}_{t,u}}\left\{\breve{X}\mathscr{Z}_*^{N;\leq 1}(\uprho-v^1)\right\}\left\{\breve{X}(\uprho-v^1)\right\}Y^N\mathrm{tr}_{\not g}\upchi\,d\varpi\right|,$$

$$2\left|\int_{\mathcal{M}_{t,u}}\left\{\breve{X}\mathscr{Z}_*^{N;\leq 1}(\uprho-v^1)\right\}\left\{\breve{X}(\uprho-v^1)\right\}Y^{N-1}\breve{X}\mathrm{tr}_{\not g}\upchi\,d\varpi\right|$$

$$\begin{aligned}
&\lesssim \varepsilon\int_{t'=0}^{t}\frac{1}{\upmu_\star(t',u)}\mathbb{Q}_N(t',u)\,dt'\\
&\quad+\varepsilon\int_{t'=0}^{t}\frac{1}{\upmu_\star(t',u)}\sqrt{\mathbb{Q}_N}(t',u)\int_{s=0}^{t}\frac{1}{\upmu_\star(s,u)}\sqrt{\mathbb{Q}_N}(s,u)\,ds\,ds\\
&\quad+\varepsilon\int_{t'=0}^{t}\frac{1}{\upmu_\star(t',u)}\sqrt{\mathbb{Q}_N}(t',u)\int_{s=0}^{t'}\sqrt{\mathbb{V}_{N+1}}(s,u)\,ds\,dt'\\
&\quad+\varepsilon\int_{t'=0}^{t}\frac{1}{\upmu_\star(t',u)}\sqrt{\mathbb{Q}_N}(t',u)\int_{s=0}^{t'}\frac{1}{\upmu_\star^{1/2}(s,u)}\sqrt{\mathbb{V}_{\leq N}}(s,u)\,ds\,dt'\\
&\quad+\varepsilon\int_{t'=0}^{t}\frac{1}{\upmu_\star^{5/2}(t',u)}\mathbb{Q}_{[1,N-1]}(t',u)\,dt'+\mathring{\upepsilon}\frac{1}{\upmu_\star^{3/2}(t,u)}.
\end{aligned}$$

Proof. We first prove (15.7.4). We treat only the first integral on the LHS since the second one can be treated using identical arguments. To proceed, we first use Cauchy-Schwarz and (12.3.2a) to bound it by

$$\leq 2\int_{t'=0}^{t}\sqrt{\mathbb{Q}_N}(t',u)\left\|(\breve{X}v^1)Y^N\mathrm{tr}_{\not g}\upchi\right\|_{L^2(\Sigma_{t'}^u)}\,dt'. \tag{15.7.6}$$

We now substitute the estimate (15.7.1) (with t in (15.7.1) replaced by t') for the second factor in the integrand (15.7.6). Following this substitution, the desired bound of RHS (15.7.6) by $\leq$ RHS (15.7.4) follows easily with the help of simple estimates of the form $ab\lesssim a^2+b^2$, the fact that $\mathbb{Q}_N$ is increasing in its arguments, and the estimate (11.3.3), which we use to bound the error integral $\mathring{\upepsilon}\int_{t'=0}^{t}\frac{1}{\upmu_\star^{3/2}(t',u)}\sqrt{\mathbb{Q}_N}(t',u)\,dt'$ by $\lesssim\mathring{\upepsilon}^2\int_{t'=0}^{t}\frac{1}{\upmu_\star^{5/2}(t',u)}\,dt'+\int_{t'=0}^{t}\frac{1}{\upmu_\star^{1/2}(t',u)}\mathbb{Q}_N(t',u)\,dt'\lesssim\mathring{\upepsilon}^2\frac{1}{\upmu_\star^{3/2}(t,u)}+\int_{t'=0}^{t}\frac{1}{\upmu_\star^{1/2}(t',u)}\mathbb{Q}_N(t',u)\,dt'$.

The proof of (15.7.5) is similar but simpler. To bound the first integral on LHS (15.7.5), we first argue as above to deduce that it is bounded by RHS (15.7.6), but with $\breve{X}v^1$ on the RHS replaced by $\breve{X}v^2$. We then use the estimate (15.7.2) in place of the estimate (15.7.1) used in the proof of (15.7.4). The remainder of the proof now proceeds as in the proof of (15.7.4). The remaining three integrals on LHS (15.7.5) can be bounded in the same way. □

15.8. L^2 **bounds for less degenerate top-order error integrals in terms of** $\mathbb{Q}_{[1,N]}$. In this section, we bound some top-order error integrals for which we need the fully modified quantities defined in Sect. 7 to avoid losing a derivative. However, the error integrals contain a helpful factor of $\upmu$. For this reason, the estimates are easier to derive and less degenerate.

Lemma 15.14 (**Bounds for less degenerate top-order error integrals**). *Assume that* $\Psi \in \{\uprho - v^1, v^1, v^2\}$ *and that* $N = 20$. *Recall that* y *is the scalar-valued function appearing in Lemma 3.8. Under the data-size and bootstrap assumptions of Sects. 8.1-8.4 and the smallness assumptions of Sect. 8.6, the following integral estimates hold for* $(t,u) \in [0, T_{(Boot)}) \times [0, U_0]$*:*

$$\left| \int_{\mathcal{M}_{t,u}} y(\breve{X}\mathscr{Z}_*^{N;\leq 1}\Psi)(\breve{X}\Psi)(\not{d}^{\#}\Psi) \cdot \begin{pmatrix} \upmu \not{d} Y^{N-1}\mathrm{tr}_{\not{g}}\upchi \\ \upmu \not{d} Y^{N-2}\breve{X}\mathrm{tr}_{\not{g}}\upchi \end{pmatrix} d\varpi \right| \tag{15.8.1a}$$

$$\lesssim \int_{t'=0}^{t} \left\{ \ln \upmu_\star^{-1}(t',u) + 1 \right\}^2 \mathbb{Q}_{[1,N]}(t',u)\, dt' + \int_{t'=0}^{t} \mathbb{V}_{\leq N+1}(t',u)\, dt' + \mathring{\upepsilon}^2,$$

$$\left| \int_{\mathcal{M}_{t,u}} (1+2\upmu) y(L\mathscr{Z}_*^{N;\leq 1}\Psi)(\breve{X}\Psi)(\not{d}^{\#}\Psi) \cdot \begin{pmatrix} \upmu \not{d} Y^{N-1}\mathrm{tr}_{\not{g}}\upchi \\ \upmu \not{d} Y^{N-2}\breve{X}\mathrm{tr}_{\not{g}}\upchi \end{pmatrix} d\varpi \right| \tag{15.8.1b}$$

$$\lesssim \int_{t'=0}^{t} \left\{ \ln \upmu_\star^{-1}(t',u) + 1 \right\}^2 \mathbb{Q}_{[1,N]}(t',u)\, dt' + \int_{u'=0}^{u} \mathbb{Q}_{[1,N]}(t,u')\, du' + \int_{t'=0}^{t} \mathbb{V}_{\leq N+1}(t',u)\, dt' + \mathring{\upepsilon}^2.$$

Proof. We prove (15.8.1b) only for the first product on the LHS since the proof for the second term is identical. To proceed, we first use the schematic identity (3.19.2c) for y, the L^∞ estimates of Prop. 9.12, Young's inequality, Lemma 12.5, and inequality (15.7.3) to obtain

$$\left| \int_{\mathcal{M}_{t,u}} (1+2\upmu) y(L\mathscr{Z}_*^{N;\leq 1}\Psi)(\breve{X}\Psi)(\not{d}^{\#}\Psi) \cdot (\upmu \not{d} Y^{N-1}\mathrm{tr}_{\not{g}}\upchi)\, d\varpi \right| \tag{15.8.2}$$

$$\begin{aligned}
&\lesssim \int_{u'=0}^{u} \left\| L\mathscr{Z}_*^{N;\leq 1}\Psi \right\|_{L^2(\mathcal{P}_{u'}^t)}^2 du' + \int_{t'=0}^{t} \left\| \upmu Y^N \mathrm{tr}_{\not{g}}\upchi \right\|_{L^2(\Sigma_{t'}^u)}^2 dt' \\
&\lesssim \int_{u'=0}^{u} \mathbb{Q}_{[1,N]}(t,u')\, du' + \int_{t'=0}^{t} \mathbb{Q}_{[1,N]}(t',u)\, dt' + \int_{t'=0}^{t} \left\{ \int_{s=0}^{t'} \frac{1}{\upmu_\star(s,u)} \sqrt{\mathbb{Q}_N}(s,u)\, ds \right\}^2 dt' \\
&\quad + \int_{t'=0}^{t} \left\{ \int_{s=0}^{t'} \sqrt{\mathbb{V}_{N+1}}(s,u)\, ds \right\}^2 dt' + \int_{t'=0}^{t} \left\{ \int_{s=0}^{t'} \frac{1}{\upmu_\star^{1/2}(s,u)} \sqrt{\mathbb{V}_{\leq N}}(s,u)\, ds \right\}^2 dt' \\
&\quad + \mathring{\upepsilon}^2 \int_{t'=0}^{t} \left\{ \ln \upmu_\star^{-1}(t,u) + 1 \right\}^2 dt'.
\end{aligned}$$

Using inequalities (11.3.5) and (11.3.6) and the fact that $\mathbb{Q}_M$ and $\mathbb{V}_M$ are increasing in their arguments, we conclude that RHS (15.8.2) is $\lesssim$ RHS (15.8.1b) as desired.

The proof of (15.8.1a) is similar, except in the first step we bound LHS (15.8.1a) by $\lesssim \int_{t'=0}^{t} \left\| \breve{X} \mathscr{Z}_*^{N;\leq 1} \Psi \right\|_{L^2(\Sigma_{t'}^u)}^2 \, dt' + \int_{t'=0}^{t} \left\| \upmu Y^N \mathrm{tr}_{\not g} \chi \right\|_{L^2(\Sigma_{t'}^u)}^2 \, dt'$. □

15.9. **Error integrals requiring integration by parts with respect to L.** In deriving our top-order energy estimates for the wave variables $\{\uprho - v^1, v^1, v^2\}$, we encounter some difficult error integrals that we can control only by integrating by parts with respect to L; see the error integral (15.14.3) and the discussion below it. It turns out that in carrying out this procedure, we must use the partially modified quantities of Sect. 7 in order to avoid generating error terms that are too large to control. This results in the presence of two types of error integrals, which we bound in this section: spacetime error integrals, some of which involve the partially modified quantities, and Σ_t^u "boundary" error integrals, some of which also involve the partially modified quantities. We remark that we treat the most difficult of these error integrals in Lemmas 15.15 and 15.17.

Lemma 15.15 (A difficult top-order hypersurface L^2 estimate). *Assume that $N = 20$ and let $\mathscr{Z}_*^{N-1;\leq 1} \in \{Y^{N-1}, Y^{N-2}\breve{X}\}$. Let ${}^{(\mathscr{Z}_*^{N-1;\leq 1})}\widetilde{\mathscr{X}}$ be the corresponding partially modified quantity defined by (7.2.2a). There exist constants $C > 0$ and $C_* > 0$ such that under the data-size and bootstrap assumptions of Sects. 8.1-8.4 and the smallness assumptions of Sect. 8.6, the following L^2 estimate holds for $(t,u) \in [0, T_{(Boot)}) \times [0, U_0]$:*

$$\begin{aligned}
\left\| \frac{1}{\sqrt{\upmu}} (\breve{X} v^1) L^{(\mathscr{Z}_*^{N-1;\leq 1})}\widetilde{\mathscr{X}} \right\|_{L^2(\Sigma_t^u)} \leq \boxed{\sqrt{2}} \frac{\left\| [L\upmu]_- \right\|_{C^0(\Sigma_t^u)}}{\upmu_\star(t,u)} \sqrt{\mathbb{Q}_N}(t,u) & \qquad (15.9.1a) \\
+ C_* \frac{1}{\upmu_\star(t,u)} \sqrt{\mathbb{Q}_N^{(Partial)}}(t,u) & \\
+ C \frac{1}{\upmu_\star^{1/2}(t,u)} \sqrt{\mathbb{Q}_{[1,N]}}(t,u) + C\varepsilon \frac{1}{\upmu_\star(t,u)} \sqrt{\mathbb{Q}_{[1,N]}}(t,u) & \\
+ C \frac{1}{\upmu_\star(t,u)} \sqrt{\mathbb{Q}_{[1,N-1]}}(t,u) + C \mathring{\upepsilon} \frac{1}{\upmu_\star^{1/2}(t,u)}, &
\end{aligned}$$

$$\begin{aligned}
\left\| \frac{1}{\sqrt{\upmu}} (\breve{X} v^1) {}^{(\mathscr{Z}_*^{N-1;\leq 1})}\widetilde{\mathscr{X}} \right\|_{L^2(\Sigma_t^u)} \leq \boxed{\sqrt{2}} \left\| L\upmu \right\|_{L^\infty({}^{(-)}\Sigma_{t;t}^u)} \frac{1}{\upmu_\star^{1/2}(t,u)} \int_{t'=0}^{t} \frac{1}{\upmu_\star^{1/2}(t',u)} \sqrt{\mathbb{Q}_{[1,N]}}(t',u) \, dt' & \qquad (15.9.1b) \\
+ C_* \frac{1}{\upmu_\star^{1/2}(t,u)} \int_{t'=0}^{t} \frac{1}{\upmu_\star^{1/2}(t',u)} \sqrt{\mathbb{Q}_{[1,N]}^{(Partial)}}(t',u) \, dt' & \\
+ C \int_{t'=0}^{t} \frac{1}{\upmu_\star^{1/2}(t',u)} \sqrt{\mathbb{Q}_{[1,N]}}(t',u) \, dt' & \\
+ C\varepsilon \frac{1}{\upmu_\star^{1/2}(t,u)} \int_{t'=0}^{t} \frac{1}{\upmu_\star^{1/2}(t',u)} \sqrt{\mathbb{Q}_{[1,N]}}(t',u) \, dt' & \\
+ C \mathring{\upepsilon} \frac{1}{\upmu_\star^{1/2}(t,u)}. &
\end{aligned}$$

Moreover, we have the following less precise estimates:

$$\left\| L^{(\mathscr{Z}_*^{N-1;\leq 1})}\widetilde{\mathscr{X}} \right\|_{L^2(\Sigma_t^u)} \lesssim \frac{1}{\upmu_\star^{1/2}(t,u)} \sqrt{\mathbb{Q}_{[1,N]}}(t,u) + \mathring{\upepsilon}, \tag{15.9.2a}$$

$$\left\| {}^{(\mathscr{Z}_*^{N-1;\leq 1})}\widetilde{\mathscr{X}} \right\|_{L^2(\Sigma_t^u)} \lesssim \int_{t'=0}^t \frac{1}{\upmu_\star^{1/2}(t',u)} \sqrt{\mathbb{Q}_{[1,N]}}(t',u)\, dt' + \mathring{\upepsilon}. \tag{15.9.2b}$$

Proof. We start by proving (15.9.1a). We multiply inequality (14.6.1a) by $\frac{1}{\sqrt{\upmu}}\breve{X}v^1$. We first consider the difficult product generated by the first term on RHS (14.6.1a). To proceed, we multiply the identity (14.5.25) by $\sqrt{\upmu}$ and use the schematic identity (3.19.2b) and the L^∞ estimates of Prop. 9.12 (in particular (9.6.3a) and (9.6.4)) to obtain

$$\frac{1}{2}\frac{1}{\sqrt{\upmu}}\left|\sum_{\imath=0}^{1} G_{LL}^{\imath}\breve{X}v^1\right| = \frac{L\upmu}{\sqrt{\upmu}} + \mathcal{O}(\varepsilon)\frac{1}{\sqrt{\upmu}}. \tag{15.9.3}$$

Inserting (15.9.3) into the product of $\frac{1}{\sqrt{\upmu}}\breve{X}v^1$ and the first product on RHS (14.6.1a), we obtain the terms

$$\frac{[L\upmu]_-}{\upmu}\left|\sqrt{\upmu}\Delta\!\!\!/\mathscr{Z}_*^{N-1;\leq 1}v^1\right| + \frac{[L\upmu]_+}{\upmu}\left|\sqrt{\upmu}\Delta\!\!\!/\mathscr{Z}_*^{N-1;\leq 1}v^1\right| + \frac{\mathcal{O}(\varepsilon)}{\upmu}\left|\sqrt{\upmu}\Delta\!\!\!/\mathscr{Z}_*^{N-1;\leq 1}v^1\right|. \tag{15.9.4}$$

Using Lemma 12.5, we see that the norm $\|\cdot\|_{L^2(\Sigma_t^u)}$ of the first product in (15.9.4) is bounded by the first term on RHS (15.9.1a) as desired. Next, we again use Lemma 12.5 and the estimate (11.2.1) to bound the norm $\|\cdot\|_{L^2(\Sigma_t^u)}$ of the second and third products on RHS (15.9.4) by the terms on the third line of RHS (15.9.1a) as desired. In proving the remaining estimates, we use (9.6.3c) to bound $\left\|\frac{1}{\sqrt{\upmu}}\breve{X}v^1\right\|_{L^\infty(\Sigma_t^u)} \leq C\frac{1}{\sqrt{\upmu_\star(t,u)}}$ and thus it remains for us to bound the norm $\|\cdot\|_{L^2(\Sigma_t^u)}$ of the remaining two terms on RHS (14.6.1a) and to multiply those bounds by $C\frac{1}{\sqrt{\upmu_\star(t,u)}}$. To handle the product generated by the second term on RHS (14.6.1a), we use Lemma 12.5, which implies that its norm $\|\cdot\|_{L^2(\Sigma_t^u)}$ is bounded by the $\mathbb{Q}_N^{(Partial)}$-involving term on RHS (15.9.1a) (which has the coefficient C_*). To bound the product generated by the next-to-last term $C\varepsilon\cdots$ on RHS (14.6.1a), we use Lemma 12.5 (the product under consideration is bounded by $\leq$ the term $C\varepsilon\frac{1}{\upmu_\star(t,u)}\sqrt{\mathbb{Q}_{[1,N]}}(t,u)$ on RHS (15.9.1a)). To bound the product generated by the last term on RHS (14.6.1a), we use the estimate (15.4.2b), inequality (11.3.6), and the fact that $\mathbb{Q}_{[1,N]}$ is increasing in its arguments. We have thus proved (15.9.1a).

We now prove (15.9.1b). We multiply inequality (14.6.1b) by $\frac{1}{\sqrt{\upmu}}\breve{X}v^1$. The most difficult product is generated by the second term on RHS (14.6.1b):

$$\frac{1}{2}\frac{1}{\sqrt{\upmu}}\left|\sum_{\imath=0}^{1} G_{LL}^{\imath}\breve{X}v^1\right|(t,u,\vartheta)\int_{t'=0}^{t}\left|\Delta\!\!\!/\mathscr{Z}_*^{N-1;\leq 1}v^1\right|(t',u,\vartheta)\,dt'. \tag{15.9.5}$$

We now substitute RHS (15.9.3) for the product $\frac{1}{\sqrt{\upmu}}\left|\sum_{\imath=0}^{1} G_{LL}^{\imath}\breve{X}v^1\right|$ in (15.9.5) and take the norm $\|\cdot\|_{L^2(\Sigma_t^u)}$ of the resulting expression. With the help of Lemma 12.3 and Lemma 12.5, we see that the norm $\|\cdot\|_{L^2(\Sigma_t^u)}$ of the product generated by the second product on RHS (15.9.3) is bounded by the next-to-last term $C\varepsilon\cdots$ on RHS (15.9.1b). To handle the remaining product (corresponding to the term $\frac{L\upmu}{\sqrt{\upmu}}$ on RHS (15.9.3)), we first decompose $\Sigma_t^u = {}^{(+)}\Sigma_{s;t}^u \cup {}^{(-)}\Sigma_{s;t}^u$ as in (11.1.9b) and again use Lemmas 12.3 and 12.5 as well as the simple estimate $\|L\upmu\|_{L^2(\Sigma_t^u)} \lesssim 1$ (see (9.6.5b)) to bound it by

$$\begin{aligned} &\leq \sqrt{2}\,\|L\upmu\|_{L^\infty({}^{(-)}\Sigma_{t;t}^u)} \frac{1}{\upmu_\star^{1/2}(t,u)} \int_{t'=0}^{t} \frac{1}{\upmu_\star^{1/2}(t',u)} \sqrt{\mathbb{Q}_{[1,N]}}(t',u)\,dt' \\ &\quad + \sqrt{2}\left\|\frac{L\upmu}{\sqrt{\upmu}}\right\|_{L^\infty({}^{(+)}\Sigma_{t;t}^u)} \int_{t'=0}^{t} \frac{1}{\upmu_\star^{1/2}(t',u)} \sqrt{\mathbb{Q}_{[1,N]}}(t',u)\,dt' \\ &\quad + C\varepsilon \frac{1}{\upmu_\star^{1/2}(t,u)} \int_{t'=0}^{t} \frac{1}{\upmu_\star^{1/2}(t',u)} \sqrt{\mathbb{Q}_{[1,N]}}(t',u)\,dt'. \end{aligned} \tag{15.9.6}$$

The first and third products on RHS (15.9.6) are manifestly bounded by RHS (15.9.1b). To bound the second product on RHS (15.9.6) by RHS (15.9.1b), we need only to use the following estimate to bound the factor multiplying the time integral:

$$\left\|\frac{L\upmu}{\sqrt{\upmu}}\right\|_{L^\infty({}^{(+)}\Sigma_{t;t}^u)} \leq C\left\|\frac{[L\upmu]_+}{\upmu}\right\|_{L^\infty({}^{(+)}\Sigma_{t;t}^u)} + C\left\|\frac{[L\upmu]_-}{\upmu}\right\|_{L^\infty({}^{(+)}\Sigma_{t;t}^u)} \leq C. \tag{15.9.7}$$

The estimate (15.9.7) is a straightforward consequence of the estimates (9.6.5a) and (9.6.5b) (with $M=0$), (11.2.1), and (11.2.9). We now bound the norm $\|\cdot\|_{L^2(\Sigma_t^u)}$ of the product of $\frac{1}{\sqrt{\upmu}}\breve{X}v^1$ and the remaining four terms on RHS (14.6.1b). In all of the remaining estimates, we rely on the bound $\left\|\frac{1}{\sqrt{\upmu}}\breve{X}v^1\right\|_{L^\infty(\Sigma_t^u)} \leq C\frac{1}{\sqrt{\upmu_\star(t,u)}}$ noted in the proof of (15.9.1a); it therefore remains for us to bound the norm $\|\cdot\|_{L^2(\Sigma_t^u)}$ of the remaining four terms on RHS (14.6.1b) and to multiply those bounds by $C\frac{1}{\sqrt{\upmu_\star(t,u)}}$. To bound the product corresponding to the first term $\left|{}^{(\mathscr{Z}_*^{N;\leq 1})}\widetilde{\mathscr{X}}\right|(0,u,\vartheta)$ on RHS (14.6.1b), we first use (12.2.3) with $s=0$ to deduce $\left\|\left|{}^{(Y^{N-1})}\widetilde{\mathscr{X}}\right|(1,\cdot)\right\|_{L^2(\Sigma_t^u)} \lesssim \left\|{}^{(Y^{N-1})}\widetilde{\mathscr{X}}\right\|_{L^2(\Sigma_0^u)}$. Next, from definition (7.2.2a), the simple inequality $|G_{(Frame)}| = |\mathrm{f}(\upgamma, \not{d}\vec{x})| \lesssim 1$ (which follows from Lemmas 3.19 and 9.5 and the L^∞ estimates of Prop. 9.12), the estimate (9.4.1c), and our assumptions on the data, we find that $\left\|{}^{(\mathscr{Z}_*^{N;\leq 1})}\widetilde{\mathscr{X}}\right\|_{L^2(\Sigma_0^u)} \lesssim \mathring{\upepsilon}$. In total, we conclude that the product under consideration is bounded in the norm $\|\cdot\|_{L^2(\Sigma_t^u)}$ by the last term on RHS (15.9.1b) as desired. To bound the norm $\|\cdot\|_{L^2(\Sigma_t^u)}$ of the second time integral $C_*\cdots$ on RHS (14.6.1b), we use Lemmas 12.3

and 12.5. Multiplying by $C\frac{1}{\sqrt{\upmu_\star(t,u)}}$, we find that the term of interest is bounded by the second term $C_* \cdots$ on RHS (15.9.1b). Similarly, we see that the product generated by the time integral $C\varepsilon \cdots$ on RHS (14.6.1b) is bounded by $\leq$ the $C\varepsilon \cdots$ term on RHS (15.9.1b). To bound the product generated by the last time integral on RHS (14.6.1b), we use a similar argument together with (15.4.2b), except that as a preliminary step, we bound the time integral on RHS (15.4.2b) by $\lesssim \sqrt{\mathbb{Q}_{[1,N]}}(t,u)$ with the help of (11.3.6). We have thus proved (15.9.1b).

The proofs of (15.9.2a) and (15.9.2b) are based on a subset of the above arguments and are much simpler; we therefore omit the details, noting only that the main simplification is that we do *not* have to rely on the estimate (15.9.3), which played a fundamental role in our proofs of (15.9.1a) and (15.9.1b). □

Lemma 15.16 (Bounds connected to easy top-order error integrals requiring integration by parts with respect to L). *Let $\Psi \in \{\uprho - v^1, v^1, v^2\}$. Assume that $N = 20$ and $\varsigma > 0$. For $i = 1, 2$, let $\mathrm{Error}_i[\mathscr{Z}_*^{N;\leq 1}\Psi; {}^{(\mathscr{Z}_*^{N-1;\leq 1})}\widetilde{\mathscr{X}}]$ be the error integrands defined in* (4.4.3a) *and* (4.4.3b)*, where the partially modified quantity ${}^{(\mathscr{Z}_*^{N-1;\leq 1})}\widetilde{\mathscr{X}}$ defined in* (7.2.2a) *is in role of $\upeta$ and we are assuming no relationship between the operators $\mathscr{Z}_*^{N;\leq 1}$ and $\mathscr{Z}_*^{N-1;\leq 1}$ (see Sect. 6.2 regarding the vectorfield operator notation). Under the data-size and bootstrap assumptions of Sects. 8.1-8.4 and the smallness assumptions of Sect. 8.6, the following estimates hold for $(t,u) \in [0, T_{(Boot)}) \times [0, U_0]$, where the implicit constants are independent of ς:*

$$\int_{\mathcal{M}_{t,u}} \left| \mathrm{Error}_1[\mathscr{Z}_*^{N;\leq 1}\Psi; {}^{(\mathscr{Z}_*^{N-1;\leq 1})}\widetilde{\mathscr{X}}] \right| d\varpi \tag{15.9.8a}$$
$$\lesssim (1+\varsigma^{-1}) \int_{s=0}^t \frac{1}{\upmu_\star^{1/2}(s,u)} \mathbb{Q}_{[1,N]}(s,u)\, ds + \int_{s=0}^t \frac{1}{\upmu_\star^{3/2}(s,u)} \mathbb{Q}_{[1,N-1]}(s,u)\, ds$$
$$+ \varsigma \mathbb{K}_{[1,N]}(t,u) + (1+\varsigma^{-1})\mathring{\upepsilon}^2,$$

$$\int_{\Sigma_t^u} \left| \mathrm{Error}_2[\mathscr{Z}_*^{N;\leq 1}\Psi; {}^{(\mathscr{Z}_*^{N-1;\leq 1})}\widetilde{\mathscr{X}}] \right| d\varpi \lesssim \mathring{\upepsilon}^2 + \varepsilon \mathbb{Q}_{[1,N]}(t,u), \tag{15.9.8b}$$

$$\int_{\Sigma_0^u} \left| \mathrm{Error}_2[\mathscr{Z}_*^{N;\leq 1}\Psi; {}^{(\mathscr{Z}_*^{N-1;\leq 1})}\widetilde{\mathscr{X}}] \right| d\varpi \lesssim \mathring{\upepsilon}^2, \tag{15.9.8c}$$

$$\int_{\Sigma_0^u} \left| (1+2\upmu)(\breve{X}\Psi)(Y \mathscr{P}^N \Psi)^{(\mathscr{Z}_*^{N-1;\leq 1})}\widetilde{\mathscr{X}} \right| d\varpi \lesssim \mathring{\upepsilon}^2. \tag{15.9.8d}$$

Proof. See Sect. 9.2 for some comments on the analysis. We first prove (15.9.8a). All products on RHS (4.4.3a) contain a quadratic factor of $(\not{d}\mathscr{Z}_*^{N;\leq 1}\Psi)^{(\mathscr{Z}_*^{N-1;\leq 1})}\widetilde{\mathscr{X}}$, $(Y\mathscr{Z}_*^{N;\leq 1}\Psi)^{(\mathscr{Z}_*^{N-1;\leq 1})}\widetilde{\mathscr{X}}$, $(\mathscr{Z}_*^{N;\leq 1}\Psi)^{(\mathscr{Z}_*^{N-1;\leq 1})}\widetilde{\mathscr{X}}$, or $(\mathscr{Z}_*^{N;\leq 1}\Psi)L^{(\mathscr{Z}_*^{N-1;\leq 1})}\widetilde{\mathscr{X}}$. Using inequalities (9.4.3a) and (9.4.5) and the L^∞ estimates of Prop. 9.12, we find that the remaining factors in the products are bounded in the norm $\|\cdot\|_{L^\infty(\Sigma_t^u)}$ by $\lesssim 1$. Hence, it suffices to bound the magnitude of the spacetime integrals of the four quadratic terms by $\lesssim$ RHS (15.9.8a). To bound the spacetime integral of $\left|(Y\mathscr{Z}_*^{N;\leq 1}\Psi)^{(\mathscr{Z}_*^{N-1;\leq 1})}\widetilde{\mathscr{X}}\right|$, we use spacetime Cauchy-Schwarz, Lemmas 12.4 and

12.5, inequalities (11.3.6) and (15.9.2b), simple estimates of the form $ab \lesssim a^2 + b^2$, and the fact that $\mathbb{Q}_{[1,N]}$ is increasing in its arguments to deduce

$$\begin{aligned}
&\int_{\mathcal{M}_{t,u}} \left| (Y \mathscr{Z}_*^{N;\leq 1}\Psi)^{(\mathscr{Z}_*^{N-1;\leq 1})}\widetilde{\mathscr{X}} \right| \, d\varpi \qquad (15.9.9)\\
&\lesssim \varsigma \mathring{\updelta}_* \int_{\mathcal{M}_{t,u}} \left| \not{d} \mathscr{Z}_*^{N;\leq 1}\Psi \right|^2 \, d\varpi + \varsigma^{-1}\mathring{\updelta}_*^{-1} \int_{s=0}^{t} \left\| {}^{(\mathscr{Z}_*^{N-1;\leq 1})}\widetilde{\mathscr{X}} \right\|_{L^2(\Sigma_s^u)}^2 \, ds \\
&\lesssim \varsigma \mathbb{K}_{[1,N]}(t,u) + \varsigma^{-1} \int_{s=0}^{t} \left\{ \int_{t'=0}^{s} \frac{1}{\upmu_\star^{1/2}(t',u)} \mathbb{Q}_{[1,N]}^{1/2}(t',u)\, dt' \right\}^2 + \varsigma^{-1}\mathring{\upepsilon}^2 \, ds \\
&\lesssim \varsigma \mathbb{K}_{[1,N]}(t,u) + \varsigma^{-1} \int_{s=0}^{t} \mathbb{Q}_{[1,N]}(s,u)\, ds + \varsigma^{-1}\mathring{\upepsilon}^2,
\end{aligned}$$

which is $\leq$ RHS (15.9.8a) as desired. We clarify that in passing to the last inequality in (15.9.9), we have used the fact that $\mathbb{Q}_{[1,N]}$ is increasing in its arguments and the estimate (11.3.6) to deduce that $\int_{t'=0}^{s} \frac{1}{\upmu_\star^{1/2}(t',u)} \mathbb{Q}_{[1,N]}^{1/2}(t',u)\, dt' \lesssim \mathbb{Q}_{[1,N]}^{1/2}(s,u)$, as we did in passing to the last line of (15.5.10).

The spacetime integral of $\left| (\not{d} \mathscr{Z}_*^{N;\leq 1}\Psi)^{(Y^{N-1})}\widetilde{\mathscr{X}} \right|$ can be bounded in the same way.

The spacetime integral of $\left| (\mathscr{Z}_*^{N;\leq 1}\Psi)^{(\mathscr{Z}_*^{N-1;\leq 1})}\widetilde{\mathscr{X}} \right|$ can be bounded by $\leq$ RHS (15.9.8a) by using essentially the same arguments; we omit the details.

To bound the spacetime integral of $\left| (\mathscr{Z}_*^{N;\leq 1}\Psi) L^{(\mathscr{Z}_*^{N-1;\leq 1})}\widetilde{\mathscr{X}} \right|$, by $\leq$ RHS (15.9.8a), we first use Cauchy-Schwarz, Lemma 12.5, and inequality (15.9.2a) to deduce

$$\begin{aligned}
\int_{\mathcal{M}_{t,u}} \left| (\mathscr{Z}_*^{N;\leq 1}\Psi) L^{(\mathscr{Z}_*^{N-1;\leq 1})}\widetilde{\mathscr{X}} \right| \, d\varpi &\lesssim \int_{s=0}^{t} \left\| \mathscr{Z}_*^{N;\leq 1}\Psi \right\|_{L^2(\Sigma_s^u)} \left\| L^{(\mathscr{Z}_*^{N-1;\leq 1})}\widetilde{\mathscr{X}} \right\|_{L^2(\Sigma_s^u)} \, ds \qquad (15.9.10)\\
&\lesssim \int_{s=0}^{t} \frac{1}{\upmu_\star(s,u)} \mathbb{Q}_{[1,N-1]}^{1/2}(s,u) \mathbb{Q}_{[1,N]}^{1/2}(s,u)\, ds \\
&\quad + \mathring{\upepsilon} \int_{s=0}^{t} \frac{1}{\upmu_\star^{1/2}(s,u)} \mathbb{Q}_{[1,N-1]}^{1/2}(s,u)\, ds.
\end{aligned}$$

Finally, using simple estimates of the form $ab \lesssim a^2 + b^2$, the estimate (11.3.6), and the fact that $\mathbb{Q}_{[1,N]}$ is increasing in its arguments, we bound RHS (15.9.10) by $\lesssim$ RHS (15.9.8a) as desired. This concludes the proof of (15.9.8a).

We now prove (15.9.8b) and (15.9.8c). Using the estimate (9.4.5) and the L^∞ estimates of Prop. 9.12, we see that RHS (4.4.3b) is bounded in magnitude by $\lesssim \varepsilon \left| \mathscr{Z}_*^{N;\leq 1}\Psi \right| \left| {}^{(\mathscr{Z}_*^{N-1;\leq 1})}\widetilde{\mathscr{X}} \right|$. Next, we use Cauchy-Schwarz on Σ_t^u, Lemma 12.5, (15.9.2b), and the estimate (11.3.6) to deduce that

$$\varepsilon \int_{\Sigma_t^u} \left| \mathscr{Z}_*^{N;\leq 1} \Psi \right| \left| {}^{(\mathscr{Z}_*^{N-1;\leq 1})}\widetilde{\mathscr{X}} \right| \, d\underline{\varpi} \lesssim \varepsilon \left\| \mathscr{Z}_*^{N;\leq 1} \Psi \right\|_{L^2(\Sigma_t^u)} \left\| {}^{(\mathscr{Z}_*^{N-1;\leq 1})}\widetilde{\mathscr{X}} \right\|_{L^2(\Sigma_t^u)} \tag{15.9.11}$$
$$\lesssim \varepsilon \left\{ \mathbb{Q}_{[1,N]}^{1/2}(t,u) + \mathring{\mathsf{e}} \right\}^2 \lesssim \text{RHS (15.9.8b)}$$

as desired. We clarify that in passing to the second inequality of (15.9.11), we have used (11.3.6) and the fact that $\mathbb{Q}_{[1,N]}$ is increasing in its arguments to bound the time integral on RHS (15.9.2b) by $\lesssim \mathbb{Q}_{[1,N]}^{1/2}(t,u)$. (15.9.8c) then follows from (15.9.8b) with $t = 0$ and Lemma 15.2.

The proof of (15.9.8d) is similar. The difference is that the L^∞ estimates of Prop. 9.12 imply only that LHS (15.9.8d) is $\lesssim \int_{\Sigma_0^u} \left| \mathscr{P}^{N+1} \Psi \right| \left| {}^{(\mathscr{Z}_*^{N-1;\leq 1})}\widetilde{\mathscr{X}} \right| \, d\underline{\varpi}$, without a gain of a factor ε. However, this integral is quadratically small in the data-size parameter $\mathring{\mathsf{e}}$, as is easy to verify using the arguments given in the previous paragraph. We have thus proved (15.9.8d). □

Lemma 15.17 (**Bounds for difficult top-order spacetime error integrals connected to integration by parts involving** L)**.** *Assume that* $N = 20$ *and* $\varsigma > 0$*. Let* ${}^{(Y^{N-1})}\widetilde{\mathscr{X}}$ *and* ${}^{(Y^{N-2}\breve{X})}\widetilde{\mathscr{X}}$ *be the partially modified quantities defined in* (7.2.2a)*. There exists a constant* $C > 0$*, independent of* ς*, such that under the data-size and bootstrap assumptions of Sects. 8.1-8.4 and the smallness assumptions of Sect. 8.6, the following integral estimates hold for* $(t,u) \in [0, T_{(Boot)}) \times [0, U_0]$ *(see Sect. 6.2 regarding the vectorfield operator notation):*

$$\left| \int_{\mathcal{M}_{t,u}} (1 + 2\upmu)(\breve{X} v^1)(Y \mathscr{Z}_*^{N;\leq 1} v^1) L^{(Y^{N-1})}\widetilde{\mathscr{X}} \, d\varpi \right|, \tag{15.9.12}$$
$$\left| \int_{\mathcal{M}_{t,u}} (1 + 2\upmu)(\breve{X} v^1)(Y \mathscr{Z}_*^{N;\leq 1} v^1) L^{(Y^{N-2}\breve{X})}\widetilde{\mathscr{X}} \, d\varpi \right|$$
$$\leq \boxed{2} \int_{t'=0}^t \frac{\left\| [L\upmu]_- \right\|_{L^\infty(\Sigma_{t'}^u)}}{\upmu_\star(t',u)} \mathbb{Q}_{[1,N]}(t',u) \, dt'$$
$$+ C\varepsilon \int_{t'=0}^t \frac{1}{\upmu_\star(t',u)} \mathbb{Q}_{[1,N]}(t',u) \, dt' + C \int_{t'=0}^t \frac{1}{\upmu_\star^{1/2}(t',u)} \mathbb{Q}_{[1,N]}(t',u) \, dt'$$
$$+ C \mathring{\mathsf{e}}^2,$$

$$\left| \int_{\Sigma_t^u} (1+2\upmu)(\breve{X} v^1)(Y \mathscr{Z}_*^{N;\leq 1} v^1) {}^{(Y^{N-1})}\widetilde{\mathscr{X}} \, d\varpi \right|, \tag{15.9.13}$$

$$\left| \int_{\Sigma_t^u} (1+2\upmu)(\breve{X} v^1)(Y \mathscr{Z}_*^{N;\leq 1} v^1) {}^{(Y^{N-2}\breve{X})}\widetilde{\mathscr{X}} \, d\varpi \right|$$
$$\leq \boxed{2} \frac{1}{\upmu_\star^{1/2}(t,u)} \sqrt{\mathbb{Q}_{[1,N]}}(t,u) \, \|L\upmu\|_{L^\infty({}^{(-)}\Sigma_{t;t}^u)} \int_{t'=0}^t \frac{1}{\upmu_\star^{1/2}(t',u)} \sqrt{\mathbb{Q}_{[1,N]}}(t',u) \, dt'$$
$$+ C\varepsilon \frac{1}{\upmu_\star^{1/2}(t,u)} \sqrt{\mathbb{Q}_{[1,N]}}(t,u) \int_{t'=0}^t \frac{1}{\upmu_\star^{1/2}(t',u)} \sqrt{\mathbb{Q}_{[1,N]}}(t',u) \, dt'$$
$$+ C \sqrt{\mathbb{Q}_{[1,N]}}(t,u) \int_{t'=0}^t \frac{1}{\upmu_\star^{1/2}(t',u)} \sqrt{\mathbb{Q}_{[1,N]}}(t',u) \, dt'$$
$$+ C\varsigma \mathbb{Q}_{[1,N]}(t,u) + C\varsigma^{-1} \mathring{\upepsilon}^2 \frac{1}{\upmu_\star(t,u)}.$$

Moreover, we have the following less degenerate estimates:

$$\left| \int_{\mathcal{M}_{t,u}} (1+2\upmu) \begin{pmatrix} (\breve{X} v^2)(Y \mathscr{Z}_*^{N;\leq 1} v^2) \\ \left\{ \breve{X}(\uprho - v^1) \right\} \left\{ Y \mathscr{Z}_*^{N;\leq 1} (\uprho - v^1) \right\} \end{pmatrix} L^{(Y^{N-1})}\widetilde{\mathscr{X}} \, d\varpi \right|, \tag{15.9.14}$$

$$\left| \int_{\mathcal{M}_{t,u}} (1+2\upmu) \begin{pmatrix} (\breve{X} v^2)(Y \mathscr{Z}_*^{N;\leq 1} v^2) \\ \left\{ \breve{X}(\uprho - v^1) \right\} \left\{ Y \mathscr{Z}_*^{N;\leq 1} (\uprho - v^1) \right\} \end{pmatrix} L^{(Y^{N-2}\breve{X})}\widetilde{\mathscr{X}} \, d\varpi \right|$$
$$\leq C\varepsilon \int_{t'=0}^t \frac{1}{\upmu_\star(t',u)} \mathbb{Q}_{[1,N]}(t',u) \, dt' + C\mathring{\upepsilon}^2,$$

$$\left| \int_{\Sigma_t^u} (1+2\upmu) \begin{pmatrix} (\breve{X} v^2)(Y \mathscr{Z}_*^{N;\leq 1} v^2) \\ \left\{ \breve{X}(\uprho - v^1) \right\} \left\{ Y \mathscr{Z}_*^{N;\leq 1} (\uprho - v^1) \right\} \end{pmatrix} {}^{(Y^{N-1})}\widetilde{\mathscr{X}} \, d\varpi \right|, \tag{15.9.15}$$

$$\left| \int_{\Sigma_t^u} (1+2\upmu) \begin{pmatrix} (\breve{X} v^2)(Y \mathscr{Z}_*^{N;\leq 1} v^2) \\ \left\{ \breve{X}(\uprho - v^1) \right\} \left\{ Y \mathscr{Z}_*^{N;\leq 1} (\uprho - v^1) \right\} \end{pmatrix} {}^{(Y^{N-2}\breve{X})}\widetilde{\mathscr{X}} \, d\varpi \right|$$
$$\leq C\varepsilon \frac{1}{\upmu_\star^{1/2}(t,u)} \sqrt{\mathbb{Q}_{[1,N]}}(t,u) \int_{t'=0}^t \frac{1}{\upmu_\star^{1/2}(t',u)} \sqrt{\mathbb{Q}_{[1,N]}}(t',u) \, dt'$$
$$+ C\varepsilon \mathbb{Q}_{[1,N]}(t,u) + C\mathring{\upepsilon}^2 \frac{1}{\upmu_\star(t,u)}.$$

Proof. We prove (15.9.12) only for the first term on the LHS since the second term can be treated in an identical fashion. To proceed, we first use Cauchy-Schwarz, and the estimates $|Y| \leq 1 + C\varepsilon$, $\left\| \breve{X}\Psi \right\|_{L^\infty(\Sigma_t^u)} \lesssim 1$, and $\|\upmu\|_{L^\infty(\Sigma_t^u)} \lesssim 1$ (which follow from (9.4.2a) and the L^∞

estimates of Prop. 9.12) to bound the LHS by

$$\leq (1+C\varepsilon)\int_{t'=0}^{t}\left\|\sqrt{\upmu}\,\text{d}\!\!\!/\mathscr{Z}_*^{N;\leq 1}\Psi\right\|_{L^2(\Sigma_{t'}^u)}\left\|\frac{1}{\sqrt{\upmu}}(\breve{X}\Psi)L^{(Y^{N-1})}\widetilde{\mathscr{X}}\right\|_{L^2(\Sigma_{t'}^u)}\,dt' \qquad (15.9.16)$$

$$+\,C\int_{t'=0}^{t}\left\|\sqrt{\upmu}\,\text{d}\!\!\!/\mathscr{Z}_*^{N;\leq 1}\Psi\right\|_{L^2(\Sigma_{t'}^u)}\left\|L^{(Y^{N-1})}\widetilde{\mathscr{X}}\right\|_{L^2(\Sigma_{t'}^u)}\,dt'.$$

The desired estimate (15.9.12) now follows from (15.9.16), Lemma 12.5, and inequalities (15.9.1a) and (15.9.2a). We clarify that to bound the integral $\int_{t'=0}^{t} C\mathring{\upepsilon}\frac{1}{\upmu_\star^{1/2}(t',u)}\mathbb{Q}_{[1,N]}^{1/2}(t',u)\,dt'$, which is generated by the last term on RHS (15.9.1a), we first use Young's inequality to bound the integrand by $\lesssim \frac{\mathring{\upepsilon}^2}{\upmu_\star^{1/2}(t',u)}+\frac{\mathbb{Q}_{[1,N]}(t',u)}{\upmu_\star^{1/2}(t',u)}$. We then bound the time integral of the first term in the previous expression by $\lesssim \mathring{\upepsilon}^2$ with the help of the estimate (11.3.6) and the time integral of the second by $\leq$ the third term on RHS (15.9.12).

The proof of (15.9.14) is similar but simpler and is based on the estimates $\left\|\breve{X}v^2\right\|_{L^\infty(\Sigma_t^u)}\lesssim\varepsilon$ and $\left\|\breve{X}(\uprho-v^1)\right\|_{L^\infty(\Sigma_t^u)}\lesssim\varepsilon$ (see (9.6.3a) and (9.6.4)) and the estimate (15.9.2a); we omit the details.

The proof of (15.9.13) is similar to the proof of (15.9.12) but relies on (15.9.1b) and (15.9.2b) in place of (15.9.1a) and (15.9.2a); we omit the details, noting only that we encounter the term $C\mathring{\upepsilon}\frac{1}{\upmu_\star^{1/2}(t,u)}\mathbb{Q}_{[1,N]}^{1/2}(t,u)$ generated by the last term on RHS (15.9.1b). We bound this term by using Young's inequality as follows: $C\mathring{\upepsilon}\frac{1}{\upmu_\star^{1/2}(t,u)}\mathbb{Q}_{[1,N]}^{1/2}(t,u)\leq C\varsigma^{-1}\mathring{\upepsilon}^2\frac{1}{\upmu_\star(t,u)}+C\varsigma\mathbb{Q}_{[1,N]}(t,u)$.

The proof of (15.9.15) is similar to the proof of (15.9.13) but is simpler. It is based on the estimates $\left\|\breve{X}v^2\right\|_{L^\infty(\Sigma_t^u)}\lesssim\varepsilon$ and $\left\|\breve{X}(\uprho-v^1)\right\|_{L^\infty(\Sigma_t^u)}\lesssim\varepsilon$ noted above and the estimate (15.9.2b); we omit the details.

□

15.10. **Estimates for the most degenerate top-order transport equation error integrals.** In the next lemma, we bound the most degenerate error integrals appearing in the top-order energy estimates for the specific vorticity, which are generated by the main terms from Prop. 14.2. These error integrals are responsible for the large blowup-exponent 6.4 in the factor $\upmu_\star^{-6.4}(t,u)$ on RHS (15.1.1c).

Lemma 15.18 (**Estimates for the most degenerate top-order transport equation error integrals**). *Assume that $N=20$ and recall that y is the scalar-valued appearing in Lemma 3.8. Under the data-size and bootstrap assumptions of Sects. 8.1-8.4 and the smallness assumptions of Sect. 8.6, the following integral estimates hold for* $(t,u)\in$

$[0, T_{(Boot)}) \times [0, U_0]$:

$$\left| \int_{\mathcal{M}_{t,u}} \begin{pmatrix} (Y\omega) Y^{N-1} \breve{X} \mathrm{tr}_{\not{g}} \chi \\ g(Y,Y)(L\omega) Y^{N-1} \breve{X} \mathrm{tr}_{\not{g}} \chi \\ y(Y\omega) Y^{N-1} \breve{X} \mathrm{tr}_{\not{g}} \chi \end{pmatrix} \mathscr{P}^{N+1} \omega \, d\varpi \right| \tag{15.10.1}$$
$$\lesssim \varepsilon^2 \frac{1}{\mu_\star(t,u)} \mathbb{Q}_N(t,u) + \varepsilon^2 \int_{t'=0}^t \frac{1}{\mu_\star^2(t',u)} \left\{ \int_{s=0}^{t'} \frac{1}{\mu_\star(s,u)} \sqrt{\mathbb{Q}_N}(s,u) \, ds \right\}^2 dt'$$
$$+ \varepsilon^2 \int_{t'=0}^t \frac{1}{\mu_\star^2(t',u)} \left\{ \int_{s=0}^{t'} \sqrt{\mathbb{V}_{N+1}}(s,u) \, ds \right\}^2 dt'$$
$$+ \varepsilon^2 \int_{t'=0}^t \frac{1}{\mu_\star^2(t',u)} \left\{ \int_{s=0}^{t'} \frac{1}{\mu_\star^{1/2}(s,u)} \sqrt{\mathbb{V}_{\leq N}}(s,u) \, ds \right\}^2 dt'$$
$$+ \int_{u'=0}^u \mathbb{V}_{N+1}(t,u') \, du' + \mathring{\epsilon}^2 \frac{1}{\mu_\star^{3/2}(t,u)}.$$

Proof. We prove (15.10.1) for the term $\left| \int_{\mathcal{M}_{t,u}} (Y\omega)(Y^{N-1} \breve{X} \mathrm{tr}_{\not{g}} \chi) \mathscr{P}^{N+1} \omega \, d\varpi \right|$ in detail. The other two error integrals on LHS (15.10.1) can be handled using nearly identical arguments, the schematic relations (3.19.2a) and (3.19.2c), and the L^∞ estimates of Prop. 9.12; we omit those details. To proceed, we use the bound $\|Y\omega\|_{L^\infty(\Sigma_t^u)} \lesssim \varepsilon$ (see (9.6.9)) and Young's inequality to deduce that the error integral under consideration is

$$\lesssim \varepsilon^2 \int_{\mathcal{M}_{t,u}} (Y^{N-1} \breve{X} \mathrm{tr}_{\not{g}} \chi)^2 \, d\varpi + \int_{\mathcal{M}_{t,u}} (\mathscr{P}^{N+1} \omega)^2 \, d\varpi \tag{15.10.2}$$
$$\lesssim \varepsilon^2 \int_{t'=0}^t \|Y^{N-1} \breve{X} \mathrm{tr}_{\not{g}} \chi\|_{L^2(\Sigma_{t'}^u)}^2 \, dt' + \int_{u'=0}^u \|\mathscr{P}^{N+1} \omega\|_{L^2(\mathcal{P}_{u'}^t)}^2 \, du'.$$

Using Lemma 12.5, we bound the last integral on RHS (15.10.2) by $\lesssim \int_{u'=0}^u \mathbb{V}_{N+1}(t,u') \, du'$ as desired. To handle the remaining time integral on RHS (15.10.2), we use the estimate

(15.7.3) to bound it as follows:

$$\begin{aligned}\varepsilon^2 \int_{t'=0}^{t} \|Y^{N-1}\breve{X}\mathrm{tr}_{\not g}\upchi\|_{L^2(\Sigma_{t'}^u)}^2\, dt' &\lesssim \varepsilon^2 \int_{t'=0}^{t} \frac{1}{\upmu_\star^2(t',u)} \|\upmu Y^{N-1}\breve{X}\mathrm{tr}_{\not g}\upchi\|_{L^2(\Sigma_{t'}^u)}^2\, dt' \\ &\lesssim \varepsilon^2 \int_{t'=0}^{t} \frac{1}{\upmu_\star^2(t',u)} \mathbb{Q}_N(t',u)\, dt' \\ &\quad + \varepsilon^2 \int_{t'=0}^{t} \frac{1}{\upmu_\star^2(t',u)} \left\{ \int_{s=0}^{t'} \frac{1}{\upmu_\star(s,u)} \sqrt{\mathbb{Q}_N}(s,u)\, ds \right\}^2 dt' \\ &\quad + \varepsilon^2 \int_{t'=0}^{t} \frac{1}{\upmu_\star^2(t',u)} \left\{ \int_{s=0}^{t'} \sqrt{\mathbb{V}_{N+1}}(s,u)\, ds \right\}^2 dt' \\ &\quad + \varepsilon^2 \int_{t'=0}^{t} \frac{1}{\upmu_\star^2(t',u)} \left\{ \int_{s=0}^{t'} \frac{1}{\upmu_\star^{1/2}(s,u)} \sqrt{\mathbb{V}_{\leq N}}(s,u)\, ds \right\}^2 dt' \\ &\quad + \varepsilon^2 \mathring{\upepsilon}^2 \int_{t'=0}^{t} \frac{1}{\upmu_\star^2(t',u)} \left\{ \ln \upmu_\star^{-1}(t',u) + 1 \right\}^2 dt'. \end{aligned} \tag{15.10.3}$$

Using the fact that $\mathbb{Q}_N$ is increasing in its arguments and the estimate (11.3.3), we find that RHS (15.10.3) $\lesssim$ RHS (15.10.1) as desired. □

15.11. **Estimates for transport equation error integrals involving a loss of one derivative.** In the next lemma, we estimate some error integrals that arise when bounding the below-top-order derivatives of the specific vorticity. We allow the estimates to lose one derivative. The advantage is that the right-hand sides of the estimates are much less singular with respect to powers of $\upmu_\star^{-1}$ compared to the estimates we would obtain in an approach that avoids derivative loss. This fact is crucially important for our energy estimate descent scheme, in which the below-top-order energy estimates become successively less singular with respect to powers of $\upmu_\star^{-1}$.

Lemma 15.19 (**Estimates for transport equation error integrals involving a loss of one derivative**). *Assume that* $2 \leq N \leq 20$ *and recall that* y *is the scalar-valued appearing in Lemma 3.8. Under the data-size and bootstrap assumptions of Sects. 8.1-8.4 and the smallness assumptions of Sect. 8.6, the following integral estimates hold for* $(t,u) \in [0, T_{(Boot)}) \times [0, U_0]$ *(see Sect. 6.2 regarding the vectorfield operator notation):*

$$\left| \int_{\mathcal{M}_{t,u}} \begin{pmatrix} (Y\upomega) Y^{N-2}\breve{X}\mathrm{tr}_{\not g}\upchi \\ g(Y,Y)(L\upomega) Y^{N-2}\breve{X}\mathrm{tr}_{\not g}\upchi \\ y(Y\upomega) Y^{N-2}\breve{X}\mathrm{tr}_{\not g}\upchi \end{pmatrix} \mathscr{P}^N \upomega\, d\varpi \right| \tag{15.11.1}$$
$$\lesssim \varepsilon^2 \int_{t'=0}^{t} \left\{ \int_{s=0}^{t'} \frac{\sqrt{\mathbb{Q}_{[1,N]}}(s,u)}{\upmu_\star^{1/2}(s,u)}\, ds \right\}^2 dt' + \int_{u'=0}^{u} \mathbb{V}_{\leq N}(t,u')\, du' + \varepsilon^2 \mathring{\upepsilon}^2.$$

Proof. The proof is the same as the proof of Lemma 15.18 except for one key difference: we use the estimate (15.4.2b) to bound the term $\|Y^{N-2}\breve{X}\mathrm{tr}_{\not g}\upchi\|_{L^2(\Sigma_{t'}^u)}^2$, in place of the estimate (15.7.3) used in bounding the term $\|Y^{N-1}\breve{X}\mathrm{tr}_{\not g}\upchi\|_{L^2(\Sigma_{t'}^u)}^2$ on the first line of RHS (15.10.3). □

15.12. **Estimates for wave equation error integrals involving a loss of one derivative.** We now provide an analog of Lemma 15.19 for the wave equations. Specifically, in the next lemma, we estimate some error integrals that arise when bounding the below-top-order derivatives of the elements of $\{\rho - v^1, v^1, v^2\}$. As in Lemma 15.19, we allow the estimates to lose one derivative, and the gain is that the right-hand sides of the estimates are much less singular with respect to powers of $\mu_\star^{-1}$ compared to the estimates we would obtain in an approach that avoids derivative loss.

Lemma 15.20 (**Estimates for wave equation error integrals involving a loss of one derivative**)**.** *Let* $\Psi \in \{\rho - v^1, v^1, v^2\}$ *and assume that* $2 \leq N \leq 20$*. Recall that* y *is the scalar-valued function appearing in Lemma 3.8. Under the data-size and bootstrap assumptions of Sects. 8.1-8.4 and the smallness assumptions of Sect. 8.6, the following integral estimates hold for* $(t, u) \in [0, T_{(Boot)}) \times [0, U_0]$ *(see Sect. 6.2 regarding the vectorfield operator notation):*

$$\int_{\mathcal{M}_{t,u}} \left| \begin{pmatrix} \breve{X} \mathscr{Z}_*^{N-1;\leq 1} \Psi \\ (1 + 2\mu) L \mathscr{Z}_*^{N-1;\leq 1} \Psi \end{pmatrix} \right| \left| \begin{pmatrix} (\breve{X}\Psi) \mathscr{Z}_*^{N-1;\leq 1} \mathrm{tr}_{\not{g}} \chi \\ -(\mu \not{d}^{\#} \Psi) \cdot (\mu \not{d} \mathscr{Z}^{N-2;\leq 1} \mathrm{tr}_{\not{g}} \chi) \\ y(\not{d}^{\#} \Psi) \cdot (\mu \not{d} \mathscr{Z}^{N-2;\leq 1} \mathrm{tr}_{\not{g}} \chi) \\ (\not{d}^{\#} \Psi) \cdot (\mu \not{d} \mathscr{Z}^{N-2;\leq 1} \mathrm{tr}_{\not{g}} \chi) \end{pmatrix} \right| d\varpi \tag{15.12.1}$$

$$\lesssim \int_{t'=0}^{t} \frac{\sqrt{\mathbb{Q}_{[1,N-1]}}(t', u)}{\mu_\star^{1/2}(t', u)} \left\{ \int_{s=0}^{t'} \frac{\sqrt{\mathbb{Q}_N}(s, u)}{\mu_\star^{1/2}(s, u)} \, ds \right\} dt' + \int_{t'=0}^{t} \frac{\mathbb{Q}_{[1,N-1]}(t', u)}{\mu_\star^{1/2}(t', u)} \, dt' + \mathring{\epsilon}^2.$$

Proof. It suffices to consider only the first term $(\breve{X}\Psi) \mathscr{Z}_*^{N-1;\leq 1} \mathrm{tr}_{\not{g}} \chi$ in the second array on LHS (15.12.1) since the other three terms in the array can be bounded using the same arguments. They are in fact smaller in view of the estimates $\|y\|_{L^\infty(\Sigma_t^u)} \lesssim \varepsilon$, $\|\not{d}\Psi\|_{L^\infty(\Sigma_t^u)} \lesssim \varepsilon$, and $\|\mu\|_{L^\infty(\Sigma_t^u)} \lesssim 1$, which are simple consequences of (3.19.2c) and the L^∞ estimates of Prop. 9.12. To proceed, we use Cauchy-Schwarz along $\Sigma_{t'}^u$, the L^∞ estimates of Prop. 9.12, Lemma 12.5, the estimate (15.4.2b), the simple estimate $\mathring{\epsilon} \sqrt{\mathbb{Q}_{[1,N-1]}}(t', u) \leq \mathring{\epsilon}^2 + \mathbb{Q}_{[1,N-1]}(t', u)$, and inequality (11.3.6) to bound the spacetime integral under consideration as follows:

$$\int_{\mathcal{M}_{t,u}} \left| \begin{pmatrix} \breve{X} \mathscr{Z}_*^{N-1;\leq 1} \Psi \\ (1 + 2\mu) L \mathscr{Z}_*^{N-1;\leq 1} \Psi \end{pmatrix} \right| \left| (\breve{X}\Psi) \mathscr{Z}_*^{N-1;\leq 1} \mathrm{tr}_{\not{g}} \chi \right| d\varpi \tag{15.12.2}$$

$$\lesssim \int_{t'=0}^{t} \left\{ \left\| \breve{X} \mathscr{Z}_*^{N-1;\leq 1} \Psi \right\|_{L^2(\Sigma_{t'}^u)} + \left\| L \mathscr{Z}_*^{N-1;\leq 1} \Psi \right\|_{L^2(\Sigma_{t'}^u)} \right\} \left\| \mathscr{Z}_*^{N-1;\leq 1} \mathrm{tr}_{\not{g}} \chi \right\|_{L^2(\Sigma_{t'}^u)} dt'$$

$$\lesssim \int_{t'=0}^{t} \frac{\sqrt{\mathbb{Q}_{[1,N-1]}}(t', u)}{\mu_\star^{1/2}(t', u)} \left\{ \mathring{\epsilon} + \int_{s=0}^{t'} \frac{\sqrt{\mathbb{Q}_N}(s, u)}{\mu_\star^{1/2}(s, u)} \, ds \right\} dt'$$

$$\lesssim \int_{t'=0}^{t} \frac{\sqrt{\mathbb{Q}_{[1,N-1]}}(t', u)}{\mu_\star^{1/2}(t', u)} \left\{ \int_{s=0}^{t'} \frac{\sqrt{\mathbb{Q}_N}(s, u)}{\mu_\star^{1/2}(s, u)} \, ds \right\} dt' + \int_{t'=0}^{t} \frac{\mathbb{Q}_{[1,N-1]}(t', u)}{\mu_\star^{1/2}(t', u)} \, dt'$$

$$+ \mathring{\epsilon}^2 \int_{t'=0}^{t} \frac{1}{\mu_\star^{1/2}(t', u)} \, dt'$$

$$\lesssim \int_{t'=0}^{t} \frac{\sqrt{\mathbb{Q}_{[1,N-1]}}(t', u)}{\mu_\star^{1/2}(t', u)} \left\{ \int_{s=0}^{t'} \frac{\sqrt{\mathbb{Q}_N}(s, u)}{\mu_\star^{1/2}(s, u)} \, ds \right\} dt' + \int_{t'=0}^{t} \frac{\mathbb{Q}_{[1,N-1]}(t', u)}{\mu_\star^{1/2}(t', u)} \, dt' + \mathring{\epsilon}^2.$$

□

15.13. **Proof of Prop. 15.4.** We first prove (15.2.4a). Let $\vec{I}$ be a $\mathscr{P}$ multi-index with $|\vec{I}| = 21$. From (4.3.6), we deduce that

$$\mathbb{E}^{(Vort)}[\mathscr{P}^{\vec{I}}\upomega](t,u) + \mathbb{F}^{(Vort)}[\mathscr{P}^{\vec{I}}\upomega](t,u) = \mathbb{E}^{(Vort)}[\mathscr{P}^{\vec{I}}\upomega](0,u) + \mathbb{F}^{(Vort)}[\mathscr{P}^{\vec{I}}\upomega](t,0) \tag{15.13.1}$$

$$+ \int_{\mathcal{M}_{t,u}} \left\{ L\upmu + \upmu \mathrm{tr}_{g\!\!/} k\!\!\!/ \right\} (\mathscr{P}^{\vec{I}}\upomega)^2 \, d\varpi$$

$$+ 2\int_{\mathcal{M}_{t,u}} (\mathscr{P}^{\vec{I}}\upomega)\upmu B \mathscr{P}^{\vec{I}}\upomega \, d\varpi.$$

We will show that the magnitude of RHS (15.13.1) is $\leq$ RHS (15.2.4a). Then taking the max of that inequality over all $\vec{I}$ with $|\vec{I}| = 21$ and appealing to Def. 12.1, we arrive at (15.2.4a). The first integral on RHS (15.13.1) was treated in Lemma 15.7. To bound the last integral on RHS (15.13.1), we first use Prop. 14.2 to express the integrand factor $\upmu B \mathscr{P}^{\vec{I}}\upomega$ as the products explicitly indicated on either RHS (14.2.3a) or RHS (14.2.3b) plus $Harmless^{\leq 21}_{(Vort)}$ error terms. The error integrals $\int_{\mathcal{M}_{t,u}} (\mathscr{P}^{\vec{I}}\upomega) Harmless^{\leq 21}_{(Vort)} \, d\varpi$ were treated in Lemma 15.10. The remaining three error integrals, which correspond to the products explicitly indicated on either RHS (14.2.3a) and RHS (14.2.3b), were treated in Lemma 15.18. We have thus proved (15.2.4a).

The proof of (15.2.4b) in the cases $2 \leq N \leq 20$ is similar. The only difference is that we bound the explicitly listed products on RHS (14.2.3a) and RHS (14.2.3b) (with $N-1$ in the role of N in (14.2.3a)-(14.2.3b)) in a different way: by using the derivative-losing Lemma 15.19 in place of Lemma 15.18. The proof of (15.2.4b) in the case $N = 1$ is similar but simpler and relies on equation (14.2.5). The proof of (15.2.4b) when $N = 0$ is even simpler since, by (3.3.11c), the last integral on RHS (15.13.1) completely vanishes.

□

15.14. **Proof of Prop. 15.3.**

Proof of (15.2.1a): We set $N = 20$ (which corresponds to the top-order number of commutations of the wave equations (3.3.11a)-(3.3.11b)). Let $\mathscr{Z}_*^{N;\leq 1}$ be an N^{th}-order vectorfield operator involving at most one $\breve{X}$ factor and let $\Psi \in \{\uprho - v^1, v^1, v^2\}$. From (4.3.1) with $\mathscr{Z}_*^{N;\leq 1}\Psi$ in the role of Ψ, the decomposition (4.3.2) with $\mathscr{Z}_*^{N;\leq 1}\Psi$ in the role of Ψ, and definition (12.1.2a), we have

$$\mathbb{E}^{(Wave)}[\mathscr{Z}_*^{N;\leq 1}\Psi](t,u) + \mathbb{F}^{(Wave)}[\mathscr{Z}_*^{N;\leq 1}\Psi](t,u) + \mathbb{K}[\mathscr{Z}_*^{N;\leq 1}\Psi](t,u) \tag{15.14.1}$$

$$= \mathbb{E}^{(Wave)}[\mathscr{Z}_*^{N;\leq 1}\Psi](0,u) + \mathbb{F}^{(Wave)}[\mathscr{Z}_*^{N;\leq 1}\Psi](t,0)$$

$$- \int_{\mathcal{M}_{t,u}} \left\{ (1+2\upmu)(L\mathscr{Z}_*^{N;\leq 1}\Psi) + 2\breve{X}\mathscr{Z}_*^{N;\leq 1}\Psi \right\} \upmu \square_g(\mathscr{Z}_*^{N;\leq 1}\Psi) \, d\varpi$$

$$+ \sum_{i=1}^{5} \int_{\mathcal{M}_{t,u}} {}^{(T)}\mathfrak{P}_{(i)}[\mathscr{Z}_*^{N;\leq 1}\Psi] \, d\varpi.$$

We will show that RHS (15.14.1) $\leq$ RHS (15.2.1a). Then, taking the max over that estimate for all such operators of order precisely N and over $\Psi \in \{\rho - v^1, v^1, v^2\}$ and appealing to Defs. 12.1 and 12.2, we conclude (15.2.1a).

To show that RHS (15.14.1) $\leq$ RHS (15.2.1a), we first use Lemma 15.2 to deduce that $\mathbb{E}^{(Wave)}[\mathscr{Z}_*^{N;\leq 1}\Psi](0,u)+\mathbb{F}^{(Wave)}[\mathscr{Z}_*^{N;\leq 1}\Psi](t,0) \lesssim \mathring{\epsilon}^2$, which is $\leq$ the first term on RHS (15.2.1a) as desired.

To bound the last integral $\sum_{i=1}^5 \int_{\mathcal{M}_{t,u}} {}^{(T)}\mathfrak{P}_{(i)}[\cdots]$ on RHS (15.14.1) by $\leq$ RHS (15.2.1a), we use Lemma 15.8.

We now address the first integral $-\int_{\mathcal{M}_{t,u}} \cdots$ on RHS (15.14.1). If $\mathscr{Z}_*^{N;\leq 1}$ is *not* of the form $Y^{N-1}L$, Y^N, $Y^{N-1}\breve{X}$, $\mathscr{Z}_*^{N-1;1}L$, or $\mathscr{Z}_*^{N-1;1}Y$, where $\mathscr{Z}_*^{N-1;1}$ contains exactly one factor of $\breve{X}$ and $N-2$ factors of Y, then it is easy to see that $\mathscr{Z}_*^{N;\leq 1}$ must be of the form of one of the operators on LHSs (14.2.2a)-(14.2.2d). The desired bound thus follows from (14.2.2a)-(14.2.2d), (14.2.2e), (15.5.7), and (15.6.1). Note that these bounds do not produce any of the difficult "boxed-constant-involving" terms on RHS (15.2.1a).

We now address the first integral $-\int_{\mathcal{M}_{t,u}} \cdots$ on RHS (15.14.1) when $\Psi = v^1$ and $\mathscr{Z}_*^{N;\leq 1}$ is one of the five operators not treated in the previous paragraph, that is, when $\mathscr{Z}_*^{N;\leq 1}$ is one of $Y^{N-1}L$, Y^N, $Y^{N-1}\breve{X}$, $\mathscr{Z}^{N-1;1}L$, or $\mathscr{Z}^{N-1;1}Y$, where $\mathscr{Z}^{N-1;1} = \mathscr{Z}_*^{N-1;1}$ contains exactly one factor of $\breve{X}$ and $N-2$ factors of Y. We consider in detail only the case $\mathscr{Z}_*^{N;\leq 1} = Y^N$; the other four cases can be treated in an identical fashion (with the help of Prop. 14.1) and we omit those details. Moreover, the estimates for the wave variables $\Psi = v^2$ and $\Psi = \rho - v^1$ are less degenerate and easier to derive; we will briefly comment on them below. To proceed, we substitute RHS (14.2.1b) (in the case $i = 1$) for the integrand factor $\mu\square_g(Y^N v^1)$ on RHS (15.14.1). It suffices for us to bound the integrals corresponding to the terms $(\breve{X}v^1)Y^N \mathrm{tr}_{\not{g}}\chi$ and $y(\not{d}^{\#}v^1)\cdot(\mu\not{d}Y^{N-1}\mathrm{tr}_{\not{g}}\chi)$ from RHS (14.2.1b); the integrals generated by the ω-involving terms on RHS (14.2.1b) were suitably bounded in Lemma 15.11, while the above argument has already addressed how to bound the integrals generated by $Harmless_{(Wave)}^{\leq N}$ terms (via (15.5.7)). To bound the difficult integral

$$-2\int_{\mathcal{M}_{t,u}} (\breve{X}Y^N v^1)(\breve{X}v^1)Y^N \mathrm{tr}_{\not{g}}\chi \, d\varpi \tag{15.14.2}$$

in magnitude by $\leq$ RHS (15.2.1a), we use the estimate (15.7.4), which accounts for the portion $\boxed{4}\cdots$ of the first boxed constant integral $\boxed{6}\cdots$ on RHS (15.2.1a) and the full portion of the boxed constant integral $\boxed{8.1}\cdots$ on RHS (15.2.1a).

We now bound the magnitude of the error integral

$$-\int_{\mathcal{M}_{t,u}} (1+2\mu)(LY^N v^1)(\breve{X}v^1)Y^N \mathrm{tr}_{\not{g}}\chi \, d\varpi. \tag{15.14.3}$$

To proceed, we use (7.2.2a)-(7.2.2b) to decompose $Y^N \mathrm{tr}_{\not{g}}\chi = Y^{(Y^{N-1})}\widetilde{\mathscr{X}} - Y^{(Y^{N-1})}\widetilde{\mathfrak{X}}$. Since RHS (14.5.1d) $= Harmless^{\leq N}$, we have already suitably bounded the error integrals generated by $Y^{(Y^{N-1})}\widetilde{\mathfrak{X}}$. We therefore must bound the magnitude of

$$-\int_{\mathcal{M}_{t,u}} (1+2\mu)(LY^N v^1)(\breve{X}v^1)Y^{(Y^{N-1})}\widetilde{\mathscr{X}} \, d\varpi \tag{15.14.4}$$

by $\leq$ RHS (15.2.1a). To this end, we integrate by parts using (4.4.2) with $\eta := {}^{(Y^{N-1})}\widetilde{\mathscr{X}}$. We bound the error integrals on the last line of RHS (4.4.2) and the $\int_{\Sigma_0^u} \cdots$ integral on the second line using Lemma 15.16. It remains for us to bound the first two (difficult) integrals on RHS (4.4.2) in magnitude by $\leq$ RHS (15.2.1a). The desired bounds have been derived in the estimates (15.9.12)-(15.9.13) of Lemma 15.17. Note that these estimates account for the remaining portion $\boxed{2}\cdots$ of the first boxed constant integral $\boxed{6}\cdots$ on RHS (15.2.1a) and the full portion of the boxed constant integral $\boxed{2}\cdots$ on RHS (15.2.1a).

To finish deriving the desired estimates in the case $\Psi = v^1$, it remains for us to bound the two error integrals generated by the term $y(\not{d}^{\#} v^1)\cdot(\mu \not{d} Y^{N-1}\mathrm{tr}_{\not{g}}\chi)$ from RHS (14.2.1b). These two integrals were suitably bounded in magnitude by $\leq$ RHS (15.2.1a) in Lemma 15.14 (note that we are using the simple bound $\{\ln \mu_\star^{-1}(t',u)+1\}^2 \lesssim \mu_\star^{-1/2}(t',u)$ in order to bound the integrand factors in the first integrals on RHS (15.8.1a) and RHS (15.8.1b)). Note also that these estimates do not contribute to the difficult "boxed-constant-involving" products on RHS (15.2.1a). We have thus shown that when $\Psi = v^1$, the desired inequality RHS(15.14.1) $\leq$ RHS (15.2.1a) holds.

We now comment on the cases $\Psi = v^2$ and $\Psi = \rho - v^1$. The proofs that RHS (15.14.1) $\leq$ RHS (15.2.1a) in these cases are essentially the same as in the case $\Psi = v^1$, except that in bounding the analog of the error integral (15.14.2), we now use the less degenerate estimate (15.7.5) in place of (15.7.4) and, in bounding the analog of the error integral (15.14.4), we use the less degenerate estimates (15.9.14)-(15.9.15) in place of (15.9.12)-(15.9.13). These less degenerate estimates do not produce any of the "boxed-constant-involving" products on RHS (15.2.1a) because they all gain a smallness factor of ε via the factors $\breve{X}v^2$ and $\breve{X}(\rho - v^1))$ (which verify the smallness estimates (9.6.3a) and (9.6.4)). In total, we have proved (15.2.1a).

Proof of (15.2.2): The argument given in the previous paragraph yields (15.2.2).

Proof of (15.2.3): We repeat the proof of (15.2.1a) with M in the role of N, where $1 \leq M \leq N-1$, and with one important change: we bound the difficult error integrals such as

$$-2\int_{\mathcal{M}_{t,u}} (\breve{X}Y^N\Psi)(\breve{X}\Psi)Y^N\mathrm{tr}_{\not{g}}\chi \, d\varpi, \qquad -\int_{\mathcal{M}_{t,u}} (1+2\mu)(LY^N\Psi)(\breve{X}\Psi)Y^N\mathrm{tr}_{\not{g}}\chi \, d\varpi$$

in a different way: by using Lemma 15.20. More precisely, we replace N with M in (14.2.1a)-(14.2.1e) and consider the explicitly listed products on the RHSs that involve the derivatives of $\mathrm{tr}_{\not{g}}\chi$ (see also (14.2.2e) in the case $\Psi = \rho - v^1$). We bound the corresponding error integrals by using the derivative-losing Lemma 15.20 in place of the arguments used in proving (15.2.1a).

□

15.15. **The main vorticity a priori energy estimates.** The energy estimates for the specific vorticity are easy to derive with the help of the bootstrap assumptions. We provide them in the next lemma.

Lemma 15.21 (**The main a priori energy estimates for the specific vorticity**)**.** *Under the data-size and bootstrap assumptions of Sects. 8.1-8.4 and Sect. 15.3 and the smallness* *assumptions of Sect. 8.6, the a priori energy estimates* (15.1.1c)-(15.1.1e) *for the vorticity hold on* $\mathcal{M}_{T_{(Boot)},U_0}$.

Proof. We start by deriving the desired estimates (15.1.1c)-(15.1.1e) for $\mathbb{V}_{21}(t,u)$ and $\mathbb{V}_{\leq 20}(t,u)$. Below we will use inequalities (15.2.4a)-(15.2.4b) and the bootstrap assumptions (15.3.1a)-(15.3.1e) to obtain the following inequalities:

$$\mathbb{V}_{21}(t,u) \leq C\left(\mathring{\epsilon}^2+\varepsilon^3\right)\mu_\star^{-12.8}(t,u)+C\int_{u'=0}^{u}\mathbb{V}_{21}(t,u')\,du'+C\int_{u'=0}^{u}\mathbb{V}_{\leq 20}(t,u')\,du', \tag{15.15.1}$$

$$\mathbb{V}_{\leq 20}(t,u) \leq C\left(\mathring{\epsilon}^2+\varepsilon^3\right)\mu_\star^{-9.8}(t,u)+C\int_{u'=0}^{u}\mathbb{V}_{\leq 20}(t,u')\,du'. \tag{15.15.2}$$

Then from (15.15.2) and Gronwall's inequality in u, we obtain $\mathbb{V}_{\leq 20}(t,u) \leq C\left(\mathring{\epsilon}^2+\varepsilon^3\right)\mu_\star^{-9.8}$. Inserting this estimate into the last integral on RHS (15.15.1), we find that $\mathbb{V}_{21}(t,u)$ obeys inequality (15.15.1) but with the last integral deleted. Hence, from Gronwall's inequality in u, we obtain $\mathbb{V}_{21}(t,u) \leq C\left(\mathring{\epsilon}^2+\varepsilon^3\right)\mu_\star^{-12.8}(t,u)$. Recalling the assumption $\varepsilon^{3/2} \leq \mathring{\epsilon}$ (see (8.6.1)), we see that we have shown (15.1.1c) and the estimate (15.1.1d) for $\sqrt{\mathbb{V}_{20}}(t,u)$.

It remains for us to derive (15.15.1)-(15.15.2). To derive (15.15.1) we set $N = 20$ in (15.2.4a), which yields an integral inequality for $\mathbb{V}_{21}(t,u)$. We then insert the bootstrap assumptions (15.3.1a)-(15.3.1e) into all terms on RHS (15.2.4a) except for the last integral $C\int_{u'=0}^{u}\mathbb{V}_{\leq N+1}(t,u')\,du'$. It immediately follows that all of the terms generated by the bootstrap assumptions, except for the ones involving time integrals, are $\leq$ the $C\left(\mathring{\epsilon}^2+\varepsilon^3\right)\mu_\star^{-12.8}(t,u)$ term on RHS (15.15.1) as desired. We now explain how to handle the terms generated by the time integrals on RHS (15.2.4a). We consider in detail only the term $C\varepsilon^2\int_{t'=0}^{t}\frac{1}{\mu_\star^2(t',u)}\left\{\int_{s=0}^{t}\frac{1}{\mu_\star(s,u)}\sqrt{\mathbb{Q}_{20}}(s,u)\,ds\right\}^2 dt'$; the remaining time integrals on RHS (15.2.4a) can be bounded in a similar fashion and we omit the details. To proceed, we use the bootstrap assumptions, the estimate (11.3.3), and the assumption (8.6.1) to deduce that the double time integral under consideration is

$$\begin{aligned}&\leq C\varepsilon^3\int_{t'=0}^{t}\frac{1}{\mu_\star^2(t',u)}\left\{\int_{s=0}^{t}\frac{1}{\mu_\star^{6.9}(s,u)}\,ds\right\}^2 dt' \\ &\leq C\varepsilon^3\int_{t'=0}^{t}\frac{1}{\mu_\star^{13.8}(t',u)}\,dt' \leq C\varepsilon^3\mu_\star^{-12.8}(t,u) \leq C\mathring{\epsilon}^2\mu_\star^{-12.8}(t,u)\end{aligned} \tag{15.15.3}$$

as desired. We have thus proved (15.15.1). The proof of (15.15.2) is based on inequality (15.2.4b) with $N = 20$ but is otherwise similar to the proof of (15.15.1); we omit the details. We have thus obtained the desired estimates for $\mathbb{V}_{21}(t,u)$ and $\mathbb{V}_{20}(t,u)$.

We now explain how to derive the estimates (15.1.1d)-(15.1.1e) for $\sqrt{\mathbb{V}_{\leq 19}}$, $\sqrt{\mathbb{V}_{\leq 18}}$, $\cdots$, $\sqrt{\mathbb{V}_0}$. The desired estimates can be derived from inequality (15.2.4b), the bootstrap assumptions (15.3.1b)-(15.3.1e), Gronwall's inequality in u, and the assumption (8.6.1) by using essentially the same arguments that we used to derive the estimates for $\sqrt{\mathbb{V}_{20}}$. However, there is one minor new feature that is needed to obtain the estimates (15.1.1e) for $\sqrt{\mathbb{V}_{\leq 15}}(t,u)$: in carrying out the above procedure, we encounter a term that needs to be treated using a slightly different argument: the term

$$C\varepsilon^3 \int_{t'=0}^{t} \left\{ \int_{s=0}^{t'} \frac{1}{\upmu_\star^{1.4}(s,u)}\, ds \right\}^2 dt' \tag{15.15.4}$$

generated by the double time integral on RHS (15.2.4b). The new part of the argument is that in addition to inequality (11.3.3), we must also use inequality (11.3.6); *inequality* (11.3.6) *is what allows us to break the* $\upmu_\star^{-1}$ *degeneracy.* More precisely, to bound the term (15.15.4), we use inequalities (11.3.3) and (11.3.6) and the assumption (8.6.1) to deduce that

$$(15.15.4) \leq C\varepsilon^3 \int_{t'=0}^{t} \frac{1}{\upmu_\star^{.8}(s,u)}\, dt' \leq C\varepsilon^3 \leq C\mathring{\upepsilon}^2 \tag{15.15.5}$$

as desired, where RHS (15.15.5) *does not involve the singular factor* $\upmu_\star^{-1}$! We have thus obtained the desired estimates (15.1.1c)-(15.3.1e), which completes the proof of the lemma. □

15.16. **Proof of Prop. 15.1.** To simplify the proof, we assume that the energy bootstrap assumptions (15.3.1a)-(15.3.1e) hold for $(t,u) \in [0, T_{(Boot)}) \times [0, U_0]$. To prove the proposition, it suffices to derive, under the energy bootstrap assumptions, the estimates (15.1.1a)-(15.1.1e) for $(t,u) \in [0, T_{(Boot)}) \times [0, U_0]$. We can then use a standard continuity-in-t argument for the fundamental L^2-controlling quantities to deduce that the estimates (15.1.1a)-(15.1.1e) do in fact hold for $(t,u) \in [0, T_{(Boot)}) \times [0, U_0]$ and, in view of our assumption $\mathring{\upepsilon} \leq \varepsilon$, that the bootstrap assumptions are never saturated (for $\mathring{\upepsilon}$ sufficiently small). Note that this argument relies on Lemma 15.2, which implies that the fundamental L^2-controlling quantities do not saturate inequalities (15.3.1a)-(15.3.1e) at the initial time 0.

We now recall that in Lemma 15.21, we derived, with the help of the energy bootstrap assumptions, the a priori vorticity energy estimates (15.1.1c)-(15.1.1e). Hence, it remains only for us to derive the wave variable energy estimates (15.1.1a)-(15.1.1b). We are of course free to use the vorticity energy estimates (15.1.1c)-(15.1.1e) in the remainder of the proof.

Estimates for $\mathbb{Q}_{20}$, $\mathbb{K}_{20}$, $\mathbb{Q}_{[1,19]}$, **and** $\mathbb{K}_{[1,19]}$: These estimates are highly coupled and must be treated as a system featuring also $\mathbb{Q}_{20}^{(Partial)}$ and $\mathbb{K}_{20}^{(Partial)}$. To proceed, we set

$$F(t,u) := \sup_{(\hat{t},\hat{u})\in[0,t]\times[0,u]} \iota_F^{-1}(\hat{t},\hat{u}) \max\left\{\mathbb{Q}_{20}(\hat{t},\hat{u}), \mathbb{K}_{20}(\hat{t},\hat{u})\right\}, \tag{15.16.1}$$

$$G(t,u) := \sup_{(\hat{t},\hat{u})\in[0,t]\times[0,u]} \iota_G^{-1}(\hat{t},\hat{u}) \max\left\{\mathbb{Q}_{20}^{(Partial)}(\hat{t},\hat{u}), \mathbb{K}_{20}^{(Partial)}(\hat{t},\hat{u})\right\}, \tag{15.16.2}$$

$$H(t,u) := \sup_{(\hat{t},\hat{u})\in[0,t]\times[0,u]} \iota_H^{-1}(\hat{t},\hat{u}) \max\left\{\mathbb{Q}_{[1,19]}(\hat{t},\hat{u}), \mathbb{K}_{[1,19]}(\hat{t},\hat{u})\right\}, \tag{15.16.3}$$

where for $1 \leq t' \leq \hat{t} \leq t$ and $0 \leq u' \leq \hat{u} \leq U_0 \leq 1$, we define

$$\iota_1(t') := \exp\left(\int_{s=0}^{t'} \frac{1}{\sqrt{T_{(Boot)} - s}}\, ds\right) = \exp\left(2\sqrt{T_{(Boot)}} - 2\sqrt{T_{(Boot)} - t'}\right), \tag{15.16.4}$$

$$\iota_2(t', u') := \exp\left(\int_{s=0}^{t'} \frac{1}{\mu_\star^{9/10}(s, u')}\, ds\right), \tag{15.16.5}$$

$$\iota_F(t', u') = \iota_G(t', u') := \mu_\star^{-11.8}(t', u')\iota_1^c(t')\iota_2^c(t', u')e^{ct'}e^{cu'}, \tag{15.16.6}$$

$$\iota_H(t', u') := \mu_\star^{-9.8}(t', u')\iota_1^c(t')\iota_2^c(t', u')e^{ct'}e^{cu'}, \tag{15.16.7}$$

and c is a sufficiently large positive constant that we choose below. The functions (15.16.4)-(15.16.7) are approximate integrating factors that we will use to absorb all error terms on the RHSs of the inequalities of Prop. 15.3 back into the LHSs. We claim that to obtain the desired estimates for $\mathbb{Q}_{20}$, $\mathbb{K}_{20}$, $\mathbb{Q}_{20}^{(Partial)}$, $\mathbb{K}_{20}^{(Partial)}$, $\mathbb{Q}_{[1,19]}$, and $\mathbb{K}_{[1,19]}$, it suffices to prove

$$F(t, u) \leq C\mathring{\epsilon}^2, \qquad G(t, u) \leq C\mathring{\epsilon}^2, \qquad H(t, u) \leq C\mathring{\epsilon}^2, \tag{15.16.8}$$

where C in (15.16.8) is allowed to depend on c. To justify the claim, we use the fact that for a fixed c, the functions $\iota_1^c(t)$, $\iota_2^c(t, u)$, e^{ct}, and e^{cu} are uniformly bounded from above by a positive constant for $(t, u) \in [0, T_{(Boot)}) \times [0, U_0]$; all of these estimates are simple to derive, except for (15.16.5), which relies on (11.3.6).

To prove (15.16.8), it suffices to show that there exist positive constants α_1, α_2, α_3, β_1, β_3, γ_1, and γ_3 with

$$\alpha_1 + \alpha_2\sqrt{\beta_1 + \beta_3\frac{\gamma_1}{1 - \gamma_3}} + \frac{\alpha_3\gamma_1}{1 - \gamma_3} < 1, \qquad \gamma_3 < 1 \tag{15.16.9}$$

such that if c is sufficiently large, then

$$F(t, u) \leq C\mathring{\epsilon}^2 + \alpha_1 F(t, u) + \alpha_2 F^{1/2}(t, u)G^{1/2}(t, u) + \alpha_3 H(t, u), \tag{15.16.10}$$

$$G(t, u) \leq C\mathring{\epsilon}^2 + \beta_1 F(t, u) + \beta_3 H(t, u), \tag{15.16.11}$$

$$H(t, u) \leq C\mathring{\epsilon}^2 + \gamma_1 F(t, u) + \gamma_3 H(t, u). \tag{15.16.12}$$

One key reason that we will be able to obtain (15.16.9) is that we will be able to make α_3, β_1, and β_3 as small as we want by choosing ς and ε to be sufficiently small. Once we have obtained (15.16.10)-(15.16.12), we easily deduce from those estimates that

$$H(t, u) \leq C\mathring{\epsilon}^2 + \frac{\gamma_1}{1 - \gamma_3}F(t, u), \tag{15.16.13}$$

$$G(t, u) \leq C\mathring{\epsilon}^2 + \left\{\beta_1 + \beta_3\frac{\gamma_1}{1 - \gamma_3}\right\}F(t, u), \tag{15.16.14}$$

$$F(t, u) \leq C\mathring{\epsilon}^2 + C\mathring{\epsilon}F^{1/2}(t, u) + \left\{\alpha_1 + \alpha_2\sqrt{\beta_1 + \beta_3\frac{\gamma_1}{1 - \gamma_3}} + \frac{\alpha_3\gamma_1}{1 - \gamma_3}\right\}F(t, u). \tag{15.16.15}$$

The desired bounds (15.16.8) (for $\mathring{\epsilon}$ sufficiently small) now follow easily from (15.16.9) and (15.16.13)-(15.16.15).

It remains for us to derive (15.16.10)-(15.16.12). To this end, we will use the critically important estimates of Prop. 11.3 as well as the following estimates, which are easy to derive:

$$\int_{t'=0}^{\hat{t}} \iota_1^c(t') \frac{1}{\sqrt{T_{(Boot)} - t'}} \, dt' = \frac{1}{c} \int_{t'=0}^{\hat{t}} \frac{d}{dt'} \left\{ \iota_1^c(t') \right\} dt' \leq \frac{1}{c} \iota_1^c(\hat{t}), \tag{15.16.16}$$

$$\int_{t'=0}^{\hat{t}} \iota_2^c(t', \hat{u}) \frac{1}{\mu_\star^{9/10}(t', \hat{u})} \, dt' = \frac{1}{c} \int_{t'=0}^{\hat{t}} \frac{d}{dt'} \left\{ \iota_2^c(t', \hat{u}) \right\} dt' \leq \frac{1}{c} \iota_2^c(\hat{t}, \hat{u}), \tag{15.16.17}$$

$$\int_{t'=0}^{\hat{t}} e^{ct'} \, dt' \leq \frac{1}{c} e^{c\hat{t}}, \tag{15.16.18}$$

$$\int_{u'=0}^{\hat{u}} e^{cu'} \, du' \leq \frac{1}{c} e^{c\hat{u}}. \tag{15.16.19}$$

We will close the estimates by taking c to be large and ε to be small.

Remark 15.1. *We stress that from now through inequality* (15.16.58)*, the constants C can be chosen to be independent of c.*

In our analysis, we will often use the fact that $\iota_1^c(\cdot)$, $\iota_2^c(\cdot)\, e^{c\cdot}$, and $e^{c\cdot}$ are non-decreasing in their arguments. Also, we will often use the estimate (11.2.13), which implies that for $t' \leq \hat{t}$ and $u' \leq \hat{u}$, we have the approximate monotonicity inequality

$$(1 + C\varepsilon)\mu_\star(t', u') \geq \mu_\star(\hat{t}, \hat{u}). \tag{15.16.20}$$

We use these monotonicity properties below without explicitly mentioning them each time.

We now set $N = 20$, multiply both sides of inequality (15.2.1a) by $\iota_F^{-1}(t, u)$ and then set $(t, u) = (\hat{t}, \hat{u})$. Similarly, we multiply both sides of the inequality described in (15.2.2) by $\iota_G^{-1}(t, u)$ and the inequality (15.2.3) by $\iota_H^{-1}(t, u)$ and, in both cases, set $(t, u) = (\hat{t}, \hat{u})$. To deduce (15.16.10)-(15.16.12), the difficult step is to obtain suitable bounds for the terms generated by the terms on RHSs (15.2.1a)-(15.2.3). Once we have obtained suitable bounds, we can then take $\sup_{(\hat{t}, \hat{u}) \in [0,t] \times [0,u]}$ of both sides of the resulting inequalities, and by virtue of definitions (15.16.1)-(15.16.3), we will easily conclude (15.16.10)-(15.16.12).

We start by showing how to obtain suitable bounds for the terms on RHS (15.2.1a) that involve the vorticity energies. These estimates are easy to derive because we have already derived suitable estimates for the vorticity energies. Specifically, we must handle the terms

$$C \sup_{(\hat{t}, \hat{u}) \in [0,t] \times [0,u]} \iota_F^{-1}(\hat{t}, \hat{u}) \int_{t'=0}^{\hat{t}} \frac{1}{\mu_\star^{3/2}(t', \hat{u})} \left\{ \int_{s=0}^{t'} \sqrt{\mathbb{V}_{21}}(s, \hat{u}) \, ds \right\}^2 dt', \tag{15.16.21}$$

$$C \sup_{(\hat{t}, \hat{u}) \in [0,t] \times [0,u]} \iota_F^{-1}(\hat{t}, \hat{u}) \int_{t'=0}^{\hat{t}} \frac{1}{\mu_\star^{3/2}(t', \hat{u})} \left\{ \int_{s=0}^{t'} \frac{1}{\mu_\star^{1/2}(s, \hat{u})} \sqrt{\mathbb{V}_{\leq 20}}(s, \hat{u}) \, ds \right\}^2 dt', \tag{15.16.22}$$

$$C \sup_{(\hat{t}, \hat{u}) \in [0,t] \times [0,u]} \iota_F^{-1}(\hat{t}, \hat{u}) \int_{t'=0}^{\hat{t}} \mathbb{V}_{\leq 21}(t', u) \, dt', \tag{15.16.23}$$

$$C \sup_{(\hat{t}, \hat{u}) \in [0,t] \times [0,u]} \iota_F^{-1}(\hat{t}, \hat{u}) \int_{u'=0}^{\hat{u}} \mathbb{V}_{\leq 20}(\hat{t}, u') \, du' \tag{15.16.24}$$

generated by the integrals on the last three lines of RHS (15.2.1a). To proceed, we insert the already proven vorticity estimates (15.1.1c)-(15.1.1e) into the integrands in (15.16.21)-(15.16.24). With the help of inequality (11.3.3), we obtain

$$\begin{aligned} & C \int_{t'=0}^{\hat{t}} \frac{1}{\upmu_\star^{3/2}(t',\hat{u})} \left\{ \int_{s=0}^{t'} \sqrt{\mathbb{V}_{21}}(s,\hat{u})\, ds \right\}^2 dt' && (15.16.25) \\ & \leq C \mathring{\upepsilon}^2 \int_{t'=0}^{\hat{t}} \upmu_\star^{-3/2}(t',\hat{u}) \left\{ \int_{s=0}^{t'} \upmu_\star^{-6.4}(s,\hat{u})\, ds \right\}^2 dt' \\ & \leq C \mathring{\upepsilon}^2 \int_{t'=0}^{\hat{t}} \frac{1}{\upmu_\star^{12.3}(t',\hat{u})}\, dt' \leq C \mathring{\upepsilon}^2 \upmu_\star^{-11.3}(\hat{t},\hat{u}), \end{aligned}$$

$$\begin{aligned} & \int_{t'=0}^{\hat{t}} \frac{1}{\upmu_\star^{3/2}(t',\hat{u})} \left\{ \int_{s=0}^{t'} \frac{1}{\upmu_\star^{1/2}(s,\hat{u})} \sqrt{\mathbb{V}_{\leq 20}}(s,\hat{u})\, ds \right\}^2 dt' && (15.16.26) \\ & \leq C \mathring{\upepsilon}^2 \int_{t'=0}^{\hat{t}} \upmu_\star^{-3/2}(t',\hat{u}) \left\{ \int_{s=0}^{t'} \upmu_\star^{-5.4}(s,\hat{u})\, ds \right\}^2 dt' \\ & \leq C \mathring{\upepsilon}^2 \int_{t'=0}^{\hat{t}} \upmu_\star^{-10.3}(t',\hat{u})\, dt' \leq C \mathring{\upepsilon}^2 \upmu_\star^{-9.3}(\hat{t},\hat{u}), \end{aligned}$$

$$C \int_{t'=0}^{\hat{t}} \mathbb{V}_{\leq 21}(t',\hat{u})\, dt' \leq C \mathring{\upepsilon}^2 \int_{t'=0}^{\hat{t}} \upmu_\star^{-12.8}(t',\hat{u})\, dt' \leq C \mathring{\upepsilon}^2 \upmu_\star^{-11.8}(\hat{t},\hat{u}), \quad (15.16.27)$$

$$C \int_{u'=0}^{\hat{u}} \mathbb{V}_{\leq 20}(\hat{t},u')\, du' \leq C \mathbb{V}_{\leq 20}(\hat{t},\hat{u}) \leq C \mathring{\upepsilon}^2 \upmu_\star^{-9.8}(\hat{t},\hat{u}). \quad (15.16.28)$$

Multiplying (15.16.25)-(15.16.28) by $\iota_F^{-1}(\hat{t},\hat{u})$ and then taking $\sup_{(\hat{t},\hat{u})\in[0,t]\times[0,u]}$, we conclude

$$(15.16.21) \leq C \mathring{\upepsilon}^2 \sup_{(\hat{t},\hat{u})\in[0,t]\times[0,u]} \upmu_\star^{.5}(\hat{t},\hat{u}) \iota_1^{-c}(\hat{t}) \iota_2^{-c}(\hat{t},\hat{u}) e^{-c\hat{t}} e^{-c\hat{u}} \leq C \mathring{\upepsilon}^2, \quad (15.16.29)$$

$$(15.16.22) \leq C \mathring{\upepsilon}^2 \sup_{(\hat{t},\hat{u})\in[0,t]\times[0,u]} \upmu_\star^{2.5}(\hat{t},\hat{u}) \iota_1^{-c}(\hat{t}) \iota_2^{-c}(\hat{t},\hat{u}) e^{-c\hat{t}} e^{-c\hat{u}} \leq C \mathring{\upepsilon}^2, \quad (15.16.30)$$

$$(15.16.23) \leq C \mathring{\upepsilon}^2 \sup_{(\hat{t},\hat{u})\in[0,t]\times[0,u]} \iota_1^{-c}(\hat{t}) \iota_2^{-c}(\hat{t},\hat{u}) e^{-c\hat{t}} e^{-c\hat{u}} \leq C \mathring{\upepsilon}^2, \quad (15.16.31)$$

$$(15.16.24) \leq C \mathring{\upepsilon}^2 \sup_{(\hat{t},\hat{u})\in[0,t]\times[0,u]} \upmu_\star^{2}(\hat{t},\hat{u}) \iota_1^{-c}(\hat{t}) \iota_2^{-c}(\hat{t},\hat{u}) e^{-c\hat{t}} e^{-c\hat{u}} \leq C \mathring{\upepsilon}^2 \quad (15.16.32)$$

as desired. We have thus accounted for the influence of the vorticity in the top-order wave energies.

We now show how to obtain suitable bounds for the terms generated by the "borderline" terms $\boxed{6} \int \cdots$, $\boxed{8.1} \int \cdots$, and $\boxed{2} \frac{1}{\upmu_\star^{1/2}(t,u)} \sqrt{\mathbb{Q}_{20}}(t,u) \left\| L \upmu \right\|_{L^\infty({}^{(-)}\Sigma_{t;t}^u)} \int \cdots$ on RHS (15.2.1a) (where we recall that $N = 20$ in this part of the proof). The terms generated by the remaining "non-borderline" terms on RHS (15.2.1a) are easier to treat. We start with the term $\boxed{6} \iota_F^{-1}(\hat{t},\hat{u}) \int_{t'=0}^{\hat{t}} \cdots$. Multiplying and dividing by $\upmu_\star^{11.8}(t',\hat{u})$ in the integrand, taking

$\sup_{t' \in [0,\hat{t}]} \upmu_{\star}^{11.8}(t',\hat{u})\mathbb{Q}_{20}(t',\hat{u})$, pulling the sup-ed quantity out of the integral, and using the critically important integral estimate (11.3.1) with $b = 12.8$, we find that

$$\begin{aligned}
&\boxed{6}\iota_F^{-1}(\hat{t},\hat{u}) \int_{t'=0}^{\hat{t}} \frac{\|[L\upmu]_-\|_{L^\infty(\Sigma_{t'}^{\hat{u}})}}{\upmu_\star(t',\hat{u})} \mathbb{Q}_{20}(t',\hat{u})\,dt' \qquad (15.16.33)\\
&\leq \boxed{6}\iota_F^{-1}(\hat{t},\hat{u}) \sup_{t'\in[0,\hat{t}]} \left\{\upmu_\star^{11.8}(t',\hat{u})\mathbb{Q}_{20}(t',\hat{u})\right\} \int_{t'=0}^{\hat{t}} \|[L\upmu]_-\|_{L^\infty(\Sigma_{t'}^{\hat{u}})}\, \upmu_\star^{-12.8}(t',\hat{u})\,dt'\\
&\leq \boxed{6}\upmu_\star^{11.8}(\hat{t},\hat{u}) \sup_{t'\in[0,\hat{t}]} \left\{\iota_1^{-c}(t')\iota_2^{-c}(t',\hat{u})e^{-ct'}e^{-c\hat{u}}\upmu_\star^{11.8}(t',\hat{u})\mathbb{Q}_{20}(t',\hat{u})\right\}\\
&\quad \times \int_{t'=0}^{\hat{t}} \|[L\upmu]_-\|_{L^\infty(\Sigma_{t'}^{\hat{u}})}\, \upmu_\star^{-12.8}(t',\hat{u})\,dt'\\
&\leq \frac{6 + C\sqrt{\varepsilon}}{11.8} F(\hat{t},\hat{u}) \leq \frac{6 + C\sqrt{\varepsilon}}{11.8} F(t,u).
\end{aligned}$$

To handle the integral $\boxed{8.1}\iota_F^{-1}(\hat{t},\hat{u})\int\cdots$, we use a similar argument, but this time taking into account that there are two time integrations. We find that

$$\begin{aligned}
&\boxed{8.1}\iota_F^{-1}(\hat{t},\hat{u}) \int_{t'=0}^{\hat{t}} \frac{\|[L\upmu]_-\|_{L^\infty(\Sigma_{t'}^{\hat{u}})}}{\upmu_\star(t',\hat{u})} \sqrt{\mathbb{Q}_{20}}(t',\hat{u}) \int_{s=0}^{t'} \frac{\|[L\upmu]_-\|_{L^\infty(\Sigma_{s}^{\hat{u}})}}{\upmu_\star(s,\hat{u})} \sqrt{\mathbb{Q}_{20}}(s,\hat{u})\,ds\,dt' \\
&\qquad (15.16.34)\\
&\leq \frac{8.1 + C\sqrt{\varepsilon}}{5.9 \times 11.8} F(t,u).
\end{aligned}$$

To handle the integral $\boxed{2}\iota_F^{-1}(\hat{t},\hat{u})\int\cdots$, we use a similar argument based on the critically important estimate (11.3.2). We find that

$$\begin{aligned}
&\boxed{2}\iota_F^{-1}(\hat{t},\hat{u}) \frac{1}{\upmu_\star^{1/2}(\hat{t},\hat{u})} \sqrt{\mathbb{Q}_{20}}(\hat{t},\hat{u})\, \|L\upmu\|_{L^\infty({}^{(-)}\Sigma_{\hat{t};\hat{t}}^{\hat{u}})} \int_{t'=0}^{t} \frac{1}{\upmu_\star^{1/2}(t',\hat{u})} \sqrt{\mathbb{Q}_{20}}(t',\hat{u})\,dt' \qquad (15.16.35)\\
&\leq \frac{2 + C\sqrt{\varepsilon}}{5.4} F(t,u).
\end{aligned}$$

The important point is that for small ε, the factors $\frac{6 + C\sqrt{\varepsilon}}{11.8}$ on RHS (15.16.33), $\frac{8.1 + C\sqrt{\varepsilon}}{5.9 \times 11.8}$ on RHS (15.16.34), and $\frac{2 + C\sqrt{\varepsilon}}{5.4}$ on RHS (15.16.35) sum to $\frac{6}{11.8} + \frac{8.1}{5.9 \times 11.8} + \frac{2}{5.4} + C\sqrt{\varepsilon} < 1$. This sum is the main contributor to the constant $\upalpha_1$ on RHS (15.16.10).

We now derive suitable bounds for the three terms on RHS (15.2.1a) that are multiplied by the large constant C_*. We bound these terms using essentially the same reasoning that we used in proving (15.16.33), (15.16.34), and (15.16.35), but we use only the crude inequality

(11.3.3) in place of the delicate inequalities (11.3.1) and (11.3.2). We find that

$$
\begin{aligned}
& C_* \iota_F^{-1}(\hat{t},\hat{u}) \int_{t'=0}^{\hat{t}} \frac{1}{\upmu_\star(t',\hat{u})} \sqrt{\mathbb{Q}_{20}}(t',\hat{u}) \sqrt{\mathbb{Q}_{20}^{(Partial)}}(t',\hat{u})\, dt' \\
& \quad \leq C F^{1/2}(t,u) G^{1/2}(t,u),
\end{aligned} \tag{15.16.36}
$$

$$
\begin{aligned}
& C_* \iota_F^{-1}(\hat{t},\hat{u}) \int_{t'=0}^{\hat{t}} \frac{1}{\upmu_\star(t',\hat{u})} \sqrt{\mathbb{Q}_{20}}(t',\hat{u}) \int_{s=0}^{t'} \frac{1}{\upmu_\star(s,\hat{u})} \sqrt{\mathbb{Q}_{20}^{(Partial)}}(s,\hat{u})\, ds\, dt' \\
& \quad \leq C F^{1/2}(t,u) G^{1/2}(t,u),
\end{aligned} \tag{15.16.37}
$$

$$
\begin{aligned}
& C_* \iota_F^{-1}(\hat{t},\hat{u}) \frac{1}{\upmu_\star^{1/2}(\hat{t},\hat{u})} \sqrt{\mathbb{Q}_{20}}(\hat{t},\hat{u}) \int_{t'=0}^{\hat{t}} \frac{1}{\upmu_\star^{1/2}(t',\hat{u})} \sqrt{\mathbb{Q}_{20}^{(Partial)}}(t',\hat{u})\, dt' \\
& \quad \leq C F^{1/2}(t,u) G^{1/2}(t,u),
\end{aligned} \tag{15.16.38}
$$

where the constants C on RHSs(15.16.36)-(15.16.38) are large and sum to the large constant α_2 on RHS (15.16.10). We remark that the largeness of α_2 will not preclude us from closing the estimates because we will gain smallness in $G(t,u)$ by using a separate argument given below.

The remaining integrals on RHS (15.2.1a) are easier to treat. We now show how to bound the term arising from the integral on the 14^{th} line of RHS (15.2.1a), which involves three time integrations. The term arising from the integrals on the 13^{th} line of RHS (15.2.1a) can be handled using similar arguments, so we do not provide those details. We claim that the following sequence of inequalities holds for the term of interest, which yields the desired bound:

$$
\begin{aligned}
& C\iota_F^{-1}(\hat{t},\hat{u}) \int_{t'=0}^{\hat{t}} \frac{1}{\upmu_\star(t',\hat{u})} \sqrt{\mathbb{Q}_{20}}(t',\hat{u}) \int_{s=0}^{t'} \frac{1}{\upmu_\star(s,\hat{u})} \int_{s'=0}^{s} \frac{1}{\upmu_\star^{1/2}(s',\hat{u})} \sqrt{\mathbb{Q}_{20}}(s',\hat{u})\, ds'\, ds\, dt' \qquad (15.16.39) \\
& \leq \frac{C}{c} \iota_F^{-1}(\hat{t},\hat{u}) \iota_2^{c/2}(\hat{t},\hat{u}) \int_{t'=0}^{\hat{t}} \frac{1}{\upmu_\star(t',\hat{u})} \sqrt{\mathbb{Q}_{20}}(t',\hat{u}) \\
& \quad \times \int_{s=0}^{t'} \frac{1}{\upmu_\star(s,\hat{u})} \sup_{(s',u')\in[1,s]\times[0,\hat{u}]} \left\{ \iota_2^{-c/2}(s',u') \sqrt{\mathbb{Q}_{20}}(s',u') \right\} ds\, dt' \\
& \leq \frac{C}{c} \iota_F^{-1}(\hat{t},\hat{u}) \iota_2^{c/2}(\hat{t},\hat{u}) \int_{t'=0}^{\hat{t}} \frac{1}{\upmu_\star(t',\hat{u})} \sqrt{\mathbb{Q}_{20}}(t',\hat{u}) \\
& \quad \times \sup_{(s',u')\in[0,t']\times[0,\hat{u}]} \left\{ \upmu_\star(s',u') \iota_2^{-c/2}(s',u') \sqrt{\mathbb{Q}_{20}}(s',u') \right\} \int_{s=0}^{t'} \frac{1}{\upmu_\star^2(s,\hat{u})}\, ds\, dt' \\
& \leq \frac{C}{c} \iota_F^{-1}(\hat{t},\hat{u}) \iota_2^{c/2}(\hat{t},\hat{u}) \sup_{(s',u')\in[0,\hat{t}]\times[0,\hat{u}]} \left\{ \upmu_\star(s',u') \iota_2^{-c/2}(s',u') \sqrt{\mathbb{Q}_{20}}(s',u') \right\} \\
& \times \sup_{(s',u')\in[0,\hat{t}]\times[0,\hat{u}]} \left\{ \sqrt{\mathbb{Q}_{20}}(s',u') \right\} \int_{t'=0}^{\hat{t}} \frac{1}{\upmu_\star(t',\hat{u})} \int_{s=0}^{t'} \frac{1}{\upmu_\star^2(s,\hat{u})}\, ds\, dt' \\
& \leq \frac{C}{c} \upmu_\star(\hat{t},\hat{u}) \sup_{(s',u')\in[0,\hat{t}]\times[0,\hat{u}]} \left\{ \iota_F^{-1}(s',u') \mathbb{Q}_{20}(s',u') \right\} \times \int_{t'=0}^{\hat{t}} \frac{1}{\upmu_\star(t',\hat{u})} \int_{s=0}^{t'} \frac{1}{\upmu_\star^2(s,\hat{u})}\, ds\, dt' \\
& \leq \frac{C}{c} F(\hat{t},\hat{u}) \leq \frac{C}{c} F(t,u),
\end{aligned}
$$

which yields the desired smallness factor $\frac{1}{c}$. We now explain how to derive (15.16.39). To deduce the first inequality, we multiplied and divided by $\iota_2^{c/2}(t',\hat{u})$ in the integral $\int \cdots ds'$, then pulled $\sup_{(s',u')\in[0,s]\times[0,\hat{u}]} \left\{ \iota_2^{-c/2}(s',u') \sqrt{\mathbb{Q}_{20}}(s',u') \right\}$ out of the integral, and finally used (15.16.17) to gain the smallness factor $\frac{1}{c}$ from the remaining terms $\int_{s'=0}^{s} \frac{1}{\upmu_\star^{1/2}(s',\hat{u})} \iota_2^{c/2}(s',\hat{u})\, ds'$. To derive the second inequality in (15.16.39), we multiplied and divided by $\upmu_\star(s,\hat{u})$ in the integral $\int \cdots ds$, and used the approximate monotonicity property (15.16.20) to pull the factor $\sup_{(s',u')\in[0,t']\times[0,\hat{u}]} \left\{ \upmu_\star(s',u') \iota_2^{-c/2}(s',u') \sqrt{\mathbb{Q}_{20}}(s',u') \right\}$ out of the ds integral, which costs us a harmless multiplicative factor of $1 + C\varepsilon$. The third inequality in (15.16.39) follows easily. To derive the fourth inequality, we use the monotonicity of $\iota_1^c(\cdot)$, $\iota_2^c(\cdot)$ and $e^{c\cdot}$, and the approximate monotonicity property (15.16.20). To derive the fifth inequality, we use inequality (11.3.3) twice. The final inequality follows easily.

Similarly, we claim that we can bound the terms on the 8^{th} through 11^{th} lines of RHS (15.2.1a) and the second term on the 12^{th} line of RHS (15.2.1a) as follows:

$$\iota_F^{-1}(\hat{t},\hat{u})C\varepsilon\int_{t'=0}^{\hat{t}}\frac{1}{\upmu_\star(t',\hat{u})}\mathbb{Q}_{20}(t',\hat{u})\,dt'\leq C\varepsilon F(\hat{t},\hat{u})\leq C\varepsilon F(t,u), \tag{15.16.40}$$

$$\iota_F^{-1}(\hat{t},\hat{u})C\varepsilon\int_{t'=0}^{\hat{t}}\frac{1}{\upmu_\star(t',\hat{u})}\sqrt{\mathbb{Q}_{20}}(t',\hat{u})\int_{s=0}^{t'}\frac{1}{\upmu_\star(s,\hat{u})}\sqrt{\mathbb{Q}_{20}}(s,\hat{u})\,ds\,dt'\leq C\varepsilon F(\hat{t},\hat{u})\leq C\varepsilon F(t,u), \tag{15.16.41}$$

$$\iota_F^{-1}(\hat{t},\hat{u})C\varepsilon\frac{1}{\upmu_\star^{1/2}(\hat{t},\hat{u})}\sqrt{\mathbb{Q}_{20}}(\hat{t},\hat{u})\int_{t'=0}^{\hat{t}}\frac{1}{\upmu_\star^{1/2}(t',\hat{u})}\sqrt{\mathbb{Q}_{20}}(t',\hat{u})\,dt'\leq C\varepsilon F(\hat{t},\hat{u})\leq C\varepsilon F(t,u), \tag{15.16.42}$$

$$\iota_F^{-1}(\hat{t},\hat{u})C\sqrt{\mathbb{Q}_{20}}(\hat{t},\hat{u})\int_{t'=0}^{\hat{t}}\frac{1}{\upmu_\star^{1/2}(t',\hat{u})}\sqrt{\mathbb{Q}_{20}}(t',\hat{u})\,dt'\leq \frac{C}{c}F(\hat{t},\hat{u})\leq \frac{C}{c}F(t,u), \tag{15.16.43}$$

$$\iota_F^{-1}(\hat{t},\hat{u})C(1+\varsigma^{-1})\int_{t'=0}^{\hat{t}}\frac{1}{\upmu_\star^{1/2}(t',\hat{u})}\mathbb{Q}_{20}(t',\hat{u})\,dt'\leq \frac{C}{c}(1+\varsigma^{-1})F(\hat{t},\hat{u})\leq \frac{C}{c}(1+\varsigma^{-1})F(t,u). \tag{15.16.44}$$

To derive (15.16.40), we use arguments similar to the ones we used in deriving (15.16.33), but in place of the delicate estimate (11.3.1), we use the estimate (11.3.3), whose imprecision is compensated for by the availability of the smallness factor ε. Similar remarks apply to (15.16.41), but we rely on the fact that there are two time integrations. The proof of (15.16.44) is similar, but we multiply and divide by by $\iota_2^{-c}(t',\hat{u})$ in the integrand and use the estimate (15.16.17) to gain the smallness factor $\frac{1}{c}$. To derive (15.16.42), we use arguments similar to the ones we used above, but we now multiply and divide by $\upmu_\star^{5.9}(t',\hat{u})$ in the time integral on LHS (15.16.42) and use (11.3.3). To derive (15.16.43), we use similar arguments based on multiplying and dividing by $\iota_2^{c/2}(t',\hat{u})$ in the time integral and using (15.16.17).

Similarly, we derive the bound

$$C\iota_F^{-1}(\hat{t},\hat{u})\int_{t'=0}^{\hat{t}}\frac{1}{\sqrt{T_{(Boot)}-t'}}\mathbb{Q}_{20}(t',\hat{u})\,dt'\leq \frac{C}{c}F(\hat{t},\hat{u})\leq \frac{C}{c}F(t,u) \tag{15.16.45}$$

for the first term on the 12^{th} line of RHS (15.2.1a) by multiplying and dividing by $\iota_1^c(t')$ in the integrand and using (15.16.16) to gain the smallness factor $\frac{1}{c}$.

Similarly, we derive the bound

$$C(1+\varsigma^{-1})\iota_F^{-1}(\hat{t},\hat{u})\int_{u'=0}^{\hat{u}}\mathbb{Q}_{20}(\hat{t},u')\,du'\leq \frac{C}{c}F(\hat{t},\hat{u})\leq \frac{C}{c}F(t,u) \tag{15.16.46}$$

for the first term on the 15^{th} line of RHS (15.2.1a) by multiplying and dividing by $e^{cu'}$ in the integrand and using (15.16.19) to gain the smallness factor $\frac{1}{c}$.

It is easy to see that the terms arising from the term on the first line and the last three terms on the 15^{th} line of RHS (15.2.1a), namely $C(1+\varsigma^{-1})\mathring{\epsilon}^2\upmu_\star^{-3/2}(t,u)$, $C\varepsilon\mathbb{Q}_{20}(t,u)$, $C\varsigma\mathbb{Q}_{20}(t,u)$, and $C\varsigma\mathbb{K}_{20}(t,u)$, are respectively bounded (after multiplying by ι_F^{-1} and taking the relevant sup) by $\leq C(1+\varsigma^{-1})\mathring{\epsilon}^2$, $\leq C\varepsilon F(t,u)$, $\leq C\varsigma F(t,u)$, and $\leq C\varsigma F(t,u)$.

To bound the term arising from the first term on the 16^{th} line of RHS (15.2.1a) (where we recall that $N=20$), we argue as follows with the help of (15.16.17) and (15.16.20):

$$\begin{aligned} & C(1+\varsigma^{-1})\iota_F^{-1}(\hat{t},\hat{u})\int_{t'=0}^{\hat{t}}\frac{1}{\upmu_\star^{5/2}(t',\hat{u})}\mathbb{Q}_{[1,19]}(t',\hat{u})\,dt' \qquad (15.16.47)\\ & \leq C(1+\varsigma^{-1})\upmu_\star^{9.8}(\hat{t},\hat{u})\iota_1^{-c}(\hat{t})\iota_2^{-c}(\hat{t},\hat{u})e^{-c\hat{t}}e^{-c\hat{u}}\sup_{t'\in[0,\hat{t}]}\left(\frac{\upmu_\star(\hat{t},\hat{u})}{\upmu_\star(t',\hat{u})}\right)^2\\ & \quad\times\left\{\sup_{t'\in[0,\hat{t}]}\iota_2^{-c}(t')\mathbb{Q}_{[1,19]}(t',\hat{u})\right\}\times\int_{t'=0}^{\hat{t}}\frac{\iota_2^{c}(t')}{\upmu_\star^{1/2}(t',\hat{u})}\,dt'\\ & \leq C(1+\varsigma^{-1})\iota_2^{-c}(\hat{t})\left\{\sup_{t'\in[0,\hat{t}]}\iota_H^{-1}(t',\hat{u})\mathbb{Q}_{[1,19]}(t',\hat{u})\right\}\times\int_{t'=0}^{\hat{t}}\frac{\iota_2^{c}(t')}{\upmu_\star^{1/2}(t',\hat{u})}\,dt'\\ & \leq\frac{C}{c}(1+\varsigma^{-1})H(\hat{t},\hat{u})\leq\frac{C}{c}(1+\varsigma^{-1})H(t,u). \end{aligned}$$

Using a similar argument based on (15.16.19), we bound the term arising from the second term on the 16^{th} line of RHS (15.2.1a) as follows:

$$\begin{aligned} & C(1+\varsigma^{-1})\iota_F^{-1}(\hat{t},\hat{u})\int_{u'=1}^{u}\mathbb{Q}_{[1,19]}(\hat{t},u')\,du' \qquad (15.16.48)\\ & \leq C(1+\varsigma^{-1})\upmu_\star^{11.8}(\hat{t},\hat{u})\iota_1^{-c}(\hat{t})\iota_2^{-c}(\hat{t},\hat{u})e^{-c\hat{t}}e^{-c\hat{u}}\left\{\sup_{u'\in[0,\hat{u}]}e^{-cu'}\mathbb{Q}_{[1,19]}(\hat{t},u')\right\}\times\int_{u'=0}^{\hat{u}}e^{cu'}\,du'\\ & \leq C(1+\varsigma^{-1})\upmu_\star^{2}(\hat{t},\hat{u})e^{-c\hat{u}}\left\{\sup_{u'\in[0,\hat{u}]}\iota_H^{-1}(\hat{t},\hat{u'})\mathbb{Q}_{[1,19]}(\hat{t},u')\right\}\times\int_{u'=0}^{\hat{u}}e^{cu'}\,du'\\ & \leq\frac{C}{c}(1+\varsigma^{-1})H(\hat{t},\hat{u})\leq\frac{C}{c}(1+\varsigma^{-1})H(t,u). \end{aligned}$$

To bound the terms arising from the three terms on the 17^{th} line of RHS (15.2.1a), we argue as follows (again recalling that $N=20$):

$$\begin{aligned} C\varepsilon\iota_F^{-1}(\hat{t},\hat{u})\mathbb{Q}_{[1,19]}(\hat{t},\hat{u}) & = C\varepsilon\upmu_\star^2(\hat{t},\hat{u})\iota_H^{-1}(\hat{t},\hat{u})\mathbb{K}_{[1,N-1]}(\hat{t},\hat{u}) \qquad (15.16.49)\\ & \leq C\varepsilon\upmu_\star^2(\hat{t},\hat{u})H(\hat{t},\hat{u})\leq C\varepsilon H(\hat{t},\hat{u})\leq C\varepsilon H(t,u),\\ C\varsigma\iota_F^{-1}(\hat{t},\hat{u})\mathbb{Q}_{[1,19]}(\hat{t},\hat{u}) & = C\varsigma\upmu_\star^2(\hat{t},\hat{u})\iota_H^{-1}(\hat{t},\hat{u})\mathbb{K}_{[1,N-1]}(\hat{t},\hat{u}) \qquad (15.16.50)\\ & \leq C\varsigma\upmu_\star^2(\hat{t},\hat{u})H(\hat{t},\hat{u})\leq C\varsigma H(\hat{t},\hat{u})\leq C\varsigma H(t,u),\\ C\varsigma\iota_F^{-1}(\hat{t},\hat{u})\mathbb{K}_{[1,N-1]}(\hat{t},\hat{u}) & = C\varsigma\upmu_\star^2(\hat{t},\hat{u})\iota_H^{-1}(\hat{t},\hat{u})\mathbb{K}_{[1,N-1]}(\hat{t},\hat{u}) \qquad (15.16.51)\\ & \leq C\varsigma\upmu_\star^2(\hat{t},\hat{u})H(\hat{t},\hat{u})\leq C\varsigma H(\hat{t},\hat{u})\leq C\varsigma H(t,u). \end{aligned}$$

Inserting all of these estimates into the RHS of $\iota_F^{-1}(\hat{t},\hat{u})\times$ (15.2.1a)$(\hat{t},\hat{u})$ and taking $\sup_{(\hat{t},\hat{u})\in[0,t]\times[0,u]}$ of both sides, we deduce that

$$F(t,u) \leq C(1+\varsigma^{-1})\mathring{\epsilon}^2 + \left\{\frac{6}{11.8} + \frac{8.1}{5.9\times 11.8} + \frac{2}{5.4} + C\sqrt{\varepsilon} + C\varsigma + \frac{C}{c}(1+\varsigma^{-1})\right\} F(t,u) \tag{15.16.52}$$
$$+ C\left\{\varepsilon + \varsigma + \frac{1}{c}(1+\varsigma^{-1})\right\} H(t,u) + CF^{1/2}(t,u)G^{1/2}(t,u).$$

We now bound the terms $\iota_G^{-1}(\hat{t},\hat{u})\times\cdots$ arising from the terms described in (15.2.2). We claim that the following analog of (15.16.52) holds:

$$G(t,u) \leq C(1+\varsigma^{-1})\mathring{\epsilon}^2 + C\left\{\varepsilon + \varsigma + \frac{1}{c}(1+\varsigma^{-1})\right\} F(t,u) + C\left\{\varepsilon + \varsigma + \frac{1}{c}(1+\varsigma^{-1})\right\} H(t,u). \tag{15.16.53}$$

The proof of (15.16.53) is similar to the proof of (15.16.52), but with the following key changes: **i)**: in view of (15.2.2), the terms corresponding to (15.16.33)-(15.16.35) and (15.16.36)-(15.16.38) are absent from RHS (15.16.53). To treat the terms $\iota_G^{-1}(\hat{t},\hat{u})\times\cdots$ corresponding to the three new terms explicitly listed in (15.2.2), we argue as in the proof of (15.16.33), (15.16.34), and (15.16.35), but using the cruder inequality (11.3.3) in place of the delicate inequalities (11.3.1) and (11.3.2). We find that

$$C\varepsilon\iota_G^{-1}(\hat{t},\hat{u})\int_{t'=0}^{\hat{t}}\frac{1}{\upmu_\star(t',\hat{t})}\sqrt{\mathbb{Q}_N}(t',\hat{t})\int_{s=0}^{t'}\frac{1}{\upmu_\star(s,\hat{t})}\sqrt{\mathbb{Q}_N}(s,\hat{t})\,ds\,dt' \leq C\varepsilon F(t,u), \tag{15.16.54}$$

$$C\varepsilon\iota_G^{-1}(\hat{t},\hat{u})\int_{t'=0}^{\hat{t}}\frac{1}{\upmu_\star(t',\hat{u})}\mathbb{Q}_N(t',\hat{u})\,dt' \leq C\varepsilon F(t,u), \tag{15.16.55}$$

$$C\varepsilon\iota_G^{-1}(\hat{t},\hat{u})\frac{1}{\upmu_\star^{1/2}(\hat{t},\hat{u})}\sqrt{\mathbb{Q}_N}(\hat{t},\hat{u})\int_{t'=0}^{\hat{t}}\frac{1}{\upmu_\star^{1/2}(t',\hat{u})}\sqrt{\mathbb{Q}_N}(t',\hat{u})\,dt' \leq C\varepsilon F(t,u). \tag{15.16.56}$$

The desired estimate (15.16.53) now follows from the same arguments used to prove (15.16.52), where (15.16.54)-(15.16.56) contribute to the second product on RHS (15.16.53).

We now bound the terms $\iota_H^{-1}(\hat{t},\hat{u})\times\cdots$ arising from the terms on RHS (15.2.3). All terms except the one arising from the integral involving the top-order factor $\sqrt{\mathbb{Q}_{20}}$ (featured in the ds integral on RHS (15.2.3)) can be bounded by $\leq C\mathring{\epsilon}^2 + \frac{C}{c}(1+\varsigma^{-1})G(t,u) + C\varsigma H(t,u)$ by using essentially the same arguments given above. In particular, we use the already proven specific vorticity energy estimates (15.1.1d)-(15.1.1e) to handle the terms generated by the integrals on the last line of RHS (15.2.3). To handle the remaining term involving the top-order factor $\sqrt{\mathbb{Q}_{20}}$, we use arguments similar to the ones we used to prove (15.16.39) (in particular, we use inequality (11.3.3) twice) to bound it as follows:

$$
\begin{aligned}
& C\iota_H^{-1}(\hat{t},\hat{u}) \int_{t'=0}^{\hat{t}} \frac{1}{\upmu_\star^{1/2}(t',\hat{u})} \sqrt{\mathbb{Q}_{[1,19]}}(t',\hat{u}) \int_{s=0}^{t'} \frac{1}{\upmu_\star^{1/2}(s,\hat{u})} \sqrt{\mathbb{Q}_{20}}(s,\hat{u})\,ds\,dt' \qquad (15.16.57) \\
& \leq C\iota_H^{-1}(\hat{t},\hat{u}) \sup_{(t',u')\in[0,\hat{t}]\times[0,\hat{u}]} \left\{ \upmu_\star^{4.9}(t',\hat{u}) \sqrt{\mathbb{Q}_{[1,19]}}(t',u') \right\} \times \sup_{(t',u')\in[0,\hat{t}]\times[0,\hat{u}]} \left\{ \upmu_\star^{5.9}(t',\hat{u}) \sqrt{\mathbb{Q}_{20}}(t',u') \right\} \\
& \times \int_{t'=0}^{\hat{t}} \frac{1}{\upmu_\star^{5.4}(t',\hat{u})} \int_{s=0}^{t'} \frac{1}{\upmu_\star^{6.4}(s,\hat{u})}\,ds\,dt' \\
& \leq CF^{1/2}(\hat{t},\hat{u}) H^{1/2}(\hat{t},\hat{u}) \leq CF(t,u) + \frac{1}{2} H(t,u).
\end{aligned}
$$

Inserting all of these estimates into the RHS of $\iota_H^{-1}(\hat{t},\hat{u})\times$ (15.2.3)$(\hat{t},\hat{u})$ and taking $\sup_{(\hat{t},\hat{u})\in[0,t]\times[0,u]}$ of both sides, we deduce that

$$
H(t,u) \leq C\mathring{\upepsilon}^2 + C\left\{1+\frac{1}{c}\right\} F(t,u) + C\left\{\frac{1}{2} + \varsigma + (1+\varsigma^{-1})\frac{1}{c}\right\} H(t,u). \qquad (15.16.58)
$$

We now consider the system of three inequalities (15.16.52), (15.16.53), and (15.16.58), and we remind the reader that the constants C in these inequalities can be chosen to be independent of c. The desired estimates (15.16.10)-(15.16.12) now follow from first choosing ς to be sufficiently small, then choosing c to be sufficiently large, then choosing ε to be sufficiently small, and using the aforementioned fact that $\frac{6}{11.8} + \frac{8.1}{5.9\times 11.8} + \frac{2}{5.4} + C\sqrt{\varepsilon} < 1$.

Estimates for $\max\left\{\mathbb{Q}_{[1,18]},\mathbb{K}_{[1,18]}\right\}$, $\max\left\{\mathbb{Q}_{[1,17]},\mathbb{K}_{[1,17]}\right\}$, $\cdots$, $\max\{\mathbb{Q}_1,\mathbb{K}_1\}$ **via a descent scheme**: We now explain how to use inequality (15.2.3) to derive the estimates for $\max\left\{\mathbb{Q}_{[1,18]},\mathbb{K}_{[1,18]}\right\}$, $\max\left\{\mathbb{Q}_{[1,17]},\mathbb{K}_{[1,17]}\right\}$, $\cdots$, $\max\{\mathbb{Q}_1,\mathbb{K}_1\}$ by downward induction. Unlike our analysis of the strongly coupled pair $\max\{\mathbb{Q}_{20},\mathbb{K}_{20}\}$ and $\max\left\{\mathbb{Q}_{[1,19]},\mathbb{K}_{[1,19]}\right\}$, we can derive the desired estimates for $\max\left\{\mathbb{Q}_{[1,18]},\mathbb{K}_{[1,18]}\right\}$ by using only inequality (15.2.3) and the already derived estimates for $\max\left\{\mathbb{Q}_{[1,19]},\mathbb{K}_{[1,19]}\right\}$. At the end of the proof, we will describe the minor changes needed to derive the desired estimates for $\max\left\{\mathbb{Q}_{[1,17]},\mathbb{K}_{[1,17]}\right\}$, $\cdots$, $\max\{\mathbb{Q}_1+\mathbb{K}_1\}$.

To begin, we define the following analogs of (15.16.7) and (15.16.3):

$$
\iota_{\widetilde{H}}(t',u') := \upmu_\star^{-7.8}(t',u')\iota_1^c(t')\iota_2^c(t',u')e^{ct'}e^{cu'}, \qquad (15.16.59)
$$

$$
\widetilde{H}(t,u) := \sup_{(\hat{t},\hat{u})\in[0,t]\times[0,u]} \iota_{\widetilde{H}}^{-1}(\hat{t},\hat{u}) \max\left\{\mathbb{Q}_{[1,18]}(\hat{t},\hat{u}),\mathbb{K}_{[1,18]}(\hat{t},\hat{u})\right\}. \qquad (15.16.60)
$$

Note that the power of $\upmu_\star^{-1}$ in the factor $\upmu_\star^{-7.8}$ has been reduced by two in (15.16.59) compared to (15.16.7), which corresponds to less singular behavior of $\max\left\{\mathbb{Q}_{[1,18]},\mathbb{K}_{[1,18]}\right\}$ near the shock. As before, to prove the desired estimate (15.1.1a) (now with $K=3$), it suffices to prove

$$
\widetilde{H}(t,u) \leq C\mathring{\upepsilon}^2. \qquad (15.16.61)
$$

We now set $N=19$, multiply both sides of inequality (15.2.3) by $\iota_{\widetilde{H}}^{-1}(t,u)$ and then set $(t,u)=(\hat{t},\hat{u})$. With one exception, we can bound all terms arising from the integrals on

RHS (15.2.3) by $\leq C\mathring{\epsilon}^2 + \frac{C}{c}(1+\varsigma^{-1})\widetilde{H}(t,u) + \varsigma\widetilde{H}(t,u)$ (where C is independent of c) by using the same arguments that we used in deriving the estimate for $\max\left\{\mathbb{Q}_{[1,19]}, \mathbb{K}_{[1,19]}\right\}$. The exceptional term is the one arising from the integral involving the above-present-order factor $\sqrt{\mathbb{Q}_{[1,19]}}$. We bound the exceptional term as follows by using inequality (11.3.3), the approximate monotonicity of $\iota_{\widetilde{H}}$, and the estimate $\sqrt{\mathbb{Q}_{[1,19]}} \leq C_c \mathring{\epsilon}\upmu_\star^{-4.9}(t,u)$ (which follows from the already proven estimate (15.16.8) for $H(t,u)$):

$$
\begin{aligned}
& C\iota_{\widetilde{H}}^{-1}(\hat{t},\hat{u}) \int_{t'=0}^{\hat{t}} \frac{1}{\upmu_\star^{1/2}(t',\hat{u})} \sqrt{\mathbb{Q}_{[1,18]}}(t',\hat{u}) \int_{s=0}^{t'} \frac{1}{\upmu_\star^{1/2}(s,\hat{u})} \sqrt{\mathbb{Q}_{[1,19]}}(s,\hat{u})\, ds\, dt' \qquad (15.16.62) \\
& \leq C_c \mathring{\epsilon} \iota_{\widetilde{H}}^{-1/2}(\hat{t},\hat{u}) \sup_{(t',u')\in[0,\hat{t}]\times[0,\hat{u}]} \left\{ \iota_{\widetilde{H}}^{-1/2}(t',u') \sqrt{\mathbb{Q}_{[1,18]}}(t',u') \right\} \times \int_{t'=0}^{\hat{t}} \frac{1}{\upmu_\star^{1/2}(t',\hat{u})} \int_{s=0}^{t'} \frac{1}{\upmu_\star^{5.4}(s,\hat{u})}\, ds\, dt' \\
& \leq C_c \mathring{\epsilon} \iota_{\widetilde{H}}^{-1/2}(\hat{t},\hat{u}) \upmu_\star^{-3.9}(\hat{t},\hat{u}) \sup_{(t',u')\in[0,\hat{t}]\times[0,\hat{u}]} \left\{ \iota_{\widetilde{H}}^{-1/2}(t',u') \sqrt{\mathbb{Q}_{[1,18]}}(t',u') \right\} \\
& \leq C_c \mathring{\epsilon} \widetilde{H}^{1/2}(\hat{t},\hat{u}) \leq C_c \mathring{\epsilon}^2 + \frac{1}{2}\widetilde{H}(t,u).
\end{aligned}
$$

In total, we have obtained the following analog of (15.16.58):

$$
\widetilde{H}(t,u) \leq C_c \mathring{\epsilon}^2 + \frac{C}{c}(1+\varsigma^{-1})\widetilde{H}(t,u) + \frac{1}{2}\widetilde{H}(t,u) + C\varsigma\widetilde{H}(t,u), \qquad (15.16.63)
$$

where C_c is the only constant that depends on c. The desired bound (15.16.61) easily follows from (15.16.63) by first choosing ς to be sufficiently small and then c to be sufficiently large so that we can absorb all factors of $\widetilde{H}$ on RHS (15.16.63) into the LHS.

The desired bounds (15.1.1b) for $\max\left\{\mathbb{Q}_{[1,17]}, \mathbb{K}_{[1,17]}\right\}$, $\max\left\{\mathbb{Q}_{[1,16]}, \mathbb{K}_{[1,16]}\right\}$ $\cdots$ can be (downward) inductively derived by using an argument similar to the one we used to bound $\max\left\{\mathbb{Q}_{[1,18]}, \mathbb{K}_{[1,18]}\right\}$, which relied on the already proven bounds for $\max\left\{\mathbb{Q}_{[1,19]}, \mathbb{K}_{[1,19]}\right\}$. The only difference is that we define the analog of the approximating integrating factor (15.16.59) to be $\upmu_\star^{-P}\iota_1^c(t')\iota_2^c(t',u')e^{ct'}e^{cu'}$, where $P = 5.8$ for the $\max\left\{\mathbb{Q}_{[1,17]}, \mathbb{K}_{[1,17]}\right\}$ estimate, $P = 3.8$ for the $\max\left\{\mathbb{Q}_{[1,16]}, \mathbb{K}_{[1,16]}\right\}$ estimate, $P = 1.8$ for the $\max\left\{\mathbb{Q}_{[1,15]}, \mathbb{K}_{[1,15]}\right\}$ estimate, and $P = 0$ for the $\max\left\{\mathbb{Q}_{[1,\leq 14]}, \mathbb{K}_{[1,\leq 14]}\right\}$ estimates; these latter estimates *do not involve any singular factor* of $\upmu_\star^{-1}$. There is one important new detail relevant for these estimates: in deriving the analog of the inequalities (15.16.62) for $\max\left\{\mathbb{Q}_{[1,\leq 14]}, \mathbb{K}_{[1,\leq 14]}\right\}$, we use the estimate (11.3.6) in place of the estimate (11.3.3); as in our proof of Lemma 15.21, the estimate (11.3.6) allows us to break the $\upmu_\star^{-1}$ degeneracy. This completes the proof of Prop. 15.1.

16. The Main Theorem

We now state and prove the main theorem.

Theorem 16.1 (Stable shock formation). *Let* $(\uprho, v^1, v^2, \upomega)$ *be a solution to the* 2D *compressible Euler equations in the form* (3.3.11a)-(3.3.11c) *under any physical*[112] *barotropic equation of state except for that of a Chaplygin gas (see* (3.3.4)*) and let* u *be a solution to the eikonal equation* (3.6.1). *Let* $\vec{\Psi} = (\Psi_0, \Psi_1, \Psi_2) := (\uprho, v^1, v^2)$ *denote the array of the*

[112]Physical in the sense described below equation (3.3.2).

difference between the wave variables and the constant state solution $(0,0,0)$*. Assume that the solution verifies the size assumptions on* Σ_0^1 *and* $\mathcal{P}_0^{2\mathring{\delta}_*^{-1}}$ *stated in Sects. 8.1 and 8.2 as well as the smallness assumptions of Sect. 8.6. In particular, let* $\mathring{\epsilon}$*,* $\mathring{\delta}$*, and* $\mathring{\delta}_*$ *be the data-size parameters from* (8.1.1)*,* (8.1.2)*-*(8.1.7)*, and* (8.2.1a)*-*(8.2.8)*. Assume the genericity condition*[113]

$$\bar{c_s}' + 1 \neq 0, \tag{16.0.64}$$

where $\bar{c_s}' := c_s'(\rho = 0)$ *denotes the value of* c_s' *corresponding to the background constant state. Let* Υ *be the change of variables map from geometric to Cartesian coordinates (see Def. 3.12). For each* $U_0 \in [0,1]$*, let*

$$\begin{aligned} &T_{(Lifespan);U_0} \\ &:= \sup\Big\{t \in [1,\infty) \mid \textit{the solution exists classically on } \mathcal{M}_{t;U_0} \\ &\qquad\qquad \textit{and } \Upsilon \textit{ is a diffeomorphism from } [1,t)\times[0,U_0]\times\mathbb{T} \textit{ onto its image}\Big\} \end{aligned}$$

(see Figure 2 on pg. 10). If $\mathring{\epsilon}$ *is sufficiently small*[114] *relative to* $\mathring{\delta}^{-1}$ *and* $\mathring{\delta}_*$ *(in the sense explained in Sect. 8.6), then the following conclusions hold, where all constants can be chosen to be independent of* U_0*.*

Dichotomy of possibilities. *One of the following mutually disjoint possibilities must occur, where* $\mu_\star(t,u)$ *is defined in* (11.1.2)*.*

- **I)** $T_{(Lifespan);U_0} > 2\mathring{\delta}_*^{-1}$*. In particular, the solution exists classically on the spacetime region* $\mathrm{cl}\mathcal{M}_{2\mathring{\delta}_*^{-1},U_0}$*, where* cl *denotes closure. Furthermore,* $\inf\{\mu_\star(s,U_0) \mid s \in [0,2\mathring{\delta}_*^{-1}]\} > 0$*.*
- **II)** $0 < T_{(Lifespan);U_0} \leq 2\mathring{\delta}_*^{-1}$*, and*

$$T_{(Lifespan);U_0} = \sup\Big\{t \in [1,2\mathring{\delta}_*^{-1}) \mid \inf\{\mu_\star(s,U_0) \mid s\in[1,t)\} > 0\Big\}. \tag{16.0.65}$$

*In addition, case **II)** occurs when* $U_0 = 1$*. In this case, we have*

$$T_{(Lifespan);1} = \{1 + \mathcal{O}(\mathring{\epsilon})\}\,\mathring{\delta}_*^{-1}. \tag{16.0.66}$$

What happens in Case I). *In case **I)**, all bootstrap assumptions, the estimates of Props. 9.12 and 10.1, and the energy estimates of Prop. 15.1 hold on* $\mathrm{cl}\mathcal{M}_{2\mathring{\delta}_*^{-1},U_0}$ *with all factors* ε *on the RHS of all inequalities replaced by* $C\mathring{\epsilon}$*. Moreover, for* $0 \leq K \leq 5$*, the following estimates*

[113] For any barotropic equation of state except for that of the Chaplygin gas (see (3.3.4)), there exist choices of the background density $\bar{\rho}$ such that the condition (16.0.64) holds.

[114] Recall that in Subsect. 8.7, we show that there exists an open set of solutions satisfying the desired smallness conditions.

hold for $(t,u) \in [0, 2\mathring{\delta}_*^{-1}] \times [0, U_0]$*:*

$$\left\| \mathscr{Z}_*^{[1,14];\leq 1} \upmu \right\|_{L^2(\Sigma_t^u)}, \left\| \mathscr{Z}_*^{\leq 14;\leq 2} L_{(Small)}^i \right\|_{L^2(\Sigma_t^u)}, \left\| \mathscr{Z}_*^{\leq 13;\leq 2} \mathrm{tr}_{\not{g}} \chi \right\|_{L^2(\Sigma_t^u)} \leq C \mathring{\epsilon}, \tag{16.0.67a}$$

$$\left\| \mathscr{Z}_*^{15+K;\leq 1} \upmu \right\|_{L^2(\Sigma_t^u)}, \left\| \mathscr{Z}_*^{15+K;\leq 2} L_{(Small)}^i \right\|_{L^2(\Sigma_t^u)}, \left\| \mathscr{Z}_*^{14+K;\leq 2} \mathrm{tr}_{\not{g}} \chi \right\|_{L^2(\Sigma_t^u)} \leq C \mathring{\epsilon} \upmu_\star^{-(K+.4)}(t,u), \tag{16.0.67b}$$

$$\left\| L \mathscr{Z}_*^{20;\leq 1} \upmu \right\|_{L^2(\Sigma_t^u)}, \left\| L \mathscr{Z}_*^{20;\leq 2} L_{(Small)}^i \right\|_{L^2(\Sigma_t^u)}, \left\| L \mathscr{Z}_*^{19;\leq 2} \mathrm{tr}_{\not{g}} \chi \right\|_{L^2(\Sigma_t^u)} \leq C \mathring{\epsilon} \upmu_\star^{-6.4}(t,u), \tag{16.0.67c}$$

$$\left\| \upmu Y^{20} \mathrm{tr}_{\not{g}} \chi \right\|_{L^2(\Sigma_t^u)}, \left\| \upmu Y^{19} \breve{X} \mathrm{tr}_{\not{g}} \chi \right\|_{L^2(\Sigma_t^u)} \leq C \mathring{\epsilon} \upmu_\star^{-5.9}(t,u). \tag{16.0.67d}$$

What happens in Case II). *In case* ***II)****, all bootstrap assumptions, the estimates of Props. 9.12 and 10.1, and the energy estimates of Prop. 15.1 hold on* $\mathcal{M}_{T_{(Lifespan);U_0},U_0}$ *with all factors* ε *on the RHS of all inequalities replaced by* $C\mathring{\epsilon}$*. Moreover, for* $0 \leq K \leq 5$*, the estimates* (16.0.67a)-(16.0.67d) *hold for* $(t,u) \in [1, T_{(Lifespan);U_0}) \times [0, U_0]$*. In addition, the scalar-valued functions* $\mathscr{Z}^{\leq 11;\leq 2}\vec{\Psi}$*,* $\breve{X}\breve{X}\breve{X}\vec{\Psi}$*,* $\mathscr{Z}^{\leq 10;\leq 2}L^i$*,* $\mathscr{Z}^{\leq 11;\leq 1}\upmu$*, and* $\breve{X}\breve{X}\upmu$ *extend to* $\Sigma_{T_{(Lifespan);U_0}}^{U_0}$ *as functions of the geometric coordinates* (t,u,ϑ) *that are uniformly bounded in* L^∞*. Furthermore, the Cartesian component functions* $g_{\alpha\beta}(\vec{\Psi})$ *verify the estimate* $g_{\alpha\beta} = m_{\alpha\beta} + \mathcal{O}(\mathring{\epsilon})$ *(where* $m_{\alpha\beta} = \mathrm{diag}(-1,1,1)$ *is the standard Minkowski metric) and have the same extension properties as* $\vec{\Psi}$ *and its derivatives with respect to the vectorfields mentioned above.*

Moreover, let $\Sigma_{T_{(Lifespan);U_0}}^{U_0;(Blowup)}$ *be the (non-empty) subset of* $\Sigma_{T_{(Lifespan);U_0}}^{U_0}$ *defined by*

$$\Sigma_{T_{(Lifespan);U_0}}^{U_0;(Blowup)} := \left\{ (T_{(Lifespan);U_0}, u, \vartheta) \mid \upmu(T_{(Lifespan);U_0}, u, \vartheta) = 0 \right\}. \tag{16.0.68}$$

Then for each point $(T_{(Lifespan);U_0}, u, \vartheta) \in \Sigma_{T_{(Lifespan);U_0}}^{U_0;(Blowup)}$*, there exists a past neighborhood containing it such that the following lower bound holds in the neighborhood:*

$$\left| X \uprho(t,u,\vartheta) \right|, \left| X v^1(t,u,\vartheta) \right| \geq \frac{\mathring{\delta}_*}{8\left| \bar{c}_s{}' + 1 \right|} \frac{1}{\upmu(t,u,\vartheta)}. \tag{16.0.69}$$

In (16.0.69)*,* $\frac{\mathring{\delta}_*}{8\left| \bar{c}_s{}' + 1 \right|}$ *is a* ***positive*** *data-dependent constant (see* (16.0.64)*), and the* $\ell_{t,u}$*-transversal vectorfield* X *is near-Euclidean-unit length:* $\delta_{ab}X^aX^b = 1 + \mathcal{O}(\mathring{\epsilon})$*. In particular,* $X\uprho$ *and* Xv^1 *blow up like* $1/\upmu$ *at all points in* $\Sigma_{T_{(Lifespan);U_0}}^{U_0;(Blowup)}$*. Conversely, at all points in* $(T_{(Lifespan);U_0}, u, \vartheta) \in \Sigma_{T_{(Lifespan);U_0}}^{U_0} \backslash \Sigma_{T_{(Lifespan);U_0}}^{U_0;(Blowup)}$*, we have*

$$\left| X \uprho(T_{(Lifespan);U_0}, u, \vartheta) \right|, \left| X v^1(T_{(Lifespan);U_0}, u, \vartheta) \right| < \infty. \tag{16.0.70}$$

Proof. Let $C' > 1$ be a constant (we will enlarge it as needed throughout the proof). We define

$$T_{(Max);U_0} := \text{ The supremum of the set of times } T_{(Boot)} \in [0, 2\mathring{\delta}_*^{-1}] \text{ such that:} \tag{16.0.71}$$

- $\vec{\Psi}$, ω, u, μ, $L^i_{(Small)}$, and all of the other quantities defined throughout the article exist classically on $\mathcal{M}_{T_{(Boot)},U_0}$.
- The change of variables map Υ from Def. 3.12 is a (global) $C^{1,1}$ diffeomorphism from $[0, T_{(Boot)}) \times [0, U_0] \times \mathbb{T}$ onto its image $\mathcal{M}_{T_{(Boot)},U_0}$.
- $\inf \left\{ \mu_\star(t, U_0) \mid t \in [0, T_{(Boot)}) \right\} > 0$.
- The fundamental L^∞ bootstrap assumptions $(\mathbf{BA}\vec{\Psi})$ and $(\mathbf{BA}\omega)$ hold with $\varepsilon := C'\mathring{\epsilon}$ for $(t, u) \in \times[0, T_{(Boot)}) \times [0, U_0]$.
- The following L^2-type energy bounds hold for $(t, u) \in \times[0, T_{(Boot)}) \times [0, U_0]$:

$$\mathbb{Q}^{1/2}_{15+K}(t,u) + \mathbb{K}^{1/2}_{15+K}(t,u) \leq C'\mathring{\epsilon}\mu_\star^{-(K+.9)}(t,u), \qquad (0 \leq K \leq 5), \tag{16.0.72}$$

$$\mathbb{Q}^{1/2}_{[1,14]}(t,u) + \mathbb{K}^{1/2}_{[1,14]}(t,u) \leq C'\mathring{\epsilon}, \tag{16.0.73}$$

$$\mathbb{V}^{1/2}_{21}(t,u) \leq C'\mathring{\epsilon}\mu_\star^{-6.4}(t,u), \tag{16.0.74}$$

$$\mathbb{V}^{1/2}_{16+K}(t,u) \leq C'\mathring{\epsilon}\mu_\star^{-(K+.9)}(t,u), \qquad (0 \leq K \leq 4), \tag{16.0.75}$$

$$\mathbb{V}^{1/2}_{\leq 15}(t,u) \leq C'\mathring{\epsilon}. \tag{16.0.76}$$

It is a standard result that if $\mathring{\epsilon}$ is sufficiently small and C' is sufficiently large, then $T_{(Max);U_0} > 0$ (this is a standard local well-posedness result combined with the initial smallness of the L^2-controlling quantities obtained in Lemma 15.2).

We now show that the energy bounds (16.0.72)-(16.0.76) and the fundamental L^∞ bootstrap assumption $(\mathbf{BA}\vec{\Psi})$ and $(\mathbf{BA}\omega)$ are not saturated for $(t, u) \in [1, T_{(Max);U_0}) \times [0, U_0]$. The non-saturation of the energy bounds (for C' sufficiently large) is provided by Prop. 15.1. The non-saturation of the fundamental L^∞ bootstrap assumptions $(\mathbf{BA}\vec{\Psi})$ then follows from Cor. 13.2. Consequently, we conclude that all of the estimates proved throughout the article hold on $\mathcal{M}_{T_{(Boot)},U_0}$ with the smallness parameter ε replaced by $C\mathring{\epsilon}$. We use this fact throughout the remainder of the proof without further remark.

Next, we show that (16.0.67a)-(16.0.67d) hold for $(t, u) \in [1, T_{(Max);U_0}) \times [0, U_0]$. To obtain (16.0.67a)-(16.0.67c), we insert the energy estimates of Prop. 15.1 into the RHS of the inequalities of Lemma 15.6 and use inequalities (11.3.3) and (11.3.6) as well as the fact that $\mathbb{Q}_{[1,M]}$ is increasing in its arguments. To obtain inequality (16.0.67d), we also insert the energy estimates of Prop. 15.1 into RHS (15.7.3) and use inequality (11.3.3).

We now establish the dichotomy of possibilities. We first show that if

$$\inf \left\{ \mu_\star(t, U_0) \mid t \in [1, T_{(Max);U_0}) \right\} > 0,$$

then $T_{(Max);U_0} = 2\mathring{\delta}_*^{-1}$. This fact can be established using the same arguments given in the proof of [30, Theorem 15.1] (for $\mathring{\epsilon}$ sufficiently small), which were based on analogs of the fundamental L^∞ bootstrap assumptions (now known to be non-saturated) and the L^∞ estimates of Props. 9.12 and 10.1. We will not repeat the (straightforward but tedious) proof here; we note only that the above assumption for $\upmu_\star$ can be combined with other simple estimates to yield that Υ extends as a global $C^{1,1}$ diffeomorphism from $[1, T_{(Max);U_0}] \times [0, U_0] \times \mathbb{T}$ onto its image and moreover, that neither the solution nor its derivatives can blow up with respect to geometric or Cartesian coordinates for times in $[0, 2\mathring{\updelta}_*^{-1}]$. We have thus shown that **I)** $T_{(Max);U_0} = 2\mathring{\updelta}_*^{-1}$ or **II)** $\inf \left\{ \upmu_\star(t, U_0) \mid t \in [1, T_{(Max);U_0}) \right\} = 0$.

We now show that case **II)** corresponds to a singularity and that the classical lifespan is characterized by (16.0.65). To this end, we first use (9.6.4), (10.2.7), (11.2.2), and the identity $\breve{X} = \upmu X$ to deduce that inequality (16.0.69) holds. Furthermore, from (3.5.1)-(3.5.3), (3.19.2c), and the L^∞ estimates of Prop. 9.12, we deduce that $|X| := \sqrt{g_{ab} X^a X^b} = 1 + \mathrm{f}(\upgamma)\upgamma = 1 + \mathcal{O}(\mathring{\upepsilon})$. From this estimate and (16.0.69), we deduce that at points in $\Sigma_{T_{(Max);U_0}, U_0}$ where $\upmu$ vanishes, $|X\Psi|$ must blow up like $1/\upmu$. Hence, $T_{(Max);U_0}$ is the classical lifespan. That is, we have $T_{(Max);U_0} = T_{(Lifespan);U_0}$ as well as the characterization (16.0.65) of the classical lifespan. The estimate (16.0.70) is an immediate consequence of the estimates (9.6.3b)-(9.6.3c) and the identity $\breve{X} = \upmu X$.

To obtain (16.0.66), we use (11.2.5a) and (11.2.6b) to deduce that $\upmu_\star(t, 1)$ vanishes for the first time when $t = \mathring{\updelta}_*^{-1} + \mathcal{O}(\mathring{\upepsilon})$.

We now derive the statements regarding the quantities that extend to $\Sigma_{T_{(Lifespan);U_0}}^{U_0}$ as L^∞ functions. Let q denote any of the quantities $\mathscr{Z}^{\leq 11;\leq 2}\Psi, \cdots, \breve{X}\breve{X}\upmu$ that, in the theorem, are stated to extend to $\Sigma_{T_{(Lifespan);U_0}}^{U_0}$ as an L^∞ function of the geometric coordinates. The L^∞ estimates of Props. 9.12 and 10.1 imply that $\|Lq\|_{L^\infty(\Sigma_t^{U_0})}$ is uniformly bounded for $0 \leq t < T_{(Lifespan);U_0}$. Recalling that $L = \frac{\partial}{\partial t}$, we conclude that q extends to $\Sigma_{T_{(Lifespan);U_0}}^{U_0}$ as an element of $L^\infty(\Sigma_{T_{(Lifespan);U_0}}^{U_0})$ as desired. The estimate $g_{\alpha\beta}(\vec{\Psi}) = m_{\alpha\beta} + \mathcal{O}(\mathring{\upepsilon})$ and the extension properties of the $\mathscr{Z}$-derivatives of the scalar-valued functions $g_{\alpha\beta}(\vec{\Psi})$ then follow from (3.5.1), the already proven bound $\|\vec{\Psi}\|_{L^\infty(\Sigma_t^{U_0})} \lesssim \mathring{\upepsilon}$, and the extension properties of the $\mathscr{Z}$-derivatives of $\vec{\Psi}$ obtained just above. This completes the proof of the theorem. $\square$

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