Median and mode in first passage under restart

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Restart—interrupting a stochastic process followed by a new start—is known to improve the mean time to its completion, and the general conditions under which such an improvement is achieved are now well understood. Here, we explore how restart affects other important metrics of first-passage phenomena, namely, the median and the mode of the first-passage time distribution. Our analysis provides a general criterion for when restart lowers the median time and demonstrates that restarting is always helpful in reducing the mode. Additionally, we show that simple nonuniform restart strategies allow to optimize the mean and the median first-passage times, regardless of the characteristic timescales of the underlying process. These findings are illustrated with the canonical example of a diffusive search with resetting.

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I. INTRODUCTION

The *mean first-passage time* is widely used to quantify performance in diverse applications ranging from randomized search algorithms to kinetics of chemical reactions [1]. Remarkably, this metrics can be significantly improved by implementation of *restart*, i.e., by interrupting the first-passage process just to start it anew. Speedup by restart was first noted in computer science more than two decades ago [2] with a new wave of current interest triggered by the seminal work of Evans and Majumdar [3] who demonstrated that stochastic resetting hastens diffusive search. More recently, the development of a general renewal approach has provided a unified and model-independent treatment of first passage under restart [4–6]. In particular, it has furnished a simple criterion for when restart helps to lower the expected completion time of first-passage processes and revealed universality in the behavior of the optimally restarted processes [5].

Although the mean completion time plays a central role for some applications, in many other settings, it does not capture the relevant timescale of the task, and other metrics may be more appropriate to quantify performance. More specifically, there are cases when the *median* time should be used instead of the mean. For example, in an enzymatic reaction with an excess of substrate molecules [see Fig. 1(b)], the time taken for a significant change in concentration of the substrate is long compared with the expected catalysis time, and thus, the latter is a more natural measure of how fast the reaction proceeds. Indeed, in the limit of large substrate concentration, the number of substrate molecules converted to products in a unit volume per second is proportional to inverse mean catalysis time [7,8]. However, in the opposite case when concentration of substrate molecules is low compared to that of the enzyme molecules [Fig. 1(c)], one deals with the highly nonstationary situation characterized by fast substrate depletion [9–11], and the mean catalysis time may be not informative. The natural metric of the reaction speed in such a nonstationary situation is given by the median turnover time—the time required to convert half of the initial amount of substrate into product. Also, in the contexts of randomized search algorithms and Internet tasks, one may be interested in the typical behavior captured by the median completion time rather than in the average values which may be dominated by rare but extreme runs [2,12–14].

Here, we analyze the effect of restart on the median time of a generic first-passage process. Namely, our analysis provides a condition for when the introduction of a small restart rate reduces the median first-passage time, or more generally, a given *quantile* of the first-passage-time density (FPTD). Using diffusive search as an illustrative example, we show that, similar to the mean first-passage time, the median can be optimized by a careful choice of the restart rate. Since the optimal restart rate is determined by the timescale of the first-passage process, which is often unknown *a priori*, we also explore restart protocols whose performance is weakly sensitive to such details. In addition, we describe the effect of restart on the *mode* of the first-passage time distribution, i.e., on the value of the process completion time which occurs most often. It turns out that in contrast to the mean and median, which may increase or decrease in response to the introduction of restart (depending on process details), this metric cannot be increased by restart.

II. EFFECT OF RESTART ON QUANTILES OF THE FPTD

Consider a first-passage time process characterized by the random completion time T with the FPTD $P(T)$. By definition, the qth quantile is the time T_q such that the process has the probability $0 < q < 1$ to finish before T_q. Say, for $q=1/2$, $T_{1/2}$ represents the median first-passage time. Clearly, this

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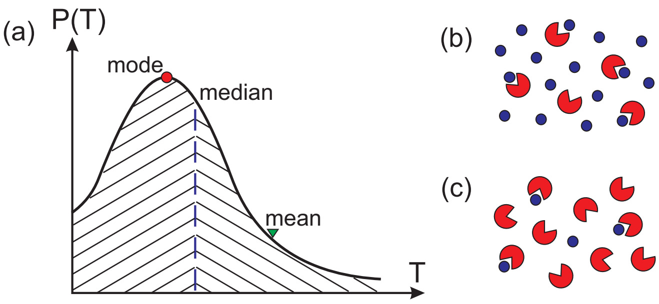

FIG. 1. (a) The mean $\langle T\rangle$, median $T_{1/2}$, and mode T_m of some first-passage-time probability distribution $P(T)$. (b) Enzymatic reaction with an excess of substrate (schematically shown in blue). The reaction speed is determined by the mean turnover time. (c) Enzymatic reaction with excess enzyme (schematically shown in red); the relevant timescale of the reaction is the median turnover time.

quantity obeys the following integral equation,

$$\int_0^{T_q} dT\, P(T) = q. \tag{1}$$

Now, let us assume that the process becomes subject to stochastic restart at the infinitesimally small rate $\delta r \to +0$. How does this affect the qth quantile of this process? In the presence of restart, Eq. (1) takes the form $\int_0^{T_q+\delta T_q} dT\, P_{\delta r}(T) = q$, where δT_q represents the change in the qth quantile and $P_{\delta r}(T)$ is the FPTD modified by the restart. It is straightforward to show that

$$\delta T_q \approx -\frac{1}{P(T_q)}\int_0^{T_q} dT[P_{\delta r}(T) - P(T)]\,. \tag{2}$$

To proceed, we need to know the difference $P_{\delta r}(T) - P(T)$ which determines the response of the FPTD to the introduction of rare Poisson restarts. As shown in Refs. [4,5], the Laplace transform of the FPTD $\tilde{P}_r(s) = \int_0^\infty dT\, e^{-sT} P_r(T)$ of any stochastic process under constant restart rate r is given by

$$\tilde{P}_r(s) = \frac{\tilde{P}(s+r)}{\frac{s}{s+r} + \frac{r}{s+r}\tilde{P}(s+r)}. \tag{3}$$

In the limit of small restart rate δr, this equation yields

$$\tilde{P}_{\delta r}(s) \approx \tilde{P}(s) + \left(\frac{\partial \tilde{P}(s)}{\partial s} - \frac{1}{s}\tilde{P}(s)^2 + \frac{1}{s}\tilde{P}(s)\right)\delta r. \tag{4}$$

Using the identities,

$$\int_0^\infty dT\, P(T) T e^{-sT} = -\partial_s \tilde{P}(s), \tag{5}$$

$$\int_0^\infty dT\, e^{-sT}\int_T^{+\infty} dt\, P(t) = \frac{1-\tilde{P}(s)}{s}, \tag{6}$$

$$\int_0^\infty dT\, e^{-sT}\int_T^{+\infty} dt\, P_2(t) = \frac{1-\tilde{P}^2(s)}{s}, \tag{7}$$

where $P_2(t) \equiv \langle \delta(T_1 + T_2 - t)\rangle_{T_1,T_2} = \int_0^t d\tau\, P(\tau)P(t-\tau)$ is the probability distribution for the sum of two independent variables T_1 and T_2 sampled from $P(T)$, we perform the inverse Laplace transform to obtain to the linear order in δr,

$$P_{\delta r}(T) - P(T) \approx \left(\int_T^\infty dt[P_2(t) - P(t)] - P(T)T\right)\delta r. \tag{8}$$

Next, substitution of Eq. (8) into Eq. (2) gives

$$\delta T_q = -\frac{\delta r}{P(T_q)}\int_0^{T_q} dT\left(\int_T^\infty dt[P_2(t) - P(t)] - P(T)T\right). \tag{9}$$

It is easy to check that Eq. (9) gives $\delta T_q = 0$ when $P(T) = \alpha e^{-\alpha T}$. This is in accord with the general argument that restart does not affect the completion of the first-passage-time processes with exponential statistics due to their memoryless property.

From Eq. (9), we may conclude that a rigorous criterion of when restart reduces the qth quantile (i.e., when $\delta T_q < 0$) is provided by the inequality,

$$\int_0^{T_q} dT\left(P(T)T + \int_0^T dt[P_2(t) - P(t)]\right) < 0, \tag{10}$$

which after integration by part becomes

$$2\langle T\rangle_{T_q} - \langle T_1 + T_2\rangle_{T_q} - T_q q + T_q K_2(T_q) < 0, \tag{11}$$

where $\langle T\rangle_{T_q} = \int_0^{T_q} dT\, P(T)T$ and $\langle T_1 + T_2\rangle_{T_q} = \int_0^{T_q} dT\, P_2(T)T$ are the truncated moment, and $K_2(T) = \int_0^{T_q} dT\, P_2(T)$ is the cumulative distribution function of the random variable $T_1 + T_2$. Due to generality of the framework adopted here, this result is applicable to any first-passage process including the situations when the process is already subject to some restart mechanism.

Given the FPTD of any process of interest, one can then readily check if the inequality in Eq. (11) is fulfilled. Generally, this should be performed numerically, but the limiting cases of $q \to 0$ and $q \to 1$ allow the analytical exploration. Let us first investigate the effect of restarting on the small-q quantiles. Assuming that $P(T) \approx CT^m$ at $T \to 0$, where $m > -1$ and $C > 0$, we immediately find $\langle T\rangle_{T_q} = \frac{C}{m+2}T_q^{m+2}$, $\langle T_1 + T_2\rangle_{T_q} = \frac{C^2\Gamma^2(m+1)}{(2m+3)\Gamma(2m+2)}T_q^{2m+3}$, $q = \frac{C}{m+1}T_q^{m+1}$, $K_2(T_q) = \frac{C^2\Gamma^2(m+1)}{(2m+2)\Gamma(2m+2)}T_q^{2m+2}$, and, therefore, Eq. (11) is satisfied only if $m < 0$. Thus, restart inhibits the short-time completion of a first-passage-time process unless the FPTD $P(T)$ is singular at $T = 0$. In the opposite limit, the response of the large-q quantiles to restart is determined by the behavior of the tails of the FPTD. Assume that we deal with the heavy-tailed first-passage-time process having finite mean completion time $\langle T\rangle$, i.e., $P(T) \approx \frac{C}{T^m}$ at $T \to \infty$, where $m > 2$ and $C > 0$. Then, $\langle T\rangle_{T_q} = \langle T\rangle - \frac{C}{m-2}T_q^{2-m}$, $\langle T_1 + T_2\rangle_{T_q} = 2\langle T\rangle - \frac{2C}{m-2}T_q^{2-m}$, $q = 1 - \frac{C}{m-1}T_q^{1-m}$, $K_2(T_q) = 1 - \frac{2C}{m-1}T_q^{1-m}$ and, thus, according to Eq. (11), restart reduces the large-q quantiles of the heavy-tailed FPTD. The opposite effect is observed in the case of the fast decaying FPTD. As an example, let us consider the first-passage-time density with the Gaussian tail: $P(T) \approx C\exp(-\frac{T^2}{T_0^2})$ at $T \to \infty$. Then, $\langle T\rangle_{T_q} \approx \langle T\rangle - \frac{CT_0^2}{2}\exp(-\frac{T_q^2}{T_0^2})$, $\langle T_1 + T_2\rangle_{T_q} \approx 2\langle T\rangle - \sqrt{\frac{\pi}{2}}C^2T_0^3\exp(-\frac{T_q^2}{2T_0^2})$, $q = 1 - \frac{\sqrt{\pi}}{2}CT_0[1 - \mathrm{erf}(\frac{T_q}{T_0})]$, $K_2(T_q) = 1 - \frac{\pi}{2}C^2T_0^2[1 - \mathrm{erf}(\frac{T_q}{2T_0})]$, and Eq. (11) is not fulfilled.

It is worth noting that the above analysis is also relevant to the "deadline meeting problem" [12–15]. The probability that a first-passage process having the FPTD $P(T)$ will finish, before the prescribed deadline T_d has passed, is determined by $p_d = \int_0^{T_d} P(T) dT$. The variational calculus based on Eq. (8) shows that, when the process is subjected to a small restart rate δr, the deadline meeting probability obtains a correction $\delta p_d = (\int_0^{T_d} dT \int_0^T dt[P(t) - P_2(t)] - \int_0^{T_d} dT\, P(T)T)\delta r$. Therefore, restart helps to increase the chance to meet deadline whenever $\int_0^{T_d} dT(P(T)T + \int_0^T dt[P_2(t) - P(t)]) < 0$ or, equivalently, $2\langle T\rangle_{T_d} - \langle T_1 + T_2\rangle_{T_d} - T_d p_d + T_d K_2(T_d) < 0$.

III. NUMERICAL RESULTS FOR ONE-DIMENSIONAL BROWNIAN SEARCH

For the sake of illustration, let us next consider a one-dimensional Brownian motion in search of an immobile (absorbing) target. In this case, the first-passage time density is given by the Levy-Smirnov distribution: $P(T) = \frac{L}{\sqrt{4\pi DT^3}} \exp{(-\frac{L^2}{4DT})}$, where L is the initial distance to the target and D is the diffusion coefficient. It is straightforward to verify that the qth quantile of this probability density is $T_q = \frac{L^2}{4D[\text{erfc}^{-1}(q)]^2}$. Substituting these expressions for $P(T)$ and T_q into Eq. (10), and performing integration numerically, we find that the inequality is satisfied for $q > q_0$, where $q_0 \approx 0.4123$. Thus, the introduction of restart, which occasionally returns the particle to its initial position, reduces the median completion time $T_{1/2}$ of diffusive search. Stochastic simulations indicate that the median time attains a minimum at the optimal restart rate $r_0 \approx 1.5D/L^2$ which is smaller than the rate $r^* \approx 2.5D/L^2$ minimizing the mean search time [3]. Clearly, since restart works by avoiding the tail of the FPTD, it becomes more potent for larger q as visible in Fig. 2(a). We also note in passing that restart decreases the deadline meeting probability of diffusive search for sufficiently short deadlines $T_d < T_{d_0}$ whereas increasing it for $T_d > T_{d_0}$, where $T_{d_0} \approx 0.7439L^2/D$ [see Fig. 2(b)]. This is in accord with the analysis reported in Ref. [15] where restart is shown to increase the chance of finding a target in the presence of sufficiently small mortality rate, whereas reducing this chance if the mortality rate is large.

Clearly, to be effective, restart requires prior knowledge of the characteristic timescale of the underlying process. In the case of the median search time for diffusion, the relevant rates are measured with respect to the (inverse) diffusive timescale $\tau_{\text{dif}} = L^2/D$. Restart rates chosen without taking this characteristic time into account may well lead to performance that is worse than without any restart. Previously, a similar challenge motivated the development of the restart strategies improving the mean first-passage time without introducing any timescale [2,16]. The strategy proposed in Ref. [16] is make restarts separated by random scale-free time intervals. We numerically investigated the effect of such stochastic scale-free restarts on the quantiles of the diffusive search. Specifically, we implement a nonuniform restart rate which is inversely proportional to the time elapsed since the start of the process, i.e., $r(t) = \frac{\alpha}{t}$, where α is a dimensionless constant. Figure 3 demonstrates that such a nonuniform restart protocol allows to minimize the large-q quantiles of the FPTD without being sensitive to the parameters of the problem. Indeed, the quantiles attain extrema at the optimal values α_q^* which do not

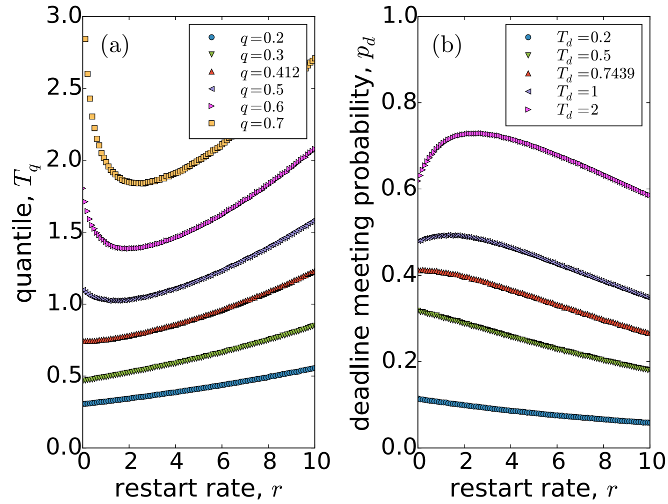

FIG. 2. (a) The qth quantile T_q of the search time distribution as a function of the rate r for Poisson restarts of a one-dimensional diffusive search for an immobile target with $D = 1$ and $L = 1$. We see that, in agreement with theoretical predictions extracted from numerical analysis of Eq. (10), restart is helpful for $q > q_0 \approx 0.412$. (b) The deadline meeting probability p_d as a function of the rate r of Poisson restarts for a one-dimensional diffusive search for an immobile target with $D = 1$ and $L = 1$. Restart is helpful for a sufficiently large time margin $T_d > T_{d_0} \approx 0.7439L^2/D$. The numerical data are based on statistics including 10^6 independent runs.

depend on the diffusive time τ_{dif} in contrast to the optimal rate of uniform restart which scales proportionally to τ_{dif}.

We also propose and explore a nonuniform deterministic restart protocol with restart times chosen from a geometric sequence, i.e., with the nth restart event at $t_n = \tau\beta^{n-1}$, where $\tau \ll \tau_{\text{dif}}$ is a microscopic cutoff, and β is a dimensionless

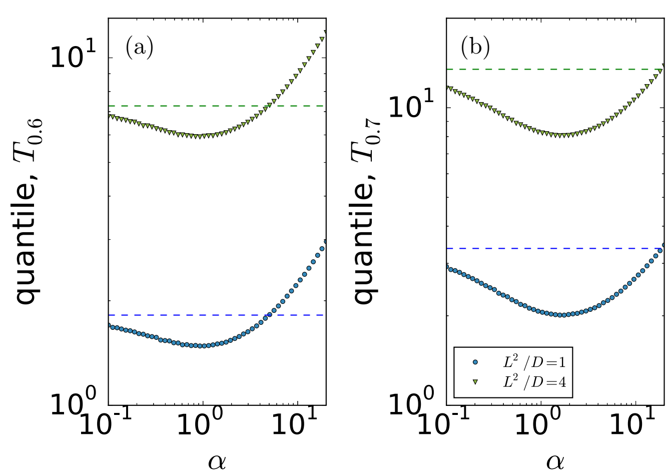

FIG. 3. Dependence of the quantiles of the search time distribution on the dimensionless parameter α of the stochastic scale-free restart protocol for different values of the diffusive timescale L^2/D. The dashed lines represent the unperturbed quantiles in the absence of restart. In numerical simulations, we set a short time cutoff $\tau = 10^{-3}$ to regularize the divergence of the restart rate at $t = 0$ and collected statistics from 10^6 independent realizations of the search process.

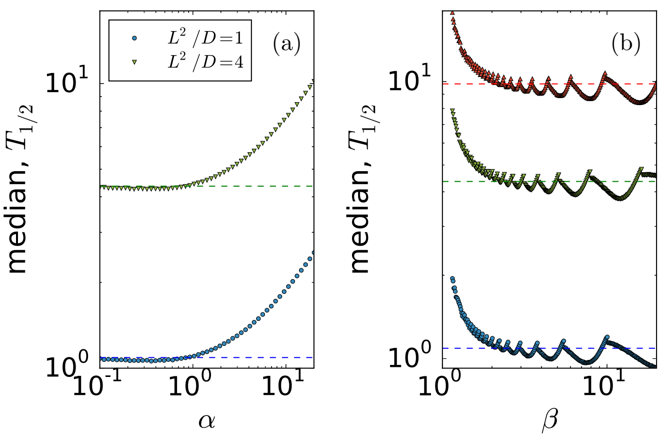

FIG. 4. The median $T_{1/2}$ of the search time distribution in the presence of stochastic scale-free (a), and the geometric (b) restart protocols, for different values of the diffusive timescale L^2/D. The dashed lines represent median search times in the absence of restart.

constant. In a long run, the time interval between successive restarts approaches the elapsed time since the start of the process so that there is no characteristic restart frequency. As depicted in Fig. 4(b), we find that T_q obtains an oscillatory dependence on β, which renders practical implementation of this strategy of quantile optimization problematic. Note, however, that the geometric restart protocol is actually quite efficient for reducing the mean search time, see Fig. 5. What is more, the performance of the geometric restart in terms of the MFPT is better than that of the above-mentioned stochastic scale-free strategy. The former achieves a minimal mean search time of $\langle T_{\beta^*}\rangle \approx 1.8\tau_{\text{dif}}$ at the optimal value of $\beta_* \approx 1.6$, that is, not sensitive to the diffusive time, whereas the latter gives $\langle T_{\alpha^*}\rangle \approx 1.97\tau_{\text{dif}}$ at $\alpha_* \approx 3.5$ [16].

IV. EFFECT OF RESTART ON THE MODE OF THE FPTD

Finally, let us discuss another interesting metric of first-passage processes: The mode of the FPTD, i.e., the time T_m at which the probability distribution $P(T)$ takes its maximum

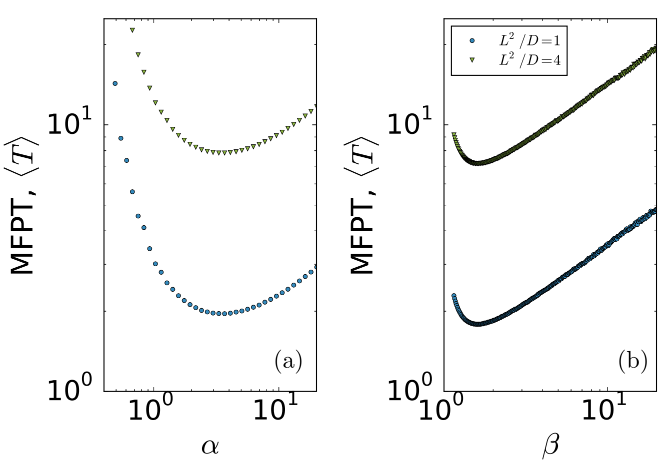

FIG. 5. The mean search time $\langle T\rangle$ in the presence of stochastic scale-free (a) and the geometric (b) restart protocols for different values of the diffusive timescale L^2/D.

value. In other words, T_m is the value of the completion time T that occurs most often, which must, thus, satisfy $\frac{dP(T)}{dT}\big|_{T=T_m}=0$, together with $\frac{d^2P(T)}{dT^2}\big|_{T=T_m}<0$. In the presence of restarts at the small rate δr, we have $\frac{dP_{\delta r}(T)}{dT}\big|_{T=T_m+\delta T_m}=0$, leading to

$$\delta T_m = -\frac{1}{\frac{d^2P(T)}{dT^2}\big|_{T=T_m}}\frac{d}{dT}[P_{\delta r}(T)-P(T)]\bigg|_{T=T_m}. \qquad (12)$$

Next, using Eq. (8), one obtains

$$\delta T_m = \frac{\delta r}{\frac{d^2P(T)}{dT^2}\big|_{T=T_m}}\int_0^{T_m} d\tau\, P(\tau)P(T_m-\tau). \qquad (13)$$

Since $\frac{d^2P(T)}{dT^2}|_{T=T_m}<0$ and $\int_0^{T_m} d\tau\, P(\tau)P(T_m-\tau)=P_2(T_m)\geqslant 0$, we immediately find that $\delta T_m \leqslant 0$. Thus, introduction of restart decreases or leave unchanged the mode of any FPTD. Although this conclusion is based on the assumption of the infinitesimally small restart rate, the same result remains valid for any FPTD. Indeed, based on the splitting rule of the Poisson process, we can safely assume that $P(T)$ in the above formulas represents the FPTD of the process that is already subject to restart at some nonvanishing rate. Then Eq. (13) indicates that the mode of this process is a nonincreasing function of the restart rate r.

V. CONCLUSION AND OUTLOOK

In conclusion, the effectiveness of restarts in reducing the *mean first-passage time* has been demonstrated in a number of studies [3–6,16–32]. However, less has been known about how restart affects other characteristic time metrics of the first-passage completion. To fill this gap, we have explored the advantages of restarting to optimization of the median and mode of a generic first-passage-time density.

The ubiquity of restarts in natural and artificial systems encourages us to think that ideas presented here will find diverse applications. Say, in the context of enzymatic reactions, restarts correspond to unbinding of substrate from enzyme prior to completion of the catalytic step. Indeed, a previous analysis in the limit of large substrate concentration has shown that an increase in the substrate unbinding rate can accelerate the reaction [7]. Our results suggest that a similar effect can potentially be achieved in the opposite limit of large enzyme concentration when the reaction speed is determined by the substrate "half-life" as mentioned in the introductory part of this paper.

A range of issues call for further theoretical investigation. As proven in Refs. [2,6], deterministic (regular) restart is the universal winning strategy in the problems when one needs to minimize the mean completion time of randomized task. It remains unclear if the deterministic restart protocol is also universally optimal for the median time optimization. Another unsettled issue is if there are any rigorous bounds on the performance of the geometric restart protocol and the nonuniform scale-free stochastic restart similar to those previously obtained for the Luby's universal strategy in computer science applications [2].

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