MATRIX OPTICS

by

NATHAN GRIER PARKE III

A.B. Princeton University

(1934)

SUBMITTED IN PARTIAL FULFILLMENT OF THE

REQUIREMENTS FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

(1948)

Signature of Author ______________________
Department of Physics, May 1, 1948

Certified by ______________________
Thesis Supervisor

____
Chairman, Department Committee on Graduate Studies

ACKNOWLEDGEMENTS

First and foremost, I owe a heavy intellectual debt to Prof. Hans Mueller. During my stay at M.I.T. I attended four different courses given by him from which I have drawn much inspiration and knowledge. In the spring of 1946 he suggested the thesis subject and since then has been a constant source of encouragement and guidance.

Secondly I wish to express my appreciation of the Research Associateship in the Research Laboratory of Electronics and for the time available for this research. Without these it would have been financially impossible for me to have completed the work for my degree.

TABLE OF CONTENTS

ABSTRACT

Mueller's phenomenological theory of optics is framed in matrix-vector language, i.e., Stokes' vectors L, L' represent the incident and emergent radiation and Mueller Matrix M represents the "instrument." The basic relation between L and L' is L' = ML. Mueller has given operational definitions of L, L' and M. This allows a comparison between atomic theory and macroscopic experiment, if one can compute L, L' and M theoretically. The general technique for such computations is the central problem of our thesis.

Jones had developed the elementary aspects of a matrix theory of optics. In the first chapter we have refined and extended this theory to the point where it is now capable of representing a statistical assemblage of scattering centers (spheres with propagation constant k, or molecules.) The resulting theory will be called generalized Jones algebra.

Wiener had developed his generalized harmonic analysis and defined the notion of a coherency matrix of a set of functions, $f_1(t),\ldots,f_n(t)$. In chapter II we show the relation between the coherency matrix of the electric vector and the Stokes vector L, used by Mueller. With this clue we discovered that we could use the Wiener correlation process, i.e., the first step of his generalized harmonic analysis, to pass from generalized Jones algebra to a

theoretical construct which may be identified with Mueller's phenomenological theory. This construct is called Mueller's generalized algebra.

In the refinement and generalization of Jones work it was necessary to consider random functions of time and generalized harmonic analysis of same. This led to a bifurcation of both Jones and Mueller algebra, depending on whether frequency or time are taken as the basic variables. Thus one speaks of t-algebras and ω-algebras of the Mueller or Jones types. Jones algebras are formulated in terms of theoretical quantities coming either from classical electrodynamics and classical statistical mechanics or from quantum statistical electrodynamics. We have not explored this latter possibility. Mueller algebras are formulated in terms of experimentally observable quantities.

Chapter II contains the intuitive material which is necessary for the interpretation and application of our theory.

Chapter III reinterprets the Wiener correlation as a phase average. It was originally defined as a time average. It is an ergodic theorem that the two kinds of averages are identical almost always. This new viewpoint is applied to give a statistical interpretation of the Stokes vector L of a partially polarized wave of the radiation from a single spectral line. The result is a new and interesting view of the nature of partial polarization and to a definition of the notion of partial coherence and its interpretation in terms of an interference experiment.

In chapter IV we turn from radiation statistics to the statistics of a set of scattering centers. We confine ourselves to a quasi-stationary problem, i.e., to the cases in which the time average of the positions of the particles can be computed independently of the Wiener correlation of the resulting radiation. This leads to the definition of a general rule for the addition of mueller matrices. In fact it leads to the extension of the notion of a Mueller matrix from auto-matrices, correlated with a single scattering center i, to cross-matrices, correlated with a pair of scattering centers i and j; thus

$$\mathfrak{M}^{ij} = \begin{matrix} \text{auto-matrices, } i=j \\ \text{cross-matrices, } i\neq j. \end{matrix}$$

We then show that there exists a function N_{ij} such that

$$\mathfrak{M}(\omega) = \sum_{i,j} N_{ij}(\omega)\,\mathfrak{M}^{ij}(\omega)$$

is the Mueller matrix of the scattering system as a whole. The matrix N_{ij} is sufficiently universal to properly describe the degree of order existing in a range of states: liquids, gases, electrolytes, and crystals. In the case of scattering by liquid elements the N_{ij} is closely related to an expression of Debye's. We have therefore called it the Debye Distribution Function.

As a demonstration of the applicability of our general law of addition we have evaluated N_{ij} for an electrolyte with s species of ions, using the Poisson-Boltzmann distribution. To show the way in which the Jones matrix J is obtained we have given a brief resume of the Mie theory of the J's for the case of scattering by spheres with propogation constant k. In particular we have showed that the relation between the nation used by Stratton and the notation used by LaMer. LaMer's notation is the basis of the MTP tables of scattering factors and Mueller's extension of same. The thesis closes with a few brief illustrative examples, including a means of measuring the mean square valance of the ions of an electrolyte.

LOGICAL INTRODUCTION TO THE PROBLEM

The Problem.

The BASIC OPTICAL PROBLEM in which we are interested may be formulated in the following general terms:

> GIVEN SOME INCIDENT RADIATION AND SOME INTERVENING MATTER, TO PREDICT THE EMERGENT RADIATION.

At the beginning of the thesis research there were in existence three lines of attack on this problem which will be called Jones Algebra[1], Mueller's Algebra[2], and Wiener's Generalized Harmonic Analysis.[3] For reasons that will become

---

1) Jones, R.C. "A New Calculus for the Treatment of Optical Systems." Jour. Opt. Soc. Am. 31 488-503 (1941); 37 107-112 (1947).

2) Mueller, H. "Foundations of Optics." Unpublished Manuscript. Presented at the N.Y., 1948 Meeting of the Optical Society of America.

__________. "Theory of Polarimetric Investigations of Light Scattering." Parts I, II. Contract W-18-035-CWS-1304. D.I.C. 2-6467. M.I.T. (1946-47).

__________. Lectures. Course 8.262. M.I.T. Fall Term (1946-47).

3) Wiener, N. "Generalized Harmonic Analysis." Acta Math. 55 117-258 (1930).

---

evident during the course of the exposition, none of these lines of attack, taken separately, constituted an entirely satisfactory solution of the basic optical problem. In

Jones algebra, the incident and emergent radiations were characterized by vectors E and E' which we shall call Maxwell vectors. The intervening matter was characterized by a 2x2 matrix, J, which we shall call a Jones matrix. The solution of the basic problem was represented by the calculation E' = JE.

In Mueller algebra, the radiation was characterized by vectors L and L' and the intervening matter represented by a 4x4 matrix, M, which we shall call a Mueller matrix. The solution of the basic problem was represented by the calculation L' = ML.

In his generalized harmonic analysis, Wiener introduced a matrix, $S_{ij}(u)$ which he calls a "coherency matrix." It is a more general mathematical entity than E or L and is capable of representing the most general type of radiation.

The THESIS PROBLEM may now be formulated in the following general terms:

> FIND A GENERAL SOLUTION OF THE BASIC OPTICAL PROBLEM. THIS GENERAL SOLUTION SHOULD INCLUDE JONES ALGEBRA, MUELLER ALGEBRA, AND WIENER GENERALIZED HARMONIC ANALYSIS AS ESSENTIAL ELEMENTS, AND SHOW CLEARLY THE ROLE THAT EACH PLAYS. THE SOLUTION SHOULD BE CAPABLE OF HANDLING BY ONE UNIFORM METHOD ANY SPECIAL CASE OF THE BASIC PROBLEM. ITS SUCCESS IS TO BE DEMONSTRATED BY OUTLINING THE METHOD OF HANDLING THE PROBLEM OF SCATTERING OF LIGHT FROM AN ELECTROLYTE IN WHICH THE IONS BUT NOT THE SOLVENT ARE ACTIVE AS SCATTERING ELEMENTS.

The Method.

The METHOD OF ATTACK consists of representing the radiation by a set of elements of a mathematical system and the matter by a group of transformations or operators which act on these elements. This is the act of mapping the physical problem into a mathematical one, the details of which involve physical insight and intuition. The solution then reduces to a mathematical activity. In this light, the choice of mathematical methods is of the essence and it becomes a point of practical importance that there is a sense in which algebra and topology encompasses the entirety of mathematical activity and hence the entirety of the mathematical part of theoretical physics. This is true in the sense that algebra is concerned with suitably restricted finite operations and relations; topology is concerned with suitably restricted infinite operations and relations.[1]

---

1) Tukey, J.W.: "Convergence and Uniformity in Topology," Princeton University Press, 1940. Ch. IX.

Our method of attack consisted, therefore, of making contact as rapidly as possible with current mathematical theory, thus translating the basic optical problem into a mathematical one so framed that it would be easy to use, "chapter and verse," well known general mathematical theorems, definitions and constructions. Having thus established a correspondence, we were able to play physical and mathematical intuition against each other and develop a general tensor algebra. The elements of this are points in Hilbert Space (L_2 functions) or such functions, when considered on a finite interval.

The general algebra was constructed by first making such adjustments in Jones' conception that it became an algebra in the technical sense of the word. The result was a dual algebra, i.e., a pair of equivalent algebras, one in terms of functins of time and the other in terms of functins of frequency. They are related by a Fourier transformation. It was then possible to generalize from an algebra capable only of hangling elementary cases to on e capable of handling the general case by assigning a Jones algebra to each elementary component of the problem and then taking the DIRECT SUM of these component algebras to obtain a GENERALIZED JONES ALGEBRA.

Before continuing the construction, it is necessary to notice that the elements of Jones algebra are theoretical quantities, e.g., functions appearing in Maxwell's equations. The elements of Mueller algebra, on the other hand are observables, i.e., intensities. We need a mapping from theoretical quantities to observables similar to the operation $\int \psi F \psi^* d\tau$ in quantum theory. Wiener generalized harmonic analysis is the required mapping. It appears in our tensor algebra as a generalized contraction called "Wiener correlation." The use of the Wiener correlation depends on the use of the time-version of Jones algebra. In the time-version the operation of "multiplication" is the "convolution." This is the second aspect of our generalized contraction process.

It is now possible to complete the structure of our algebra. By taking the KRONECKER PRODUCT of a generalized Jones algebra with its complex conjugate and CONTRACTING ON TIME, t, one constructs the time-version of a GENERALIZED MUELLER ALGEBRA. The Fourier transform of this Mueller algebra is the required frequency version of GENERALIZED MUELLER ALGEBRA. The algebraic structure which is completed with this step is called GENERALIZED OPTICAL ALGEBRA.

One final remark, Lebesgue integration plays an essential role in the theory because it is the necessary tool for modern statistical theory, Wiener generalized harmonic analysis, the Plancherel theory of Fourier transforms, the Riesz-Fisher theory of Fourier series. It is the natural definition of integral for the class of functions which appear to be required for the representation of radiation and matter.

The Results.

Chapter I is mathematical and presents the essential steps leading to the construction of the theory. Chapter II is physical and presents the intuitive material necessary in order to make connection with previous work and for the interpretation of the mathematics. In Chapter III we introduce the underlying statistical concepts which, up to this point, have remained in the background, although implicit in Wiener generalized harmonic analysis. The ideas are illustrated by considering the statistical representation of a spectral line having collision, natural and Doppler broadening. We arrive at a statistical interpretation of partial polarization. Finally, we combine these results to obtain a statistical interpretation of the meaning of the components of the Stokes vector L. In Chapter IV we take up the statistical properties of the matter through which the radiation passes. In the process we consider the problem of scattering in crystals, liquids, and electrolytes. This leads to a generalization of the notion of a Mueller matrix, to a simple relation between these Mueller matrices and the underlying Jones matrices, and to a general law for the addition of the auto- and cross-Mueller matrices associated with the scattering particles to obtain the resultant Mueller matrix of the system as a whole.

It is for this resultant Mueller matrix that one has the basic law: $L' = ML$. The technique of obtaining the resultant Mueller matrix is illustrated by a detailed consideration of the problem of scattering by an electrolyte.

Conclusion.

The above survey leads us to believe that we have obtained a complete solution of the thesis problem. It appears to us that we have built, on the work of Jones, Mueller, and Wiener, the foundations of a statistical theory of optics which bears to Mueller's phenomenological theory the same relation that statistical mechanics bears to thermodynamics. The practical advantages are the same for both theories, namely the possibility of an a priori computation of the macroscopic quantities in terms of microscopic ones. Such an inference from detailed knowledge to statistical knowledge is called inference in the Abelian direction. An inference in the reverse direction, in the sense of Wiener, is Tauberian. We believe that our generalized algebra is strong enough for the systematic carrying out of Abelian and Tauberian physical inferences.

We wished first to give an introduction which displayed the logical structure of our theory. We shall also give a brief historical review in the first few pages of Chapter I.

CHAPTER I

GENERALIZED OPTICAL ALGEBRA

1.0 Historical Introduction.

The title of the thesis, "Matrix Optics," and the heading of this chapter, "Generalized Optical Algebra," have been chosen to convey the fact that we are describing the results of theoretical research on a basic optical problem in algebraic language. We have also used mathematical techniques from the modern theories of integration, statistics, and harmonic analysis. But, they have been subsumed as definitions of generalized algebraic operations. The practical advantage of "algebraizing" a branch of theoretical physics lies in the resulting systematization and simplification of the calculations which one finds it necessary to make in any application of the thoery. These considerations motivated both Jones and Mueller in the development of their optical algebras. Wiener, on the other hand, was more deeply concerned with interrelations between statistics and harmonic analysis and in the analytic problem of representing "white light." It is the purpose of this chapter to sho the logical interrelation between the works of Jones, Mueller, and Wiener. In the process we shall develop a statistical theory of optics which will form the natural bridge between the microscopic and the macroscopic aspects of the subject.

Although, in a very broad sense, the historical roots of this research lie in the "Traite de la lumiere" of Huygens (1678), i.e., in the first systematic wave theory of light, it will be sufficient to begin by noting the long neglected paper of Stokes, "On the Composition and Resolution of Streams of Polarized Light from Different Sources."[1] In this

1) Stokes, G.G. Trans. Camb. Phil. Soc., 9 399 (1852)

paper Stokes considered both the wave and the statistical aspects of light and showed that an arbitrary beam of light of given "refrangability" could be completely characterized by a set of four quantities which we shall denote by I,M,C,S and call, collectively, a Stokes vector.

We note, in passing, that it was not until 1873, in his great treatise, that Maxwell elaborated the Electromagnetic theory of light. This was the precursor of a great deal of theoretical and experimental research culminating in the body of knowledge known as the classical theory of radiation, beautifully summarized in Born's "Optik " (1933). At the turn of the century there was a major change in outlook beginning with Planck's theory of black body radiation (1900), leading to the older quantum theory of Bohr (1913), and culminating in the wave mechanics (1925-1931) associated with names like Heisenberg, Born, Schroedinger and Dirac. These results are summarized in Heitler's "Quantum Theory of Radiation" (1945).

The need for the present research was called to the writer's attention by Mueller whose "Foundations of Optics" (1948) develops a phenomenological theory of light based on the Stokes vector representation of the radiation and a 4x4 matrix representation of the instrument through which the radiation passes. Mueller is the first to notice the need for a phenomenological approach to the subject giving operational definitions of the basic quantities.

In 1941 there appeared the first of a series of papers by Jones presenting "A New Calculus for the Treatment of Optical Systems."[1] The radiation discussed is a monochromatic

1) Jones, R.C. Jour. Opt. Soc. Am., <u>31</u> 488-503 (1941); <u>37</u> 107-112 (1947). We shall refer to this calculus as "Jones Algebra."

plane wave characterized by the components E_x and E_y of the electric amplitude. We shall call them, collectively, the Maxwell vector. Jones studied in detail the class of 2x2 matrices characterizing the effect of the optical instrument on the radiation. We shall call such instruments elementary, denote their matrices by J and call them Jones matrices.

In 1942 Perrin published a paper on the "Polarization of Light Scattered by Isotropic Opalescent Media."[2] In this

2) Perrin, F. Jour. Chem. Phys. <u>10</u> 415-417 (1942)

paper he used the Stokes vector and introduced a 4x4 matrix to characterize the scattering media. The results of this paper were established from wave optics and on the basis of a non too elegant or satisfying treatment of the statistical aspects of the problem. Mueller realized that the Stokes vector and this 4x4 matrix were observables and capable of being made the basis of a phenomenological treatment of optics. He further realized that an algebra bases on this vector and matrix would be a direct and powerful tool for the solution of problems of interference and scattering which were engaging his attention. The result has been an extremely effective algebra dealing directly with observable quantities. The 4x4 matrices which represent the instrument and which are the elements of this algebra will be denoted by M and called Mueller matrices.

There is a connection between the Maxwell vector, E, and the Stokes vector, L. This connection for the case of a monochromatic plane wave was first established by Mueller. He also established the analogous relation between J and M for the case of elementary instruments.

Mueller's fundamental contribution is his phenomenological "Foundations of Optics" which he sketched briefly at the New York 1948 meeting of the Optical Society of America and which will appear in full in a forthcoming paper. The fundamental point of his theory is the recognition of the fact that the quantities of his algebra are observables and that those of Jones algebra are not. He then proceeds to give his quantities operational definitions (in the sense of Bridgeman), thus erecting a phenomenological theory. This brings us to the point at which our research began. Mueller made the suggesion that there should be a simple relation between Wiener's coherency matrix and Stokes vector and that Wiener's generalized harmonic analysis would probably be the key to the relation between wave optics and phenomenological optics. His conjectures have been born out and the results are the subject matter of our thesis.

1.1 Jones Algebra.

Jones algebra is a mathematical system in which the vector

(1) $$F(\omega) = (F_1(\omega), F_2(\omega))$$

characterizes the radiation. The F_1 and F_2 are the spectral resolutions of the electric amplitude components of a plane electromagnetic wave. Jones confines his attention to this pair of densities at a single frequency under the classical idealization of a monochromatic plane wave. Such a wave is an elementary solution of the electromagnetic wave equation in free space in rectangual coordinates and will be called an "elementary" wave. The effect of the instrument which transforms the wave is characterized by a matrix

(2) $$J(\omega) = [J_{ij}(\omega)]$$

with components which are complex functions of frequency. Instruments which can be so characterized will be called simple or better "elementary" instruments. The definition of J is given by the formula

$$E' = J \cdot E \tag{3}$$

where E' is a plane wave component of the emerging radiation.

In connection with the representation of radiation by vector plane waves and instruments by matrices, Mueller calls attention to a fundamental point. The incoming radiation may always be resolved into plane waves. The E, E' of Jones are the result of selecting a single one of these plane wave components from the incoming and emerging radiation. Since E, E' each satisfy the linear homogeneous wave equation in free space, the principle of superposition applies. But the Jones matrix is entirely a relative concept, depending not only on the instrumental agent affecting the radiation but also on the E and E' being compared. This is what Mueller terms the decomposition of the agent into elementary instruments. This is a concept correspondong to the decomposition of the incoming and emerging radiation into elementary waves. In many paractical problems with sufficiently simple instruments, such as a polarizer or analyzer, the directions of the incoming and emerging wave are fixed and the action of the instrument can be characterized by a single Jones matrix. In the problem of scattering by an electrolyte or a colloidal suspension this simplification is untenable. The agent in this case is characterized by a statistical set of Jones matrices.

The other fundamental operation of Jones algebra is the addition of Maxwell vectors, i.e.,

$$E = E_1 + E_2 \tag{4}$$

Physically this corresponds to coherent superposition of the two elementary waves E_1 and E_2.

It follows immediately from eq. (3) that if the incoming radiation passes through two instruments J_1 and J_2 in series, first J_1 and then J_2, the effect is the same as if it had passed through the single instrument

$$\boxed{J = J_2 J_1} \tag{5}$$

It follows from eq. (4) that if the radiation is passed in equal amounts through two instruments J_1 and J_2 in parallel and if the emerging radiation can be coherently superimposed, the effect is the same as if the radiation had passed through the single instrument

$$\boxed{J = J_1 + J_2} \tag{6}$$

where eqs. (5) and (6) involve the usual definitions of matrix addition and multiplication.

Three examples of Jones matrices will be useful for illustrative and reference purposes. Their derivation and other examples of them will be found in Jones' papers, cited above.

(7) $$J = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \quad ; \quad a, b, \text{ real}; \quad \text{IDEAL POLARIZER}$$

(8) $$J = \begin{bmatrix} e^{iB_1} & 0 \\ 0 & e^{iB_2} \end{bmatrix}; \quad B_1, B_2 \text{ real}; \quad \text{IDEAL WAVE PLATE}$$

(9) $$J = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}; \quad \alpha \text{ real}; \quad \text{IDEAL ROTATOR}$$

Four examples of Maxwell vectors will be useful for reference. The components are densities at an unspecified frequency.

(10) $E = (1,0)$ HORIZONTAL LINEAR POLARIZED

(11) $E = (0,1)$ VERTICAL LINEAR POLARIZED

(12) $E = (1,i)$ RIGHT CIRCULAR POLARIZED

(13) $E = (i,1)$ LEFT CIRCULAR POLARIZED

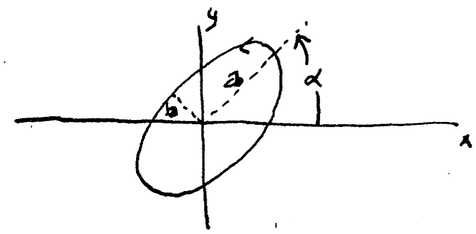
The correctness of these examples may be verified by computing the "polarization ratio"

(14) $$R(\omega) = F_2(\omega)/F_1(\omega) = e^{i\delta}\tan\gamma \quad .$$

The significance of the polarization ratio may be understood by considering the graph

(15)
$$x(t) = \mathrm{Re}(F_1(\omega)e^{-i\omega t})$$
$$y(t) = \mathrm{Re}(F_2(\omega)e^{-i\omega t})$$

This graph is easily proved to be an ellipse,



as shown in the figure. We note the geometrical significance of the angle α and define β

(16)
$$\beta = \tan^{-1} b/a$$

where a, b are the semi-major and semi-minor axes. It is demonstrated in Born's "Optik" that,

(17)
$$\tan 2\alpha = \tan 2\gamma \cos\delta$$
$$\sin 2\beta = \sin 2\gamma \sin\delta$$

We give another derivation in Sec. 2.6.

1.2 Mueller Algebra.

Mueller algebra is a mathematical system in which the vector

(1')
$$L = (L_1, L_2, L_3, L_4) = (I, M, C, S) \qquad \text{Stokes Vector}$$

characterizes the radiation. The vector L was originally introduced by Stokes for the representation of the general monochromatic plane wave. Mueller also uses the second notation, (I,M,C,S), for the vector relative to a special coordinate system occuring in his operational definitions.

One of the discoveries of the present research is the fact that, if $S'_{ij}(\omega)$ is the coherency matrix (to be defined later) corresponding to a given Maxwell vector (random), then

$$S'_{ij}(\omega) = \begin{bmatrix} I+M & C-iS \\ C+iS & I-M \end{bmatrix} \qquad \text{Coherency Matrix}$$

where I,M,C,S are measured by Mueller's operational methods. The physical meaning of these quantities and the proof of this result are given in Chapter II.

The effect of the instrument is represented by a real 4x4 matrix

$$(2') \qquad M = [m_{ij}] \qquad \text{Mueller Matrix}$$

The definition of M is given by the formula

$$(3') \qquad \boxed{L' = ML} \qquad \text{Basic Rule}$$

where L' is the Stokes vector of the emerging radiation.

The other fundamental operation of Mueller algebra is the addition of Stokes vectors, i.e.,

$$(4') \qquad L = L_1 + L_2 \qquad \text{Incoherent Superposition}$$

Physically, this operation corresponds to the incoherent superposition of the two radiations L_1 and L_2.

It follows immediately that the effect of two instruments M_1 and M_2 in series is given by

(5') $$\boxed{M = M_2 M_1}$$ Instruments in Series

It likewise follows that the effect of two instruments in parallel is given by the formula

(6') $$\boxed{M = M_1 + M_2}$$ Instruments in Parallel

Mueller has shown that, under a very wide class of circumstances, there exists an M such that eq. (5') is valid for ALL **L**.

The corresponding operation $E' = JE$ in Jones algebra is valid only for completely polarized monochromatic waves and elementary instruments.

The operation $L = L_1 + L_2$ is valid only if the two radiations are incoherent, i.e., statistically independent. The corresponding operation $E = E_1 + E_2$ is valid only if the radiations are coherent, i.e., functionally dependent and completely polarized.

The range of validity of the operations $M_2 M_1$ and $M_1 + M_2$ cannot be satisfactorily ascertained without recourse to the general theory of this thesis. The difficulty arises on points like the following:

Suppose the incoming beam is split into two parts L and L and one beam is passed through an instrument M and emerges as ML. These beams are then superposed. The question is, when does ML+L represent the resulting radiation? This and many similar questions lead to the necessity for an enveloping algebra in which Mueller algebra, Jones algebra, and Wiener generalized harmonic analysis play their parts and in which the special role of each is clearly evident.

In building up the general theory it was found desirable to make contact with many branches of modern mathematics: abstract algebra, topology, Lebesgue integration, transformations in Hilbert space, and especially with Wiener generalized harmonic analysis and modern statistical theory. Thus the general results of the research are presented on a broader and more abstract base than is usually the custom. We believe, however, that the generality of the results justify such a formulation.

For the reader desiring only an understanding of the specific results on particular problems solved in the thesis, we feel that it is unnecessary to do more than scan the remainder of the chapter. In doing the research, we have worked from the particular to the general; the results in Chapters II, III, and IV were obtained first. It was only at the end that we saw, all at once, the possibility of a general abstract theory which would be uniformly effective and which would displace some of the ad hoc hypotheses we made on the way. This chapter is mathematical and a record of this new insight. It will be referred to in the later chapters for specific formulae and may be referred to in more detail as the need arises.

1.3 Some Abstract Algebra.

Jones algebra and Mueller algebra are "algebras" in the technical sense used in the modern abstract theory of linear associative algebras. We have found it a help toward the understanding of our problem to make contact with this theory. In algebraic structure theory we can add and multiply algebras, much as one does numbers. These products and additions as reflected in the elements of the algebras are called KRONECKER PRODUCT and DIRECT SUM. These operations lie very deep in the structure of mathematics. In tensor analysis the Kronecker product appears as a tensor product giving tensors whose order is the sum of the orders of the tensors appearing in the product. In set theory the direct sum is often called the cartesian product. When the elements being so multiplied are ordered sets of coordinates, e.g., (x,y), this operation leads to the coordinates of a space whose dimensions has the sum of the dimensions of the spaces entering into the cartesian product (or direct sum).

When we make a systematic use of these concepts we find that Mueller algebra is the square of Jones algebra and the the Wiener coherency matrix is an element in the square of the sum of a set of elementary Jones algebras. Before we can undertake the task of algebraizing our theory in the sense given above we must discuss the algebraic concepts which we shall need and also discuss some of the underlying problems of analysis surrounding the functions which represent the radiation. The present section is devoted to a summary of some of the ideas from abstract algebra.[1]

1) A full list of references will be found in: Parke, N.G.: "Guide to the Literature of Mathematics and Physics." McGraw-Hill, N.Y., 1947. Part II, under "Algebra-Abstract." Where detailed references are not given they may be completed by reference tothe appropriate section of the above work.

The concept of a "group" is familiar, we list the following definition for reference (Cf. A.A.Albert,"Modern Higher Algebra")

<u>Definition.</u> A non-empty set $\mathfrak{G}$ of elements a,b,... is said to form a group with respect to an operation O if:

I. $\mathfrak{G}$ is closed with respect to O, i.e., if aOb always belongs to $\mathfrak{G}$.

II. The associative law holds in $\mathfrak{G}$, i.e.,

$$aO(bOc) = (aOb)Oc$$

for every a,b,c of $\mathfrak{G}$.

III. For every a and b of $\mathfrak{G}$ there exists a solution x and a solution y of $\mathcal{G}$ of the equations

$$a0x = b \,, \quad y0a = b$$

The group $\mathcal{G}$ is "commutative" or abelian if, in addition,

$$a0b = b0a$$

for every a and b of $\mathcal{G}$.

The Maxwell vectors and Jones matrices are examples of groups under the operation + which corresponds to coherent superposition. They are in fact abelian groups under this operation.

The next important concept is that of a RING.

<u>Definition.</u> A ring is an additive abelian group $\mathfrak{A}$ such that postulates I, II, and III hold,

I. The set $\mathfrak{A}$ is closed with respect to a second operation designated by multiplication; i.e., every a and b of $\mathfrak{A}$ define a unique element ab of $\mathfrak{A}$.

II. Multiplication is associative; i.e.,

$$a(bc) = (ab)c$$

for every a,b,c of $\mathfrak{A}$.

III. The distributive laws

$$a(b+c) = ab + ac, \quad (b+c)a = ba + ca$$

hold for every a,b,c of $\mathfrak{A}$.

The Jones matrices furnish an example of a ring.

A ring $\mathfrak{A}$ is said to have a "unity element" if there exists an e in $\mathfrak{A}$ such that

$$ae = ea = a$$

for every a in $\mathfrak{A}$.

It is possible in a ring to have ab = 0 even though a $\neq$ 0. Such elements are called " divisors of zero."

Some Jones matrices are divisors of zero.

<u>Definition.</u> A field $\mathfrak{F}$ is an additive abelian group whose non-zero elements form a multiplicative abelian group such that a(b+c) = ab + ac, a0 = 0a = 0 for every a,b,c of $\mathfrak{F}$.

The Jones matrices do not form a field, nor does the underlying set of complex functions of t or ω. The reason is that one can have divisors of zero. The complex numbers and the real numbers do form fields. The complex numbers are unique in being closed under algebraic operations, i.e., taking a square root of -1.

We shall need the concept of a linear set[1] over a ring. Consider an additive abelian group $\mathfrak{L}$ of elements $\alpha, \beta, \ldots$ (e.g., Maxwell vectors) and a ring $\mathfrak{A}$ of elements a,b,c,... (e.g., Jones matrices) and an operation a giving an element in $\mathfrak{L}$ (e.g., eq.(8)). Then $\mathfrak{L}$ is a linear set if

$$a(b\alpha) = (ab)\alpha \quad , \quad (a+b)\alpha = a\alpha + b\alpha \quad , \quad a(\alpha+\beta) = a\alpha + a\beta$$

for every a,b of $\mathfrak{A}$ and α,β of $\mathfrak{L}$.

THE MAXWELL VECTORS ARE A LINEAR SET OVER THE RING OF JONES MATRICES.

---

1) Alternative terms for linear set are : vector space, linear space, -module.

We are writing Jones matrices to the left of the Maxwell vector. In such cases one would usually write the components of the Maxwell vector in a vertical column. The matrices would appear in order of application from right to left. Many algebraists are now writing the vectors as horizontal arrays and to the left of the matrix. The matrices are then written in order of application from left to right. We shall be careless in this matter because physicists are used to the first scheme involving vertical vectors. On the other hand, for writing convenience we shall often resort to horizontal vectors without resorting to "right multiplication." We shall be careless of these matters but we shall be specific when there is any real possibility of confusion.

The Jones matrices are linear sets over the field of complex numbers if we deal with the frequency representation (to be defined) and concentrate on an unspecified frequency. In the general case the Jones matrices are linear sets over the ring of complex functions forming a Hilbert Space in which the definition of multiplication is ordinary multiplication in ω-space, the convolution in t-space. The Jones matrices, with this definition of multiplication, form a ring. A linear set which is also a ring is called an ALGEBRA. THE SET OF JONES MATRICES IS THEREFORE AN ALGEBRA OVER THE RING OF COMPLEX FUNCTIONS (HILBERT SPACE). This brings up an important historical remark. Wedderburn in thesis (1907) developed the structure of algebras over a field. In 1927 Artin extended Wedderburn's theory to algebras over a ring.[1] This generalization is necessary

1) The most recent exposition of the subject is: Jacobson, N.: "The Theory of Rings." (Mathematical Surveys No.II.) American Mathematical Society, N.Y., 1943.

for our theory of optical algebras just as it is necessary for quantum theory which requires rings of operators in Hilbert Space.

It is in the above sense that we speak of Jones algebra. The fact of fundamental importance here is that these algebras can be added and multiplied and that these are the operations by which we pass from Jones algebra to Muelller algebra.

1.4 Another Jones Algebra. The Time-Frequency Duality.

The notion of "equivalence" (isomorphism) of mathematical systems plays an essential role in our theory. We define this notion for groups and rings.

Definition. Let $\mathfrak{G}$ and $\mathfrak{G}'$ be groups with respective operations O and O' and let there be a (1-1) correspondence

$a \leftrightarrow a'$; a in $\mathfrak{G}$, a' in $\mathfrak{G}'$ such that

$$(aOb)' , a'O'b'$$

for all a,b, of $\mathfrak{G}$. The we call $\mathfrak{G}$ and $\mathfrak{G}'$ equivalent groups.

Definition. Two rings $\mathfrak{A}$, $\mathfrak{A}'$ are called equivalent if there is a (1-1) correspondence $a \leftrightarrow a'$ between them such that

$$(a+b)' = a' + b'$$

for every a,b of $\mathfrak{A}$ and corresponding a',b' of $\mathfrak{A}'$.

Jones algebra, as he formulated it, was constructed over the ring of complex functions $F(\omega)$ of frequency ω. In order to develop a generalization of Jones algebra which will yield a satisfying statistical theory of optics it is necessary to set up an equivalent algebra over the ring of complex functions $f(t)$ of time t.

Our definition of integration is that due to Lebesgue. What this involves and why it is necessary will be discussed in the next section. The point of importance in our present consideration is that we shall not distinguish between two functions $f(t)$ and $g(t)$ or $F(\omega)$ and $G(\omega)$ if they are equal "almost everywhere." The meaning of this phrase will be clarified in the next section.

We shall refer to the algebra over the $F(\omega)$ as the frequency-representation and the one over the $f(t)$ as the time-representation.

Strictly speaking it would have been better to have started the theory with the time representation and write down the Maxwell vector,

$$e(t) = (f_1(t), f_2(t)) \tag{18}$$

where $f_1(t), f_2(t)$ are the time dependent components of an elementary plane wave. We then would establish a (1-1) correspondence with $E(\omega)$ by means of the Fourier transform and its inverse defined

$$\mathcal{F}(\ ) = \frac{1}{2\pi}\int_{-\infty}^{\infty}(\ )\, e^{i\omega t}\, dt \tag{19}$$

$$\mathcal{F}^{-1}(\ ) = \int_{-\infty}^{\infty}(\ )\, e^{-i\omega t}\, d\omega$$

In order to be able to carry out the Fourier transforms in every case, we take e(t) to be zero outside the time interval (-T,T). Physically this restriction means that we are always dealing with finite amounts of energy and coincides with the fact that any observation is of finite duration. However, at convenient points in the analysis we shall pass to the limit $T\to\infty$, with or without taking averages. The standard reason for this will be analytical simplicity. It is in this sense that we take

$$f(t) = e^{-i\omega_0 t}$$

and

$$F(\ ) = \delta(\omega-\omega_0) \quad \text{Dirac Delta Function.}$$

to be Fourier transforms of each other

$$\delta(\omega-\omega_0) = \mathcal{F}(e^{-i\omega_0 t}) \tag{20}$$

The Fourier transform supplies the required correspondence between the two representations

$$E(\omega) = \mathcal{F}(e(t)) \tag{21}$$

$$e(t) = \mathcal{F}^{-1}(E(\omega))$$

An important operation for our theory is the "convolution," also called "resultant" and "faltung."

$$h(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t-t_o)g(t_o)dt_o \tag{22}$$

If $H = \mathfrak{F}h$, $F = \mathfrak{F}f$, and $G = \mathfrak{F}g$, it is a theorem that

$$\boxed{H(\omega) = F(\omega)G(\omega).} \tag{23}$$

If one writes the convolution as a symbolic product,[1]

---

1) This research is essentially on the structure and application of optical algebras. We shall, in general, relegate some of the delicate conversion questions and questions of inversion of limiting processes to a later paper by the device of introducing an algebraic operation with certain formal properties which can be arrived at by heuristic reasoning but which require deeper justification of the extent of their validity. If the physical results of the algebra prove interesting, it will be time enough to clean up the analysis.

$$\boxed{h(t) = f(t-t_o) * g(t_o)} \tag{24}$$

we can define the time representation of the Jones matrix by the equation

$$\boxed{e'(t) = j(t-t_o) * e(t_o)} \tag{25}$$

and have a (1-1) correspondence between the Jones matrices in the two systems given by

$$\begin{aligned} j(t) &= \mathfrak{F}^{-1}(J(\omega)) \\ J(\omega) &= \mathfrak{F}(j(t)) \end{aligned} \tag{26}$$

Thus the operation of multiplication, in the usual sense in the frequency representation, becomes the operation of convolution in the time representation. Since is a linear transformation, the operation + remains unaltered. Thus

$$e(t) = e_1(t) + e_2(t) \tag{27}$$

It has been important to establish this duality of representation on a fairly clear cut basis. Quantum mechanics gives us $J(\omega)$ in a straightforward manner. However, for theoretical purposes, we need to find $j(t)$ and "add" in the time representation the effect of the time variation of the position and orientation of the instrument. An example of such an application is the case in which our elementary instrument is a scattering anisotropic molecule.

Other equivalent Jones algebras which will be useful will be obtained by taking the complex conjugates of the two algebras we have displayed above. We shall need these when we define the Wiener correlation process and use it for constructing Mueller algebra.

1.5 Some Abstract Analysis (Topology).

We have taken the first steps toward showing how modern algebraic theory bears on the problem of representing radiation and matter. In short, we have made contact between our problem of representing the theory of physical optics and the body of mathematics known as abstract algebra. We have used the notion of a Fourier transform to pass from Jones algebra based on the frequency representation (monochromatic plane waves) to an equivalent Jones algebra based on the time representation. Time is more fundamental if we are using interference techniques. Frequency is more fundamental if we are using spectroscopic techniques. However, having introduced Maxwell vectors which are functions of time we are immediately confronted with some problems of analysis which it is wise to clarify before we continue with the algebraic side of the problem. The word "topology" is included in the section heading because there is a sense in which algebra and topology encompass the entirety of mathematical activity and therefore the entirety of the activity of using mathematics for theoretical physics. Modern algebra is concerned with suitably restricted finite operations and relations; topology is concerned with suitably restricted infinite operations and relations (functions, limits, integrals, derivatives, etc.,). In addition one gets the hybrids, algebraic topology and topological algebra when they are used together.[1]

---

1) Cf. Tukey, J.W.:"Convergence and Uniformity in Topology," Princeton University Press, 1940.

In this light, we have proceeded to a point in the algebraic side of our theory and it is necessary to consider some of the topological aspects. The elements of our algebra are functions and it is necessary to say something about their character. This necessity is all the more pressing because it turns out in the later chapters of the thesis that they will be "random" functions and it is desirable to be clear as to the meaning of this term.

Physically, the functions $f(t)$ are real and have properties on any finite interval which lead us to characterize them as elements of Hilbert Space. They also will generally have physical properties which are invariant under a translation in time. For this reason the natural basis for representing them is the set of exponentials $e^{-i\omega t}$. This immediately leads to a complex representation of essentially real quantities, such as occurs in most wave theory and in electrical engineering. A complex representation causes no trouble in the realm of linear operations. It is the products of complex representations which require care. As we shall be using some rather unusual types of product operations in passing from Jones algebra to Mueller algebra, it seems desirable to make a few remarks about the matter from time to time.

Let us first examine the Maxwell vector $e(t)$ and ask to what class of functions its components should belong. this point can be settled on physical grounds. The branch of analysis which we shall use is variously known as "Real Variable Theory," "Modern Theories of Integration,"[1]

---

1) Cf. Kestelman, H.: "Modern Theories of Integration," Oxford University Press, 1937. For the more fundamental topological viewpoint toward the problems of analysis, cf. the very interesting work: Pontrjagin, L.: "Topological Groups," Princeton University Press, 1939.

"Lebesgue Integrals."

The f(t) are real functions of t or, if complex, it is Re(f(t)) which is taken to represent a component of the incoming radiation. The domain of definition of f(t) is assumed to be $(-\infty, \infty)$ but they are chopped off to zero outside the interval (-T,T) which represents the period of observation. By the total power in such a component is meant

$$P = \int_{-\infty}^{\infty} f(t)f(t)^{*}dt$$

i.e., f(t) is assumed to be a r.m.s representation. The reasonable physical assumption is that P is finite, i.e., that

$$\int_{-\infty}^{\infty} |f(t)|^2 dt$$

where we are using Lebesgue integration. From a practical standpoint, the use of Lebesgue integration means that the functions can be badly behaved at an ∞ of points provided their total interval of bad behavior is zero. Functions with finite total power are said to be Lebesgue integrable square, in symbols, f(t) L_2, read "f(t) belongs to L-two." Plancherel's theory of Fourier transforms applies to this class of functions.[1] We shall list for reference

1) Cf. Titchmarsh, E.C.:" Fourier Integrals." Oxford University Press, 1937. Chapter III.

some of the results of the theory which are needed here.

If

$$\text{(A)} \qquad \int_a^b f(t)\,dt < \infty$$

we write that $f(t) \varepsilon L_2(a,b)$.

<u>Plancherel's Theorem.</u> Let f(x) be a (real or complex) function of class $L_2(-\infty, \infty)$, and let

$$\text{(B)} \qquad \mathfrak{F}_a: \quad F(x,a) = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} f(y)\, e^{ixy}\, dy \quad .$$

Then, as $a \to \infty$, F(x,a) converges "in mean" over $(-\infty, \infty)$ to a function F(x) of $L_2(-\infty, \infty)$; and reciprocally

$$\text{(C)} \qquad \mathfrak{F}_a^{-1}: \quad f(x,a) = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} F(y)\, e^{-ixy}\, dy$$

converges "in mean" to f(x). The transforms f(x), F(x) are connected by the formulae

$$\text{(D)} \qquad \mathfrak{F}: \quad F(x) = \frac{1}{\sqrt{2\pi}} \frac{d}{dx} \int_{-\infty}^{\infty} f(y)\, \frac{e^{ixy}-1}{iy}\, dy$$

$$\text{(E)} \qquad \mathfrak{F}^{-1}: \quad f(x) = \frac{1}{\sqrt{2\pi}} \frac{d}{dx} \int_{-\infty}^{\infty} F(y)\, \frac{e^{-ixy}-1}{-iy}\, dy$$

for almost all values of x.

If $F = \mathfrak{F}f$ and $G = \mathfrak{F}g$, then

(F) $$\int_{-\infty}^{\infty} F(x)\,G(x)\,dx = \int_{-\infty}^{\infty} f(t)\,g(-t)\,dt$$

(G) $$\int_{-\infty}^{\infty} F(x)\,G^*(x)\,dx = \int_{-\infty}^{\infty} f(t)\,g^*(t)\,dt$$

(H) $$\int_{-\infty}^{\infty} |F(\omega)|^2\,d\omega = \int_{-\infty}^{\infty} |f(t)|^2\,dt$$

The physical interpretation of eq.(H) is that we can compute the total energy in either the frequency or the time representation.

We have used such expressions as "in mean" and "almost all" or "almost everywhere" and shall use them often in what follows; these phrases are all connected with the notinn of Lebesgue integration. However, it does not appear necessary for the physicist to embark on a long and uneconomic excursion into the details of this phase of modern mathematics. Rather it is sufficient that he be aware of the increased freedom which such concepts allow over and above the tight restrictions of classical formulations and definitions which which he is familiar and under which he usually labors to express the results of his experiments.

In order to make this viewpoint clearer, let it be said at the outset that a mathematical proof may be looked upon as a logical survey of the extent to which a desirable algorithm is valid. A mathematical generalization or abstraction is, in this light, a relaxation of the conditions under which the desired result may be considered true. In particular, Lebesgue integration is a generalization of Riemann integration. It extends the class of functions for which the process of integration may be considered valid. The extension is from the class of functions which have only a finite number of properly behaved discontinuities to those having an infinite number of such. Even in classical integration theory one has the extension to "improper" integrals, i.e., to those situations for which integration should certainly be meaningful but which exceed the original narrow limits of definition.

The generalization from Riemann integration to Lebesgue integration is a generalization from conditions which ALWAYS HOLD to conditions which ALMOST ALWAYS HOLD. This generalization is necessary for a symmetric theory of Fourier series and Fourier transforms. It is necessary for the von Neumann theory of rings of operators in Hilbert space and hence for a rigorous theory of quantum mechanics. It is absolutely necessary for the modern theory of probability and statistics and for the Wiener generalized harmonic analysis. In a very broad sense, the modern tendencies of mathematics toward algebra and topology are essential for the development of mathematical systems which are both logically consistant and structurally adequate for the deeper needs of theoretical physics. This is the direction in which one must go to get a sufficiently close correspondence between the mathematical formalism and physical intuition.

The generalization to Lebesgue integration is not obtained without cost. It is paid for by getting results which are true "almost always" rather than always, and expansions which are "convergent in the mean" instead of uniformly convergent. However this relaxation of strict determinism is not out of keeping with similar trends in quantum theory or statistical mechanics where analogous relaxations have been introduced for practical and logical reasons.

We now turn to the definition of such phrases as "almost always." For the sake of argument we consider the "domain", $(-\infty, \infty)$, the time-axis of our problem. Suppose that $f(t)$ has property P on some set of points on this time axis. We define a characteristic function, $E(t)$, which has the value 1 where $f(t)$ has property P and the value 0 elsewhere. If

$$\int_{-\infty}^{\infty} E(t)dt = 1$$

one says that $f(t)$ has property P "almost everywhere."

Consider the specific problem which we face in our theory. The function $f(t)$ represents a component of the Maxwell vector. We apply the operation

$$\mathfrak{F}\mathfrak{F}^{-1}f(t) = g(t)$$

and obtain the Fourier integral representation of f. We then take as property P the relation

$$f(t) = g(t) \quad .=. \quad \text{Property P at t}$$

It turns out that g(t) has property P "almost everywhere" and this fact is symbolized by writing

$$f(t) \;(=)\; g(t)$$

From a double application of Parseval's theorem it follows that

$$\text{(I)} \qquad \int_{-\infty}^{\infty} |f(t)|^2 dt = \int_{-\infty}^{\infty} |g(t)|^2 dt$$

i.e., the power associated with f and the power associated with g are equal. Since it is only the power which can be observed, i.e., the blackening of a photographic plate or some such means, it is physically impossible to distinguish between functions which are equal almost everywhere, and hence the symbol (=) might well be read "physically identical."

Another important concept is convergence in the mean. Let $f_n(t)$ be a sequence of functions and suppose that there exists a function f(t) such that

$$\text{(J)} \qquad \lim_{n\to\infty} \int_{-\infty}^{\infty} |f_n(t) - f(t)|^2 dt = 0$$

then the sequence is said to converge in the mean to f(t). The technical advantage of this concept lies in the fact that f(t) can be quite bad but $f_n(t)$ can be functions whose integrals are easily handled with the aid of Peirce's integral table. The notion is important for the symmetric Riesz-Fisher theory of Fourier series.

1.6 Hilbert Space.[1]

We have discussed Lebesgue integration and some of the general atmosphere surrounding modern analysis. We now wish to become more specific. In the discussion of Jones

1) Cf. v.Neumann, J.: "Mathematische Grundlagen der Quantenmechanik," Springer, 1932. Ch. II.
Stone, M.H.:"Linear Transformations in Hilbert Space," American Mathematical Society, N.Y., 1932.

algebra we used the Fourier transform to generate an equivalent Jones algebra. The Fourier transform is an example of a linear transformation in Hilbert space. The operators of quantum theory furnish many other examples. We wish to consider some of the purely algebraic aspects of this transformation theory. The functions $f(t)$ or $F(\omega)$ are vectors in Hilbert space and the linear transformations like are the matrices. In fact, The Riesz-Fisher theorem states that $f(t)$ can be represented by a doubly infinite sequence of coefficients a_n, the Fourier coefficients of $f(t)$. The Fourier transform can similarly be represented by a doubly infinite matrix which carries a_n into b_n, the Fourier coefficients of $F(\omega)$. The resulting algebra is quite similar to the algebra of the Maxwell vector and Jones matrix or the algebra of the Stokes vector and Mueller matrix.[1]

1) For an interesting and helpful exposition of these

analogies, cf.:

Halmos, P.R.: "Finite Dimensional Vector Spaces." Princeton University Press, 1942.

See also:

Murray, F.J.: "Linear Transformations in Hilbert Space." Princeton University Press, 1941.

We give the axiomatic definition of Hilbert space due to von Neumann. The advantage of such a systematic list lies in the fact that when an axiomatic system is applied to physics the interpretative bridge can be made to the primitive definitions and theorems and all subsequent interpretation becomes logically related to this nucleus.

Definition. A class H of elements f,g,... is called a HILBERT SPACE if it satisfies the following postulates

Postulate A. H is a linear space, i.e.,

(1) There exists a commutative and associative operation denoted by +, applicable to every pair f,g of elements of H, with the property that f+g is also an element of H.

(For our application it will be convenient to think of f,g,... as horizontal linear polarized plane waves. The operation + is to be interpreted as superposition.)

(2) There exists a distributive and associative operation denoted by $\cdot$ applicable to every pair, a and f, where a is a complex number and f is an element of H, with the properties that $1.f = f$ and $a.f$ is an element of H.

Postulate B. There exists a numerically-valued function (f,g) defined for ever pair f,g of elements of H, with the properties

(1) $$(af,g) = a(f,g)$$

(2) $(f_1+f_2,g) = (f_1,g) + (f_2,g)$

(3) $(g,f) = (f,g)^*$

(4) $(f,f) \geq 0$

(5) $(f,f) = 0$ if and only if f (=) 0.

The non-negative real number $\sqrt{(f,f)}$ will be denoted by $|f|$ for convenience. (Physically, (f,f) is the self-energy and (f,g) is the mutual-energy of waves f and g.)

Postulate C. For every n, n = 1.2.,,, ., there exists a set of n linearly independent elements of H; that is, elements $f_1,\ldots,f_n$ such that $\Sigma a_n f_n = 0$ is true if and only if $a_1 = \ldots = a_n = 0$.

Postulate D. H is separable; i.e., there exists a denumerably infinite set of elements of H, $f_1,f_2,\ldots,$ such that , for every $g \varepsilon H$ and every $\epsilon > 0$, there exists an $n(g,\epsilon)$ for which $|f_n - g| < \epsilon$.

Postulate E. H is complete, i.e., if a sequence $[f_n]$ of elements of H satisfy the condition

$$|f_m - f_n| \rightarrow 0 \qquad m,n, \rightarrow \infty$$

there exists an element f of H, such that

$$|f - f_n| \rightarrow 0 \qquad n \rightarrow \infty$$

The transforms of Hilbert space can be integral operators like Fourier or Laplace transforms. The whole theory is quite extensive as the size of Stone's book would indicate. For the present, however, we are interested in only the most elementary considerations.

We shall symbolize a transform of an element of Hilbert space by T and write

(K) $$Tf = g$$

By the SUM of two transforms we mean

(L) $$T = T_1 + T_2$$

where

$$Tf = T_1 f + T_2 f.$$

By the PRODUCT of two transforms we mean

(M) $$T = T_2 T_1$$

where

$$Tf = T_2(T_1 f)$$

Transforms can be multiplied by a complex number a, thus

(N) $$T = aT_o$$

where

$$Tf = a(T_o f)$$

Thus the transforms in Hilbert space form an algebra over the set of functions f,g,... the elements of H. The Maxwell vectors form a group over the same space.

Now consider the Fourier transform $\mathfrak{F}$. Of special interest is the set of functions which are left unaltered (up to a phase factor) by the transform, $\mathfrak{F}$, i.e., such φ_n that

(O) $$\lambda_n \varphi_n = \mathfrak{F}\,\varphi_n$$

These are the eigen-functions of the transform in the same sense that the wave functions of the quantum mechanics are the eigen-functions of the transform H,

$$E_n \psi_n = H_n \psi_n$$

In titchmarsh, loc. cit., it is shown that the eigen-functions of the Fourier transform are the Hermite functions,

(P) $$\varphi_n(x) = e^{-\frac{1}{2}x^2} H_n(x)$$
$$H_n(x) = (-1)^n e^{x^2} \left(\frac{d}{dx}\right)^n e^{-x^2}$$

where $H_n(x)$ are Hermite polynomials. The functions

(Q) $$\psi_n(x) = \frac{\phi_n(x)}{(2^n n! \sqrt{\pi})^{1/2}}$$

form an orthonormal set over $(-\infty, \infty)$. Any function $f(x) \in L_2$ can be expanded in a series of $\psi_n(x)$ in the sense

(R) $$f(x) \;(=)\; \sum_0^\infty a_n \psi_n(x)$$

Furthermore

(S) $$(f,f) = \sum_0^\infty a_n a_n^*$$

and finally

(T) $$\mathfrak{F}\phi_n(x) = i^n \phi_n(x)$$

These results will be useful in the sequal. They show the intimate mathematical relationship which exists between the complex exponentials and the Gaussian distribution functions. The statistical part of our theory, as it applies to spectral line shape, involves Gaussian distributions exclusively. At that time via the generalized harmonic analysis we will find an equally interesting connection between the Gaussian distribution of phase and the resonance type of response.

We shall need the theorem of the resultant. It tells how products transform under the Fourier transformation. Use has already been made of the theorem in constructing the time-representation of Jones algebra.

<u>Theorem.</u> If $f(t)$, $f(t) \in L_2(-\infty,\infty)$, then

(U) $$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(u)f(t-u)\,du \quad \text{and} \quad F(\omega)G(\omega)$$

are Fourier transforms, i.e., using the previously introduced notation,

$$\mathcal{F}(g*f) = FG$$

A further useful theorem is that: if $f \in L_2$ and $g \in L$, then $g*f$ and GF are transforms of class L_2. This may prove to be an important relaxation of the requirements on the Jones matrix $j(t)$.

<u>1.7 Complex Maxwell Vectors.</u>

It is desirable to make a few remarks about the use of the complex exponentials $e^{-i\omega t}$ as a basis for the $f(t)$

occuring as components of the Maxwell vector e(t),

(28) $$e(t) = (f_1(t), f_2(t))$$

To make f(t) an element of Hilbert space we assume that it is zero except on some finite interval (-T,T) and of class L_2. Physically this means the experiment is of finite duration and involves finite energy. We now represent f(t) on a complex exponential basis

(29) $$f(t) \;(=)\; \mathfrak{F}^{-1} F(\omega)$$

where

(30) $$F(\omega) = \mathfrak{F} f(t).$$

We therefore have the expression

(31) $$\mathfrak{F}^{-1}\mathfrak{F} f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} d\omega$$

which involves complex exponentials as a basis and is physically indistinguishable from f(t) itself. However $\mathfrak{F}^{-1}\mathfrak{F} f(t)$ is rather well behaved, the double integration having ironed out all the places where f(t) was badly behaved.

It will be shown in Sec. 2.4 that it is the complex exponentials and not the sines and cosines which lead to the simplest spectra under a generalized harmonic analysis.

It is worth noticing, in passing, the corresponding Fourier series representation of f(t) used by Wiener in his proof of the Plancherel theorem. The trick is to put

$$f_T(t) = \begin{cases} f(t) & \text{if } |t| < T \\ 0 & \text{if } |t| \geq T \end{cases} \tag{32}$$

and represent $f_T(t)$ over the interval $(-2T,2T)$ by a Fourier series,

$$f_T(t) \sim \sum_{-\infty}^{\infty} a_n e^{i(n\,t/2T)} \tag{33}$$

In chapter III we shall use the random complex function

$$Ae^{-i(\omega t - \varphi(t))} \tag{34}$$

where $\varphi(t)$ is a random function to describe the components of a Maxwell vector corresponding to a spectral line with collision broadening and partial polarization.

Because all the functions of time which occur may be represented by a superposition of exponentials, we can resort to the same trick used in electrical engineering and let $f(t)$ be the r.m.s. amplitude of the electric component i.e., $f(t) = E(t)/\sqrt{2}$, which avoids cluttering up the calculations with factors, 1/2. Self and mutual energies are then given by the bracket functions introduced in the postulates for Hilbert space

$$(f_i, f_j) = \begin{matrix} \text{self-energy when } i = j \\ \text{mutual-energy when } i \neq j \end{matrix} \tag{35}$$

The final and most important reason for the use of the complex exponentials as a basis is the fact that they act like stationary time series, i.e., their spectra are invariant under linear transformations in time.

1.8 A Delimitation.

As already indicated, recognition of the possibility of a generalized algebra came late in the research, after most of the particular results had been obtained. It came while observing from an abstract standpoint the mathematical meaning of the particular operations we had been using. Once we had adopted our algebraic-structural viewpoint we saw clearly many new possibilities for future research suggested by parallelisms to be found in the work of Weyl, Eddington, and von Neumann. Weyl built up a general representation theory in his "Group Theory and Quantum Mechanics." Eddington in his "Relativity Theory of Protons and Electrons" indroduced as a basis of Dirac's theory of the electron, the notion of "wave tensors." von Neumann in his "Rings of Operators," is extending the ideas of tensor analysis to operators in Hilbert space. In looking over the mathematical work of the algebraists and topologists we saw the germs of further possibilities which invite careful examination and offer hope not only in the limited optical problem we are discussing here but also in the direction of a more unified exposition of the results of theoretical physics in general.

We recognize these possibilities, but for the present we resolutely return to the task of recording the tentative form of our generalized optical algebra. At present it looks desirable to adopt a generalized tensor notation in which the time variable, t, also acts as a summation index. This departure from the notation used in algebraic structure theory, i.e., a departure from matrix-vector notation but it appears currently to be the simplest and most meaningful one to work with. However, after we have transformed a Jones algebra into a Mueller algebra via the Wiener correlation process, we shall recast the result in matrix-vector notation. It is only while crossing the Wiener statistical bridge that the tensor notation is more tractable. A second reason for the adoption of the tensor notation is the fact in it the notions of "direct" sum and "Kronecker" product achieve their simplest definition. The definitions given in algebra books look more complicated because the process of defining and recasting in vector-matrix form are carried on simultaneously.

1.9 Wiener Generalized Harmonic Analysis.

Wiener has pointed out that in the electromagnetic theory of light the field vectors E and B are not observables as they are at lower frequencies.[1] This same point has

1) Wiener, N. "Harmonic Analysis and the Quantum Theory." Jour. of the Franklin Institute. 207 525-539 (1929).

been made by Mueller in connection with his phenomenological foundations of optics. Optical observations always end with intensity measurements. These measurements being made with the aid of the blackening of a photographic plate, the use of a photometer, the use of a photoelectric cell, estimations by the naked eye, or the use of a bolometer. The quantities of the Maxwell theory which most nearly correspond to observations are the ENERGY DENSITY and the POYNTING VECTOR, expressions which depend QUADRATICALLY on the electric and magnetic vectors,

$(1/2)(E.D + H.B)$	Energy Density
ExH	Poynting Vector

This LINEAR-QUADRATIC DUALITY is the relation between the algebras of Jones and Mueller, the definition of "product" being Wiener correlation instead of scalar and vector products used above. This linear-quadratic duality between theoretical quantities and observables is also exhibited in the quantum mechanics. Schroedinger's equation governs the wave functions ψ. The observables are the quadratically related quantities $\int \psi F \psi^* d\tau$. It is again a statistical average which provides the bridge between the linear theoretical quantities and the corresponding expectations for the observables. Such a bridge for our problem is furnished by the Wiener generalized harmonic analysis.

Short Bibliography of papers by Wiener and Others.

For purposes of reference, we list a series of papers by Wiener and others on generalized harmonic analysis and related topics.

Wiener, N.: "Generalized Harmonic Analysis." Acta Math. 55 117-258 (1930).

________.: "Harmonic Analysis of Irregular Motion, I,II." Jour. Math. Phys. M.I.T. 5 99-122, 158-191 (1926).

________.: "The Spectrum of an Arbitrary Function." Proc. London Math. Soc. 27 487-496 (1928).

________: "Coherency Matrices and Quantum Theory." J. Math. Phys. M.I.T. 109-125 (1928).

________.: "The Historical Background of Harmonic Analysis." Amer. Math. Soc. Semicentennial Addresses. 1938.

________ and A. Wintner.:"The Discrete Chaos." Am. J. Math. 65 279-298 (1943).

________ and ________.: "On the Ergodic Dynamics of Almost Periodic Systems." Am. J. Math. 63 794-824 (1941).

Wiener, N.: "The Homogeneous Chaos." Am. J. Math. 60 897-824 (1941).

________.: "The Ergodic Theorem." Duke Math. Jour. 5 1-18 (1939).

________.: "The Extrapolation, Interpolation and Smoothing of Stationary Time Series with Engineering Applications." D.I.C. Contract 6037 Mass. Inst. of Tech. 1942.

Weeks. D.W.: "Three Mathematical Methods of Analysing Polarized Light." J. Math. Phys. M.I.T. 13 371-379 (1934)

__________.: "A Study of Sixteen Coherency Matrices." J. Math. Phys. M.I.T. 13 380-386 (1934).

Cameron, R.H. and W.T.Martin.: "Transformations of Wiener Integrals under Translations." Annals of Math. 45 386-396 (1944).

________ and ______.: "An Expression for the Solution of a Class of Non-Linear Integral Equations." Am. Jour. Math. 46 281-298 (1944).

________ and ______.: "The Wiener Measure of Hilbert Neighborhoods in the Space of Real Continuous Functions." Jour. of Math. and Phys. M.I.T. 23 195-209 (1944).

________ and ______.: "Evaluations of Various Wiener Integrals by use of Certain Sturm-Liouville Differential Equations." Bull. Am. Math. Soc. 51 73-89 (1945).

Cameron, R.H. "Some Examples of Fourier-Wiener Transforms of Analytic Functionals." Duke Math. Jour. 12 485-488 (1945).

Cameron, R.H. and W.T.Martin." Fourier-Wiener Transforms of Analytic Functionals." Duke Math. Jour. 12 489-567 (1945).

_______ and _______.: "Transformation of Wiener Integrals Under a General Class of Linear Transformations." Trans. Am. Math. Soc. 58 184-219 (1945).

_______ and _______.: "Fourier-Wiener Transforms of Functionals Belonging to L_2 over the Space C." Duke Math. Jour. 14 99-107 (1947).

_______ and _______.: "The Behavior of Measure and Measurability under Change of Scale in Wiener Space." Bull. Am. Math. Soc. 53 130-137 (1947).

_______ and _______."The Orthogonal Development of Non-Linear Functionals in a Series of Fourier-Hermite Functionals." Annals of Math. 48 385-392 (1947).

See also, the references in:

Volume 25, Radiation Laboratory Series, McGraw-Hill, N.Y., 1948. Ch. VI. "Statistical Properties of Time Variable Data." by R.S.Phillips. p.266.

Wiener Generalized Harmonic Analysis.

As introduced by Wiener, the generalized harmonic analysis of a function f(t) divides in a natural manner into two steps. The first and essential step is the computation of the auto-correlation function

$$\varphi(t) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(t+\tau) f^*(\tau)\, d\tau \tag{36}$$

The second step is the computation of the integrated Fourier transform of this function

$$S(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\varphi(t)\,\frac{e^{i\omega t}-1}{i\omega}\,dt \tag{37}$$

We wish to recall that we have already introduced a quite similar looking operation, the convolution, eq. (22),

$$h(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t-\tau)\,g(\tau)\,d\tau \tag{38}$$

The Wiener operation applies to functions for which the required integral exists for every t. The convolution applies to functions of class L_2. In the Wiener correlation process the variable of integration has the SAME sign in both factors of the integrand. In the convolution the variable of integration has OPPOSITE signs in the factors.

In passing from Jones algebra in the frequency representation to Jones algebra, time representation, we made the assumption that f(t) was zero outside the interval (-T,T) in order to be able to stay in class L_2 and give a neat (1-1) correspondence between these equivalent Jones algebras. At this stage of the exposition we will take the time representation as fundamental and remove boundedness restriction on the non-zero interval of f(t), assuming only that f(t) is such that the Wiener correlation is valid. This indeed is Wiener's only assumption. We then redefine the convolution as

(39) $$h(t) = \int_{-\infty}^{\infty} f(t-t_o)g(t_o)dt_o$$

and notice that when $f(t) \varepsilon L_2$ this leads to the product

(39') $$H(\omega) = 2\pi F(\omega)G(\omega)$$

It is assumed that we are using the unsymmetric Fourier transforms

(40) $$\mathcal{F}(\ ) = \frac{1}{2\pi}\frac{d}{d\omega}\int_{-\infty}^{\infty}(\ )\frac{e^{i\omega t}-1}{it}dt$$
$$\mathcal{F}^{-1}(\ ) = \frac{d}{dt}\int_{-\infty}^{\infty}(\ )\frac{e^{-i\omega t}-1}{-i\omega}d\omega$$

A word should be said about $j(t-t_o)$. It is zero for $t_o > t$ and is the response of the system to a unit impulse at time t_o. It is called a "weighting function." A similar function appearing in electrical engineering is called an "indicial admittance." It is of class L_2 if the effect of a unit impulse dies out as it will in the physical problem we are considering. This is due to the law of conservation of energy.

The factor 2π appearing in eq.(39) can be confusing. By definition $2\pi G(\omega)\delta(\omega-\omega_o)$ is the response of the instrument to the input $e^{-i\omega_o t}$. If one had used $e^{-i\omega t}$ and quantum mechanical methods he would come out with the response $2\pi G(\omega)$. But it is $G(\omega)$ which is the transform of $g(t)$ in the time representation. Our Jones matrix will therefore be smaller by a factor 2π than that defined by Jones. The transform of our $J(\omega)$ is $j(t)$ which we call the time-Jones matrix and which is zero for $t < 0$. The relation between the incoming and outgoing radiation is then given by the convolution

$$e'(t) = \int_{-\infty}^{\infty} j(t-t_o)e(t_o)dt_o$$

or, as we shall find more convenient

$$e'(t) = \int_{-\infty}^{\infty} j(t_o)e(t-t_o)dt_o.$$

1.10 Some Primitive Concepts from General Topology.

It will be useful in the remainder of the chapter to use some of the primitive concepts of general topology. They allow brief, direct, and logical introduction of several new operations which are required. They have been an integral part of our thinking technique for at least ten years, they have motivated some of the generalizations of this chapter. It seems more practical to list them in one place and use them without comment than to attempt to dissect them out of the exposition.[1]

---

1) This particular list is abstracted from: Lefschetz, S.: "Algebraic Topology." American Mathematical Society, Colloquium Publications, N.Y., 1942. Ch.I.

Formal Abbreviations:

$A \rightarrow B$	.=.	A implies B
$A \leftrightarrow B$	.=.	A equivalent to B (logical)
$A \cong B$	.=.	A equivalent to B (set theoretic)

Λ	.=.	Null Set
V	.=.	Domain of Discourse
$X \subset Y$	.=.	Set X included in set Y
$x \varepsilon X$	.=.	Element x belongs to set X
$X = \{x\}$	.=.	X is the set of elements in the bracket
$\{X_a\}$	.=.	A collection of Sets indexed by a
$\bigcup_a X_a$	.=.	The set of x_a for all a; the <u>union</u> of^a sets X_a
$\bigcap_a X_a$	.=.	The set of all elements in in every set X_a; the <u>intersection</u> of sets X_a.

We also write (when a are integers)

$X_1 \cup X_2 \cup \ldots \cup X_n$ for "uhion"

$X_1 \cap X_2 \cap \ldots \cap X_n$ for "intersection".

Given X,Y, the set of all $x \varepsilon X$ which do not belong to Y is called the <u>compliment</u> of Y in X. This set is also called the difference of X and Y, written X-Y.

If P is a property and X = x , the totality of x which satisfy P is denoted by

$$\{ x \mid x \text{ has property } P \}$$

the symbol "$|$" is read "such that."

Negation of any relation is written by means of a bar through the positive symbol, e.g., $x \not\varepsilon Y$ means that x does not belong to set Y.

The equation $X_a \cap X_b = \Lambda$ means that sets X_a and X_b are disjoint, i.e., have no common elements.

Transformations and Functions.

The following general definition of function and of (1-1) correspondence is especially important for the rest of the chapter. Let $X = \{x\}$, $Y = \{y\}$ be two sets and G a subset of the set of ordered pairs $\{(x,y)\}$. Suppose G has the property that for every x there is precisely one pair $(x,y_x) \varepsilon G$. There results then an assignment to each $x \varepsilon X$ of a definite element $y_x \varepsilon Y$. This assignment is known as a TRANSFORMATION OF Y INTO X or a FUNCTION ON X TO Y. This transformation is written symbolically in several forms

$$T: \ X \to Y$$
$$T: \ x \to y_x$$
$$x \to y_x \quad \text{defines } T$$
$$y_x = Tx.$$

The set X is called the RANGE of T; the y_x is the VALUE of T at x. The set Y' of all Tx for all $x \varepsilon X$ is a subset of Y called the TRANSFORM or IMAGE of X under T. We write $Y' = TX$.

If $Y' = Y$, T is said to transform X ONTO Y. The transformation T is said to be UNIVALENT when $x \neq x' \longrightarrow Tx \neq Tx'$. It is called (1-1) when it is both "univalent" and a transformation "onto." The set G of pairs (x,y) serving to define T is known as the GRAPH of T.

Cartesian Products.

The set of pairs (x,y) where $x \varepsilon X$ and $y \varepsilon Y$ used in the general definition of a function or transformation is the CARTESIAN PRODUCT of X and Y, written XxY. This primitive notion is exceedingly important and fundamental being one of the essential ideas in every generalized product. Lefschetz, loc. cit. gives the following general definition.

Definition of Cartesian Product. Let $\{X_a\}$ be a system of sets indexed by $A = \{a\}$, with $X_a = \{x_a\}$. The cartesian product of X_a is the set of all single-valued functions $\xi(a)$ on A to $\bigcup_a X_a$ such that $\xi(a) \varepsilon X_a$ for every a. The product is denoted by $\mathbb{P} X_a$; when $A = \{1,2,\ldots,n\}$ it is written $X_1 x X_2 x \ldots x X_n$.

The following three examples will clarify the definition:

Ex.1. Take two disjoint sets $X_1 = \{x_1\}$, $X_2 = \{x_2\}$ then $X_1 x X_2$ is the set of pairs (x_1,x_2), $x_i \varepsilon X_i$.

Ex.2. Let $X_1 = X_2 = X = \{x\}$ then XxX or X^2 is the set of all all ordered pairs (x',x'') where x' and $x'' \varepsilon X$.

Ex.3. By a function f on sets X_a (or variables x_a) to set Y is meant a function on $\mathbb{P} X_a$ to Y, e.g., $f(x_1,x_2,\ldots,x_n)$ when $A = \{1,2,\ldots,n\}$.

1.11 Application. Generalized Jones Algebra.

The groundwork has now been laid and we can proceed rapidly with the construction of our algebra. We shall call algebras for which time is the underlying variable, t-algebras, and those for which frequency is the underlying variable, ω-algebras. The algebras we deal with before applying the Wiener correlation process are called Jones algebras. Those obtained after applying the correlatinn will be called Mueller algebras. By an elementary Jones algebra we mean one involving a single resultant Jones matrix. By an elementary Mueller algebra we mean one for which the resultant Mueller matrix can be expressed in terms of a single Jones matrix.

Generalized Jones algebra involves passing from an elementary instrument and elementary wave to a collection of such. We tacitly include in the generalization the introduction of random functions of time. These random functions describe the incoming radiation and the position and orientation of the elementary instruments (atoms and molecules). The significance of the randomness of the functions does not affect the theory this chapter except at one point and that is the actual evaluation of the correlation. It is when we use statistical information about a set of random functions to change the correlation from a "time" average to a "phase" average that the term "random" takes on significance. This point is the subject matter of Chapters III and IV and is called statistical harmonic analysis.

The principal concept we shall need in this section is DIRECT SUM.

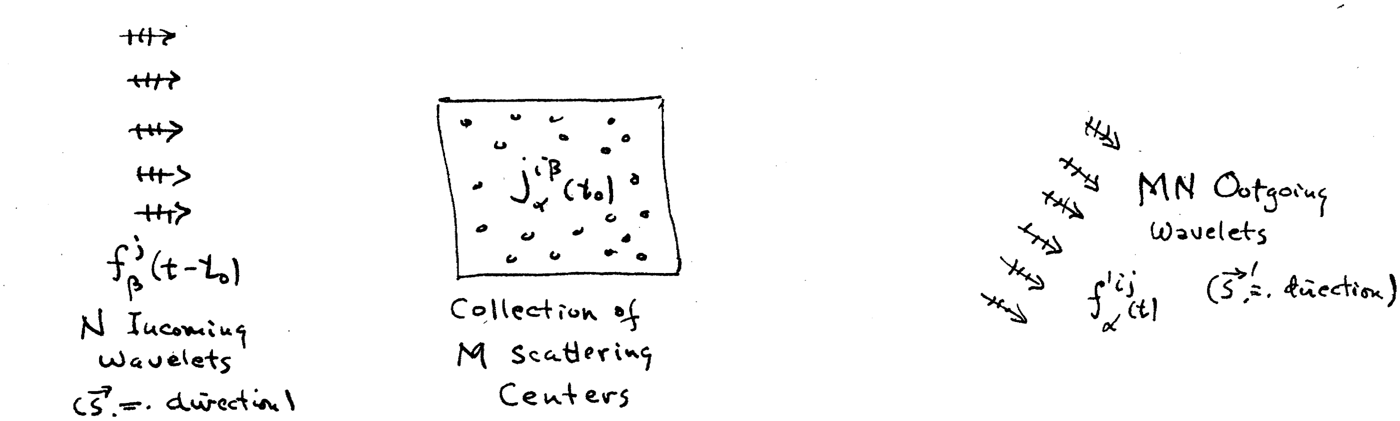
<u>Definition.</u> Let $f_1, f_2, \ldots, f_n$ be a set of functions on T to F. Let $A = \{1,2,\ldots,n\}$ be the set of subscripts. The <u>direct sum</u> is a function $f_i(t)$ on AxT to F.

<u>Examples</u>. The components, $u(t) = v_1(t)$, $v(t) = v_2(t)$, $w(t) = v_3(t)$, of a velocity are, separately, functions on T to U, T to V, and T to W. By introducing an indexing set $A = \{1,2,3\}$ to index the sets U,V,W, one may consider the vector $v_i(t)$ to be a function on AxT to V. Direct summs appear in tensor analysis under the guise of introducing new labling indices. Thus, $v_i(t)$ is a vector, i.e., a function of an index and a parameter. The set of quantities $U_{im}(t)$ where $m = 1,2,\ldots,N$ may be considered the direct sum of the velocities of each of a set of N particles, <u>it is a function of two indices and a parameter.</u> In fact, let $A = \{1,2,3,\}$, $N = \{1,2,\ldots,N,\}$ then $u_{im}(t)$ is a function on AxNxT to U. Tensors are most easily handled in a multi-index notation together with summation conventions. The essential part about a tensor lies in its law of transformation and not in its characteristic multi-index notation. Yet, the index notation is most familiar in its use in tensor analysis and the term "tensor notation" is rather firmly established.

We may, rather loosely, fall into the habit and call the multi-index functions of our algebra tensors instead of just functions. If we do, we do not imply that we have a group of transformations under which our functions are invariant. It is quite probable that in later research we will study these functions relative to groups of transformations. At that time we will be quite careful to state which indices have tensor character and which do not. For the present we shall use the index notation and the very great convenience of the summation convention, even extending it to parameters (in this chapter t, but not ω, will be so treated).

Present Application.

The notion of DIRECT SUM was introduced in order to generalize Jones algebra in a systematic manner, making it general enough to handle scattering problems, schematically indicated in the following figure.



Scattering, A Basic Optical Problem.

A set of wavelets $f^{j}_{\beta}(t)$ enter a general optical instrument consisting of a set of scattering centers $j^{i\beta}_{\alpha}(t)$ from which emerge a set of scattered wavelets, $f'^{ij}_{\alpha}(t)$. The quantities are, loosely, called tensors but better called multi-index functions or, briefly, just plain <u>functions</u>.

The algebra surrounding a single one of the incoming wavelets, say $f^{5}_{\beta}(t)$, and a single scattering particle, say $j^{3\beta}_{\alpha}(t)$, and a single emergen wavelet, say $f'^{35}_{\alpha}(t)$ is an elementary Jones algebra, i.e., a t-algebra for which the basic calculation is,

$$f'^{35}_{\alpha}(t) = \sum_{\beta=1}^{2} \int_{-\infty}^{\infty} j^{3\beta}_{\alpha}(t_o) f^{5}_{\beta}(t-t_o)\,dt_o \tag{41}$$

By a generalized Jones t-algebra we mean the direct sum of these elementary algebras. We shall speak of the incoming Maxwell functions $f^{j}_{\beta}(t)$, the Jones function $j^{i\beta}_{\alpha}(t)$ characterizing the set of scattering particles, and the outgoing Maxwell function $f'^{ij}_{\alpha}(t)$. After the introduction of a few summation conventions, the basic calculation for this generalized algebra will have the appearence

$$\boxed{f'^{ij}_{\alpha}(t) = j^{i\beta}_{\alpha}(\underline{t}_o) f^{j}_{\beta}(t-\underline{t}_o)} \quad . \tag{42}$$

It is important to notice that if N wavelets come in and strike M scattering centers, MN wavelets will emerge.

These are then indexed by the set of ordered pairs

$$MxN = \{(i,j)\}.$$

After passing through a second instrument, one will get

$$f''^{ijk}_{\alpha}(t) \text{ or better } f''^{i_1 i_2 i_3}_{\alpha}(t)$$

indexed by the set of ordered triples $M_1xM_2xM_3 = \{(i_1,i_2,i_3)\}$ where $i_i = 1,2,\ldots,M_i$. In this case there are $M_1M_2M_3$ emerging wavelets. There are two courses of action which one can take as he passes from one instrument to the next: 1) he can let the indices accumulate, 2) he can replace the ordered pairs (i,j) at each stage by their position in inverse dictionary order, i.e., by a single index, say k, which is defined

$$\boxed{(i,j) \implies k = i + M(j-1)} \tag{43}$$

Note, that equation (43) is a function on MxN to K, in fact, and this is the important point, it is a one-one correspondence. This enables us to index with K instead of MxN. The significance of the new ordering is the following: the first M terms represent the wavelets emerging from the M scattering centers as a result of the first incoming wavelet, the second M terms represent the wavelets emerging from the M scattering centers as a result of the second incoming wavelet, etc. In the sequal we shall either accumulate indices or reduce their number, depending on convenience. As a general rule it is easier to let indices accumulate until one has reached the desired Mueller ω-algebra.

First Part of the Generalized Summation Convention.

Equation (42) was stated with the proviso that a few conventions were necessary for its interpretation. The usual convention of tensor algebra will be used, i.e., a repeated index, one upper and one lower, is to be summed, i.e.,

$$a_\sigma x^\sigma \Longrightarrow \underset{\substack{\text{over range}\\ \text{of index } \sigma}}{\Sigma a_\sigma x^\sigma} \tag{44}$$

In our application, an important phenomena of the Fourier transform is the fact that in passing from ω-algebra to t-algebra, ordinary multiplication of ω-functions becomes a <u>convolution</u> on corresponding functions of t. We therefore make constant use of the operation

$$\int_{-\infty}^{\infty} j(t_o)e(t-t_o)dt_o$$

which we abbreviated

$$j(t_o) * e(t-t_o)$$

in Sec.1.3. It will be a further convenience to drop even the symbol "$*$" and write

$$\boxed{j(\underline{t}_o)e(t-\underline{t}_o) \Longleftrightarrow \int_{-\infty}^{\infty} j(t_o)e(t-t_o)dt_o} \tag{45}$$

and make a

<u>Generalized Summation Convention: Convolution.</u> Repetition of a time variable, $t, t_o, t_1, \ldots$ with a single bar under both occurances implies <u>convolution.</u>

Convolution can be thought of as a generalization of the notion of contraction. From this viewpoint, t is treated as an index. We are essentially using a continuous variable as an index. The whole purpose of introducing the few primitive concepts of general topology, Sec. 1.9 was to be able to give a general definition of the concept <u>function</u> and bring out with it such ideas as the fact that the sequence of real numbers $a_1, a_2, \ldots$ is a function on the set of integers, I, to the set of real numbers R. From this standpoint it is not a difficult step to the idea that, in the case of f(t), a function on T to F, T can be treated as an indexing set. This kind of generalization is an example of the freedom one gains through the adoption of the current abstract viewpoint in mathematics.

<u>Kronecker Product.</u>

one can view the Wiener correlation as taking place in two steps, a Kronecker product of a generalized Jones algebra by its complex conjugate, followed by a contraction on t, used as a summation index. The convolution can be viewed in this manner.

We have already dropped any real distinction between the variables t and the indices i,j, or α, β. We are using the term <u>function</u> for such entities as

$$f^{i}_{\alpha}(t), \quad j^{i\beta}_{\alpha}(t), \quad f^{ij\ldots k}_{\alpha}(t)$$

In fact, when it is convenient (as it will be in the application of the definition of Kronecker product) we may think of them as written in the form

$$f(i,\alpha,t), \quad j(i,\alpha,\beta;t), \quad f(i,j,\ldots,k;\alpha;t) .$$

<u>Definition.</u> By the <u>Kronecker Product</u> of two functions $f(x_1,x_2,\ldots,x_n)$, $g(y_1,y_2,\ldots,y_m)$ we mean the function

$$h(x_1,\ldots,x_n,y_1,\ldots,y_m) = f(x_1,\ldots,x_n)g(y_1,\ldots,y_m)$$

or, more briefly, $h(x,y) = f(x)g(y)$.

The Kronecker product h is a function on XxY to H = FG, f being a function on X to F and g a function on Y to G. FG is the direct product of sets F and G, i.e., the set with elements fg. Usually F = G and fg is ordinary multiplication.

Applying the above definition to the convolution, one first takes the Kronecker product

$$j(t_1)e(t-t_2)$$

and then "contracts" by identifying t_1 and t_2 and using the summation convention for convolution. This is the direct analog of tensor multiplication followed by contraction.

In eq.(42) we took the Kronecker product of the incoming Maxwell function and the Jones function of the set of scattering centers. We then contracted on t_o, writing

$$(42) \qquad f'^{ij}_{\alpha}(t) = j^{i\beta}_{\alpha}(\underline{t}_o)f^{j}_{\beta}(t-t_o)$$

We have left the indices i,j open, thus leaving the question of superposition until later in the analysis.

Summary.

The introduction of the DIRECT SUM, the KRONECKER PRODUCT, and the convolution aspect of the GENERALIZED SUMMATION CONVENTION, enabled us to construct a generalized Jones t-algebra in a convenient notation. The important point to recognize about t-algebra of the Jones variety is the universal validity of the principle of superposition which follows from the linearity of Maxwell's equations and the possibility of representing ANY incoming radiation in terms of solutions of these equations. This point was made in Sec. 1.7 . The details of the superposition in Jones t-algebra require some care. Tacitly it is assumed, when writing $f^{i}_{\beta}(t)$, that one is considering a set of functions of time AT A FIXED POINT. Maxwell functions represent plane waves, $f^{i}_{\beta}(t;x,y,z)$. The variables x,y,z have been suppressed on the tacit assumption that all superpositions are to be carried out for given (x,y,z) at all instants t. The net result is that in generalized Jones algebra we have the operation (in fact, the group operation for the Maxwell functions)

(46) $$f_\alpha(t) = \sum_i f^i_\alpha(t)$$

WHICH IS ALWAYS MEANINGFUL. It is only in Jones -algebra that the addition of Maxwell functions can only be interpreted as coherent superposition. This is the reason for omitting any consideration of generalized Jones -algebra.

One may summarize the rules of the "algebra" by pointing out that eq.(42) is the basic fule for computing the effect of the instrument on the incoming radiation. One takes the DIRECT SUM OF INSTRUMENTS IN PARALLEL, $j^{i\beta}_\alpha(t)$, and the KRONECKER PRODUCT, $j^{i\beta}_\alpha(t_1)j^{j\delta}_\gamma(t_2)$ of INSTRUMENTS IN SERIES followed by contractions to include the necessary convolutions.

1.12 Elementary Mueller Algebra.

Logically, it would not be necessary to introduce elementary Mueller algebra at this point. Pedagogically, we shall gain considerable understanding from a consideration of this case first.

A Logical Recapitulation.

As far as the logical structure of the theory is concerned· the elementary t-algebra, introduced in Sec. 1.4 is taken as primitive. The remaining algebras are defined in terms of it. It was necessary to begin the discussion, Sec. 1.1, with elementary Jones ω-Algebra. This algebra is closest to

Jones original formulation. The fact that addition in an ω-algebra corresponds to coherent superposition only, rules it out as a logical primitive.

On the other hand, Jones elementary ω-algebra is of the greatest practical importance for the application of our theory. Classical and quantum theory gives the Jones matrices of elementary ω-algebra. It is into this algebra that the raw theoretical material is fed. It is out of the Mueller ω-algebra that phenomenological predictions emerge.

In Sec. 1.2 we introduced Mueller's algebra in an ad hoc fashion in order to emphasize the parallelism in the two approaches to the basic problem, his and Jones'. In his forthcoming paper, "Foundations of Optics," Mueller introduces his algebra on a phenomenological basis. We introduce it in this section as a logical construction over elementary Jones t-algebra.

The term <u>elementary</u> has been used somewhat loosely up to this point. It refers to the class of radiations and instruments the algebra is capable of representing, the more elementary the smaller the class. The fact that the Maxwell vectors, in t-algebra are only restricted by the assumption that the Wiener correlation will be possible, leads to the situation that only for transients, real or artificial (by defining f(t) to be zero outside (-T,T)), is there a corresponding Jones ω-algebra. If one carries out the Wiener correlation in two steps, first averaging over an interval (-T,T) and then passing to the limit $T \longrightarrow \infty$, one may speak of a Mueller t-algebra as the limit of a sequence of squares (Kronecker products of transient Jones algebras by their complex conjugates) of Jones algebras obtained by using a sequence of intervals of definition $(-T_1,T_1), (-T_2,T_2), \ldots$ for which $T_n \longrightarrow \infty$, as $n \longrightarrow \infty$. We wish to avoid a deeper penetration into this limiting process in this algebraic chapter, but it is well to be aware of its existence. Elementary Mueller algebras arise from Maxwell vectors for which the Wiener correlation exists.

Generalized Summation Convention. Correlation.

We begin with the Jones t-algebra in which the Maxwell vectors are $f'_\beta(t)$ and $f_\alpha(t)$ and in which $j^\alpha_\beta(t)$ is the Jones matrix of the instrument. The basic law is

$$\boxed{f'_\beta(t) = j^\alpha_\beta(t_o) f_\alpha(t-t_o)} \tag{47}$$

For the purpose of introducing the CORRELATION PROCESS, consider the incoming radiation $f_\alpha(t)$. From it construct the pair of radiations $f_\alpha(t+t_1)$, $f^*_\beta(t_2)$ and take their Kronecker product $f_\alpha(t+t_1)f^*_\beta(t_2)$. The Wiener correlation of the pair is

$$\boxed{\varphi_{\alpha\beta}(t) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} f_\alpha(t+t_o) f^*_\beta(t_o)\,dt_o} \tag{48}$$

called the INTERFERENCE of $f_\alpha(t)$ and $f_\beta(t)$. It is the "time" example of a Wiener coherency matrix. Its Fourier transform is the "frequency" example and will be called the SPECTRUM of the same pair of functions. The physical significance of the interference, $\varphi_{\alpha\beta}(t)$, is discussed in Secs. 2.4 and 2.10. As a more convenient notation for the operation, eq.(48), we introduce the

<u>Generalized Summation Convention. Correlation.</u> Repetition of a time variable $t, t_0, t_1, \ldots$ with a double bar under the t in both occurances implies <u>correlation.</u>

This enables us to write

$$(49)\quad \boxed{f_\alpha(t+\underline{\underline{t}}_0)f_\beta^*(\underline{\underline{t}}_0) \quad \Longleftrightarrow \quad \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} f_\alpha(t+t_0)f_\beta^*(t_0)\,dt_0}$$

<u>The "Statistical Bridge."</u>

When, in Chapter III, we discuss generalized harmonic analysis from a statistical standpoint, it will be clear that it is the statistical bridge from the Jones algebra of theoretical quantities to the Mueller algebra of observables, i.e., phenomenological quantities. Applying the correlation process, one crosses the statistical bridge and obtains

$$(51)\quad \begin{aligned} f_\alpha'(t+\underline{\underline{t}}_0)f_\beta'(\underline{\underline{t}}_0) &= j_\alpha^\gamma(\underline{t}_1)j_\beta^{\delta*}(\underline{t}_2)f_\gamma(t+\underline{\underline{t}}_0-\underline{t}_1)f_\delta^*(\underline{\underline{t}}_0-\underline{t}_2) \\ &= j_\alpha^\gamma(\underline{t}_1)j_\beta^{\delta*}(\underline{t}_2)f_\gamma(t-\underline{t}_1+\underline{t}_2+\underline{\underline{t}}_0)f_\delta^*(\underline{\underline{t}}_0) \end{aligned}$$

i.e., the elementary Mueller t-algebra with the basic law

$$(52)\quad \boxed{\varphi'_{\alpha\beta}(t) = j_\alpha^\gamma(\underline{t}_1)j_\beta^{\delta*}(\underline{t}_2)\;\varphi_{\gamma\delta}(t-\underline{t}_1+\underline{t}_2)}$$

connecting the incoming and outgoing radiation. It is convenient to define the MUELLER FUNCTION

(52') $$\boxed{m_{\alpha\beta}^{\gamma\delta}(t_1,t_2) = j_\alpha^\gamma(t_1)j_\beta^{\delta *}(t_2)}$$

as the Kronecker product of two Jones functions. In terms of the Mueller function, the basic law has the form

(52") $$\varphi'_{\alpha\beta}(t) = m_{\alpha\beta}^{\gamma\delta}(\underline{t}_1,\underline{t}_2)\varphi_{\gamma\delta}(t-\underline{t}_1+\underline{t}_2)$$

involving a double convolution. Making the Fourier transform, $\mathfrak{F}$, the convolutions turn into multiplications of the usual sort. One obtains the basic law

(53) $$S'_{\alpha\beta}(\omega) = (2\pi J_\alpha^\gamma(\omega))(2\pi J_\beta^\delta(\omega))^* S_{\gamma\delta}(\omega)$$

in Mueller ω-algebra. If we now define the corresponding Mueller function

(54) $$\boxed{M_{\alpha\beta}^{\gamma\delta}(\omega) = (2\pi J_\alpha^\gamma(\omega))(2\pi J_\beta^\delta(\omega))^*}$$

the basic law in elementary Mueller ω-algebra becomes

(55) $$\boxed{S'_{\alpha\beta}(\omega) = M_{\alpha\beta}^{\gamma\delta}(\omega)S_{\gamma\delta}(\omega)}$$

Thus the relation between incoming radiation $S_{\alpha\beta}(\omega)$, and the intervening instrument $M_{\alpha\beta}^{\gamma\delta}(\omega)$, and the emerging radiation $S'_{\alpha\beta}(\omega)$ is in the required standard form. The functions $S'_{\alpha\beta}(\omega)$, $S_{\alpha\beta}(\omega)$ are called SPECTRA. The $\varphi'_{\alpha\beta}(t)$, $\varphi_{\alpha\beta}(t)$ are called INTERFERENCES. Collectively they are called WIENER COHERENCY MATRICES.

Transformation to Complex Mueller Algebra.

As indicated previously, after reaching the Mueller ω-algebra it is practical to convert the multi-index functions to matrices and vectors. The transformation is

$$(56)\qquad \begin{aligned} \mathcal{M}_{ij}(\omega) &= M_{\alpha\beta}^{\gamma\delta}(\omega) \\ \mathcal{L}'_i(\omega) &= S'_{\alpha\beta}(\omega) \\ \mathcal{L}_j(\omega) &= S_{\gamma\delta}(\omega) \end{aligned} \qquad \text{where:} \quad \begin{aligned} i &= \alpha + 2(\beta - 1) \\ j &= \gamma + 2(\delta - 1) \end{aligned}$$

Equation (54) may now be rewritten

$$(57)\qquad \boxed{\mathcal{L}'(\omega) = \mathcal{M}(\omega)\mathcal{L}(\omega)}$$

where $\mathcal{L}$, $\mathcal{L}'$ are called COMPLEX STOKES VECTORS. The $\mathcal{M}$ is called a COMPLEX MUELLER MATRIX.

Transformation to Real Mueller Algebra.

From a theoretical standpoint, Complex Mueller Algebra is the most convenient form. We had arrived at it even before we were aware of the connection between the Stokes vector and the coherency matrix. On the other hand, all Mueller's work, including his phenomenological definitions, has been based on the original (I,M,C,S) form of the Stokes vector. The transformation of the complex algebra to the real algebra is therefore of practical importance. Using eq.(39), Ch. II, one may write

$$(58)\qquad \mathcal{L}(\omega) = (1/2)\begin{bmatrix} I(\omega)+M(\omega) \\ C(\omega)+iS(\omega) \\ C(\omega)-iS(\omega) \\ I(\omega)-M(\omega) \end{bmatrix} = \begin{bmatrix} \mathcal{L}_1 \\ \mathcal{L}_2 \\ \mathcal{L}_3 \\ \mathcal{L}_4 \end{bmatrix}$$

If one introduces an index notation an writes

$$(59) \qquad L(\omega) = \begin{bmatrix} L_1(\omega) \\ L_2(\omega) \\ L_3(\omega) \\ L_4(\omega) \end{bmatrix} = \begin{bmatrix} I(\omega) \\ M(\omega) \\ C(\omega) \\ S(\omega) \end{bmatrix}$$

One may introduce the transformation T from the real to the complex algebra, defined:

$$(60) \qquad \boxed{\mathcal{L} = TL}$$

It is evident from this definition that

$$(61) \qquad T = (1/2) \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & i \\ 0 & 0 & 1 & -i \\ 1 & -1 & 0 & 0 \end{bmatrix}$$

and that the inverse is

$$(62) \qquad T^{-1} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & -i & i & 0 \end{bmatrix}$$

To find the corresponding transformation on the M's we notice that

$$\mathcal{L}' = \mathcal{M}\mathcal{L}$$

From which it follows that

$$\mathcal{L}' = TL' = TML = TMT^{-1}\mathcal{L}$$

and hence that

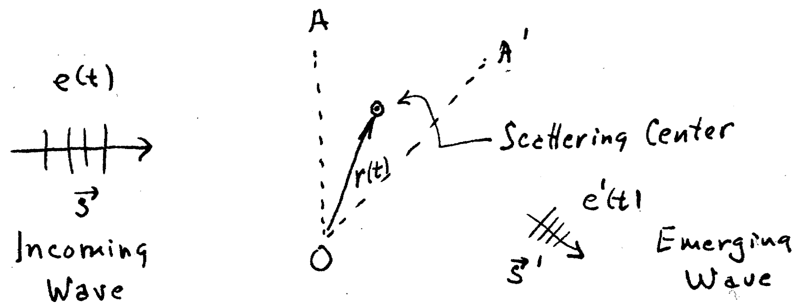
$$(63) \qquad \boxed{\mathcal{M} = TMT^{-1}}$$

which is the required transformation from the real Mueller matrix M to the complex Mueller matrix $\mathcal{M}$.

Equatinns (54), (56), and (63) enables us to calculate the relation between the Jones matrix J and the real Mueller matrix M. This derivation of the relation between J and M is original with us. Mueller arrived at the same result with a more physical type of reasoning using the notion of a transmission coefficient.

1.13 Retarded Maxwell Vectors.

In the application of our algebra to the scattering problem it is necessary to account for the positions of the various scattering centers which are random functions of time, i.e., $r_i(t)$. The incoming wave is assumed



to have direction $\vec{s}$ and the emerging wave is assumed to have direction $\vec{s}'$. Phases of the incoming wave are referred to plane OA perpindicular to s and phases of the emerging wave are referred to plane OA' perpindicular to s'. One then writes (making an approximation which is discussed in Sec.4.1)

$$(64)\qquad \begin{aligned} e(t - \vec{s}.\vec{r}_i(t)/v) &= e_i(t) \\ e'(t+\vec{s}'.\vec{r}_i(t)/v) &= e'_i(t) \end{aligned}$$

for the incoming and outgoing Maxwell vectors. If the $r_i(t)$ happen to be constants we can take the Fourier transform of eq.(64) and obtain

$$(65)\qquad \begin{aligned} \mathcal{F}\, e_i(t) &= P_i(\omega)E(\omega) \\ \mathcal{F}\, e'_i(t) &= P'_i(\omega)E'(\omega) \end{aligned}$$

where $P_i = e^{is.r_i\omega/v}$, $P'_i = e^{-is'.r_i\omega/v}$ are phase factors. It is in ω-algebra and not in t-algebra that one has phase factors.

1.14 Generalized Mueller Algebra.

We now approach the end of our structural theory. It began with Jones t-algebra and will be completed with the construction of generalized Mueller ω-algebra. We will obtain expressions for the STOKES VECTORS and MUELLER MATRICES which require for their evaluation

1. The Maxwell vectors characterising the underlying set of completely polarized plane waves.
2. The Jones matrices characterixing the underlying set of elementary instruments (atoms, molecules, colloidal particles).
3. The Statistical Distribution of the Waves.
4. The Statistical Distribution of the Instruments.

We will then have achieved a STATISTICAL THEORY OF OPTICS which bears the same relation to MUELLER'S PHENOMENOLOGICAL THEORY that STATISTICAL MECHANICS bears to THERMODYNAMICS. The detailed working out of our theory for special cases is the subject matter of Chapters III and IV.

We begin with the basic law of generalized Jones algebra,

$$f'_\alpha(t) = j_\alpha^{j\beta}(\underline{t}_o)f_{j\beta}(t-\underline{t}_o) \tag{42'}$$

where

$$f_{j\beta}(t) = f_\beta(t - (\vec{s}-\vec{s}').\vec{r}_j(t)/v) \tag{66}$$

In eq.(42'), the summation on j is possible because all retardation effects have been accounted for, eq.(66). Furthermore it should be noticed that we are restricting the instrument statistics to the description of their position, $r_j(t)$. In order to account for the orientation or, better, the statistical distribution of the instrument over its "states" it is necessary to look into the quantum theory leading to the form of the Jones matrices in elementary ω-algebra. This study is beyond the scope of this paper and is an interesting and important avenue for further research. In the present problem the instrument statistics are included as a retardation in the expression for the Maxwell vector.

Application of the Wiener correlation gives

(67) $$\varphi'_{\alpha\beta}(t) = (j_\alpha^{j\gamma}(\underline{t}_1)j_\beta^{k\delta*}(\underline{t}_2))f_{j\gamma}(t+\underline{t}_0-\underline{t}_1)f^*_{k\delta}(\underline{t}_0-\underline{t}_2)$$

(68) $$\boxed{\varphi'_{\alpha\beta}(t) = m_{\alpha\beta}^{jk\gamma\delta}(\underline{t}_1,\underline{t}_2)\varphi_{jk\gamma\delta}(t-\underline{t}_1+\underline{t}_2)}$$

The Fourier transform of this "interference" gives

(69) $$\boxed{S'_{\alpha\beta}(\omega) = M_{\alpha\beta}^{jk\gamma\delta}(\omega)S_{jk\gamma\delta}(\omega)}$$

where

(70) $$\boxed{M_{\alpha\beta}^{jk\gamma\delta}(\omega) = (2\pi J_\alpha^{j\gamma}(\omega))(2\pi J_\beta^{*k\delta}(\omega))}$$

when j = k, the M's are called AUTO-MATRICES, when j $\neq$ k they are called CROSS-MATRICES.

<u>The General Law of Composition of Mueller Matrices.</u>

The difficulties of the problem are now reduced to the evaluation of the interference

(71) $$\varphi_{jk\gamma\delta}(t) = f_{j\gamma}(t+\underline{t}_o)f^*_{k\delta}(\underline{t}_o)$$

In Chapter IV, three cases are worked out in detail and lead to the general result

(72) $$S_{jk\gamma\delta}(\omega) = N_{jk}(\omega)S_{\gamma\delta}(\omega)$$

where $N_{jk}(\omega)$ is characteristic of the distribution of the scattering centers. This allows us to write

(73) $$\boxed{M^{\gamma\delta}_{\alpha\beta}(\omega) = N_{jk}(\omega)M^{jk\gamma\delta}_{\alpha\beta}(\omega)}$$

as the GENERAL LAW FOR THE COMPOSITION OF MUELLER MATRICES. In complex Mueller algebra this law has the form

(74) $$\mathcal{M}(\omega) = N_{jk}(\omega)\mathcal{M}^{jk}(\omega)$$

<u>Standard Forms for $N_{jk}(\omega)$.</u>

The importance of the N_{jk} is well illustrated by the following three examples which embody a unification of the treatment of scattering from liquids, solids and gases, i.e., coherent scattering (solids), partially coherent scattering (liquids, electrolytes), and incoherent scattering (gases).

1. Coherent Scattering.

(75) $$N_{jk}(s,s',\omega) = e^{i\frac{\omega}{v}(s-s').(r_j-r_k)/V}$$

2. Incoherent Scattering.

(76) $$N_{jk} = \delta_{jk}$$

3. Partially Coherent Scattering.

(77) $$N_{jk}(s,s',\omega) = \begin{cases} 1 & \text{when } j = k \\ i(s)/nV & \text{when } j \neq k \end{cases}$$

$$s = \omega|s-s'|/v \; ; \quad i(s) = \mathfrak{F}(\rho(r) - n)$$

where: N = concentration

$\rho(r)$ = radial concentration density.

In the case of an electrolyte:

(78) $$\boxed{i(s) = \pm 1/2(1+(s/\kappa)^2)}$$

where + applies to ions with opposite sign, - applies to ions with the same sign.

$$\kappa^2 = 2nz^2|e|^2/\varepsilon kT \; .$$

The details are worked out in Chapter IV.

Generalized Mueller Algebra.

This completes the construction of generalized Mueller algebra. In the cases for which eq.(72) is valid we have a three step rule for computing the MUELLER MATRIX $\mathcal{M}$ FOR THE GENERAL INSTRUMENT.

Step 1. Compute $\mathcal{M}^{jk}(\omega)$ from eq.(70), followed by the transformation, eq.(56).

Step 2. Compute $N_{jk}(\omega)$ using appropriate eq.(75)-(77).

Step 3. $\mathcal{M}(\omega) = N_{jk}(\omega)\mathcal{M}^{jk}(\omega)$.

The question of adding $\vec{\mathcal{L}}_1 + \vec{\mathcal{L}}_2$ never arises as such. It corresponds to incoherent superposition. In most cases of interest to us, the radiations are not incoherent and the addition must be carried out (conceptually) with the aid of instruments and is therefore included in the above rule. Therefore the one and only universally valid law of Mueller algebra is

$$\boxed{\mathcal{L}'(\omega) = \mathcal{M}(\omega)\,\mathcal{L}(\omega)} \tag{79}$$

All the difficulties of the problem are concentrated in the computation of the matrix, $\mathcal{M}(\omega)$, which are the elements of the "algebra" and are the mathematical representation of the instrumental setup intervening between $\mathcal{L}$ and $\mathcal{L}'$.

1.15 Statistical Composition of Mueller Matrices.

We now return to eq.(71) and to a difficulty which was bypassed at that point. When written in slightly less abbreviated form, this equation becomes

(80) $$\varphi_{jk\gamma\delta}(t) = f_\gamma(t+\underline{\underline{t}}_o - \frac{(s-s').r_j(t+\underline{\underline{t}}_o)}{v})f^*_\delta(t_o - \frac{(s-s')r_k(\underline{\underline{t}}_o)}{v})$$

where correlation on $\underline{\underline{t}}_o$ is assumed. Even this form of the retardation makes use of the assumption that $r_j(t)$ fluctuates insuch a way that this t and the parameter t in the function f(t) may be treated independently. One way of making this appear reasonable is treated in Section 4.1 We feel however that this step has more universal validity than the special treatment would indicate. This is a statistical problem and beyond the scope of the algebraic part of the subject. A deeper study of the matter is a desirable avenue for ruther research. Specifically we therefore make the important assumption that the t appearing in f(t) and the t appearing in $r_j(t)$ are independent variables and that we can apply the correlation process to the one and average the other separately. It is also necessary to assume that the random functions $r_j(t)$ are homogeneous in time, which is quite reasonable physically. This assumption takes the form $r_j(t+t_o)$ (=) $r_j(t)$.

With this preamble we rewrite eq.(80)

(81) $$\varphi_{jk\gamma\delta}(t) = f_\gamma(t+\underline{\underline{t}}_1 - \frac{(s-s').(r_j(\underline{\underline{t}}_2) - r_k(\underline{\underline{t}}_2))}{v})f^*_\delta(\underline{\underline{t}}_1)$$

involving one correlation on t_1 and one average on t_2.

Independent averages are always to be viewed with suspicion and as an approximation which requires deeper justification. The interesting point is that this assumption when valid leads to a reasonable and remarkable law for the composition of Mueller matrices which applies to liquids, solids, gases, electrolytes and colloidal suspensions.

We may interpret eq.(81)

$$\varphi_{jk\gamma\delta}(t) = \varphi_{\gamma\delta}(t - (s-s').(r_j(\underline{\underline{t}}_2)-r_k(\underline{\underline{t}}_2))/v) \tag{82}$$

Taking the Fourier transform gives

$$S_{jk\gamma\delta}(\omega) = S_{\gamma\delta}(\omega)\ e^{i\omega(s-s').(r_j(\underline{\underline{t}}_2)-r_k(\underline{\underline{t}}_2))/v} \tag{83}$$

This is the required decomposition. We define

$$\boxed{N_{jk}(\omega) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} e^{i\omega\frac{(s-s')\cdot(r_j(t)-r_k(t))}{v}}\, dt} \tag{84}$$

and call it the DEBYE DISTRIBUTION FUNCTION. In terms of it one may write

$$S_{jk\gamma\delta}(\omega) = N_{jk}(\omega)S_{\gamma\delta}(\omega) \tag{85}$$

which is eq.(72). The evaluation of eq.(84) in special cases has been carried out in Chapter IV.

1.16 Summary. Block Diagram.

The structure of our generalized optical algebra is complete. We now have a bridge from the Jones ω-algebra, into which elementary theoretical data is fed, to the Mueller ω-algebra, the basic elements of which are observables. In order to make more clear what the structure of the bridge is like we have concluded the chapter with a block diagram which may be opened out and viewed in conjunction with the reading of the chapter. We ourselves have found it a useful visual aid and mnemonic device.

A study of the block diagram reveals, rather quickly, a number of things about the chapter which could, at best, be conveyed rather slowly and tediously in words alone. The block diagram can itself be viewed as a 4x2 matrix with:

Row indices: 1 .=. Jones ω-algebra
2 .=. Jones t-algebra
3 .=. Mueller t-algebra
4 .=. Mueller ω-algebra

Column indices: 1 .=. Elementary algebra
2 .=. Generalized algebra

At the beginning of the research the elements (1,1) and (4,1) were in existence, the work of Jones and Mueller. We constructed the remaining elements. Element (1,2) is omitted as of no practical importance. Element (1,1) is chiefly useful as the input for the theoretical results of quantum mechanics for the case of simple harmonic incident radiation. The path (2,1) to (2,2) to (2,3) is as far as one can proceed easily without simplifying assumptions. We have made the step (2,3) to (2,4) by assuming that one can average independently over the incoming waves and the positions of the scattering particles. Physically, this is quite reasonable for a large class of situations. By doing so we reach Mueller's phenomenological algebra. We reach it with two important gains: we have generalized Mueller matrices to include cross- as well as auto-matrices and we have thus been able to state a general law for the composition of Mueller matrices in terms of a general distribution function N of a type associated with the name of Debye. One can now break down an optical system into a "network" of elementary instruments. We have a simple correspondence between the Jones matrix and the Mueller matrix for an elementary instrument and a general law for the composition of these matrices to give the Mueller matrix of the network.

On the diagram there are nine optical algebras. The most usuful classes for analysis are the Jones t-algebras and the Mueller ω-algebras. In the blocks of the diagram each of the algebras is characterized by its form & the statement of the basic law, i.e., the relation between the incoming and outgoing radiation. The first step in the solution is to set up the elementary ω-algebra of the parts. This is done either classically or quantum mechanically. These are transformed into a set of t-algebras under $\mathfrak{F}^{-1}$. The next step is to take the direct sum of the elementary algebras to obtain the generalized Jones t-algebra representing the network. The next step is to take the Kronecker product of this generalized Jones algebra with its complex conjugate. Contracting this on the two open t-variables with the aid of Wiener correlation yields a generalized Mueller t-algebra. The last step is to transform this into a Mueller ω-algebra with a $\mathfrak{F}$-transform.

It should should be remarked that one can always transform , the interference of the outgoing radiation, and obtain $S' = \mathfrak{F}\varphi'$ or $\int S = \int\mathfrak{F}\varphi'$, the spectrum or integrated spectrum of the outgoing radiation. One can do this even when it is not feasible to perform separate averaging and obtain the corresponding Mueller algebra. In fact our analysis provides a general criterion for the existence of a Mueller ω-algebra. The S's are Wiener coherency matrices for the out coming radiation.

It has been convenient to classify the mathematical operations by which one goes from one algebra to the next into algebraic and topological. The block diagram has been so arranged that vertical transitions are topological and horizontal ones are algebraic. These transitions have been lettered and may be summarized as follows:

<u>Transition A:</u> This is a Fourier transform, $\mathfrak{F}$. It is discussed in Sec. 1.4. In particular see eqs.(19), (21),(24),(25), and (26). Under this transformation multiplication becomes convolution.

<u>Transition B:</u> This transformation is a direct sum of

elementary algebras. Just before transition C one takes the Kronecker product of elementary algebras. See Sec. 1.11.

Transition C: This transformation is the Wiener correlation. it takes the form of a generalized contractinn following a Kronecker product. See Sec.L.12. See eqs. (49), (50),(51),(52),(67),(68); see also, Sec. 1.9.

Transition D: This is a Fourier transform.

Transition E: This is a simple algebraic transformation which divides the order of the indexed functions by two. It has the practical advantage of reducing a highly "indexed" algebra to a simple matrix-vector form. See eqs. (43),(56),(57).

Transition F: This is the same as Transition C, a Wiener correlation.

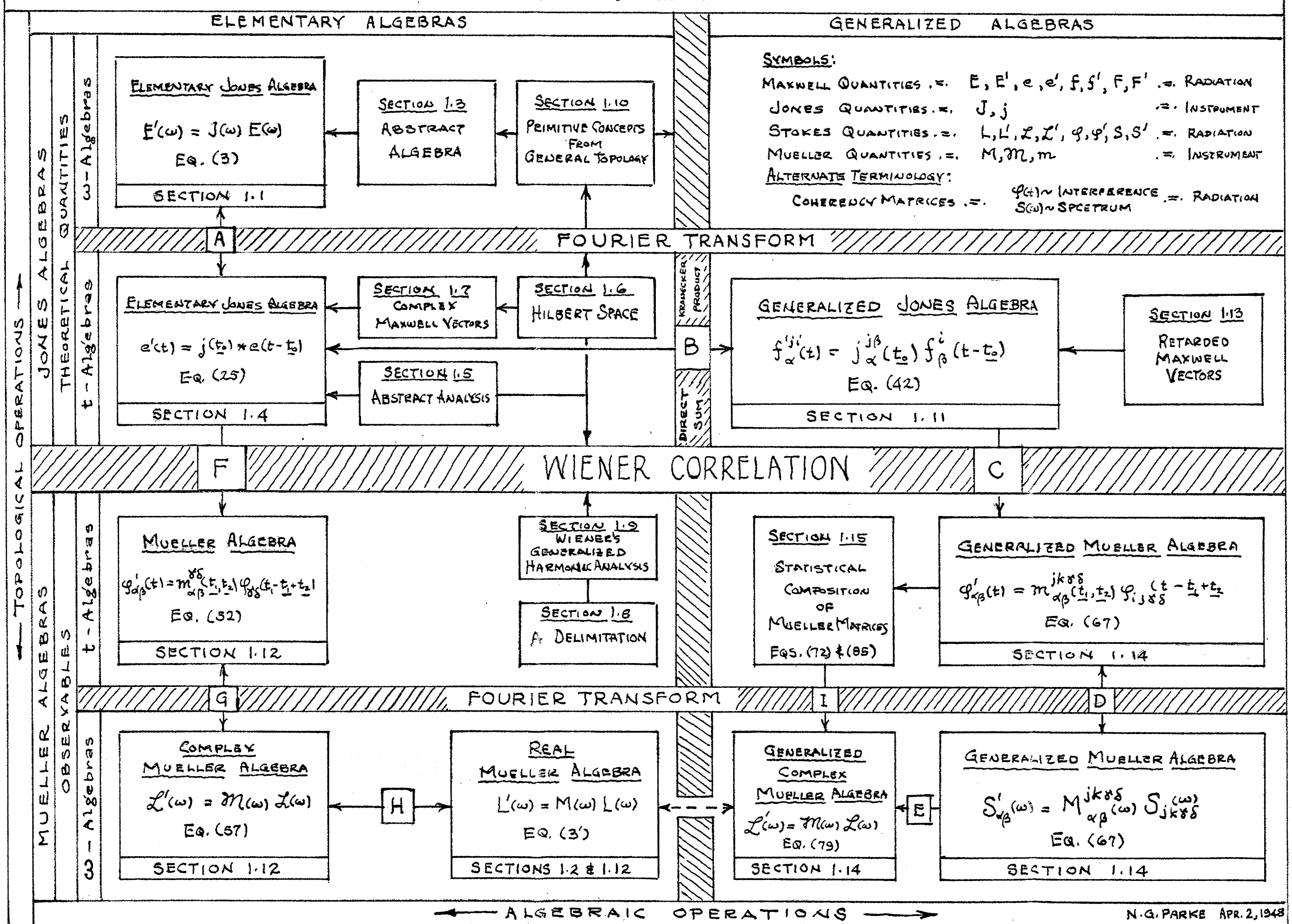
Transition G: This is a Fourier transform followed by a transition like E.

Transition H: This is a matrix transformation, T, which takes our form of matrix algebra into Mueller's form. Real Mueller algebra is the one which goes with the original form of the Stokes vector. This is the form in which Mueller developed his operational definitions. The theoretical disadvantage of real Mueller algebra is that the expression for M in terms of J is much more complicated than the corresponding expression for $\mathcal{M}$ in terms of J in our complex algebra. Mueller derived the J - M relationship with the aid of the notion of transmission coefficients. This relationship is given in Sec. 2.12. See also eqs. (58),(59),(60),(61),(62),(63).

Sections 1.3, 1.5, 1.6, 1.7, 1.8, 1.9,1.10 are general rather than specific and have been included as mathematical background to make this exposition as self contained as possible. The places at which these sections illuminate our structure theory are indicated in the block diagram.

THE STRUCTURE OF GENERALIZED OPTICAL ALGEBRA

BLOCK DIAGRAM OF CHAPTER I



CHAPTER II

ILLUSTRATIONS AND EXAMPLES

The material in the remaining three chapters is an expsition of the heuristic research which forms the physical backgroud of the generalized optical algebra. It is the research which was completed prior to the complete conception of that algebra. It contains all the applications of the theory which have been made to date. We suggested, in Chapter I, at the end of Sec. 1.2 that one might wish to read the remaining three chapters prior to continuing with the general theory which from that point onward is largely mathematical. However, it will be necessary to refer to Chapter I for specific formulae and mathematical ideas.

2.1 A Physical Interpretation of Generalized Harmonic Analysis.

A list of references to Wiener's and related papers on the generalized harmonic analysis is given in Sec. 1.9. The process, applied to a function f(t), takes place in two steps.

Step I: $$\varphi(\tau) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} f(t+\tau)f(t)\,dt$$

Step II: $$S(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\varphi(\tau)\,\frac{e^{iu\tau}-1}{i\tau}\,d\tau$$

These steps have a vivid and simple physical interpretation. For, consider an Idealized Young Interference Experiment. Let the incident radiation be linearly polarized, not necessarily monochromatic, plane wave represented by the component $f(t)$ of its electric vector. We take $f(t)$ to be real. Back of the slits one places a photographic plate, parallel to the plane of the slits. On this plate, each line

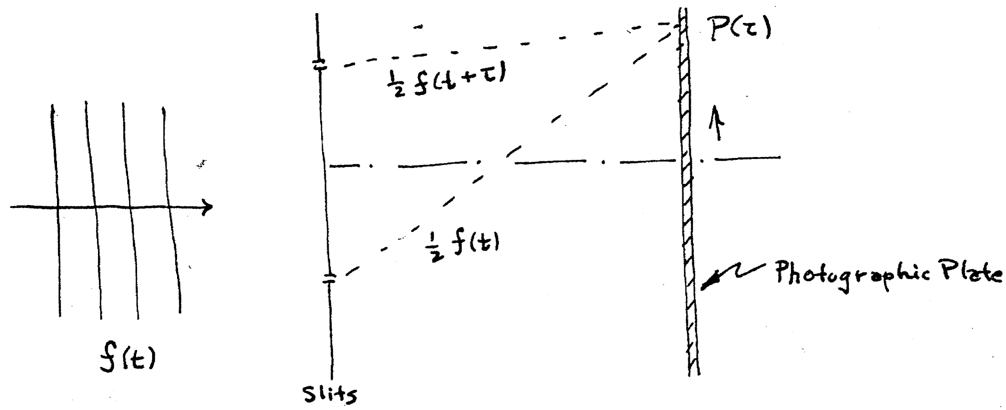


Figure 1. An Idealized Young Interference Experiment.

On this plate, each line parallel to the slits is the correspondent of a time difference τ between the two interfering components of the original wave. The resultant electric vector at position $P(\tau)$ is

$$F(t;\tau) = (1/2)[f(t+\tau) + f(t)]$$

since we have coherent superposition.

The average intensity of the incoming beam is

$$(1) \qquad I = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} f(t)f(t)\,dt = \varphi(0)$$

The average intensity at the photographic plate is

$$(2) \quad I(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} (F(t+\tau))^2 dt$$
$$= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} [f(t+\tau) + f(t)]^2 dt$$
$$= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(t)f(t)dt + \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(t)f(t+\tau)dt$$
$$= \tfrac{1}{2}\varphi(0) + \tfrac{1}{2}\varphi(\tau)$$

From this it follows that

$$(6) \qquad \boxed{\varphi(\tau) = 2I(\tau) - I(0)}$$

We see, therefore, that $\varphi(\tau)$ is the variable part of the interference pattern. We shall refer to $\varphi(\tau)$ resulting from the first step of the GHA[1] as CORRELATIONS or, more usually INTERFERENCES.

---

1) GHA = Generalized Harmonic Analysis.

To gain an interpretation of Step II of the GHA the first point to notice is that this step gives the integrated Fourier transform of $\varphi(\tau)$, i.e., it is an analytical expression similar to the one appearing in Plancherel's theorem (Sec.1.5, eqs. (D),(E)) except that the derivative d/du is not taken here. The use of the integrated transform is a matter of technical convenience. It is a generalization which makes it possible to work with a wider class of functions, in particular, with those having sharp frequency spectra. This last is an idealization but a useful one. Throughout the paper we shall use the expressions

$$(4) \qquad S'(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\varphi(\tau)e^{iu\tau}d\tau = \frac{1}{2\pi}\frac{d}{du}\int_{-\infty}^{\infty}\varphi(\tau)\,\frac{e^{iu\tau}-1}{i\tau}\,d\tau$$

and

$$(5) \qquad S(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\varphi(\tau)\,\frac{e^{iu\tau}-1}{i\tau}\,d\tau$$

interchangably. $S'(u)$ is called the SPECTRUM AND $S(u)$ is called the INTEGRATED SPECTRUM of $f(t)$. We shall often omit the prime and still mean the spectrum.

Let us now apply the GHA to the elementary function $e^{-i\omega t}$. The first step gives

$$\varphi(\tau) = \frac{\omega}{2\pi}\int_{-\pi/\omega}^{\pi/\omega} e^{-i\omega\tau}\,dt = e^{-i\omega\tau}$$

The second step gives

$$(7) \qquad S(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega\tau}\,\frac{e^{iu\tau}-1}{i\tau}\,d\tau$$

This leads to a consideration of the integral[1]

1) L.A.Pipes: "Applied Mathematics for Physicists and Engineers." McGraw-Hill, N.Y., 1946. p.473.

$$(8) \qquad \sigma(a) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{ia\tau}}{i\tau}d\tau = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\cos a\tau}{i\tau}d\tau + \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\sin a\tau}{\tau}\,d\tau$$

or

$$(9) \qquad \sigma(a) = \begin{cases} 1/2 & \text{when } a > 0 \\ 0 & \text{when } a = 0 \\ -1/2 & \text{when } a < 0 \end{cases}$$

In terms of this integral

(10) $$S(u) = \sigma(u-\omega) - \sigma(-\omega) = \sigma(u-\omega) + \sigma(\omega)$$

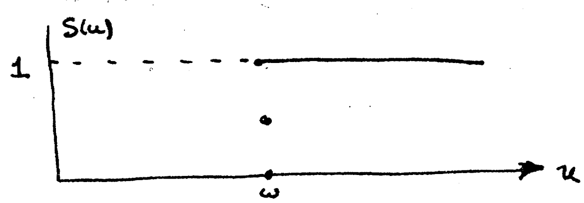
But $\omega > 0$ and hence $\sigma(\omega) = 1/2$. As a result

(11) $$S(u) = \mathbb{1}(u-\omega)$$

where

(12) $$\mathbb{1}(x) = \begin{cases} 1 \text{ when } x > 0 \\ 1/2 \text{ when } x = 0 \\ 0 \text{ when } x < 0 \text{.} \end{cases}$$

This is the Heaviside unit step function. If one makes a graph of $S(u)$, one sees that it is



The Integrated Spectrum: $S(u)$

the total or integrated energy in the spectrum up to the frequency u. As might be expected, there is no energy until one reaches frequency ω at which frequency one gets the full intensity of unity, the whole intensity being concentrated at this frequency. This is the justification of the term "integrated spectra" and the significance of the second step of the GHA.

Now consider a more complex example, a sharp line spectrum containing several frequencies, amplitudes, and phases. Let

(13) $$f(t) = \sum_{1}^{n} a_k e^{-i\omega_k t}$$

where ω_k = frequency, $|a_k|$ = amplitude, and $\arg(a_k)$ = phase. The first step of GHA yields

(14)
$$\varphi(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \sum_{km} a_k a_k^* e^{-i\omega_k \tau} e^{-i(\omega_k-\omega_m)t}\, dt$$
$$= \sum_1^n a_k a_k^* e^{-i\omega_k \tau}$$

There are seveal remarkable features about the last expression. It differs from eq.(13) only by virtue of the fact that the amplitudes a_k in the former have been replaced by their quadratic duals, $a_k a_k^*$. The absolute phase of a_k cancels. The $a_k a_k^*$ are intensities and observables. The transformatin from theoretical quantities to observables is the major service of the analysis.

Now consider the result of step II.

(15)
$$S(u) = \sum_1^n a_k a_k^* \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega_k \tau} \frac{e^{iu\tau}-1}{i\tau}\, d\tau$$

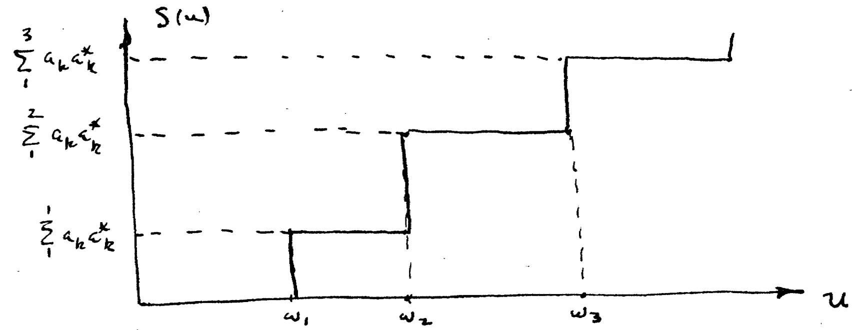
Using the Heaviside step function, the integrated spectrum is

(16)
$$S(u) = \sum_1^n a_k a_k^* \mathbb{1}(u-\omega)$$

In terms of the Dirac δ-functions the spectrum is

(17)
$$S'(u) = \sum_1^n a_k a_k^* \delta(u-\omega_k)$$

i.e., it consists of n sharp lines of intensity $a_k a_k^*$ at frequency ω_k. The graph of the integrated spectrum S(u), is



One important point is being bypassed for the moment. The function f(t) in eq.(13) is complex. We have referred here to $a_k a_k^*$ as the intensity of the line having frequency ω_k represented by this function. The function f(t) represents some real component of an electric vector and the question arises as to the steps one must take to get such a complex representation. If the real part of f(t) is that which has physical significance, what law determines the corresponding complex part? The GHA associates unit spectral intensity with the function e^{-iu_0t}, i.e., a step function which is zero up to u_1 and 1 thereafter. The relation between the coherency matrix hinges on this point which is discussed in detail in Sec. 2.4.

2.2 Wiener's Coherency Matrix.

In this section we shall confine our attention to the mathematical aspects of the coherency matrix. Its physical significance will come out later in two respects, in its relation to the Stokes vector and in the task of describing the very large collection of wavelets emerging from the scattering centers of a crystal, liquid, etc. It is such a situation which is described by a set of functions

$$f_1(t), f_2(t), \ldots, f_n(t)$$

We begin our analysis by considering a linear combination of these functions

$$f(t) = \Sigma\, a_n f_n(t)$$

The first step of the GHA gives

$$\varphi(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f(t+\tau) f^*(t)\, dt \tag{20}$$
$$= \Sigma\, a_i a_j^* \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f_i(t+\tau) f_j^*(t)\, dt$$

If one defines

$$\varphi_{ij}(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} f_i(t+\tau) f_j^*(t)\, dt \qquad \text{(interference)} \tag{21}$$

eq.(20) may be written

$$\varphi(\tau) = \Sigma\, a_i \varphi_{ij} a_j^* \tag{22}$$

The significance of this result is that, if one has a set of functions, it is possible to infer the first step of a GHA of ANY linear combination of this set from the n^2 GHA of this set in pairs. The required result is the Hermitian form, eq.(22). This result marks out clearly the depth to which one must push the statistical part of the analysis.

The second step of the GHA gives

$$S(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \varphi(\tau) \frac{e^{iu\tau} - 1}{i\tau}\, d\tau \tag{23}$$
$$= \Sigma\, a_i \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} \varphi_{ij}(\tau) \frac{e^{iu\tau} - 1}{i\tau}\, d\tau \right) a_j^*$$

The expression

(24) $$S_{ij}(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\varphi_{ij}(\tau)\,\frac{e^{iu\tau}-1}{i\tau}\,d\tau \qquad \text{(Spectrum)}$$

is WIENER'S COHERENCY MATRIX. In terms of it, eq.(23) may be written as the Hermitian form

(25) $$S(u) = \Sigma\, a_i S_{ij} a_j^*$$

The following relations are important

(26) $$\varphi_{ij}(\tau) = \varphi^*_{ji}(-\tau)$$

(27) $$S_{ij}(u) = S^*_{ji}(u)$$

The matrix $\varphi_{ij}(\tau)$ is almost Hermitian and will be called the INTERFERENCE of the set of functions $f_i(t)$. The matrix $S_{ij}(u)$ is Hermitian and will be called the SPECTRUM of the set of functions $f_i(t)$.

Linear Transformations of the Spectrum and Interference.

If the original set of functions is subjected to the linear transformation

(28) $$g_i(t) = \Sigma\, a_{ij} f_j(t)$$

then

(29) $$\psi_{ij}(\tau) = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} g_{i\ell}(t+\tau)g_j^*(t)\,dt = \Sigma\, a_{i\ell}\varphi_{\ell m}(\tau)\, a^*_{jm}$$

and

(30) $$T_{ij}(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\psi_{ij}(\tau)\,\frac{e^{iu\tau}-1}{i\tau}\,d\tau$$

or

(31) $$T_{ij}(u) = \sum a_{il}\,S_{lm}(u)\,a_{jm}^{*}$$

Thus, if one has determined the coherency matrix of a set of functions, one amy, without resort to further GHA, determine the coherency matrix of any set of functions derivable from the original set by a linear transformation.

2.3 Stokes Vector.

To obtain the physical significance of the coherency matrix, which we now call the "spectrum" of the set of functions, we turn to Mueller's phenomenological definition of Stokes vector.

The essence of Mueller's definition may be summarized by saying that a plane wave may be characterized by the effect of an ideal wave plate and an ideal polarizer on the intensity. The polarizer has a preferred axis which may be taken to have angle A with respect to the x-axis. The wave plate has two equally preferrable axes which are mutually perpendicular and oriented along the x- and y-axes of the chosen coordinate system. The differential retardation effect of the wave plate is characterized by an angle B. It turns out experimentally that for any settings A and B of the polarizer and wave plate, the intensity of the emergent beam is given by the formula

(32) $I' = (1/2)[I+M\cos 2A+(C\cos B+S\sin B)\sin 2A]$

The four coefficients, I,M,C,S characterized the light and taken as an array are called the Stokes vector (I,M,C,S).

An insight into the meaning of the Stokes vector can be obtained by looking at the above operational definition for the case of a monochromatic plane wave represented as a Maxwell vector (E_x,E_y), i.e., in terms of theoretical quantities.

Let the plane wave pass through the polarizer at angle A. The resultant Maxwell vector, resolved relative to the preferred axis of the polarizer is

$$E_{x'} = E_x\cos A + E_y\sin A$$
$$E_{y'} = 0$$

The intensity I' of the emergent beam is

$$I' = E_{x'}E^*_{x'} = E_xE^*_x \cos^2 A + E_yE^*_y \sin^2 A + (E_xE^*_y + E^*_xE_y) \sin A \cos A$$

The intensity of the incoming beam is

$$I = E_xE^*_x + E_yE^*_y$$

Using this fact and the identity

$$I' = (E_xE^*_x + E_yE^*_y)/2 + (E_xE^*_x - E_yE^*_y)(\cos 2A)/2 + (E_xE^*_y + E^*_xE_y)(\sin 2A)/2.$$

leads to the result, $(B = 0)$

$$I = E_x E_x^* + E_y E_y^*$$
$$M = E_x E_x^* - E_y E_y^*$$
$$C = E_x E_y^* + E_y E_x^*$$

Now consider the case in which the wave passes through both the polarizer and the wave plate. The wave plate is assumed to retard E_x by phase B_x and E_y by phase B_y. In this case the Maxwell vector of the emerging beam is given by

$$E_{x'} = E_x e^{iB_x} \cos A + E_y e^{iB_y} \sin A$$
$$E_{y'} = 0$$

Let $B = B_x - B_y$. Then

$$I' = E_{x'} E_{x'}^* = E_x E_x^* \cos^2 A + E_y E_y^* \sin^2 A + (E_x E_y^* e^{iB} + E_x^* E_y e^{-iB}) \sin A \cos A$$

or

$$I' = \tfrac{1}{2} \left(I + M \cos 2A + \left(C \cos B + i \frac{E_x E_y^* - E_x^* E_y}{2} \sin B \right) \sin 2A \right)$$

With the above identity one finds that

$$S = i(E_x E_y^* - E_y E_x^*)$$

Therefore, the Maxwell and Stokes vectors for the case of a monochromatic plane wave are related by the formula

$$\begin{bmatrix} I \\ M \\ C \\ S \end{bmatrix} = \begin{bmatrix} E_x E_x^* + E_y E_y^* \\ E_x E_x^* - E_y E_y^* \\ E_x E_y^* + E_y E_x^* \\ i(E_x E_y^* - E_y E_x^*) \end{bmatrix} \tag{33}$$

E_x, E_y are rms amplitudes

2.4 Relation Between the Coherency Matrix and Stokes Vector.

Consider the pair of functions

$$f_1(t) = e^{-i\omega t}, \; f_2(t) = e^{i\omega t}$$

The "interference" for this pair is

$$(33) \qquad \varphi_{ij}(\tau) = \begin{bmatrix} e^{-i\omega\tau} & 0 \\ 0 & e^{i\omega\tau} \end{bmatrix}$$

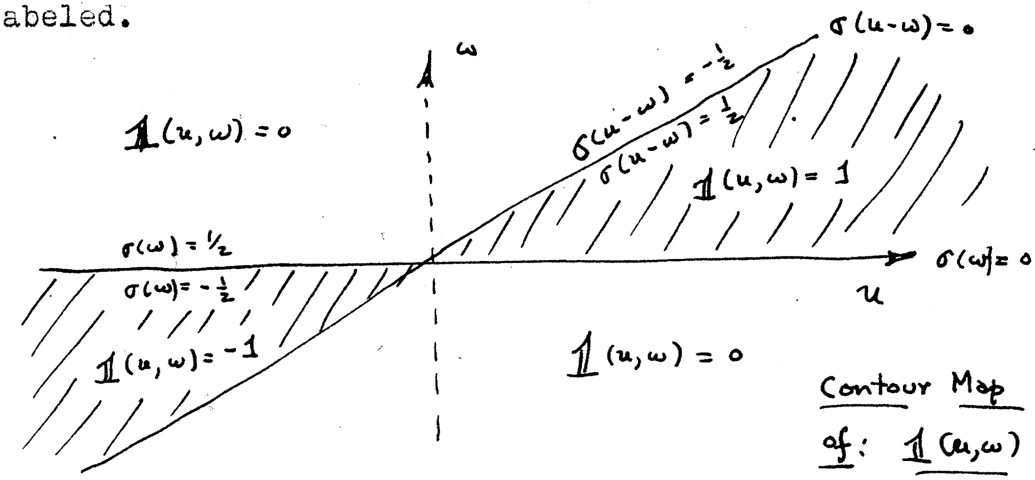
Before computing the "spectrum" consider the special function ,

$$\mathbb{1}(u,\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-i\omega\tau}\, \frac{e^{iu\tau}-1}{i\tau}\, d\tau$$

From a previous section

$$\mathbb{1}(u,\omega) = \sigma(u-\omega)+\sigma(\omega)$$

This function is most easily vizualized by dividing the (u,ω)-plane into areas in which the value of the functions are labeled.



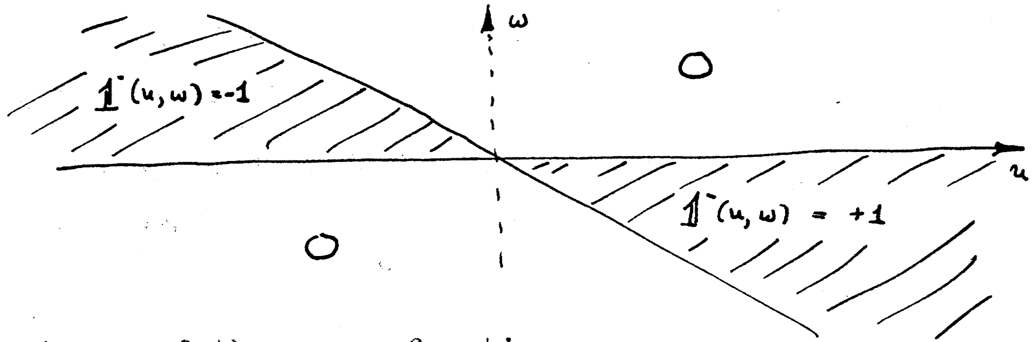
Contour Map of: $\mathbb{1}(u,\omega)$

It will be convenient to write

$$(34) \qquad \begin{aligned} \mathbb{1}^+(u,\omega) &= \mathbb{1}(u,\omega) \\ \mathbb{1}^-(u,\omega) &= \mathbb{1}(u,-\omega) \end{aligned}$$

Step Functions

It should be noticed that $\mathbb{1}^-(u,\omega)$ has the contour map



In terms of these new functions

$$S_{ij}(u) = \begin{bmatrix} \mathbb{1}^+(u,\omega) & 0 \\ 0 & \mathbb{1}^-(u,\omega) \end{bmatrix} \tag{35}$$

Let us now determine the spectrum of

$$g_1(t) = \cos\omega t, \quad g_2(t) = \sin\omega t$$

These functions are linearly related to the $f_i(t)$ by the transformation

$$\begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ i/2 & -i/2 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

The "interference" for this new pair of functions is

$$\psi_{ij}(\tau) = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{i}{2} & -\frac{i}{2} \end{bmatrix} \begin{bmatrix} e^{-i\omega\tau} & 0 \\ 0 & e^{i\omega\tau} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{i}{2} \\ \frac{1}{2} & \frac{i}{2} \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} \cos\omega\tau & -\sin\omega\tau \\ \sin\omega\tau & \cos\omega\tau \end{bmatrix}$$

The "spectrum" for this new pair is

$$T_{ij}(u) = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{i}{2} & -\frac{i}{2} \end{bmatrix} \begin{bmatrix} \mathbb{1}^+ & 0 \\ 0 & \mathbb{1}^- \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{i}{2} \\ \frac{1}{2} & \frac{i}{2} \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix} \mathbb{1}^+ + \frac{1}{4} \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix} \mathbb{1}^-$$

The "spectrum" of $\cos \omega t$ is

$$S(u) = \tfrac{1}{4}[1,0]\,[T_{ij}]\begin{bmatrix}1\\0\end{bmatrix} = \tfrac{1}{4}(\mathbf{1}^{+}+\mathbf{1}^{-})$$

The "spectrum" of $\sin \omega t$ is

$$S(u) = \tfrac{1}{4}[0,1]\,[T_{ij}]\begin{bmatrix}0\\1\end{bmatrix} = \tfrac{1}{4}(\mathbf{1}^{+}+\mathbf{1}^{-})$$

These elementary calculations emphasize that the complex exponentials $e^{-i\omega t}$ and $e^{i\omega t}$ are the natural functions for GHA, using the exponential kernal. Another important point in favor of the exponentials is the fact that, using them, phase shifts and attenuation correspond to multiplication by complex numbers. The complex exponentials form a "basis" in our space of time series and hence ideal for theoretical purposes. Wiener has shown that it is possible to make an approximate harmonic analysis by first dissecting the function $f(t)$, i.e., make it zero outside the interval $(-A,A)$ and then expand it as a periodic function over the interval $(-2A,2A)$. This leads to the approximate "interference" $\varphi_A(\tau)$. The "limit in the mean" of this approximate interference is $\varphi(\tau)$, the required interference. The approximate interference leads to the spectrum $S_A(u)$, which is a step function of the type we have just studied.

With this preamble, we can make a GHA of the Maxwell vector

$$(36) \qquad e(t) = (f_1(t), f_2(t)) = (E_x, E_y)e^{-i\omega t}$$

which was used for the purpose of introducing the Stokes vector. The first step of the GHA gives

$$(37) \qquad \varphi_{ij}(\tau) = \begin{bmatrix} E_x E_x^* & E_x E_y^* \\ E_y E_x^* & E_y E_y^* \end{bmatrix} e^{-i\omega\tau}$$

The second step gives

$$(38) \qquad S_{ij}(u) = \begin{bmatrix} E_x E_x^* & E_x E_y^* \\ E_y E_x^* & E_y E_y^* \end{bmatrix} \mathbb{1}(u,\omega)$$

From inspection of eq.(33) it follows that

$$(39) \qquad \boxed{S_{ij}(u) = \frac{1}{2}\begin{bmatrix} I+M & C-iS \\ C+iS & I-M \end{bmatrix} \mathbb{1}(u,\omega)}$$

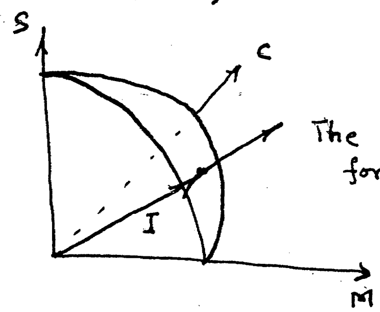
This is the desired elementary relation between the coherency matrix and the Stokes vector. The Stokes vector is only defined for monochromatic waves. The coherency matrix therefore allows one to generalize the notion, in fact, it IS is the generalization of the notinn of a Stokes vector to the polychromatic case. It is even more of a generalization than this. It is the natural mathematical entity for representing not only a single plane wave but also a statistical set of plane waves which are interrelated by different amounts of coherency. The general treatment of this point is given in chapter I.

2.5 The Poincare Sphere.

The Stokes vector of a completely polarized wave satisfies an important equation on its components

$$(41) \qquad I^2 - M^2 - C^2 - S^2 = \det(S'_{ij}(u))/4 = 0$$

Thus, the I,M,C,S for such a wave are not independent. The wave can be pictured genmetrically as a vector in M,C,S space having length I. Waves of intensity I lie on a sphere of radius I, called the POINCARE SPHERE.



The vector is longer than $\sqrt{M^2+C^2+S^2}$ for partially polarized light.

Poincare Sphere

2.6 The Effect of a Rotation of Coordinate Axes. Two Important Equations.

The components of the Maxwell vector depend upon the (x,y)-coordinate system to which the linearly polarized components are referred. A rotation of coordinates is represented by the equation

$$(42) \qquad \begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

and under such a transformation, the Maxwell vector is transformed

$$(43) \qquad \begin{bmatrix} E_{x'} \\ E_{y'} \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} E_x \\ E_y \end{bmatrix}$$

It will be convenient to investigate the corresponding transformation of the Stokes vector. The transformation of the coherency matrix is

(44) $$T_{ij} \quad \Sigma R_{i\ell}(\alpha)R_{jm}(\alpha)S_{\ell m}$$

where

$$R_{ij}\ (\alpha) = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$

In matrix notation

(45) $$[T_{ij}] = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

Carrying out the indicated computation

(46)
$$T_{11} = \frac{S_{11}+S_{22}}{2} + \frac{S_{11}-S_{22}}{2}\cos 2\alpha + \frac{S_{12}+S_{21}}{2}\sin 2\alpha$$
$$T_{22} = \frac{S_{11}+S_{22}}{2} - \frac{S_{11}-S_{22}}{2}\cos 2\alpha - \frac{S_{12}+S_{21}}{2}\sin 2\alpha$$
$$T_{12} = -\frac{S_{11}-S_{22}}{2}\sin 2\alpha + \frac{S_{12}+S_{21}}{2}\cos 2\alpha + \frac{S_{12}-S_{21}}{2}$$
$$T_{21} = -\frac{S_{11}-S_{22}}{2}\sin 2\alpha + \frac{S_{12}+S_{21}}{2}\cos 2\alpha - \frac{S_{12}-S_{21}}{2}$$

Noting that

(47) $$(1/2)\begin{bmatrix} I'+M' & C'-iS' \\ C'+iS' & I'-M' \end{bmatrix} = [T_{ij}]$$

the transformation may be interpreted

(48)
$$(I'+M') = I + M\cos 2\alpha + C\sin 2\alpha$$
$$(I'-M') = I - M\cos 2\alpha - C\sin 2\alpha$$
$$(C'-iS') = -M\sin 2\alpha + C\cos 2\alpha - iS$$
$$(C'+iS') = -M\sin 2\alpha + C\cos 2\alpha + iS$$

adding and rearranging

$$
\begin{aligned}
I' &= I \\
M' &= M \cos 2\alpha + C \sin 2\alpha \\
C' &= -M \sin 2\alpha + C \cos 2\alpha \\
S' &= S
\end{aligned} \tag{49}
$$

also

$$M'^2 + C'^2 = M^2 + C^2 \tag{50}$$

and

$$\begin{bmatrix} M' \\ C' \end{bmatrix} = \begin{bmatrix} \cos 2\alpha & \sin 2\alpha \\ -\sin 2\alpha & \cos 2\alpha \end{bmatrix} \begin{bmatrix} M \\ C \end{bmatrix} \tag{51}$$

Thus a rotation of the coordinate system used by the observer corresponds to a rotation of twice the angle about the S-axis in the Poincare representation. Mueller has given the interpretation of the general rotation. We shall not need it here.

Two Important Equations.

Two important equations can now be easily derived. Suppose that the x,y-coordinate system is so chosen that the polarization ellipse is in canonical form, i.e., that its principle axes coincide with the coordinate axes. Then the Maxwell vector may be written

$$\begin{bmatrix} E_x \\ E_y \end{bmatrix} = \begin{bmatrix} a \\ ib \end{bmatrix} e^{-i\phi} \quad ; \quad \begin{aligned} x &= a \cos \phi \\ y &= b \sin \phi \end{aligned} \tag{52}$$

The coherency matrix is

$$\begin{bmatrix} a^2 & -iab \\ iab & b^2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} I'+M' & C'-iS' \\ C'+iS' & I'-M' \end{bmatrix} \tag{53}$$

and

(54) $\quad I' = a^2+b^2, \quad M' = a^2-b^2, \quad C' = 0, \quad S' = 2ab.$

In this particular coordinate system

(55) $\quad \tan 2\beta' = 2ab/(a^2-b^2) = S'/M'$

But $2\beta'$ is the angle the vector representing the wave makes with the MC-plane and this magnitude does not change with rotation of coordinates. In particular

(56) $\quad \sin 2\beta = S/\sqrt{(M^2+C^2+S^2)} = S/I \qquad (\beta' = \beta)$

Indeed, expressing the Stokes vector in terms of A_1, A_2, δ where $f_1(t) = A_1 \, e^{-i\delta/2} e^{-i\omega t}$, $f_2(t) = A_2 e^{i\delta/2} e^{-i\omega t}$

one has

(57)
$$\begin{bmatrix} I \\ M \\ C \\ S \end{bmatrix} = \begin{bmatrix} A_1{}^2 + A_2{}^2 \\ A_1{}^2 - A_2{}^2 \\ 2A_1A_2\cos\delta \\ 2A_1A_2\sin\delta \end{bmatrix} \qquad \text{Rms values}$$

which yields on substitution into eq.(56)

(58)
$$\boxed{\sin 2\beta = \sin 2\gamma \sin \delta}$$

This is the FIRST EQUATION. Likewise

$$\tan 2\alpha = C/M = \frac{2A_1A_2}{A_1^2 - A_2^2} \cos\delta$$

from which follows the SECOND EQUATION

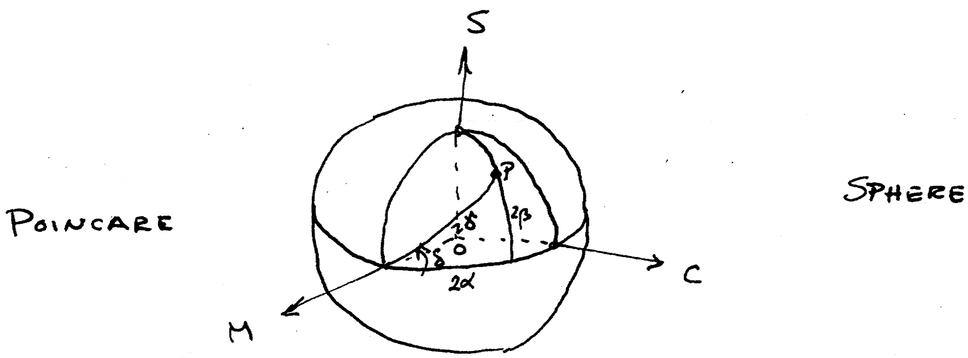
(59)
$$\boxed{\tan 2\alpha = \tan 2\gamma \cos \delta}$$

These are the equations quoted in Chapter I for the purpose of computing α, β, characterizing the polarization ellipse when one knows γ, δ which are easily obtained from the polarization ratio R(E)

$$R(E) = E_y/E_x = (A_2/A_1)e^{i\delta}$$

of the components of the Maxwell vector.

2.7 Interpretation of the Poincare Sphere Representation.



In terms of 2α and 2β, polar coordinates on the Poincare sphere one can write

$$\text{(60)} \qquad \begin{aligned} I &= I \\ M &= I\cos 2\beta \cos 2\alpha \\ C &= I \cos 2\beta \sin 2\alpha \\ S &= I \sin 2\beta \end{aligned}$$

Calling the MC-plane the equitorial plane, the positive S-axis where it pierces the sphere the north pole, one has

dcp = right circular polarization = north pole

lcp = left circular polarization = south pole

lp = linear polarization = the equator

dep = right elliptical polarization = N. hemisph.

lep = left elliptical polarization = S. Hemisph.

hlp = horizontal linear polarization = 0° Long.

vlp = vertical linear polarization = 180° Long.

$+45^{\circ}$lp = 45° linear polarization = 90° Long.

-45°lp = -45° linear polarization = 270° Long.

The angles γ and δ have the interpretation shown in the figure. For,

(61) $$\tan\delta = S/C = \frac{A_1 A_2 \sin\delta}{A_1 A_2 \cos\delta}$$

and

(62) $$\cos 2\gamma = M/I = \frac{A_1^2 - A_2^2}{A_1^2 + A_2^2} = \frac{1-\tan^2\gamma}{1+\tan^2\gamma} = \cos^2\gamma - \sin^2\gamma$$

Thus one may also write

(63)
$$\begin{aligned} I &= A_1^2 + A_2^2 = E_x E_x^* + E_y E_y^* \\ M &= I\cos 2\gamma \\ C &= I\sin 2\gamma\cos\delta \\ S &= I\sin 2\gamma\sin\delta \end{aligned}$$

where $R(E) = E_x/E_y = \dfrac{A_2 e^{i\delta/2}}{A_1 e^{-i\delta/2}} = \tan\gamma\, e^{i\delta}$

is the polarization ratio. Mueller makes use of the notion of OPPOSITELY POLARIZED WAVES. These are waves with oppositely directed vectors in the Poincare sphere. Common examples of oppositely polarized pairs are :

(dcp, lcp), (hlp, vlp), (45°lp, -45°lp).

It can be shown that if two Maxwell vectors are oppositely polarized and if R_1 is the polarization ratio of one and R_2 is the polarization ratio of the other, then

(64) $$\boxed{R_1 R_2^* = -1}$$

2.8 Eigen-Vectors of an Optical Instrument

Algebraically the Jones matrices are endomorphisms of the group of Maxwell vectors, i.e., they are transformations of the group of Maxwell vectors into itself. It is a question of interest to ask what vectors are invariant under such a transformation. Rotation of the coordinate system is another transformation of interest. This will be considered first. Let a rotation be represented by

(65) $$R(\alpha) = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$

The eigen-vectors satisfy the equation

(66) $$R(\alpha)E = \lambda E$$

which is, written out

(67) $$\begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} E_y \\ E_x \end{bmatrix} = \lambda \begin{bmatrix} E_y \\ E_x \end{bmatrix}$$

Such an equation has a non-trivial solution if and only if

(68) $$\begin{vmatrix} \cos\alpha - \lambda & \sin\alpha \\ -\sin\alpha & \cos\alpha - \lambda \end{vmatrix} = 0$$

or

$$\lambda^2 - 2\lambda\cos\alpha + 1 = 0$$

i.e.,

$$\lambda = \cos\alpha \pm i\sin\alpha = e^{\pm i\alpha}$$

or

(69) $$\lambda = e^{\pm i\alpha}$$

In this case the amplitudes satisfy

$$[\cos\alpha - (\cos\alpha \pm i\sin\alpha)]E_x + (\sin\alpha)E_y = 0$$

or

$$\pm i E_x = E_y \qquad \text{When } \alpha \neq 0, \pi$$

i.e.,

(70) $$R(E) = \pm i = \tan\gamma\, e^{i\delta}.$$

and $\gamma = \pm \pi/4$, $\delta = \pi/2$

then
$$\tan 2\alpha = \tan 2\gamma \cos \delta = (\pm\infty)(0) = \text{indet.}$$
$$\sin 2\beta = \sin 2\gamma \sin \delta = \pm 1$$

i.e., the two invariant radiations are dcp and lcp, i.e., oppositely polarized radiation. This is obvious from the Poincare sphere, for the rotation is about the S-axis.

Next consider the ideal polarizer making an angle of 0° with the x-axis. The matrix for such an instrument is

(71) $$P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

This instrument obviously has the eigen-values 0 and 1 corresponding to hlp and vlp. The vlp is trivially invariant, being multiplied by zero.

As a final instrument, consider the wave plate with its principle axes horizontal and vertical. The Jones matrix for such an instrument is

(72) $$W = \begin{bmatrix} e^{iB_1} & 0 \\ 0 & e^{iB_2} \end{bmatrix}$$

Which has the eigen-values
$$\lambda = e^{iB_1}, \quad \lambda = e^{iB_2}$$

Again, the eigen-vectors are hlp and vlp.

2.9 Scalar Product. Symmetry Properties.

In this chapter we are essentially working with the frequency representation of the Jones algebra. If E_1 and E_2 are two Maxwell vectors corresponding to a monochromatic wave of some unspecified frequency i.e.,

$$E_1 = (F_{11}, F_{12}), \quad E_2 = (F_{21}, F_{22})$$

then, by the SCALAR PRODUCT (E_1, E_2) is meant

(73) $$(E_1, E_2) = (F_{11}F_{21}^* + F_{12}F_{22}^*).$$

A vector is said to be NORMAL if

$$(E_1, E_1) = 1 \quad \text{(Rms values)}$$

A pair of vectors are said to be ORTHOGONAL if

$$(E_1, E_2) = 0$$

A pair of vectors for which

$$(E_i, E_j) = \delta_{ij}$$

is said to be a COMPLETE ORTHOGONAL NORMAL SET.

Orthogonal vectors represent oppositely polarized lights. The pair of eigen vectors of a Jones matrix are orthogonal and can be normalized. The completeness means that two such vectors can be used as base vectors for the group of Maxwell vectors. The interpretation of the scalar product is:

$$(E_i, E_j) = \begin{matrix} \text{SELF-INTENSITY} & i = j \\ \text{MUTUAL-INTENSITY} & i \neq j \end{matrix}$$

Another point of interest is the effect of a transformation of coordinates T on the Jones matrices. Let primes denote the new coordinate system. Then

$$E' = TE \qquad \text{Transformation of Coordinates}$$

Let small letters represent the effect of the Jones matrix J or J' on the Maxwell vector; thus

$$e = JE, \qquad e' = J'E'$$

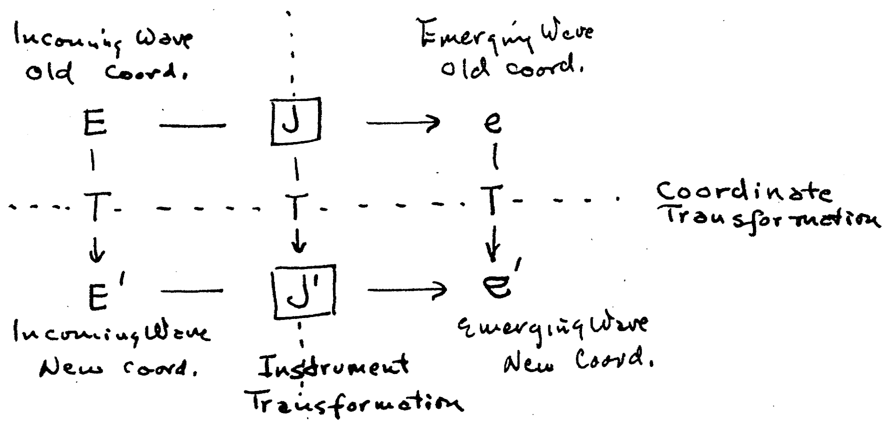
But

$$e' = Te = J'TE \qquad \text{or } e = T^{-1}J'TE$$

from which it follows that

$$(74) \qquad J = T^{-1}J'T \qquad \text{or} \qquad \boxed{J' = TJT^{-1}}$$

as the required laws of transformation of the Jones matrices. Diagramatically this may be vizualized as follows



Equation (74) gives the natural method of talking about the symmetry of a Jones matrix. If $\{A\}$ is the group of rotations and J has rotational symmetry, then

$$J = AJA^{-1} \qquad \text{for ALL } A$$

or $\qquad JA = AJ \qquad$ i.e., J commutes with all A.

In expanded form , one must have the identity

$$(75) \qquad \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \overset{\alpha}{\equiv} \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}$$

Expanding

$$\begin{bmatrix} J_{11}\cos\alpha - J_{12}\sin\alpha & J_{11}\sin\alpha + J_{12}\cos\alpha \\ J_{21}\cos\alpha - J_{22}\sin\alpha & J_{21}\sin\alpha + J_{22}\cos\alpha \end{bmatrix} \overset{\alpha}{\equiv} \begin{bmatrix} J_{11}\cos\alpha + J_{21}\sin\alpha & J_{12}\cos\alpha + J_{22}\sin\alpha \\ -J_{11}\sin\alpha + J_{21}\cos\alpha & -J_{12}\sin\alpha + J_{22}\cos\alpha \end{bmatrix}$$

whereupon, equating corresponding coefficients,

(76) $$J = \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \quad .=. \quad \text{Form of J for Rotational Symmetry}$$

Consider another case. This time the vertical plane is to be a plane of symmetry. In this case the Jones matrix is to be invariant under the two element group of reflextions in the vertical plane, i.e.,under the identity and the transformation

Thus $$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$JA = \begin{bmatrix} -J_{11} & J_{12} \\ -J_{21} & J_{22} \end{bmatrix} = \begin{bmatrix} -J_{11} & -J_{12} \\ J_{21} & J_{22} \end{bmatrix} = AJ$$

or J must have the form

(77) $$J = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \quad .=. \quad \text{Form of J with vertical and Horizontal Planes of Symmetry}$$

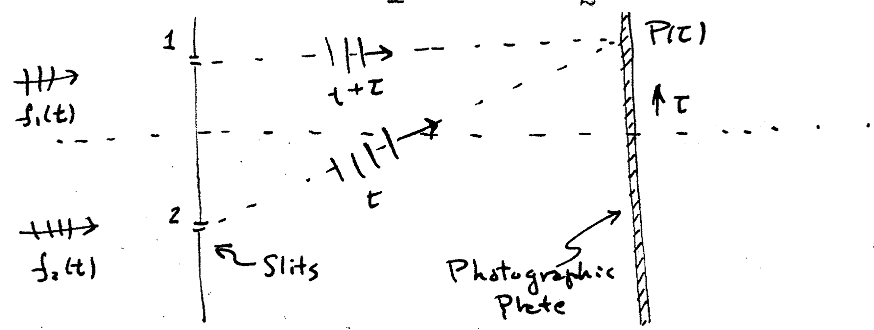
which is also invariant under the transformation

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

i.e., such an instrument is also symmetric about the horizontal plane. These examples show how, in general, with both Jones and Mueller matrices, one sets about the problem of determining their most general form with the restriction that they be invariant under a specified group of transformations. The invariant matrices under a given group of transformations are just those which commute with the transformations.

2.10 Physical Interpretation of $\varphi_{ij}(\tau)$ and $S_{ij}(u)$.

The most intuitive approach to a physical interpretation of the φ's is a Young interference experiment. Let us consider two vlp waves $f_1(t)$ and $f_2(t)$.



At point $P(\tau)$, after passing through slits 1 and 2 respectively, the resultant wave is

(79) $$f(t;\tau) = f_1(t+\tau) + f_2(t)$$

The intensity $I(\tau)$ at $P(\tau)$ is

(80) $$\begin{aligned} I(\tau) &= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \{f(t,\tau)\}^2 dt \\ &= \varphi_{11}(0) + \varphi_{22}(0) + \varphi_{12}(\tau) + \varphi_{21}(-\tau) \\ &= \varphi_{11}(0) + \varphi_{22}(0) + 2\varphi_{12}(\tau) \end{aligned}$$

Thus, one may say that

$\varphi_{11}(\tau), \varphi_{22}(\tau)$.=. variable part of the SELF INTERFERENCE of $f_1(t)$ and $f_2(t)$

$\varphi_{12}(\tau) = \varphi_{21}(-\tau)$.=. variable part of the MUTUAL INTERFERENCE of $f_1(t)$ and $f_2(t)$.

If, on the other hand, we had considered the Maxwell vector

$$e(t) = (f_1(t), f_2(t))$$

and had measured the Stokes vector of this radiation as a function of frequency, taking the inverse transform $\mathcal{F}^{-1}$ of the spectrum

$$\varphi_{ij}(\tau) = \frac{1}{2}\mathcal{F}^{-1}\begin{bmatrix} I(\omega)+M(\omega) & C(\omega)-i\,S(\omega) \\ C(\omega)+i\,S(\omega) & I(\omega)-M(\omega) \end{bmatrix} \tag{81}$$

we would have

$$\varphi_{11}(\tau) = \frac{1}{2}\mathcal{F}^{-1}\left(I(\omega)+M(\omega)\right)$$
$$\varphi_{22}(\tau) = \frac{1}{2}\mathcal{F}^{-1}\left(I(\omega)-M(\omega)\right)$$
$$\varphi_{12}(\tau) = \varphi^{*}_{21}(-\tau) = \frac{1}{2}\mathcal{F}^{-1}\left(C(\omega)-i\,S(\omega)\right)$$

If we look at the interference pattern of a single incoming polarization component we get

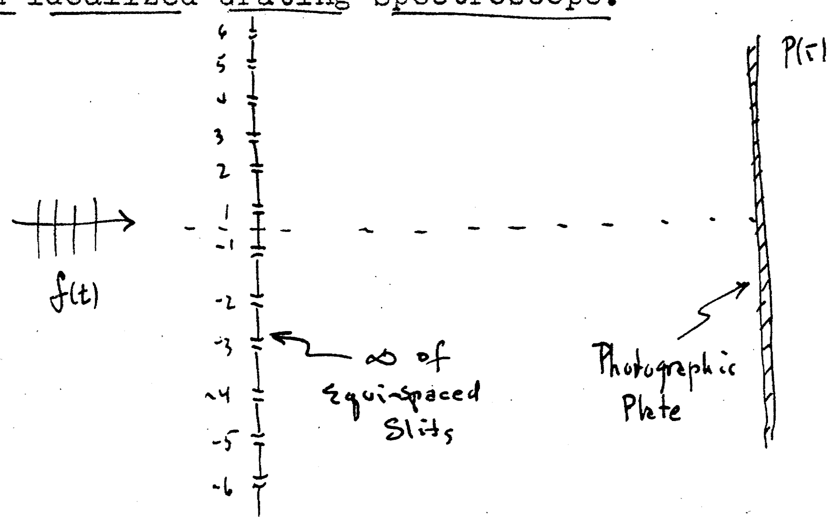
$$I(\tau) = 2\varphi(0) + 2\varphi(\tau) \tag{82}$$

Taking the Fourier transform $\mathcal{F}$ of this pattern gives

$$\mathcal{F}\,I(\tau) = 2\varphi(0)\delta(\omega) + 2S(\omega) \tag{83}$$

Thus except for a Dirac δ-function at frequency u = 0, the Fourier transform of the self interference of a wave f(t) is its spectrum

<u>An Idealized Grating Spectroscope.</u>



In this case the wave at point $P(\tau)$ would be

(84) $$f(t;\tau) = \lim_{n\to\infty} f(t-n\tau) + \cdots + f(t) + \cdots + f(t+n\tau) = \lim_{n\to\infty} f_n(t;\tau)$$

The intensity at $P(\tau)$ would be

(85) $$I(\tau) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T}\{f(t;\tau)\}^2 dt$$

When one squares (84) for fixed n one gets the array of $(2n+1)^2$ products

(86) $$[f(t+i\tau)f(t+j\tau)] \ (=) \ [f(t+(i-j)\tau)f(t)]$$

On substitution into eq.(85) one obtains

(87) $$I_n(\tau) = \sum_{k,l=-n}^{n} \varphi((k-l)\tau)$$

We now notice that

$$\varphi((k-l)\tau) = \int_{-\infty}^{\infty} S(u)e^{-i(k-l)u\tau} du$$

(89) $$I(\tau) = \lim_{n\to\infty}\int_{-\infty}^{\infty} S(u)\left(\sum_{k,l=-n}^{n} e^{-i(k-l)u\tau}\right)du$$

But $$\sum_{k,l=-n}^{n} e^{-i(k-l)\tau} = \left(\sum_{k=-n}^{n} e^{-iku\tau}\right)\left(\sum_{l=-n}^{n} e^{-ilu\tau}\right) = \left(\frac{\sin(n+\frac{1}{2})u\tau}{\sin\frac{1}{2}u\tau}\right)^2$$

which is the Fejer kernel. The principal maxima of this kernel occur when

$$u\tau/2 = \pm N\pi$$

and the result is that

(90) $$I(\tau) = \sum_{-\infty}^{\infty} S\left(\frac{2\pi N}{\tau}\right)$$

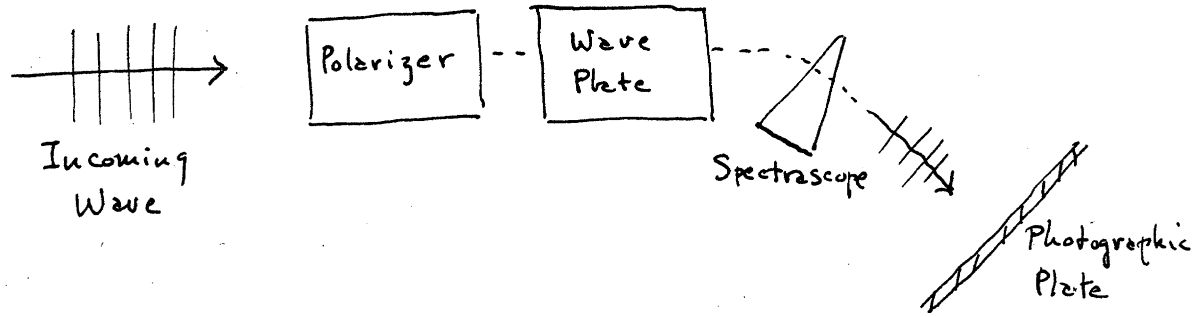
If we limit our consideration to first order spectra,

(91) $$I(\tau) = S(u) \quad \text{where} \quad u = 2\pi/\tau$$

This shows that $S(u)$ can be interpreted as the pattern of intensity obtained from a grating spectroscope. The point of interest is that the result was obtained by the same approach as that used for the physical interpretation of interference. The mathematical treatment has been heuristic because the result is intended to be illustrative rather than a logical part of the structure of our theory. We see that an infinity of equally spaced slits gives the SPECTRUM $S(u)$ and a pair of slits gives the INTERFERENCE $\varphi(\tau)$ of the incoming wave $f(t)$. Furthermore, these two patterns are Fourier transforms of each other

(92) $$\boxed{S(u) = \mathfrak{F}\varphi(\tau)}$$

To obtain a physical picture of $S_{ij}(u)$ we set up the following experiment,



We make use of the fact that

$$(93) \qquad S_{ij}(u) = (1/2)\begin{bmatrix} I(u)+M(u) & C(u)-iS(u) \\ C(u)+iS(u) & I(u)-M(u) \end{bmatrix}$$

From eq.(32) we have

$$(94) \qquad I_p(u) = (1/2)[I(u)+M(u)\cos 2A + (C(u)\cos B+S(u)\sin B)\sin 2A]$$

as the expression for $I_p(u)$, the intensity of the spectrum as a function of the setting of the polarizer and wave plate. We may tabulate

A	B	$I_p(u)$
0°	0°	$(1/2)(I(u)+M(u)) = S_{11}(u)$
90°	0°	$(1/2)(I(u)-M(u)) = S_{22}(u)$
45°	0°	$(1/2)(I(u)+C(u)) = \sum_{ij} S_{ij}(u)$
45°	90°	$(1/2)(I(u)+S(u)) = S_{11}(u)+S_{22}(u)$ $\mp i(S_{12}(u)-S_{21}(u))$

which furnishes the required interpretation. The matrix $S_{ij}(u)$ controls the intensity of the spectral lines in the above spectroscopic experiment. This is the justification of the term SPECTRA for the S's. The Young interference experiment furnishes the justification of the term INTERFERENCE for the φ's.

2.11 Complex Mueller Matrices and Stokes Vectors. Relation to Jones Algebra.

In Chapter I, eqs. (54),(56) we derived the relation between the Jones matrix, the spectra and the complex Mueller matrix and the Stokes vector. It will be of some help to the intuition to carry out this series of operations in full. It should be noted that, after eq.(40) we denote by J Jones matrices which are smaller by a factor 2π than those used in this chapter. If we now revert to the larger J's we have

$$\boxed{M^{\gamma\delta}_{\alpha\beta}(\omega) = J^{\gamma}_{\alpha}(\omega) J^{*\delta}_{\beta}(\omega)} \tag{95}$$

Carrying out the transformation indicated in eq.(56) we have

$$\left[\begin{pmatrix} \gamma & \delta \\ \alpha & \beta \end{pmatrix}\right] = \left[\begin{array}{c|c} \begin{pmatrix} 1 & 1 \\ \alpha & \beta \end{pmatrix} & \begin{pmatrix} 1 & 2 \\ \alpha & \beta \end{pmatrix} \\ \hline \begin{pmatrix} 2 & 1 \\ \alpha & \beta \end{pmatrix} & \begin{pmatrix} 2 & 2 \\ \alpha & \beta \end{pmatrix} \end{array}\right] \tag{96}$$

or

$$[(i,j)] = \left[\begin{array}{cc|cc} (1,1) & (3,1) & (1,3) & (3,3) \\ (2,1) & (4,1) & (2,3) & (4,3) \\ \hline (1,2) & (3,2) & (1,4) & (3,4) \\ (2,2) & (4,2) & (2,4) & (4,4) \end{array}\right] \tag{96'}$$

from which it follows that

$$(97) \qquad \mathcal{M} = \begin{bmatrix} M_{11}^{11} & M_{11}^{21} & M_{11}^{12} & M_{11}^{22} \\ M_{21}^{11} & M_{21}^{21} & M_{21}^{12} & M_{21}^{22} \\ M_{12}^{11} & M_{12}^{21} & M_{12}^{12} & M_{12}^{22} \\ M_{22}^{11} & M_{22}^{21} & M_{22}^{12} & M_{22}^{22} \end{bmatrix} = [m_{ij}]$$

or

$$(97') \qquad \mathcal{M} = \begin{bmatrix} J_1^1 J_1^{*1} & J_1^2 J_1^{*1} & J_1^1 J_1^{*2} & J_1^2 J_1^{*2} \\ J_2^1 J_1^{*1} & J_2^2 J_1^{*1} & J_2^1 J_1^{*2} & J_2^2 J_1^{*2} \\ J_1^1 J_2^{*1} & J_1^2 J_2^{*1} & J_1^1 J_2^{*2} & J_1^2 J_2^{*2} \\ J_2^1 J_2^{*1} & J_2^2 J_2^{*1} & J_2^1 J_2^{*2} & J_2^2 J_2^{*2} \end{bmatrix}.$$

An easy mnemonic device is to be found in the recognition of the fact that

$$(98) \qquad i \text{ or } j = (1,2,3,4) \longleftrightarrow (\alpha,\beta) \text{ or } (\gamma,\delta) = ((1,1),(2,1),(1,2),(2,2))$$

or that

$$(98') \qquad \begin{array}{ccc} 1 & \leftrightarrow & (1,1) \\ 2 & \leftrightarrow & (2,1) \\ 3 & \leftrightarrow & (1,2) \\ 4 & \leftrightarrow & (2,2) \end{array}$$

The relationship between the spectra and the Stokes vector is

$$(99) \qquad \mathcal{L} = [\mathcal{L}_i] = \begin{bmatrix} S_{11} \\ S_{21} \\ S_{12} \\ S_{22} \end{bmatrix} \sim \begin{bmatrix} \overline{F_1 F_1^*} \\ \overline{F_2 F_1^*} \\ \overline{F_1 F_2^*} \\ \overline{F_2 F_2^*} \end{bmatrix}$$

and

$$(100) \qquad \mathcal{L}' = [\mathcal{L}'_i] \quad \text{similarly defined.}$$

where the primes now indicate emerging radiation. We shall only use primes to distinguish between spectra and integrated spectra when it is absolutely necessary to avoid confusion.

The Reversed Instrument

If we indicate the transpose of a matrix with a tilda, e.g., $\tilde{J}$ = transpose of J, and if we use a left handed coordinate system when passing through the instrument in the reverse direction, then the Jones matrix of the reverded instrument is the transpose of the Jones matrix for the original instrument

(101) $\boxed{J \;.\!=\!.\; \text{original instrument right handed system} \longleftrightarrow \tilde{J} \;.\!=\!.\; \text{reversed instrument left handed system}}$

This result is given by Jones in his series of papers cited in Chapter I. It follows immediately from eq.(97) that the same rule holds for ANY Mueller matrix.

Thus,

(102) $\boxed{\mathcal{M} \;.\!=\!.\; \text{original instrument right handed system} \leftrightarrow \tilde{\mathcal{M}} \;.\!=\!.\; \text{reversed instrument left handed system}}$

because the subsitution $J^i_j \rightarrow J^j_i$ transposes $\mathcal{M}$, i.e., i.e., transposing J transposes $\mathcal{M}$.

The other way of loosing at the matter is to introduce left and right multiplication. Then the basic rule becomes:

Direct Instrument

(103) $$\mathcal{L}'_r = \mathcal{M}\mathcal{L}_r$$ right handed system

Reversed Instrument

(104) $$\tilde{\mathcal{L}}'_\ell = \tilde{\mathcal{L}}_\ell \mathcal{M}$$ left handed system

One may carry out the transformation from a right to a left handed system with the aid of the matrix

(105) $$S = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Then treating S as a J matrix, one can apply eq.(97') and obtain the corresponding $\mathcal{S}$ matrix for transforming the $\mathcal{L}$'s from a right handed to a left handed coordinate system

(106) $$\mathcal{S} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \tilde{\mathcal{S}}$$

It is more convenient in practice to have a matrix $\mathcal{M}_\ell$ for the reversed instrument, corresponding to a right handed coordinate system. To obtain this one has only to transform eq.(104)

(107) $$\tilde{\mathcal{L}}''_r = \tilde{\mathcal{L}}''\mathcal{S} = \tilde{\mathcal{L}}_r \mathcal{S}\mathcal{M}\mathcal{S} = \tilde{\mathcal{L}}_r \tilde{\mathcal{M}}_\ell$$

from which it follows that

(108) $$\mathcal{L}''_r = \mathcal{S}\tilde{\mathcal{M}}\mathcal{S}\mathcal{L}_r = \mathcal{M}_\ell \mathcal{L}_r$$

where

(109) $$\boxed{\mathcal{M}_R = \mathcal{S}\tilde{\mathcal{M}}\mathcal{S}}$$ Mueller Matrix of Reversed Instrument

is the correct matrix for the reversed instrument in the same coordinate system as that of the original matrix.

The transformation $\mathcal{S}(\ )\mathcal{S}$ has the effect of putting the following signs on the elements of the transpose $\tilde{\mathcal{M}}$ of $\mathcal{M}$.

(110) $$\mathcal{S}(\ )\mathcal{S} = \begin{bmatrix} + & - & - & + \\ - & + & + & - \\ - & + & + & - \\ + & - & - & + \end{bmatrix}$$

It is not the function of this chapter or even of this thesis to go beyond laying the algebraic and statistical foundations of Mueller algebra. As an illustration of the manner in which the fundamental theory could be applied we have derived a rather general rule, eq.(74), for the composition of "parallel" Mueller matrices[1] for a system of statistically dependent scattering centers. It is an important task for further research to carry through the construction of Mueller algebra for different typical statistical situations. Out of such research there should emerge a systematic generalized Mueller algebra in "matrix-vector" form for which there have been explicitly stated under various approximations, laws for the composition of STATISTICALLY DEPENDENT OPTICAL NETWORKS.

---

1) "Series" occurs in multiple scattering considerations.

2.12 The Transformation from Complex to Real Mueller Algebra.

Theoretically, because of the simplicity and elegance of eqs.(97'),(99),(100) relating Jones matrices and Maxwell vectors to the corresponding complex Mueller matrices and Stokes vectors, the complex Mueller algebra has many advantages.

Phenomenologically, however, both from the standpoint of geometric representation and experimental interpretation the real Mueller algebra has many advantages. It is this algebra which Mueller develops in his "Foundation" and for which he he has found many beautiful geometric pictures of the Poincare sphere type. For these results we refer the reader to Mueller's paper. Here we confine our attention to the transformation from Complex to Real Mueller algebra and to the expressions for the real Mueller matrix M and the real Stokes vector L in terms of the J and E of the corresponding elementary Jones ω- algebra.

It follows from eqs. (58) and (59) that

$$(111) \qquad L = \begin{bmatrix} \mathcal{L}_1 + \mathcal{L}_4 \\ \mathcal{L}_1 - \mathcal{L}_4 \\ \mathcal{L}_2 + \mathcal{L}_3 \\ -i(\mathcal{L}_2 - \mathcal{L}_3) \end{bmatrix}$$

where

$$(112) \qquad T^{-1} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & -i & i & 0 \end{bmatrix} \quad ; \quad T^{-1}: \begin{array}{l} \mathcal{L} \to L \\ \mathcal{M} \to M \end{array}$$

From eq.(111) it follows that

$$(113) \qquad M = T^{-1}\mathcal{M}\,T$$

where

$$T = 1/2 \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & i \\ 0 & 0 & 1 & -i \\ 1 & -1 & 0 & 0 \end{bmatrix} \tag{114}$$

Hence

$$M = T^{-1} \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & i \\ 0 & 0 & 1 & -i \\ 1 & -1 & 0 & 0 \end{bmatrix} \tag{115}$$

or

$$M = \frac{1}{2} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & -i & i & 0 \end{bmatrix} \begin{bmatrix} m_{11}+m_{14} & m_{11}-m_{14} & m_{12}+m_{13} & +i(m_{12}-m_{13}) \\ m_{21}+m_{24} & m_{21}-m_{24} & m_{22}+m_{23} & i(m_{22}-m_{23}) \\ m_{31}+m_{34} & m_{31}-m_{34} & m_{32}+m_{33} & i(m_{32}-m_{33}) \\ m_{41}+m_{44} & m_{41}-m_{44} & m_{42}+m_{43} & i(m_{42}-m_{43}) \end{bmatrix} \tag{116}$$

or

$$2M = \begin{bmatrix} \begin{matrix} m_{11}+m_{14} \\ +m_{41}+m_{44} \end{matrix} & \begin{matrix} m_{11}-m_{14} \\ +m_{41}-m_{44} \end{matrix} & \begin{matrix} m_{12}+m_{13} \\ +m_{42}+m_{43} \end{matrix} & \begin{matrix} i(m_{12}-m_{13} \\ +m_{42}-m_{43}) \end{matrix} \\ \\ \begin{matrix} m_{11}+m_{14} \\ -m_{41}-m_{44} \end{matrix} & \begin{matrix} m_{11}-m_{14} \\ -m_{41}+m_{44} \end{matrix} & \begin{matrix} m_{12}+m_{13} \\ -m_{42}-m_{43} \end{matrix} & \begin{matrix} i(m_{12}-m_{13} \\ -m_{42}+m_{43}) \end{matrix} \\ \\ \begin{matrix} m_{21}+m_{24} \\ +m_{31}+m_{34} \end{matrix} & \begin{matrix} m_{21}-m_{24} \\ +m_{31}-m_{34} \end{matrix} & \begin{matrix} m_{22}+m_{23} \\ +m_{32}+m_{33} \end{matrix} & \begin{matrix} i(m_{22}-m_{23} \\ +m_{32}-m_{33}) \end{matrix} \\ \\ \begin{matrix} -i(m_{21}+m_{24}) \\ -i(m_{31}-m_{34}) \end{matrix} & \begin{matrix} i(m_{21}-m_{24} \\ -m_{31}+m_{34}) \end{matrix} & \begin{matrix} -i(m_{22}+m_{23} \\ -m_{32}-m_{33}) \end{matrix} & \begin{matrix} (m_{22}-m_{23} \\ -m_{32}+m_{33}) \end{matrix} \end{bmatrix} \tag{117}$$

The inverse relation, obtained by the formula

$$\boxed{\mathcal{M} = TMT^{-1}} \tag{118}$$

yields

$$2\mathcal{M} = \begin{bmatrix} (m_{00}+m_{01}) + (m_{10}+m_{11}) & (m_{02}+m_{12}) - i(m_{03}+m_{13}) & (m_{02}+m_{12}) + i(m_{03}+m_{13}) & (m_{00}-m_{01}) + (m_{10}-m_{11}) \\ (m_{20}+m_{21}) + i(m_{30}+m_{31}) & (m_{22}+m_{33}) + i(m_{32}-m_{23}) & (m_{22}-m_{33}) + i(m_{32}+m_{23}) & (m_{20}-m_{21}) + i(m_{30}-m_{31}) \\ (m_{20}+m_{21}) - i(m_{30}+m_{31}) & (m_{22}-m_{33}) - i(m_{32}-m_{23}) & (m_{22}+m_{33}) - i(m_{32}-m_{23}) & (m_{20}-m_{21}) - i(m_{30}-m_{31}) \\ (m_{00}+m_{01}) - (m_{10}+m_{11}) & (m_{02}-m_{12}) + i(m_{03}-m_{13}) & (m_{02}-m_{12}) + i(m_{03}-m_{13}) & (m_{00}-m_{01}) - (m_{10}-m_{11}) \end{bmatrix} \tag{119}$$

where, in Mueller's notation $i,j = 0,1,2,3$ and

$$M = \begin{bmatrix} m_{00} & m_{01} & m_{02} & m_{03} \\ m_{10} & m_{11} & m_{12} & m_{13} \\ m_{20} & m_{21} & m_{22} & m_{23} \\ m_{30} & m_{31} & m_{32} & m_{33} \end{bmatrix} \tag{120}$$

Since the elements of M can be computed experimentally eq.(119) gives the matrix $\mathcal{M}$ for comparison with theory. In practice it is easier to put the experimental values into eq.(120) and apply eq.(118) directly to obtain the corresponding numerical values of $\mathcal{M}$. Eq.(117) shows that the computation of M from the J is rather laborious. Eq.(97') shows that the corresponding computation of is relatively easy.

CHAPTER III

STATISTICAL HARMONIC ANALYSIS

3.1 Introduction.

We have now completed the algebraic part of our theory and turn to the statistical significance of the Wiener correlation process. Eq.(17), Chapter II, showed that a function like

$$f(t) = \sum_{1}^{n} a_k e^{-i\omega_k t}$$

had a sharp line spectrum,

$$S(u) = \sum_{1}^{n} a_k a_k^* \,\delta(u-\omega_k)$$

We made no explicit use of the fact that f(t) of Jones t-algebra could be one of a class of random functions. As far as the analysis up to this chapter is concerned the f(t) were simply arbitrary functions of t for which the Wiener correlation was a valid analytic process.

We intend to accomplish in this chapter a shift in viewpoint from arbitrary functions to relatively simple sets of random functions and to obtain a statistical description of an actual spectrial line which is meaningfully related to the components of the Stokes vector.

Actual spectral lines have a shape which is to be described by the intensity

$$I(\nu) = S_{11}(\nu) + S_{22}(\nu) \tag{1}$$

where $S_{ij}(\nu)$ is the "spectrum" of $e(t) = (f_1(t)+f_2(t))$. A survey of the literature[1] shows that there are three

---

1) White, H.E.: "Introduction to Atomic Spectra," McGraw-Hill, N.Y., 1934. p.420.
Margenau, H., and W.W.Watson: "Pressure Effects on Spectral Lines" Rev.Mod.Phys. 8 22-53 (1936).
Van Vleck, J.H., and Weisskopf, V.: Rev.Mod.Phys. 17 227 (1945).
Jablonski, A.: Phys.Rev. 68 78 (1945)
Foley, H.M.: Phys.Rev. 69 616 (1946).

---

sources of broadening which it is desirable to include in the present study: 1) natural broadening, 2) collision broadening, and 3) Doppler broadening. Natural broadening is included for theoretical reasons. In practice, as one goes from the microwave region to the x-ray region, it is masked either by collision or Doppler broadening. Other types of broadening, due to all kinds of interaction, are discussed in the article of Margenau and Watson, cited above, but will not be considered here.

Natural broadening, due to the life time of the atomic states gives rise to the spectrum

$$I(\nu) = \frac{\gamma}{2\pi}\,\frac{1}{4\pi^2(\nu_0-\nu)^2+(\gamma/2)^2} \tag{2}$$

when attention is confined to a transition between exactly two states.

Collision broadening, due to the shortening of the average life time in a state gives rise to a similar spectrum

$$(3) \qquad I(\nu) = \frac{\Gamma}{2\pi} \frac{1}{4\pi^2(\nu_0-\nu)^2+(\frac{\Gamma}{2})^2}$$

Doppler broadening, due to the thermal motion of the radiating atoms relative to the observer gives rise to the characteric spectrum

$$(4) \qquad I(\nu) = \text{Const. } e^{-\frac{Mc^2}{2RT}\frac{(\nu-\nu_0)^2}{\nu_0^2}}$$

The current theoretical description of these phenomena makes use of two statistical theories, the quantum mechanics i.e., the statistical study of the characteristic states of an atom or molecule, andthe quantum statistical mechanics i.e., the statistical study of assemblies of atoms or molecules. Thus one would expect a theoretical knowledge of the associated radiation to consist not of a particular pair of time functions $(f_1(t),f_2(t))$ but instead to be less detailed, i.e., limited to statistical information about a class of such pairs. In the light of the linear-quadratic duality, one wishes to compute the physical effects associated with a "typical " pair of the class. This leads to a consideration of the class effect associated with the class of pairs. It is to this class that one applies the statistical harmonic analysis to get such results as the Stokes vector, which, in this light, is the net statistical information about the class.

Before looking for the statistical description of a class of Maxwell vectors which will generate the required line shapes it is necessary to make a short excursion into the theory of random processes.[1]

---

1) A survey of the mathematical aspects of the subject together with a rather complete bibliography will be found in:
Cramer, H.: "Problems in Probability Theory," Annals Math. Stat. 18 165-193 (1947).
A very readable summary of statistical harmonic analysis has appeared in the Radiation Laboratory Series, M.I.T, Volume 25. "Theory of Servomechanism," Ch.VI. "Statistical Properties of Time-Variable Data," by R.S.Phillips.
On p.266 of this work will be found a footnote reference to the important papers of Wang and Uhlenbeck, Chandrasekhar, andRice, as well as the fundamental Wiener, paper, Acta Math. 55 118 (1930).

3.2 Random Processes.

Following Wang and Uhlenbeck[2], a random process, (function), y(t), is a process in which the variable y

---

2) Wang, M.S., and G.E.Uhlenbeck: "On the Theory of the Brownian Motion, II," Rev.Mod.Phys. 17 323 (1945).

---

does not depend in a completely definite way on the indepent variable t, i.e., if one repeats the experiment again and again, one gets NOT ONE y(t) BUT A CLASS OF THEM, so that it is only probability distributions that are reproducable. In fact the random process y(t) is completely defined by the following set of probability distributions

$w_1(y;t)$ = probability density for y at time t

$w_2(y_1,t_1;y_2,t_2)$ = joint probability density for y_1 at t_1 and at y_2 at t_2

(5) $w_3(y_1,t_1;y_2,t_2;y_3,t_3)$ = joint probability density for $(y_1 \text{ at } t_1) \cap (y_2 \text{ at } t_2) \cap (y_3 \text{ at } t_3)$

where "$\cap$" is read "and" and w_n similarly defined. Thus we have a heirarchy of distribution data. The set of functions, eq.(5), must fulfill the following conditions

(a) $$w_n \geqslant 0$$

(b) $w_n(y_1,t_1;\ldots;y_n,t_n)$ is a symmetric function of the pairs of variables $(y_1,t_1),\ldots,(y_n,t_n)$.

(c) $$w_k(y_1,t_1;\ldots;y_k,t_k) = \int\ldots\int dy_{k+1}\ldots dy_n w_n(y,t)$$

The heirarchy of w_k describe, successively, the random process in more and more detail. To obtain the w_k experimentally requires, in general, a great number of records. In the present application to the theory of light as in the case of Brownian motion one can make an important simplification because the process is STATIONARY IN TIME (homogeneous), i.e., the underlying mechanism which causes the fluctuations is assumed not to change in the course of time. A shift of the time axis will not influence the w_k. As a result, eq.(5) reduces to

$$w_1(y)dy = \text{probability for } y \text{ at any } t$$

(6) $$w_2(y_1, y_2, t) = \text{joint probability for } y_1 \text{ and } y_2 \text{ at time interval } t,$$

and so on.

These w_k can be obtained experimentally from ONE sufficiently long record, $y(t)$.

3.3 Statistical Harmonic Analysis and the Ergodic Theorem.

There is a close connection between random processes andthe ergodic theory. Without getting into the mathematical complications,[1] the situation can be described in the

1) Cramer, H.: Annals of Math. Stat. 18 165 (1947)
von Neumann, J.: "Proof of the Quasi-Ergodic Theorem." Proc. Nat. Acad. Sci. 18 70-82 (1932).

following manner. The first step of the GHA requires the computation of

(7) $$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} y(t+\tau)y^*(t)dt = \varphi(\tau)$$

In the applications contemplated, $y(t)$ is not an isolated function but a member of a class of functions. The process is defined if one knows the probability distribution Π over the space Ω of these functions. One way of giving Π is by means of w_k, defined in eq.(5). Eq.(7) associates with each y(t) a $\varphi(\tau)$ i.e., is a mapping of the space Ω of functions y(t) on the space Φ of functions $\varphi(\tau)$. The ergodic theorem, required here, says that in the case of a process stationary in time, if one picks a y(t) at random and applies eq.(7) to obtain $\varphi(\tau)$, there is UNIT PROBABILITY that

(8) $$\varphi(\tau) = \int w_2(y_1, y_2^*, \tau)\, y_1 y_2^* \, dy_1 \, dy_2^*$$

That is, in the sense of Gibbs, one can "almost always" replace the time average, eq.(7), by the ensemble average, eq.(8). If y(t) is an electrical signal, one can use the electronic equipment being developed by Y.W.Lee to determine $\varphi(\tau)$ under eq.(7). If, on the other hand, as in the present study, one is generating a stochastic theory, one makes a hypothesis about $w_2(y_1, y_2; t)$ and determines $\varphi(\tau)$ under eq.(8). The term STATISTICAL HARMONIC ANALYSIS refers to the GENERALIZED HARMONIC ANALYSIS OF A CLASS OF RANDOM FUNCTIONS.

3.4 Classification of Random Processes.

A process is called PURELY RANDOM if

(9) $$w_2(y_1, t_1; y_2, t_2) = w_1(y_1, t_1) w_1(y_2, t_2)$$

an so on for all w_K. In this case all the information about the process is contained in $w_1(y, t)$. A purely random process is a limiting case. In any actual situation, y_1 and y_2 are correlated if $t_2 - t_1$ are small enough.

A more general process, one required for the present study, is the MARKOFF PROCESS,[1] i.e., on in which all the

---

1) The fundamental paper on the subject is: Kolmogoroff, A.: Math.Ann. 104 415-458 (1931). He uses the term STOCHASTICALL-DEFINITE PROCESS. For the histor of later developments and reference to the important work of Khintchine, Feller, and Doob, cf. Cramer, loc. cit.

---

information about the process is contained in w_2. It is useful, in connection with the Markoff process, to define the notion of CONDITIONAL PROBABILITIES

(10) $P_2(y_1 \mid y_2, t) =$ probability density on y_2, given y_1 at time t earlier [2]

---

2) A better notation would be $P_2(y_2, t \mid y_1, 0)$ where the symbol " $\mid$ " is read "such that" as in Sec. 1.10.

It is evident that

(11) $$w_2(y_1, y_2, t) = w_1(y_1) P_2(y_1 \; y_2, t)$$

and P_2 must satisfy the relations

(12a) $$P_2(y_1 \; y_2, t) \geqslant 0$$

(12b) $$\int dy_2 P_2(y_1 \mid y_2, t) = 1$$

(12c) $$w(y_2) = \int w_1(y_1) P_2(y_1 \mid y_2, t) dy_1$$

In Brownain motion

$$\lim_{t\to\infty} P_2(y_1 \mid y_2,t) = w_1(y_2)$$

a property which excludes the existence of hidden periodicities

It is important to know that P_2 cannot be an arbitrary function of y_1 and y_2, and t but must satisfy the SMOLUCHOWSKI EQUATION

$$P_2(y_1 \mid y_2,t) = \int dy\, P_2(y_1 \mid y,t_0)\, P_2(y \mid y_2, t-t_0), \tag{13}$$

for all t_0 between 0 and t.

One could go on to the next class of process, i.e., to those defined by giving w_3. However, physical applications very rarely require a higher order of complexity than the definition of w_2. Wiener correlation requires just w_2.

3.5 Gaussian Processes.[1]

A very important class of processes are those for which the basic distributions occuring in eq.(5) are Gaussian. The spectrum for Doppler broadening involves a Gaussian distribution of frequency by birtue of the expression

$$e^{-mv^2/2kT}$$

the Maxwellian velocity distribution arising from the insertion of the kinetic energy in the Boltzmann factor.

An analysis of the set of random functions

$$\{f(t)\} = \{A\, e^{i(\omega t - \varphi(t))}\}$$

corresponding to the set of random phases $\{\varphi(t)\}$ leads to the correct spectrum for natural and collision broadening

1) Doob, J.L.: "The Elementary Gaussian Processes," Annals of Math. Stat. 15 229 (1944).

if one assumes that the $\{\varphi(t)\}$ result from a Gaussian Markoff process. This analysis will be given in detail in the next section. The physical reasonableness of the assumption can best be seen by the following considerations:

Imagine an ensemble of atoms, in a cavity, which are radiating and feeding a standing wave of frequency ν equal to $\Delta E/h$ where ΔE is the energy difference between the two atomic levels under consideration. The atoms are excited by some external source of energy. Radiation seeps out of the cavity at the same rate and may be analyzed. In any event the standing wave has intensity $Nh\nu$ which is large compared with the rate of the transitions and leakage. These transitions cause an infinitude of infinitessimal shifts in the absolute phase of the standing wave and one gets a Brownian motion of the phase φ at time t relative to the initial values of the phase at time t_o. The phase, as a result has a Gaussian distribution with dispersion proportional to $\sqrt{(t-t_o)}$. These two examples indicate the importance of Gaussian processes in our problem.

3.6 A Statistical Description of Natural Line Breadth.

Consider a plane polarized wave described by the class of random functions

$$\{f(t)\} = \left\{ A e^{-i(\omega t - \varphi(t))} \right\} \tag{14}$$

related to the class of random phases $\{\varphi(t)\}$. This class is assumed to be Gauss-Markoffian with the distribution

$$W_2(\varphi, \varphi_0, \tau) = C(\tau) e^{-\xi(\tau)(\varphi - \varphi_0)^2} \tag{15}$$

The computation of $\varphi(\tau)$ yields

$$\varphi(\tau) = AA^* C(\tau) e^{-i\omega\tau} \int_{-\infty}^{\infty} e^{-\xi(\tau)\psi^2} e^{i\psi} d\psi \tag{16}$$

where $\psi = \varphi - \varphi_0$ i.e., $\varphi(t+\tau) - \varphi(t)$. If we examine the integral, we find

$$I(\tau) = \int_{-\infty}^{\infty} e^{-\xi(\tau)\psi^2} e^{i\psi} d\psi = \frac{1}{\sqrt{2\xi(\tau)}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} e^{i\frac{x}{\sqrt{2\xi(\tau)}}} dx$$

where $x = \sqrt{2\xi(\tau)}\,\psi$, $dx = \sqrt{2\xi(\tau)}\,d\psi$. This is a Fourier transform of the type contemplated in Ch.I, eq.(T), in fact using this equation

$$I(\tau) = \sqrt{\frac{\pi}{\xi(\tau)}}\, e^{-\frac{1}{4\xi(\tau)}}$$

and hence

$$\boxed{\varphi(\tau) = AA^* e^{-i\omega\tau} e^{-\frac{1}{4\xi(\tau)}}} \tag{17}$$

where we eliminated $C(\tau)$ by the normalizing condition

$$C(\tau) = \int_{-\infty}^{\infty} e^{-g(\tau)\psi^2} d\psi = 1$$

We now suppose that one has BROWNIAN MOTION OF PHASE, i.e.,

(18) $$\boxed{g(\tau) = \frac{1}{2\gamma|\tau|}}$$

the coefficient being chosen in anticipation of the fact that γ will turn out to be the number of transitions per second per atom. Then the spectral intensity is given by

(19) $$S(u) = \frac{AA^*}{2\pi} \int_{-\infty}^{\infty} e^{-i(\omega-u)\tau} e^{-\frac{|\tau|\gamma}{2}} d\tau$$

Splitting the integral into two parts to deal with $|\tau|$ one gets

$$S(u) = \frac{AA^*}{2\pi} \int_0^{\infty} e^{-[i(\omega-u)+\frac{\gamma}{2}]\tau} d\tau + \frac{AA^*}{2\pi} \int_0^{\infty} e^{[i(\omega-u)-\frac{\gamma}{2}]\tau} d\tau$$

$$= \frac{AA^*}{2\pi} \left| \frac{e^{-[i(\omega-u)+\frac{\gamma}{2}]\tau}}{-[i(\omega-u)+\frac{\gamma}{2}]\tau} \right|_0^{\infty} + \frac{AA^*}{2\pi} \left| \frac{e^{[i(\omega-u)-\frac{\gamma}{2}]\tau}}{[i(\omega-u)-\frac{\gamma}{2}]\tau} \right|_0^{\infty}$$

which reduces to

(20) $$\boxed{S(u) = \frac{AA^*}{2\pi} \frac{\gamma}{(\omega-u)^2+\gamma^2/4}}$$ NATURAL LINE SHAPE

This is the desired formula for the natural line shape. The proper expression for collision broadening is obtained by substituting

$$\Gamma + \gamma \qquad \text{for} \qquad \gamma$$

where Γ is the number of transitions per second per atom induced by collisions.

3.7 An Interference Experiment and the Stokes Vector.

It will be instructive to consider the result of a Young interference experiment performed with this radiation Eq.(3),Ch.II, states that the interference pattern, as a function of τ should have intensity

(21) $$I_p(\tau) = \tfrac{1}{2}\varphi(0) + \tfrac{1}{2}\,\mathrm{Re}\,(\varphi(\tau))$$

i.e.,

(21') $$\boxed{I_p(\tau) = \frac{AA^*}{2}\left(1 + e^{-\frac{2|\tau|}{\gamma}}\cos\omega\tau\right)}$$

"Fading" of Interference Pattern

One notes that the central maximum has intensity AA^* which is what one would get for coherent superposition, whereas as $\tau \to \infty$ one gets intensity $(1/2)AA^*$ which is the result of incoherent superposition. It is important not to be misled by this result. Suppose that one "manufactures" light with horizontal and vertical polarization components.

(22) $$\begin{aligned} f_1(t) &= f(t) \\ f_2(t) &= f(t+\alpha) \end{aligned}$$

This might be done by splitting the beam and running one path with a retardation α . For this light one has the spectral density matrix

$$(23) \qquad S_{ij}(u) = S(u)\begin{bmatrix} 1 & e^{iu\alpha} \\ e^{-iu\alpha} & 1 \end{bmatrix}$$

and the corresponding Stokes vector

$$(24) \qquad \begin{bmatrix} I(u) \\ M(u) \\ C(u) \\ S(u) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \cos u\alpha \\ -\sin u\alpha \end{bmatrix} \left(\frac{AA^*}{2\pi} \frac{\gamma}{(\omega-u)^2+\gamma^2/4} \right.$$

which, in any small frequency interval, is completely polarized, i.e.,

$$I^2 = M^2 + C^2 + S^2$$

The different colors have different polarizations, ranging from the circular to 45° linear, i.e., they lie on the (C,S)-plane of the Poincare representation.

Suppose on the other hand that one makes an experimental analysis of this light with the aid of a wave plate, a polarizer and a intensity measuring instrument which has uniform sensitivity over the entire spectrum (an idealization) of course). Analytically this means integrating the spectral matrix, eq.(23) from $-\infty$ to $+\infty$ with respect to u. The result

$$(25) \qquad \begin{bmatrix} I \\ M \\ C \\ S \end{bmatrix}_{\omega} = AA^* \begin{bmatrix} 1 \\ 0 \\ e^{-\gamma\alpha/2}\cos\alpha\omega \\ -e^{-\gamma\alpha/2}\sin\alpha\omega \end{bmatrix}$$

The light will LOOK mixed because

(26) $$I^2 - (M^2 + C^2 + S^2) = I^2(1 - e^{-\gamma\alpha}) \geq 0$$

and as $\alpha \to \infty$ one gets "NATURAL LIGHT." If one, on the other hand, has an intensity measuring device with "normalized" sensitivity w(u), such as the eye, the observed Stokes vector will be given by the matrix expression

(27) $$\begin{bmatrix} I+M & C-iS \\ C+iS & I-M \end{bmatrix}_{\substack{Exp.\\Obsv.}} = \int_{-\infty}^{\infty} S_{ij}(u)\, w(u)\, du$$

which can be evaluated easily when w(u) is known. Thus, the experimental Stokes vector will depend upon the spectral sensitivity and resolution of the intensity measuring device.

One can speak of the PARTIAL COHERENCE of the TOTAL INTENSITY of time separated parts of the same beam, if and only if one is speaking aoubt integrated intensities with respect to some w(u).

In the particular example considered f_1 and f_2 were of the same intensity. If one takes

(28) $$f_1(t) = \rho_1 f(t) \qquad \rho_1, \rho_2 \text{ real } > 0$$
$$f_2(t) = \rho_2 f(t+\alpha)$$

then one gets

(30) $$\begin{bmatrix} I \\ M \\ C \\ S \end{bmatrix}_{\substack{Exp.\\Obsv.}} = \begin{bmatrix} \frac{1}{2}(\rho_1^2 + \rho_2^2) \\ \frac{1}{2}(\rho_1^2 - \rho_2^2) \\ \rho_1\rho_2 e^{-\gamma\alpha/2}\cos\alpha\omega \\ -\rho_1\rho_2 e^{-\gamma\alpha/2}\sin\alpha\omega \end{bmatrix}$$ (should be eq.(29))

and the resulting integrated Stokes vector will be

$$S_{ij}(u) = S(u)\begin{bmatrix} \rho_1^2 & \rho_1\rho_2 e^{iu\alpha} \\ \rho_1\rho_2 e^{-iu\alpha} & \rho_2^2 \end{bmatrix} \tag{29}$$

(this equ. out of place)

i.e., by mixing two parts of a given beam of linearly polarized light (after rotating one of the parts 90^o)[1]

1) This is probably the joker, experimentally, for the rotator must give 90^o twist toevery frequency component This is an idealized conceptual picture use here for illustrative purposes.

one can synthesize any desired integrated Stokes vector. It should not be forgotten that we are dealing with the radiation corresponding to a single spectral line. It should be noticed that since spectral lines are quite narrow, it would be difficult if not impossible to observe much else with the aid of filters but the integrated stokes vector.

3.8 The Statistical Analysis of a Maxwell Vector for the Case of Natural and Collision Broadening of a Single Spectral Line. The Mixture Constant.

Up to this point, attention has been confined to a single polarization component, thought of, for convenience as vertical polarization. A new feature enters the problem when one considers a plane wave containing both polarizations, say horizontal and vertical.

Statistically, this involves a class of pairs of functions $\{f_1(t), f_2(t)\}$. Since attention is again restricted to a single spectral line of "basic" frequency ω and to a stochastic drift of phase, it is convenient to write

$$(31) \qquad \begin{aligned} f_1(t) &= \rho_1 e^{i(\omega t - \varphi_1(t))} \\ f_2(t) &= \rho_2 e^{i(\omega t - \varphi_2(t))} \end{aligned} \qquad \rho_1 \rho_2 \text{ real} > 0$$

and apply the statistical analysis to the pair of random phases $\{\varphi_1(t), \varphi_2(t)\}$ for which one has the following Gaussian distributions

$$(32) \qquad W_{ij}(\varphi_{i\beta}, \varphi_{j\alpha}, \tau) = K_{ij} e^{-\frac{1}{2} a_{ij} (\varphi_{i\beta} - \varphi_{j\alpha} - \varphi \varepsilon_{ij})^2}$$

where $\varphi_{i\alpha} = \varphi_i(t) \;;\; \varphi_{i\beta} = \varphi_i(t+\tau)$ and

$$(33) \qquad \begin{aligned} a_{ij} &= \frac{1}{a|\varepsilon_{ij}| + \gamma|\tau|} \qquad a > 0 \\ \varepsilon_{11} &= \varepsilon_{22} = 0 \;,\; \varepsilon_{12} = -\varepsilon_{21} = 1 \end{aligned}$$

For reasons which will come out later, a is called the MIXTURE CONSTANT, and φ the RELATIVE PHASE.

The mixture constant represents the tendency of the polarizations to become independent. When $a = 0$, the relative phases are dependent in the same sense that the relative phase of the same component approaches zero as $\tau \to 0$. In this case when $a=0$, the relative phase approaches φ as $\tau \to 0$. If $a > 0$, the relative phase has a certain distribution about φ even when $\tau = 0$. Thus φ represents the relative phase that the two polarizations tend to maintain due to external causes and the amount by which $a > 0$ measures the extent to which these tendencies are not effective.

It is presumed that isotropic radiation in a natural source corresponds to the limit $a \to 0$ and that the present of even the slightest external causes, such as the presence of glass surrounding a discharge tube, fields, etc., causes a to take on some finite value. Thus the larger the mixture constant, the more "mixed" the light.

It should be noted that, statistically, light is now defined by six constants

ω = basic frequency
γ = emission probability
ρ_1 = strength of horizontal polarization
ρ_2 = strength of vertical polarization
φ = relative phase
a = mixture constant.

Actually, one more bit of data will be required and that is the Doppler distribution of ω, i.e.,

$$\omega = \left(1 + \frac{v}{c}\right)\omega_0 \tag{34}$$

where v is governedby the Boltzmann factor

$$e^{-Mv^2/2kT}$$

The treatment of this added complication is reserved for the next section. The immediate task is to carry out the statistical harmonic analysis of the above radiation.

The only new computation required is the evaluatinn of the integral

(35) $$\varphi_{12}(\tau) = \rho_1\rho_2 e^{-i(\omega\tau-\varphi)} \int_{-\infty}^{\infty} e^{i\psi} e^{-\frac{\frac{1}{2}\psi^2}{a+|\tau|\gamma}} d\psi$$

or

(35') $$\boxed{\varphi_{12}(\tau) = \rho_1\rho_2 e^{-i(\omega\tau-\varphi)} e^{-\frac{a+|\tau|\gamma}{2}}}$$

and the integral

(36) $$\int_0^u S_{12}(u) = \rho_1\rho_2 e^{-i\varphi} \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{iu\tau}-1}{i\tau} e^{-\left(\frac{a+|\tau|\gamma}{2}+i\omega\tau\right)} d\tau$$

or

(36') $$S_{12}(u) = \rho_1\rho_2 e^{-\left(\frac{a}{2}+i\varphi\right)} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(u-\omega)\tau} e^{-\frac{|\tau|\gamma}{2}} d\tau$$

or

(36") $$S_{12}(u) = \rho_1\rho_2 e^{-\left(\frac{a}{2}+i\varphi\right)} \frac{1}{2\pi} \frac{\gamma}{(\omega-u)^2+\frac{\gamma^2}{4}}$$

As a result one can immediately write for the "total" Stokes vector, the expression.

(37) $$\begin{bmatrix} I \\ M \\ C \\ S \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(\rho_1^2+\rho_2^2) \\ \frac{1}{2}(\rho_1^2-\rho_2^2) \\ \rho_1\rho_2 e^{-a/2}\cos\varphi \\ \rho_1\rho_2 e^{-a/2}\sin\varphi \end{bmatrix}$$

The equations for the STATISTICAL CONSTANTS are

(38)
ρ_1^2 = I+M .=. hlp amplitude
ρ_2^2 = I-M .=. vlp amplitude
a = ln $(I^2-M^2)/(C^2+S^2)$.=. mixture constant
φ = $\tan^{-1}S/C$.=. relative phase (mean)

Thus one can determine the statistical constants from the experimental measurement of the Stokes Vector and vice versa.

Note that

(39) $$e^{a} = \frac{I^2 - M^2}{C^2 + S^2}$$

and that

(40) $$C^2 + S^2 = \rho_1^2 \rho_2^2 e^{-a}$$

hence $$I^2 - M^2 - C^2 - S^2 = (e^{a} - 1)\rho_1^2\rho_2^2 e^{-a} = \rho_1^2\rho_2^2(1 - e^{-a})$$

and hence that the light is perfectly polarized if

$$\rho_1 = 0 \ , \ \rho_2 = 0 \ , \ \text{or} \ a = 0$$

The case a = 0 is the only non-trivial one and means, statistically, that the polarizations are perfectly correlated.

3.9 The General Statistical Representation of a Plane Wave. The Interference.

The point has now been reached in the theory to develop a random Maxwell vector $\{(f_1(t), f_2(t))\}$ which has a statistical character capable of describing a spectral line with natural, collision, and Doppler broadening.

Consider the following class of functions -

(41) $$\{f_i(t)\} = \left\{\rho_i e^{-i\left[\left(1 + \frac{v(t)}{c}\right)\omega t - \varphi_i(t)\right]}\right\}$$

These random functions $\{f_i(t)\}$ are themselves functions of the random functions $\{v(t)\}$, $\{\varphi(t)\}$ about which one postulates the following statistical information:

The function v(t) is a purely random function, homogeneous in time and therefore characterized by the distribution

$$W(v) = \sqrt{\frac{M}{2\pi kT}}\, e^{-\frac{Mv^2}{2kT}} \tag{42}$$

a Gaussian distribution, the Maxwell distribution of the velocities in the direction of observation.

As before, the $\varphi_1(t)$ and $\varphi_2(t)$ arise from a Markoff process and are governed by the Gaussian distribution

$$W_{ij}(\varphi_i, \varphi_j; \tau) = K_{ij}\, e^{-\frac{1}{2} a_{ij} (\varphi_i - \varphi_j - \varepsilon_{ij} \varphi)^2} \tag{43}$$

$$\varphi = \langle \varphi_1(t) - \varphi_2(t) \rangle_{Avg.}$$

where

$$\begin{aligned} \varphi_i &= \varphi_i(t+\tau) \\ \varphi_j &= \varphi_j(t) \\ a_{ij} &= \frac{1}{a|\varepsilon_{ij}| + \gamma|\tau|} \\ K_{ij} &= \sqrt{\frac{a_{ij}}{2\pi}} = \sqrt{\frac{1}{2\pi(a|\varepsilon_{ij}| + \gamma|\tau|)}} \end{aligned} \tag{44}$$

In terms of these distributions the first step of the statistical harmonic analysis may be written

$$\varphi_{ij}(\tau) = \rho_i \rho_j \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} W(v)\, W_{ij}(\psi_{ij}, \tau)\, e^{-i[(1+\frac{v}{c})\omega\tau - \psi_{ij}]}\, dv\, d\psi_{ij} \tag{45}$$

where, $$\psi_{ij} = \varphi_i(t+\tau) - \varphi_j(t)$$

By an appropriate change of variable one can write the phase φ explicitly

(46) $$\varphi_{ij}(\tau) = (\rho_i \rho_j e^{-i\varepsilon_{ij}\varphi})(e^{-i\omega\tau})\, D(\tau)\, N_{ij}(\tau)$$

where

(47) $$D(\tau) = \int_{-\infty}^{\infty} W(v)\, e^{-i\frac{v\omega}{c}}\, dv$$

(48) $$N_{ij}(\tau) = \int_{-\infty}^{\infty} W_{ij}(\psi,\tau)\, e^{-i\psi}\, d\psi$$

and

(49) $$W_{ij}(\psi,\tau) = \sqrt{\frac{1}{2\pi\,(a|\varepsilon_{ij}|+\gamma|\tau|)}}\; e^{-\frac{\frac{1}{2}\psi^2}{(a|\varepsilon_{ij}|+\gamma|\tau|)}}$$

Physically, one may make the following interpretation of the factors in the elements of the interference,

$e^{-i\omega\tau}$.=.	interference due to sharp line of frequency , the pure interference pattern.
$\rho_1\rho_2 e^{-\varepsilon_{ij}\varphi}$.=.	polarization characteristics of the interference
$N_{ij}(\tau)$.=.	broadening and depolarization due to the radiation process and external causes.
$D(\tau)$.=.	Doppler effect on the interference.

It may be said that the Doppler effect and "transition" effect modulate the pure interference pattern.

A point woth mentioning is the fact that the INTEGRATED SPECTRUM is given by

(50) $$\varphi_{ij}(0) = \int_{-\infty}^{\infty} S_{ij}(u)\, du$$

and hence the "total" Stokes vector is given by

(51) $$\left[\varphi_{ij}(0)\right] = \begin{bmatrix} I+M & C-iS \\ C+iS & I-M \end{bmatrix}_{\text{Exp. Obsv.}}$$

without the necessity of passing by a Fourier transform from $\varphi_{ij}(\tau)$ to $S_{ij}(u)$. This result may be proved

(52) $$\begin{aligned} \int_{-\infty}^{\infty} S(u)\, du &= \frac{1}{2\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{iu\tau} \varphi_{ij}(\tau)\, du\, d\tau \\ &= \lim_{A \to \infty} \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin \tau A}{\tau} \varphi_{ij}(\tau)\, d\tau \\ &= \varphi_{ij}(0) \end{aligned}$$

The expression $\frac{\sin \tau A}{\tau}$ is known as Dirichlet's discontinuous kernel.[1] Since it is the total intensity which is observed

1) Margenau and Murphy: "Mathematics of Physics and Chemistry," Van Nostrand, N.Y., 1938. p.247.

in an experiment, the evaluations of the "observed" Stokes vector is already completed when one has evaluated the interference matrix and set $\tau = 0$.

The next problem is the evaluations of $N_{ij}(\tau)$ and $D(\tau)$ Before doing this it will be helpful to evaluate the following general definite integral.

(53) $$F(a,b,c) = \int_{-\infty}^{\infty} e^{-(ax^2+bx+c)}\,dx$$

The technique is the one of completing the square

$$ax^2+bx = (\sqrt{a}x+A)^2 - A^2 = ax^2 + 2A\sqrt{a}\,x$$

from which it follows that

(54) $$A = \frac{b}{2\sqrt{a}} \quad ; \quad A^2 = \frac{b^2}{4a}$$

and one may write

$$F(a,b,c) = e^{\frac{b^2-4ac}{4a}} \int_{-\infty}^{\infty} e^{-(\sqrt{a}x+A)^2}\,dx$$

$$= \frac{1}{\sqrt{a}}\, e^{\frac{b^2-4ac}{4a}} \int_{-\infty}^{\infty} e^{-\xi^2}\,d\xi$$

or

(55) $$\boxed{F(a,b,c) = \sqrt{\frac{\pi}{a}}\, e^{\frac{b^2-4ac}{4a}}}$$

Applying this to the evaluation of $N_{ij}(\tau)$

(56) $$N_{ij}(\tau) = \sqrt{\frac{1}{2\pi\,(a|\varepsilon_{ij}|+\gamma|\tau|)}} \int_{-\infty}^{\infty} e^{-\left(\frac{1/2}{a|\varepsilon_{ij}|+\gamma|\tau|}\psi^2 + i\psi\right)}\,d\psi$$

where we take $\text{"}a\text{"} = \frac{1}{2(a|\varepsilon_{ij}|+\gamma|\tau|)}$, $b = i$, $c = 0$

(57) $$\boxed{N_{ij}(\tau) = e^{-\frac{a|\varepsilon_{ij}|+\gamma|\tau|}{2}}}$$

Radiative and Depolarization Effect

Applying it to the evaluation of $D(\tau)$

(58) $$D(\tau) = \sqrt{\frac{M}{2\pi kT}} \int_{-\infty}^{\infty} e^{-\left(\frac{Mv^2}{2kT} + i\frac{\omega\tau v}{c}\right)}\,dv$$

where

$$a = \frac{M}{2kT} \quad , \quad b = i\,\frac{\omega\tau}{c} \quad , \quad c = 0$$

and hence

$$(59) \qquad \boxed{D(\tau) = e^{-\left(\frac{kT}{Mc^2}\omega^2\right)\tau^2}}$$

Doppler Effect

Note that $\underset{T\to 0}{D(\tau)\to 1}$ as, in absence of thermal motion, it should.

The case for which one observes the "total" intensity of the spectral line is very simple, the Doppler and radiation broadening effects drop out i.e.,

$$(60) \qquad N_{ij}(0) = e^{-\frac{a}{2}|\varepsilon_{ij}|}$$

$$(61) \qquad D(0) = 1$$

and hence

$$(62) \qquad \varphi_{ij}(0) = \rho_i\rho_j e^{-\left(\frac{a}{2}|\varepsilon_{ij}| + i\varphi\,\varepsilon_{ij}\right)} = \rho_i\rho_j e^{-\psi_{ij}}$$

which involves a new use of the symbol ψ_{ij} , and where

ρ_1 = amplitude of vertical polarization

ρ_2 = amplitude of horizontal polarization

$\psi_{ij} = \frac{a}{2}|\varepsilon_{ij}| + i\varphi\varepsilon_{ij}$ = complex phase

$\mathcal{R}e(\psi_{ij}) = \frac{a}{2}|\varepsilon_{ij}|$ = depolarization (phase dispersion)

$\mathcal{I}m(\psi_{ij}) = \varphi\,\varepsilon_{ij}$ = mean relative phase

$$(63) \qquad \rho_1^2 = I + M \quad , \quad \rho_2^2 = I - M$$

$$\psi_{ij} = |\varepsilon_{ij}|\,\ln\sqrt{\frac{I^2-M^2}{C^2+S^2}} + i\,\varepsilon_{ij}\tan^{-1}\frac{S}{C}$$

and

(64) $$\begin{bmatrix} I \\ M \\ C \\ S \end{bmatrix}_{\substack{\text{Exp.} \\ \text{obsv.}}} = \begin{bmatrix} \frac{1}{2}(\rho_1^2+\rho_2^2) \\ \frac{1}{2}(\rho_1^2-\rho_2^2) \\ \rho_1\rho_2 e^{-a/2}\cos\varphi \\ -\rho_1\rho_2 e^{-a/2}\sin\varphi \end{bmatrix}$$ "total" Stokes vector independent of "broadening " effects

as before.

3.10. The Spectral Matrix for the General Plane Wave.

From eq.(46) it is evident that the computation of $S_{ij}(u)$ involves the computation of

(65) $$I_{ij}(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iu\tau} e^{-i\omega\tau} D(\tau) N_{ij}(\tau)\, d\tau$$

for then one may write

$$S_{ij}(u) = \rho_i \rho_j e^{-i\varphi\varepsilon_{ij}} I_{ij}(u)$$

The $|\tau|$ involved in $N_{ij}(\tau)$ requires the consideration of

(66) $$f^+(a,b,c) = \int_0^{\infty} e^{-(ax^2+bx+c)}\, dx = \frac{1}{\sqrt{a}} e^{\frac{b^2-4ac}{4a}} \int_A^{\infty} e^{-\xi^2} d\xi$$
$$f^-(a,b,c) = \int_{-\infty}^{0} e^{-(ax^2+bx+c)}\, dx = \frac{1}{\sqrt{a}} e^{\frac{b^2-4ac}{4a}} \int_{-\infty}^{A} e^{-\xi^2} d\xi$$

If one defines, for complex $z = x+iy$

(67) $$\mathrm{Erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-\xi^2} d\xi$$

where the path of integration is the line segment from 0 to z, then, in terms of this function

(68) $$f^+(a,b,c) = \frac{1}{2} F(a,b,c)\,(1 - \mathrm{Erf}(A))$$
$$f^-(a,b,c) = \frac{1}{2} F(a,b,c)\,(1 + \mathrm{Erf}(A))$$

where

$$A = b/2\sqrt{a}$$

and b will turn out to be complex. Now,

$$I_{ij} = \frac{1}{2\pi}\int_0^\infty e^{-\left(\frac{kT\omega^2}{Mc^2}\tau^2 + \left(\frac{\gamma}{2} + i(\omega-u)\right)\tau + \frac{a}{2}|\varepsilon_{ij}|\right)}\,d\tau$$

(69)

$$+ \frac{1}{2\pi}\int_{-\infty}^0 e^{-\left(\frac{kT\omega^2}{Mc^2}\tau^2 + \left(-\frac{\gamma}{2} + i(\omega-u)\right)\tau + \frac{a}{2}|\varepsilon_{ij}|\right)}\,d\tau$$

Identifying coefficients

(70)

$$a_1 = \frac{kT\omega^2}{Mc^2} \qquad a_2 = \frac{kT\omega^2}{Mc^2}$$

$$b_1 = \frac{\gamma}{2} + i(\omega-u) \qquad b_2 = -\frac{\gamma}{2} + i(\omega-u)$$

$$c_1 = \frac{a}{2}|\varepsilon_{ij}| \qquad c_2 = \frac{a}{2}|\varepsilon_{ij}|$$

$$A_1 = \frac{b_1}{2\sqrt{a_1}} = \sqrt{\frac{Mc^2}{4kT\omega^2}}\left(\frac{\gamma}{2} + i(\omega-u)\right)$$

$$A_2 = \frac{b_2}{2\sqrt{a_2}} = \sqrt{\frac{Mc^2}{4kT\omega^2}}\left(-\frac{\gamma}{2} + i(\omega-u)\right)$$

one may write

(71)

$$I_{ij}(u) = \frac{1}{4\pi}\left(F(a_1,b_1,c_1) + F(a_2,b_2,c_2)\right)$$
$$- \frac{1}{2\pi}\left(\mathrm{Erf}(A_1)\,F(a_1,b_1,c_1) - \mathrm{Erf}(A_2)\,F(a_2,b_2,c_2)\right)$$

where $F(a,b,c) = \sqrt{(\pi/a)}\ e^{(A^2-C)}$

Equation (71) is in closed form but it behaves badly when one attempts to pass to the limit $T \to 0$ to obtain the case of no Doppler broadening. It is worth while to note that

(72) $$I_{ij}(u) = e^{\frac{\alpha}{2}|\varepsilon_{ij}|}\ I(u;T,\gamma)$$

where

(73) $$I(u;T,\gamma) = \frac{1}{2\pi}\int_0^{\infty} e^{-\left(\frac{kT\omega^2}{Mc^2}\tau^2 + \left(\frac{\gamma}{2} + i(\omega-u)\right)\tau\right)}\,d\tau + \frac{1}{2\pi}\int_{-\infty}^{0} e^{-\left(\frac{kT\omega^2}{Mc^2}\tau^2 + \left(-\frac{\gamma}{2} + i(\omega-u)\right)\tau\right)}\,d\tau$$

Here $I(u;T,\gamma)$ is the spectral distribution factor common to the "spectrum." From previous work

(74) $$\boxed{I(u;0,\gamma) = \frac{\gamma/2\pi}{(\omega-u)^2+\gamma^2/4}}$$ Radiation Broadening

The other interesting approximation is the neglect of radiation broadening in favor of Doppler broadening,

(75) $$I(u;T,0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\left(\frac{kT\omega^2}{Mc^2}\tau^2 + i(\omega-u)\tau\right)}\,d\tau$$

which has the result

(76) $$\boxed{I(u;T,0) = \sqrt{\frac{Mc^2}{4\pi kT\omega^2}}\ e^{-\frac{Mc^2}{4kT}\left(\frac{\omega-u}{\omega}\right)^2}}$$ Doppler Broadening

which checks eq.(4). Analytically it appears quite possible to expand $I(u;T,\gamma)$ as a power series in T and γ to get the general line shape. This, however, will not be carried out here. In terms of $I(u;T,\gamma)$ the general spectrum is

(77) $$\boxed{S_{ij}(u) = \rho_i\rho_j\, e^{-\psi_{ij}}\, I(u;T,\gamma)}$$ Radiation with Depolarization Doppler and Radiation Broadening

CHAPTER IV

QUASI-STATIONARY SCATTERING

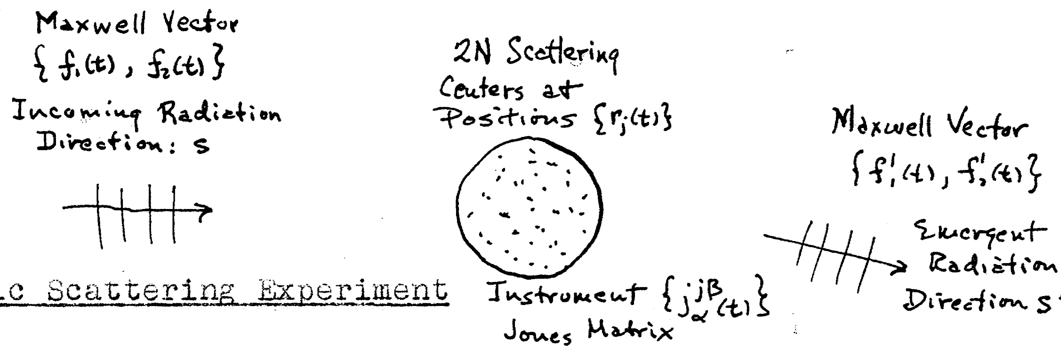
4.1 Introduction.

All the logical and intuitive ground work has been laid. It is now possible to turn to a class of problems that the theory was designed to handle and dispose of it compactly and briefly. The class is best described as single scattering by a group of particles whose relative positions may have any degree of order ranging from the complete regularity of a perfect crystial to the complete disorder of a gas. The term quasi-stationary refers to the fact that we are averaging the positions of the particles independently of the Wiener correlation of the incoming radiation.

In the heuristic process leading to the general theory it is true that much of the material of this chapter was arrived at by a process of trial and error on particular examples. It is the end result of this process that the theory was concieved. But, one concieved the theory continued to develop under its own momentum and the mathematical technique was improved and refined. The chapter is being rewritten deductively even though it was first evolved inductively.

The theory represents a gain over previous optical algebras in that they were limited to two extremes, the Jones algebra of coherent scattering and the Mueller algebra of incoherent scattering.

Following the theory of Chapter I, we begin our discussion with Jones t-algebra. The incoming wave is represented by



Basic Scattering Experiment

the Maxwell vector $\{f_1(t), f_2(t)\}$ having direction $\vec{s}$. As this wave impinges on the 2N scattering centers, 2N wavelets emerge. These wavelets are resolved into plane wave components and we are interested in the ones having direction $\vec{s}\,'$. We denote them by the generalized Maxwell functions $\{f'^{j}_1(t), f'^{j}_2(t)\}$. They are computed with aid of the Jones functions $\{j^{j\beta}_{\alpha}(t)\}$. The centers are at positions $\{r_j(t)\}$. The basic law for the relation between the incoming and outgoing radiation is

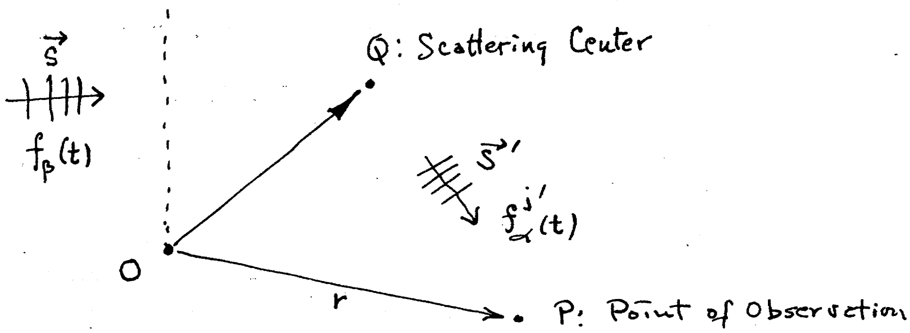
$$(1) \qquad \boxed{f'^{j}_{\alpha}(t) = j^{j\beta}_{\alpha}(\underline{t}_o) f_{\beta}(t-\underline{t}_o)}$$

4.2 Maxwell Vectors in Space and Time.

The expression eq.(1) is not quite correct as it stands. We must be a little careful about the time t.

We are at all times dealing with plane waves. The basic expression $f(t)$, for the component of such a wave, when the retardation is omitted, refers to an arbitrary origin O of the chosen coordinate system. The basic law for the wavelet from the jth scattering center is correctly written

$$f'^{j}_{\alpha}\left(t - \frac{s'\cdot r}{v}\right) = j^{j\beta}_{\alpha}(\underline{t}_0)\, f_{\beta}\left(t - \frac{s'\cdot(r - r_j(t))}{v} - \frac{s\cdot r_j(t)}{v} - \underline{t}_0\right) \tag{2}$$



<u>Retardation Geometry</u>

Or, resorting to the summation convention, i.e., superposing wavelets one obtains

$$f'_{\alpha}\left(t - \frac{s'\cdot r}{v}\right) = j^{j\beta}_{\alpha}(\underline{t}_0)\, f_{\beta}\left(t - \frac{s'\cdot r}{v} - \underline{t}_0 - \frac{(s - s')\cdot r_j(t)}{v}\right) \tag{3}$$

Making a change in the epoch, gives

$$f'_{\alpha}(t) = j^{j\beta}_{\alpha}(\underline{t}_0)\, f_{\beta}\left(t - \underline{t}_0 - \frac{(s - s')\cdot r_j(t - \underline{t}_0)}{v}\right) \tag{4}$$

as the solution of our problem in generalized Jones t-algebra We have the summation of j, β and convolution on t_0.

Even eq.(4) leaves something to be desired if the particles are moving rapidly. The times t used in the evaluation of r_j should be $t + \tau^j$ where

$$\upsilon\tau^{j} = s \cdot r^{j}(t+\tau^{j}) \tag{5}$$

But if we consider this refinement we are forced to reconsider the expressions

$$f'_{\alpha}(t) = j_{\alpha}^{\beta}(\underline{t}_o) f_{\beta}(t-\underline{t}_o) = j_{\alpha}^{\beta}(t-\underline{t}_o) f_{\beta}(\underline{t}_o)$$

which are quite correct for stationary scattering centers and inquire as to their correctness and equivalence for a moving center. This endeavor quickly leads to some very difficult research which should wait until some of the simpler situations have been explored. We shall therefore treat the scattering centers as QUASI-STATIONARY and assume that we can average separately over the positions of the scattering centers and over the Maxwell vectors. We shall therefore write eq.(4)

$$\boxed{f'_{\alpha}(t) = j_{\alpha}^{j\beta}(\underline{t}_o)\, f_{\beta}\left(t - \underline{t}_o - \frac{(s-s')\cdot r_j(\tau)}{\upsilon}\right)} \tag{6}$$

where now we use τ as the index for the "independent" time series of positions of the scattering centers and give up any thought of including $r_j(\tau)$ in the Wiener correlation. This simplifying assumption is tantemount to neglecting the effect of the motion of the scattering centers on the power spectrum of the scattered radiation. We are forced to make it because we have not sufficient information about $r_j(\tau)$ from statistical mechanics to calculate the required for the relative positions r_{1j}, r_{2k} under the action of the inter-scattering-center forces, e.g., ionic forces in the case of an electrolyte. We know, at best, $w_1(r_j)$ and in most cases less, e.g., the Debye-Huckel theory of electrolytes. We shall refer to our theory as a <u>quasi-stationary scattering theory depending on the distribution</u>

$$w_1(r_j) = w_1(r_1, r_2, \ldots, r_{2N})$$

which is independent of τ since we assume $r_j(\tau)$ to be homogeneous (stationary) in time. This theory may be expected to agree more and more closely with experiment the more sluggish the particles.

It may be counted one of the virtues of generalized optical algebra that it is written in a notation which forces one to be explicit about the simplifying assumptions which it is necessary to make. As a result it becomes comparatively easy to retrace our steps and find possible sources of dissagreement with experiment. As an example of one such disagreement. It is found that in the scattering of light from smokes there is a small depolarization effect. The light used is the mercury green line. We predict no such depolarization when we make the quasi-stationary assumption. It seems reasonable in view of the results of Chapter III to conclude that by not making the quasi-stationary assumption the effect of the different optical paths making up the scattered radiation andthe finite width of the line would account for the slight depolarization effect. This is a good point for further research.

4.3 Evaluation of the Debye Distribution Function.

We may now use the theory outlined in Sec.1.14 and pass to generalized Mueller ω-algebra in which the basic law is

(7) $$\mathcal{L}'(\omega) = \mathcal{M}(\omega)\,\mathcal{L}(\omega)$$

where the MUELLER MATRIX is given by

(8) $$\mathcal{M}(\omega) = N_{jk}(\omega)\,\mathcal{M}^{jk}(\omega)$$

andthe DEBYE DISTRIBUTION FUNCTION is given by

(9) $$N_{jk}(\omega) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} e^{i\omega\frac{(s-s')\cdot(r_j(\tau)-r_k(\tau))}{v}}\,d\tau$$

(10) $$S = \frac{\omega\,|\vec{s}-\vec{s}'|}{v} = 2\frac{\omega}{v}\cos\gamma/2$$

the AUTO AND CROSS MUELLER MATRICES are

(11) $$\mathcal{M}^{jk}(\omega) = \left[M_{\alpha\beta}^{\gamma\delta jk}(\omega)\right]$$

where $\alpha + 2(\beta-1)$ = col. and $\gamma + 2(\delta-1)$ = row. of the given element in $\mathcal{M}^{jk}$.

The MUELLER FUNCTIONS are related to the JONES FUNCTIONS by

(12) $$M_{\alpha\beta}^{\gamma\delta jk}(\omega) = J_{\alpha}^{\gamma j}(\omega)\,J_{\beta}^{\delta k\,*}(\omega)$$

The matrices $\mathcal{M}$, N etc., depend on the scattering angle and frequency. The first task is the evaluation of the Debye distribution function.

$N_{jk}(s)$ for Coherent Scattering.

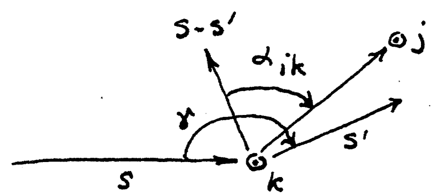
if the scattering centers are members of a crystal lattice, the $r_j(\tau) = r_j$, i.e., contents in the quasi-stationary approximation eq.(9) becomes

$$N_{jk} = e^{i\omega \frac{(s-s')\cdot(r_j-r_k)}{v}} = e^{i\omega \tau_{jk}(\gamma)} = e^{i\varphi_{jk}} \tag{13}$$

where

$$\varphi_{jk} = \frac{(s-s')\cdot(r_j-r_k)\,\omega}{v} \tag{14}$$

The geometry of the situation is



Let $\quad r_{jk} = |r_j - r_k| \;;\; \alpha_{jk} = \cos^{-1} \dfrac{(s-s')\cdot(r_j-r_k)}{\sqrt{(s-s')^2(r_j-r_k)^2}}$

$$S = \frac{\omega|s-s'|}{v} = 2\frac{\omega}{v}\cos\frac{\gamma}{2} \tag{15}$$

Then

$$\varphi_{jk}(S) = S\, r_{jk}\cos\alpha_{jk} = 2\frac{\omega}{v} r_{jk}\cos\alpha_{jk}\cos\frac{\gamma}{2} = \omega\,\tau_{jk}(\gamma) \tag{16}$$

$$\tau_{jk}(\gamma) = \frac{2 r_{jk}\cos\alpha_{jk}\cos\gamma/2}{v} \tag{17}$$

N_{jk}(s) for Incoherent Scattering.

In this case we have (j,k not summed)

(18) $$N_{jk}(s) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} e^{isr_{jk}(\tau)\cos\alpha_{jk}(\tau)}\, d\tau$$

Making use of the ergodic hypothesis we have

(19) $$N_{jk}(s) = \int w(r,\alpha)e^{isr\cos\alpha}\, dr d\alpha$$

It is usual to assume that the distribution of the scattering particles has the form

(20) $$w(r,\alpha) = \frac{2\pi\rho(r)r^2\sin\alpha}{nV}$$

where

n = average concentration
V = volume of scattering region
$\rho(r)$ = local concentration

i.e., that the distribution is isotropic as viewed from any particle. This is the first approximation usually made in the scattering of x-rays by liquids.[1] To make a a better approximation would lead into a detailed study of

1) Gingrich, N.S.: "Diffraction of X-rays by liquid Elements." Rev. Mod. Phys. 15 90 (1943).

the statistical effects of intermolecular forces and their angular dependence.

Using eq.(20), α integrates out. One obtains

$$(21) \qquad N_{jk}(s) = \begin{cases} \dfrac{4\pi}{nV} \displaystyle\int_0^R \rho(r)\, r^2 \frac{\sin sr}{sr}\, dr & j \neq k \\ \dfrac{4\pi}{nV} \displaystyle\int_0^R \rho(r)\, r^2\, dr = 1 & j = k \end{cases}$$

Before continuing the evaluation of eq.(21) it is desirable to consider the integral

$$(22) \qquad \lim_{R\to\infty} \frac{4\pi}{Vs} \int_0^R r \sin sr\, dr = \lim_{R\to\infty} \frac{3}{R^3 s^3} \int_0^{Rs} x \sin x\, dx$$
$$= \lim_{R\to\infty} \frac{3 \sin Rs}{R^3 s^3} - \frac{3 \cos Rs}{R^2 s^2} = 0 \qquad \text{when } s \neq 0$$

In the case s = 0, the integral becomes

$$(23) \qquad \lim_{R\to\infty} \frac{3}{R^3} \int_0^R r^2 dr = 1, \quad \text{for all } R$$

With this in mind, one has

$$(24) \qquad N_{jk}(s) = \begin{cases} \dfrac{4\pi}{nV}\left[\displaystyle\int_0^R r^2(\rho(r) - n) \frac{\sin sr}{sr}\, dr\right] + \begin{cases} 0 & j \neq k,\ s \neq 0 \\ 1 & j \neq k,\ s = 0 \end{cases} \\ 1, \quad j = k, \quad \text{all } s \end{cases}$$

Now we specialize to the INCOHERENT CASE, i.e., scattering from a uniform distribution of scatterers, i.e.,

$$(25) \qquad \boxed{\rho(r) = n}$$

For this case

$$(26) \qquad N_{jk}(s) = \begin{cases} 0 & j \neq k,\ s \neq 0 \\ 1 & j \neq k,\ s = 0 \\ 1 & j = k,\ \text{all } s \end{cases}$$

Thus the special of COHERENT FORWARD SCATTERING is given by

$$(27) \qquad \boxed{N_{jk}(0) = 1}$$

The case of INCOHERENT SCATTERING ($s \neq 0$) is given by

(28) $$\boxed{N_{jk}(s) = \delta_{jk}, \quad s \neq 0}$$

$N_{jk}(s)$ for Partially Coherent Scattering.

The integral

(29) $$I(s) = \frac{4\pi}{nV} \int_0^R r^2(\rho(r) - n) \frac{\sin sr}{sr} dr$$

occuring in eq.(24) occurs in the theory of scattering of x-rays by liquid elements (Gingrich, loc. cit. p.93). He introduces the integral

(30) $$\boxed{i(s) = 4\pi \int_0^\infty r^2(\rho(r) - n) \frac{\sin sr}{sr} dr}$$

We note in anticipation that for an electrolyte

(31) $$\boxed{\rho(r) - n = \pm \frac{\kappa^2}{8\pi} \frac{e^{-\kappa r}}{r}}$$

the derivation of which will be sketched later. In terms of i(s) one may write for any $\rho(r)$ distribution

(32) $$\boxed{N_{jk}(s) = \begin{matrix} i_{jk}(s)/nV & j \neq k \\ 1 & j = k \end{matrix} \quad s \neq 0}$$

4.4 Poisson-Boltzmann Distribution. i(s) for an Electrolyte.[1]

1) The derivation is patterned after Falkenhagen;"Electrolytes" Oxford, 1934. p.101.

One begins by trying to find the time average potential in the neighborhood of a given ion called the central ion. In our problem this is the ion which was given position vector $r_k(\tau)$. Let point P be at fixed distance from the central ion, assumed positive, and

$$\Psi(P) \;.=.\; \text{mean electrostatic potential at P}$$

If we consider a small volume dV surrounding P we have

$$n_+ dV = n e^{-q\Psi/kT} dV = \text{avg. no. + ions in } dV$$

(33) $$n_- dV = n e^{q\Psi/kT} dV = \text{avg. no. - ions in } dV$$

where q = charge per ion

n = N/V = number of pos. or neg. ions per cm^3

Eq.(33) must satisfy the condition

(34) $$\lim_{T \to \infty} n_+ dV = n dV$$

i.e., uniform distribution at a sufficiently high temperature.

The average Ψ is given by the Poisson differential equation

(35) $$\boxed{\nabla^2 \Psi = -\rho/\epsilon}$$

where ρ = true charge. The expression for ρ is obtained from eq.(33) and is

$$\rho = nq\left(e^{-q\Psi/kT} - e^{q\Psi/kT}\right)$$

(36) $$= -2nq \sinh(q\Psi/kT)$$

Then Ψ satisfies the equation

(37) $$\nabla^2\Psi = \frac{2nq}{\epsilon}\sinh\frac{q\Psi}{kT}$$

or approximating $\left|\frac{q\Psi}{kT}\right| \ll 1$

(38) $$\boxed{\nabla^2\Psi = K^2\Psi}$$ Poisson-Boltzmann Equation

where

(39) $$\boxed{K^2 = \frac{2nq^2}{\epsilon kT}}$$

By symmetry we look for a radial solution of the equation

(40) $$\frac{1}{r}\frac{d^2}{dr^2}(r\Psi) = K^2\Psi$$

which is

(41) $$\boxed{\Psi = \frac{q}{4\pi\epsilon}\,\frac{e^{-Kr}}{r}}$$

in order that

(42) $$\Psi \rightarrow \frac{q}{4\pi\epsilon r} \quad \text{when} \quad r \rightarrow 0$$

Using this value of Ψ and making the same approximation

(43) $$n_+ = n\left(1 - \frac{q^2}{4\pi\epsilon kT}\,\frac{e^{-Kr}}{r}\right) = n\left(1 - \frac{K^2}{8\pi n}\,\frac{e^{-Kr}}{r}\right)$$
$$n_- = n\left(1 + \frac{q^2}{4\pi\epsilon kT}\,\frac{e^{-Kr}}{r}\right) = n\left(1 + \frac{K^2}{8\pi n}\,\frac{e^{-Kr}}{r}\right)$$

Now using $\rho(r)$ in the original sense

(44) $$\boxed{\rho(r) - n = \pm\frac{K^2}{8\pi}\,\frac{e^{-Kr}}{r}}$$

where the plus sign refers to a like ions and the minus sign refers to unlike ions. i(s) may now be computed

(45)
$$
\begin{aligned}
i_{\pm}(s) &= \pm \frac{\kappa^2}{2s} \int_0^\infty e^{-\kappa r} \sin sr \, dr \\
&= \pm \operatorname{Im} \frac{\kappa^2}{2s} \int_0^\infty e^{-(\kappa - is)r} \, dr \\
&= \pm \operatorname{Im} \frac{\kappa^2}{2s} \left. \frac{e^{-(\kappa - is)r}}{-(\kappa - is)} \right|_0^\infty \\
&= \pm \operatorname{Im} \frac{\kappa^2}{2s} \frac{1}{\kappa - is} \\
&= \pm \operatorname{Im} \frac{\kappa^2}{2s} \frac{\kappa + is}{\kappa^2 + s^2}
\end{aligned}
$$

or

(46)
$$\boxed{i(s) = 1/2(1+(s/\kappa)^2)}$$

If we number the ions in such a way that j = 1,2,...,N are the indices of plus ions and j = N+1, ..., 2N are the indices of minus ions, then

(47)
$$
N_{jk}(s) = \begin{cases} -i(s)/nV , & j \neq k \text{ , } j, k \text{ either } 1,\dots,N \text{ or } N+1,\dots,2N \\ +i(s)/nV , & j \text{ or } k \text{ } 1,2,\dots,N \text{ and } k \text{ or } j = N+1,\dots,2N \\ 1 & j = k \end{cases}
$$

for a simple electrolyte.

As a check on the coefficient of eq.(44) we integrate

(48)
$$
\begin{aligned}
\int_0^\infty (\rho(r) - n) 4\pi r^2 dr &= \pm \frac{\kappa^2}{2} \int_0^\infty r e^{-\kappa r} dr \\
&= \mp \frac{\kappa^2}{2} \left. \frac{e^{-\kappa r}}{\kappa^2} (\kappa r + 1) \right|_0^\infty = \pm \frac{1}{2}
\end{aligned}
$$

Since: plus goes with opposite signed ions and minus goes with same signed ions, eq.(48) predicts an average deficiency of 1/2 ion of the same sign as the central ion and an excess of 1/2 ion of the opposite sign. This accounts for the net charge necessary to give zero total charge for the electrolyte.

4.5 The Mueller Matrix for a Simple Electrolyte.

The first step is to compute the auto- and cross-Mueller matrices. We denote by

$$(49)\qquad \begin{aligned} {}^{+}J^{\beta}_{\alpha}(\omega) &= \text{Jones matrices of plus ions} \\ {}^{-}J^{\beta}_{\alpha}(\omega) &= \text{Jones matrices of minus ions} \end{aligned}$$

and by

$$(50)\qquad \begin{aligned} {}^{++}\mathcal{M}(\omega) &= \left[{}^{+}J^{\gamma}_{\alpha}(\omega)\,{}^{+}J^{\delta *}_{\beta}(\omega)\right] \\ {}^{+-}\mathcal{M}(\omega) &= \left[{}^{+}J^{\gamma}_{\alpha}(\omega)\,{}^{-}J^{\delta *}_{\beta}(\omega)\right] \\ {}^{-+}\mathcal{M}(\omega) &= {}^{+-}\mathcal{M}^{*}(\omega) \\ {}^{--}\mathcal{M}(\omega) &= \left[{}^{-}J^{\gamma}_{\alpha}(\omega)\,{}^{-}J^{\delta *}_{\beta}(\omega)\right] \end{aligned}$$

the basic auto-and cross-Mueller matrices. Then

$$(51)\qquad \mathcal{M}^{ij}(\omega) = \begin{cases} {}^{++}\mathcal{M} & \text{if } i,j = 1,2,\ldots,N \\ {}^{+-}\mathcal{M} & \text{if } i = 1,2,\ldots,N,\ j = N+1,\ldots,2N \\ {}^{-+}\mathcal{M} & \text{if } j = 1,2,\ldots,N,\ i = N+1,\ldots,2N \\ {}^{--}\mathcal{M} & \text{if } i,j = N+1,\ldots,2N \end{cases}$$

which is best visualized

$$(52)\qquad \mathcal{M}^{ij}(\omega) = \left[\begin{array}{c|c} {}^{++}\mathcal{M} & {}^{+-}\mathcal{M} \\ \hline {}^{-+}\mathcal{M} & {}^{--}\mathcal{M} \end{array}\right]$$

In similar fashion

$$
(53) \qquad N_{ij}(s) = \left[\begin{array}{ccc|ccc}
1 & & -\frac{i(s)}{nV} & & & \\
 & \ddots & & & \frac{i(s)}{nV} & \\
-\frac{i(s)}{nV} & & 1 & & & \\
\hline
 & & & 1 & & -\frac{i(s)}{nV} \\
 & \frac{i(s)}{nV} & & & \ddots & \\
 & & & -\frac{i(s)}{nV} & & 1
\end{array}\right]
$$

and hence

$$
\begin{aligned}
\mathcal{M} &= N_{ij} \mathcal{M}^{ij} \\
&= N\left({}^{++}\mathcal{M} + {}^{--}\mathcal{M}\right) \\
(54) \qquad &\quad - N(N-1)\frac{i(s)}{nV}\left({}^{++}\mathcal{M} + {}^{--}\mathcal{M}\right) \\
&\quad + N^2\left({}^{+-}\mathcal{M} + {}^{-+}\mathcal{M}\right)\frac{i(s)}{nV}
\end{aligned}
$$

or

$$
\begin{aligned}
\mathcal{M} &= N\left(1 - (N-1)\frac{i(s)}{nV}\right)\left({}^{++}\mathcal{M} + {}^{--}\mathcal{M}\right) \\
(55) \qquad &\quad - N^2\left({}^{+-}\mathcal{M} + {}^{-+}\mathcal{M}\right)\frac{i(s)}{nV}
\end{aligned}
$$

or the Mueller matrix per ion is

$$
(56) \qquad \boxed{\frac{\mathcal{M}}{N} = \left({}^{++}\mathcal{M} + {}^{--}\mathcal{M}\right) - \left({}^{++}\mathcal{M} + {}^{-+}\mathcal{M} + {}^{+-}\mathcal{M} + {}^{--}\mathcal{M}\right) i(s)}
$$

If we write

$$
(57) \qquad \boxed{\begin{aligned}
\mathcal{M}_0 &= {}^{++}\mathcal{M} + {}^{--}\mathcal{M} \\
\mathcal{M}_1 &= \mathcal{M}_0 + {}^{+-}\mathcal{M} + {}^{-+}\mathcal{M}
\end{aligned}}
$$

then

$$\boxed{\frac{\mathcal{M}}{N} = \mathcal{M}_0 - \frac{\mathcal{M}_1}{2(1+(\frac{s}{\kappa})^2)}} \tag{58}$$

is the Mueller matrix per ion. This completes the calculation for simple electrolytes containing N ions of each sign.

4.6 Mueller Matrix for a General Electrolyte.

Suppose now that the solution contains the ionic species $1,\ldots,i,\ldots,s$. The number of these species is respectively

$$n_1,\ldots,n_i,\ldots,n_s$$

and the corresponding valencies

$$z_1,\ldots,z_i,\ldots,z_s$$

which may be positive or negative and must satisfy

$$n_i z_i = 0 \qquad \text{(summation convention)} \tag{59}$$

Again consider a given central ion, j. The density of the ions of the ith species is

$$n_i e^{-z_i e \Psi_j / kT} \tag{60}$$

The total density in a volume element near the central ion is

$$\Pi_j = e \sum_i n_i z_i e^{-z_i e \Psi_j / kT} \tag{61}$$

and, as before, we get the Poisson-Boltzmann equation

$$\nabla^2 \Psi_j = \kappa^2 \Psi_j \tag{62}$$

where

(63) $$\kappa^2 = \frac{e^2}{\epsilon kT} \sum_i n_i z_i^2$$

If the ion in question has charge $z_j e$ the potential around it is

(64) $$\Psi_j = \frac{z_j e}{4\pi\epsilon} \frac{e^{-\kappa r}}{r}$$

The density of ions of species j about a central ion of species i is

(65) $$n_j \left(1 - \frac{z_i z_j e^2}{4\pi\epsilon} \frac{e^{-\kappa r}}{r}\right)$$

We write

(66) $$\rho_{ij}(r) - n_j = - \frac{z_i z_j n_j e^2}{4\pi\epsilon kT} \frac{e^{-\kappa r}}{r}$$

and find

(67) $$i_{ij}(s) = - \frac{z_i z_j n_j e^2}{\epsilon kT} \frac{1}{\kappa^2 + s^2}$$

or

(68) $$i_{ij}(s) = - \frac{z_i z_j n_j}{\sum_\ell n_\ell z_\ell^2} \frac{1}{(1 + (\frac{s}{\kappa})^2)}$$

in the particular case

$$n_1 = N \quad , \quad n_2 = N$$
$$z_1 = 1 \quad , \quad z_2 = -1$$

this becomes

$$i_{ij}(s) = \pm \frac{1}{2(1+(s/\kappa)^2)} \qquad \begin{matrix} - & i = j \\ + & i \neq j \end{matrix}$$

as before.

Now let

$$(69) \qquad x_j = n_j/\Sigma n_k$$

the fraction of the particles of species i

$$(70) \qquad z^2 = (\Sigma n_k z_k^2)/(\Sigma n_k)$$

the root mean square charge, and

$$(71) \qquad \zeta_i = z_i/z$$

then

$$(72) \qquad i_{\ell m}(s) = -X_m \frac{\zeta_\ell \zeta_m}{1+(\frac{s}{\kappa})^2}$$

then (changing the sign of $i_{\ell m}(s)$ for convenience)

$$(72') \qquad i_{\ell m}(s) = X_m \frac{\zeta_\ell \zeta_m}{1+(\frac{s}{\kappa})^2}$$

where we use ℓ for the central ion of the ℓth species and m for the neighboring ion of the mth species.

The matrix for N_{jk} now has the form

$$(73) \qquad N_{jk} = \begin{array}{cc} & \begin{array}{ccccc} \ell = 1 & 2 & \cdots & \cdots & s \end{array} \\ \begin{array}{c} m= \\ 1 \\ 2 \\ \vdots \\ \vdots \\ s \end{array} & \left[\begin{array}{c|c|c|c|c} \begin{matrix} 1 & -\frac{i_{11}}{n_1 V} \\ -\frac{i_{11}}{n_1 V} & 1 \end{matrix} & & & & -\frac{i_{1s}}{n_s V} \\ \hline & 1 \ddots 1 & & & \\ \hline & & 1 \ddots 1 & & \\ \hline & & & 1 \ddots 1 & \\ \hline -\frac{i_{s1}}{n_1 V} & & & & \begin{matrix} 1 & -\frac{i_{ss}}{n_s V} \\ -\frac{i_{ss}}{n_s V} & 1 \end{matrix} \end{array}\right] \end{array}$$

One forms the corresponding Mueller matrices

(74) $$\boxed{\mathcal{M}_{\ell m} = J_{\alpha}^{\ell\gamma} J_{\beta}^{m\delta\,*}}$$

where

$$J_{\alpha}^{\ell\beta} = \text{ the Jones matrices of the } \ell \text{ th species.}$$

The resultant Mueller matrix is

$$\mathcal{M} = \sum \left(n_\ell - \left(\frac{n_\ell (n_\ell - 1)}{n_\ell V}\, i_{\ell\ell}\right)\mathcal{M}_{\ell\ell}\right. - \sum_{m \neq \ell} n_\ell n_m \mathcal{M}_{\ell m} \frac{i_{\ell m}(s)}{n_m V}$$

or better, the resultant Mueller matrix is

(75) $$\mathcal{M} = \mathcal{M}_0 - \sum_{\ell, m} n_\ell\, i_{\ell m} \mathcal{M}_{\ell m}$$

where

(76) $$\boxed{\mathcal{M}_0 = \sum n_\ell \mathcal{M}_{\ell\ell}}$$

is the MUELLER MATRIX FOR INCOHERENT SUPERPOSITION.

In more detail

(77) $$\mathcal{M} = \mathcal{M}_0 - \sum_{\ell, m} \frac{n_\ell n_m \varsigma_\ell \varsigma_m \mathcal{M}_{\ell m}}{(\sum n_\ell)(1 + (\frac{s}{k})^2)}$$

Finally, one may write

(78) $$\frac{\mathcal{M}}{N} = \frac{\mathcal{M}_0}{N} - \sum_{\ell, m} \frac{x_\ell \varsigma_\ell x_m \varsigma_m}{1 + (\frac{s}{k})^2} \mathcal{M}_{\ell m}$$

where

(79) $$\frac{\mathcal{M}_0}{N} = \sum x_\ell \mathcal{M}_{\ell\ell}$$

as the resultant Mueller matrix per particle, where

$\zeta_\ell = z_\ell / z$.=. specific valance

$z^2 = \Sigma x_\ell z_\ell^2$.=. rms. valance $= \dfrac{\epsilon k T K^2}{N e^2}$

$x_\ell = \dfrac{n_\ell}{\Sigma n_\ell}$.=. fraction of particle of type ℓ .

$N = \Sigma n_\ell$.=. total number of particles

and hence x_ℓ , ζ_ℓ are non-dimensional.

z = root mean square (rms) valance per particle

If now we write

$$\mathcal{M}_1 / N = \sum_{\ell, m} x_\ell \zeta_\ell x_m \zeta_m M_{\ell m} \tag{81}$$

we have

$$\frac{\mathcal{M}}{N} = \frac{\mathcal{M}_0}{N} - \frac{1}{1+(\frac{\xi}{K})^2} \frac{\mathcal{M}_1}{N} \tag{82}$$

i.e., the resultant Mueller matrix is a linear combination of two matrices, $\mathcal{M}_0$ the incoherent matrix and $\mathcal{M}_1$ the coherence matrix. As the concentration increases, only K^2 changes. In fact

$$K^2 = \frac{N}{T} \left(\frac{e^2}{\epsilon k} \Sigma x_i z_i^2 \right) = \frac{N}{T} \left(\frac{e^2 z^2}{\epsilon k} \right) \tag{83}$$

One computes the scattered radiation from the formula

$$\frac{\mathcal{L}'}{N} = \frac{\mathcal{M}}{N} \mathcal{L} \tag{84}$$

and the question arises, what is the variation of I/N vs., where I/N is the scattered intensity per molecule.

N is the number of molecules in the scattering volume. The answer to such questions involves more knowledge about $\mathcal{M}_0$ and $\mathcal{M}_1$.

4.7 The Matrices $\mathcal{M}_0$ and $\mathcal{M}_1$ for Diagonal J.

Our theory reduces the theoretical calculation of the Mueller matrix to the determination of the Debye distribution function N_{jk} andthe determination of the Jones matrices $J_{\alpha}^{j\beta}$ of the individual instruments (molecules or scattering centers.) It is this part of the analysis which furnishes the real difficulties. As a means of illustration we shall suppose that the J's are given by the Mie theory . This will be discussed in detail in the next section. For the present the important point is that the J matrix is diagonal

$$(85) \qquad J_{\alpha}^{j\beta} = \begin{bmatrix} J_1^{j1} & 0 \\ 0 & J_2^{j2} \end{bmatrix}$$

where J_1^{j1} and J_2^{j2} are complex. Tables of these functions had aready been prepared for Dr. LeMer by the Mathematical Tables Project (MTP) of the National Bureau of Standards under the direction of Dr. Arnold N. Lowan. They were tabulated in terms of

γ .=. scattering angle measured from the backward directinn

m .=. index of refraction of the scattering spheres relative to the surrounding medium.

α .=. $2\pi r/\lambda$

LeMer required only $|J_1^{j1}|$ and $|J_2^{j2}|$ for his work but Mueller, in his work on scattering from smokes, required J_1^{j1} and J_2^{j2} in toto, just as we do. These biproducts of the computation for LeMer, Mueller obtained directly from Lowan. Dr. LeMer's work is reported in OSRD Report No. 1857, Sept. 29, 1945. The reference to Mueller's work is cited in the logical introduction to our theses.

It is an important fact that diagonal J implies diagonal $\mathcal{M}$. Reference to eq.(97'), Ch. II, shows that

$$(86) \qquad \mathcal{M}^{jk} = \begin{bmatrix} J_1^{j1}J_1^{k1*} & & & 0 \\ & J_2^{j2}J_1^{k1*} & & \\ & & J_1^{j1}J_2^{k2*} & \\ 0 & & & J_2^{j2}J_2^{k2*} \end{bmatrix}$$

Mueller extended the MTP tables to give the coefficients a,b,c,d of the auto-matrix

$$(87) \qquad \mathcal{M}^{jj} = \begin{bmatrix} a_j & & & 0 \\ & c_j - i d_j & & \\ & & c_j + i d_j & \\ 0 & & & b_j \end{bmatrix}$$

He called a,b,c,d "scattering factors." We observe from eq.(86) and the theory of scattering from electrolytes that these scattering factors are sufficient if and only if the Jones matrices of all the scatterers are identical or, as in Mueller's work, we are only discussing incoherent scattering.

It is an immediate consequence of eqs. (78),(79) that $\mathcal{M}_0$, $\mathcal{M}_1$, and hence $\mathcal{M}$ are likewise diagonal; i.e.,

$$\mathcal{M} = \mathcal{M}_0 - \frac{1}{1+(\frac{\xi}{k})^2}\mathcal{M}_1 \tag{88}$$

$$= \begin{bmatrix} a_0 - \frac{1}{1+(\frac{\xi}{k})^2} a_1 & & 0 \\ & (c_0 - id_0) - \frac{1}{1+(\frac{\xi}{k})^2}(c_1 - id_1) & \\ 0 & & \text{etc} \end{bmatrix}$$

where

$$a_0 = N \sum x_\ell a_{\ell\ell}, \quad b_0 = N \sum x_\ell b_{\ell\ell}, \text{ etc.} \tag{89}$$

$$a_1 = N \sum x_\ell \xi_\ell x_m \xi_m a_{\ell m}, \text{ etc.} \tag{90}$$

Thus Mueller's calculations supply $a_{\ell\ell}$, $b_{\ell\ell}$, $c_{\ell\ell}$, $d_{\ell\ell}$ and hence a_0, b_0, c_0, d_0, after the indicated averaging. But it is necessary to work directly from the MTP tables to compute $a_1 b_1 c_1 d_1$.

4.8 The Mie Theory of J for Scattering by Spherical Particles.

The MTP tables are based on the following definitions. We have made a few editorial corrections of points which seemed inconsistant.

$$i_1 = \sum_{n=1}^{\infty} \left\{ A_n \pi_n + P_n [x \pi_n - (1-x^2)\pi_n'] \right\} \tag{91}$$

$$i_2 = \sum_{n=1}^{\infty} \left\{ A_n [x\pi_n - (1-x^2)\pi_n'] + P_n \pi_n \right\}$$

$$A_n = \frac{a_n}{n(n+1)}; \quad P_n = \frac{p_n}{n(n+1)}; \quad x = \cos\delta \tag{92}$$

$$a_n = (-1)^{n+\frac{1}{2}}(2n+1)\,\frac{S_n'(\beta)\,S_n(\alpha) - m\,S_n'(\alpha)\,S_n(\beta)}{S_n'(\beta)\,\varphi_n(\alpha) - m\,\varphi_n'(\alpha)\,S_n(\beta)} \tag{93}$$

$$p_n = (-1)^{n+\frac{1}{2}}(2n+1)\,\frac{m\,S_n(\alpha)\,S_n'(\beta) + S_n(\beta)\,S_n'(\alpha)}{m\,\varphi_n(\alpha)\,S_n'(\beta) - S_n(\beta)\,\varphi_n'(\alpha)}$$

$$S_n(x) = \left(\frac{\pi x}{2}\right)^{1/2} J_{n+\frac{1}{2}}(x) \tag{94}$$

$$C_n(x) = (-1)^n \left(\frac{\pi x}{2}\right)^{1/2} J_{-(n+\frac{1}{2})}(x)$$

$$\pi_n = \pi_n(x) = \frac{\partial P_n(x)}{\partial x} \;;\quad \pi_n' = \pi_n'(x) = \frac{\partial^2 P_n(x)}{\partial x^2} \tag{95}$$

$$\varphi_n(x) = S_n(x) + i\,C_n(x) \tag{96}$$

$$\beta = m\alpha \;,\quad \alpha = 2\pi r/\lambda \tag{97}$$

(98) $J_{n+\frac{1}{2}}(x)$, $J_{-(n+\frac{1}{2})}(x)$.=. Bessel Functions half integral order.

$P_n(x)$.=. Legendre Polynomials.

It is necessary to give a brief review of the Mie[1]

---

1) Stratton, J.A.:"Electromagnetic Theory." 1941. pp.563-67.

---

theory in order to properly identify these quantities with the J's as defined by Jones. Let us consider a plane wave falling on a sphere of radius a and propogation constant k_1 imbeded in an infinite homogenous medium k_2. The electric vector is linearly polarized in the x-direction and propogated in the positive z-direction.

The incident field is expanded in spherical waves

$$E_i = \vec{a}_x E_o e^{ik_2 z - i\omega t} = E_o e^{-i\omega t} \sum_{n=1}^{\infty} i^n \frac{2n+1}{n(n+1)} \left(\vec{m}^{(1)}_{o1n} - i\, \vec{n}^{(1)}_{e1n} \right)$$

(99)
$$H_i = \vec{a}_y \frac{k_2}{\mu_2 \omega} E_o e^{ik_2 z - i\omega t}$$
$$= -\frac{k_2 E_o}{\mu_2 \omega} e^{-i\omega t} \sum_{n=1}^{\infty} i^n \frac{2n+1}{n(n+1)} \left(\vec{m}^{(1)}_{e1n} + i\, \vec{n}^{(1)}_{o1n} \right)$$

where E_o is the amplitude and

(100)
$$\vec{m}^{(1)}_{\substack{o\\e}1n} = \left[\pm \frac{1}{\sin\theta} j_n(k_2 R)\, P_n'(\cos\theta) \begin{matrix}\cos\\ \sin\end{matrix} \phi \right] \vec{i}_2 - \left[j_n(k_2 R) \frac{\partial P_n'}{\partial\theta} \begin{matrix}\sin\\ \cos\end{matrix} \phi \right] \vec{i}_3$$

(101)
$$\vec{n}^{(1)}_{\substack{o\\e}1n} = \left[\frac{n(n+1)}{k_2 R} j_n(k_2 R)\, P_n'(\cos\theta) \begin{matrix}\sin\\ \cos\end{matrix} \phi \right] \vec{i}_1 + \left[\frac{1}{k_2 R} \left[k_2 R\, j_n(k_2 R) \right]' \frac{\partial P_n'}{\partial\theta} \begin{matrix}\sin\\ \cos\end{matrix} \phi \right] \vec{i}_2 + \left[\pm \frac{1}{k_2 R \sin\theta} \left[k_2 R\, j_n(k_2 R) \right]' P_n'(\cos\theta) \begin{matrix}\cos\\ \sin\end{matrix} \phi \right] \vec{i}_3$$

Primes denote differentiation with respect to $k_2 R$. The emergent field is given by

(102)
$$E_r = E_o e^{-i\omega t} \sum_{n=1}^{\infty} i^n \frac{2n+1}{n(n+1)} \left[a_n^r \vec{m}^{(3)}_{o1n} - i\, b_n^r n^{(3)}_{e1n} \right]$$
$$H_r = -\frac{k_2}{\omega \mu_2} E_o e^{-i\omega t} \sum_{n=1}^{\infty} i^n \frac{2n+1}{n(n+1)} \left(b_n^r \vec{m}^{(3)}_{e1n} + i\, a_n^r n^{(3)}_{o1n} \right)$$

where " (3) " means the use of $h^{(1)}$ in place of $j^{(1)}$. Stratton also gives the expansion for the internal field, ($R < a$), and then applies the boundary conditions. The net result is that

(103)
$$a_n^r = -\frac{\mu_1 j_n(N\rho) \left[\rho j_n(\rho) \right]' - \mu_2 j_n(\rho) \left[N\rho\, j_n(N\rho) \right]'}{\mu_1 j_n(N\rho) \left[\rho h_n^{(1)}(\rho) \right]' - \mu_2 h_n^{(1)}(\rho) \left[N\rho\, j_n(N\rho) \right]'},$$

(104) $$b_n^r = -\frac{\mu_1 j_n(\rho)\,[N\rho\, j_n(N\rho)]' - \mu_2 N^2 j_n(N\rho)\,[\rho\, j_n(\rho)]'}{\mu_1 h_n^{(1)}(\rho)\,[N\rho\, j_n(N\rho)]' - \mu_2 N^2 j_n(N\rho)\,[\rho\, h_n^{(1)}(\rho)]'}$$

where

(105) $$k_1 = N k_2\ , \quad \rho = k_2 a\ , \quad k_1 a = N\rho$$

and

(106) $$j_n(\rho) = \sqrt{\frac{\pi}{2\rho}}\, J_{n+\frac{1}{2}}(\rho)$$

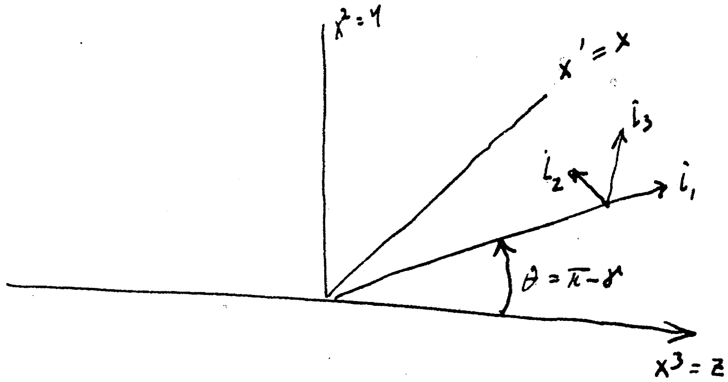
(107) $$h_n^{(1)}(\rho) = \sqrt{\frac{\pi}{2\rho}}\, H^{(1)}_{n+\frac{1}{2}}(\rho)$$

(108) $$H^{(1)}_{n+\frac{1}{2}}(\rho) = J_{n+\frac{1}{2}}(\rho) + i\, N_{n+\frac{1}{2}}(\rho)$$

(109) $$N_{n+\frac{1}{2}}(\rho) = -(-1)^n J_{-(n+\frac{1}{2})}(\rho)$$ (Jahnke and Emde, p.31)

(108') $$H_{n+\frac{1}{2}}(\rho) = J_{n+\frac{1}{2}}(\rho) + i\,(-1)^{n+1} J_{-(n+\frac{1}{2})}(\rho)$$

The unit vectors i_1, i_2, i_3 require a word of explanation. If one is discussing hlp radiation, the diagram is

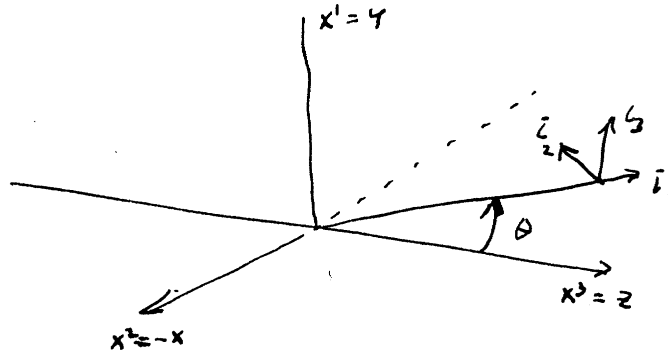


hlp Diagram

where

$$\phi = 0$$
(110)
$$\gamma = \pi - \theta$$
$$\vec{i}_1 = \text{ direction of scattering, } z'$$
$$\vec{i}_2 = \text{ } x'\text{-direction}$$
$$\vec{i}_3 = \text{ } y'\text{-direction}$$

The primes refer to the observers coordinates. In the case of vlp waves one has the diagram



vlp-diagram

where

$$\phi = 3\pi/2$$
$$\gamma = \pi - \theta$$
(111)
$$\vec{i}_1 = \text{ direction of scattering, } z'$$
$$\vec{i}_2 = x' \text{ direction}$$
$$\vec{i}_3 = y' \text{ direction}$$

It is now necessary to write down $\vec{m}_{o1n}$ and $\vec{n}_{e1n}$ for these cases

(112)
$$\vec{m}^{h}_{o1n} = \frac{1}{\sin\theta}\, h_n^{(1)}(k_2R)\, P_n^1(\cos\theta)\,\vec{i}_2 \quad ; \text{ hlp}$$
$$\vec{m}^{v}_{o1n} = h_n^{(1)}(k_2R)\,\frac{\partial P_n^1}{\partial\theta}\,\vec{i}_3 \quad ; \text{ vlp}$$

(113)
$$\vec{n}^{h}_{e1n} = \frac{1}{k_2R}\left[k_2R\, h_n^{(1)}(k_2R)\right]'\frac{\partial P_n^1}{\partial\theta}\,\vec{i}_2 \quad ; \text{ hlp}$$
$$\vec{n}^{v}_{e1n} = \frac{1}{k_2R\sin\theta}\left[k_2R\, h_n^{(1)}(k_2R)\right]' P_n^1(\cos\theta)\,\vec{i}_3 \;; \text{ vlp.}$$

The propogation constant k_2 satisfies

(114) $$k_2^2 = \mu_2 \epsilon_2 \omega^2 + i \mu_2 \sigma_2 \omega$$

and, in our case, $\sigma = 0$, which implies

(115) $$k_2 = \frac{\omega}{v_2}$$

By comparing coefficients in eq.(102), it is clear that

(116) $$\boxed{\begin{aligned} \vec{l}_2 \, J_1^1 &= \sum_{n=1}^{\infty} i^n \frac{2n+1}{n(n+1)} \left(a_n^r \vec{m}^h_{o1n} - i b_n^r \vec{n}^h_{e1n}\right) \\ \vec{l}_3 \, J_2^2 &= \sum_{n=1}^{\infty} i^n \frac{2n+1}{n(n+1)} \left(a_n^r \vec{m}^v_{o1n} - i b_n^r \vec{n}^v_{e1n}\right) \end{aligned}}$$

We are primarily interested in what the factors the $R(i_1)$, $I(i_1)$, $R(i_2)$, and $I(i_2)$, given in the MTP tables, must be multiplied by to give the real and imaginary parts of the J's. The Stratton coefficient a_n^r is

(117) $$a_n^r = - \frac{\mu_1 j_n(N\rho)\left[\rho j_n(\rho)\right]' - \mu_2 j_n(\rho)\left[N\rho j_n(N\rho)\right]'}{\mu_1 j_n(N\rho)\left[\rho h_n^{(1)}(\rho)\right]' - \mu_2 \cancel{j} \, h_n^{(1)}(\rho)\left[N\rho j_n(N\rho)\right]'}$$

Comparing with the Lowan coefficient we conclude that

(118) $$\begin{aligned} \rho j_n(\rho) &= \sqrt{\frac{\pi\rho}{2}} \, J_{n+\frac{1}{2}}(\rho) = S_n(\rho) \\ \rho n_n(\rho) &= (-1)^{n+1} \sqrt{\frac{\pi\rho}{2}} \, J_{-(n+\frac{1}{2})}(\rho) = -C_n(\rho) \\ \rho h_n^{(1)}(\rho) &= S_n(\rho) - i C_n(\rho) = \tilde{\varphi}_n(\rho) \end{aligned}$$

hence if

(119) $$\mu_1 = \mu_2 = 1 \qquad \frac{1}{N} = k_2/k_1 = v_1/v_2 = m$$
$$\rho = \alpha, \quad N\rho = \beta \qquad \text{"}\sim\text{" complex conjugate}$$
$$a_n^r = - \frac{S_n(\alpha) S_n'(\beta) - m S_n(\beta) S_n'(\alpha)}{\tilde{\varphi}_n(\alpha) S_n'(\beta) - m S_n(\beta) \tilde{\varphi}_n'(\alpha)}$$

Thus

(120) (Lowan) $a_n = (-1)^{n+\frac{1}{2}}(2n+1)(-\tilde{a}_n^r)$ (Stratton)

and in similar fashion

(121) $$b_n^r = -\frac{m\, S_n(\alpha)\, S_n'(\beta) - S_n(\beta)\, S_n'(\alpha)}{m\, \tilde{\varphi}_n(\alpha)\, S_n'(\beta) - S_n(\beta)\, \tilde{\varphi}_n'(\alpha)}$$

and

(122) (Lowan) $b_n = (-1)^{n+\frac{3}{2}}(2n+1)(-\tilde{b}_n^r)$ (Stratton)

Letting x = cos θ

$$P_n'(\cos\theta) \sim -P_n'(x)\sin\theta = -\Pi_n \sin\theta$$

(123) $$\frac{\partial P_n'}{\partial\theta} \sim \Pi_n'(1-x^2) - \Pi_n(x)$$

Letting $\delta = k_2 R$ where R = radial distance to observer

(124) $$m_{o1n}^h = -\Pi_n \frac{\tilde{\varphi}_n(\delta)}{\delta} \simeq -\Pi_n(-i)^{n+1}\frac{e^{i\delta}}{\delta}$$

(125) $$n_{e1n}^h = \frac{\tilde{\varphi}_n'(\delta)}{\delta}\Pi_n'(1-x^2) \simeq [\Pi_n'(1-x^2) - \Pi_n(x)](-i)^n\frac{e^{i\delta}}{\delta}$$

(126) $$m_{o1n}^v = \frac{\tilde{\varphi}_n(\delta)}{\delta}\Pi_n'(1-x^2) \simeq [\Pi_n'(1-x^2) - \Pi_n(x)](-i)^n\frac{e^{i\delta}}{\delta}$$

(127) $$n_{e1n}^v = -\frac{\tilde{\varphi}_n'(\delta)}{\delta}\Pi_n \simeq -\Pi_n(-i)^n\frac{e^{i\delta}}{\delta}$$

From eq. (120), (121)

(128) $$i^n a_n^r = \frac{n(n+1)}{2(n+1)} (-i)^{n+1} \tilde{A}_n$$

(129) $$-i^{n+1} b_n^r = \frac{n(n+1)}{2(n+1)} \frac{(-i)^{n+1} \tilde{P}_n}{i}$$

Substituting into eq.(116) yields

(130) $$J_1^1 \simeq \frac{e^{i\delta}}{\delta} \sum \left(\tilde{A}_n \pi_n + \tilde{P}_n (\pi_n x - \pi_n'(1-x^2)] \right)$$
$$J_2^2 \simeq \frac{e^{i\delta}}{\delta} \sum \left(\tilde{A} [\pi_n x - \pi_n'(1-x^2)] + \tilde{P}_n \pi_n \right)$$

From this it follows that

(131) $$\boxed{\begin{aligned} J_1^1 &= (e^{i\delta}/\delta)(R(i_1) - iI(i_1)) = i_1 e^{i\delta}/\delta \\ J_2^2 &= (e^{i\delta}/\delta)(R(i_2) - iI(i_2)) = i_2\, e^{i\delta}/\delta \end{aligned}}$$

for outgoing waves, i.e., the time factor is $e^{-i\omega t}$.

Muellers scattering factors are computed

(132) $$a = R(i_1)^2 + I(i_1)^2 = J_1^1 J_1^{1*}$$
$$b = R(i_2)^2 + I(i_2)^2 = J_2^2 J_2^{2*}$$
$$c = R(i_1)R(i_2) + I(i_1)I(i_2) = \mathrm{Re}\, J_1^1 J_2^{2*}$$
$$d = R(i_1)I(i_2) - R(i_2)I(i_1) = \mathrm{Im}\, J_1^1 J_2^{2*}$$

Hence

(133) $$J_1^1 J_2^{2*} = c+id$$
$$J_2^2 J_1^{1*} = c-id$$

and we may now use the MTP tables and Mueller's extension of them with confidence that we know their origin and can extend them to larger particles if necessary.

4.9 Scattering by Particles of Uniform Valance.

It will bring out an important point to consider the case of scattering by particles having a Poisson-Boltzmann distribution in which only one of the "ionic" components scatter appreciably.

From eq.(78),(79), and (87)

(134) $$\frac{\mathcal{M}}{N} = \mathcal{M}_{11}\frac{s^2}{s^2+K^2} = \begin{bmatrix} a_{11} & & & \\ & c_{11}-id_{11} & 0 & \\ & 0 & c_{11}+id_{11} & \\ & & & b_{11} \end{bmatrix}\frac{s^2}{s^2+K^2}$$

where a_{11}, b_{11}, c_{11}, d_{11} are the Mueller scattering factors

and

(83) $$K^2 = \frac{N}{T}\left(\frac{e^2z^2}{\epsilon k}\right)$$

z^2 = mean square valance, considering all "ions"

(15) $$S = 2\frac{\omega}{v}\cos\frac{\gamma}{2}$$

Computing the scattered intensity I' gives (dropping subscripts)

(135) $$2I'/IN = \left[(a+b) + (a-b)\,M/I\right]\frac{s^2}{s^2+K^2}$$

where I,M,C,S, is the Stokes vector of the incoming radiation.

For $K^2 << s^2$

(136) $$2I'/IN = \left[(a+b) + (a-b)\,M/I\right]\left(1-\left(\frac{K}{s}\right)^2\right)$$

For $s^2 = 0$, $N_{jk} = 1$

(136') $$2I'/IN = N\left[(a+b) + (a-b)\,M/I\right]$$

We consider first the case $s^2 \neq 0$. Then the initial slope of the intensity curve is $d(I'/IN)/dN$ or

$$(137) \qquad d(I'/IN)/dN = -\frac{I'}{IN}\frac{1}{1-(\frac{K}{S})^2}\frac{1}{S^2}\frac{dK^2}{dN} = -\frac{I'}{IN}\frac{1}{1-(\frac{K}{S})^2}\left(\frac{K}{S}\right)^2\frac{1}{N}$$

evaluated for N = 0, gives

$$(138) \qquad \left(\frac{d(I'/IN)}{dN}\right)_{N=0} = -\frac{[(a+b)+(a-b)M/I]}{S^2}\left[\frac{e^2z^2}{\epsilon kT}\right]$$

where $[(a+b) + (a-b)M/I]$ is the value of I'/IN at $K = 0$ obtained by extrapolation. Thus if one picks a scattering angle, not forward, one can obtain an estimate of the mean square valance of the scattering centers and other ions present. Note that the slope of the I'/IN vs. N curve is NEGATIVE. Eq.(78) seems to admit the possibility of positive slopes for a sufficiently complicated electrolyte. We have not tried to construct such a case.

4.10. Scattering Due to a Uniform Distribution.

For this case $N_{jk} = \delta_{jk}$ and

$$(139) \qquad \mathcal{M} = \delta_{jk}\, m^{jk}$$

i.e., the total intensity ratio

$$(140) \qquad I'/IN = (1/2)[(\bar{a} + \bar{b}) + (\bar{a} - \bar{b})M/I]$$

where $\bar{a}, \bar{b}$ are mean values computed with the aid of eq.(79).

4.11 Scattering from a Lattice.

For this case $N_{jk} = e^{i\varphi_{jk}}$ where

$$\varphi_{jk} = \frac{(s-s')\cdot(r_j - r_k)\,\omega}{v} \tag{141}$$

and

$$\mathcal{M} = \mathcal{M}_{11}\,\Sigma e^{i\varphi_{jk}} \tag{142}$$

or one gets scattering only at those points for which the Bragg law is satisfied. In the case of lattices with a uniform distribution of imperfections, one gets

$$\mathcal{M} = \mathcal{M}_{11}\,(N + \Sigma e^{i\varphi_{jk}}) \tag{143}$$

Eq.(143) could be more rigorously derived. But its correctness seems apparent if one thinks of the array as an incoherent superposition of N rigidly arranged scatterers and N randomly distributed scatterers. In the Bragg directions the $\Sigma e^{i\varphi_{jk}}$ outweighs the random scattering. At all other points, only the random scattering exists.

4.12 Epilogue.

We now come to the end of our thesis, the main topic of which was the study of optical algebras. The logical structure of the theory was complete at the end of Chapter I. The remaining three chapters have been devoted to more special considerations illustrating the theory and its applications. These illustrations have necessarily been brief, idealized, and somewhat artificial. Otherwise we should have been involved in the careful exposition and verification of particulars which, while of great interest in themselves, add nothing to our understanding of the technique so far evolved.

The last two chapters have been particularly important in showing the way one goes about making estimates of the statistical values of the time averages involved in the Wiener correlation process. In the interests of simplicity we have confined our attention to the cases for which it appeared legitimate to average separately over the positions of the scattering particles. Actually there is a very promising field for further research in a study of the stochastic problems involved in obtaining a more detailed description of the Jones t-functions and the relative positions of the scatterers. The quasi-stationary assumption Sec. 4, side stepped that problem.

Another avenue of inquiry which merits further research is the quantum mechanical determination and subsequent application of the Jones matrices to molecular scattering. The really difficult and detailed labor involved in the actual application of our theory is to be found at precisely this point.

We may summarize by saying that we have reached a point from which it is clearer than before how one can make use of our detailed theoretical knowledge of the microscopic properties of matter and radiation to predict the macroscopic observables L,L' and M, definied phenomenologically by Mueller.

Biographical Sketch

Nathan Grier Parke III. Spencer Brook Road, Concord, Mass. INDUSTRIAL PHYSICIST; born, Woodstock, Vt., Jan. 27, 1912; son of Nathan Grier, and Olive B.W. Parke. Education: Wyoming Seminary, Kingston, Pa., 1927-1930. Princeton University, A.B.,cum laude in Mathematical Physics, 1934; Graduate Study: Princeton University, Mathematics, 1934-35, Astronomy 1935-36. University of Maryland, Mathematics 1940-41. Johns Hopkins, Mathematics, 1941-42. Massachusetts Institute of Technology, Physics 1945-48. Married: Nov. 12, 1938, Sally Lara, Phila.,Pa. Children: Nathan Grier IV, Jonathan Lara. Teaching Experience: Master in Mathematics, John Hun School, 1934-35. Assistant in Astronomy, Princeton 1935-36. Teaching Fellow, University of Maryland, 1940-41, Night Course Instruction, Mathematics, Johns Hopkins, 1941-42. Research Associate in Physics, Research Laboratory of Electronics, M.I.T., 1945-48. Business Experience: Research Engineer, Glenn L. Martin Co., Baltimore, Md. 1936-40. Principal Mathematician and Head of Applied Mathematics Section, Aviation Design Research Branch, Bureau of Aeronautics Navy Department, 1942-45. Author: "Guide to the Literature of Mathematics and Physics," McGraw-Hill, 1947. Affiliations: Assoc.F.,I.Ae.S., Mem.A.M.S.,A.P.S.,O.S.A.,A.A.A.S.,Inst. of Math.Stat.,Sigma Xi. Clubs: Princeton Club of N.Y., Soc. Mayflower Descendents. Biographical Sketches: "Who's Who in Aviation","Who's Who in Engineering".