Cohomologically proper stacks over $\mathbb{Z}_p$: algebra, geometry and representation theory

by

Dmitry Kubrak

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Author .....
Department of Mathematics
April 20, 2020

Certified by .....
Roman Bezrukavnikov
Professor of Mathematics
Thesis Supervisor

Accepted by .....
Roman Bezrukavnikov
Chairman, Department Committee on Graduate Theses

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Abstract

In this thesis, we study a class of so-called cohomologically proper stacks from various perspectives, mainly concentrating on the p-adic context. Cohomological properness is a relaxed properness condition on a stack which roughly asks the cohomology of any coherent sheaf to be finitely generated over the base. We extend some of the techniques available for smooth proper schemes to smooth cohomologically proper stacks, featuring in particular recently developed theory of prismatic cohomology and the classical Deligne-Illusie method for the Hodge-to-de Rham degeneration. As main applications we prove the Totaro's conjectural inequality between the dimensions of the de Rham and the singular $\mathbb{F}_p$-cohomology of the classifying stack of a reductive group, compute the ring of prismatic characteristic classes at non-torsion primes and give some new examples of the Hodge-to-de Rham degeneration for stacks in characteristic 0. We also study some descent properties of certain Brauer group classes on conical resolutions, a question having some applications to the theory of Fedosov quantizations in characteristic p. Some surprising results about the $\mathbb{G}_m$-weights of differential 1-forms that are obtained along the way, originally motivated the attempt to generalize the integral p-adic Hodge theory to the setting of cohomologically proper stacks.

Thesis Supervisor: Roman Bezrukavnikov
Title: Professor of Mathematics

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Chapter 1

Introduction

This thesis studies a class of so-called "cohomologically proper" stacks, mainly in the p-adic context. Its general goal is to generalize various techniques and results available for smooth proper schemes to smooth cohomologically proper Artin stacks. My thesis consists of three relatively independent parts with their brief descriptions below:

- (Chapter 2, [77]) The descent of certain Brauer group classes on conical resolutions that come from differential operators in characteristic p (joint with R.Travkin) — a step in a program of Bezrukavnikov-Okounkov concerning the relation between quantum cohomology of symplectic resolutions and their non-commutative quantizations.

- (Chapter 3, [75]) Integral p-adic Hodge theory for stacks (joint with A.Prikhodko) — a (partial) extension of integral p-adic Hodge theory ([17],[19]) to Hodge-proper stacks over $\mathbb{Z}_p$. We get a proof of Totaro's conjectural inequality

$$\dim_{\mathbb{F}_p} H^n_{\mathrm{dR}}(BG_{\mathbb{F}_p}) \geq \dim_{\mathbb{F}_p} H^n_{\mathrm{sing}}(BG(\mathbb{C}), \mathbb{F}_p)$$

  for classifying stacks of reductive groups as an application.

- (Chapter 4, [76]) Hodge-to-de Rham degeneration for smooth n-Artin stacks (joint with A.Prikhodko) — a generalization of the method of Deligne-Illusie and a proof of the degeneration of the Hodge-to-de Rham spectral sequence for a certain class of stacks in characteristic 0 which admit a Hodge-proper spreading over a finitely generated $\mathbb{Z}$-algebra. This seems to give the most general class of smooth Artin stacks for which the degeneration is known to

hold (giving new examples even in the case of schemes).

The order of the chapters agrees with the chronological order of the corresponding collaborate projects (first one with R.Travkin [77] and the other two with A.Prikhodko ([75],[76]). This also corresponds to the logical order of how the projects have been developed, [77] raised a question about the dimension of the space of $\mathbb{G}_m$-invariant differential 1-forms on conical resolutions which then lead to the attempt to generalize the results of Bhatt, Morrow and Scholze ([17]) to the setting of stacks which are not exactly proper, but share some natural properties that proper schemes have. The right notion appeared to be the cohomological properness, however some new issues (namely the comparison of the étale cohomology of rigid analytic and algebraic generic fibers) specific exclusively to the context of stacks emerged. Meanwhile the work [111] of Totaro was suggesting that one could use the p-adic Hodge theory to deduce the inequality $\dim_{\mathbb{F}_p} H^n_{\mathrm{dR}}(BG_{\mathbb{F}_p}) \geq \dim_{\mathbb{F}_p} H^n_{\mathrm{sing}}(BG(\mathbb{C}), \mathbb{F}_p)$ for reductive G which he observed by hand in some examples. This motivated me and Artem to find a way around the mentioned issue specifically in the case of BG (in fact more generally for $[X/G]$ with X proper). This lead to the proof of Totaro's inequality for a general reductive group G. The generalization of the Deligne-Illusie method for the Hodge-to-de Rham degeneration was first going to be a part of the p-adic Hodge theory project, however, together with the discussion of various applications, (see Section 4.4) it turned out logically independent enough to become a separate project after all.

Below we give a brief overview of each of the parts, trying to make clear interaction between the material in them. A more detailed introduction to each of the parts then can be found in the beginning of the corresponding chapters.

1.0.1 Quantizations in characteristic p and descent for certain Brauer group classes on conical resolutions

A quantization of an affine Poisson algebraic variety is a non-commutative deformation of its ring of functions such that the "classical limit" of the commutator gives back the Poisson structure. It turns out that many important algebras (e.g. universal envelopping algebras, W-algebras, Cherednik algebras, and many others) are in fact quantizations of some well-known Poisson varieties (like Lie algebras with the Kostant-Kirillov Poisson bracket, Slodowy slices, Nakajima quiver varieties, and others). Realization of such algebras as quantizations often allows to reveal certain structures either of the algebras themselves or their categories of representations, that are not visible otherwise.

More surprisingly, one can also learn some subtle things about the geometry of the Poisson variety itself using its quantizations.

One instance of this powerful interaction between representation theory and algebraic geometry is given by the program of Bezrukavnikov, Okounkov, and their collaborators (as outlined in [3]). It concerns conical symplectic resolutions:

Definition 1.0.1.1. A resolution of singularities $\pi\colon X \to Y$ is called *conical* if X and Y are endowed with $\mathbb{G}_m$-actions that are intertwined by π and such that Y is contracted to a single point $y_0 \in Y$ when $t \in \mathbb{G}_m$ goes to 0. It is called *symplectic* if X has a nondegenerate symplectic form $\omega \in \Gamma(X, \Omega^2_X)$ that is rescaled by the action ($t^*\omega = t^n\omega$ for some $n \in \mathbb{Z}_{\geq 0}$ and all $t \in \mathbb{G}_m$).

The program relates certain braid group action on the $\mathbb{G}_m$-equivariant derived category $\mathrm{D}^b\mathrm{Coh}(X)^{\mathbb{G}_m}$ of X with the monodromy of the quantum connection on the cohomology $H^*(X(\mathbb{C}), \mathbb{C})$. The (in general still conjectural) construction of this action is rather nontrivial: one takes a reduction of the resolution to a big enough characteristic p and uses some natural equivalences between the categories of the representations of its Fedosov quantizations. This way, the realization of the program gives an expression for certain curve counts on X (basically the number of curves passing through given set of cycles inside X) in terms of the representation theory of the Fedosov quantizations of its mod p reductions.

Due to the work of Bezrukavnikov and Kaledin [12], Fedosov quantizations in characteristic p are described as follows: there is a canonical quantization $\mathcal{A}_0$ and all others are obtained as its twists by line bundles $\mathcal{O}(\lambda) \in \mathrm{Pic}(X)$, their isomorphism classes being in bijection with $\mathrm{Pic}(X)/p \cdot \mathrm{Pic}(X)$. They also come with a new feature: all Fedosov quantizations are Azumaya algebras on the Frobenius twist $X^{(1)}$. Moreover the class $[\mathcal{A}_\lambda] \in \mathrm{Br}(X^{(1)})$ in the Brauer group does not depend on $\lambda \in \mathrm{Pic}(X)$. The way to get an action of the braid group by derived autoequivalences on $\mathrm{D}^b\mathrm{Coh}(X)^{\mathbb{G}_m}$ from certain equivalences between the categories of $\mathcal{A}_\lambda$-modules is to prove that the class $[\mathcal{A}_\lambda] \in \mathrm{Br}(X^{(1)})$ is 0 on the formal neighbourhood of the central fiber $\pi^{-1}(y_0)$. In the case of the Springer resolution $T^*G/B \to \mathcal{N}$ this was shown by constructing an explicit splitting bundle in [14] by Bezrukavnikov, Mirkovic, and Rumynin. As is explained below, in general one can use the fact that the class $[\mathcal{A}_0]$ comes from a differential form on $X^{(1)}$. A similar strategy was used in the case of quiver varieties in [13].

Namely, let X be a smooth variety over an algebraically closed field $\Bbbk$ of characteristic p. To each differential 1-form α on the Frobenius twist $X^{(1)}$ one can associate an Azumaya algebra $\mathcal{D}_{X,\alpha}$

on $X^{(1)}$ in the following way. The sheaf of crystalline differential operators $\mathcal{D}_X$ on X defines an Azumaya algebra on the cotangent bundle $T^*X^{(1)}$. Given a differential 1-form α on $X^{(1)}$, restricting $\mathcal{D}_X$ to the graph $\Gamma_\alpha \subset T^*X^{(1)}$ defines an Azumaya algebra $\mathcal{D}_{X,\alpha}$ on $X^{(1)}$. In all known examples of conical symplectic resolutions the class $[\mathcal{A}_0] \in \mathrm{Br}(X^{(1)})$ is of the form $[\mathcal{D}_{X,\alpha}]$ for an explicit α[1].

In the joint work [77] (or Chapter 2) with Roman Travkin we proved the following general result:

Theorem 1.0.1.2 (Theorem 2.0.2.1). *Let $\pi\colon X \to Y$ be a proper map of algebraic varieties over an algebraically closed field of characteristic p. Assume that X is smooth and $R^1\pi_*\mathcal{O}_X = R^2\pi_*\mathcal{O}_X = 0$. Then for any $\alpha \in H^0(X^{(1)}, \Omega^1_{X^{(1)}})$ there exists an étale cover $U \to Y$ such that the pull-back of $[\mathcal{D}_{X,\alpha}] \in \mathrm{Br}(X^{(1)})$ to $(U \times_Y X)^{(1)}$ is trivial.*

Since symplectic resolutions are rational, Theorem 1.0.1.2 applies in particular to this case, thus giving a splitting of the Fedosov quantization $\mathcal{A}_0$ on an étale neighbourhood of any given fiber. But as we see such a splitting in fact holds in much greater generality.

We note that the Azumaya algebra $\mathcal{A}_0$ is almost never split globally. Nevertheless, one can try to globalize the statement of Theorem 1.0.1.2 in the following way: is it true that the classes $[\mathcal{D}_{X,\alpha}] \in \mathrm{Br}(X^{(1)})$ come as pull-backs of certain classes in $\mathrm{Br}(Y^{(1)})$? This turns out to be a much more difficult question which we adress in Chapter 2 as well. We define a notion of a *resolution with conical slices*: roughly speaking this is a conical resolution $\pi : X \to Y$ together with an étale conical slice at each point $y \in Y$. Then we prove the following:

Theorem 1.0.1.3 (Theorem 2.0.2.3). *Let $\pi\colon X \to Y$ be a resolution with conical slices in characteristic 0, such that $\pi_*\mathcal{O}_X = \mathcal{O}_Y$ and $R^1\pi_*\mathcal{O}_X = R^2\pi_*\mathcal{O}_X = 0$. Then for any general reduction $\pi_{\Bbbk}\colon X_{\Bbbk} \to Y_{\Bbbk}$ to a field $\Bbbk$ of characteristic $p \gg 0$*

- *$\pi_{\Bbbk} : X_{\Bbbk} \to Y_{\Bbbk}$ is a resolution with conical slices;*
- *the classes $[\mathcal{D}_{X_{\Bbbk},\alpha}]$ descend to $Y^{(1)}_{\Bbbk}$ for all α.*

If one just takes a resolution with conical slices in characteristic p then it is not necessarily true that all the classes $[\mathcal{D}_{X_{\Bbbk},\alpha}]$ descend to $Y^{(1)}_{\Bbbk}$ (see Section 2.5.2): it turns out that this has a surprising explanation via integral p-adic Hodge theory (see Chapter 3). Thus it is very important to choose p big enough in the statement of Theorem 1.0.1.3.

[1] Given by the contraction of the symplectic form ω by the Euler field corresponding to the $\mathbb{G}_m$-action.

In Chapter 2 (more precisely, Section 2.7) we also show that more or less all known examples of conical symplectic resolutions (in particular Nakajima quiver varieties, hypertoric varieties and Slodowy slices) in characteristic 0 are resolutions with conical slices. It is also worth pointing out that the global descent of the quantization class $[\mathcal{A}_0]$ in these cases allows to formulate a global analogue of conjectures in [3].

1.0.2 p-adic Hodge theory for Hodge-proper stacks

While establishing the main results of [77] Roman Travkin and me proved a curious result about differential 1-forms on conical resolutions. It concerns the grading on $H^0(X, \Omega^1_X)$ induced by the $\mathbb{G}_m$-action on X.

Theorem 1.0.2.1 (Lemma 2.5.1.2 and Proposition 2.5.1.4)**.** *Let $\pi\colon X \to Y$ be a conical resolution over k, such that $\pi_*\mathcal{O}_X = \mathcal{O}_Y$ and $R^1\pi_*\mathcal{O}_X = R^2\pi_*\mathcal{O}_X = 0$. Then*

- *If $\operatorname{char} k = 0$, the $\mathbb{G}_m$-weights of $H^0(X, \Omega^1_X)$ are strictly positive;*
- *If $\operatorname{char} k > 0$, the $\mathbb{G}_m$-weights of $H^0(X, \Omega^1_X)$ are nonnegative.*

It took us some time to realize that the weaker result for the case of positive characteristic is not an artefact of the proof and that there are indeed some conical resolutions for which the weight zero subspace (aka $\mathbb{G}_m$-invariants) $H^0(X, \Omega^1_X)^{\mathbb{G}_m} \subset H^0(X, \Omega^1_X)$ is non-empty.

From the results of [77] it also follows that, under the conditions above, $H^0(X, \Omega^1_X)^{\mathbb{G}_m}$ can be identified with the first de Rham cohomology $H^1_{dR}([X/\mathbb{G}_m])$ of the quotient stack $[X/\mathbb{G}_m]$. In the case of proper schemes, Bhatt, Morrow, and Scholze [17] explained certain dimension jumps of the de Rham cohomology of a mod p reduction by the presence of extra p-torsion in the singular cohomology:

Theorem 1.0.2.2 ([16], Theorem 1.1)**.** *Let X be a proper smooth scheme over $\mathbb{Z}_p$. Fix an embedding $\mathbb{Z}_p \hookrightarrow \mathbb{C}$. Then*

$$\dim_{\mathbb{F}_p} H^i_{\mathrm{dR}}(X_{\mathbb{F}_p}) \geq \dim_{\mathbb{F}_p} H^i_{\mathrm{sing}}(X(\mathbb{C}), \mathbb{F}_p)$$

We note that this is just one of the simplest applications of integral p-adic Hodge theory ([17],[18],[19]). Theorem 1.0.2.2 raises a natural question: does a similar inequality hold for the quotient stack $[X/\mathbb{G}_m]$ in the case of a conical resolution $\pi : X \to Y$ over $\mathbb{Z}_p$?

Independently, in [111] Totaro computed the de Rham cohomology of the classifying stack BG for some split reductive groups G over $\mathbb{F}_p$. In particular he showed the following:

Theorem 1.0.2.3 ([111], Theorem 0.2)**.** *Let G be a split reductive group and let p be a non-torsion prime for G. Then there exists a (non-canonical) isomorphism of algebras*

$$H^*_{\mathrm{dR}}(BG_{\mathbb{F}_p}) \simeq H^*_{\mathrm{sing}}(BG(\mathbb{C}), \mathbb{F}_p)$$

Recall that p is called torsion for G if the p-torsion in $H^*(BG(\mathbb{C}), \mathbb{Z})$ is nontrivial. The number of torsion primes for a given split reductive group G is finite and the structure of de Rham cohomology of $BG_{\mathbb{F}_p}$ at torsion primes remains a mystery in general. Some things are known, but only in the case $p = 2$: Totaro proved an analogous isomorphism for $G = SO(n)$ or $O(2n)$. An isomorphism for $G = G_2$ was also shown recently by Primozic in [101]. In the case $G = O(2n + 1)$ by [111] the dimensions of individual cohomology groups on both sides are the same, but the algebra structure is different. Finally for $G = \mathrm{Spin}(11)$ Totaro showed that

$$H^{32}_{\mathrm{dR}}(B\mathrm{Spin}(11)_{\mathbb{F}_2}) > H^{32}_{\mathrm{sing}}(BSpin(11)(\mathbb{C}), \mathbb{F}_2),$$

thus the equality does not hold in general. This motivated a similar question in [111]: does the inequality of Theorem 1.0.2.2 hold for $X = BG$?

Theorem 1.0.2.2 extends formally to all proper stacks, however both $[X/\mathbb{G}_m]$ and BG are (almost) never proper. Nevertheless, they are "cohomologically proper": cohomology of certain complexes of sheaves are finite-dimensional. In our joint work [75] with Artem Prikhodko (or Chapter 3) we used this observation and tried to apply the theory of prismatic cohomology [19] in this context.

Definition 1.0.2.4. We call a smooth n-Artin stack $\mathcal{Y}$ over $\mathbb{Z}_p$ *Hodge-proper* if the Hodge cohomology $H^i(\mathcal{Y}, \wedge^j \mathbb{L}_{\mathcal{Y}/\mathbb{Z}_p})$ are finitely generated over $\mathbb{Z}_p$ for all i and j.

Here $\mathbb{L}_{\mathcal{Y}/\mathbb{Z}_p}$ denotes the cotangent complex of $\mathcal{Y}$. Let $\mathfrak{S}/p \coloneqq \mathbb{F}_p[[T]]$ be the mod p Kisin ring and let $C^\flat$ be the T-adic completion of the algebraic closure $\mathbb{F}_p((T))^{\mathrm{alg}}$.

Proposition 1.0.2.5 (A corrollary of Propositions 3.5.5.1 and 3.5.5.3)**.** *Let $\mathcal{Y}/\mathbb{Z}_p$ be a Hodge-proper smooth n-Artin stack. There is a "mod p prismatic cohomology" complex $R\Gamma_{\mathbb{\Delta}/p}(\mathcal{Y}) \coloneqq R\Gamma_{\mathbb{\Delta}}(\mathcal{Y}/\mathfrak{S})/p$ of $\mathfrak{S}/p$-modules, such that*

- *each cohomology group* $H^i_{\mathbb{\Delta}/p}(\mathcal{Y})$ *is finitely generated over* $\mathfrak{S}/p$*;*
- *the generic fiber* $R\Gamma_{\mathbb{\Delta}/p}(\mathcal{Y})\otimes_{\mathfrak{S}/p}C^\flat$ *is isomorphic to the adic étale cohomology* $R\Gamma_{\text{ét}}(\widehat{\mathcal{Y}}_{\mathbb{C}_p},\mathbb{F}_p)\otimes_{\mathbb{F}_p} C^\flat$ *of the Raynaud generic fiber;*
- *the reduction* $R\Gamma_{\mathbb{\Delta}/p}(\mathcal{Y}_{\mathbb{F}_p})\otimes^{\mathbb{L}}_{\mathbb{\Delta}/p}\mathbb{F}_p$ *is isomorphic to the de Rham cohomology* $R\Gamma_{dR}(\mathcal{Y}_{\mathbb{F}_p})$ *of the reduction* $\mathcal{Y}_{\mathbb{F}_p}$*.*

Corollary 1.0.2.6 (Theorem 3.5.6.1)**.** *This gives the inequality*

$$\dim_{\mathbb{F}_p} H^i_{dR}(\mathcal{Y}_{\mathbb{F}_p}) \geq \dim_{\mathbb{F}_p} H^i_{\text{ét}}(\widehat{\mathcal{Y}}_{\mathbb{C}_p},\mathbb{F}_p).$$

for all $i>0$

In fact the results of Chapter 3 are much stronger, one has a similar statement with $\mathbb{Z}_p$-coefficients, comparison with the cristalline cohomology of $\mathcal{Y}_{\mathbb{F}_p}$ in the sense of [92], Breuil-Kisin module structure on $H^i_{\mathbb{\Delta}}(\mathcal{Y})$, and some other results. This level of generality is kept to make the connection with the above questions more clear.

However, there is a caveat! Unlike in the case of a smooth proper scheme there is no clear way to compare the étale cohomology of the Raynaud generic fiber with the singular cohomology of the topological space of its complex points. The question of which geometric (or cohomological) assumptions on the stack should guarantee such a comparison looks quite challenging, in particular we do not expect that the Hodge-properness assumption suffices in general. In Chapter 3 we prove the following

Theorem 1.0.2.7 (Theorem 3.6.5.2)**.** *Let* G *be a reductive group over* $\mathbb{Z}_p$ *and let* X *be a smooth proper scheme with an action of* G*. Then*

$$H^i_{\text{ét}}(\widehat{[X/G]}_{\mathbb{C}_p},\mathbb{F}_p)\simeq H^i_{\text{sing}}(X(\mathbb{C})_{hG(\mathbb{C})},\mathbb{F}_p)$$

the latter denoting the $G(\mathbb{C})$*-equivariant cohomology of* $X(\mathbb{C})$*.*

The proof is a multi-step reduction to the case of $B\mathbb{G}_m$, which we then approximate by the projective spaces $\mathbb{P}^n$. Some of the reduction steps took quite a lot of work, for example we developed the theory of algebraic and adic local systems on stacks, adapting certain classical base change results for étale cohomology of schemes [6] and adic spaces [57] to this context.

For $X = \operatorname{Spec} \mathbb{Z}_p$ this gives a positive answer to the question of Totaro. Namely

Corollary 1.0.2.8 (Corollary 3.6.5.3)**.** *Let G be a reductive group over $\mathbb{Z}_p$. Then*

$$\dim_{\mathbb{F}_p} H^i_{\mathrm{dR}}(BG_{\mathbb{F}_p}) \geq \dim_{\mathbb{F}_p} H^i_{\mathrm{sing}}(BG(\mathbb{C}), \mathbb{F}_p).$$

Unfortunately, Theorem 1.0.2.7 does not apply to the case of a conical resolution, since in this case X is usually not proper. Here, for $i = 1$ solely (which was the original question in [77]) we are going to try a different method, see updates in the near future!

Besides that, in Section 3.7.2, using the results of [111], we compute the prismatic cohomology rings of the classifying stacks of reductive groups at non-torsion primes. Similarly one can also make a mod 2 prismatic cohomology computation in the case of $SO(n)$ and $O(n)$ (not included in the thesis). The latter is used in a crucial way in my ongoing project with Anlong Chua to compute the de Rham cohomology of $B\mathrm{Spin}(n)$ over $\mathbb{F}_2$ for all n via prismatic cohomology, see e.g. [2].

1.0.3 Deligne-Illusie method and the Hodge-to-de Rham degeneration for stacks

Let X be a smooth scheme over $\mathbb{C}$, and let $X(\mathbb{C})$ be the topological space of its complex points. In [50] Grothendieck showed that there is a formula for the singular cohomology of $X(\mathbb{C})$ in purely algebraic terms, namely

$$H^n_{\mathrm{sing}}(X(\mathbb{C}), \mathbb{C}) \simeq H^n_{\mathrm{dR}}(X/\mathbb{C}),$$

If, moreover, X is assumed to be projective, one obtains the Hodge decomposition

$$H^n_{\mathrm{sing}}(X(\mathbb{C}), \mathbb{C}) \simeq \bigoplus_{p+q=n} H^q(X, \Omega^p_X).$$

It is possible to get such a decomposition only using some transcendental methods (like Hodge theory). However, for X proper, there still is a functorial filtration $F^\bullet H^n_{\mathrm{dR}}(X/\mathbb{C})$, defined purely in algebraic terms and whose associated graded is given by the sum above. Namely, the de Rham

complex has a natural filtration $\Omega^{\geq p}_{X,\mathrm{dR}}$ given by subcomplexes

$$\Omega^{\geq p}_{X,\mathrm{dR}} := \ \ldots \longrightarrow 0 \longrightarrow \Omega^p_X \xrightarrow{d} \Omega^{p+1}_X \xrightarrow{d} \ldots \xrightarrow{d} \Omega^{\dim X}_X \ .$$

This filtration induces the Hodge-to-de Rham spectral sequence

$$E_1^{p,q} = H^q(X, \Omega^p_{X/R}) \ \Rightarrow \ H^{p+q}_{\mathrm{dR}}(X/\mathbb{C}).$$

which degenerates by a purely algebraic argument of Deligne-Illusie [36] that uses a reduction to a large enough characteristic p.

In our other joint work [76] with Artem Prikhodko (see also Chapter 4) we generalize the method of Deligne-Illusie to the case of n-Artin stacks. The analogue of Hodge cohomology for a smooth n-Artin stack $\mathcal{Y}$ over a base ring R is given by

$$H^n_{\mathrm{H}}(\mathcal{Y}/R) := \oplus_{i=0}^n H^{n-i}(\mathcal{Y}, \wedge^i \mathbb{L}_{\mathcal{Y}/R}),$$

where $\mathbb{L}_{\mathcal{Y}/R}$ is the cotangent complex of $\mathcal{Y}$. Similarly to the case of schemes there is the Hodge-to-de Rham spectral sequence

$$E_1^{p,q} = H^q(\mathcal{Y}, \wedge^p \mathbb{L}_{\mathcal{Y}/R}) \ \Rightarrow \ H^{p+q}_{\mathrm{dR}}(\mathcal{Y}/R).$$

For $R = k$ a perfect field of characteristic p, we prove the following analogue of a theorem of Deligne-Illusie:

Theorem 1.0.3.1 (Theorem 4.2.1.16)**.** *Let $\mathcal{Y}$ be a smooth n-Artin stack over k. Then a smooth lift $\widetilde{\mathcal{Y}}$ of $\mathcal{Y}$ to $W_2(k)$ gives a functorial equivalence*

$$R\Gamma(\mathcal{Y}, \tau^{\leq p-1}\Omega^\bullet_{\mathcal{Y}/k,\mathrm{dR}}) \simeq R\Gamma\left(\mathcal{Y}^{(1)}, \bigoplus_{i=0}^{p-1} \wedge^i \mathbb{L}_{\mathcal{Y}^{(1)}/k}[-i]\right).$$

In particular for $n \leq p-1$ one has $H^n_{\mathrm{dR}}(\mathcal{Y}/k) \simeq H^n_{\mathrm{H}}(\mathcal{Y}^{(1)}/k)$.

The existence of a lift $\widetilde{\mathcal{Y}}$ to $W_2(k)$ is a nontrivial condition even for schemes, but it is satisfied automatically if $\mathcal{Y}$ comes as a mod p reduction of some stack over $\mathbb{Z}$ or, more generally, some finitely generated $\mathbb{Z}$-algebra R. The proof of Theorem 1.0.3.1 follows ideas of [44] and uses in a

crucial way the quasi-syntomic descent for the crystalline cohomology (given by [18]).

Let F be a field of characteristic 0. To apply the Deligne-Illusie method to a given stack over F its Hodge cohomology groups need to be finite-dimensional, in other words it should be Hodge-proper. However, unlike in the case of proper schemes, there does not necessarily exist a Hodge-proper spreading $\mathcal{Y}_R$ (i.e. $\mathcal{Y} = \mathcal{Y}_R \times_R F$) over some finitely generated $\mathbb{Z}$-algebra R. But, if such a spreading exists, the degeneration holds:

Theorem 1.0.3.2 (Theorem 4.2.2.2)**.** *Let $\mathcal{Y}$ be a Hodge-properly spreadable smooth n-Artin stack over F. Then the Hodge-to-de Rham sprectral sequence for $\mathcal{Y}$ degenerates at the first page. In particular for all n*

$$H^n_{\mathrm{dR}}(\mathcal{Y}) \simeq \bigoplus_{p+q=n} H^q(\mathcal{Y}, \wedge^p \mathbb{L}_{\mathcal{Y}/F}).$$

In other words, Theorem 1.0.3.2 reduces the question of the equality of two vector spaces (namely de Rham and Hodge cohomology) to a question of finite-generatedness of certain modules (namely the Hodge cohomology of a spreading), which in some cases is easier to establish. In particular we automatically deduce the Hodge-to-de Rham degeneration for all proper stacks. These include proper Deligne-Mumford stacks for which the degeneration was proved in [104].

Most fun though is finding some non-proper examples. Using some strong cohomological vanishing results of [46] we prove the following theorem:

Theorem 1.0.3.3 (a particular case of Theorem 4.4.1.3)**.** *Let X be a smooth scheme and let* $\operatorname{Spec} A$ *be a finite type affine scheme over F, both endowed with an action of a reductive group G. Assume that*

- *There is a proper G-equivariant map $\pi\colon X \to \operatorname{Spec} A$;*
- $\dim A^G < \infty$.

Then the quotient stack $[X/G]$ is Hodge-properly spreadable.

In [108], using Hodge theory, Teleman proved the Hodge-to-de Rham degeneration in this case under the assumption that π is projective. Our proof is purely algebraic and does not require projectivity. More generally Teleman proves the degeneration for so-called "KN-completely stratified" spaces; a generalized version of those can also be covered by Theorem 1.0.3.2, but using some yet unpublished results of Halpern-Leistner [52].

Chapter 2

Descent for certain Brauer group classes on conical resolutions

2.0.1 Motivation

Given a singular variety Y over a field of characteristic 0, one wants to study its resolutions of singularities $\pi\colon X \to Y$. Usually there are too many of them, but one can impose certain additional conditions on the resolutions. In particular for a normal singular conical Poisson variety Y, one can study its symplectic resolutions $\pi\colon X \to Y$. One can then try to classify all of them, and in some form this was done recently by Namikawa [88]. Namely, provided there exists at least one symplectic resolution $\pi\colon X \to Y$, the vector space $V_{\mathbb{R}} = \mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R}$ can be partitioned into a union of rational cones, and there is an action of a finite group W on $V_{\mathbb{R}}$ that maps cones to cones. The set of symplectic resolutions $\pi\colon X \to Y$ is then identified with the set of cones modulo the action of W.

One can ask, given two symplectic resolutions $\pi\colon X \to Y$, $\pi'\colon X' \to Y$, if their derived categories of coherent sheaves are equivalent. This is a particular case of "K-equivalence implies D-equivalence" conjecture, due to Kawamata [70] (see also Conjecture 5.1 in the ICM talk by Bondal and Orlov [22]). Étale locally on the base this was proved first by Bezrukavnikov and Kaledin for quotient singularities in [11], and then separately by Kaledin for general symplectic resolution in [64]. Both proofs are based on the notion of a Fedosov quantization in characteristic p, and this notion is what connects the algebras $\mathcal{D}_{X,\alpha}$ and symplectic resolutions. We will first sketch Kaledin's proof, then we will show how classes $[\mathcal{D}_{X,\alpha}]$ naturally pop up in the context and

why it can be important to study their descent.

Results of [12] imply the existence of a canonical Frobenius-constant quantization $\mathcal{A}_0$ of X. By definition of being Frobenius-constant, $\mathcal{A}_0$ possesses an isomorphism $\mathcal{Z}(\mathcal{A}_0) = \mathcal{O}_X^p$ and $(\mathrm{Fr}_X)_*\mathcal{A}_0$ is an Azumaya algebra on $X^{(1)}$. For a class $\lambda \in \mathrm{Pic}(X)$, one can define $\mathcal{A}_\lambda$ as the twist of $\mathcal{A}_0$ by the corresponding line bundle $\mathcal{O}(\lambda)$. It is known that the isomorphism class of $\mathcal{A}_\lambda$ depends only on $\overline{\lambda} \in \mathrm{Pic}(X)/p\cdot\mathrm{Pic}(X)$ and that all $\mathcal{A}_\lambda$ have the same class in the Brauer group. One can prove (see Theorem 2.2.2.6) that $(\mathrm{Fr}_X)_*\mathcal{A}_\lambda$ splits on some étale neighbourhood $j\colon U \to Y$ of each point $y \in Y$, namely $j^*((\mathrm{Fr}_X)_*\mathcal{A}_\lambda) = \underline{\mathrm{End}}_{\mathcal{O}_{X^{(1)}}}(\mathcal{E}_U)$ for some vector bundle $\mathcal{E}_U$ on $(U\times_Y X)^{(1)}$. Using the fact that $\mathcal{A}_\lambda$ was a quantization one can prove that $\mathcal{E}_U$ is a tilting object, namely $R^\bullet\mathrm{Hom}(\mathcal{E}_U, \mathcal{E}_U) = \Gamma(j^*\mathcal{A}_\lambda)$ and the functor $R^\bullet\mathrm{Hom}(\mathcal{E}_U, \bullet)$ gives an equivalence between the derived category of coherent sheaves on $(U \times_Y X)^{(1)}$ and finitely generated $\Gamma(j^*\mathcal{A}_\lambda)$-modules. We have $\mathrm{Pic}(X) = \mathrm{Pic}(X')$, and the quantization $\mathcal{A}'_\lambda$ of X' is compatible with $\mathcal{A}_\lambda$, in the sense that it has a splitting bundle $\mathcal{E}'_U$, such that $\mathcal{E}_U$ is isomorphic to $\mathcal{E}'_U$ on the smooth part of Y. Finally, one has $\Gamma(j^*\mathcal{A}_\lambda) = \Gamma(j^*\mathcal{A}'_\lambda)$ and we obtain derived equivalences

$$D^b(\mathrm{Coh}((X \times_Y U)^{(1)})) \cong D^b(\Gamma(j^*\mathcal{A}_\lambda) - \mathrm{mod}^{\mathsf{fg}}) \cong D^b(\mathrm{Coh}((X' \times_Y U)^{(1)})).$$

Bundles $\mathcal{E}_U$, $\mathcal{E}'_U$ are rigid and can be lifted to tilting objects in characteristic 0, providing a derived equivalence there. However, the resulting equivalence, that we will obtain, will depend on the original choice of $\lambda \in \mathrm{Pic}(X)$, and one could try to keep track of this. One could also try to extend this sort of equivalence to a global one. In the same paper Kaledin proves that, if $\pi\colon X \to Y$ is conical, the corresponding tilting object on étale neighbourhood of the central point $y_0 \in Y$ comes as a pull-back of a $\mathbb{G}_m$-equivariant sheaf $\mathcal{E}_0$ on X, which then produces a global equivalence. However the Azumaya algebra $\mathcal{A}_\lambda$ is not globally split and it is not clear what is the global relation between $\mathcal{A}_\lambda$ and $\mathcal{E}_0$, as well as the relation between $\mathcal{E}_0$ and splitting bundles for $\mathcal{A}_X$ at the other points.

This is the place where one could need the Azumaya algebras $\mathcal{D}_{X,\alpha}$. Namely one can give another description of the class $[\mathcal{A}_\lambda]$ in $\mathrm{Br}(X^{(1)})$. In the case when X admits an open embedding to the cotangent bundle to a stack it is known that $[\mathcal{A}_\lambda] = [\mathcal{D}_{X,\alpha}]$ for a 1-form α. The form α can be written down, namely it is given by the contraction of the symplectic 2-form $\omega \in H^0(X, \Omega^2_X)$ with the Euler vector field ξ provided by the contracting $\mathbb{G}_m$-action (see e.g. the proof of Proposition 10.3 in [13]). In particular, this is true for all Hamiltonian reductions of the cotangent bundle of a

vector space, which is the main set of examples we consider. This is also expected to be true for any conical symplectic resolution $\pi\colon X \to Y$.

Though the Azumaya algebra $(\mathrm{Fr}_X)_*\mathcal{A}_\lambda$ is usually not split even for the simplest examples, it is reasonable to expect that the class $[\mathcal{A}_\lambda]$ in the Brauer group $\mathrm{Br}(X^{(1)})$ (which does not depend on the choice of λ) descends to $Y^{(1)}$: namely that there exists a class $c \in \mathrm{Br}(Y^{(1)})$, such that $[\mathcal{A}_\lambda] = \pi^*(c)$. Let $\mathcal{C}$ be an Azumaya algebra on $Y^{(1)}$, such that $c = [\mathcal{C}]$. Then $\mathcal{A}_\lambda \otimes_{\mathcal{O}_{X^{(1)}}} \pi^*\mathcal{C}^{op}$ is split for any $\lambda \in \mathrm{Pic}(X)$, and so we obtain an equivalence

$$\mathrm{Coh}(X^{(1)}) \cong \mathrm{Coh}(\mathcal{A}_\lambda \otimes \pi^*\mathcal{C}^{op}),$$

where $\mathrm{Coh}(\mathcal{A}_\lambda \otimes \pi^*\mathcal{C}^{op})$ is the abelian category of $\mathcal{A}_\lambda \otimes \pi^*\mathcal{C}^{op}$-modules which are $\mathcal{O}_{X^{(1)}}$-coherent. We have $\Gamma(\mathcal{A}_\lambda \otimes \pi^*\mathcal{C}^{op}) = \Gamma(\mathcal{A}_\lambda) \otimes \mathcal{C}^{op}$ and for each $\lambda \in \mathrm{Pic}(X)$ we obtain the composite functor

$$R\Gamma_{\lambda,\mathcal{C}}\colon D^b(\mathrm{Coh}(X^{(1)})) \cong D^b(\mathrm{Coh}(\mathcal{A}_\lambda \otimes \pi^*\mathcal{C}^{op})) \xrightarrow{R\Gamma_\lambda} D^b(\Gamma(\mathcal{A}_\lambda) \otimes \mathcal{C}^{op} - \mathrm{mod}^{\mathsf{fg}}),$$

which is a derived equivalence whenever $R\Gamma_\lambda$ is. **To summarise:** if we want to define some derived equivalences between $D^b(\mathrm{Coh}(X^{(1)}))$ and $D^b(R_\lambda - \mathrm{mod}^{\mathsf{fg}})$, for some non-commutative resolutions R_λ indexed by $\lambda \in \mathrm{Pic}(X)$, there is a natural way to do that, if the class $[\mathcal{A}_\lambda]$ descends to $Y^{(1)}$. The case of $[\mathcal{A}_\lambda]$ being of the form $[\mathcal{D}_{X\alpha}]$ returns us to the question formulated in the beginning.

We prove that all classes $[\mathcal{D}_{X\alpha}]$ descend étale locally on the base for all α, if $R^1\pi_*\mathcal{O}_X = R^2\pi_*\mathcal{O}_X = 0$, and we define a class of resolutions (which we call resolutions with conical slices) for which the descent is true globally (for a reduction to big enough characteristic p). We also prove that there are a lot of examples of such, e.g. any resolution of singularities of a Hamiltonian reduction of a vector space by the GIT-quotient with a non-trivial character, is a resolution with conical slices (provided it is a resolution of singularities). In particular, resolutions of singularities of hypertoric and Nakajima quiver varieties are resolutions with conical slices.

Remark 2.0.1.1. Following what we have written above, for regular λ (ones for which $R\Gamma_\lambda$ is an equivalence) we obtain equivalences

$$R\Gamma_{\lambda,\mathcal{C}}\colon D^b(\mathrm{Coh}(X^{(1)})) \xrightarrow{\sim} D^b(\Gamma(\mathcal{A}_\lambda) \otimes \mathcal{C}^{op} - \mathrm{mod}^{\mathsf{fg}})$$

for some Azumaya algebra $\mathcal{C}$. In this case, $\mathcal{A}_\lambda$ also splits on some étale neighbourhoods of the

fibers and so, for any $y \in Y$, $R\Gamma_{\lambda,\mathcal{C}}$ provides an equivalence between the category of sheaves with support at $\pi^{-1}(y)$ and the category of modules with support at y:

$$R\Gamma_{\lambda,\mathcal{C},y}\colon D^b(\mathrm{Coh}_{\pi^{-1}(y)}(X^{(1)})) \xrightarrow{\sim} D^b(\Gamma(\mathcal{A}_\lambda)_y - \mathrm{mod}^{\mathsf{fg}}).$$

It is known that for any symplectic resolution (over $\mathbb{C}$) $K^0(\mathrm{Coh}_{\pi^{-1}(y)}(X)) \otimes_{\mathbb{Z}} \mathbb{Q} \cong H^\bullet(X_y, \mathbb{Q})$ (see the sketch in Corollary 1.10 in [64]), and, since K^0 remains the same for generic reduction, we get a canonical basis in $H^\bullet(X_y, \mathbb{Q})$, corresponding to the basis of irreducible modules in $K^0(\Gamma(\mathcal{A}_\lambda)_y - \mathrm{mod}^{\mathsf{fg}})$ (depending on λ). This can be considered as a partial generalisation of the results of [14].

The program, outlined by Bezrukavnikov and Okounkov and partially written down in Conjecture 1 of [3], proposes that there is some interesting structure behind equivalences $R\Gamma_\lambda$. As a part of the conjecture, categories $D^b(\mathrm{Coh}(\mathcal{A}_\lambda \otimes \pi^*\mathcal{C}^{op})$ indexed by $\lambda \in \mathrm{Pic}(X)$, categories $D^b(\mathrm{Coh}(X_C^{(1)}))$ indexed by Namikawa's cones C, and derived equivalences $R\Gamma_{\lambda,\mathcal{C}}$ (for various X_C), should give rise to a representation of the Poincaré groupoid of a certain space $V_{\mathbb{C}}^\circ$, which is a complement of $V_{\mathbb{C}} = \mathrm{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{C}$ to a certain affine hyperplane arrangement $\Sigma \subset V_{\mathbb{C}}$. However, the precise statement of this conjecture is about the sheaves with support at the central fiber $\pi^{-1}(y_0)$, and our results also provide a way to reformulate it for the global situation.

2.0.2 Plan of the chapter

Logically the chapter can be divided in two parts. The first one is short and relatively simple, it studies the local side of the story. The second one takes the remaining space of the chapter and is more technical. It gives a positive answer for the global splitting in some cases.

In Section 2.1 we give a brief tour through the theory of "crystalline differential operators" in characteristic p and define Azumaya algebras $\mathcal{D}_{X,\alpha}$ on the Frobenius twist $X^{(1)}$. The content of this section can intersect in a significant way with sections 4.1-4.3 of [91]. Algebras $\mathcal{D}_{X,\alpha}$ are the main objects that we study in this chapter and in this section we try to put them into a context of some more widely known objects in algebraic and differential geometry in characteristic p. In Section 2.1.1 we remind what the relative Frobenius twist $X^{(S)}$ is. In Section 2.1.2 we recall the Cartier isomorphism (Theorem 2.1.2.1) and define the Cartier operator $\mathsf{C}\colon (\mathrm{Fr}_X)_*\Omega^1_{X,cl} \to \Omega^1_{X^{(1)}}$. In Section 2.1.3 we give the definition of "crystalline differential operators" and discuss its Azumaya property (Proposition 2.1.3.1): namely the sheaf of differential operators $\mathcal{D}_X$ defines an Azumaya

algebra on $T^*X^{(1)}$. In Section 2.1.4, to each differential form α on $X^{(1)}$, we associate an Azumaya algebra $\mathcal{D}_{X,\alpha}$ on $X^{(1)}$ as the restriction of $\mathcal{D}_X$ to the graph $\Gamma_\alpha \subset T^*X^{(1)}$ (see Definition 2.1.4.1). This way we obtain a map $c_X\colon H^0(X^{(1)}, \Omega^1_{X^{(1)}}) \to \mathrm{Br}(X^{(1)})$. In Section 2.1.5 we remind the definition of the p-curvature of a flat connection and relate it to algebras $\mathcal{D}_{X,\alpha}$ (see e.g. Proposition 2.1.5.4 and Proposition 2.1.5.7). In particular we show that the Azumaya property of $\mathcal{D}_X$ implies the classical Cartier theorem (see Remark 2.1.5.8). For each differential 1-form α we define two groupoids: the groupoid $\mathrm{Spl}(X,\alpha)$ of splittings of $\mathcal{D}_{X,\alpha}$ and the groupoid $LIC(X,\alpha)$ of line bundles with p-curvature α, and show that they are equivalent (Proposition 2.1.5.7). In Section 2.1.6 we provide a brief introduction to the theory of gerbes and show that the groupoids $\mathrm{Spl}(U,\alpha)$ and $LIC(U,\alpha)$, for various opens $U \to X$, give rise to a pair of isomorphic gerbes $\mathcal{S}(X,\alpha)$ and $\mathcal{L}(X,\alpha)$ on $X_{\text{ét}}$. In Section 2.1.7 we recall an a priori different from c_X map $\overline{c}_X\colon H^0(X^{(1)}, \Omega^1_{X^{(1)}}) \to \mathrm{Br}(X^{(1)})$ that comes from a certain 4-term exact sequence (see Proposition 2.1.7.1). Finally, in Section 2.1.8 using a certain Picard groupoid we show that $c_X = \overline{c}_X$ and in particular that c_X is additive (Proposition 2.1.8.5).

In Section 2.2 we study if given a proper morphism $\pi\colon X \to Y$ from a smooth variety X, the classes $[\mathcal{D}_{X,\alpha}] \in \mathrm{Br}(X^{(1)})$ descend to $Y^{(1)}$ étale locally on Y. We prove the following result (Theorem 2.2.2.6), which answers the question in some generality:

Theorem 2.0.2.1. *Let $\pi\colon X \to Y$ be a proper map from a smooth variety X, such that $R^1\pi_*\mathcal{O}_X = R^2\pi_*\mathcal{O}_X = 0$. Then there exists an étale cover $U \to Y$, such that the pull-back of $\mathcal{D}_{X,\alpha}$ to $U \times_Y X$ is split.*

The core of our proof is a well-known lemma (see Lemma 2.2.2.1), which says that a sum of a Frobenius-linear operator A and a surjective linear operator B between finite-dimensional vector spaces, is a surjective map of abelian groups. Another important input is Lemma 2.2.2.2, where we prove that under the assumption $R^1\pi_*\mathcal{O}_X = R^2\pi_*\mathcal{O}_X = 0$ the Cartier operator $\pi_*\mathsf{C}\colon \pi_*\Omega^1_{X,cl} \to \pi_*\Omega^1_X$ is a surjection. We then use Lemma 2.2.2.1 for $A = \bullet^{(1)}$ and $B = \mathsf{C}$ to prove that the restriction of any differential 1-form α on $X^{(1)}$ on the formal neighbourhood of any fiber $\pi^{-1}(y)$ is of the form $\omega^{(1)} - \mathsf{C}(\omega)$ for some ω. From this we deduce that $\mathcal{D}_{X,\alpha}$ is split on the formal neighbourhood of any fiber of π. Finally, Popescu's theorem allows us to extend the splitting to some étale neighbourhood of the fiber.

Section 2.3 presents two approaches for studying the global descent. The first is the *Picard obstruction*, it is defined in Section 2.3.1 and is a class in $H^1_{\text{ét}}(Y, \mathrm{Pic}_{X/Y})[p]$. This obstruction is

very natural, but seems somewhat useless for us, or at least we were not able to prove anything interesting about it. Instead, in Section 2.3.2, for the case of an affine $Y = \operatorname{Spec} A$, we define another type of obstructions, which we call $Q_{\pi,N}$ (see Definition 2.3.2.4). By definition for each N there is a natural projection $H^0(X, \Omega^1_X) \twoheadrightarrow Q_{\pi,N}$ and spaces $Q_{\pi,N}$ for different N are organised into a chain of surjections:

$$H^0(X, \Omega^1_X) \twoheadrightarrow Q_{\pi,0} \twoheadrightarrow Q_{\pi,1} \twoheadrightarrow \cdots \twoheadrightarrow Q_{\pi,N} \twoheadrightarrow \cdots$$

Moreover, if a differential 1-form α maps to 0 in some $Q_{\pi,N}$, the corresponding class $[\mathcal{D}_{X,\alpha}] \in \operatorname{Br}(X^{(1)})$ descends to $Y^{(1)}$ (see remark in the Definition 2.3.2.4). The idea of the definition is based on the following observation: the Cartier operator $\mathsf{C}\colon H^0(X, \Omega^1_{X,cl}) \to H^0(X^{(1)}, \Omega^1_{X^{(1)}})$ does not necessarily map closed Kähler differentials $H^0(Y, \Omega^1_{Y,cl})$ to Kähler differentials $H^0(Y^{(1)}, \Omega^1_{Y^{(1)}})$. Moreover, sometimes it can even happen that any globally defined 1-form on $X^{(N)}$ is obtained from a pull-back of a Kähler differential on Y by applying Cartier operator big enough number of times. Spaces $Q_{\pi,N}$ are defined as the obstruction for this to be true. The nice thing about spaces $Q_{\pi,N}$ is that they have a natural structure of $(A)^{p^N}$-module, such that the projection $H^0(X, \Omega^1_X) \twoheadrightarrow Q_{\pi,N}$ is $(A)^{p^N}$-linear. This allows to study the corresponding coherent sheaves $\mathcal{Q}_{\pi,N}$ on $Y^{(N)}$ using the means of algebraic geometry. The relation between the descent of $[\mathcal{D}_{X,\alpha}]$ and Kähler differentials is given by Lemma 2.3.2.1: namely, if $\alpha = \pi^*\theta$ for some Kähler differential θ, then $[\mathcal{D}_{X,\alpha}]$ descends to Y.

In Section 2.4 we develop some tools, namely we define *the category of étale germs of resolutions* (Section 2.4.2), *étale equivalences* between *pointed resolutions* (Definition 2.4.2.3 and Definition 2.4.2.10) and *resolutions with conical slices* (Definition 2.4.4.3). The rough idea is to define a class of resolutions for which we could apply Lemma 2.3.2.5 and proceed by induction: we want any (or almost any) point of the base of the resolution to have a neighbourhood in a suitable topology, which decomposes as a product of some smooth variety and a resolution of the same sort. The starting point is to take conical resolutions (Definition 2.4.4.1), then, by Luna's étale slice theorem, every non-central point has an étale decomposition into a product of its orbit of $\mathbb{G}_m$-action and a slice to the orbit. But for our purposes this is not enough, as we will not have any information about the slice. However our definition is not very far from this one, in addition we just ask for a contracting $\mathbb{G}_m$-action in an étale neighbourhood of each non-central point (Definition 2.4.4.3). Some technical work is needed to make this definition work. We define the notion of *étale equiva-*

lence (Definition 2.4.2.10) between two pointed resolutions as an invertible morphism in a certain category Res^{et}_*, which we call the category of *étale germs of resolutions*. We define a *resolution with conical slices* as a conical resolution $\pi\colon X \to Y$, such that $\mathcal{O}_Y \xrightarrow{\sim} \pi_*\mathcal{O}_X$, $R^1\pi_*\mathcal{O}_X = R^2\pi_*\mathcal{O}_X = 0$ and at every non-central point y it is étale equivalent to a conical resolution $\pi'\colon X' \to Y'$. **Warning:** our definition of conical resolution by default includes a separability assumption on the action of $\mathbb{G}_m$! In Section 2.4.5 we prove that at each non-central point a resolution with conical slices has a slice which is again a resolution with conical slices (Proposition 2.4.5.4). This then will allow us to make the induction step in Section 2.6, while studying $Q_{\pi,N}$ for resolutions with conical slices. Finally, in Section 2.4.6 we prove that given a resolution with conical slices in characteristic 0, a general reduction to characteristic p is again a resolution with conical slices (Proposition 2.4.6.5).

The notion of a resolution with conical slices seems to be interesting even outside of the context of this thesis. In a certain sense, for this class of resolutions it is enough to prove any statement, that is local in étale topology and $\mathbb{A}^1$-homotopy invariant, just for the neighbourhoods of the central points. As we will see, quite a few well-known resolutions are resolutions with conical slices (Section 2.7). The application of this structure is demonstrated on the example of spaces $Q_{\pi,N}$ (and sheaves $\mathcal{Q}_{\pi,N}$) and it looks plausible that it can be useful in some other situations too.

The reason why we need étale slices and say not formal ones is that we need a statement like Remark 2.4.5.5, which says that there is an étale correspondence between the complement to a central point in $\pi\colon X \to Y$ and a finite union of products of resolutions with conical slices with $\mathbb{A}^1$. Finiteness is very important here, if we want to prove something like Proposition 2.4.6.5 or have a universal bound for N, such that $Q_{\pi,N} = 0$ on each slice.

In Section 2.5 we study weights of differential 1-forms on a conical resolution. The $\mathbb{G}_m$-action on $\pi\colon X \to Y$ produces a positive grading on the ring of global functions on X and Y. A sheaf of differential 1-forms Ω^1_X is $\mathbb{G}_m$-equivariant giving a grading on $H^0(X, \Omega^1_X)$, which turns it into a graded module over the ring of functions Y. Interesting question is: what can we say about this grading? For example: is it true that the grading is positive as well? Assuming that $R^i\pi_*\mathcal{O}_X = 0$ for $i = 1, 2$ this is almost true. Here is one part of the result (Proposition 2.5.1.4):

Theorem 2.0.2.2. *Let $\pi\colon X \to Y$ be a conical resolution of singularities over an algebraically closed field of characteristic 0 with $\mathcal{O}_Y \xrightarrow{\sim} \pi_*\mathcal{O}_X$ and $R^1\pi_*\mathcal{O}_X = R^2\pi_*\mathcal{O}_X = 0$. Then all $\mathbb{G}_m$-weights of $H^0(X, \Omega^1_X)$ are strictly positive.*

The proof of Theorem 2.5.1.1 is quite interesting and uses both reductions to characteristic

p and analytification of X over $\mathbb{C}$. We first prove, using the surjectivity of Cartier operator (Lemma 2.2.2.2), that for big enough p all weights of differential 1-forms on the reduction are non-negative, and that all $\mathbb{G}_m$-invariant 1-forms are closed (Lemma 2.5.1.2). Then we use the comparison of de Rham and singular cohomology to show that there are no non-zero closed $\mathbb{G}_m$-invariant 1-forms over $\mathbb{C}$. In Section 2.5.2 we give an example of a conical resolution in characteristic 2 which has a non-zero $\mathbb{G}_m$-invariant 1-form. We also state a conjecture, which (if true) gives a bound from below on the dimension of the space of invariant forms similar to the specialization result of [16]. In Section 2.5.3 we play around a little bit and try to cope with the existence of $\mathbb{G}_m$-invariant forms in characteristic p. Namely we give a somewhat complicated definition of totally positive forms (Definition 2.5.3.3), which morally are differential 1-forms on X that have positive weight at a conical neighbourhood of each point. However, there is a choice of the conical neighbourhood involved, and this makes the actual definition more complicated. The space $H^0(X, \Omega^1_X)^{\gg 0}$ of totally positive forms is invariant under multiplication by functions and we define the sheaf $\mathcal{M}_{\pi,\underline{\text{inv}}}$ as the coherent sheaf on Y, corresponding to the quotient $H^0(X, \Omega^1_X)/H^0(X, \Omega^1_X)^{\gg 0}$. In Definition 2.5.3.10 we define a strictly positive resolution with conical slices: namely, it is a resolution with conical slices for which $\mathcal{M}_{\pi,\underline{\text{inv}}} = 0$. Finally, we end the section with Proposition 2.5.4.2, which states that a general reduction of a resolution with conical slices is strictly positive.

In Section 2.6 we prove our main theorem (Theorem 2.6.2.4):

Theorem 2.0.2.3. *Let $\pi\colon \mathcal{X} \to \mathcal{Y}$ be a map of schemes over $S = \operatorname{Spec} R$ finite type and flat over $\mathbb{Z}$, such that the generic fiber $\pi_\eta\colon X_\eta \to Y_\eta$ is a resolution with conical slices. Then there exists an étale open $S' \to S$, such that for any geometric point $s\colon \operatorname{Spec} \mathsf{k}_s \to S'$*

- *the corresponding fiber $\pi_s\colon X_s \to Y_s$ is a strictly positive resolution with conical slices,*
- *the classes $[\mathcal{D}_{X_s,\alpha}]$ descend to $Y_s^{(1)}$ for all $\alpha \in H^0(X_s^{(1)}, \Omega^1_{X_s^{(1)}})$.*

The theorem follows directly from the Proposition 2.5.4.2 and Proposition 2.6.2.2, which states that for a strictly positive resolution with conical slices, the sheaf $Q_{\pi,N}$ is 0. The proof of Proposition 2.6.2.2 proceeds by induction on $\dim X$, using Remark 2.4.5.5, while for the central point it uses the same trick with surjectivity of Cartier operator as in Theorem 2.5.1.1.

In Section 2.7 we prove that some symplectic resolutions are resolutions with conical slices. The existence of formal slices for symplectic resolutions was proved by Kaledin in [63]. However the existence of a contracting $\mathbb{G}_m$-action on the slice is still a conjecture (Conjecture 1.8 in [66])

and here we would need étale slices anyways. For quiver varieties $N_Q(\lambda, \alpha)$ in [30] Crawley-Boevey showed that at any point the base $N_Q(\lambda, \alpha)$ looks étale locally like the central point of $N_{Q'}(0, \alpha')$ which then has a natural contracting $\mathbb{G}_m$-action. It is remarkable that Crawley-Boevey's result fits perfectly into the framework of resolutions with conical slices: he does not prove the existence of a slice, but provides an étale equivalence with some other variety which has a contracting $\mathbb{G}_m$-action. Given a quiver Q with the set of vertices I, the dimension vector $\alpha \in \mathbb{N}^I$ and a vector $\lambda \in \mathbb{K}^I$, affine variety $N_Q(\lambda, \alpha)$ is defined as the Hamiltonian reduction of the space of representations of the double quiver $\overline{Q}$ with dimension vector α and with the level of the moment map equal to λ. Geometric invariant theory provides a projective map $\pi^\theta_\lambda \colon N_Q(\lambda, \alpha)^\theta \to N_Q(\lambda, \alpha)$ from the GIT quotient $N_Q(\lambda, \alpha)^\theta$. The sufficient conditions for the map π^θ_λ to be a resolution of singularities were given by Crawley-Boevey as well and we summarise them in Proposition 2.7.2.11. Unfortunately, étale equivalence in [30] is proved only for the bases of resolutions and not for the resolutions themselves. We do some work to extend the equivalence to this level (see Proposition 2.7.2.18). Since for $\lambda = 0$ the resolution $\pi^\theta_0 \colon N_Q(0, \alpha)^\theta \to N_Q(0, \alpha)$ is conical, we get that the resolution π^θ_0 is a resolution with conical slices (Theorem 2.7.2.19). From Theorem 2.6.2.4 we then get that for a general reduction to characteristic p all classes $[\mathcal{D}_{N_Q(0,\alpha)^\theta, \alpha}]$ descend to $N_Q(0, \alpha)$. In particular, following Section 2.7.2 this is true for Nakajima quiver varieties, so this answers the question raised in [13], which originally motivated this research.

In Section 2.7.3 we generalise the result of Crawley-Boevey to an arbitrary Hamiltonian reduction of a cotangent bundle to vector space. Namely given a symplectic vector representation of a reductive group G in a vector space (V, ω) we can take its GIT Hamiltonian reduction $\mathfrak{M}(G, V)^\theta_\lambda = \mu^{-1}(\lambda) /\!\!/_\theta G$ and consider a natural map $\pi^\theta_\lambda \colon \mathfrak{M}(G, V)^\theta_\lambda \to \mathfrak{M}(G, V)^0_\lambda = \mu^{-1}(\lambda) /\!\!/ G$. We restrict to the case of $V \oplus V^*$ and a symplectic representation of G induced by some representation $G \to \mathrm{GL}(V)$. Closely following the argument of Crawley-Boevey, we prove that if π^θ_0 is a resolution of singularities and the base is normal, it is a resolution with conical slices (see Theorem 2.7.3.11). We then apply this to obtain the result for hypertoric vrieties. A hypertoric variety is a Hamiltonian reduction of the cotangent bundle of a representation of an algebraic torus $\mathbb{G}^k_m$. Hypertoric varieties are encoded by the combinatorial data of a weighted, cooriented, affine hyperplane arrangement $\mathcal{A}$ in some lattice that depends on the representation. The case $\theta = 0$ corresponds to arrangements where all hyperplanes pass through 0. If such an arrangement is unimodular, it possesses a simplification $\widetilde{\mathcal{A}}$, which differs from $\mathcal{A}$ only by the choice of a non-trivial

character θ. Hypertoric variety $\mathfrak{M}(\widetilde{\mathcal{A}})$ is smooth, and the natural map $\pi_{\mathcal{A}}\colon \mathfrak{M}(\widetilde{\mathcal{A}}) \to \mathfrak{M}(\mathcal{A})$ is known to be a resolution of singularities. Applying Theorem 2.7.3.11, we get that resolutions of singularities of hypertoric varieties are resolutions with conical slices.

Finally, in Section 2.7.4 we prove that the Slodowy slices (or rather their resolutions of singularities) are resolutions with conical slices. Here the proof is different and in a way simplier, it relies mostly on Lemma 2.4.5.1. In all these cases we obtain that for general reduction to characteristic p the classes $[\mathcal{D}_{X,\alpha}]$ descend to the base of the resolution.

2.0.3 Remaining questions

There are several natural questions which one can ask and which we did not cover in this writeup. They remain to be the subject of a future research.

Question 1. Is it true that any conical symplectic resolution is a resolution with conical slices?

This question is similar to the conjecture due to Kaledin (Conjecture 1.8 in [66]) which claims that the transversal slice to a Poisson leaf has a contracting $\mathbb{G}_m$-action. Though Kaledin's conjecture is about the formal slice, may be using some finiteness results recently proved by Namikawa (see [89]) one can obtain étale slices as well. Anyways, it is not yet clear at all how to attack any of these two statements.

Question 2 For a conical symplectic resolution $\pi\colon X \to Y$ is it true that the class $[\mathcal{A}_\lambda]$ in $\mathrm{Br}(X^{(1)})$ comes from a differential 1-form (meaning $[\mathcal{A}_\lambda] = [\mathcal{D}_{X,\alpha}]$ for some α)?

One even has a candidate for α — it should be the contraction of the symplectic form ω with the Euler vector field corresponding to the contracting $\mathbb{G}_m$-action. This is a computation, which still needs to be done in the case of general conical symplectic resolution $\pi\colon X \to Y$.

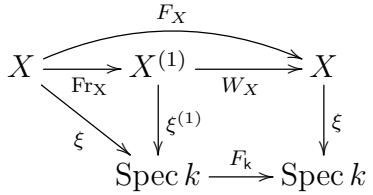
2.1 Frobenius twist and differential operators

2.1.1 Frobenius twist

Let S be a scheme over $\mathbb{F}_p$. The *absolute Frobenius* $F_S\colon S \to S$ is given by

- id_S on underlying topological space,
- $f \mapsto f^p$ for $f \in \mathcal{O}_S$ on the level of structure sheaves.

Let k be an algebraically closed field of characteristic p. For any k-scheme $X \xrightarrow{\xi} \operatorname{Spec}\mathsf{k}$ the *relative Frobenius twist* $X^{(1)} \xrightarrow{\xi^{(1)}} \operatorname{Spec}\mathsf{k}$ is defined as the pull-back of X with respect to F_{k}:



Let $W_X \colon X^{(1)} \to X$ be the natural morphism in the corresponding Cartesian square. Absolute Frobenius is functorial on X and from the universal property of the pull-back we obtain a unique decomposition $F_X = \mathrm{Fr}_X \circ W_X$.

Morphism $\mathrm{Fr}_X \colon X \to X^{(1)}$ is called the *relative Frobenius morphism*. We denote by $\bullet^{(1)}$ the pull-back of any object $\bullet$ on X to $X^{(1)}$ under W_X. One can inductively define $\bullet^{(k)}$ by $(\bullet^{(k-1)})^{(1)}$. Also note that W_X is an isomorphism of abstract schemes (since F_{k} is).

For a smooth k-scheme X we denote $\Omega^i_{X/\operatorname{Spec}\mathsf{k}}$ simply by Ω^i_X. By the flat base change, the natural maps

$$\mathcal{T}_{X^{(1)}} \xrightarrow{\sim} W_X^* \mathcal{T}_X = \mathcal{T}_X^{(1)} \text{ and } (\Omega^i_X)^{(1)} = W_X^* \Omega^i_X \xrightarrow{\sim} \Omega^i_{X^{(1)}}$$

are isomorphisms. For a vector space V over k we denote by $V^{(1)}$ the vector space V with the twisted k-structure:

$$V^{(1)} \coloneqq V \underset{\mathsf{k},F_{\mathsf{k}}}{\otimes} \mathsf{k} \text{ or equivalently } \lambda \overset{(1)}{\cdot} v = F_{\mathsf{k}}^{-1}(\lambda) \cdot v \text{ for } a \in \mathsf{k} \text{ and } v \in V.$$

where $\overset{(1)}{\cdot}$ denotes the new linear structure. Note that $V^{(1)}$ is exactly $F_{\mathsf{k}}^* V$, if we consider V as a quasi-coherent sheaf on $\operatorname{Spec}\mathsf{k}$. For every vector space V there is the corresponding affine space $\mathbb{A}(V)$ considered as a scheme that represents a functor with values in k-vector spaces (k-vector space scheme). Then on the level of k-points the two twists coinside:

$$(\mathbb{A}(V))^{(1)}(\mathsf{k}) = V^{(1)}.$$

Returning to the commutative diagram above, from the base change and the isomorphism

above we obtain natural isomorphisms

$$H^0(X,\Omega^i_X)^{(1)} \simeq H^0\big(X^{(1)},(\Omega^i_X)^{(1)}\big) \simeq H^0(X^{(1)},\Omega^i_{X^{(1)}}).$$

This way to any differential i-form $\omega \in H^0(X,\Omega^i_X)$ on X one can associate a differential form $\omega^{(1)} \in H^0(X^{(1)},\Omega^i_{X^{(1)}})$ on $X^{(1)}$. The map $\omega \mapsto \omega^{(1)}$ is by definition F_{k}-linear.

Note that, following the definitions, the subsheaf $\mathcal{O}_{X^{(1)}} \subset (\mathrm{Fr}_X)_*\mathcal{O}_X$ can be identified with the subsheaf $(\mathrm{Fr}_X)_*(\mathcal{O}^p_X) \subset (\mathrm{Fr}_X)_*\mathcal{O}_X$ of p-th powers of functions on X.

2.1.2 Cartier operator

Let $\Omega^\bullet_X$ be the algebraic de Rham complex of X. The differential of $(\mathrm{Fr}_X)_*\Omega^\bullet_X$ is $\mathcal{O}_{X^{(1)}}$-linear: $d(f^p\omega) = f^p d\omega - pf^{p-1}\cdot df\wedge\omega = f^p d\omega$, so $(\mathrm{Fr}_X)_*\Omega^\bullet_X$ defines a coherent sheaf of DG-algebras on $X^{(1)}$. In particular, cocycle, coboundary and cohomology sheaves of $(\mathrm{Fr}_X)_*\Omega^\bullet_X$ have natural structures of coherent sheaves on $X^{(1)}$. By definition the i-th cocycle sheaf $Z^i((\mathrm{Fr}_X)_*\Omega^\bullet_X)$ is identified with $(\mathrm{Fr}_X)_*\Omega^i_{X,cl}$, where $\Omega^i_{X,cl}$ is the sheaf of closed differential forms of degree i on X. The direct sum of the cohomology sheaves $\bigoplus_i \mathcal{H}^i((\mathrm{Fr}_X)_*\Omega^\bullet_X)$ then forms a coherent sheaf of graded algebras on $X^{(1)}$.

Theorem 2.1.2.1 (Cartier isomorphism, [69], Theorem 7.2)**.** *Let X be a smooth scheme over an algebraically closed field k of characteristic p. Then there is a unique isomorphism of coherent sheaves of graded algebras on $X^{(1)}$:*

$$\bigoplus_i C_i^{-1}\colon \bigoplus_i \Omega^i_{X^{(1)}}[-i] \xrightarrow{\sim} \bigoplus_i \mathcal{H}^i\left((\mathrm{Fr}_X)_*\Omega^\bullet_X\right)[-i],$$

such that $C_0^{-1}(1) = 1$ and $C_1^{-1}(dg^{(1)}) = [g^{p-1}dg]$ for $g \in \mathcal{O}_X$.

Cartier isomorphism provides an isomorphism $C_1\colon \mathcal{H}^1((\mathrm{Fr}_X)_*\Omega^\bullet_X) \xrightarrow{\sim} \Omega^1_{X^{(1)}}$ in the other direction. We define the Cartier operator $\mathsf{C}\colon (\mathrm{Fr}_X)_*\Omega^1_{X,cl} \to \Omega^1_{X^{(1)}}$ as the composition of the natural projection $(\mathrm{Fr}_X)_*\Omega^1_{X,cl} \to \mathcal{H}^1((\mathrm{Fr}_X)_*\Omega^\bullet_X)$ and C_1. It is easy to see that $\mathrm{Ker}\mathsf{C}$ is exactly the subsheaf of exact 1-forms, and so it is isomorphic to the image of $d\colon (\mathrm{Fr}_X)_*\mathcal{O}_X \to (\mathrm{Fr}_X)_*\Omega^1_{X,cl}$. Finally, $\mathrm{Ker}d = \mathcal{O}_{X^{(1)}}$, and as a result we obtain the following short exact sequence:

$$0 \to (\mathrm{Fr}_X)_*\mathcal{O}_X/\mathcal{O}_{X^{(1)}} \xrightarrow{d} (\mathrm{Fr}_X)_*\Omega^1_{X,cl} \xrightarrow{\mathsf{C}} \Omega^1_{X^{(1)}} \to 0.$$

Remark 2.1.2.2. Further we will sometimes omit $(\mathrm{Fr}_X)_*$ before Ω^1_X, $\Omega^1_{X,cl}$, $\mathcal{O}_X$ and other quasi-coherent sheaves on X and will automatically consider them as sheaves on $X^{(1)}$. For example, in this convention, the short exact sequence above becomes even shorter, and looks like this:

$$0 \to \mathcal{O}_X/\mathcal{O}_{X^{(1)}} \xrightarrow{d} \Omega^1_{X,cl} \xrightarrow{\mathsf{C}} \Omega^1_{X^{(1)}} \to 0.$$

We tried our best to avoid a possible confusion specifying whether the sheaf is on X or $X^{(1)}$ where necessary.

2.1.3 Crystalline differential operators

Let X be a smooth scheme over an algebraically closed field k of characteristic p. The sheaf of "crystalline differential operators" on X is defined as the universal enveloping algebra $\mathcal{D}_X$ of the Lie algebroid $\mathcal{T}_X$ of vector fields on X. In other words, $\mathcal{D}_X$ is a sheaf of associative algebras containing $\mathcal{O}_X$, such that for a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$, an extension of the $\mathcal{O}_X$-action on $\mathcal{F}$ to a $\mathcal{D}_X$-action is equivalent to endowing $\mathcal{F}$ with a flat connection. More explicitly, $\mathcal{D}_X$ is a sheaf of algebras generated by the algebra of functions $\mathcal{O}_X$ and the $\mathcal{O}_X$-module of vector fields $\mathcal{T}_X$ subject to the module and commutator relations $f\cdot\partial = f\partial,\ \partial\cdot f - f\cdot\partial = \partial(f)$ for $\partial \in \mathcal{T}_X, f \in \mathcal{O}_X$, and the Lie algebroid relations $\partial'\cdot\partial'' - \partial''\cdot\partial' = [\partial', \partial'']$ for $\partial', \partial'' \in \mathcal{T}_X$. If we fix a local frame $\{\partial_1, \ldots, \partial_n\}$ of vector fields on X, we obtain a decomposition $\mathcal{D}_X = \oplus_{I\in\mathbb{N}^n}\ \mathcal{O}_X \cdot \partial^I$ as a left $\mathcal{O}_X$-module. For $X = \mathbb{A}^n$ the global sections of $\mathcal{D}_X$ are given by the Weyl algebra

$$W_n \coloneqq \mathsf{k}[x_1, \ldots, x_n, \partial_1, \ldots \partial_n]/\langle [\partial_i, x_i] = 1, [\partial_i, x_j]_{i\neq j} = [x_i, x_j] = [\partial_i, \partial_j] = 0\rangle.$$

Sheaf $\mathcal{O}_X$ is naturally a sheaf of left modules over $\mathcal{D}_X$. The corresponding action by functions is defined by the multiplication and by differentiation for vector fields: $g \circ f = gf$ and $\partial \circ f = \partial(f)$. It is easy to see that this defines a homomorphism $\theta\colon \mathcal{D}_X \to \underline{\mathrm{End}}_{\mathsf{k}}(\mathcal{O}_X)$ to the sheaf of k-linear endomorphisms of $\mathcal{O}_X$. While in the case of $\mathrm{char}\,\mathsf{k} = 0$ this map is an embedding, it turns out that for $\mathrm{char}\,\mathsf{k} > 0$ this map has a huge kernel and consequently has a relatively small image.

By the binomial formula, given a vector field $\partial \in \mathcal{T}_X$, the differential operator $\partial^p \in \mathcal{D}_X$ acts

on functions as a derivation:

$$\partial^p(f\cdot g)=\sum_{i=0}^{p}\binom{p}{i}\partial^i(f)\cdot\partial^{p-i}(g)=\partial^p(f)\cdot g+f\cdot\partial^p(g).$$

This way it defines another vector field $\partial^{[p]}\in\mathcal{T}_X$, which is called the *p-th restricted power of* ∂. For $\partial\in\mathcal{T}_X$ we set $\iota(\partial)\coloneqq\partial^p-\partial^{[p]}\in\mathcal{D}_X$. By the definition of $\bullet^{[p]}$, the map ι lands in the kernel of the k-linear action $\theta\colon\mathcal{D}_X\to\underline{\mathrm{End}}_{\mathsf{k}}(\mathcal{O}_X)$.

It is not difficult to show that ι is additive and that $\iota(f\partial)=f^p\iota(\partial)$ (e.g. see [69], Proposition 5.3 or a simpler argument in [14], Lemma 1.3.1). So it defines a map $\iota\colon\mathcal{T}_{X^{(1)}}\to(\mathrm{Fr}_X)_*\mathcal{D}_X$ of quasi-coherent sheaves on $X^{(1)}$ ([14], Lemma 1.3.1): namely $\iota(\partial^{(1)})=\partial^p-\partial^{[p]}$. Moreover, one can show that $\mathrm{Ker}\theta$ is given exactly by the two-sided ideal $\mathcal{D}_X\cdot\iota(\mathcal{T}_X)\cdot\mathcal{D}_X$.

Let now $T^*X^{(1)}$ be the total space of the cotangent bundle of $X^{(1)}$, and let $q\colon T^*X^{(1)}\to X^{(1)}$ be the natural projection. Then $q_*\mathcal{O}_{T^*X^{(1)}}$ is naturally a quasi-coherent sheaf of commutative algebras on $X^{(1)}$. Note, that since the map q is affine, the functor q_* induces an equivalence between the category of quasi-coherent sheaves of $q_*\mathcal{O}_{T^*X^{(1)}}$-modules on $X^{(1)}$ and the category of quasi-coherent sheaves on $T^*X^{(1)}$. The natural pairing $\langle\cdot,\cdot\rangle\colon\mathcal{T}_{X^{(1)}}\otimes\Omega^1_{X^{(1)}}\to\mathcal{O}_{X^{(1)}}$ defines an embedding $\mathcal{T}_{X^{(1)}}\hookrightarrow q_*\mathcal{O}_{T^*X^{(1)}}$, which extends to a natural isomorphism $\mathrm{Sym}^\bullet_{X^{(1)}}\mathcal{T}_{X^{(1)}}\simeq q_*\mathcal{O}_{T^*X^{(1)}}$.

Proposition 2.1.3.1 ([14], Lemma 1.3.2, Proposition 2.3.3)**.** *The map* $\iota\colon\mathcal{T}_{X^{(1)}}\to\mathcal{D}_X$ *lands in the center* $Z(\mathcal{D}_X)\subset\mathcal{D}_X$*. Moreover, its extension to* $\mathrm{Sym}^\bullet_{X^{(1)}}\mathcal{T}_{X^{(1)}}$ *provides an isomorphism of the center* $Z(\mathcal{D}_X)$ *with* $q_*\mathcal{O}_{T^*X^{(1)}}$*. So* $\mathcal{D}_X$ *is a* $q_*\mathcal{O}_{T^*X^{(1)}}$*-central quasi-coherent sheaf of algebras on* $X^{(1)}$ *and in this way it defines a quasi-coherent sheaf of algebras on* $T^*X^{(1)}$*. As a sheaf of algebras on* $T^*X^{(1)}$*,* $\mathcal{D}_X$ *is an Azumaya algebra. It has rank* $p^{2\dim X}$ *and is non-trivial if* $\dim X>0$*.*

2.1.4 Azumaya algebras $\mathcal{D}_{X,\alpha}$

According to the Proposition 2.1.3.1 there is a sheaf $\mathcal{D}_X$ of Azumaya algebras on $T^*X^{(1)}$. Let $\alpha\in H^0(X^{(1)},\Omega^1_{X^{(1)}})$ be a 1-form on $X^{(1)}$. Its graph $i_\alpha\colon\Gamma_\alpha\hookrightarrow T^*X^{(1)}$ is by definition the section given by α and considered as a subvariety of $T^*X^{(1)}$. Projection to $X^{(1)}$ provides an isomorphism $q\colon\Gamma_\alpha\simeq X^{(1)}$.

Definition 2.1.4.1. The Azumaya algebra $\mathcal{D}_{X,\alpha}$ on $X^{(1)}$ is defined as the restriction $i_\alpha^*\mathcal{D}_X$ of $\mathcal{D}_X$ to $\Gamma_\alpha\simeq X^{(1)}$.

This restriction can be described more explicitly in terms of α. The graph $\Gamma_\alpha \subset T^*X^{(1)}$ is defined by the system of equations $\partial^{(1)} - \langle \partial^{(1)}, \alpha \rangle = 0$ for all $\partial^{(1)} \in \mathcal{T}_{X^{(1)}}$, where the brackets denote the natural pairing $\langle \cdot, \cdot \rangle \colon \mathcal{T}_{X^{(1)}} \otimes \Omega^1_{X^{(1)}} \to \mathcal{O}_{X^{(1)}}$. Recall that we have an isomorphism $\iota \colon q_* \mathcal{O}_{T^*X^{(1)}} \simeq Z(\mathcal{D}_X)$. The restriction of $\mathcal{D}_X$ to Γ_α as a sheaf of algebras is obtained by taking the quotient of $\mathcal{D}_X$ by the ideal generated by $\iota(\partial - \langle \partial^{(1)}, \alpha \rangle)$. In other words, $\mathcal{D}_{X,\alpha} = \mathcal{D}_X / \langle \partial^p - \partial^{[p]} - \langle \partial^{(1)}, \alpha \rangle \rangle_{\partial \in \mathcal{T}_{X^{(1)}}}$, where $\langle \partial^{(1)}, \alpha \rangle$ lies in $\mathcal{O}_{X^{(1)}} = (\mathcal{O}_X)^p \subset \mathcal{O}_X \subset \mathcal{D}_X$.

Note that the association $\alpha \mapsto \mathcal{D}_{X,\alpha}$ defines a map $c_X \colon H^0(X^{(1)}, \Omega^1_{X^{(1)}}) \to \mathrm{Br}(X^{(1)})$. For now we only know that it is a well-defined map of sets.

2.1.5 Azumaya algebras $\mathcal{D}_{X,\alpha}$ and p-curvature

As we have seen, the algebras $\mathcal{D}_{X,\alpha}$ are some objects of differential-geometric nature which exist exclusively in characteristic p. There is another object of this sort, namely the *p-curvature* of a flat connection on X. In this subsection we are going to relate them to each other.

Let $\mathcal{E}$ be a quasi-coherent sheaf on X and let $\nabla \colon \mathcal{E} \to \mathcal{E} \otimes \Omega^1_X$ be a connection on $\mathcal{E}$, which by deinition means that locally for any $s \in \mathcal{E}(U)$ and $f \in \mathcal{O}_X(U)$ one has $\nabla(fs) = s \otimes df + f\nabla(s)$. Note that ∇ is not $\mathcal{O}_X$-linear, but the difference of two connections $\nabla_1 - \nabla_2$ is, so connections on $\mathcal{E}$ form a torsor over the sheaf of $\mathcal{O}_X$-linear maps from $\mathcal{E}$ to $\mathcal{E} \otimes \Omega^1_X$. Dually a connection ∇ defines an $\mathcal{O}_X$-linear map $\nabla \colon \mathcal{T}_X \to \underline{\mathrm{End}}_{\mathsf{k}}(\mathcal{E})$, and $\nabla_1 - \nabla_2$ lands in $\underline{\mathrm{End}}_{\mathcal{O}_X}(\mathcal{E})$ for any ∇_1, ∇_2.

Sheaves $\mathcal{T}_X$ and $\underline{\mathrm{End}}_{\mathsf{k}}(\mathcal{E})$ have a natural structure of a sheaf of Lie algebras of k-vector spaces. This structure is given by the commutator of vector fields for $\mathcal{T}_X$ and the commutator $[a, b] = ab - ba$ for $\underline{\mathrm{End}}_{\mathsf{k}}(\mathcal{E})$. The classical curvature $\mathrm{curv}(\nabla) \colon \bigwedge^2_{\mathcal{O}_X} \mathcal{T}_X \to \underline{\mathrm{End}}_{\mathsf{k}}(\mathcal{E})$ measures how far ∇ is from being a homomorphism of Lie algebras, namely

$$\mathrm{curv}(\nabla)(\partial_1, \partial_2) = [\nabla(\partial_1), \nabla(\partial_2)] - \nabla([\partial_1, \partial_2]).$$

A connection ∇ is called *integrable* or *flat* if the map $\mathrm{curv}(\nabla)$ is identically equal to 0. In this case $\nabla \colon \mathcal{T}_X \to \underline{\mathrm{End}}_{\mathsf{k}}(\mathcal{E})$ is a Lie algebra homomorphism.

In characteristic p both $\mathcal{T}_X$ and $\underline{\mathrm{End}}_{\mathsf{k}}(\mathcal{E})$ possess a richer structure of p-th restricted Lie algebra (see Sect. 3.1 in [23]), given by $\bullet^{[p]}$ and $\bullet^p$ correspondingly. Analogously to the classical case, the *p-curvature* map $\mathrm{curv}_p(\nabla) \colon \mathcal{T}_{X^{(1)}} \to (\mathrm{Fr}_X)_* \underline{\mathrm{End}}_{\mathsf{k}}(\mathcal{E})$ measures how much ∇ fails to preserve this operation. It is defined by the following formula:

$$\mathrm{curv}_p(\nabla)(\partial^{(1)}) = \nabla(\partial)^p - \nabla(\partial^{[p]}).$$

By the same computation as before ([69], Proposition 5.3), $\mathrm{curv}_p(\nabla)$ is p-linear: $\mathrm{curv}_p(\nabla)(f^{(1)}\partial^{(1)}) = f^p\mathrm{curv}_p(\nabla)(\partial^{(1)})$.

Another feature of characteristic p is that any connection $\nabla\colon \mathcal{E} \to \mathcal{E}\otimes\Omega^1_X$ is in fact $\mathcal{O}_{X^{(1)}}$-linear since $d(f^p) = 0$. So, $\nabla\colon \mathcal{T}_X \to \underline{\mathrm{End}}_{\mathsf{k}}(\mathcal{E})$ actually lands in $\mathcal{O}_{X^{(1)}}$-linear endomorphisms $\underline{\mathrm{End}}_{\mathcal{O}_{X^{(1)}}}((\mathrm{Fr}_X)_*\mathcal{E})$. It follows that p-curvature lands in $\underline{\mathrm{End}}_{\mathcal{O}_{X^{(1)}}}((\mathrm{Fr}_X)_*\mathcal{E})$ too. However, it is true that $\mathrm{curv}_p(\nabla)(\partial^{(1)})$ is even $\mathcal{O}_X$-linear:

Lemma 2.1.5.1. *p-curvature $\mathrm{curv}_p(\nabla)$ lands in $\mathcal{O}_X$-linear endomorphisms of $\mathcal{E}$. This way it defines a map $\mathrm{curv}_p(\nabla)\colon \mathcal{T}_X^{(1)} \to (\mathrm{Fr}_X)_*\underline{\mathrm{End}}_{\mathcal{O}_X}(\mathcal{E})$.*

Proof. By the Leibniz rule,

$$\nabla(\partial)^p(f\cdot e) = \partial^p(f)\cdot e + f\cdot\nabla(\partial)^p(e).$$

On the other hand,

$$\nabla(\partial^{[p]})(f\cdot e) = \partial^{[p]}(f)\cdot e + f\cdot\nabla(\partial^{[p]})(e).$$

Now, since $\partial^p(f) = \partial^{[p]}(f)$, subtracting second equality from the first one, we get that

$$\nabla(\partial)^p(f\cdot e) - \nabla(\partial^{[p]})(f\cdot e) = f\cdot(\nabla(\partial)^p(e) - \nabla(\partial^{[p]})(e))$$

or, in other words,

$$\mathrm{curv}_p(\nabla)(\partial^{(1)})(f\cdot e) = f\cdot\mathrm{curv}_p(\nabla)(\partial^{(1)})(e).$$

□

Remark 2.1.5.2. In particular, if $\mathcal{E}$ is a line bundle, $\underline{\mathrm{End}}_{\mathcal{O}_X}(\mathcal{E}) = \mathcal{O}_X$, and $\mathrm{curv}_p(\nabla)$ defines a map $\mathrm{curv}_p(\nabla)\colon \mathcal{T}_{X^{(1)}} \to (\mathrm{Fr}_X)_*\mathcal{O}_X$. Moreover, one can show (see Proposition 5.2, [69]) that p-curvature is not only $\mathcal{O}_X$-linear, but is also flat with respect to the induced connection ∇_{End} on $\underline{\mathrm{End}}_{\mathcal{O}_X}(\mathcal{E})$. In the case when $\mathcal{E}$ is a line bundle, this connection is trivial, namely $\nabla_{\mathrm{End}}(\partial) = \theta(\partial)$, where $\theta(\partial)\colon \mathcal{O}_X \to \mathcal{O}_X$ is the derivation corresponding to ∂. The flat sections are given by $\mathcal{O}_{X^{(1)}} \subset \mathcal{O}_X$, and so p-curvature defines a map $\mathrm{curv}_p(\nabla)\colon \mathcal{T}_{X^{(1)}} \to \mathcal{O}_{X^{(1)}}$. Such a map is the same as a differential 1-form on $X^{(1)}$, which in this case we will call $\mathrm{curv}_p(\nabla)$ as well.

Let $MIC(X)$ be the category of pairs $(\mathcal{E}, \nabla)$, where $\mathcal{E}$ is a quasi-coherent sheaf on X and ∇ is an integrable connection on $\mathcal{E}$. ∇ defines a natural $\mathcal{D}_X$-action on $\mathcal{E}$, namely functions act by multiplication and vector fields act through ∇: $\partial \mapsto \nabla(\partial) \in \mathrm{End}_{\mathcal{O}_{X^{(1)}}}(\mathcal{E}) \subset \mathrm{End}_{\mathsf{k}}(\mathcal{E})$. The integrability of the connection guarantees that this action is well-defined. As was mentioned before, the converse is also true: an action of $\mathcal{D}_X$ which extends the $\mathcal{O}_X$-module structure on $\mathcal{E}$ endows $\mathcal{E}$ with a flat connection ($\nabla(\partial)$ acts as ∂). In other words we have an equivalence of categories between $MIC(X)$ and the category of $\mathcal{O}_X$-quasi-coherent $\mathcal{D}_X$-modules. It is easy to see from this point of view that $\mathrm{curv}_p(\nabla)(\partial^{(1)})$ is just equal to $\nabla(\iota(\partial)) = \nabla(\partial^p - \partial^{[p]})$, where $\iota(\partial) = \partial^p - \partial^{[p]} \in \mathcal{D}_X$.

The p-curvature is a local invariant of the integrable connection. One has the following description of the situation when $\mathrm{curv}_p(\nabla) = 0$:

Theorem 2.1.5.3 (Cartier, [69], Theorem 5.1)**.** *Let X be a smooth scheme of finite type over k. Let $MIC(X)$ denote the category of quasi-coherent sheaves with integrable connection $(\mathcal{E}, \nabla)$. Remember also that we have the relative Frobenius morphism $\mathrm{Fr}_X \colon X \to X^{(1)}$ to the Frobenius twist of X. There is an equivalence of categories between the category of quasi-coherent sheaves on $X^{(1)}$ and the subcategory of $MIC(X)$ consisting of pairs $(\mathcal{E}, \nabla)$, such that $\mathrm{curv}_p(\nabla)$ is equal to 0.*

Then the natural question is: what do non-zero p-curvatures correspond to? We can formulate an answer in terms of algebras $\mathcal{D}_{X,\alpha}$.

A differential 1-form $\alpha \in \Omega^1_{X^{(1)}}$ gives a map $\tilde{\alpha} \colon \mathcal{T}_X^{(1)} \to \mathcal{O}_{X^{(1)}}$. For any quasi-coherent sheaf $\mathcal{E}$ on X we have a natural map $\mathcal{O}_{X^{(1)}} \to (\mathrm{Fr}_X)_* \mathcal{O}_X \to (\mathrm{Fr}_X)_* \underline{\mathrm{End}}_{\mathcal{O}_X}(\mathcal{E})$. Taking the composition, we obtain a well-defined map $\tilde{\alpha} \colon \mathcal{T}_X^{(1)} \to (\mathrm{Fr}_X)_* \underline{\mathrm{End}}_{\mathcal{O}_X}(\mathcal{E})$ for any quasi-coherent sheaf $\mathcal{E}$. Let's denote by $MIC_\alpha(X)$ the full subcategory of $MIC(X)$ consisting of pairs $(\mathcal{E}, \nabla)$, such that $\mathrm{curv}_p(\nabla) = \tilde{\alpha}$.

Proposition 2.1.5.4. *There is an equivalence of categories between the category $MIC_{\tilde{\alpha}}(X)$ and the category of $\mathcal{D}_{X,\alpha}$-modules that are $\mathcal{O}_X$-quasi-coherent.*

Proof. Recall the presentation of $\mathcal{D}_{X,\alpha}$ as a quotient of $\mathcal{D}_X$. Thus $\mathcal{D}_{X,\alpha}$-modules are $\mathcal{D}_X$-modules for which the action map factorises through $\mathcal{D}_{X,\alpha}$. So it is enough to check that $(\mathcal{E}, \nabla) \in MIC_{\tilde{\alpha}}(X)$ if and only if $\partial^p - \partial^{[p]} - \langle \partial^{(1)}, \alpha \rangle$ acts as 0 (under the $\mathcal{D}_X$-action on $\mathcal{E}$ defined by ∇) for any $\partial \in \mathcal{T}_X$. But

$$\nabla(\partial^p - \partial^{[p]} - \langle \partial^{(1)}, \alpha \rangle) = \mathrm{curv}_p(\nabla)(\partial^{(1)}) - \langle \partial^{(1)}, \alpha \rangle.$$

This expression is equal to 0 for any ∂ if and only if $\mathrm{curv}_p(\nabla) = \tilde{\alpha}$. Note that here again we have $\langle \partial^{(1)}, \alpha \rangle \in \mathcal{O}_{X^{(1)}} = (\mathcal{O}_X)^p \subset \mathcal{O}_X$. □

Remark 2.1.5.5. In fact this proposition can be considered as a direct generalization of Cartier's theorem (see Remark 2.1.5.8).

Remark 2.1.5.6. Let $(\mathcal{E}, \nabla)$ be a line bundle with a flat connection. In local coordinates $\mathcal{E} \simeq \mathcal{O}_X$, and a connection ∇ is given by a closed 1-form ω, namely $\nabla = d + \omega$. Then the explicit formula for the p-curvature is given by $\mathrm{curv}_p(d + \omega) = \omega^{(1)} - \mathsf{C}(\omega)$.

Let's fix $\alpha \in \Omega^1_{X^{(1)}}$ and assume that $\mathrm{curv}_p(\nabla)$ is equal to α. By Proposition 2.1.5.4 we have a $\mathcal{D}_{X,\alpha}$-action on $\mathcal{E}$, or, in other words, a homomorphism $\theta_\nabla \colon \mathcal{D}_{X,\alpha} \to \mathrm{End}_{\mathcal{O}_{X^{(1)}}}((\mathrm{Fr}_X)_*\mathcal{E})$. By Proposition 2.1.3.1, $\mathcal{D}_{X,\alpha}$ has rank $p^{2\dim X}$ over $\mathcal{O}_{X^{(1)}}$. Moreover, since $\mathcal{D}_{X,\alpha}$ is an Azumaya algebra its fiber at each point is a matrix algebra of size $p^{\dim X}$. Bundle $(\mathrm{Fr}_X)_*\mathcal{E}$ has rank $p^{\dim X}$, so $\mathrm{End}_{\mathcal{O}_{X^{(1)}}}((\mathrm{Fr}_X)_*\mathcal{E})$ has rank $p^{2\dim X}$, and the fiber at each point is a matrix algebra of the same size. Map θ_∇ is an isomorphism if and only if it is an isomorphism on each fiber, which is necessarily the case, since the matrix algebra is simple. This gives an isomorphism $\theta_\nabla \colon \mathcal{D}_{X,\alpha} \to \mathrm{End}_{\mathcal{O}_{X^{(1)}}}((\mathrm{Fr}_X)_*\mathcal{E})$, or in other words $(\mathrm{Fr}_X)_*\mathcal{E}$ is a splitting bundle for the Azumaya algebra $\mathcal{D}_{X,\alpha}$. On the other hand, if we have a splitting $\theta \colon \mathcal{D}_{X,\alpha} \to \mathrm{End}_{\mathcal{O}_{X^{(1)}}}(\mathcal{E}')$ we get an embedding $(\mathrm{Fr}_X)_*\mathcal{O}_X \hookrightarrow \mathrm{End}_{\mathcal{O}_{X^{(1)}}}(\mathcal{E}')$ which shows that $\mathcal{E}'$ is in fact of the form $(\mathrm{Fr}_X)_*\mathcal{E}$ for some line bundle $\mathcal{E}$ on X (namely $\mathcal{E}$ is just $\mathcal{E}'$ considered as a module over $\mathcal{O}_X$). The restriction of θ to $(\mathrm{Fr}_X)_*\mathcal{T}_X$ provides $\mathcal{E}$ with an integrable connection with the p-curvature α.

To summarize the discussion, let us introduce two categories. Let $\mathrm{Spl}(X, \alpha)$ be the category of splittings of $\mathcal{D}_{X,\alpha}$, namely by definition

- objects of $\mathrm{Spl}(X, \alpha)$ are pairs $(\mathcal{E}', \theta)$, consisting of a vector bundle $\mathcal{E}'$ on $X^{(1)}$ and an isomorphism $\theta \colon \mathcal{D}_{X,\alpha} \to \mathrm{End}_{\mathcal{O}_{X^{(1)}}}(\mathcal{E}')$,
- morphisms in $\mathrm{Spl}(X, \alpha)$ are usual morphisms between vector bundles that respect the splittings. Note that such a morphism is necessarily an isomorphism.

The second category $LIC(X, \alpha)$ is the category of line bundles with an integrable connection with p-curvature α, namely

- objects of $LIC(X, \alpha)$ are pairs $(\mathcal{E}, \nabla)$, consisting of a line bundle $\mathcal{E}$ on X and an integrable connection ∇, such that $\mathrm{curv}_p(\nabla) = \alpha$,

- morphisms in $LIC(X,\alpha)$ are isomorphisms of line bundles which respect the connections.

We get the following

Proposition 2.1.5.7. *The category* $\mathrm{Spl}(X,\alpha)$ *is equivalent to* $LIC(X,\alpha)$ *for any smooth variety* X *and any differential 1-form* α *on* $X^{(1)}$.

Remark 2.1.5.8. In particular taking $\alpha = 0$ we get that $(\mathcal{O}_X, d)$ produces a splitting bundle for $\mathcal{D}_{X,0}$, namely we have an isomorphism $\theta_d\colon \mathcal{D}_{X,0} \xrightarrow{\sim} \mathrm{End}_{\mathcal{O}_{X^{(1)}}}((\mathrm{Fr}_X)_*\mathcal{O}_X)$. So we see that $\mathcal{D}_{X,0}$ is canonically split for any X and is Morita-equivalent to $\mathcal{O}_{X^{(1)}}$. This canonical Morita-equivalence is given by tensor product of a quasi-coherent sheaf on $X^{(1)}$ with $(\mathrm{Fr}_X)_*\mathcal{O}_X$, where the $\mathcal{D}_{X,0}$-module structure is given by d.

2.1.6 The gerbes $\mathcal{S}(X,\alpha)$ and $\mathcal{L}(X,\alpha)$

Gerbes are 2-categorical analogues of principal G-bundles. They provide a geometrization of certain cohomology classes and, similar to the situation with principal bundles, we can glue them. $\mathbb{G}_m$-gerbes play a rather important role in classical differential geometry, they serve as geometrizations (categorifications) of certain interesting characteristic classes (lying in $H^3(X,\mathbb{Z})$). $\mathbb{G}_m$-gerbes show up naturally in our story as well, namely the data of $\mathrm{Spl}(U,\alpha)$ and $LIC(U,\alpha)$ for étale open $U \to X$ glues exactly into a $\mathbb{G}_m$-gerbe.

Let's fix a sheaf of abelian groups μ on $X_{\text{ét}}$. We look at stacks as categories fibered over $X_{\text{ét}}$ in groupoids (satisfying the descent properties). In particular if we have an object $S \in \mathcal{G}(U)$ for a stack $\mathcal{G}$ then we have a well-defined sheaf of groups $\underline{\mathrm{Aut}}_S$ on $U_{\text{ét}}$. Let $p\colon \mathcal{G} \to X_{\text{ét}}$ denote the natural projection functor, explicitely for any object $S \in \mathcal{G}(U)$ we have $p(S) = U$.

Definition 2.1.6.1. A *μ-gerbe* is a data of a stack in groupoids $\mathcal{G}$ over $X_{\text{ét}}$ and an isomorphism $\iota_S\colon \mu|_{p(S)} \xrightarrow{\sim} \underline{\mathrm{Aut}}_S$ for each $S \in \mathcal{G}$ such that

- $\mathcal{G}$ is locally non-empty;
- Any two objects in $\mathcal{G}$ are locally isomorphic;
- For a morphism $f\colon T \to S$ in $\mathcal{G}$ let $p(f)\colon p(T) \to p(S)$ be the image of f under p. Then for

any $f\colon T \to S$ the following diagram should commute

$$\begin{array}{ccc} p(f)^*\left(\mu|_{p(S)}\right) & \xrightarrow{p(f)^*} & \mu|_{p(T)} \\ \downarrow{\scriptstyle \iota_s} & & \downarrow{\scriptstyle \iota_t} \\ f^*\underline{\mathrm{Aut}}_S & \xrightarrow{f^*} & \underline{\mathrm{Aut}}_T \end{array}$$

Given a 1-form α on $X^{(1)}$, to each $U \in X_{\text{ét}}$ we can associate two groupoids: $\mathrm{Spl}(U, \alpha|_U)$ and $LIC(U, \alpha|_U)$. We will show that this data defines two fibered categories $\mathcal{S}(X, \alpha)$ and $\mathcal{L}(X, \alpha)$ which are in fact $\mathbb{G}_m$-gerbes.

Let $f\colon U \to X$ be an étale open in $X_{\text{ét}}$. Since f is étale, it induces an isomorphism $f^*\Omega^1_X \xrightarrow{\sim} \Omega^1_U$ and this way provides a map $f\colon T^*U \to T^*X$ between total spaces of cotangent bundles. Abusing notation let $f\colon T^*U^{(1)} \to T^*X^{(1)}$ also denote the corresponding map between the Frobenius twists.

Lemma 2.1.6.2. *Let $f\colon U \to X$ be an étale morphism. Then $f^*\mathcal{D}_X$ is canonically isomorphic to $\mathcal{D}_U$.*

Proof. We use the presentation of $\mathcal{D}_U$ as universal enveloping algebras of the Lie algebroid $\mathcal{T}_U$ of vector fields. The statement of the lemma then follows immediately from the isomorphism $\mathcal{T}_U \xrightarrow{\sim} f^*\mathcal{T}_V$ and functoriality of Frobenius. □

Corollary 2.1.6.3. *Let $f\colon U \to X$ be an étale open and let α be a differential 1-form on $X^{(1)}$. Then $f^*\mathcal{D}_{X,\alpha}$ is canonically isomorphic to $\mathcal{D}_{U,f^*\alpha}$.*

Proof. This follows from a simple observation that $f^{-1}(\Gamma_\alpha) = \Gamma_{f^*\alpha}$. □

Remark 2.1.6.4. In what follows we identify the small étale sites $X_{\text{ét}} \simeq X^{(1)}_{\text{ét}}$ via a map sending $(f\colon U \to X) \in X_{\text{ét}}$ to $(f\colon U^{(1)} \to X^{(1)}) \in X^{(1)}_{\text{ét}}$

Let now $g\colon V \to X$ and $f\colon U \to X$ be two objects in X_{et}, and $h\colon V \to U$ be a morphism between them. Such a morphism produces natural functors $\mathcal{S}(h)\colon \mathrm{Spl}(U, f^*\alpha) \to \mathrm{Spl}(V, g^*\alpha)$ and $\mathcal{L}(h)\colon LIC(U, f^*\alpha) \to LIC(V, g^*\alpha)$. Indeed, if $(\mathcal{E}, \theta)$ is a splitting bundle for $\mathcal{D}_{U,f^*\alpha}$, then $(h^*\mathcal{E}, h^*\theta)$ is a splitting bundle for $h^*\mathcal{D}_{V,h^*(f^*\alpha)}$, which by Corollary 2.1.6.3 is canonically isomorphic to $\mathcal{D}_{V,h^*(f^*\alpha)} = \mathcal{D}_{V,g^*\alpha}$. If now we have an object $(\mathcal{E}, \nabla)$ of $LIC(U, f^*\alpha)$, then

$$\begin{aligned} \mathrm{curv}_p(h^*\nabla)(\partial) = (h^*\nabla(\partial))^p - h^*\nabla(\partial^{[p]}) = \nabla(h_*\partial)^p - \nabla(h_*(\partial^{[p]})) = \\ = \nabla(h_*\partial)^p - \nabla((h_*\partial)^{[p]}) = \mathrm{curv}_p(\nabla)(h_*\partial) = f^*\alpha(h_*\partial) = h^*(f^*\alpha)(\partial) = g^*\alpha(\partial), \end{aligned}$$

so $\mathrm{curv}_p(h^*\nabla) = h^*\alpha$, and $(h^*(\mathcal{E}), h^*\nabla)$ defines an object of $LIC(V, g^*\alpha)$. It is easy to see that these maps on objects extend to honest functors, and that the data $\{U \mapsto \mathrm{Spl}(U, f^*\alpha)\}$, $\{U \mapsto LIC(U, f^*\alpha)\}$ defines two presheaves of groupoids (or equivalently categories fibered in groupoids over $X_{\text{ét}}$), which we call $\mathcal{S}(X, \alpha)$ and $\mathcal{L}(X, \alpha)$ correspondingly. The fact that $\mathcal{S}(X, \alpha)$ and $\mathcal{L}(X, \alpha)$ are in fact sheaves of groupoids follows from faithfully flat descent.

It remains to construct isomorphisms $\iota_S \colon \mathbb{G}_m \to \underline{\mathrm{Aut}}_S$. Let $(\mathcal{E}, \theta) \in \mathcal{S}(X, \alpha)(U)$ be a splitting bundle for $f^*\mathcal{D}_{X,\alpha} \simeq \mathcal{D}_{U,f^*\alpha}$ on $f \colon U \to X$, namely we have $\theta \colon \mathcal{D}_{U,f^*\alpha} \simeq \mathrm{End}_{\mathcal{O}_{U^{(1)}}}((\mathrm{Fr}_X)_*\mathcal{E})$ for some étale cover $f \colon U \to X$. The automorphisms of such splitting bundle are given by maps $(\mathrm{Fr}_X)_*\mathcal{E} \to (\mathrm{Fr}_X)_*\mathcal{E}$ which are invertible and induce a trivial map on $\mathrm{End}_{\mathcal{O}_{U^{(1)}}}((\mathrm{Fr}_X)_*\mathcal{E})$. Restricting further to some cover $V \to U$ on which $\mathcal{E}$ trivializes we see that locally these automorphisms are given by the kernel of the map $\mathrm{GL}_{p^{\dim U}}(\mathcal{O}_{U^{(1)}}) \to \mathrm{PGL}_{p^{\dim X}}(\mathcal{O}_{U^{(1)}})$ which is equal to $\mathbb{G}_m(\mathcal{O}_{U^{(1)}})$ identified with the scalar matrices inside $\mathrm{GL}_{p^{\dim U}}(\mathcal{O}_{U^{(1)}})$. The natural map $\mathbb{G}_m \to \mathrm{End}_{\mathcal{O}_{U^{(1)}}}((\mathrm{Fr}_X)_*\mathcal{E})$ then gives the desired isomorphism $\iota_{(\mathcal{E},\theta)} \colon \mathbb{G}_m \to \underline{\mathrm{Aut}}_{(\mathcal{E},\theta)}$.

On the line bundles' side, given any $(\mathcal{E}, \nabla) \in \mathcal{S}(X, \alpha)(U)$ for $f \colon U \to X$ we can restrict further to some cover $V \to U$ such that $(\mathcal{E}, \nabla)|_V$ is isomorphic to $(\mathcal{O}_U, d+\omega)$ for some closed 1-form ω. Since $\mathrm{curv}_p(\nabla) = f^*\alpha$, ω also should satisfy a condition $\omega^{(1)} - \mathsf{C}(\omega) = f^*\alpha$. An automorphism of $(\mathcal{E}, \nabla)$ is given by an automorphism of $\mathcal{E}$, preserving ∇. If $f \colon \mathcal{O}_U \xrightarrow{\sim} \mathcal{O}_U$ is an automorphism of $\mathcal{O}_U$ given by an invertible function $f \in \mathcal{O}_X^\times$, then $f^*(d+\omega) = d + d\log f + \omega$. The kernel of $d\log \colon \mathcal{O}_U^\times \to \Omega^1_U$ is given by $(\mathcal{O}_U^\times)^p = \mathcal{O}^\times_{U^{(1)}} \subset \mathcal{O}_U^\times$ which can be identified with $\mathbb{G}_m(\mathcal{O}_{U^{(1)}}) \subset \mathrm{End}_{\mathcal{O}_{U^{(1)}}}((\mathrm{Fr}_U)_*\mathcal{O}_U)$. This shows that a map $\mathbb{G}_m \to \underline{\mathrm{Aut}}_{(\mathcal{E},\nabla)}$ given by action of $\mathbb{G}_m \subset \mathrm{End}((\mathrm{Fr}_U)_*\mathcal{E})$ is an isomorphism as well.

The functoriality of maps ι under pull-backs can be easily checked as well and is left as an exercise to the reader

In this way, to a differential 1-form α on $X^{(1)}$, we can associate two $\mathbb{G}_m$-gerbes: $\mathcal{S}(X, \alpha)$ and $\mathcal{L}(X, \alpha)$. It follows from Proposition 2.1.5.7 that they are actually isomorphic:

Proposition 2.1.6.5. *There is an equivalence of gerbes $\mathcal{S}(X, \alpha) \simeq \mathcal{L}(X, \alpha)$ for any smooth X and any differential 1-form α on $X^{(1)}$.*

2.1.7 Additivity of the map c_X: cohomology

Here we present a more classical point of view on the map c_X. Namely we will obtain the homomorphism $c_X \colon H^0(X^{(1)}_{\text{ét}}, \Omega^1_{X^{(1)}}) \to H^2(X^{(1)}_{\text{ét}}, \mathcal{O}^\times_{X^{(1)}})$ from a map $\Omega^1_{X^{(1)}} \to \mathcal{O}^\times_{X^{(1)}}[2]$ in the derived

category of étale sheaves on X. Such a map is the same as an element of $\mathrm{Ext}^2(\Omega^1_{X^{(1)}}, \mathcal{O}^\times_{X^{(1)}})$.

Proposition 2.1.7.1. *There is a 4-term exact sequence*

$$0 \to \mathcal{O}^\times_{X^{(1)}} \xrightarrow{\mathrm{Fr}^*_X} \mathcal{O}^\times_X \xrightarrow{d\log} \Omega^1_{X,\mathrm{cl}} \xrightarrow{\bullet^{(1)} - \mathsf{C}} \Omega^1_{X^{(1)}} \to 0.$$

Proof. For the usual proof not using gerbes or differential operators see ([87], Proposition 4.14). We will deduce the exactness from Proposition 2.1.3.1 and Proposition 2.1.5.7. The exactness in the second term is obvious: étale locally (as well as globally) $d\log f = 0 \Leftrightarrow df = 0 \Leftrightarrow f \in \mathcal{O}^\times_{X^{(1)}}$. For the third term we need to prove that locally $\omega^{(1)} - \mathsf{C}(\omega) = 0 \Leftrightarrow \omega = d\log f$. As we have seen, the condition $\omega^{(1)} - \mathsf{C}(\omega) = 0$ for some cover U, means that $(\mathcal{O}_U, d + \omega)$ has p-curvature 0 and is a splitting bundle for $\mathcal{D}_{U,0}$. So, after refining the cover, it should be isomorphic to $(\mathcal{O}_U, d)$, and this means exactly that $\omega = d\log f$ for some function f on U. It remains only to prove that the map $\bullet^{(1)} - \mathsf{C}$ is surjective. Let α be a differential 1-form on a cover $U^{(1)} \to X^{(1)}$. Then, by the Azumaya property, there is a refinement $V^{(1)} \to U^{(1)}$, such that the pull-back of $\mathcal{D}_{X,\alpha}$ is split. So there is a line bundle with a flat connection $(\mathcal{E}, \nabla)$ on V, such that $\mathrm{curv}_p(\nabla) = \alpha$. Refining the cover, we can assume that $(\mathcal{E}, \nabla) = (\mathcal{O}_U, d + \omega)$, and so $\omega^{(1)} - \mathsf{C}(\omega) = \alpha$. □

By Yoneda's presentation for Ext-groups, such a sequence provides a class in $\mathrm{Ext}^2(\Omega^1_{X^{(1)}}, \mathcal{O}^\times_{X^{(1)}})$ which produces a map $\bar{c}_X \colon H^0(X^{(1)}_{et}, \Omega^1_{X^{(1)}}) \to H^2(X^{(1)}_{et}, \mathcal{O}^\times_{X^{(1)}})$. The image is contained in p-torsion, so c_X lands in $\mathrm{Br}(X^{(1)})[p]$. We would like to know that $\bar{c}_X$ is the same as the map c_X that we defined before.

2.1.8 Additivity of the map c_X: Picard groupoids

Let $f \colon U \to X$ be an étale open and let $(\mathcal{E}_1, \nabla_1)$, $(\mathcal{E}_2, \nabla_2)$ be line bundles with integrable connections on U. Their tensor product $\mathcal{E}_1 \otimes \mathcal{E}_2$ is endowed with a natural flat connection $\nabla \colon \mathcal{E}_1 \otimes \mathcal{E}_2 \to \mathcal{E}_1 \otimes \mathcal{E}_2 \otimes \Omega^1_X$ given by the formula $\nabla = \nabla_1 \otimes \mathrm{id} + \mathrm{id} \otimes \nabla_2$ or in dual presentation the map $\nabla \colon \mathcal{T}_{X^{(1)}} \to \mathrm{End}_{\mathcal{O}_{U^{(1)}}}(\mathcal{E}_1 \otimes \mathcal{E}_2)$ is given by $\partial \mapsto \nabla_1(\partial) \otimes \mathrm{id} + \mathrm{id} \otimes \nabla_2(\partial)$. The p-curvature of the tensor product is easy to compute, namely:

$$\mathrm{curv}_p(\nabla)(\partial) = \nabla(\partial)^p - \nabla(\partial^{[p]}) = (\nabla_1(\partial) \otimes \mathrm{id} + \mathrm{id} \otimes \nabla_2(\partial))^p - \nabla_1(\partial^{[p]}) \otimes \mathrm{id} - \mathrm{id} \otimes \nabla_2(\partial^{[p]}) =$$
$$= \nabla_1(\partial)^p \otimes \mathrm{id} - \nabla_1(\partial^{[p]}) \otimes \mathrm{id} + \mathrm{id} \otimes \nabla_2(\partial)^p - \mathrm{id} \otimes \nabla_2(\partial^{[p]}) = \mathrm{curv}_p(\nabla_1)(\partial) + \mathrm{curv}_p(\nabla_2)(\partial).$$

In particular if $\mathcal{E}_1$, $\mathcal{E}_2$ were line bundles and $\mathrm{curv}_p(\nabla_1) = \alpha_1$, $\mathrm{curv}_p(\nabla_2) = \alpha_2$, then $\mathrm{curv}_p(\nabla) = \alpha_1 + \alpha_2$. Note that this way $(\mathcal{E}_1, \nabla_1) \otimes (\mathcal{E}_2, \nabla_2) \coloneqq (\mathcal{E}_1 \otimes \mathcal{E}_2, \nabla)$ is a splitting bundle for $\mathcal{D}_{X,\alpha_1+\alpha_2}$. In local coordinates, $(\mathcal{O}_U, d + \omega_1) \otimes (\mathcal{O}_U, d + \omega_2) = (\mathcal{O}_U, d + \omega_1 + \omega_2)$.

The operation of tensor product induces a map of fibered categories $\otimes\colon \mathcal{L}(X, \alpha_1) \times \mathcal{L}(X, \alpha_2) \to \mathcal{L}(X, \alpha_1 + \alpha_2)$.

We see that the natural operation of the tensor product adds p-curvatures, so to have such an operation available we need to consider $\mathcal{L}(X, \alpha)$ for all α at the same time. Namely we define the stack of groupoids $\mathcal{L}(X)$ by $\mathcal{L}(X)(U) = \sqcup_\alpha \mathcal{L}(X, \alpha)(U)$. It is then endowed with a symmetric monoidal structure $\otimes\colon \mathcal{L}(X) \times \mathcal{L}(X) \to \mathcal{L}(X)$, which turns it into a *stack of Picard groupoids* or a *Picard stack*. In our treatment of Picard stacks we follow the classical source ([34], 1.4). In short a *Picard groupoid* is a groupoid $\mathcal{P}$ endowed with a functor $+\colon \mathcal{P} \times \mathcal{P} \to \mathcal{P}$, together with functorial isomorphisms $\tau\colon X + Y \xrightarrow{\sim} Y + X$, $\sigma\colon (X + Y) + Z \xrightarrow{\sim} X + (Y + Z)$ which turn "+" into an *associative and strictly commutative functor* (following the terminology of *op.cit.*), and such that the functor $X \mapsto X + Y$ is an isomorphism for all $Y \in Ob(\mathcal{P})$. The typical example of a Picard groupoid is given by $\mathrm{Pic}(X)$: the groupoid of line bundles on X with the "+"-operation given by the tensor product of line bundles.

Every Picard groupoid $\mathcal{P}$ has a unique unit object $e \in Ob(\mathcal{P})$, with an isomorphism $e + e \xrightarrow{\sim} e$ and isomorphisms $e + X \mapsto X$, functorial in X and satisfying some natural axioms. For $\mathrm{Pic}(X)$ unit object is given by the trivial line bundle $\mathcal{O}_X$, in our case of $\mathcal{L}(X)(U)$ by $(\mathcal{O}_U, d)$. The group $\mathrm{Aut}_{\mathcal{P}}(e)$ is abelian and $\mathrm{Aut}_{\mathcal{P}}(X)$ is isomorphic to $\mathrm{Aut}_{\mathcal{P}}(e)$ for any $X \in Ob(\mathcal{P})$, via the functor $+_X\colon Y \mapsto Y + X$ which sends e to X. The set of isomorphism classes of objects of $\mathcal{P}$ under operation $+$ also forms a group, which we call $\mathrm{Iso}(\mathcal{P})$.

Definition 2.1.8.1. A *Picard stack* is a stack of groupoids $\mathcal{P}$ on $X_{\text{ét}}$, endowed with an operation $+\colon \mathcal{P} \times \mathcal{P} \to \mathcal{P}$ together with functorial isomorphisms $\tau\colon X + Y \xrightarrow{\sim} Y + X$, $\sigma\colon (X + Y) + Z \xrightarrow{\sim} X + (Y + Z)$ which turn $\mathcal{P}(U)$ into a Picard groupoid for any $U \in X_{\text{ét}}$.

Since $\mathcal{P}$ is a stack, the presheaf $U \mapsto \mathrm{Aut}_{\mathcal{P}}(e)(U)$ which associates to U the automorphism group of the unit object in $\mathcal{P}(U)$ is actually a sheaf, which we call $\underline{\mathrm{Aut}}_{\mathcal{P}}(e)$. On the other hand the presheaf of isomorphism classes of objects $U \mapsto \mathrm{Iso}(\mathcal{P}(U))$ does not form a sheaf, and we denote by $\underline{\mathrm{Iso}}(\mathcal{P})$ its sheafification.

We denote by $C^{[-1,0]}(X)$ the category of complexes $\mathcal{K}^\bullet$ of sheaves of abelian groups on X_{et}, such that $\mathcal{K}^i = 0$ for $i \notin [-1, 0]$. To every such complex $d\colon \mathcal{K}^{-1} \to \mathcal{K}^0$ we can associate a prestack

of Picard groupoids $\mathrm{pch}(\mathcal{K}^\bullet)$, such that

- for any U in X_{et}, we have $Ob(\mathrm{pch}(\mathcal{K}^\bullet)(U)) = \mathcal{K}^0(U)$,
- for any two $x, y \in \mathcal{K}^0(U)$, a morphism from x to y is given by an element $f \in \mathcal{K}^{-1}(U)$, such that $df = y - x$,
- composition of morphisms is given by the addition in $\mathcal{K}^{-1}(U)$,
- functor $+$ is given by the addition on $\mathcal{K}^{-1}$ and $\mathcal{K}^0$.

The Picard stack $\mathrm{ch}(\mathcal{K}^\bullet)$ is defined as the sheafification of the prestack $\mathrm{pch}(\mathcal{K}^\bullet)$. The sheaf $\mathcal{H}^0(\mathcal{K}^\bullet)$ then can be interpreted as $\underline{\mathrm{Iso}}(\mathrm{ch}(\mathcal{K}^\bullet))$, while $\mathcal{H}^{-1}(\mathcal{K}^\bullet)$ is nothing but the sheaf $\underline{\mathrm{Aut}}_{\mathcal{P}}(e)$. We have a 4-term exact sequence

$$0 \to \mathcal{H}^{-1}(\mathcal{K}^\bullet) \to \mathcal{K}^{-1} \xrightarrow{d} \mathcal{K}^0 \to \mathcal{H}^0(\mathcal{K}^\bullet) \to 0,$$

which can be presented by some class $\gamma \in \mathrm{Ext}^2(\mathcal{H}^0(\mathcal{K}^\bullet), \mathcal{H}^{-1}(\mathcal{K}^\bullet))$. Note that γ produces a natural map $\overline{\gamma}\colon H^0(X_{\text{ét}}, \mathcal{H}^0(\mathcal{K}^\bullet)) \to H^2(X_{\text{ét}}, \mathcal{H}^{-1}(\mathcal{K}^\bullet))$.

Proposition 2.1.8.2 ([34], Proposition 1.4.15)**.** *The association $\mathcal{K}^\bullet \mapsto \mathrm{ch}(\mathcal{K}^\bullet)$ defines an equivalence of categories*

$$\mathrm{ch}\colon D^{[-1,0]}(X_{\text{ét}}) \to \mathsf{PicGpd}(X_{\text{ét}})$$

where $D^{[-1,0]}(X_{\text{ét}})$ is the subcategory of the derived category of sheaves on $X_{\text{ét}}$ with non-trivial cohomology only in degree -1 and 0 and $\mathsf{PicGpd}(X_{\text{ét}})$ is the 1-category underlying the 2-category of Picard stacks on $X_{\text{ét}}$.

Let's now fix a global section $\alpha \in H^0(X_{\text{ét}}, \mathcal{H}^0(\mathcal{K}^\bullet)) = \underline{\mathrm{Iso}}(\mathrm{ch}(\mathcal{K}^\bullet))(X)$ and consider the full substack $\mathrm{ch}_\alpha(\mathcal{K}^\bullet) \subset \mathrm{ch}(\mathcal{K}^\bullet)$, given by the property that objects $\mathrm{ch}_\alpha(\mathcal{K}^\bullet)(U)$ are objects of $\mathrm{ch}(\mathcal{K}^\bullet)(U)$ that are isomorphic to $\alpha|_U$. Substacks $\mathrm{ch}_\alpha(\mathcal{K}^\bullet)$ for all α are the connected components of the Picard stack $\mathrm{ch}(\mathcal{K}^\bullet)$. It is easy to see that by definition each substack $\mathrm{ch}_\alpha(\mathcal{K}^\bullet)$ is an $\mathcal{H}^{-1}(\mathcal{K}^\bullet)$-gerbe. We can associate to it an element in $H^2(X_{\text{ét}}, \mathcal{H}^{-1}(\mathcal{K}^\bullet))$ by taking $\overline{\gamma}(\alpha)$.

Proposition 2.1.8.3 ([94], Theorem 12.2.8)**.** *Let μ be a sheaf of groups on $X_{\text{ét}}$. Then there is a natural bijection*

$$\{\textit{Isomorphism classes of } \mu\textit{-gerbes}\} \longleftrightarrow H^2(X_{\text{ét}}, \mu)$$

Under this bijection given $\mathcal{K}^\bullet$ *the* $\mathcal{H}^{-1}(\mathcal{K}^\bullet)$*-gerbe* $\mathrm{ch}_\alpha(\mathcal{K}^\bullet)$ *goes to* $\overline{\gamma}(\alpha) \in H^2(X_{\text{ét}}, \mathcal{H}^{-1}(\mathcal{K}^\bullet))$.

We denote the cohomology class corresponding to a gerbe $\mathcal{G}$ by $[\mathcal{G}]$.

The complex that we should consider is given by the middle arrow in the 4-term exact sequence from Proposition 2.1.7.1:

$$\mathcal{O}^\bullet := \mathcal{O}_X^\times \xrightarrow{d\log} \Omega^1_{X,cl}.$$

We claim that $\mathrm{ch}(\mathcal{O}^\bullet) = \mathcal{L}(X)$. Indeed, for any $U \in X_{\text{ét}}$ we have a map from $\Omega^1_{X,cl}(U)$ to the set of objects of $\mathcal{L}(X)(U)$ given by $\omega \mapsto (\mathcal{O}_U, d + \omega)$ and, since any line bundle is locally trivial, this map is locally an isomorphism. Isomorphisms between bundles with connection are given by invertible functions f, which change the connection by $d\log f$. Finally, $(\mathcal{O}_U, d+\omega_1)\otimes(\mathcal{O}_U, d+\omega_2) = (\mathcal{O}_U, d + \omega_1 + \omega_2)$, as well as $f_1 \otimes f_2$ acts on the tensor product by $f_1 f_2$, so the tensor product of line bundles corresponds to the addition on the complex $\mathcal{O}^\bullet$. Note that from the 4-term exact sequence we have $\underline{\mathrm{Aut}}(e) = \mathcal{O}^\times_{X^{(1)}}$ and $\underline{\mathrm{Iso}}(\mathcal{L}(X)) = \Omega^1_{X^{(1)}}$, where the last equality is given by the p-curvature. From this we also get that $\mathrm{ch}_\alpha(\mathcal{O}^\bullet) = \mathcal{L}(X, \alpha)$ for any $\alpha \in H^0(X^{(1)}_{et}, \Omega^1_{X^{(1)}})$ and using Proposition 2.1.8.3 that $[\mathcal{L}(X, \alpha)] = \overline{c}_X(\alpha)$.

Proposition 2.1.8.4 ([94], Lemma 12.3.9)**.** *Let* $\mathcal{A}$ *be an Azumaya algebra over* X*. Then the corresponding class* $[\mathcal{A}] \in H^2(X_{\text{ét}}, \mathbb{G}_m)$ *coincides with the class corresponding to the* $\mathbb{G}_m$*-gerbe of splittings* $\mathcal{S}(\mathcal{A})$ *in* $H^2(X_{\text{ét}}, \mathbb{G}_m)$ *via Proposition2.1.8.3.*

Using that $\mathcal{S}(\mathcal{D}_{X,\alpha}) \simeq \mathcal{L}(X, \alpha)$ we get that

$$c_X(\alpha) = [\mathcal{S}(\mathcal{D}_{X,\alpha})] = [\mathcal{L}(X, \alpha)] = \overline{c}_X(\alpha)$$

This proves the following proposition:

Proposition 2.1.8.5. *Let* X *be a smooth scheme over an algebraically closed field* k *of characteristic* p*, and let* $c_X \colon H^0(X^{(1)}, \Omega^1_{X^{(1)}}) \to \mathrm{Br}(X^{(1)})$ *be the map given by* $\alpha \mapsto [\mathcal{D}_{X,\alpha}]$*. Let also* $\overline{c}_X \colon H^0(X^{(1)}, \Omega^1_{X^{(1)}}) \to \mathrm{Br}(X^{(1)})$ *be the map obtained from the 4-term exact sequence*

$$0 \to \mathcal{O}^\times_{X^{(1)}} \xrightarrow{\mathrm{Fr}_X^*} \mathcal{O}_X^\times \xrightarrow{d\log} \Omega^1_{X,\mathrm{cl}} \xrightarrow{\bullet^{(1)} - \mathsf{C}} \Omega^1_{X^{(1)}} \to 0.$$

Then c_X *is equal to* $\overline{c}_X$ *and, consequently,* c_X *is a homomorphism of abelian groups.*

Corollary 2.1.8.6. *Let* $\alpha = \omega^{(1)} - \mathsf{C}(\omega)$*. Then* $c_X(\alpha) = 0$ *and Azumaya algebra* $\mathcal{D}_{X,\alpha}$ *is split.*

Proof. By definition of the map $\overline{c}_X\colon H^0(X^{(1)},\Omega^1_{X^{(1)}})\to \mathrm{Br}(X^{(1)})$, coming from the 4-term exact sequence, we have $\mathrm{Im}(\bullet^{(1)}-\mathsf{C})\subset \mathrm{Ker}(\overline{c}_X)$. The statement of the corollary then follows from $c_X=\overline{c}_X$. It can also be seen directly, since, if $\alpha=\omega^{(1)}-\mathsf{C}(\omega)$, by Proposition 2.1.5.7 $((\mathrm{Fr}_X)_*\mathcal{O}_X, d+\omega)$ is a splitting bundle for $\mathcal{D}_{X,\alpha}$. □

2.2 Descent étale locally

2.2.1 Setup

Now we will concentrate on a very particular question about algebras $\mathcal{D}_{X,\alpha}$, which shows up naturally in quantizations of symplectic resolutions in characteristic p.

Let $\pi\colon X\to Y$ be a resolution of singularities over an algebraically closed field k of characteristic $p>0$, such that $\mathcal{O}_Y\xrightarrow{\sim}\pi_*\mathcal{O}_X$ and $R^i\pi_*\mathcal{O}_X=0$, for $i=1,2$. Let α be a 1-form on $X^{(1)}$ and let $\mathcal{D}_{X,\alpha}$ be the associated Azumaya algebra (see Definition 2.1.4.1).

We would like to investigate the following question: can the class $c_X(\alpha)=[\mathcal{D}_{X,\alpha}]$ in the Brauer group $\mathrm{Br}(X^{(1)})$ of $X^{(1)}$ be descended to a class in $\mathrm{Br}(Y^{(1)})$? In other words, does the class $c_X(\alpha)$ lie in the image of the natural map $\mathrm{Br}(Y^{(1)})\to\mathrm{Br}(X^{(1)})$?

2.2.2 Splitting étale locally on Y

In this subsection we prove that the classes $c_X(\alpha)$ descend to Y étale locally for any α, provided $\mathcal{O}_Y\xrightarrow{\sim}\pi_*\mathcal{O}_X$ and $R^i\pi_*\mathcal{O}_X=0$, for $i=1,2$. By "descends étale locally" we mean that there exists an étale cover $f\colon U\to Y$, such that the pull-back $f^*c_X(\alpha)$ in $\mathrm{Br}((X\times_Y U)^{(1)})$ descends to U, namely $f^*c_X(\alpha)=\pi^*[\mathcal{C}]$ for some Azumaya algebra $\mathcal{C}$ on U. Shrinking U, we can assume that $\mathcal{C}$ is split, so equivalently we can ask if there exists $f\colon U\to Y$, such that the Azumaya algebra $f^*\mathcal{D}_{X,\alpha}=\mathcal{D}_{X\times_Y U,f^*\alpha}$ is split. The idea is to use Corollary 2.1.8.6 and to prove that the operator $\bullet^{(1)}-\mathsf{C}$ is surjective locally on Y (in the same sense).

We start with the following linear algebraic lemma:

Lemma 2.2.2.1. *Let $A,B\colon V\to W$ be two maps between finite-dimensional vector spaces over an algebraically closed field k of characteristic $p>0$, such that A is Frobenius-linear and B is linear and surjective. Then $A-B\colon V\to W$ is surjective as a map of abelian groups.*

Proof. We consider V and W as the sets of k-points of the corresponding affine spaces $\mathbb{A}(V)$ and $\mathbb{A}(W)$. Any linear operator T from V to W induces a natural map $\tilde{T}$ between corresponding affine spaces. There also exists a linear map $A'\colon V \to W$, such that the map $\mathrm{Fr}_W \circ \tilde{A}'\colon \mathbb{A}(V) \to \mathbb{A}(W)$ coincides with A on the level of k-points. Since the differential of the Frobenius map at any point is equal to 0, we get that so does the differential of $\mathrm{Fr}_W \circ \tilde{A}'$. From this we see that $d(\mathrm{Fr}_W \circ \tilde{A}' - \tilde{B}) = -d\tilde{B}$. After identifying the tangent spaces at any point with V or W correspondingly, the differential $d\tilde{B}$ is given by B on the tangent space of any point. We get that $\mathrm{Fr}_W \circ \tilde{A}' - \tilde{B}$ is a submersion and so is an open map. It follows that $\mathrm{Im}(\mathrm{Fr}_W \circ \tilde{A}' - \tilde{B})$ is open in $\mathbb{A}(W)$.

Let $U \subset \mathbb{A}(W)$ be an open set. Then for any closed point $w \in W(\mathsf{k})$ the subset $w - U \subset \mathbb{A}(W)$ is also open and so the intersection $U \cap (w - U)$ is non-empty. In other words w lies in $U(\mathsf{k}) + U(\mathsf{k}) \subset W(\mathsf{k})$. Taking $U = \mathrm{Im}(\tilde{A} - \tilde{B})$, which is closed under addition, we get that $w \in U(\mathsf{k})$, and that $A - B$ is surjective. □

Next lemma proves that the Cartier operator is surjective:

Lemma 2.2.2.2. *Let $\pi\colon X \to Y$ be a proper morphism of algebraic varieties over k, such that X is smooth and $R^i\pi_*\mathcal{O}_X = 0$ for $i = 1, 2$. Then the Cartier operator $\pi_*\mathsf{C}\colon \pi_*\Omega^1_{X,cl} \to \pi_*\Omega^1_{X^{(1)}}$ is surjective.*

Proof. From $R^1\pi_*\mathcal{O}_X = R^2\pi_*\mathcal{O}_X = 0$ and the short exact sequence $0 \to \mathcal{O}_{X^{(1)}} \to \mathcal{O}_X \to \mathcal{O}_X/\mathcal{O}_{X^{(1)}} \to 0$, applying $R^\bullet\pi_*$ we see that $R^1\pi_*(\mathcal{O}_X/\mathcal{O}_{X^{(1)}}) = 0$.

By Section 2.1.2 we also have the following short exact sequence

$$0 \to \mathcal{O}_X/\mathcal{O}_{X^{(1)}} \xrightarrow{d} \Omega^1_{X,\mathrm{cl}} \xrightarrow{\mathsf{C}} \Omega^1_{X^{(1)}} \to 0$$

and so applying $R^\bullet\pi_*$ we get that the map $\pi_*\mathsf{C}\colon \pi_*\Omega^1_{X,cl} \to \pi_*\Omega^1_{X^{(1)}}$ is a surjection. □

Note that if Y is affine, the induced map on global sections $\mathsf{C}\colon H^0(Y^{(1)}, \pi_*\Omega^1_{X,cl}) \to H^0(Y^{(1)}, \pi_*\Omega^1_{X^{(1)}})$ is also a surjection. Note also that the spaces of global sections $H^0(Y^{(1)}, \pi_*\Omega^1_{X,cl})$, $H^0(Y^{(1)}, \pi_*\Omega^1_{X^{(1)}})$ and the map $\bullet^{(1)} - \mathsf{C}$ fit well in the conditions of Lemma 2.2.2.1. Indeed, $\bullet^{(1)}$ is Frobenius-linear, while C is linear and surjective. The only problem is that if Y is affine X is typically not proper, and both spaces $H^0(Y^{(1)}, \pi_*\Omega^1_{X,cl})$ and $H^0(Y^{(1)}, \pi_*\Omega^1_{X^{(1)}})$ are infinite dimensional.

Nevertheless if $Y = \mathrm{Spec}\,\mathsf{k}$, then X is proper and we obtain the following:

Proposition 2.2.2.3. *Let X be a proper smooth variety over k, such that $H^1(X,\mathcal{O}_X) = H^2(X,\mathcal{O}_X) = 0$. Then for any differential 1-form $\alpha \in H^0(X,\Omega^1_X)$ the Azumaya algebra $\mathcal{D}_{X,\alpha}$ is split.*

Proof. We apply Lemma 2.2.2.1 to the map $\bullet^{(1)} - \mathsf{C}\colon H^0(X,\Omega^1_{X,cl}) \to H^0(X^{(1)},\Omega^1_{X^{(1)}})$. The image of this map lies in the kernel of $c_X\colon H^0(X^{(1)},\Omega^1_{X^{(1)}}) \to \mathrm{Br}(X^{(1)})$, so it is enough to prove that $\bullet^{(1)} - \mathsf{C}$ is surjective. Map C is k-linear, $\bullet^{(1)}$ is Frobenius-linear and, by Lemma 2.2.2.1, C is surjective. So we are in the context of Lemma 2.2.2.1, $\bullet^{(1)} - \mathsf{C}$ is surjective, and we are done. □

If Y is non-trivial, we slightly generalise the argument to prove that $\mathcal{D}_{X,\alpha}$ splits on the formal neighbourhood of any fiber of π.

Proposition 2.2.2.4. *Let $\pi\colon X \to Y$ be a proper morphism of algebraic varieties over k, such that X is smooth and $R^i\pi_*\mathcal{O}_X = 0$ for $i = 1,2$. Then for any $\alpha \in H^0(X,\Omega^1_X)$ the algebra $\mathcal{D}_{X,\alpha}$ splits on the formal neighbourhood of any fiber of π.*

Proof. Let's take any closed point $y \in Y(\mathsf{k})$. The question is Zariski local on Y, so we can assume that Y is affine. Let $\widehat{\mathcal{O}}_y$ be the completion of the ring $\Gamma(Y,\mathcal{O}_Y)$ at the point y and let $\widehat{X_y}$ denote the formal neighbourhood of the fiber $\pi^{-1}(y)$. Let also Z denote the formal fiber: $Z = X \times_Y \operatorname{Spec}\widehat{\mathcal{O}}_y$. By the Grothendieck's existence theorem ([38], III, Theorem 5.1.4) one has an equivalence of categories $\mathrm{Coh}(\widehat{X_y}) \simeq \mathrm{Coh}(Z)$. In particular, $\mathcal{D}_{X,\alpha}$ is split on $\widehat{X_y}^{(1)}$ if and only if it is split on $Z^{(1)}$. The Frobenius twist $Z^{(1)}$ is naturally identified with $X^{(1)} \times_{Y^{(1)}} \operatorname{Spec}\widehat{\mathcal{O}}^{(1)}_{\mathrm{Fr}_Y(y)}$. We have a natural map of schemes $\mathrm{Fr}_Z\colon Z \to Z^{(1)}$ and the maps $\pi_*(\bullet^{(1)})\colon \pi_*(\mathrm{Fr}_Z)_*\Omega^1_Z \to \pi_*\Omega^1_{Z^{(1)}}$ and $\pi_*\mathsf{C}\colon \pi_*(\mathrm{Fr}_Z)_*\Omega^1_{Z,\mathrm{cl}} \to \pi_*\Omega^1_{Z^{(1)}}$. As before, to split $\mathcal{D}_{X,\alpha}$ on $Z^{(1)}$ it is enough to find a closed 1-form ω on Z such that $\omega^{(1)} - \mathsf{C}(\omega) = \alpha|_{Z^{(1)}}$.

Since π is proper, globally defined closed 1-forms on Z form a finitely generated module $M = \pi_*(\mathrm{Fr}_Z)_*\Omega^1_{Z,\mathrm{cl}}$ over $\widehat{\mathcal{O}}^{(1)}_{\mathrm{Fr}_Y(y)}$. So does the module $N = \pi_*\Omega^1_{Z^{(1)}}$ of all globally defined 1-forms on $Z^{(1)}$. Let $\mathfrak{m}_{\mathrm{Fr}_Y(y)} \subset \widehat{\mathcal{O}}^{(1)}_{\mathrm{Fr}_Y(y)}$ be the maximal ideal of the point $\mathrm{Fr}_Y(y) \in Y^{(1)}(\mathsf{k})$. We endow M and N with natural descending filtrations: $F_iM := \mathfrak{m}^i_{\mathrm{Fr}_Y(y)}M$, $F_iN := \mathfrak{m}^i_{\mathrm{Fr}_Y(y)}N$. Modules M and N are complete with respect to the topologies induced by $F_\bullet$.

The map $\omega \mapsto \omega^{(1)} - \mathsf{C}(\omega)$ is continuous with respect to these topologies. Namely $\mathsf{C}(f^{(1)}\cdot\omega) = f^{(1)}\mathsf{C}(\omega)$ and $(f^{(1)}\cdot\omega)^{(1)} = (f^{(1)})^p\omega^{(1)}$, so $(\bullet^{(1)} - \mathsf{C})(F_iM) \subset F_iN$. We get that it is enough to solve the equation $\omega^{(1)} - \mathsf{C}(\omega) = \alpha|_{Z^{(1)}}$ on the level of the associated graded modules $\mathrm{gr}_{F^\bullet}(M)$ and $\mathrm{gr}_{F^\bullet}(N)$.

Note that the twist $\bullet^{(1)}$ maps F_iM not just to F_iN, but to $F_{pi}N$. It follows that for $i > 0$ the map $\mathrm{gr}_i(\bullet^{(1)})\colon \mathrm{gr}_i(M) \to \mathrm{gr}_i(N)$ is 0, and so $\mathrm{gr}_i(\bullet^{(1)} - \mathsf{C}) = \mathrm{gr}_i(-\mathsf{C})$ is surjective by Lemma 2.2.2.2. It remains only to prove that $\mathrm{gr}_0(\bullet^{(1)} - \mathsf{C})$ is surjective. But the spaces $\mathrm{gr}_0 M = M/F_1M$ and $\mathrm{gr}_0 N = M/F_1N$ are finite-dimensional, $\mathrm{gr}_0(\bullet^{(1)})$ is Frobenius-linear and $\mathrm{gr}_0(\mathsf{C})$ is k-linear and surjective. Lemma 2.2.2.1 can be applied and so $\mathrm{gr}_0(\bullet^{(1)} - \mathsf{C})$ is surjective too. We are done.

□

Remark 2.2.2.5. As we will see soon, in fact it is more important that $\mathcal{D}_{X,\alpha}$ splits on $Z^{(1)}$, than that it splits on the formal neighbourhood of the fiber.

Finally, using Popescu's theorem (a fancy version of Artin approximation, which was introduced to the first author by Brian Conrad), we prove that $\mathcal{D}_{X,\alpha}$ also splits on some étale neighbourhood of each fiber.

Theorem 2.2.2.6. *Under the assumptions of Proposition 2.2.2.4, there exists an étale cover $U \to Y$, such that the pullback of $\mathcal{D}_{X,\alpha}$ to $U \times_Y X$ splits.*

Proof. Since $Y^{(1)}$ is quasi-compact, it is enough to prove that for each $y \in Y^{(1)}(\mathsf{k})$ Azumaya algebra $\mathcal{D}_{X,\alpha}$ splits on some étale neighbourhood of y. In particular, we can assume that Y is affine, $Y = \operatorname{Spec}\Gamma(Y, \mathcal{O}_Y)$. Let $\mathcal{O}_y$ be the usual Zariski localisation of $\mathcal{O}_Y$ at the point y and let $\widehat{\mathcal{O}_y}$ be its formal completion. The field k is excellent like any other field, $\Gamma(Y, \mathcal{O}_Y)$ is a finitely generated algebra over k and is excellent, finally $\mathcal{O}_y$ is a localisation of $\mathcal{O}_Y$ and so is excellent too. Any excellent ring is a G-ring, namely if A is an excellent local ring, then the natural map $A \to \widehat{A}$ to its formal completion is regular. We get that the map $\mathcal{O}_y \to \widehat{\mathcal{O}_y}$ is regular. So does the natural map $\Gamma(Y, \mathcal{O}_Y) \to \mathcal{O}_y$ and consequently the composition $\Gamma(Y, \mathcal{O}_Y) \to \widehat{\mathcal{O}_y}$.

Popescu's theorem ([96],[97] and [98]) states that a map between two Noetherian rings $A \to B$ is regular if and only if B is a filtered colimit of finitely generated smooth A-algebras. Since the map $\mathcal{O}_Y \to \widehat{\mathcal{O}_y}$ is regular, we get that $\widehat{\mathcal{O}_y}$ can be obtained as a filtered colimit $\varinjlim A_i$ of smooth finitely generated $\mathcal{O}_Y$-algebras. Functor Spec sends colimits to limits, so we get that $\operatorname{Spec}\widehat{\mathcal{O}_y} = \varprojlim \operatorname{Spec} A_i$.

Let's now assume that the pull-back of $\mathcal{D}_{X,\alpha}$ to $\operatorname{Spec}\widehat{\mathcal{O}_y}^{(1)} \times_{Y^{(1)}} X^{(1)}$ splits. We have

$$\operatorname{Spec}\widehat{\mathcal{O}_y}^{(1)} \times_{Y^{(1)}} X^{(1)} \simeq (\varprojlim \operatorname{Spec} A_i^{(1)}) \times_{Y^{(1)}} X^{(1)} \simeq \varprojlim (\operatorname{Spec} A_i^{(1)} \times_{Y^{(1)}} X^{(1)}),$$

and, since $\mathcal{D}_{X,\alpha}$ is a coherent sheaf of algebras, it splits over some $\operatorname{Spec} A_i^{(1)} \times_{Y^{(1)}} X^{(1)}$ as well. Smooth map $\operatorname{Spec} A_i \to Y$ admits an étale quasi-section: there exists a surjection $A_i[f^{-1}] \twoheadrightarrow B_i$ for some $\mathcal{O}_Y$-algebra B_i and a function $f \in \mathcal{O}_Y$, such that the corresponding map $\operatorname{Spec} B_i \to Y$ is étale. This is the étale neighbourhood we are looking for: $\operatorname{Spec} B_i \hookrightarrow \operatorname{Spec} A_i$ and $\mathcal{D}_{X,\alpha}$ splits on $\operatorname{Spec} B_i^{(1)} \times_{Y^{(1)}} X^{(1)}$.

So it is enough to prove that the pull-back of $\mathcal{D}_{X,\alpha}$ to $\operatorname{Spec} \widehat{\mathcal{O}_y}^{(1)} \times_{Y^{(1)}} X^{(1)}$ splits. But this was already done in Proposition 2.2.2.4. □

2.3 Obstructions to global descent

As we saw in Theorem 2.2.2.6, if $\mathcal{O}_Y \xrightarrow{\sim} \pi_*\mathcal{O}_X$ and $R^1\pi_*\mathcal{O}_X = R^2\pi_*\mathcal{O}_X = 0$, the classes $[\mathcal{D}_{X,\alpha}]$ descend to $Y^{(1)}$ étale locally for all α. The way to prove it was to split them étale locally on $Y^{(1)}$, namely, we proved that the pull-back of $\mathcal{D}_{X,\alpha}$ to $(U \times_Y X)^{(1)}$ splits for some étale cover $U \to Y$. Obviously, this kind of argument will not help in the global situation, unless the algebra $\mathcal{D}_{X,\alpha}$ is globally split. Nevertheless, one could try to study obstructions to the global descent. In this section we define two different types of obstructions.

2.3.1 Picard obstruction

From the local-to-global principle it is natural to expect that obstructions to global descent should lie in some cohomology group of some natural sheaf on $Y^{(1)}$. It is true indeed and the corresponding sheaf is easy to describe, but on the other hand it seems to be almost impossible to work with. This approach does not take into account that our classes come from differential forms.

Let $\mathcal{A}$ be an Azumaya algebra on $X^{(1)}$ and let $[\mathcal{A}]$ be its class in $\operatorname{Br}(X^{(1)}) = H^2(X^{(1)}_{\text{ét}}, \mathbb{G}_m)$. We have a natural map $\varphi\colon \operatorname{Br}(X^{(1)}) = H^2(X^{(1)}_{\text{ét}}, \mathbb{G}_m) \to H^0(Y^{(1)}_{\text{ét}}, R^2\pi_*\mathbb{G}_m)$. It is easy to see that $\varphi([\mathcal{A}]) = 0$ if and only if $\mathcal{A}$ splits on $(U \times_Y X)^{(1)}$ for some étale cover $U \to Y$. In particular Theorem 2.2.2.6 says exactly that $\varphi([\mathcal{D}_{X,a}]) = 0$.

Note that $R^0\pi_*\mathbb{G}_m = \mathbb{G}_m$ and $R^1\pi_*\mathbb{G}_m = \underline{\operatorname{Pic}}^{(1)}_{X/Y}$. From Leray spectral sequence $H^p_{\text{ét}}(Y^{(1)}, R^q\pi_*\mathbb{G}_m) \Rightarrow H^{p+q}_{\text{ét}}(X^{(1)}, \mathbb{G}_m)$ we get a short exact sequence

$$0 \to \operatorname{Br}(Y^{(1)}) \to \operatorname{Ker}(\varphi) \to H^1_{\text{ét}}(Y^{(1)}, \underline{\operatorname{Pic}}^{(1)}_{X/Y}) \to 0,$$

and, if $\varphi([\mathcal{A}]) = 0$, this way we get a well-defined class $[\widetilde{\mathcal{A}}]$ in $H^1_{\text{ét}}(Y^{(1)}, \underline{\mathrm{Pic}}^{(1)}_{X/Y})$, which is zero if and only if $[\mathcal{A}]$ lies in the image of $\mathrm{Br}(Y^{(1)})$. We call this class the *Picard obstruction* corresponding to $\mathcal{A}$.

Unfortunately, the only thing we can say about this class in general, is that $[\widetilde{\mathcal{A}}]$ lies in the p-torsion subgroup $H^1_{\text{ét}}(Y^{(1)}, \underline{\mathrm{Pic}}^{(1)}_{X/Y})[p]$. In particular, we do not know when this subgroup is zero, unless $Y = \operatorname{Spec} \mathsf{k}$, in that case we obtain the statement of Proposition 2.2.2.3.

2.3.2 Sheaves $\mathcal{Q}_{\pi,N}$

This class of obstructions works only for affine $Y = \operatorname{Spec} A$ and a proper map $\pi\colon X \to Y$ from a smooth variety X, that satisfies $\mathcal{O}_Y \xrightarrow{\sim} \pi_*\mathcal{O}_X$ and $R^1\pi_*\mathcal{O}_X = R^2\pi_*\mathcal{O}_X = 0$. We will construct a sequence of sheaves $\mathcal{Q}_{\pi,N}$ on $Y^{(N)}$ (or equivalently a sequence of A^{p^N}-modules) endowed with natural maps $\psi_N\colon (\mathrm{Fr}^N_Y)_*\pi_*\Omega^1_X \to \mathcal{Q}_{\pi,N}$, such that if $\psi_N(\alpha^{(N-1)}) = 0$ for some N, then the class $[\mathcal{D}_{X,\alpha}]$ descends to $Y^{(1)}$.

The first idea is to functorially extend the map $c_X\colon H^0(X^{(1)}, \Omega^1_{X^{(1)}}) \to \mathrm{Br}(X^{(1)})$ to the case of singular varieties, in particular Y. This can be done indeed, but only with the help of some homotopy algebra, which we decided not to use (at least in this paper). Interested reader will find more details about that in Remark 2.3.2.3.

We use the following simple, but completely satisfactory for our purposes, lemma, instead.

Lemma 2.3.2.1. *Let $\pi\colon X \to Y$ be a map from a smooth variety X to an affine variety Y. Let $\alpha \in H^0(X^{(1)}, \Omega^1_{X^{(1)}})$ be a 1-form, and suppose that there exists a Kähler differential θ on $Y^{(1)}$, such that $\alpha = \pi^*(\theta)$. Then the class $[\mathcal{D}_{X,\alpha}] \in \mathrm{Br}(X^{(1)})$ descends to $Y^{(1)}$.*

Proof. Let ι be an embedding of $Y^{(1)}$ to a smooth variety $Z^{(1)}$, such that $\theta = \iota^*\theta_Z$ for some 1-form θ_Z on $Z^{(1)}$. If θ is represented in the form $\sum_{i=1}^n f_i dg_i$ then for ι one can take the map into an affine space, constructed in the following way. Namely, let $j\colon Y \to \mathbb{A}^m$ be any closed embedding to an affine space, and let $\iota\colon Y \to \mathbb{A}^m \times \mathbb{A}^n \times \mathbb{A}^n$ be the map, sending y to the triple $(j(y), f_i(y), g_k(y))$. If we put coordinates (x_s, y_i, z_k) on $Z = \mathbb{A}^m \times \mathbb{A}^n \times \mathbb{A}^n$, taking $\theta_Z = \sum_{i=1}^n y_i dz_i$, we get that $\theta = \iota^*\theta_Z$.

We define an element $c_Y(\theta) \in \mathrm{Br}(Y^{(1)})$ (this is just a notation, the map c_Y is not a priori defined, since Y is not necessarily smooth; in particular in our definition it could depends on ι) by $c_Y = \iota^* c_Z(\theta_Z)$. Then

$$\pi^* c_Y(\theta) = \pi^*\iota^* c_Z(\theta_Z) = (\iota \circ \pi)^* c_Z(\theta_Z) = c_X((\iota \circ \pi)^*\theta_Z) = c_X(\pi^*\theta).$$

So the class $[\mathcal{D}_{X,\alpha}]$ descends to Y and we are done. $\square$

Remark 2.3.2.2. In particular, since $\mathcal{O}_Y \xrightarrow{\sim} \pi_*\mathcal{O}_X$ the class $[\mathcal{D}_{X,df}]$ for $f \in H^0(X^{(1)}, \mathcal{O}_{X^{(1)}})$ always descends to $Y^{(1)}$.

Remark 2.3.2.3. One can show that the class $c_Y(\theta) \in \mathrm{Br}(Y)$ does not depend on the choice of the embedding ι and that this construction gives a well-defined map from the space of Kähler differentials $H^0(Y^{(1)}, \Omega^1_{Y^{(1)}})$ to the Brauer group $\mathrm{Br}(Y^{(1)})$. With some modifications this kind of definition can also be extended to the case of arbitrary non-smooth X. Namely, one can construct a well-defined map $c_X \colon \mathbb{H}^0(X^{(1)}, \mathbb{L}^\bullet_{X^{(1)}}) \to \mathrm{Br}(X^{(1)})$ from the 0-th hyper-cohomology group of the cotangent complex of X. Moreover, this map is functorial in X. Note that $\mathbb{L}^\bullet_X$ has cohomology sheaves in negative degrees and a priori $\mathbb{H}^0(X, \mathbb{L}^\bullet_X)$ is not isomorphic to the space of global Kähler differentials $H^0(X, \Omega^1_X)$ (unless X is smooth or affine).

Now we return to our setting of affine $Y = \operatorname{Spec} A$ and a proper map $\pi \colon X \to Y$ from a smooth variety X with $\mathcal{O}_Y \xrightarrow{\sim} \pi_*\mathcal{O}_X$ and $R^1\pi_*\mathcal{O}_X = R^2\pi_*\mathcal{O}_X = 0$. We have a natural A-module structure on $H^0(X, \Omega^1_X)$ given by multiplication of a differential form by a function, and, since π is proper, the module $H^0(X, \Omega^1_X) = H^0(Y, \pi_*\Omega^1_X)$ is finitely generated over A. Recall that we have a map $\pi^* \colon \mathrm{Br}(Y^{(1)}) \to \mathrm{Br}(X^{(1)})$. Let $\mathrm{Br}(X^{(1)}/Y^{(1)})$ denote the quotient $\mathrm{Br}(X^{(1)})/\pi^*(\mathrm{Br}(Y^{(1)}))$ and let $c_{X/Y} \colon H^0(X^{(1)}, \Omega^1_{X^{(1)}}) \to \mathrm{Br}(X^{(1)}/Y^{(1)})$ be the map obtained from c_X by composition with the natural projection $\mathrm{Br}(X^{(1)}) \to \mathrm{Br}(X^{(1)}/Y^{(1)})$. It is clear that the class $[\mathcal{D}_{X,\alpha}]$ descends to Y if and only if $c_{X/Y}(\alpha) = 0$. Note that by Corollary 2.1.8.6 and Lemma 2.3.2.1

- $\mathbb{F}_p$-vector subspace spanned by $\omega^{(1)} - \mathsf{C}(\omega)$ for closed 1-forms ω on X,
- A-submodule generated by $\pi^*(\theta)$ for Kähler differentials θ on $Y^{(1)}$,

lie in the kernel of $c_{X/Y}$. Let $S_\pi = \langle \pi^*(\theta), \omega^{(1)} - \mathsf{C}(\omega) \rangle_{\theta,\omega}$ be the $\mathbb{F}_p$-subspace of $H^0(X^{(1)}, \Omega^1_{X^{(1)}})$ generated by these two types of 1-forms and let Q_π denote quotient $H^0(X^{(1)}, \Omega^1_{X^{(1)}})/S_\pi$. If we show that $Q_\pi = 0$, we will get that $[\mathcal{D}_{X,\alpha}]$ descends to $Y^{(1)}$ for any α. If Q_π had a structure of A-module it would define a coherent sheaf $\mathcal{Q}_\pi$ on $Y^{(1)}$ and to prove that $Q_\pi = 0$ it would be enough to show that all stalks of $\mathcal{Q}_\pi$ are zero. However, there is no structure on Q_π except that of an $\mathbb{F}_p$-vector space.

So we try to be more clever and filter S_π by a system of subspaces $S_{\pi,0} \subset S_{\pi,1} \subset \ldots \subset S_\pi$, such that $S_{\pi,N}$ has a module structure over $(A)^{p^N}$, or, in other words, $S_{\pi,N}$ defines a subsheaf

$\mathcal{S}_{\pi,N} \subset (\mathrm{Fr}_X^N)_*\pi_*\Omega^1_X$. The strategy then is to prove that $S_{\pi,N} = H^0(X^{(1)}, \Omega^1_{X^{(1)}})$ for N big enough, and that consequently $Q_\pi = 0$. Let's now define the sheaves $\mathcal{S}_{\pi,N}$.

The starting point is the following observation: let $\Omega^1_{Y,cl} \subset \pi_*\Omega^1_{X,cl}$ be the natural embedding of closed Kähler differentials to closed 1-forms on X. Let $\pi_*\mathsf{C}\colon \pi_*\Omega^1_{X,cl} \to \pi_*\Omega^1_{X^{(1)}}$ be the Cartier operator. Does $\pi_*\mathsf{C}$ map $\Omega^1_{Y,cl} \subset \pi_*\Omega^1_{X,cl}$ to $\Omega^1_Y \subset \pi_*\Omega^1_X$? If the Cartier operator were defined for singular varieties, this would be true by functoriality. Surprisingly, usually this is not true at all, moreover in some cases (of the most interest for us) we can prove that all differential 1-forms on X can be obtained from Kähler differentials on Y by applying Cartier operator sufficiently many times.

Recall that by Lemma 2.2.2.2 Cartier operator $\pi_*\mathsf{C}\colon \pi_*\Omega^1_{X,cl} \to \pi_*\Omega^1_{X^{(1)}}$ is surjective. We inductively define sheaves $\mathcal{C}_{\pi,n} \subset \pi_*\Omega^1_X$ by $\mathcal{C}_{\pi,0} = \Omega^1_X$ and $\mathcal{C}_{\pi,n} = \mathsf{C}^{-1}(\mathcal{C}^{(1)}_{\pi,n-1})$, where $\mathcal{C}^{(1)}_{\pi,n-1}$ is the analogous subsheaf of $\pi_*\Omega^1_{X^{(1)}}$ for $\pi\colon X^{(1)} \to Y^{(1)}$. Sheaf $\mathcal{C}_{\pi,n}$ is the subsheaf of differential 1-forms in $\pi_*\Omega^1_X$ on which C^n is well-defined. Moreover, from the p-linearity of Cartier operator, it is easy to see that $(\mathrm{Fr}^n_Y)_*\mathcal{C}_{\pi,n}$ defines a coherent subsheaf of $(\mathrm{Fr}^n_Y)_*\pi_*\Omega^1_X$. So, in particular, $\mathcal{C}_{\pi,0} = \pi_*\Omega^1_X$, $\mathcal{C}_{\pi,1} = \pi_*\Omega^1_{X,cl}$, and in general we have a sequence of surjections

$$(\mathrm{Fr}^n_Y)_*\mathcal{C}_{\pi,n} \xrightarrow{\mathsf{C}} (\mathrm{Fr}^{n-1}_Y)_*\mathcal{C}^{(1)}_{\pi,n-1} \xrightarrow{\mathsf{C}} \ldots \xrightarrow{\mathsf{C}} (\mathrm{Fr}_Y)_*\pi_*\Omega^1_{X^{(n-1)},cl} \xrightarrow{\mathsf{C}} \pi_*\Omega^1_{X^{(n)}}$$

of coherent sheaves on $Y^{(n)}$.

Let's now consider $\Omega^1_Y \subset \pi_*\Omega^1_X$ and the intersection $\Omega^1_Y \cap \mathcal{C}_{\pi,n}$. We have a natural map $\mathsf{C}^n\colon \Omega^1_Y \cap \mathcal{C}_{\pi,n} \to \pi_*\Omega^1_{X^{(n)}}$ and we define $\mathcal{S}_{\pi,n}$ as the image of this map. Note that $\mathcal{S}^{(1)}_{\pi,n-1} \subset \pi_*\Omega^1_{X^{(n)}}$ is embedded into $\mathcal{S}_{\pi,n}$, because $\mathsf{C}(f^p g^{p-1} dg) = f^{(1)} dg^{(1)}$ and so for any $\theta \in \Omega^1_{Y^{(1)}}$ there exists $\theta' \in \Omega^1_Y$, such that $\mathsf{C}(\theta') = \theta$. Using $\bullet^{(n-1)}$ we can identify $H^0(Y^{(n)}, \pi_*\Omega^1_{X^{(n)}}) = H^0(X^{(n)}, \Omega^1_{X^{(n)}})$ with $H^0(X^{(1)}, \pi_*\Omega^1_{X^{(1)}})$ (as $\mathbb{F}_p$-vector spaces) and images of $H^0(Y^{(n)}, \mathcal{S}_{\pi,n}) \subset H^0(X^{(n)}, \Omega^1_{X^{(n)}})$ for different n give the desired inductive system $S_{\pi,0} \subset S_{\pi,1} \subset \ldots$.

N suppose that we have $\alpha \in S_{\pi,N}$ for some N. Unraveling the definitions, this means that we can find the set of 1-forms $\omega_i \in H^0(X, \Omega^1_X)$ and a Kähler differential $\theta \in H^0(Y, \Omega^1_{Y^{(1)}})$ that satisfy the following system of equations:

$$\begin{cases} \begin{cases} d\omega_i = 0, \ \forall i = 1, \dots, N \\ \mathsf{C}\omega_1 = \alpha \\ \mathsf{C}\omega_2 = \omega_1^{(1)} \\ \dots \\ \mathsf{C}\omega_i = \omega_{i-1}^{(1)} \\ \dots \\ \mathsf{C}\omega_N = \omega_{N-1}^{(1)} \end{cases} \\ \pi^*\theta = \omega_N^{(1)}, \ \theta \in H^0(Y, \Omega^1_{Y^{(1)}}) \end{cases}$$

Taking the sum of all equalities we get that $\pi^*\theta - \alpha = (\bullet^{(1)} - \mathsf{C})(\sum_{i=1}^N \omega_i)$. This means that $\alpha \in S_\pi$ and the class $[\mathcal{D}_{X,\alpha}]$ descends to $Y^{(1)}$.

Definition 2.3.2.4. For $N \geq 0$ we define the obstruction sheaf $\mathcal{Q}_{\pi,N}$ as the quotient $\pi_*\Omega^1_{X^{(N)}}/\mathcal{S}_{\pi,N}$. It is a coherent sheaf on $Y^{(N)}$ and since Y is affine we have a short exact sequence

$$0 \to S_{\pi,N} \to H^0(X^{(1)}, \pi_*\Omega^1_{X^{(1)}}) \xrightarrow{\psi_N} Q_{\pi,N} \to 0,$$

where $Q_{\pi,N}$ is the twist $H^0(Y^{(N)}, \mathcal{Q}_{\pi,N})^{(1-N)}$ of the global sections of $\mathcal{Q}_{\pi,N}$. From the discussion above: if $\psi_N(\alpha) = 0$ for some N, the class $[\mathcal{D}_{X,\alpha}]$ descends to $Y^{(1)}$.

The following lemma is very important and gives a motivation for the upcoming definition of a resolution with conical slices (in Section 2.4):

Lemma 2.3.2.5. *Let $\pi\colon X \to Y$ be a resolution of singularities such that $\mathcal{Q}_{\pi,N} = 0$ for some N. Let S be an arbitrary smooth affine variety, then for $\pi \times \mathrm{id}_S\colon X \times S \to Y \times S$ we have $\mathcal{Q}_{\pi\times\mathrm{id}_S,N} = 0$.*

Proof. The statement of the lemma amounts to saying that for any differential 1-form $\alpha \in \Omega^1(X \times S)$, there is a solution to the system (2.3.2). We now use the formula

$$\Omega^1_{X\times S} = \Omega^1_X \boxtimes \mathcal{O}_S \oplus \mathcal{O}_X \boxtimes \Omega^1_S$$

to write α as a sum of 1-forms of the following kind: $\alpha_X \boxtimes f_S + f_X \boxtimes \alpha_S$. Moreover, we can assume

that there is only one summand, so that

$$\alpha = \alpha_X \boxtimes f_S + f_X \boxtimes \alpha_S.$$

By assumption, there is a solution $(\omega_{1,X}, \dots, \omega_{N,X}, \theta_Y)$ to the system (2.3.2) with α replaced by α_X. Since S is smooth and affine, $\mathsf{C}\colon \Omega^1_{S,cl} \to \Omega^1_{S^{(1)}}$ is surjective and induces a surjective map on global sections. So (2.3.2) must also have a solution for α_S (and id_S instead of π). We denote the corresponding sequence of 1-forms on S by $(\omega_{1,S}, \dots, \omega_{N,S}, \theta_S = \omega_{N,S})$. Also, using the assumption $\mathcal{O}_Y \xrightarrow{\sim} \pi_*\mathcal{O}_X$, we can descend f_X to Y, namely find $f_Y \in \Gamma(Y, \mathcal{O}_Y)$ such that $f_X = \pi^* f_Y$. We now take

$$\omega_i = \omega_{i,X} \boxtimes f_S^{p^i} + f_X^{p^i} \boxtimes \omega_{i,S}, \quad \theta = \theta_Y \boxtimes f_S^{p^N} + f_Y^{p^N} \boxtimes \theta_S.$$

Using $\mathsf{C}(f^p\omega) = f^{(1)}\mathsf{C}(\omega)$, we see that the resulting sequence $(\omega_1, \dots, \omega_N, \theta)$ solves the system (2.3.2) for the form α.

□

2.4 Resolutions with conical slices

2.4.1 Setup

We would like to have a class of resolutions for which we could efficiently apply Lemma 2.3.2.5. Namely, it would be great if a neighbourhood of any point $y \in Y$ could be decomposed as a product of something smooth and some other resolution $\pi'\colon X' \to Y'$ for which we already knew that $\mathcal{Q}_{\pi',N} = 0$ for some N. A good candidate for this is a conical resolution: if there is a contracting $\mathbb{G}_m$-action on $\pi\colon X \to Y$, then, by Luna's slice theorem, étale locally at any non-central point, π can be decomposed as a product of the $\mathbb{G}_m$-orbit and some slice $\pi'\colon X' \to Y'$. However we will not know much about π', in particular there is no need for π' to be again conical. It turns out though that if we add by force the condition for the existense of a conical slice satisfying the same property (plus some other mild technical conditions), everything will work out relatively well: see Section 2.6 and Theorem 2.6.2.4. This condition can be elegantly reformulated as the existence of another contracting $\mathbb{G}_m$-action in an étale neighbourhood of each point: see Definition 2.4.4.3 and Proposition 2.4.5.4 for the relation to slices. We call a resolution satisfying this a *resolution with conical slices*.

2.4.2 Category of étale germs of resolutions

In this subsection we construct what we call *the category of étale germs of resolutions*. The idea is to identify resolutions which look the same in the neighbourhood of some point of the base (where the point is a part of the data). We could probably consider something like the category of resolutions with a henselian base, but instead we just localise the category of all resolutions by the class of morphisms which are étale. We follow probably the most classical approach to localisation of categories, namely we prove that the class of étale maps satisfies the right Ore condition. In particular, as a result we get a well defined and quite natural notion of an étale equivalence between resolutions of singularities with a chosen point on the base.

Definition 2.4.2.1. By a *resolution (of singularities)* $\pi\colon X \to Y$ we mean a proper birational map from a smooth variety X to a possibly singular variety Y.

Remark 2.4.2.2. Please note that Y does not actually need to be singular in the definition, so in this sense our definition of resolution can differ from the more standard ones.

Definition 2.4.2.3. *A pointed resolution* (π, X, Y, y) over an algebraically closed field k is the data of a resolution $\pi\colon X \to Y$ over k and a distinguished point $y \in Y(\mathsf{k})$.

Fixing field k, we define Res_* to be the category with objects being pointed resolutions (π, X, Y, y) and morphisms given by pairs of maps $p\colon Y' \to Y$ and $q\colon X' \to X$, such that $p(y') = y$ and the following square is commutative:

$$\begin{array}{ccc} X' & \xrightarrow{q} & X \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ Y' & \xrightarrow{p} & Y \end{array}$$

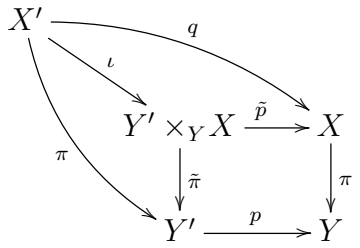
We would like to identify two pointed resolutions, which become isomorphic after taking some étale neighbourhoods of distinguished points. For this we can try to localise the category Res_* by the set of morphisms f for which the corresponding p and q are étale. This simple lemma about étale morphisms will appear to be very useful in this context.

Lemma 2.4.2.4. *Let* $f\colon (\pi, X, Y, y) \to (\pi', X', Y', y')$ *be a morphism, such that the corresponding morphisms* p *and* q *are étale. Then* $X' = Y' \times_Y X$ *and the corresponding square is Cartesian.*

Proof. By definition, we have a commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{q} & X \\ {\scriptstyle \pi'}\downarrow & & \downarrow{\scriptstyle \pi} \\ Y' & \xrightarrow{p} & Y \end{array}$$

which gives a natural map $\iota\colon X' \to Y' \times_Y X$.



We have a decomposition $q = \tilde{p} \circ \iota$, where $\tilde{p}\colon Y' \times_Y X \to X$ is the pull-back of p to X. The map $\tilde{p}$ is étale as a pull-back of p, q is étale by assumption, therefore we get that ι is étale too.

Let $Y_0 \subset Y$ and $Y_0' \subset Y'$ be the smooth loci where π and π' are 1-to-1, and let $j\colon Y_0 \to X$, $j'\colon Y_0' \to X'$ be the corresponding embeddings. We have two open embeddings $\tilde{j}\colon Y_0' \times_Y Y_0 \to Y' \times_Y X$ and $j\colon Y_0' \times_Y Y_0 \to X'$. Both Y_0 and Y_0' are open and dense in X and X', so $Y_0' \times_Y Y_0$ is open and dense in both X' and $Y' \times_Y X$. Note that the restriction of ι to $Y_0' \times_Y Y_0$ is just the identity map. We also have that ι should necessarily be surjective, since $\iota(X') = \overline{\iota(Y_0' \times_Y Y_0)} = \overline{Y_0' \times_Y Y_0} = Y' \times_Y X$. Summarising the above, we get that ι is étale, surjective, and is an isomorphism on some dense open set, therefore it is an isomorphism on the whole and the corresponding square is Cartesian. □

Definition 2.4.2.5. $\mathrm{Et}((\pi, X, Y, y), (\pi', X', Y', y')) \subset \mathrm{Mor}_{\mathsf{Res}_*}((\pi, X, Y, y), (\pi', X', Y', y'))$ is defined as the subset of morphisms for which p and q are étale.

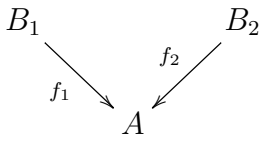
Let Sch denote the category of schemes over k and let $\mathsf{Ar}_*(\mathsf{Sch})$ be the category of pointed arrows in Sch. Namely objects of $\mathsf{Ar}_*(\mathsf{Sch})$ are maps $f\colon X \to Y$ with a distinguished point $y \in Y(\mathsf{k})$ and morphisms between two such arrows (f, X, Y, y) and (g, X', Y', y') are given by commutative squares

$$\begin{array}{ccc} X & \xrightarrow{q} & X' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ Y & \xrightarrow{p} & Y' \end{array}$$

such that $p(y) = y'$. It is easy to see that Res_* is a full subcategory of $\mathsf{Ar}_*(\mathsf{Sch})$.

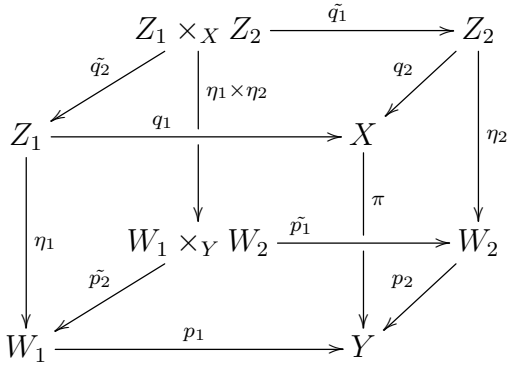
The category $\mathsf{Ar}_*(\mathsf{Sch})$ has fiber products: if $A = (f, X, Y, y)$, $B_1 = (g_1, Z_1, W_1, w_1)$, $B_2 = (g_2, Z_2, W_2, w_2)$ and $f_1 \colon B_1 \to A$, $f_2 \colon B_2 \to A$ are two morphisms, then the corresponging fiber product $B_1 \times_A B_2$ is given by the map $g_1 \times g_2 : Z_1 \times_X Z_2 \to W_1 \times_Y W_2$ with a distinguished point $w_1 \times w_2 \in (W_1 \times_Y W_2)(\mathsf{k})$. However Res_* does not have fiber products in general, the problem is that $\mathsf{Res}_* \subset \mathsf{Ar}_*(\mathsf{Sch})$ is not closed under the fiber product in $\mathsf{Ar}_*(\mathsf{Sch})$. To illustrate this we can take some smooth Y, and let $A = (\mathrm{id}, Y, Y, y)$ be the identity map on Y, $B_1 = (\pi, \mathrm{Bl}_y(Y), Y, y)$ the blow-up at the point y and $B_2 = (\mathrm{id}, \operatorname{Spec}\mathsf{k}, \operatorname{Spec}\mathsf{k}, \operatorname{Spec}\mathsf{k})$ the identity map for a point, with the maps f_1 and f_2 given by the projection $\mathrm{Bl}_y(Y) \to Y$ and the point $y \colon \operatorname{Spec}\mathsf{k} \to Y$ correspondingly. Then the fiber product $B_1 \times_A B_2$ is given by $\pi \colon \pi^{-1}(\mathrm{Bl}_y(Y)) \to \operatorname{Spec}\mathsf{k}$, which is not birational if $\dim Y \geq 2$. But for some pairs f_1, f_2 fiber product still exists, in particular when one of the morphisms lies in Et:

Lemma 2.4.2.6. *Let $A = (\pi, X, Y, y), B_1 = (\eta_1, Z_1, W_1, w_1), B_2 = (\eta_2, Z_2, W_2, w_2)$ and f_1, f_2 be objects and a pair of morphisms in Res_*:*



Assume that f_2 lies in $\mathrm{Et}(B_2, A)$. Then the fiber product $B_1 \times_A B_2$ exists and is given by $(\eta_1 \times \eta_2, Z_1 \times_X Z_2, W_1 \times_Y W_2, w_1 \times w_2)$.

Proof. We have the following diagram:



The quadruple $(\eta_1 \times \eta_2, Z_1 \times_X Z_2, W_1 \times_Y W_2, w_1 \times w_2)$ is the fiber product of (η_1, Z_1, W_1, w_1) and (η_2, Z_2, W_2, w_2) over (π, X, Y, y) in the category of arrows in the category of schemes $\underline{Sch}$. So

the only thing we need to check is that

$$\eta_1 \times \eta_2 \colon Z_1 \times_X Z_2 \to W_1 \times_Y W_2$$

is a resolution of singularities. By Lemma 2.4.2.4 Z_2 is equal to $X \times_Y W_2$, so

$$Z_1 \times_X Z_2 = Z_1 \times_X X \times_Y W_2 = Z_1 \times_Y W_2 = Z_1 \times_{W_1} (W_1 \times_Y W_2),$$

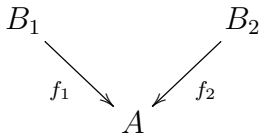
Moreover the bottom, right and top faces of the cube are Cartesian squares and it is easy to see that the left face is a Cartesian square too. Therefore, $\eta_1 \times \eta_2$ is just the pull-back of η_1 under $\tilde{p_2}$, which is étale (as a pull-back of étale map). We see that $\eta_1 \times \eta_2$ is also a resolution. □

Remark 2.4.2.7. Note that $\tilde{f_2} \colon B_1 \times_A B_2 \to B_1$ lies in $\mathrm{Et}(B_1 \times_A B_2, B_1)$. So étale maps are closed under pull-backs in Res_* as well.

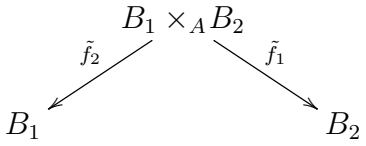
Remark 2.4.2.8. There is another important example when the fiber product exists, namely when $A = \underline{\mathrm{Spec}\,\mathsf{k}} = (\mathrm{id}, \mathrm{Spec}\,\mathsf{k}, \mathrm{Spec}\,\mathsf{k}, \mathrm{Spec}\,\mathsf{k})$. It is easy to see that $\underline{\mathrm{Spec}\,\mathsf{k}}$ is a final object of Res_*, and so the corresponding fiber product is just the product. It is given by the quadruple $B_1 \times B_2 = (\pi_1 \times \pi_2, Z_1 \times Z_2, W_1 \times W_2, w_1 \times w_2)$ which is a resolution of singularities if π_1, π_2 are.

Lemma 2.4.2.9. *Collection of subsets* $\mathrm{Et}((\pi, X, Y, y), (\pi', X', Y', y'))$ *inside the sets of morphisms* $\mathrm{Mor}_{\mathsf{Res}_*}((\pi, X, Y, y), (\pi', X', Y', y'))$ *satisfies the right Ore condition.*

Proof. This follows essentially from Lemma 2.4.2.6: having the diagram



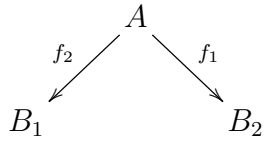
such that f_2 lies in $\mathrm{Et}(B_2, A)$, we can form another diagram



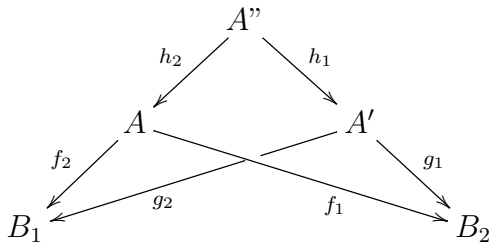
such that $\tilde{f_2}$ lies in $\mathrm{Et}(B_1 \times_A B_2, B_1)$.

□

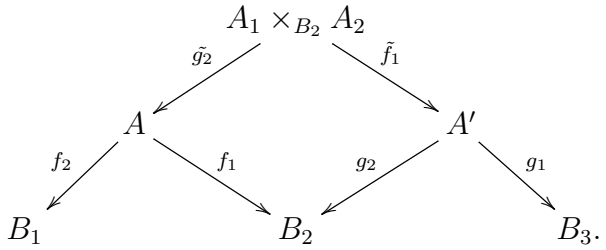
Last lemma allows us to localise Res_* at the class of morphisms Et. We call the result of localisation $\mathsf{Res}_*^{\mathsf{et}}$ *the category of étale germs of resolutions*. We remind ([48], III.2, Lemma 8) that the set of morphisms in $\mathsf{Res}_*^{\mathsf{et}}$ between objects B_1 and B_2 is given by classes of equivalences of diagrams



where f_2 is étale. An equivalence between pairs (f_1, f_2) and (g_1, g_2) is given by an object A" and a pair of morphisms $h_1 \in \mathrm{Et}(A", A')$, $h_2 \in \mathrm{Et}(A", A)$, such that the following diagram commutes:



The composition $(f_1, f_2) \circ (g_1, g_2)$ is given by the fiber product:



Morphisms in $\mathsf{Res}_*^{\mathsf{et}}$ that we are interested in the most, are isomorphisms. We call them *étale equivalences*.

Definition 2.4.2.10. Let (π, X, Y, y) and (π', X', Y', y') be two pointed resolutions. We say that (π, X, Y, y) is *étale equivalent* to (π', X', Y', y') if they are isomorphic as objects of $\mathsf{Res}_*^{\mathsf{et}}$. Unwinding the definitions, it means that there exists a pointed resolution $(\pi", X", Y", y")$ and a correspondence $(\pi, X, Y, y) \xleftarrow{f_1} (\pi", X", Y", y") \xrightarrow{f_2} (\pi', X', Y', y')$ (given by correspondences

$Y \xleftarrow{p_1} Y" \xrightarrow{p_2} Y'$, and $X \xleftarrow{q_1} X" \xrightarrow{q_2} X'$, $p_1(y") = y$, $p_2(y") = y'$, such that the diagram

$$\begin{array}{ccccc} X & \xleftarrow{q_1} & X" & \xrightarrow{q_2} & X' \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi"} & & \downarrow{\scriptstyle \pi'} \\ Y & \xleftarrow{p_1} & Y" & \xrightarrow{p_2} & Y' \end{array}$$

is commutative) with maps p_1, p_2, q_1, q_2 being étale. We denote the equivalence by $(\pi, X, Y, y) \sim_{et} (\pi', X', Y', y')$.

Remark 2.4.2.11. Note that by Lemma 2.4.2.6 we necessarily have $X \times_Y Y" = X" = Y" \times_Y X'$.

2.4.3 Separable $\mathbb{G}_m$-action

Let k be an algebraically closed field (not necessarily of finite characteristic), let X be a variety over k (possibly singular), endowed with a $\mathbb{G}_m$-action. We have the natural non-zero tangent vector $\partial_t \in T_{\{1\}}\mathbb{G}_m$. For any point $x \in X(\mathsf{k})$ we have the differential of the action map $a_x \colon T_{\{1\}}\mathbb{G}_m \to T_x X$, which extends to the map of sheaves $a \colon \mathcal{O}_X \to Der(\mathcal{O}_X)$, where $\mathrm{Der}(\mathcal{O}_X)$ is the sheaf of k-linear derivations of $\mathcal{O}_X$. Consider the dual map $a^* : \Omega^1_X \to \mathcal{O}_X$, where Ω^1_X is the sheaf of Kähler differentials on X. The image $\mathcal{I}_{a^*} = \mathrm{Im}(a^*) \subset \mathcal{O}_X$ is some sheaf of ideals on X.

Definition 2.4.3.1. The subscheme $X^{d\mathbb{G}_m} \subset X$ of *pseudo-fixed* points is the subscheme defined by the sheaf of ideals $\mathcal{I}_{a^*}$. This is a $\mathbb{G}_m$-invariant closed subscheme of X.

Remark 2.4.3.2. The subscheme $X^{d\mathbb{G}_m} \subset X$ can also be viewed in a more geometric way. The vector $\partial_t \in T_{\{1\}}\mathbb{G}_m$ defines a section $a(\partial_t)$ of the tangent bundle $TX \to X$, together with its graph $\Gamma_{a(\partial_t)} \subset TX$. The subscheme of pseudo-fixed points $X^{d\mathbb{G}_m} \subset X$ is then just the (scheme-theoretic) intersection of $\Gamma_{a(\partial_t)}$ and the zero-section $\Gamma_0 \subset TX$.

$X^{d\mathbb{G}_m}$ is the subscheme of points where the differential of the $\mathbb{G}_m$-action is zero. If $\mathrm{char}\,\mathsf{k} = 0$ it is true that a point is fixed if and only if the differential of the action is zero (e.g. see [40]), so $X^{d\mathbb{G}_m}(\mathsf{k}) = X^{\mathbb{G}_m}(\mathsf{k})$. But if $\mathrm{char}\,\mathsf{k} \neq 0$ these two subsets can easily happen to be different.

Definition 2.4.3.3. The $\mathbb{G}_m$-action on X is called *separable*, if all pseudo-fixed points are fixed:

$$X^{d\mathbb{G}_m}(\mathsf{k}) = X^{\mathbb{G}_m}(\mathsf{k}).$$

The simplest example when this property can fail is given by the composition of any non-trivial $\mathbb{G}_m$-action and $\mathrm{Fr}_{\mathbb{G}_m}\colon \mathbb{G}_m \to \mathbb{G}_m$. Then the differential of the action is 0 at any point and $X^{d\mathbb{G}_m} = X$.

Another simple example to consider is the affine line $\mathbb{A}^1 = \operatorname{Spec} \mathsf{k}[z]$ with the action of $\mathbb{G}_m$, given by some character $n \in \mathbb{Z}$: $t \circ z = t^n z$. We claim that this action is separable if and only if $(n, p) = 1$. Indeed $a_0(\partial_t) = \partial_t(t^n)|_1 \cdot \partial_z = n\partial_z$. More generally a linear diagonalised $\mathbb{G}_m$-action on an affine space $\mathbb{A}^n$ is separable if and only if p does not divide the weights of the coordinates.

Remark 2.4.3.4. If we have a contracting $\mathbb{G}_m$-action on an affine variety $Y = \operatorname{Spec} A$, then there exists a $\mathbb{G}_m$-equivariant embedding $Y \hookrightarrow \mathbb{A}^n$ (with some contracting action on $\mathbb{A}^n$) mapping y_0 to 0. Then, if the weights of the coordinates for the action on $\mathbb{A}^n$ are not divisible by p, $(\mathbb{A}^n)^{d\mathbb{G}_m} = (\mathbb{A}^n)^{\mathbb{G}_m}$. We have an embedding $Y^{d\mathbb{G}_m} \hookrightarrow (\mathbb{A}^n)^{d\mathbb{G}_m} = (\mathbb{A}^n)^{\mathbb{G}_m}$, and so $Y^{d\mathbb{G}_m} = Y^{\mathbb{G}_m}$. In other words, if there exists a set of homogeneous generators of A with degrees not divisible by p, the action is separable.

2.4.4 Definition of a resolution with conical slices

Let k be an algebraically closed field. We first define what we mean by a conical resolution. Note that in our definition there is a separability assumption on the $\mathbb{G}_m$-action, which makes the definition different from the usual one in the case when the characteristic of k is positive.

Definition 2.4.4.1. A resolution $\pi\colon X \to Y$ is called *conical*, if it can be endowed with a separable $\mathbb{G}_m$-action, contracting Y to a single fixed point $y_0 \in Y(\mathsf{k})$ which is then called the *central point* of the resolution.

Here we say that π can be endowed with a separable $\mathbb{G}_m$-action if there are separable $\mathbb{G}_m$-actions on X and Y intertwined by π.

Definition 2.4.4.2. Let $\pi\colon X \to Y$ be a resolution. We say that $y \in Y(\mathsf{k})$ has a *conical étale neighbourhood* if there exists (π', X', Y', y'), such that $\pi'\colon X' \to Y'$ is a conical resolution, y' is its central point and $(\pi, X, Y, y) \sim_{et} (\pi', X', Y', y')$.

Definition 2.4.4.3. A *resolution with conical slices* is a conical resolution $\pi\colon X \to Y$ such that $\mathcal{O}_Y \xrightarrow{\sim} \pi_*\mathcal{O}_X$, $R^1\pi_*\mathcal{O}_X = R^2\pi_*\mathcal{O}_X = 0$ and every point $y \in Y(\mathsf{k})$ has an étale conical neighbourhood.

Remark 2.4.4.4. Note that from the condition $\mathcal{O}_Y \xrightarrow{\sim} \pi_*\mathcal{O}_X$ it follows that Y is normal.

Lemma 2.4.4.5. *Let $\pi\colon X \to Y$ be a resolution with conical slices. Let $(\pi, X, Y, y) \sim_{et} (\pi', X', Y', y')$ be an étale conical neighbourhood of a point $y \in Y(\mathsf{k})$. Then $\pi'\mathcal{O}_{X'} = \mathcal{O}_{Y'}$ and $R^1\pi'_*\mathcal{O}_{X'} = R^2\pi'_*\mathcal{O}_{X'} = 0$.*

Proof. The étale equivalence $(\pi, X, Y, y) \sim_{et} (\pi', X', Y', y')$ is given by a commutative diagram

$$\begin{array}{ccccc} X & \xleftarrow{q_1} & X" & \xrightarrow{q_2} & X' \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi"} & & \downarrow{\scriptstyle \pi'} \\ Y & \xleftarrow{p_1} & Y" & \xrightarrow{p_2} & Y' \end{array}$$

where all horizontal maps are étale. From the flat base change we get that $\pi"\mathcal{O}_{X"} = \mathcal{O}_{Y"}$ and $R^1\pi"_*\mathcal{O}_{X"} = R^2\pi"_*\mathcal{O}_{X"} = 0$. Applying the flat base change one more time, we get that $\pi'\mathcal{O}_{X'} = \mathcal{O}_{Y'}$ and $R^1\pi'_*\mathcal{O}_{X'} = R^2\pi'_*\mathcal{O}_{X'} = 0$ on some open neighbourhood $U' \subset Y'$ of the point y', such that U' is contained in $p_2(Y")$. Since U' is open and contains y', the map $\mathbb{G}_m \times U' \to Y'$, given by the $\mathbb{G}_m$-action, is surjective (since the orbit of any point contains y' in its closure and so intersects U'). From this, using the natural $\mathbb{G}_m$-equivariant structure on $\mathcal{O}_{X'}$ we get that $\pi'\mathcal{O}_{X'} = \mathcal{O}_{Y'}$ and $R^1\pi'_*\mathcal{O}_{X'} = R^2\pi'_*\mathcal{O}_{X'} = 0$ on the whole Y'.

□

2.4.5 Construction of a slice

This subsection is devoted to the mentioned relation between Definition 2.4.4.3 and the existence of conical slices.

Let X be a variety over k and $x \in X(\mathsf{k})$. We denote by $\underline{X}_{\{x\}}$ the trivial pointed resolution (id_X, X, X, x). We also define $\underline{\mathbb{A}}^1$ to be $\underline{\mathbb{A}}^1_{\{0\}}$ in the previous notation. Note that $\underline{\mathbb{A}}^1 \sim_{et} \underline{\mathbb{G}_m}_{\{1\}}$, where equivalence is given by the composition of the embedding $\mathbb{G}_m \to \mathbb{A}^1$ and the translation by -1. More generally, if X is of pure dimension n and x is a smooth point of X, we have $\underline{X}_{\{x\}} \sim_{et} (\underline{\mathbb{A}}^1)^n$. We also remind that if (ξ, Z, W, w) is a pointed resolution, then $(\xi, Z, W, w) \times \underline{\mathbb{A}}^1 = (\xi \times \mathrm{id}, Z \times \mathbb{A}^1, W \times \mathbb{A}^1, w \times \{0\})$.

Lemma 2.4.5.1. *If there exists an étale equivalence $(\xi_1, Z_1, W_1, w) \times \underline{\mathbb{A}}^1 \sim_{et} (\xi_2, Z_2, W_2, w_2) \times \underline{\mathbb{A}}^1$, then there also exists an étale equivalence $(\xi_1, Z_1, W_1, w_1) \sim_{et} (\xi_2, Z_2, W_2, w_2)$.*

Proof. Let $\theta\colon T \to U$ with $u \in U(\mathsf{k})$ be a resolution which realizes the étale equivalence:

$$\begin{array}{ccccc} Z_1 \times \mathbb{A}^1 & \xleftarrow{q_1} & T & \xrightarrow{q_2} & Z_2 \times \mathbb{A}^1 \\ \downarrow{\scriptstyle \xi_1 \times \mathrm{id}} & & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \xi_2 \times \mathrm{id}} \\ W_1 \times \mathbb{A}^1 & \xleftarrow[p_1]{} & U & \xrightarrow[p_2]{} & W_2 \times \mathbb{A}^1 \end{array}$$

By definition, we have étale maps $W_1 \times \mathbb{A}^1 \xleftarrow{p_1} U \xrightarrow{p_2} W_2 \times \mathbb{A}^1$ and a point $u \in U(\mathsf{k})$ that maps to $w_1 \times \{0\}$ and $w_2 \times \{0\}$ respectively. For $i = 1, 2$ consider the natural isomorphisms $T_{w_i \times \{0\}}(W_i \times \mathbb{A}^1) \xrightarrow{\sim} T_{w_i} W_i \oplus T_{\{0\}}\mathbb{A}^1$. The maps p_1 and p_2 are étale, so for any point $z \in U(\mathsf{k})$ they give isomorphisms of Zariski tangent spaces $T_{p_1(z)}W_1 \oplus T_{\{0\}}\mathbb{A}^1 \cong T_z U \cong T_{p_2(z)}W_2 \oplus T_{\{0\}}\mathbb{A}^1$. We denote by $V_{z,1}$ and $V_{z,2}$ the two 1-dimensional subspaces of $T_z U$, which come as preimages of $T_{\{0\}}\mathbb{A}^1$ under these two isomorphisms. For any function $f\colon U \to \mathbb{A}^1$ and any $z \in U(\mathsf{k})$ we have two maps $g_{z,1}, g_{z,2}$, given by the restriction of $df_z\colon T_z U \to T_{f(z)}\mathbb{A}^1$ to subspaces $V_{z,1}$ and $V_{z,2}$. Consider the open subvariety $W_f \subset U$ of points z, such that $f(z) = 0$ and $g_{z,1}$ and $g_{z,2}$ are isomorphisms. By definition, both maps $\mathrm{pr}_i \circ p_i|_{W_f}\colon W_f \to W_i$ (where pr_i is the projection $W \times \mathbb{A}^1 \to W_i$) are unramified at any point z of W_f, therefore are étale. Choosing f such that $f(u) = 0$ and $g_{u,1}$, $g_{u,2}$ are isomorphisms, we get that u is contained in $W_f(\mathsf{k})$. We get an étale correspondence $W_1 \xleftarrow{p_1} W_f \xrightarrow{p_2} W_2$, such that $p_i(u) = w_i$.

Now, by Remark 2.4.2.11, the fiber products $Z_1 \times_{W_1} W_f$ and $Z_2 \times_{W_2} W_f$ are equal and give a resolution $\xi_f\colon Z_f \to W_f$. The natural maps $Z_1 \xleftarrow{q_1} Z_f \xrightarrow{q_2} Z_2$ are étale, since they are base changes of p_1 and p_2. We obtain an étale equivalence $(\xi_1, Z_1, W_1, w_1) \leftarrow (\xi_f, Z_f, W_f, u) \to (\xi_2, Z_2, W_2, w_2)$. □

Lemma 2.4.5.2. *Let (ξ, Z, W, w) be a pointed resolution with an étale equivalence $(\xi, Z, W, w) \times \underline{\mathbb{A}}^1 \sim_{et} (\pi', X', Y', y')$ where (π', X', Y', y') is conical. Then there exists another conical resolution (ξ', Z', W', w') with an étale equivalence $(\xi', Z', W', w') \sim_{et} (\xi, Z, W, w)$.*

Proof. Let $\theta\colon T \to U$ with $u \in U(\mathsf{k})$ be a resolution which realises the étale equivalence between $(\xi, Z, W, w) \times \underline{\mathbb{A}}^1$ and (π', X', Y', y'):

$$\begin{array}{ccccc} Z \times \mathbb{A}^1 & \xleftarrow{q_1} & T & \xrightarrow{q_2} & X' \\ \downarrow{\scriptstyle \xi \times \mathrm{id}} & & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \pi'} \\ W \times \mathbb{A}^1 & \xleftarrow[p_1]{} & U & \xrightarrow[p_2]{} & Y' \end{array}$$

We have a $\mathbb{G}_m$-action on $W \times \mathbb{A}^1$, given by the trivial action on W and the dilation on $\mathbb{A}^1$. We want to find a slice to the orbit of w, but in Y', not in $W \times \mathbb{A}^1$. For this we consider the orbit $w \times \mathbb{G}_m \hookrightarrow W \times \mathbb{A}^1$. We look at its preimage $p_1^{-1}(w \times \mathbb{G}_m) \hookrightarrow U$ and take a connected component which contains y' in its image under p_2. We denote by $S \subset Y'$ the image under p_2. S is a subvariety of Y' of dimension 1, passing through y'. For any function $f\colon Y' \to \mathbb{A}^1$, such that $f(y') = 0$, we have a map $df|_{T_{y'}S}\colon T_{y'}S \to T_0\mathbb{A}^1$.

Recall that we have a $\mathbb{G}_m$-action on Y' with a fixed point y'. So we also have a $\mathbb{G}_m$-action on the ring $\mathcal{O}_{Y'}$ and, as a representation of $\mathbb{G}_m$, it decomposes as a direct sum over all possible characters: $\mathcal{O}_{Y'} = \bigoplus_{i\in\mathbb{Z}} V_i$, where $t(v) = t^i v$ for $v \in V_i$. Remember that Y' is conical, so we have $V_i = 0$ for $i < 0$ and $V_0 = \mathsf{k}$. Let $\mathfrak{m}_{y'} \subset \mathcal{O}_{Y'}$ be the maximal ideal corresponding to the point y', then we also have an isomorphism of $\mathbb{G}_m$-representations $\mathfrak{m}_{y'} \cong \bigoplus_{i>0} V_i$. The natural projection $\mathfrak{m}_{y'} \twoheadrightarrow \mathfrak{m}_{y'}/\mathfrak{m}_{y'}^2$ to the Zariski cotangent space is $\mathbb{G}_m$-equivariant. Now choose a homogeneous element in $\mathfrak{m}_{y'}/\mathfrak{m}_{y'}^2$, which is nonzero on $T_{y'}S$ (considered as a linear function on $T_{y'}Y'$). For that, take any possibly non-homogeneous element which is nonzero, then one of its homogeneous components is nonzero too. If we take any equivariant lift $f \in \mathfrak{m}_{y'}$ under the projection $\mathfrak{m}_{y'} \twoheadrightarrow \mathfrak{m}_{y'}/\mathfrak{m}_{y'}^2$, by construction we will have that $df|_{T_{y'}S}\colon T_{y'}S \to T_0\mathbb{A}^1$ is an isomorphism. Fiber $Z' = f^{-1}(0)$ contains y' and since f was homogeneous it is also endowed with a contracting $\mathbb{G}_m$-action which lifts to $W' = \pi'^{-1}(Z')$. Putting $w' = y'$ and $\xi' = \pi'|_{W'}$ we get a conical pointed resolution (ξ', Z', W', w').

To construct an étale equivalence between (ξ', Z', W', w') and (ξ, Z, W, w), we proceed as in Lemma 2.4.5.1, using the function $f \circ p_2\colon U \to \mathbb{A}^1$ on U. Namely take $W" = f \circ p_2^{-1}(0)$ and $w" = u$. Shrinking $W"$ to $W_{f\circ p_2}$ as in Lemma 2.4.5.1 and repeating the argument we get the desired equivalence. □

Lemma 2.4.5.3. *Let $\pi\colon X \to Y$ be a resolution with conical slices and let $y \neq y_0$ be a point different from the central one. Then there exists a pointed resolution (ξ, Z, W, w) and an étale equivalence $(\pi, X, Y, y) \sim_{et} (\xi, Z, W, w) \times \underline{\mathbb{A}}^1$.*

Proof. We consider the contracting $\mathbb{G}_m$-action on Y. The point y is non-central, so the stabilizer of y is a finite group scheme over k. Since the action is separable, it is étale. Hence Luna's slice theorem [81] can be applied: there is a subvariety W in Y containing y and an étale map $\mathbb{G}_m \times W \to Y$. Resolution $\pi\colon X \to Y$ is $\mathbb{G}_m$-equivariant, so this decomposition also lifts to an étale map $\mathbb{G}_m \times Z \to X$, where $Z = \pi^{-1}(W)$. This gives an étale equivalence $(\pi, X, Y, y) \sim_{et} (\xi, Z, W, w) \times \underline{\mathbb{G}_m}_{\{1\}} \sim_{et} (\xi, Z, W, w) \times \underline{\mathbb{A}}^1$. □

Proposition 2.4.5.4 (Conical slice)**.** *Let $\pi\colon X \to Y$ be a resolution with conical slices, and let $y \neq y_0$ be a non-central point of Y. Then there exists a resolution with conical slices $\xi\colon Z \to W$ with the central point w_0 and an étale equivalence $(\pi, X, Y, y) \sim_{et} (\xi, Z, W, w_0) \times \underline{\mathbb{A}}^1$.*

Proof. By Lemma 2.4.5.3 (π, X, Y, y) is étale equivalent to $(\xi, Z, W, w_0) \times \underline{\mathbb{A}}^1$ for some (ξ, Z, W, w_0), and by Lemma 2.4.5.2 it can be chosen to be conical. We now need to prove that any point $w \neq w_0$ has a conical étale neighborhood. Let $t \in \mathbb{G}_m(\mathsf{k})$, then w has a conical étale neighbourhood if and only if the point tw has. Taking a suitable t we can assume that w is in the image of the correspondence between (π, X, Y, y) and (ξ, Z, W, w_0), and that this correspondence restricts to an étale equivalence between $(\xi, Z, W, w) \times \underline{\mathbb{A}}^1$ and $(\pi, X, Y, \tilde{y})$ for some point $\tilde{y} \in Y(\mathsf{k})$. Now again, by Lemma 2.4.5.3 and Lemma 2.4.5.2, $(\pi, X, Y, \tilde{y})$ is étale equivalent to $(\xi', Z', W', w'_0) \times \underline{\mathbb{A}}^1$ for some conical resolution (ξ', Z', W', w'_0) with the central point w'_0. We get that

$$(\xi, Z, W, w) \times \underline{\mathbb{A}}^1 \sim_{et} (\xi', Z', W', w'_0) \times \underline{\mathbb{A}}^1$$

and from Lemma 2.4.5.1 it follows that $(\xi, Z, W, w) \sim_{et} (\xi', Z', W', w'_0)$, and so any point $\xi\colon Z \to W$ admits a conical étale neighbourhood. The equalities $\xi_*\mathcal{O}_Z = \mathcal{O}_W$ and $R^1\xi_*\mathcal{O}_Z = R^2\xi_*\mathcal{O}_Z = 0$ follow from Lemma 2.4.4.5 and the flat base change (e.g. see Proposition 2.4.6.2 and apply it to $S = \mathbb{A}^1$) for the trivial family of resolutions $\pi \times \mathrm{id}\colon X \times \mathbb{A}^1 \to Y \times \mathbb{A}^1$ over $\mathbb{A}^1$.

□

We end this subsection with the following important remark.

Remark 2.4.5.5. Let $\pi\colon X \to Y$ be a resolution with conical slices and let $y_0 \in Y(\mathsf{k})$ be the central point. Then by Proposition 2.4.5.4 we have an étale correspondence between Y and $\bigsqcup_y W_y \times \mathbb{A}^1$, where we take the union over all points $y \in Y(\mathsf{k}) \setminus y_0$ and W_y is the transversal slice to the $\mathbb{G}_m$-orbit of y, constructed in Proposition 2.4.5.4. Note that it covers $Y \setminus y_0$. Moreover, it extends to an étale correspondence between resolution $\pi\colon X \setminus \pi^{-1}(y_0) \to Y \setminus y_0$ and $\bigsqcup_y \xi_y \times \mathrm{id}\colon Z_y \times \mathbb{A}^1 \to W_y \times \mathbb{A}^1$. Note that since Y is quasi-compact, the union $\bigsqcup_y \xi_y \times \mathrm{id}\colon Z_y \times \mathbb{A}^1 \to W_y \times \mathbb{A}^1$ can be replaced by a finite subunion.

2.4.6 Reductions to characteristic p

Let R be a domain which is finite type and flat over $\mathbb{Z}$. Let $p\colon \mathcal{X} \to S$ be a scheme of finite type over $S = \operatorname{Spec} R$. Let $K = \operatorname{Frac} R$ be the field of fractions and let $\mathbb{K} = \overline{K}$ be its algebraic closure.

We have the geometric generic point map $\eta\colon \operatorname{Spec}\mathbb{K} \to S$ given by the embedding $R \subset \mathbb{K}$. We have structure maps $p_S\colon S \to \operatorname{Spec}\mathbb{Z}$, $p_{\mathcal{X}}\colon \mathcal{X} \to \operatorname{Spec}\mathbb{Z}$. Let k_s be an algebraically closed field and $s\colon \operatorname{Spec}\mathsf{k}_s \to S$ be a geometric point with the function field k_s. Let X_s be the corresponding fiber: $X_s = \mathcal{X} \times_S \operatorname{Spec}\mathsf{k}_s$. Taking $s = \eta$ we obtain the geometric generic fiber X_η, whose image is dense in $\mathcal{X}$.

Consider a proper map $\pi\colon \mathcal{X} \to \mathcal{Y}$ such that $\pi_\eta\colon X_\eta \to Y_\eta$ is a resolution over $\operatorname{Spec}\mathbb{K}$ and $\mathcal{Y}$ is affine over S. Then for every $s\colon \operatorname{Spec}\mathsf{k}_s \to S$ such that X_s is smooth and π_s is birational, we have a resolution $\pi_s\colon X_s \to Y_s$ over $\operatorname{Spec}\mathsf{k}_s$.

Proposition 2.4.6.1 ([38] IV_4, 17.7.8(ii))**.** *Let $\mathcal{X}$, X_η, X_s be as before and assume that X_η is smooth. Then there is a Zariski open $S' \subset S$, such that for any geometric point $s\colon \operatorname{Spec}\mathsf{k}_s \to S'$ the corresponding fiber X_s is smooth.*

Let $Y_\eta^\circ \subset Y_\eta$ be the dense open subset on which π_η is 1-to-1. The map $\pi_\eta^{-1}\colon Y_\eta^\circ \to X_\eta$ is defined over some finite extension $K \subset L$ and can be extended to some étale open $S' \to S$. Since the image of Y_η° is dense in $\mathcal{Y} \times_S S'$, shrinking S' we can also assume that the image of Y_s° in X_s for any $s\colon \operatorname{Spec}\mathsf{k}_s \to S'$ is dense and open as well. This proves that there exists an étale open $S' \to S$ such that for any geometric point $s\colon \operatorname{Spec}\mathsf{k}_s \to S'$ the map $\pi_s\colon X_s \to Y_s$ is a resolution with singularities.

Let's now assume that for $\pi_\eta\colon X_\eta \to Y_\eta$ we have $\mathcal{O}_{Y_\eta} \xrightarrow{\sim} (\pi_\eta)_*\mathcal{O}_{X_\eta}$ and $R^1(\pi_\eta)_*\mathcal{O}_{X_\eta} = R^2(\pi_\eta)_*\mathcal{O}_{X_\eta} = 0$. Replacing S with some Zariski open $S' \subset S$, we can assume that X and Y are both flat over S. In this case we have the base change theorem:

Proposition 2.4.6.2 ([38] III, 6.9.10)**.** *Let $\pi\colon \mathcal{X} \to \mathcal{Y}$ be a morphism of schemes over S and $\mathcal{F}$ be a quasicoherent sheaf of $\mathcal{O}_{\mathcal{X}}$-modules, flat over S. Assume that $\mathcal{O}_{\mathcal{Y}}$-modules $R^i\pi_*\mathcal{F}$ are also flat over S. Let $s\colon \operatorname{Spec}\mathsf{k}_s \to S$ be a geometric point and $s_Y\colon Y_s \to \mathcal{Y}$, $s_X\colon X_s \to \mathcal{X}$ be the corresponding morphisms from fibers. Let $\pi_s\colon X_s \to Y_s$ be the pull-back of π. Then*

$$R^i(\pi_s)_*\left(s_X^*\mathcal{F}\right) \simeq s_Y^*\left(R^i\pi_*\mathcal{F}\right)$$

Remark 2.4.6.3. Returning to our situation and applying the proposition to $\mathcal{F} = \mathcal{O}_{\mathcal{X}}$, we get that $R^i(\pi_s)_*\mathcal{O}_{X_s} \simeq s_Y^*(R^i\pi_*\mathcal{O}_{\mathcal{X}})$, in particular $(\pi_s)_*\mathcal{O}_{X_s} \simeq s_Y^*\mathcal{O}_{\mathcal{X}}$. We have a natural morphism $\mathcal{O}_{\mathcal{Y}} \to \pi_*\mathcal{O}_{\mathcal{X}}$ which is an isomorphism on a dense open subscheme of $\mathcal{Y}$ (containing the image of Y_η). The set of points where this morphism is not an isomorphism is closed, so is contained in

some proper closed subscheme of $\mathcal{Y}$. Its image under projection $p_{Y/S}\colon Y \to S$ is constructible and since it does not contain the generic point, it is contained in some proper closed subscheme of S. Let $S' \subset S$ be the complement, then for any $s\colon \operatorname{Spec} \mathsf{k}_s \to S'$ we have $\mathcal{O}_{Y_s} \xrightarrow{\sim} (\pi_s)_*\mathcal{O}_{X_s}$. Similarly, shrinking S' further, we can also make $R^1(\pi_s)_*\mathcal{O}_{X_s} = R^2(\pi_s)_*\mathcal{O}_{X_s} = 0$ for all $s\colon \operatorname{Spec} \mathsf{k}_s \to S$.

Let's now deal with the $\mathbb{G}_m$-actions, namely with the separability property (see Definition 2.4.3.3). Analogously to Definition 2.4.3.1 we define the subscheme $\mathcal{X}^{d\mathbb{G}_m} \subset \mathcal{X}$ of pseudo-fixed points. Namely, let $\mathbb{G}_m = \operatorname{Spec} R[t, t^{-1}]$ be the multiplicative group scheme over S, let $e\colon S \to \mathbb{G}_m$ be the embedding of the identity, and let $\mathcal{T}_e\mathbb{G}_m = e^*\mathcal{T}_{\mathbb{G}_m}$ be the tangent space at e. We have a nowhere-zero section $\partial_t \in \Gamma(S, \mathcal{T}_e\mathbb{G}_m)$ which gives the trivialisation $\mathcal{T}_e^*\mathbb{G}_m \xrightarrow{\sim} \mathcal{O}_S$. Consider $p^*\mathbb{G}_m$ as a group scheme over $\mathcal{X}$ (namely take $\mathcal{X} \times_S \mathbb{G}_m$) and let $Lie(p^*\mathbb{G}_m)$ be the corresponding Lie algebroid (considered just as a coherent sheaf). Since $p^*\mathbb{G}_m$ is just the fiber product, it is clear that $Lie(p^*\mathbb{G}_m) = p^*(Lie(\mathbb{G}_m)) = p^*(\mathcal{O}_S) = \mathcal{O}_{\mathcal{X}}$. The derivative of the $\mathbb{G}_m$-action gives a map $a_S\colon \mathcal{O}_{\mathcal{X}} \xrightarrow{\sim} Lie(p^*\mathbb{G}_m) \to \operatorname{Der}_R(\mathcal{O}_{\mathcal{X}})$ and the dual map $a_S^*\colon \Omega^1_{\mathcal{X}/S} \to \mathcal{O}_{\mathcal{X}}$, where $\operatorname{Der}(\mathcal{O}_{\mathcal{X}})$ and $\Omega^1_{\mathcal{X}/S}$ are the sheaves of R-linear derivations of $\mathcal{O}_{\mathcal{X}}$ and relative Kähler differentials on $\mathcal{X}$ respectively. The image of a_S^* is some sheaf of ideals on $\mathcal{X}$, which we denote by $\mathcal{I}_{\mathcal{X},a_R}$. Similarly, for any geometric point s, we have a map $a_s^*\colon \Omega^1_{\mathcal{X}/\mathsf{k}_s} \to \mathcal{O}_{X_s}$ and a sheaf of ideals $\mathcal{I}_{X_s,a_s} \subset \mathcal{O}_{X_s}$. We define the subschemes of pseudo-fixed points $\mathcal{X}^{d\mathbb{G}_m} \subset \mathcal{X}$ and $X_s^{d\mathbb{G}_m} \subset X_s$ as subschemes determined by $\mathcal{I}_{\mathcal{X},a_R}$ and $\mathcal{I}_{X_s,a_s}$ correspondingly.

Let $i_s\colon X_s \to \mathcal{X}$ be the natural morphism from the fiber, then the Cartesian square

$$\begin{array}{ccc} X_s & \xrightarrow{i_s} & \mathcal{X} \\ \downarrow & & \downarrow \\ \operatorname{Spec} \mathsf{k}_s & \xrightarrow{s} & S \end{array}$$

induces an isomorphism $i_s^*\Omega^1_{\mathcal{X}/S} \xrightarrow{\sim} \Omega^1_{\mathcal{X}/\mathsf{k}_s}$ which intertwines a_R^* and a_s^*. This gives the equality $i_s^*\mathcal{I}_{\mathcal{X},a_R} = \mathcal{I}_{X_s,a_s}$ and the identification of the corresponding subschemes:

$$X_s^{d\mathbb{G}_m} = X_s \times_{\mathcal{X}} \mathcal{X}^{d\mathbb{G}_m}.$$

Recall that a $\mathbb{G}_m$-action on a scheme X over an algebraically closed field k is called separable if $X^{d\mathbb{G}_m}(\mathsf{k}) = X^{\mathbb{G}_m}(\mathsf{k})$. Recall also that in the case $\operatorname{char} \mathsf{k} = 0$ the action is always separable.

Lemma 2.4.6.4. *Let $\mathcal{X}$ be a scheme over $S = \operatorname{Spec} R$ endowed with a $\mathbb{G}_m$-action. Let $S^{non-sep} \subset S$*

be the subset, consisting of the images of geometric points $s\colon \operatorname{Spec}\mathsf{k}_s \to S$ *such that the induced action on* X_s *fails to be separable. Let* $p_S\colon S \to \operatorname{Spec}\mathbb{Z}$ *be the natural projection, then* $p_S(S^{non-sep}) \subset \operatorname{Spec}\mathbb{Z}$ *is a finite set.*

Proof. Let $\mathcal{Z} = \mathcal{X}^{d\mathbb{G}_m} \setminus \mathcal{X}^{\mathbb{G}_m}$ be the open subscheme of $\mathcal{X}^{\mathbb{G}_m}$. $S^{non-sep}$ is by definition the image of $\mathcal{Z}$ under $p\colon \mathcal{X} \to S$. Let $s\colon \operatorname{Spec}\mathsf{k}_s \to S$ be a geometric point, where k_s is of characteristic 0. We know that in this case $\mathcal{X}^{d\mathbb{G}_m}(\mathsf{k}_s) = X_s^{d\mathbb{G}_m}(\mathsf{k}_s) = X_s^{\mathbb{G}_m}(\mathsf{k}_s) = \mathcal{X}^{\mathbb{G}_m}(\mathsf{k}_s)$. This means that $\mathcal{Z}(\mathsf{k}_s) = \varnothing$ for any such s and as a conclusion the image of $\mathcal{Z}$ under the natural projection $p_{\mathcal{Z}}\colon \mathcal{Z} \to \operatorname{Spec}\mathbb{Z}$ does not contain $\operatorname{Spec}\mathbb{Q}$. On the other hand the image of $\mathcal{Z}$ is constructible subset of $\operatorname{Spec}\mathbb{Z}$, so is a finite set. □

In particular there exist an étale (as well as Zariski) open $S' \to S$ such that for any geometric point $s\colon \operatorname{Spec}\mathsf{k}_s \to S'$ the action on X_s and Y_s is separable.

Proposition 2.4.6.5. *Let* $S = \operatorname{Spec}R$*, where* R *is a domain which is finite type and flat over* $\mathbb{Z}$*. Let* $\eta\colon \operatorname{Spec}\mathbb{K} \to S$ *be the geometric generic point. Let* $\pi\colon \mathcal{X} \to \mathcal{Y}$ *be a map of schemes over* S*, such that* $\pi_\eta\colon X_\eta \to Y_\eta$ *is a resolution of singularities with conical slices. Then there exists an étale open* $S' \to S$ *such that for any geometric point* $s\colon \operatorname{Spec}\mathsf{k}_s \to S'$ *the base change* $\pi_s\colon X_s \to Y_s$ *is a resolution with conical slices.*

Proof. The $\mathbb{G}_m$-action on π_η is defined over some finite extension $\mathbb{K} \subset L$ and consequently can be defined over some étale open $S' \to S$ finite type over S. Moreover, we can assume that the action on $\mathcal{Y} \times_S S'$ is contracting. Let $y_0\colon S' \to \mathcal{X}$ be the central point.

By Proposition 2.4.6.1 and the comment about birationality below it, we can find étale open $S' \to S$, such that $\pi_s\colon X_s \to Y_s$ is a resolution of singularities for any $s\colon \operatorname{Spec}\mathsf{k}_s \to S'$. By Remark 2.4.6.3 we can shrink it further, so that $\mathcal{O}_{Y_s} \xrightarrow{\sim} (\pi_s)_*\mathcal{O}_{X_s}$ and $R^1(\pi_s)_*\mathcal{O}_{X_s} = R^2(\pi_s)_*\mathcal{O}_{X_s} = 0$. Finally, Lemma 2.4.6.4 states that the $\mathbb{G}_m$-action on π_s can also be assumed to be separable. It only remains to find étale conic neighbourhoods.

For that we proceed by induction on the dimension of $\mathcal{X}$. By Lemma 2.4.5.2 we can find a finite collection of resolutions with conical slices $\theta_{i,\eta}\colon Z_{i,\eta} \to W_{i,\eta}$ and an étale correspondence between $\pi_\eta\colon X_\eta \setminus \pi^{-1}((y_0) \to Y_\eta \setminus (y_0)$ and $\bigsqcup \theta_{i,\eta} \times \mathrm{id}\colon Z_{i,\eta} \times \mathbb{A}^1 \to W_{i,\eta} \times \mathbb{A}^1$. Moreover, we can assume that all these correspondences are defined over some étale open $S' \to S$, are étale, and moreover that each $\theta_i\colon \mathcal{Z}_i \to \mathcal{W}_i$ is a conical resolution (over S'). By induction, we can assume that each $\theta_{i,s}\colon Z_{i,s} \to W_{i,s}$ is a resolution with conical slices, and in particular that $w_s \in W_{i,s}$ has

an étale conical neighbourhood. Since $\mathcal{Y} \times_S S'$ is affine over S', shrinking S', we can also assume that any $y_s \colon \mathsf{k}_s \to \mathcal{Y} \times_S S'$ can be factored through some global point $y \colon S' \to \mathcal{Y}$. Let's choose $\mathcal{W}_i \times \mathbb{A}^1$ which covers y under the correspondence and a point $(w,t) \colon S' \to \mathcal{W}_i \times \mathbb{A}^1$ that is in the correspondence with y. Taking its product with $\mathbb{A}^1$ and the base change to $s \colon \operatorname{Spec} \mathsf{k} \to S'$, we obtain an étale conical neighbourhood of $y_s \in Y_s$. □

2.5 $\mathbb{G}_m$-weights of 1-forms on a conical resolution

2.5.1 $\mathbb{G}_m$-weights in characteristic 0

Let R be a domain which is finite type and flat over $\mathbb{Z}$, let $S = \operatorname{Spec} R$ and let $\eta \colon \operatorname{Spec} \mathbb{K} \to S$ be its geometric generic point. Let $\pi \colon \mathcal{X} \to \mathcal{Y}$ be a proper map of finite type schemes over $S = \operatorname{Spec} R$, such that $\mathcal{O}_{\mathcal{Y}} \xrightarrow{\sim} \pi_* \mathcal{O}_{\mathcal{X}}$ and $R^1 \pi_* \mathcal{O}_{\mathcal{X}} = R^2 \pi_* \mathcal{O}_{\mathcal{X}} = 0$. Assume that $\pi_\eta \colon X_\eta \to Y_\eta$ is a conical resolution. Assume moreover, that the corresponding $\mathbb{G}_m$-action on π_η extends to π, and that $\mathcal{Y}$ is contracted to a single point $y_0 \colon \operatorname{Spec} R \to \mathcal{Y}$ under this action. Consequently, $\mathcal{Y}$ is affine and is equal to $\operatorname{Spec} A$ for some finitely generated R-algebra A. A comes with a natural positive grading $A = \bigoplus_{i \geq 0} A^i$, such that $A^0 \simeq R$ (given by y_0) and each A^i is of finite rank over R.

Replacing S with some étale open, following the proof of Proposition 2.4.6.5 we can assume that for any geometric point $s \colon \operatorname{Spec} \mathsf{k}_s \to S$ we have a conical resolution $\pi_s \colon X_s \to Y_s$. The corresponding $\mathbb{G}_m$-action induces an action on the space $V = H^0(X_s, \Omega^1_{X_s})$, which decomposes into a direct sum over weights $k \in \mathbb{Z}$: $H^0(X_s, \Omega^1_{X_s}) = \bigoplus_k H^0(X_s, \Omega^1_{X_s})^k$, where $t \circ \omega = t^k \omega$ for $\omega \in H^0(X_s, \Omega^1_{X_s})^k$. Let also $H^0(X_s, \Omega^1_{X_s,cl})^k \subset H^0(X_s, \Omega^1_{X_s})^k$ denote the subspace of closed forms of weight k.

Theorem 2.5.1.1. *Let $\pi \colon \mathcal{X} \to \mathcal{Y}$ be as above. Then $H^0(X_\eta, \Omega^1_{X_\eta})^k = 0$ for $k \leq 0$, or in other words all weights of the space of 1-forms on X_η are strictly positive:*

$$H^0(X_\eta, \Omega^1_{X_\eta}) = \bigoplus_{k>0} H^0(X_\eta, \Omega^1_{X_\eta})^k.$$

Moreover, there exists an étale open $S' \to S$, such that for any $s \colon \operatorname{Spec} \mathsf{k}_s \to S'$ the same is true for X_s:

$$H^0(X_s, \Omega^1_{X_s}) = \bigoplus_{k>0} H^0(X_s, \Omega^1_{X_s})^k.$$

Proof. Let $p_{\mathcal{X}/S}\colon \mathcal{X} \to S$ and $p_{\mathcal{Y}/S}\colon \mathcal{Y} \to S$ denote the structure morphisms. For every i we have a quasi-coherent sheaf $(p_{\mathcal{X}/S})_*\Omega^i_{\mathcal{X}/S}$ on $S = \operatorname{Spec} R$ and a well-defined map $d\colon (p_{\mathcal{X}/S})_*\Omega^i_{\mathcal{X}/S} \to (p_{\mathcal{X}/S})_*\Omega^{i+1}_{\mathcal{X}/S}$. Its kernel is a subsheaf $(p_{\mathcal{X}/S})_*\Omega^i_{\mathcal{X}/S,cl} \subset (p_{\mathcal{X}/S})_*\Omega^i_{\mathcal{X}}$ of closed relative 1-forms. Recall that we had quasi-coherent sheaves $\pi_*\Omega^i_{\mathcal{X}/S}$ with subsheaves $\pi_*\Omega^i_{\mathcal{X}/S,cl} \subset \pi_*\Omega^i_{\mathcal{X}/S}$ of R-modules on $\mathcal{Y}$. The functor $(p_{\mathcal{Y}/S})_*$ induces isomorphisms:

$$(p_{\mathcal{Y}/S})_*(\pi_*\Omega^i_{\mathcal{X}/S}) \simeq (p_{\mathcal{X}/S})_*\Omega^i_{\mathcal{X}/S} \quad \text{and} \quad (p_{\mathcal{Y}/S})_*(\pi_*\Omega^i_{\mathcal{X}/S,cl}) \simeq (p_{\mathcal{X}/S})_*\Omega^i_{\mathcal{X}/S,cl}.$$

We will denote $(p_{\mathcal{X}/S})_*\Omega^1_{\mathcal{X}/S}$ by $\mathcal{V}$ and $(p_{\mathcal{X}/S})_*\Omega^1_{\mathcal{X}/S,cl}$ by $\mathcal{V}_{cl}$. The corresponding R-modules of global sections will be denoted by V and V_{cl}. V has a natural A-module structure given by the multiplication of differential 1-forms on functions. Shrinking S, we can assume that V and V_{cl} are flat over R, and that the base-change holds for any $s\colon \operatorname{Spec} \mathsf{k}_s \to S$. Namely for any $s\colon \operatorname{Spec} \mathsf{k}_s \to S$ we have isomorphisms of k_s-vector spaces (sheaves on $\operatorname{Spec} \mathsf{k}_s$):

$$(p_{X_s})_*\Omega^1_{X_s} \simeq s^*\mathcal{V} \quad \text{and} \quad (p_{X_s})_*\Omega^1_{X_s,cl} \simeq s^*\mathcal{V}_{cl},$$

where $p_{X_s}\colon X_s \to \operatorname{Spec} \mathsf{k}_s$ is the structure map.

Now consider the $\mathbb{G}_m$-action on $\mathcal{X}$. The sheaf $\Omega^1_{\mathcal{X}/S}$ has a natural $\mathbb{G}_m$-equivariant structure, which gives $\mathbb{G}_m$-equivariant structures on $\mathcal{V}$ and $\mathcal{V}_{cl}$ (with respect to trivial $\mathbb{G}_m$-action on S). In other words we have a grading $V = \bigoplus_{k\in\mathbb{Z}} V^k$ which descends to the grading on $V_{cl} = \bigoplus_{k\in\mathbb{Z}} V^k_{cl}$. The grading on V agrees with the grading on A and turns V into a graded A-module. Since π is proper, V is finitely generated over A, so each V^k is of finite rank over R. V^k_{cl} is an R-submodule of V^k, so is also finitely generated. Moreover, since A is positively graded, we have that $V^k = 0$ for all but finitely many $k < 0$. Consider the corresponding coherent sheaves $\mathcal{V}^k$ and $\mathcal{V}^k_{cl}$ on $\operatorname{Spec} R$. The function $d_k(s) = \dim_{\mathsf{k}_s} s^*\mathcal{V}^k$ on the set of geometric points $s\colon \operatorname{Spec} \mathsf{k}_s \to S$ is semi-continuous. In particular, if d_k is equal to n on generic point, then we can shrink S, such that the function $s \mapsto d_k(s)$ is constant (and equals to n). Note that due to the flat base change $V^k_s \coloneqq s^*\mathcal{V}^k$ is exactly the space $H^0(X_s, \Omega^1_{X_s})^k$. Analogously, we define $d_{k,cl}(s) \coloneqq \dim_{\mathsf{k}_s} s^*\mathcal{V}^k_{cl}$. Let $(V^k_s)_{cl} \subset V^k_s$ denote the corresponding subspace. Note that from the flat base change $(V^k_s)_{cl} = H^0(X_s, \Omega^1_{X_s,cl})^k$.

Let's assume that for some $k < 0$ we have $V^k_\eta = H^0(X_\eta, \Omega^1_{X_\eta})^k \neq 0$. Then we can also find some $s\colon \operatorname{Spec} \mathsf{k}_s \to S$ of characteristic $p > 0$ such that $d_k(s) > 0$. Since $V^k = 0$ for all but finitely many $k < 0$, there exists $N \gg 0$, such that $V^k_s = 0$ for all $k < -N$ and all s.

Lemma 2.5.1.2. *Let $\pi\colon X \to Y$ be a conical resolution over an algebraically closed field k of characteristic p, such that $R^1\pi_*\mathcal{O}_X = R^2\pi_*\mathcal{O}_X = 0$. Then all weights of $H^0(X, \Omega^1_X)$ are nonnegative and all $\mathbb{G}_m$-invariant forms are closed.*

Proof. The action of $\mathbb{G}_m$ on $\pi\colon X \to Y$ gives an action of $\mathbb{G}_m^{(1)}$ on $\pi\colon X^{(1)} \to Y^{(1)}$. This produces an action of $\mathbb{G}_m^{(1)}$ on $H^0(X^{(1)}, \Omega^1_{X^{(1)}})$. Note that $H^0(X^{(1)}, (\mathrm{Fr}_X)_*\Omega^1_X)$ is also endowed with a natural action of $\mathbb{G}_m$ (without a twist), coming from the action on X. These actions differ by $\mathrm{Fr}_{\mathbb{G}_m}$, which agrees with the k-structures on both spaces. Cartier operator C is k-linear and commutes with the action, so it maps a component of weight k to a component of weight $\frac{k}{p}$. In particular all components with weights not divisible by p are killed by C.

Let $V = H^0(X^{(1)}, \Omega^1_{X^{(1)}})$ and let $V^k = H^0(X^{(1)}, \Omega^1_{X^{(1)}})^k$. Let $\alpha \in V^k$ be some homogeneous 1-form of weight $k < 0$. By Lemma 2.2.2.1 the map $\pi_*\mathsf{C}\colon \pi_*\Omega^1_{X,cl} \to \pi_*\Omega^1_{X^{(1)}}$ is surjective and so does the induced map $\mathsf{C}\colon (V^{kp})_{cl} \to (V^k)^{(1)}$ for any k. We can identify $(V^k)^{(1)}$ with V^k using the Frobenius twist $\bullet^{(1)}$. But $V^{p^n k} = 0$ for n big enough, so we get that $V^k = 0$ for all $k < 0$.

It remains to treat the case $k = 0$. By Proposition 2.2.2.4, the map $(V^0)_{cl} \to V^0$ induced by Cartier morphism is a surjection. We get $\dim V^0 \leq \dim(V^0)_{cl}$, but since $(V^0)_{cl} \subset V^0$ the equality $(V^0)_{cl} = V^0$ follows. □

After shrinking S, Lemma 2.5.1.2 is true for all reductions $\pi_s\colon X_s \to Y_s$. From this we get that $V^k_\eta = 0$ for all $k < 0$ and that $V^0_\eta = (V^0_\eta)_{cl}$ as well.

The field $\mathbb{K}$ of functions on the geometric generic point is an algebraic closure of finitely generated field over $\mathbb{Q}$ and admits an embedding $\mathbb{K} \to \mathbb{C}$ to complex numbers. Let $\xi\colon \operatorname{Spec}\mathbb{C} \to \operatorname{Spec}\mathbb{K}$ denote the corresponding morphism and let $\pi_\xi\colon X_\xi \to Y_\xi$ be the corresponding base change of $\pi_\eta\colon X_\eta \to Y_\eta$. We have $d_i(\eta) = d_i(\xi)$, and to prove that $V^0_\eta = 0$ it is enough to prove that $H^0(X_\xi, \Omega^1_{X_\xi})^{\mathbb{G}}_m = 0$ (where $V^{\mathbb{G}}_m \subset V$ denotes the invariants of $\mathbb{G}_m$-action).

Let $\pi^{an}\colon X^{\mathrm{an}} \to Y^{\mathrm{an}}$ be the analytification of π_ξ. By assumption, we have $R^1(\pi_\xi)_*\mathcal{O}_{X_\xi} = 0$ and Serre's GAGA provides isomorphisms

$$(R^i(\pi_\xi)_*\mathcal{O}_{X_\xi})^{\mathrm{an}} \simeq R^i\pi^{\mathrm{an}}_*\mathcal{O}_{X^{\mathrm{an}}}.$$

of coherent sheaves on the complex analytic space Y^{an}. In particular, we have $R^1\pi^{an}_*\mathcal{O}_{X^{\mathrm{an}}} = 0$. Note that since Y^{an} is Stein it also follows that $H^1(X^{\mathrm{an}}, \mathcal{O}_{X^{\mathrm{an}}}) = 0$.

Lemma 2.5.1.3. *Let $\pi\colon X \to Y$ be a proper morphism of complex analytic spaces with $R^1\pi_*\mathcal{O}_X = 0$. Then $R^1\pi_*\underline{\mathbb{C}}_X = 0$.*

Proof. $R^1\pi_*\underline{\mathbb{C}}_X = R^1\pi_*\underline{\mathbb{Z}}_X \otimes \underline{\mathbb{C}}_X$ and, applying $R^\bullet\pi_*$ to the exponential sheaf sequence $0 \to \underline{\mathbb{Z}}_X \to \mathcal{O}_X \xrightarrow{exp} \mathcal{O}_X^\times \to 0$, we get that $R^1\pi_*\underline{\mathbb{Z}}_X = 0$ and, consequently, $R^1\pi_*\underline{\mathbb{C}}_X = 0$. □

In our case Y^{an} is contractible ($\mathbb{G}_m$-action contracts it to the central point). By Zariski's connectedness theorem all fibers of π_ξ (and consequently π^{an}) are connected, so $R^0\pi_*\underline{\mathbb{C}}_{X^{\mathrm{an}}} = \underline{\mathbb{C}}_{Y^{\mathrm{an}}}$. In particular $H^1(Y^{\mathrm{an}}, R^0\pi_*\underline{\mathbb{C}}_{X^{\mathrm{an}}}) = 0$ and since by lemma $R^0\pi_*\underline{\mathbb{C}}_{X^{\mathrm{an}}} = 0$ it follows from the Serre-Leray spectral sequence that $H^1(X^{\mathrm{an}}, \mathbb{C}) = 0$.

Note that $H^1(X^{\mathrm{an}}, \mathbb{C})$ has an algebraic description as the de Rham cohomology $H^1_{\mathrm{dR}}(X_\xi)$. Thus $H^1_{\mathrm{dR}}(X_\xi) = 0$. Now from the Hodge-de Rham spectral sequence, using $E_2^{0,1} \simeq H^1(X_\xi, \mathcal{O}_{X_\xi}) = 0$, the de Rham cohomology $H^1_{\mathrm{dR}}(X_\xi)$ are just equal to $E_2^{1,0}$ which can be identified with the group $H^0(X_\xi, \Omega^1_{X_\xi,cl})/d(H^0(X_\xi, \mathcal{O}_{X_\xi}))$. We get that $H^0(X_\xi, \Omega^1_{X_\xi,cl}) \simeq d(H^0(X_\xi, \mathcal{O}_{X_\xi}))$: all closed differential 1-forms are exact.

Let's now take $\alpha \in H^0(X_\xi, \Omega^1_{X_\xi})^{\mathbb{G}}_m$. It is closed, thus $\alpha = df$ and f should be $\mathbb{G}_m$-invariant. But $H^0(X_\xi, \mathcal{O}_{X_\xi})^{\mathbb{G}_m} = H^0(Y_\xi, \mathcal{O}_{Y_\xi})^{\mathbb{G}_m} \simeq \mathbb{C}$. So $\alpha = 0$. We are done.

□

In particular we get the following corollary:

Proposition 2.5.1.4. *Let $\pi\colon X \to Y$ be a conical resolution of singularities over a field of characteristic 0 with $\mathcal{O}_Y \xrightarrow{\sim} \pi_*\mathcal{O}_X$ and $R^1\pi_*\mathcal{O}_X = R^2\pi_*\mathcal{O}_X = 0$. Then all $\mathbb{G}_m$-weights of $H^0(X, \Omega^1_X)$ are strictly positive.*

2.5.2 $\mathbb{G}_m$-invariant 1-forms in characteristic 2 (and others)

In this section we construct a counterexample to Proposition 2.5.1.4 in characteristic 2, namely we use another famous pathological example given by Enriques surfaces. Following Proposition 7.3.8 in [59], for a classical Enriques surface S over an algebraically closed field k of characteristic 2 we have

- $H^1(S, \mathcal{O}_S) = H^2(S, \mathcal{O}_S) = 0$;
- $H^1_{dR}(S/\mathsf{k}) = \mathsf{k}$.

Let $\mathcal{L}$ be some very ample line bundle on S and let X be the total space of $\mathcal{L}^{\otimes -1}$. Let Y be the cone over S corresponding to $\mathcal{L}$, namely

$$Y = \operatorname{Spec} \bigoplus_{n=0}^{\infty} H^0(S, \mathcal{L}^{\otimes n}).$$

Let y_0 be the point corresponding to the maximal ideal $\mathfrak{m} = \bigoplus_{n>0}^{\infty} H^0(S, \mathcal{L}^{\otimes n})$, this is the origin of the cone.

Algebra $A = \bigoplus_{n=0}^{\infty} H^0(S, \mathcal{L}^{\otimes n})$ can be identified with the ring $\Gamma(X, \mathcal{O}_X)$ of global functions on X. We have two natural maps: the projection $p\colon X \to S$ and the affinization $\pi\colon X \to Y$ which identifies X with the blow-up of Y at y_0. We have a contracting $\mathbb{G}_m$-action on Y which is given by the natural grading on A. Together with the natural $\mathbb{G}_m$-action on X along the fibers of $\mathcal{L}^{\otimes -1}$ it turns π into a conical resolution of singularities.

Let's now compute $R^\bullet \pi_* \mathcal{O}_X$. We have an obvious commutative square

$$\begin{array}{ccc} X & \xrightarrow{p} & S \\ {\scriptstyle \pi}\downarrow & & \downarrow \\ Y & \longrightarrow & \operatorname{Spec} \mathsf{k} \end{array}$$

Y is affine, and so to compute $R^\bullet \pi_* \mathcal{O}_X$ as vector spaces, we can go the other way and compute $\mathbb{H}^\bullet(S, R^\bullet p_* \mathcal{O}_X)$ instead. The map p is affine, so $R^\bullet p_* \mathcal{O}_X = p_* \mathcal{O}_X$ and the latter is equal to the direct sum $\oplus_{n=0}^{\infty} \mathcal{L}^{\otimes n}$. $\mathcal{L}$ is very ample, so $H^i(S, \mathcal{L}^{\otimes k}) = 0$ for $k > 0$ and $i > 0$. Since S is ordinary, the same is also true for $k = 0$. We get that $\mathbb{H}^\bullet(S, R^\bullet p_* \mathcal{O}_X) = \mathbb{H}^0(S, p_* \mathcal{O}_X)$, and so $R^\bullet \pi_* \mathcal{O}_X = \mathcal{O}_Y$.

Now, since $H^1(S, \mathcal{O}_S) = 0$ the space $H^1_{\mathrm{d}R}(S/\mathsf{k})$ is identified with $H^0(S, \Omega^1_S)$. Given that $H^1_{\mathrm{d}R}(S/\mathsf{k}) = \mathsf{k}$ we obtain a non-zero differential 1-form ω on S. The pull-back $p^*\omega$ is then a $\mathbb{G}_m$-invariant form on X. This shows that Proposition 2.5.1.4 is not generally true in finite characteristic.

Remark 2.5.2.1. Note that given any projective variety S over an algebraically closed field k of characteristic p, satisfying $H^1(S, \mathcal{O}_S) = H^2(S, \mathcal{O}_S) = 0$ and $H^1_{dR}(S/\mathsf{k}) \neq 0$ the same construction provides a counterexample for Proposition 2.5.1.4 over k. Note also that this never happens in characteristic 0 due to the Hodge symmetry in cohomology.

Remark 2.5.2.2. Following recent results by Bhatt, Morrow and Sholze [17], for a proper smooth scheme X over $\operatorname{Spec} \mathbb{Z}[1/N]$ (in fact the result is stated for proper smooth formal schemes over

Spf $\mathcal{O}_{\mathbb{C}_p}$, but this case follows) one has $\dim_{\overline{\mathbb{F}}_p} H^1_{\mathrm{dR}}(X_{\overline{\mathbb{F}}_p}/\overline{\mathbb{F}}_p) \geq \dim_{\mathbb{F}_p} H^1_{\mathrm{sing}}(X(\mathbb{C}), \mathbb{F}_p)$ for $(p, N) = 1$. In our case (of a conical resolution of singularities $\pi\colon X \to Y$, let's say over over $\operatorname{Spec}\mathbb{Z}[1/N]$ as well) the scheme X is not proper over $\operatorname{Spec}\mathbb{Z}[1/N]$, but it still has some sort of "properness" if we take $\mathbb{G}_m$-action into account. Also, the action of $\mathbb{G}_m$ on the singular cohomology of $X(\mathbb{C})$ is trivial, since $\mathbb{G}_m$ is connected. So one could probably expect the same inequality to be true if we restrict to $\mathbb{G}_m$-invariants (more precisely to the de Rham cohomology (see [111]) of the quotient stack $[X/\mathbb{G}_m]$) on the left. This motivates the following conjecture:

Conjecture 2.5.2.3. Let $\pi\colon X \to Y$ be a conical resolution of singularities over $\mathbb{Z}[1/N]$ (or, more generally, one could consider the situation where X and Y are formal schemes over Spf $\mathcal{O}_{\mathbb{C}_p}$ or, even more generally, rigid analytic spaces). Let p be a prime not dividing N. Then

$$\dim_{\overline{\mathbb{F}}_p} H^1_{\mathrm{dR}}([X_{\overline{\mathbb{F}}_p}/\mathbb{G}_m]/\overline{\mathbb{F}}_p) \geq \dim_{\mathbb{F}_p} H^1_{\mathrm{sing}}(X(\mathbb{C})_{h\mathbb{C}^\times}, \mathbb{F}_p).$$

We will see in 3.7.1 that at least a variant of this conjecture indeed holds (if one replaces $H^1_{\mathrm{sing}}(X(\mathbb{C}), \mathbb{F}_p)$ with the étale cohomology $H^1_{\text{ét}}(\widehat{X}_{\mathbb{C}_p}, \mathbb{F}_p)$ of the Raynaud generic fiber).

2.5.3 Totally positive forms

Let $\pi\colon X \to Y$ be a resolution with conical slices over an algebraically closed field k of characteristic p, let $y \in Y(\mathsf{k})$ be a point and $(\pi, X, Y, y) \sim_{et} (\pi', X', Y', y')$ be an étale conic neighbourhood. We would like to have a good notion of a positive weight for 1-forms on X, even though there is no $\mathbb{G}_m$-action and grading on the space of 1-forms on X itself.

By definition the étale equivalence is given by a commutative diagram

$$\begin{array}{ccccc} X & \xleftarrow{q_1} & X" & \xrightarrow{q_2} & X' \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi"} & & \downarrow{\scriptstyle \pi'} \\ Y & \xleftarrow{p_1} & Y" & \xrightarrow{p_2} & Y' \end{array}$$

where all horizontal maps are étale. We have isomorphisms $q_1^*\Omega^1_X \simeq \Omega^1_{X"} \simeq q_2^*\Omega^1_{X'}$ since q_1, q_2 are étale, and by the flat base change we also obtain isomorphisms $p_1^*\pi_*\Omega^1_X \simeq \pi"_*\Omega^1_{X"} \simeq p_2^*\pi"_*\Omega^1_{X'}$. By Lemma 2.4.4.5, we also know that $\mathcal{O}_{Y"} \xrightarrow{\sim} \pi"_*\mathcal{O}_{X"}$ and $R^1\pi"_*\mathcal{O}_{X"} = R^2\pi"_*\mathcal{O}_{X"} = 0$, as well as $\mathcal{O}_{Y'} \xrightarrow{\sim} \pi'_*\mathcal{O}_{X'}$ and $R^1\pi'_*\mathcal{O}_{X'} = R^2\pi'_*\mathcal{O}_{X'} = 0$. In particular, $\pi'\colon X' \to Y'$ satisfies the assumptions of Lemma 2.5.1.2, and so all 1-forms on X' have positive weight.

To simplify the notations, we denote $H^0(X, \Omega^1_X)$, $H^0(X', \Omega^1_{X'})$ and $H^0(X", \Omega^1_{X"})$ by M, M' and $M"$. Let also A, A' and $A"$ denote the rings of global functions $H^0(X, \mathcal{O}_X)$, $H^0(X', \mathcal{O}_{X'})$ and $H^0(X", \mathcal{O}_{X"})$. The construction that we are going to give is Zariski local, so we can assume that $Y = \operatorname{Spec} A$ and $Y" = \operatorname{Spec} A"$ (Y' is automatically affine since it is conical).

The torus action on Y' induces a grading on A'. Since Y' is conical, the weights of the action are non-negative and $\mathfrak{m}_{y'}$ is identified with $(A')^{>0} = \oplus_{k>0}(A')^k$, while $A' = \oplus_{k\geq 0}(A')^k$, where $(A')^0 = \mathsf{k}$. The module M' also possesses a grading and by Theorem 2.5.1.1 $(M')^{<0} = \bigoplus_{k<0}(M')^k = 0$. The strictly positive part $(M')^{>0}$ is naturally a submodule, moreover $\mathfrak{m}_{y'} \cdot M \subset M^{>0}$, so the quotient $M'/M'^{>0}$ is a finite-dimensional vector space over k and is supported at the point y' as a module over A'.

The étale equivalence above provides isomorphisms $M \otimes_A A" \cong M" \cong M' \otimes_{A'} A"$. Without loss of generality we can assume that the preimage of y' under p_2 consists of the single point $y"$ (if not we can throw out some divisors from $Y"$ that pass through the other points in the preimage, but do not pass through $y"$), so that $\mathfrak{m}_{y'} \cdot A" \cong \mathfrak{m}_{y"}$. Under these assumptions we define an $A"$-module $(M")^{>0} = (M')^{>0} \otimes_{A'} A"$. Note that $\mathfrak{m}_{y"} \cdot M" \subset (M")^{>0}$, since $\mathfrak{m}_{y"} \cong \mathfrak{m}_{y'} \cdot A"$ and $\mathfrak{m}_{y'} \cdot M' \subset (M')^{>0}$. So, again, $M"/M"^{>0}$ is a finite-dimensional vector space supported at $y"$.

Since M is torsion free, the map $M \to M" \cong M \otimes_A A"$ is an embedding. We define $M^{>0}_{\sim\pi'}$ as the intersection $M \cap (M")^{>0}$. This is a submodule, and since $\mathfrak{m}_y \subset \mathfrak{m}_{y"}$, we have $\mathfrak{m}_y \cdot M \subset M^{>0}_{\sim\pi'}$. $M^{>0}_{\sim\pi'}$ is a submodule of differential 1-forms α on X, such that $q_1^*\alpha = \sum f_i \cdot q_2^*\alpha'_i$ for some 1-forms α_i of positive weight on X' and for some functions f_i on $X"$.

Submodule $M^{>0}_{\sim\pi'}$ a priori depends on the choice of a conical neighbourhood (and a choice of equivalence) and we do not have a good argument for why in fact it does not. But this is not a big deal and instead we consider all conical neighbourhoods at once, namely we define a submodule $M^{>0}_{\mathrm{at}\ y} \subset M$ as the sum of all $M^{>0}_{\sim\pi'}$ inside M. In other words, differential 1-form α lies in $M^{>0}_{\mathrm{at}\ y}$ if and only if $\alpha = \alpha_1 + \alpha_2 + \cdots + \alpha_n$, where each α_i lies in $M^{>0}_{\sim\pi'_i}$ for some conical étale neighbourhood $(\pi, X, Y, y) \sim_{et} (\pi'_i, X'_i, Y'_i, y'_i)$ (possibly different for each i).

Definition 2.5.3.1. Let $\pi\colon X \to Y$ be a resolution with conical slices. Differential 1-form $\alpha \in H^0(X, \Omega^1_X)$ is called *strictly positive at a point* $y \in Y(\mathsf{k})$ if α lies in $M^{>0}_{\mathrm{at}\ y}$.

Remark 2.5.3.2. If the point y admits a conical neighbourhood (π', X', Y', y') for which all differential 1-forms have strictly positive weight, then $M^{>0}_{\mathrm{at}\ y} = M$. Indeed, in this case $(M')^{>0} = M'$, so $(M")^{>0} = M"$ and as a result $M^{>0}_{\sim\pi'} = M$. In particular, if y belongs to the locus $Y^\circ \subset Y$

on which π is 1-to-1, then $(\pi, X, Y, y) \sim_{et} (\mathrm{id}, \mathbb{A}^n, \mathbb{A}^n, 0)$. For $\mathbb{A}^n$ with contracting $\mathbb{G}_m$-action all 1-forms have positive weight, so $M^{>0}_{\mathrm{at}\ y} = M$.

Definition 2.5.3.3. Let $\pi\colon X \to Y$ be a resolution with conical slices. Differential 1-form $\alpha \in H^0(X, \Omega^1_X)$ is called *totally positive* if $\alpha \in M^{>0}_{\mathrm{at}\ y}$ for all points $y \in Y(\mathsf{k})$. We denote the subspace of all totally positive forms on X by $H^0(X, \Omega^1_X)^{\gg 0}$.

In other words $H^0(X, \Omega^1_X)^{\gg 0} = \cap_y M^{>0}_{\mathrm{at}\ y}$. This way the subspace of totally positive 1-forms is naturally an A-submodule. Since A is Noetherian and $M = H^0(X, \Omega^1_X)$ is finitely generated, $H^0(X, \Omega^1_X)^{\gg 0}$ is finetely generated too. We denote the corresponding coherent sheaf on $Y = \operatorname{Spec} A$ by $M^{\gg 0}$ or $(\pi_*\Omega^1_X)^{\gg 0}$. We have an embedding of sheaves $(\pi_*\Omega^1_X)^{\gg 0} \subset \pi_*\Omega^1_X$ and we call the quotient sheaf $\pi_*\Omega^1_X/(\pi_*\Omega^1_X)^{\gg 0}$ by $\mathcal{M}_{\pi,\underline{\mathrm{inv}}}$ and its global sections by $M_{\pi,\underline{\mathrm{inv}}} = H^0(X, \Omega^1_X)/H^0(X, \Omega^1_X)^{\gg 0}$.

Lemma 2.5.3.4. *Let $i_y\colon \operatorname{Spec}\mathsf{k} \hookrightarrow Y$ be the morphism given by a point $y \in Y(\mathsf{k})$. Then there is a natural surjection $i_y^*(\mathcal{M}_{\pi,\underline{inv}}) \twoheadrightarrow M/M^{>0}_{at\ y}$.*

Proof. As we have seen, $\mathfrak{m}_{y'} \cdot M/M^{>0}_{\mathrm{at}\ y'} = 0$, so for each point $y' \neq y$ we have $\mathfrak{m}_y \cdot M^{>0}_{\mathrm{at}\ y'} = M^{>0}_{\mathrm{at}\ y'}$. We have a short exact sequence

$$0 \to M^{\gg 0} \to M \to M_{\pi,\underline{\mathrm{inv}}} \to 0.$$

Tensoring it up with $A/\mathfrak{m}_y$, we get an exact sequence

$$M^{\gg 0}/\mathfrak{m}_y M^{\gg 0} \to M/\mathfrak{m}_y M \twoheadrightarrow M_{\pi,\underline{\mathrm{inv}}}/\mathfrak{m}_y M_{\pi,\underline{\mathrm{inv}}} \to 0$$

The last term in the sequence is by definition $i_y^*(\mathcal{M}_{\pi,\underline{\mathrm{inv}}})$. We also have a short exact sequence

$$0 \to M^{>0}_{\mathrm{at}\ y} \to M \to M/M^{>0}_{\mathrm{at}\ y} \to 0,$$

which, being tensored with $A/\mathfrak{m}_y$, gives

$$M^{>0}_{\mathrm{at}\ y}/\mathfrak{m}_y M^{>0}_{\mathrm{at}\ y} \to M/\mathfrak{m}_y M \twoheadrightarrow M/M^{>0}_{\mathrm{at}\ y} \to 0.$$

By definition of $M^{\gg 0}$ we have an embedding $M^{\gg 0} \subset M^{>0}_{\mathrm{at}\ y}$. Moreover

$$\mathfrak{m}_y M^{\gg 0} = \mathfrak{m}_y \left(\bigcap_{y'} M^{>0}_{\mathrm{at}\ y'} \right) = \mathfrak{m}_y M^{>0}_{\mathrm{at}\ y} \cap \left(\bigcap_{y' \neq y} M^{>0}_{\mathrm{at}\ y'} \right) \subset \mathfrak{m}_y M^{>0}_{\mathrm{at}\ y},$$

so we also have an embedding $M^{\gg 0}/\mathfrak{m}_y M^{\gg 0} \hookrightarrow M^{>0}_{\mathrm{at}\ y}/\mathfrak{m}_y M^{>0}_{\mathrm{at}\ y}$. Finally, the commutative diagram

$$\begin{array}{ccccccc} M^{\gg 0}/\mathfrak{m}_y M^{\gg 0} & \longrightarrow & M/\mathfrak{m}_y M & \twoheadrightarrow & i_y^*(\mathcal{M}_{\pi,\underline{\mathrm{inv}}}) & \longrightarrow & 0 \\ \downarrow & & \| & & \downarrow s & & \\ M^{>0}_{\mathrm{at}\ y}/\mathfrak{m}_y M^{>0}_{\mathrm{at}\ y} & \longrightarrow & M/\mathfrak{m}_y M & \longrightarrow & M/M^{>0}_{\mathrm{at}\ y} & \longrightarrow & 0 \end{array}$$

given by the embedding $M^{\gg 0} \hookrightarrow M^{>0}_{\mathrm{at}\ y}$, shows that the map s should be a surjection. □

In particular if $\mathcal{M}_{\pi,\underline{\mathrm{inv}}} = 0$, then $M^{>0}_{\mathrm{at}\ y} = M$ for all points $y \in Y(\mathsf{k})$.

Lemma 2.5.3.5. *Let $j\colon U \hookrightarrow Y$ be an embedding of an open set and assume that for each $y \in U(\mathsf{k})$ we have $M/M^{>0}_{at\ y} = 0$. Then $j^*\mathcal{M}_{\pi,\underline{inv}} = 0$.*

Proof. All definitions are Zariski local on Y, so without loss of generality we can assume $U = Y$. But then by definition $M^{\gg 0} = \cap_y M^{>0}_{\mathrm{at}\ y} = M$, so $M_{\pi,\underline{\mathrm{inv}}} = 0$. □

Following the discussion in Remark 2.5.3.2, Y° is an example of such U. It follows from the lemma that $\mathcal{M}_{\pi,\underline{\mathrm{inv}}}$ is always supported on the exceptional locus $Y_{exc} = Y \setminus Y^\circ$. It is some strange invariant of the resolution with conical slices which seems to be pretty hard to compute explicitely.

While considering several resolutions at the same moment, we will freely switch between the notation M and $H^0(X, \Omega^1_X)$, hopefully avoiding any confusions. Also if the source of étale equivalence needs to be specified we use a notation $M^{>0}_{\pi\sim\pi'}$ instead of $M^{>0}_{\sim\pi'}$

Proposition 2.5.3.6. *Let $(\pi_1, X_1, Y_1, y_1) \sim_{et} (\pi_2, X_2, Y_2, y_2)$ be an étale equivalence between resolutions with conical slices. Then the stalk $(\mathcal{M}_{\pi_1,\underline{inv}})_{y_1}$ is zero if and only if the stalk $(\mathcal{M}_{\pi_2,\underline{inv}})_{y_2}$ is zero.*

Proof. Let $\pi\colon X \to Y$ be a resolution with conical slices, and let $U \to Y$ be a surjective étale morphism. Let $V = U \times_Y X$, and let $\xi\colon V \to U$ be the corresponding resolution. Despite the fact that U is not necessarily conical and so does not need to be a resolution with conical slices, every point $u \in U(\mathsf{k})$ has an étale conical neighbourhood (e.g. coming from an étale conical neighbourhood for $p(u) \in Y(\mathsf{k})$), and so the definition of $\mathcal{M}_{\xi,\underline{\mathrm{inv}}}$ still makes sense.

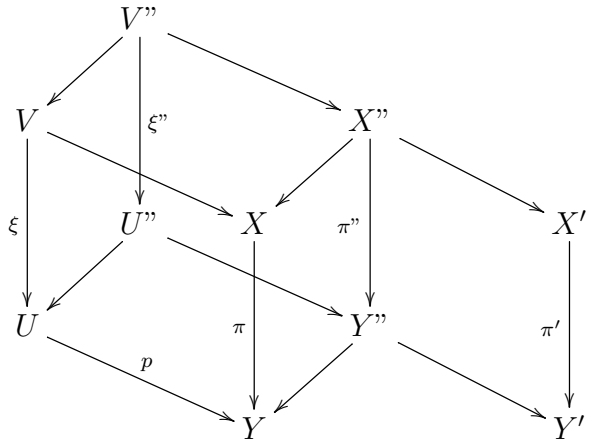
We have a commutative square

$$\begin{array}{ccc} V & \xrightarrow{q} & X \\ {\scriptstyle\xi}\downarrow & & \downarrow{\scriptstyle\pi} \\ U & \xrightarrow{p} & Y \end{array}$$

where both p and q are étale and surjective.

Lemma 2.5.3.7. *$\mathcal{M}_{\xi,\underline{inv}} = 0$ if and only if $\mathcal{M}_{\pi,\underline{inv}} = 0$*

Proof. $\Leftarrow$. We will prove that $H^0(V, \Omega^1_V) = M^{>0}_{\text{at } u}$ for all points $u \in U(\mathsf{k})$. Let $y = p(u)$. Let $(\pi, X, Y, y) \sim_{et} (\pi', X', Y', y')$ be some étale conical neighbourhood of y. Then it also gives rise to an étale neighbourhood $(\xi, V, U, u) \sim (\pi', X', Y', y')$ of u:



Here $\xi" : V" \to U"$ is the pull-back of $\pi" : X" \to Y"$ under $p : U \to Y$. Let $U = \operatorname{Spec} B$. Now, since $H^0(X, \Omega^1_X) = M^{>0}_{\text{at } y}$, we know that any 1-form α on X is a sum of forms α_i which lie in $M^{>0}_{\pi\sim\pi'_i}$ for some étale neighbourhoods $(\pi, X, Y, y) \sim (\pi'_i, X'_i, Y'_i, y'_i)$. It is easy to see from the definitions that if α_i lies in $M^{>0}_{\pi\sim\pi'_i}$, then $q^*(\alpha_i)$ lies in $M^{>0}_{\xi\sim\pi'_i}$ (taking the pull-back of the corresponding expression on $X"$ to $V"$). But $H^0(V, \Omega^1_V) = H^0(X, \Omega^1_X) \otimes_A B$, and any differential 1-form β on V is of the form $\sum_j f_j q^*(\tilde{\alpha}_j)$ for some 1-forms $\tilde{\alpha}_j$ on X. So β lies in $M^{>0}_{\text{at } u}$, and we are done.

$\Rightarrow$ We will prove that $H^0(X, \Omega^1_X) = M^{>0}_{\text{at } y}$ for all points $y \in Y(\mathsf{k})$. Let u be a point which maps to y, and let $(\xi, V, U, u) \sim_{et} (\xi', V', U', u')$ be an étale conical neighbourhood of u. It also gives rise

to a neighbourhood $(\pi, X, Y, y) \sim_{et} (\xi', X', Y', y')$.

$$\begin{array}{ccccccc} V' & \longleftarrow & V" & \longrightarrow & V & \xrightarrow{q} & X \\ \downarrow{\scriptstyle \xi'} & & \downarrow{\scriptstyle \xi"} & & \downarrow{\scriptstyle \xi} & & \downarrow{\scriptstyle \pi} \\ U' & \longleftarrow & U" & \longrightarrow & U & \xrightarrow{p} & Y \end{array}$$

We have A-submodule $M^{>0}_{\pi\sim\xi'} \subset H^0(X, \Omega^1_X)$, and B-submodule $M^{>0}_{\xi\sim\xi'} \subset H^0(X, \Omega^1_X)$. However both of them are defined as intersections with the same B"-submodule of $H^0(V", \Omega^1_{V"})$: $M^{>0}_{\pi\sim\xi'} = H^0(X, \Omega^1_X) \cap H^0(V", \Omega^1_{V"})^{>0}$ and $M^{>0}_{\xi\sim\xi'} = H^0(V, \Omega^1_V) \cap H^0(V", \Omega^1_{V"})^{>0}$. From faithfully flat descent we get that $M^{>0}_{\xi\sim\xi'} = M^{>0}_{\pi\sim\xi'} \otimes_A B$. Since $M^{>0}_{\text{at } u} = \cup_{\xi\sim\xi'} M^{>0}_{\xi\sim\xi'}$ from this we obtain that $M^{>0}_{\text{at } u} \subset M^{>0}_{\text{at } y} \otimes_A B$. But $M^{>0}_{\text{at } u} = H^0(V, \Omega^1_V)$, so again by faithfully flat descent $M^{>0}_{\text{at } y} = H^0(X, \Omega^1_X)$.

□

The statement of the proposition now easily follows from the lemma. If one of the stalks, say $(\mathcal{M}_{\pi_1,\underline{\text{inv}}})_{y_1}$, is zero, it means that $\mathcal{M}_{\pi_1,\underline{\text{inv}}}$ is zero on some Zariski neighbourhood U of y_1. As definition of $\mathcal{M}_{\pi_1,\underline{\text{inv}}}$ was Zariski local we can replace Y_1 with U. An étale correspondence that gives the equivalence $(\pi_1, X_1 \times_{Y_1} U, U, y_1) \sim_{et} (\pi_2, X_2, Y_2, y_2)$ is surjective on some Zariski neighbourhoods U_1, U_2 of points y_1 and y_2. Now, applying the lemma twice (for the first and then for the second arrow in the étale correspondence), we get that $\mathcal{M}_{\pi_2,\underline{\text{inv}}}$ is zero on U_2, so the stalk $(\mathcal{M}_{\pi_2,\underline{\text{inv}}})_{y_2}$ is zero.

□

Lemma 2.5.3.8. *Let $\pi_1 \colon X_1 \to Y_1$ and $\pi_2 \colon X_2 \to Y_2$ be two resolutions with conical slices, such that $\mathcal{M}_{\pi_1,\underline{inv}}$ and $\mathcal{M}_{\pi_2,\underline{inv}}$ are equal to 0. Let $\pi_1 \times \pi_2 \colon X_1 \times X_2 \to Y_1 \times Y_2$ be their product. Then $\mathcal{M}_{\pi\times\pi',\underline{inv}} = 0$.*

Proof. We have $\Omega^1_{X_1\times X_2} = \Omega^1_{X_1} \boxtimes \mathcal{O}_{X_2} \oplus \mathcal{O}_{X_1} \boxtimes \Omega^1_{X_2}$. We will show that $\pi_*(\mathcal{O}_{X_1} \boxtimes \Omega^1_{X_2})$ lies in $(\pi_*\Omega^1_{X_1\times X_2})^{\gg 0}$. Let α be a section of $\pi_*(\mathcal{O}_{X_1} \boxtimes \Omega^1_{X_2})$, and $(y_1, y_2) \in (Y_1 \times Y_2)(\mathsf{k})$ be a point. We can assume that α is decomposable: $\alpha = f \boxtimes \omega$. Since $\mathcal{M}_{\pi_2,\underline{\text{inv}}} = 0$, ω is equal to $\omega_1 + \cdots + \omega_n$, where $\omega_i \in M^{>0}_{\sim\pi'_i}$ for some étale conical neighbourhoods $(\pi'_i, X'_i, Y'_i, y'_i)$ of $y_2 \in Y_2(\mathsf{k})$. Now consider any étale neighbourhood $(\widetilde{\pi}_1, \widetilde{X_1}, \widetilde{Y_1}, \tilde{y_1})$ of the point $y_1 \in Y_1(\mathsf{k})$. Each differential form $f \boxtimes \omega_i$ lies in $M^{>0}_{\sim\pi'_i\times\widetilde{\pi}_1}$, so $f \boxtimes \omega$ is strictly positive at (y_1, y_2). Since the point (y_1, y_2) was arbitrary, we get that $f \boxtimes \omega$ is totally positive, and $\pi_*(\mathcal{O}_{X_1} \boxtimes \Omega^1_{X_2}) \subset (\pi_*\Omega^1_{X_1\times X_2})^{\gg 0}$. Analogously, $\pi_*(\Omega^1_{X_1} \boxtimes \mathcal{O}_{X_2}) \subset (\pi_*\Omega^1_{X_1\times X_2})^{\gg 0}$. Thus $\mathcal{M}_{\pi\times\pi',\underline{\text{inv}}} = 0$. □

Setting π_2 to be $\mathrm{id}_{\mathbb{A}^1}\colon \mathbb{A}^1 \to \mathbb{A}^1$, we get that $\mathcal{M}_\pi = 0$ implies $\mathcal{M}_{\pi\times \mathrm{id}_{\mathbb{A}^1}} = 0$. The converse is also true:

Lemma 2.5.3.9. *Let $\pi\colon X \to Y$ be a resolution with conical slices such that $\mathcal{M}_{\pi\times\mathrm{id}_{\mathbb{A}^1},\underline{inv}} = 0$. Then $\mathcal{M}_{\pi,\underline{inv}} = 0$.*

Proof. Conisder the projection $p\colon X\times\mathbb{A}^1 \to X$ and the embedding $\iota\colon X\times\{0\}\hookrightarrow X\times\mathbb{A}^1$. Let $\alpha\in H^0(X,\Omega^1_X)$ be a 1-form on X. We have $p^*\alpha\in H^0(X\times\mathbb{A}^1,\Omega^1_{X\times\mathbb{A}^1})$. Since $\mathcal{M}_{\pi\times\mathrm{id}_{\mathbb{A}^1},\underline{\mathrm{inv}}} = 0$, for any $y\in Y(\mathsf{k})$ there exist differential 1-forms $\alpha_1,\dots,\alpha_n$ on $Y\times\mathbb{A}^1$ such that $p^*\alpha = \alpha_1+\dots+\alpha_n$, and $\alpha_i\in M^{>0}_{\sim\pi'_i}$ for some étale conical neighbourhood (π'_i,X'_i,Y'_i,y'_i) of $y\times\{0\}\in(Y\times\mathbb{A}^1)(\mathsf{k})$. Following the proof of Lemma 2.4.5.2, for each i we can find a $\mathbb{G}_m$-invariant subvariety $W'_i\hookrightarrow Y'_i$, such that the conical neighbourhood above restricts to a conical étale neighbourhood $(\pi,X,Y,y)\sim_{et}(\xi'_i,Z'_i,W'_i,y'_i)$, where $Z'_i = X'_i\times_{Y'_i}W'_i$. Weights are preserved under restriction, so $\iota^*\alpha_i$ lies in $M^{>0}_{\sim\xi'_i}$ for each i. On the other hand, $\alpha = \iota^*p^*\alpha = \iota^*\alpha_1+\dots\iota^*\alpha_n$, so α is strictly positive at y. It follows that $\mathcal{M}_{\pi,\underline{\mathrm{inv}}} = 0$. □

Definition 2.5.3.10. A resolution with conical slices $\pi\colon X\to Y$ is called *strictly positive* if the sheaf $\mathcal{M}_{\pi,\underline{\mathrm{inv}}}$ is zero. In other words, the defining property of strict positivity is that all 1-forms on X should be totally positive.

Lemma 2.5.3.8 and Lemma 2.5.3.9 show that $\pi\colon X\to Y$ is strictly positive if and only if $\pi\times\mathrm{id}_{\mathbb{A}^1}\colon X\times\mathbb{A}^1\to Y\times\mathbb{A}^1$ is.

2.5.4 Application to resolutions with conical slices

Recall the statement of Proposition 2.4.6.5: for a map $\pi\colon\mathcal{X}\to\mathcal{Y}$ of schemes over $S=\operatorname{Spec}R$, such that the geometric generic fiber is a resolution of singularities with conical slices, we can find an étale open $S'\to S$, such that for any geometric point $s\colon\operatorname{Spec}\mathsf{k}_s\to S'$ the base change $\pi_s\colon X_s\to Y_s$ is a resolution with conical slices as well.

Proposition 2.5.4.1. *In the context of Proposition 2.4.6.5, we can find an étale open $S'\to S$ such that for any $s\colon\operatorname{Spec}\mathsf{k}_s\to S'$ we have $\mathcal{M}_{\pi_s,\underline{inv}} = 0$. In other words, for any $s\colon\operatorname{Spec}\mathsf{k}_s\to S'$ all differential 1-forms on X_s are totally positive.*

Proof. We proceed by induction on $\dim X_\eta$ using Remark 2.4.5.5. Following the proof of Proposition 2.4.6.5, we can find a finite collection of resolutions with conical slices $\theta_i\colon\mathcal{Z}_i\to\mathcal{W}_i$ defined

over S', such that there exists an étale correspondence between $\pi\colon \mathcal{X} \setminus \pi^{-1}((y_0) \to \mathcal{Y} \setminus (y_0)$ and $\bigsqcup \theta_i \times \mathrm{id}\colon \mathcal{Z}_i \times \mathbb{A}^1 \to \mathcal{W}_i \times \mathbb{A}^1$, defined over S' as well. So by the induction hypothesis and Lemma 2.5.3.8, we can find S', such that $\mathcal{M}_{\pi_s,\underline{\mathrm{inv}}}$ is supported at the central point y_0. Finally, for the central point the statement follows from Theorem 2.5.1.1.

□

Proposition 2.5.4.2. *Let $\pi\colon \mathcal{X} \to \mathcal{Y}$ be a map of schemes over $S = \operatorname{Spec} R$ finite type and flat over $\mathbb{Z}$, such that the generic fiber $\pi_\eta\colon X_\eta \to Y_\eta$ is a resolution with conical slices. Then there exists an étale open $S' \to S$, such that for any geometric point $s\colon \operatorname{Spec} \mathsf{k}_s \to S'$ the corresponding fiber $\pi_s\colon X_s \to Y_s$ is a strictly positive resolution with conical slices.*

Proof. Follows directly from Proposition 2.4.6.5 and Proposition 2.5.4.1. □

2.6 Descent for resolutions with conical slices

2.6.1 Sheaf $\mathcal{Q}_{\pi,N}$ and étale equivalences

Recall (Definition 2.3.2.4) the N-th obstruction sheaf $\mathcal{Q}_{\pi,N} = \pi_*\Omega^1_{X^{(N)}}/\mathcal{S}_{\pi,N}$, where $\mathcal{S}_{\pi,N}$ is the subsheaf of 1-forms that can be obtained as the N-th power of the Cartier operator applied to a Kähler differential on Y.

Recall the notion of an étale equivalence from Definition 2.4.2.10. It would be useful to know that to prove $\mathcal{Q}_{\pi,N} = 0$ it is enough to prove $\mathcal{Q}_{\pi',N} = 0$ for some (π', X', Y', y') étale equivalent to (π, X, Y, y). Some form of this is indeed true:

Lemma 2.6.1.1. *Given an étale equivalence $(\pi, X, Y, y) \sim_{et} (\pi', X', Y', y')$, $(\mathcal{Q}_{\pi,N})_y = 0$ if and only if $(\mathcal{Q}_{\pi',N})_{y'} = 0$.*

Proof. Let the equivalence be given by a commutative diagram

$$\begin{array}{ccccc} X & \xleftarrow{q_1} & X" & \xrightarrow{q_2} & X' \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi"} & & \downarrow{\scriptstyle \pi'} \\ Y & \xleftarrow{p_1} & Y" & \xrightarrow{p_2} & Y' \end{array}$$

By definition q_i's and p_i's are étale, in particular they are flat. From flat base change we have isomorphisms $p_1^*\pi_*\Omega^1_{X^{(N)}} \simeq \pi"_*\Omega^1_{X"^{(N)}} \simeq p_2^*\pi'_*\Omega^1_{X'^{(N)}}$. Moreover, by functoriality of Cartier operator, we also get isomorphisms $p_1^*\mathcal{S}_{\pi,N} \simeq \mathcal{S}_{\pi",N} \simeq p_2^*\mathcal{S}_{\pi',N}$ and, consequently, $p_1^*\mathcal{Q}_{\pi,N} \simeq \mathcal{Q}_{\pi",N} \simeq p_2^*\mathcal{Q}_{\pi',N}$.

Let $y" \in Y"(\mathsf{k})$ be a point which maps to y and y'. By faithfully flat descent, we get that $(p_1^*\mathcal{Q}_{\pi,N})_{y"} = 0$ if and only if $(\mathcal{Q}_{\pi,N})_y = 0$. The same is true if we replace y with y'. So, $(\mathcal{Q}_{\pi',N})_{y'} = 0 \Leftrightarrow (\mathcal{Q}_{\pi",N})_{y"} = 0 \Leftrightarrow (\mathcal{Q}_{\pi,N})_y = 0$.

□

Recall also that we have the subsheaf $(\pi_*\Omega^1_X)^{\gg 0} \subset \pi_*\Omega^1_X$ (see Definition 2.5.3.3) of totally positive forms. We have the natural map $(\pi_*\Omega^1_{X^{(N)}})^{\gg 0} \to \mathcal{Q}_{\pi,N}$ and we denote the image by $\mathcal{Q}^{\gg 0}_{\pi,N}$. If $\pi\colon X \to Y$ is strictly positive (see Definition 2.5.3.10) we have $(\pi_*\Omega^1_{X^{(N)}})^{\gg 0} = \pi_*\Omega^1_{X^{(N)}}$ and so $\mathcal{Q}^{\gg 0}_{\pi,N} = \mathcal{Q}_{\pi,N}$ for all N.

2.6.2 Descent for a strictly positive resolution with conical slices

Let $\pi\colon X \to Y$ be a conical resolution with the central point y_0 and $Y = \operatorname{Spec} A$. We recall the discussion about weights and Cartier operator from Lemma 2.5.1.2. The action of $\mathbb{G}_m$ on $\pi\colon X \to Y$ gives an action of $\mathbb{G}_m^{(1)}$ on $\pi\colon X^{(1)} \to Y^{(1)}$. This produces an action of $\mathbb{G}_m^{(1)}$ on $H^0(X^{(1)}, \Omega^1_{X^{(1)}})$ of which we think as the standard one. Note that $H^0(X^{(1)}, (\mathrm{Fr}_X)_*\Omega^1_X)$ is also endowed with the natural action of $\mathbb{G}_m$ (without twist) coming from the action on X. These actions differ by $\mathrm{Fr}_{\mathbb{G}_m}$, and this agrees with the natural k-structures on both spaces. Cartier operator C is k-linear and so maps the component of weight k to the component of weight $\frac{k}{p}$. In particular all components with weights not divisible by p are killed by C.

The $A^{(N)}$-submodule $S_{\pi,N} \subset H^0(X^{(N)}, \Omega^1_{X^{(N)}})$ defines a $\mathbb{G}_m^{(N)}$-invariant subspace (since the subspace $H^0(Y, \Omega^1_Y) \subset H^0(X, \Omega^1_X)$ is $\mathbb{G}_m$-invariant and C is k-linear), so $\mathcal{S}_{\pi,N}$ has a structure of $\mathbb{G}_m$-equivariant sheaf on $Y^{(N)}$. This way $\mathcal{Q}_{\pi,N}$ also obtains a $\mathbb{G}_m^{(N)}$-equivariant structure, in particular its support should be a $\mathbb{G}_m^{(N)}$-invariant subvariety of $Y^{(N)}$. Note that the central point of the resolution $\pi\colon X^{(N)} \to Y^{(N)}$ is the point $\mathrm{Fr}^N_Y(y_0)$.

We are ready to prove the key lemma:

Lemma 2.6.2.1. *Let $\pi\colon X \to Y$ be a conical resolution. Then $Q_{\pi,N}$ is finite-dimensional over k if and only if the support of $\mathcal{Q}_{\pi,N}$ is the central point. In this case, there exists $s > 0$, such that the image of positive weight 1-forms $H^0(X^{(1)}, \Omega^1_{X^{(1)}})^{>0}$ in $Q_{\pi,N+s}$ is zero.*

Proof. Since $Q_{\pi,N}$ is finite-dimensional over k, the corresponding sheaf $\mathcal{Q}_{\pi,N}$ has support only at a finite set of point. But since the support should be $\mathbb{G}_m^{(N)}$-invariant, the only option is the central point $\mathrm{Fr}^N_Y(y_0)$. This proves one implication. To prove the other, note that $\mathcal{Q}_{\pi,N}$ is coherent, so if

it is supported at the single point $\mathrm{Fr}_Y^N(y_0)$, its global sections should be finite-dimensional. We see that in this case $S_{\pi,N}$ has finite codimension in $H^0(X^N, \Omega^1_{X^{(N)}})$, in particular there is an integer r such that all forms of weight greater then r lie in $\mathcal{S}_{\pi,N}$.

Let's take a $\mathbb{G}_m^{(N)}$-equivariant lift $Q_{\pi,N} \dashrightarrow H^0(X^{(N)}, \Omega^1_{X^{(N)}})$ (as k-vector spaces). Let $\alpha_1, \dots, \alpha_n$ be a homogeneous basis of the image and let $d_i \geq 0$ be the weight of α_i. Without loss of generality we can assume that α is equal to some α_i with $d_i > 0$. Then by surjectivity of the Cartier operator there exists a homogeneous closed 1-form β_1 on $X^{(N-1)}$ of weight pd_i, such that $\mathsf{C}(\beta_1) = \alpha_i$. Repeating this procedure s_i times, where $s_i > 0$ is any number with $p^{s_i} d_i > r$, we obtain a 1-form $\beta_{s_i} \in S_{\pi,N}$, such that $\mathsf{C}^{s_i}(\beta_{s_i}) = \alpha_i$. It follows that $\alpha_i \in S_{\pi,N+s_i}$. Finding the maximum s of all s_i (for all i such that $d_i > 0$), we get that all 1-forms with positive weight lie in $S_{\pi,N+s}$ and so their image in $Q_{\pi,N+s}$ is 0. □

Proposition 2.6.2.2. *Let $\pi\colon X \to Y$ be a strictly positive resolution with conical slices. Then $\mathcal{Q}_{\pi,N} = 0$ for $N \gg 0$.*

Proof. We proceed by induction on $\dim X$. The base of induction is the case of the trivial resolution $\pi\colon \operatorname{Spec}\mathsf{k} \to \operatorname{Spec}\mathsf{k}$, where $\mathcal{Q}^{\gg 0}_{\pi,N} = \mathcal{Q}_{\pi,N} = 0$ is zero for all N. To prove that the sheaf $\mathcal{Q}^{\gg 0}_{\pi,N}$ is zero in general, we need to prove that its stalk at each point is. Let $y \neq y_0$ be a non-central point of Y. Then, by Proposition 2.4.5.4, (π, X, Y, y) is étale equivalent to the product $(\pi', X', Y', y'_0) \times \underline{\mathbb{A}}^1$ for some resolution with conical slices $\pi'\colon X' \to Y'$ with the central point y'_0. By Lemma 2.6.1.1 it is enough to prove that $(\mathcal{Q}_{\pi'\times\mathrm{id},N})_{\mathrm{Fr}^N_{Y'}(y'_0)\times\{0\}} = 0$. Since π is strictly positive, by Lemma 2.5.3.8 and Proposition 2.5.3.6 we get that π' is strictly positive too. So, by induction, we can assume that there exists N, such that $\mathcal{Q}_{\pi',N} = 0$. Now, by Lemma 2.3.2.5, we get that the stalk $(\mathcal{Q}_{\pi'\times\mathrm{id},N})_{\mathrm{Fr}^N_{Y'}(y'_0)\times\{0\}}$ is zero. Moreover, by Remark 2.4.5.5 we can choose such N uniformly. So there exists N, such that $(\mathcal{Q}_{\pi,N})_{\mathrm{Fr}^N_Y(y)} = 0$ for all $y \neq y_0$. This means that the sheaf $\mathcal{Q}_{\pi,N}$ is supported at $\mathrm{Fr}^N_Y(y_0)$. We also know that it is coherent, so we get that the space of global sections of $Q_{\pi,N}$ is finite-dimensional.

Let's take a $\mathbb{G}_m$-equivariant lift $Q_{\pi,N} \dashrightarrow H^0(X^{(N)}, \Omega^1_{X^{(N)}})$. Let $\alpha_1, \dots, \alpha_n$ be a homogeneous basis of the image of this lift. Since $\mathcal{M}_{\pi,\underline{\mathrm{inv}}} = 0$ we know that each α_i is totally positive. In particular, it is strictly positive at y_0, so $\alpha_i = \beta_{i1} + \dots + \beta_{in}$, where for each k there exists an étale conical neighbourhood $(\pi, X, Y, y_0) \sim_{et} (\pi'_{ik}, X'_{ik}, Y'_{ik}, y'_{ik})$, such that $\beta_{ik} \in M^{>0}_{\sim\pi'_{ik}}$ (see Definition 2.5.3.1). By definition it means that $q_1^*(\beta_{ik}) = \sum_j f_j \cdot q_2^*(\beta'_j)$ for some differential 1-forms $\beta'_j \in H^0(X'_{ik}, \Omega^1_{X'_{ik}})$ with positive weight and some functions f_j on $X"_{ik}$. Using the isomorphisms $q_1^* \mathcal{Q}_{\pi,N} \cong \mathcal{Q}_{\pi"_{ik},N} \cong q_2^* \mathcal{Q}_{\pi'_{ik},N}$ from the proof of Lemma 2.6.1.1 we see that it is enough to prove

that the classes $[\beta'_j] \in Q_{\pi'_{ik},N}$ are zero. Moreover from the same isomorphisms we see that the stalk $(\mathcal{Q}_{\pi'_{ik},N})_{\mathrm{Fr}_Y^N(y'_{ik})}$ is finite-dimensional. Since the support should be $\mathbb{G}_m^{(N)}$-invariant we see that it is actually supported at $\mathrm{Fr}_Y^N(y'_{ik})$. So we are in the framework of Lemma 2.6.2.1. Fixing a decomposition as above for each α_i, and a resolution for each β_{ik}, using Lemma 2.6.2.1, we see that there exists $s > 0$, such that the images in $Q_{\pi'_{ik},N+s}$ of all β'_j's for all i and k are zero. So each α_i in $Q_{\pi,N+s}$ is 0, and since this was the basis of representatives for $Q_{\pi,N}$ and $Q_{\pi,N} \twoheadrightarrow Q_{\pi,N+s}$ we get that $Q_{\pi,N+s} = 0$.

□

Recall that if $\mathcal{Q}_{\pi,N} = 0$, then $[\mathcal{D}_{X\alpha}]$ descends to $Y^{(1)}$ for all differential 1-forms α (Definition 2.3.2.4). This way from Proposition 2.6.2.2 we obtain

Theorem 2.6.2.3. *Let $\pi\colon X \to Y$ be a strictly positive resolution with conical slices. Then the classes $[\mathcal{D}_{X,\alpha}]$ descend to $Y^{(1)}$ for all $\alpha \in H^0(X^{(1)}, \Omega^1_{X^{(1)}})$.*

Finally, by Proposition 2.5.4.2 we also get the following result, which is the main theorem in this paper:

Theorem 2.6.2.4. *Let $\pi\colon \mathcal{X} \to \mathcal{Y}$ be a map of schemes over $S = \operatorname{Spec} R$ finite type and flat over $\mathbb{Z}$, such that the generic fiber $\pi_\eta\colon X_\eta \to Y_\eta$ is a resolution with conical slices. Then there exists an étale open $S' \to S$, such that for any geometric point $s\colon \operatorname{Spec} \mathsf{k}_s \to S'$*

- *the corresponding fiber $\pi_s\colon X_s \to Y_s$ is a strictly positive resolution with conical slices,*
- *the classes $[\mathcal{D}_{X_s,\alpha}]$ descend to $Y_s^{(1)}$ for all $\alpha \in H^0(X_s^{(1)}, \Omega^1_{X_s^{(1)}})$.*

2.7 Examples of resolutions with conical slices

To construct examples, we use Theorem 2.6.2.4: we prove that certain well-known resolutions in characteristic 0 are actually resolutions with conical slices. All of them are the so-called symplectic resolutions (see Section 2.7.1), and particular examples include Nakajima quiver varieties (Section 2.7.2), hypertoric varieties (Section 2.7.3) and Slodowy slices (Section 2.7.4). If we prove that all of them are resolutions with conical slices, Theorem 2.6.2.4 will say that for a general reduction $s\colon \operatorname{Spec} \mathsf{k}_s \to S'$ to characteric p, all classes $[\mathcal{D}_{X_s,\alpha}]$ in $\mathrm{Br}(X_s^{(1)})$ descend to $Y_s^{(1)}$.

2.7.1 Background on symplectic resolutions

Let $\mathbb{K}$ be a field of characteristic 0.

Definition 2.7.1.1. A *symplectic resolution* is a smooth algebraic variety X over $\mathbb{K}$ equipped with a non-degenerate differential 2-form ω, such that the affinisation map $\pi\colon X \to Y = \operatorname{Spec} H^0(X, \mathcal{O}_X)$ is a birational projective map.

Note that, since $Y = \operatorname{Spec} H^0(X, \mathcal{O}_X)$, we have $\mathcal{O}_Y \xrightarrow{\sim} \pi_*\mathcal{O}_X$ and so Y is normal. Conversely, if Y is normal and affine, then $\mathcal{O}_Y \xrightarrow{\sim} \pi_*\mathcal{O}_X$ and so $Y = \operatorname{Spec} H^0(X, \mathcal{O}_X)$.

Since Ω is non-degenerate, it is easy to see that the dimension of X is even and that $\omega^{\dim X/2}$ gives a trivialisation of the canonical bundle $K_X \coloneqq \Omega_X^{\dim X}$. Grauert-Riemenschneider theorem (Theorem 4.3.9 in [79]) states that $R^\bullet\pi_*(K_X) = \pi_*(K_X)$ for any resolution $\pi\colon X \to Y$. Since in our case $K_X = \mathcal{O}_X$, we get that every symplectic resolution is necessarily rational, namely $R^\bullet\pi_*(\mathcal{O}_X) = \mathcal{O}_Y$.

In particular $R^1\pi_*\mathcal{O}_X = R^2\pi_*\mathcal{O}_X = 0$, so to prove that $\pi\colon X \to Y$ is a resolution with conical slices it is enough to construct: 1) a contracting $\mathbb{G}_m$-action on π, 2) an étale conical neighbourhood for any point $y \in Y(\mathbb{K})$.

2.7.2 Marsden-Weinstein reductions for representations of quivers

In this subsection we mostly follow Crawley-Boevey's paper [30] on normality of Marsden-Weinstein reductions for representations of quivers. The main results we use are Theorem 4.9 and Corollary 4.10 from [30], but with a slight elaboration, see Proposition 2.7.2.18. We give a sketch of the content needed from [30], so that it becomes clear that Proposition 2.7.2.18 follows easily from the mentioned results.

Let $\mathbb{K}$ be an algebraically closed field of characteristic 0 and let A be a finite-dimensional semisimple associative $\mathbb{K}$-algebra. Since $\mathbb{K}$ is algebraically closed, we have that A is just a direct sum of matrix algebras. Any left or right A-module and any A-A-bimodule is semisimple.

Let M be a finite-dimensional A-A-bimodule. A symplectic bilinear form ω on M is called *balanced* if $\omega(xa, y) = -\omega(y, ax) = -\omega(ya, x) = \omega(x, ya)$ for all $a \in A$ and $x, y \in M$. Subspace $U \subset M$ is called Lagrangian if $U = U^\perp$ and (co)isotropic if $U \subset U^\perp$ ($U^\perp \subset U$).

Not very difficult to prove is the following analogue of Darboux Theorem for bimodules with a balanced symplectic form:

Theorem 2.7.2.1 ([30], Theorem 2.2)**.** *If M is an A-A-bimodule with a balanced symplectic form ω, then any maximal isotropic sub-bimodule S of M has an isotropic bimodule complement. Both S and the complement are Lagrangian. Moreover, there is an isomorphism of bimodules $M \cong S \oplus S^*$, under which $\omega((s,f),(s',f')) = f(s') - f'(s)$.*

Corollary 2.7.2.2 ([30], Corollary 2.3)**.** *If M is an A-A-bimodule with a balanced symplectic form ω, then any isotropic sub-bimodule S of M has a coisotropic bimodule complement.*

Definition 2.7.2.3. By *a quadruple* (A, M, ω, ζ) we will mean a data of a finite-dimensional semisimple associative $\mathbb{K}$-algebra A, a finite-dimensional A-A-bimodule M, a balanced symplectic form ω on M, and a linear map $\zeta : A \to \mathbb{K}$ satisfying $\zeta(ab) = \zeta(ba)$ for all $a, b \in A$ (in other words $\zeta \in (A/[A,A])^* \subset A^*$).

Let $G = A^\times$ be the group of units of A. Since A is a direct sum of matrix algebras, G is a product of copies of general linear groups. G acts on any A-A-bimodule M via $g \circ m = gmg^{-1}$ (where the multiplication on the right hand side of the equation is given by the action of A from the left and the right on M). Note that $\mathrm{Lie}(G) = A$, where the bracket on A is given by commutator.

Let $\mu : M \to A^*$ be the map defined by $\langle \mu(m), a \rangle = \omega(m, am)$. Note that $\omega(m, am) = \omega(ma, m) = -\omega(m, ma)$, so $\langle \mu(m), a \rangle = \frac{1}{2}\omega(m, [a, m])$, where $[a, m] = am - ma$.

Lemma 2.7.2.4 ([30], Lemma 3.1)**.** *μ is a moment map for the action of G on M.*

Proof. By the definition of a moment map we need to show that $\langle d\mu_m(v), a \rangle = \omega_m(v, a_m)$ for all $m \in M$, $v \in T_m M$ and $a \in \mathrm{Lie}(G)$. Here ω_m is the symplectic form on $T_m M$ induced by ω and $m \mapsto a_m$ is the vector field on M induced by a.

After the natural identification of $T_m M$ and $\mathrm{Lie}(G)$ with A, the vector field a_m induced by a is given by the map $M \to M$, sending m to $[a, m] = am - am$.

On the other hand, from the formula for μ we see that $\langle d\mu_m(v), a \rangle = \omega(v, am) + \omega(m, av) = \omega(v, am) + \omega(ma, v) = \omega(v, [a, m]) = \omega(v, a_m)$. So μ is a moment map. □

Definition 2.7.2.5. Categorical quotient

$$N(A, M, \omega, \zeta) = \mu^{-1}(\zeta) /\!\!/ G$$

is by definition the *Marsden-Weinstein reduction* associated to the quadruple (A, M, ω, ζ).

Let Q be a finite quiver with a vertex set I, and let $h(a)$ and $t(a)$ denote the head and tail vertices of an arrow a in Q. For $\alpha \in \mathbb{N}^I$, the space of representations of Q with dimension vector α is by definition the $\mathbb{K}$-vector space

$$\mathrm{Rep}(\overline{Q}, \alpha) = \bigoplus_{a \in \overline{Q}} \mathrm{Mat}(\alpha_{h(a)} \times \alpha_{t(a)}, \mathbb{K}).$$

Let $\overline{Q}$ be the double of Q, obtained from Q by adjoining a reverse arrow a^* for each arrow $a \in Q$. Finite-dimensional $\mathbb{K}$-algebra $\mathrm{End}(\alpha) = \prod_{i \in I} \mathrm{Mat}(\alpha_i, \mathbb{K})$ is semisimple and acts naturally on $\mathrm{Rep}(\overline{Q}, \alpha)$ both from the left and the right, turning it into $\mathrm{End}(\alpha)$-$\mathrm{End}(\alpha)$-bimodule. The space $\mathrm{Rep}(\overline{Q}, \alpha)$ can be identified with the total space of cotangent bundle to $\mathrm{Rep}(Q, \alpha)$. From this identification $\mathrm{Rep}(\overline{Q}, \alpha)$ obtains the canonical symplectic form ω_α, which in explicit form is given by

$$\omega_\alpha(x, y) = \sum_{a \in Q} \left(\mathrm{tr}(x_{a^*} y_a)) - \mathrm{tr}(x_a y_{a^*})\right).$$

It is easy to see that ω_α is balanced with respect to the action of $\mathrm{End}(\alpha)$. The group $\mathrm{GL}(\alpha) = \mathrm{End}(\alpha)^\times = \prod_{i \in I} \mathrm{GL}(\alpha_i)$ acts on $\mathrm{Rep}(\overline{Q}, \alpha)$ by symplectomorphisms and there is a moment map μ_α

$$\mu_\alpha(x) = \sum_{a \in \overline{Q}} [x_a, x_{a^*}] \in \mathrm{End}(\alpha),$$

where in the formula $\mathrm{End}(\alpha)$ is identified with its dual using the trace pairing. Finally, to a vector $\lambda \in \mathbb{K}^I$ we can associate a function $\zeta_\lambda : \mathrm{End}(\alpha) \to \mathbb{K}$, defined by $\zeta_\lambda(\theta) = \sum_{i \in I} \lambda_i \,\mathrm{tr}(\theta_i)$.

This way a quiver Q with the vertex set I, elements $\alpha \in \mathbb{N}^I$ and $\lambda \in \mathbb{K}^I$, give rise to a quadruple $(\mathrm{End}(\alpha), \mathrm{Rep}(\overline{Q}, \alpha), \omega_\alpha, \zeta_\lambda)$. We have an affine variety $N(\mathrm{End}(\alpha), \mathrm{Rep}(\overline{Q}, \alpha), \omega_\alpha, \zeta_\lambda)$ (see Definition 2.7.2.5) which in this case will be denoted simply by $N_Q(\lambda, \alpha)$.

Lemma 2.7.2.6 ([30], Lemma 3.3)**.** *Every quadruple (A, M, ω, ζ) is of the form $(\mathrm{End}(\alpha), \mathrm{Rep}(\overline{Q}, \alpha), \omega_\alpha, \zeta_\lambda)$ for some Q, α and λ.*

Proof. Using the analogue of Darboux theorem Theorem 2.7.2.1 we can find a maximal isotropic submodule S inside M and identify M with $S \oplus S^*$. The construction of the corresponding quiver Q will depend on S.

The vertex set I of Q is defined as the set of simple left A-modules up to isomorphism. The dimension vector α is defined by setting α_i to be equal to the dimension of the corresponding

simple module. Since A is a product of matrix algebras, we have $\mathrm{End}(\alpha) = A$.

The number of arrows between X and Y in Q is defined as the multiplicity of $X \otimes Y^*$ in S (this multiplicity can be 0). Note that if X and Y are simple left A-modules, then $X \otimes Y^*$ is a simple A-A-bimodule and every simple A-A-bimodule can be obtained this way. It follows that $S = \mathrm{Rep}(Q, \alpha)$ and $M = \mathrm{Rep}(\overline{Q}, \alpha)$. It is also not hard to check that the described moment maps coincide.

Finally, ζ is an Ad-invariant element of $A^* \cong \mathrm{End}(\alpha)^*$ and these are generated by the trace functions on each of the simple components of A. The coordinates of ζ in this basis define the vector $\lambda \in \mathbb{K}^I$, such that $\zeta = \zeta_\lambda$. □

Properties of the map μ_α

For any $\lambda \in \mathbb{K}^I$ we denote by $\lambda \cdot \alpha$ the sum $\sum_{i \in I} \alpha_i \lambda_i$. Note that the action of $\mathrm{GL}(\alpha)$ on $\mathrm{Rep}(\overline{Q}, \alpha)$ factors through $\mathbb{P}G(\alpha) = (\prod_{i \in I} \mathrm{GL}(\alpha_i))/\mathbb{G}_m$, where the scalars $\mathbb{G}_m$ are embedded diagonally. One-dimensional subspace $\mathbb{K} = \mathrm{Lie}(\mathbb{G}_m) \subset \mathbb{K}^I$ (where $\mathbb{K}^I$ is identified with $(\mathrm{End}(\alpha)/[\mathrm{End}(\alpha), \mathrm{End}(\alpha)])^*$ using the trace paring) is spanned by the vector $(\alpha_i)_{i \in I} \in \mathbb{K}^I$ (since the trace of a scalar matrix t of size α_i is equal to $\alpha_i \cdot t$). In particular, the image of μ_α lies in $\mathrm{Lie}(\mathbb{G}_m)^\perp \subset \mathrm{End}(\alpha)$, or, in other words, if λ is in the image of μ_α, then $\lambda \cdot \alpha = 0$.

Associated to the quiver Q, there is also a quadratic form

$$q_Q(\alpha) = \sum_{i \in I} \alpha_i^2 - \sum_{a \in Q} \alpha_{h(a)} \alpha_{t(a)}$$

on $\mathbb{Z}^I$. We denote by $(\alpha, \beta)_Q$ the symmetric bilinear form with $(\alpha, \alpha)_Q = 2q(\alpha)$ and we define $p_Q(\alpha)$ as $1 - q_Q(\alpha)$. We also define a subset $\Delta^+ \subset \mathbb{Z}^I$ of positive roots (positive roots of the corresponding Kac-Moody algebra) which corresponds exactly to the dimension vectors of indecomposable representations of Q (this is Kac theorem, see [62]).

Theorem 2.7.2.7 ([29], Theorem 1.1)**.** *For $\alpha \in \mathbb{N}^I$ the following are equivalent*

1. *μ_α is a flat morphism;*

2. *$\mu_\alpha^{-1}(0)$ has dimension $\alpha \cdot \alpha - 1 + 2p(\alpha)$;*

3. *$p(\alpha) \geq \sum_{t=1}^r p(\beta^{(t)})$ for any decomposition $\alpha = \beta^{(1)} + \ldots + \beta^{(r)}$ into positive roots;*

4. $p(\alpha) \geq \sum_{t=1}^{r} p(\beta^{(t)})$ *for any decomposition* $\alpha = \beta^{(1)} + \ldots + \beta^{(r)}$ *into nonzero* $\beta^{(t)} \in \mathbb{N}^I$.

The way to study the fibers $\mu_\alpha^{-1}(\lambda)$ and the varieties $N_Q(\lambda, \alpha)$, introduced by Crawley-Boevey, was to relate them to the representations of the certain algebra Π^λ, called the *deformed preprojective algebra*. It is defined as

$$\Pi^\lambda = \mathbb{K}\overline{Q} / \left(\sum_{a \in \overline{Q}} [a, a^*] - \sum_{i \in I} \lambda_i e_i \right),$$

where $\mathbb{K}\overline{Q}$ is the path algebra of $\overline{Q}$. For $\lambda \in \mathbb{K}^I$, such that $\lambda \cdot \alpha = 0$, the variety $\mu_\alpha^{-1}(\lambda)$ is identified with the space of representations of Π^λ with dimension vector α. Closed orbits of G-action on $\mathrm{Rep}(\overline{Q}, \alpha)$ correspond to semisimple representations of $\overline{Q}$. So points of $N_Q(\lambda, \alpha) = \mu_\alpha^{-1}(\lambda) /\!\!/ G$ correspond to isomorphism classes of semisimple representations of Π^λ of dimension α.

For $\lambda \in \mathbb{K}^I$ let Δ_λ^+ denote the set of positive roots α with $\lambda \cdot \alpha = 0$. Following Crawley-Boevey we define $\Sigma_\lambda \subset \Delta_\lambda^+$ to be the set of $\alpha \in \Delta_\lambda^+$, such that

$$p(\alpha) > \sum_{t=1}^{r} p(\beta^{(t)})$$

for any decomposition of $\alpha = \beta^{(1)} + \ldots + \beta^{(r)}$ where $r \geq 2$ and $\beta^{(t)}$ lie in Δ_λ^+.

Theorem 2.7.2.8 ([29], Theorem 1.2)**.** *For* $\lambda \in \mathbb{K}^I$ *and* $\alpha \in \mathbb{N}^I$ *the following are equivalent*

1. *There is a simple representation of* Π^λ *of dimension vector* α*;*

2. α *is an element of* Σ_λ.

In this case μ_α^{-1} *is a reduced and irreducible complete intersection of dimension* $\alpha \cdot \alpha - 1 + 2p(\alpha)$ *and the general element of* $\mu_\alpha^{-1}(\lambda)$ *is a simple representation of* Π^λ.

We also will need the following corollary:

Corollary 2.7.2.9 ([29], Corollary 1.4)**.** *If* $\alpha \in \Sigma_\lambda$*, then* $N_Q(\lambda, \alpha) = \mu_\alpha^{-1}(\lambda) /\!\!/ G$ *is a reduced and irreducible scheme of dimension* $2p(\alpha)$.

Remark 2.7.2.10. Note that if $\alpha \in \Sigma_0$, then α lies in Σ_λ for all $\lambda \in \mathbb{K}^I$, such that $\lambda \cdot \alpha = 0$, and also satisfies the conditions of Theorem 2.7.2.7.

GIT and resolutions of singularities of $N_Q(\lambda,\alpha)$

Let θ be a character of $\mathbb{P}G(\alpha) = \mathrm{GL}(\alpha)/\mathbb{G}_m = (\prod_{i=I}\mathrm{GL}(\alpha_i))/\mathbb{G}_m$. θ can be considered as an element of $\mathbb{Z}^I$, such that $\theta\cdot\alpha = 0$. Recall that the natural action of $\mathrm{GL}(\alpha)$ on $\mathrm{Rep}(\overline{Q},\alpha)$ factors through $\mathbb{P}G(\alpha)$, so we can consider open subsets of θ-semistable and θ-stable points of $\mathrm{Rep}(\overline{Q},\alpha)$. The corresponding GIT quotient $N_Q(\lambda,\alpha)^\theta = \mu_\alpha^{-1}(\lambda)/\!\!/_\theta G$ is defined as the categorical quotient $\mu^{-1}(\lambda)^{\theta-ss}/\!\!/G$, where $\mu^{-1}(\lambda)^{\theta-ss}$ is by definition the intersection of $\mu^{-1}(\lambda)$ with the set of θ-semistable points $\mathrm{Rep}(\overline{Q},\alpha)^{\theta-ss}$. This is a variety endowed with the natural projective "semisimplification" map $\pi_\lambda^\theta : N_Q(\lambda,\alpha)^\theta \to N_Q(\lambda,\alpha)$, which is better seen from the equivalent definition of $N_Q(\lambda,\alpha)^\theta$ as

$$N_Q(\lambda,\alpha)^\theta = \mathrm{Proj}\bigoplus_{m=0}^{\infty}\left\{f\in\mathcal{O}(\mu^{-1}(0)) \mid g\circ f = \theta(g)^m\cdot f, \text{ for any } g\in G\right\}.$$

The map $\pi_\lambda^\theta : N_Q(\lambda,\alpha)^\theta \to N_Q(\lambda,\alpha)$ is then just the projection from Proj to the Spec of its 0th graded component.

The stable locus $\mu^{-1}(\lambda)^{\theta-st} \subset \mu^{-1}(\lambda)^{\theta-ss}$ is defined as the subset of points with a finite stabiliser. Geometric invariant theory says that this is an open subset and if $\mu^{-1}(\lambda)^{\theta-st}$ is smooth, the quotient $\mu^{-1}(\lambda)^{\theta-st}/\!\!/G$ is smooth too.

In the classical paper by King [72] it is proven that for the space of representations of a quiver, θ-(semi)stability in the sense of GIT is the same as θ-(semi)stability in terms of slopes. In particular a simple representation is θ-stable for any character θ. It also follows that if α is indivisible, meaning that integers α_i do not have common divisors, one can find a character θ, such that the set of θ-semistable points coincides with the set of θ-stable ones ([72], Remark 5.4).

If $\alpha\in\Sigma_\lambda$, then by Theorem 2.7.2.8 $\mu_\alpha^{-1}(\lambda)$ is irreducible, and the general element is a simple representation of Π^λ and, in particular, is θ-stable. So the set of θ-semistable points is dense, and it follows that the semisimplification map

$$\pi_\lambda^\theta : N_Q^\theta(\lambda,\alpha) \to N_Q(\lambda,\alpha)$$

is a birational map of irreducible varieties. θ-stable Π^λ-modules have trivial endomorphisms, so by ([32], Lemma 10.3) stable locus is smooth. If in addition α is indivisible, taking θ, such that all θ-semistable points are θ-stable, we get that $N_Q^\theta(\lambda,\alpha)$ is smooth, and that π_λ^θ is a resolution

of singularities. Moreover $\mu_\alpha^{-1}(\lambda)/\!\!/_\theta \mathbb{P}G(\alpha)$ is a symplectic reduction, so is endowed with a natural symplectic form. Finally, in ([30], Theorem 8.2) it is proven that $N(\lambda,\alpha)$ is normal for all λ and α. This turns π_λ^θ into a symplectic resolution.

We can summarise this discussion in the following:

Proposition 2.7.2.11 ([30],[29],[32]). *Let $\alpha \in \mathbb{N}^I$ and $\lambda \in \mathbb{K}^I$ be such that*

- $\lambda \cdot \alpha = 0$,
- $\alpha \in \Sigma_\lambda$,
- α *is indivisible.*

Then the map $\pi_\lambda^\theta : N_Q^\theta(\lambda,\alpha) \to N_Q(\lambda,\alpha)$ is a symplectic resolution of singularities.

For a dimension vector $\alpha \in \mathbb{N}^I$, let $\mathfrak{h}_\alpha$ be the subspace of $\mathbb{K}^I$ consisting of λ with $\lambda \cdot \alpha = 0$. If $\lambda \in \mathfrak{h}_\alpha$ is such that $\lambda \notin \mathfrak{h}_\beta$ for all $0 < \beta < \alpha$ (meaning $\alpha - \beta$ and β are sums of positive roots), then the condition $\alpha \in \Sigma_\lambda$ is equivalent to α being a positive root. In this case all elements of $\mu_\alpha^{-1}(\lambda)$ are simple Π^λ-modules ([31], Lemma 4.1). It follows that all points in $\mu_\alpha^{-1}(\lambda)$ are θ-stable, so $N_Q(\lambda,\alpha)$ is equal to $N_Q^\theta(\lambda,\alpha)$ and is smooth. If α is indivisible, the set of such λ is non-empty and by definition is the complement of $\mathfrak{h}_\alpha$ to the union of hyperplanes $\mathfrak{h}_\alpha \cap \mathfrak{h}_\beta$ over all β such that $0 < \beta < \alpha$.

Let $\alpha \in \Sigma_0$. Then (see Remark 2.7.2.10) $\alpha \in \Sigma_\lambda$ for all $\lambda \in \mathfrak{h}_\alpha$, map μ_α is flat and produces a flat family $\mu_\alpha^{-1}(\mathfrak{h}_\alpha) \to \mathfrak{h}_\alpha$. From Corollary 2.7.2.9 the induced family $\mu_\alpha^{-1}(\mathfrak{h}_\alpha)/\!\!/G \to \mathfrak{h}_\alpha$ is equidimensional and is flat as well. If in addition α is indivisible, the general fiber is smooth, and the whole family admits a simultaneous resolution

$$\pi_{\mathfrak{h}_\alpha}^\theta : \mu_\alpha^{-1}(\mathfrak{h}_\alpha)/\!\!/_\theta \mathbb{P}G(\alpha) \to \mu_\alpha^{-1}(\mathfrak{h}_\alpha)/\!\!/\mathbb{P}G(\alpha),$$

which restricts to resolutions

$$\pi_\lambda^\theta : N_Q^\theta(\lambda,\alpha) \to N_Q(\lambda,\alpha)$$

for each $\lambda \in \mathfrak{h}_\alpha$.

Étale local structure

Let (A, M, ω, ζ) be a quadruple (see Definition 2.7.2.3). Recall that we have $G = A^\times$, acting on M with a moment map μ given by $\langle \mu(m), a\rangle = \omega(m, am)$. Points of $\mu^{-1}(\zeta)/\!\!/G$ correspond to closed

G-orbits on $\mu^{-1}(\zeta)$. Let $x \in \mu^{-1}(\zeta)$ be a point, whose G-orbit $G \cdot x$ is closed. Note that the tangent space to the orbit is given by $[A, x] \subset M$, if we identify $T_x M$ with M.

Lemma 2.7.2.12 ([30], Lemma 1)**.** *$[A, x]$ is an isotropic subspace of M. In other words $[A, x] \subseteq [A, x]^{\perp}$.*

Proof.

$$\omega([a, x], [b, x]) = \omega(ax, bx) - \omega(ax, xb) - \omega(xa, bx) + \omega(xa, xb) =$$
$$= \omega(axb, x) + \omega(xb, ax) - \omega(x, abx) + \omega(x, axb) =$$
$$= \omega(x, bax) - \omega(x, abx) = \langle \mu(x), ba - ab \rangle = \zeta(ba - ab) = 0$$

□

Moreover, by Matsushima's theorem [85] the stabiliser G_x is reductive, and from this it follows that $A_x = \{a \in A | ax - xa\}$ is a semisimple algebra ([30], Lemma 4.2).

Let now L be an A_x-A_x-bimodule complement to A_x in A: $A = A_x \oplus L$. Note that $[A, x]$ is an A_x-A_x-subbimodule of M, so by Corollary 2.7.2.2 we can choose a coisotropic A_x-A_x-bimodule complement C, so that $M = [A, x] \oplus C$. Let $W = C \cap [A, x]^{\perp}$. Let $\mu_x \colon M \to A_x^*$ be the map obtained by composing μ with the restriction map $A^* \to A_x^*$. Let $\hat{\mu}$ denote the restriction of μ_x to W.

Lemma 2.7.2.13 ([30], Lemma 4.3)**.** *The restriction of symplectic form ω to W is non-degenerate and is hence a symplectic form. It is balanced for the A_x-A_x-bimodule structure and the corresponding moment map is $\hat{\mu}$.*

This way each point $x \in M$ with a closed G-orbit provides a new quadruple (A_x, W, ω, ζ).

Crawley-Boevey then defines a map $\nu \colon C \to L^*$ by $\langle \nu(c), l \rangle = \omega(c, lx) + \omega(c, lc) + \omega(x, lc)$. The map is defined in such a way that

$$\langle \mu(x + c), a + l \rangle = \zeta(a + l) + \langle \mu_x(c), a \rangle + \langle \nu(c), l \rangle$$

for $c \in C$, $a \in A_x$ and $l \in L$. There is an important lemma:

Lemma 2.7.2.14 ([30], Lemma 4.4)**.** *The assignment $c \mapsto x + c$ induces a G_x-equivariant map $\mu_x^{-1}(0) \cap \nu^{-1}(0) \to \mu^{-1}(\zeta)$, and the induced map $(\mu_x^{-1}(0) \cap \nu^{-1}(0)) /\!\!/ G_x \to \mu^{-1}(\zeta) /\!\!/ G$ is étale at 0.*

Proof. The formula above shows that $x+c \in \mu^{-1}(\zeta)$ if and only if $c \in \mu_x^{-1}(0) \cap \nu^{-1}(0)$. The action of G on M gives a map $\varphi\colon G \times_{G_x} C \to M$ which is étale at $(1,0)$. Then Luna's Fundamental Lemma ([81],p.94) applies to φ: there exists $U \subset X = G \times_{G_x} C$ which is saturated for $\pi_X\colon X \to X/\!\!/G$, such that $\varphi|_U$ is étale, $V = \varphi(U)$ is an affine open subset of M, saturated for $\pi_M\colon M \to M/\!\!/G$, the corresponding morphism $\varphi/G\colon U/\!\!/G \to V/\!\!/G$ is étale and induces a G-equivariant isomorphism $U \xrightarrow{\sim} V \times_{V/\!\!/G} U/\!\!/G$.

The statement of the lemma is then obtained by the base change to the closed subvariety $V' = V \cap \mu^{-1}(\zeta) \subset V$ which is G-invariant. Namely $U' = V' \times_{V/\!\!/G} U/\!\!/G$ is actually $X' \cap U$, where $X' = G \times_{G_x} C'$ and $C' = \mu_x^{-1}(0) \cap \nu^{-1}(0)$. It remains to note that $X/\!\!/G \cong C/\!\!/G_x$ and that $X'/\!\!/G \cong C'/\!\!/G_x$. □

We would like to lift this étale map to the level of GIT quotients. Let us fix a character θ of G, then its restriction to G_x gives a character θ_x. Let C be a vector space with G_x-action and consider its GIT quotient

$$C/\!\!/_{\theta_x} G_x = \operatorname{Proj} \bigoplus_{n=0}^{\infty} \Gamma(C, \mathcal{O}_C)^{\theta_x^n},$$

where $\Gamma(C, \mathcal{O}_C)^{\theta_x^n}$ is the subspace of functions which are multiplied by $\theta_x(g)^n$ under the action of $g \in G_x$ for all g. We also have $X = G \times_{G_x} C$ and, since G_x is reductive, X is affine. Consider the corresponding GIT-quotient with respect to θ:

$$X/\!\!/_{\theta} G = \operatorname{Proj} \bigoplus_{n=0}^{\infty} \Gamma(X, \mathcal{O}_X)^{\theta^n}.$$

Lemma 2.7.2.15. *These two GIT quotients coincide: $X/\!\!/_{\theta} G = C/\!\!/_{\theta_x} G_x$*

Proof. $\Gamma(X, \mathcal{O}_X) = (\Gamma(G, \mathcal{O}_G) \otimes_{\mathbb{K}} \Gamma(C, \mathcal{O}_C))^{G_x}$. For each character θ of G, the subspace $\Gamma(G, \mathcal{O}_G)^{\theta}$ is one-dimensional and is spanned by the character itself, considered as a function $\theta\colon G \to \mathbb{A}^1$. We have a G-equivariant embedding $\Gamma(X, \mathcal{O}_X) \hookrightarrow \Gamma(G, \mathcal{O}_G) \otimes_{\mathbb{K}} \Gamma(C, \mathcal{O}_C)$, corresponding to the G-equivariant map $G \times C \to X$. In the tensor product G acts only on $\Gamma(G, \mathcal{O}_G)$, and so $\Gamma(X, \mathcal{O}_X)^{\theta}$ is actually a subspace of $\mathbb{K} \cdot \theta \otimes_{\mathbb{K}} \Gamma(C, \mathcal{O}_C)$. Moreover, since $\Gamma(X, \mathcal{O}_X) = (\Gamma(G, \mathcal{O}_G) \otimes_{\mathbb{K}} \Gamma(C, \mathcal{O}_C))^{G_x}$, it is equal to its G_x-invariants $(\mathbb{K} \cdot \theta \otimes_{\mathbb{K}} \Gamma(C, \mathcal{O}_C))^{G_x}$. Action of G_x on $G \times C$ is given by $g_x \circ (g, c) = (g \cdot g_x, g_x^{-1} \circ c)$. Since $g_x \circ \theta = \theta_x(g_x) \cdot \theta$, we have $g_x \circ (\theta \otimes f) = \theta_x(g_x) \cdot (\theta \otimes (g_x^{-1} \circ f))$ and so, for $(\theta \otimes f)$ to be G_x-invariant, we need f to be rescaled by $\theta_x(g_x)$ under the action of g_x. In other

words, we get that $\Gamma(X,\mathcal{O}_X)^{\theta} = \mathbb{K}\cdot\theta \otimes_{\mathbb{K}} \Gamma(C,\mathcal{O}_C)^{\theta_x}$. This way we obtain an isomorphism

$$\bigoplus_{n=0}^{\infty}\Gamma(C,\mathcal{O}_C)^{\theta_x^n} \xrightarrow{\sim} \bigoplus_{n=0}^{\infty}\Gamma(X,\mathcal{O}_X)^{\theta^n} \text{ where } f\mapsto\theta^n\otimes f, \text{ for } f\in\Gamma(C,\mathcal{O}_C)^{\theta_x^n},$$

and so $X/\!\!/_{\theta}G = C/\!\!/_{\theta_x}G_x$. □

Note that the isomorphism is constructed in a way such that the square

$$\begin{array}{ccc} X/\!\!/_{\theta}G & \longrightarrow & C/\!\!/_{\theta_x}G_x \\ \downarrow & & \downarrow \\ X/\!\!/G & \longrightarrow & C/\!\!/G_x \end{array}$$

is commutative. So Lemma 2.7.2.15 shows that in the proof of Lemma 2.7.2.14 we can replace the categorical quotient with the GIT-quotient with respect to some character θ (or θ_x correspondingly), and extend the étale map to the map of resolutions. Being even more precise, following the terminology of Section 2.4.2, we obtain an étale equivalence of pointed resolutions

$$(\pi, C'/\!\!/_{\theta_x}G_x, C'/\!\!/G_x, 0) \sim_{et} (\pi^{\theta}_{\lambda}, N_Q(\lambda,\alpha)^{\theta}, N_Q(\lambda,\alpha), x),$$

where $C' = \mu_x^{-1}(0)\cap\nu^{-1}(0)$.

Lemma 2.7.2.16 ([30], Lemma 4.8)**.** *There is a morphism $(\mu_x^{-1}(0)\cup\nu^{-1}(0))/\!\!/G_x \to \hat{\mu}^{-1}(0)/\!\!/G_x$, sending 0 to 0, which is étale at 0.*

Proof. (Sketch) One first proves that $C = C^{\perp}\oplus W$ as A_x-A_x-bimodules and that the restriction of the corresponding projection $p\colon C\to W$ to $\nu^{-1}(0)\subset C$ is étale at 0. Then again, using Luna's Fundamental Lemma, there is an open subvariety U' of $\nu^{-1}(0)/\!\!/G_x$ containing 0, such that the restriction of $p/G_x\colon \nu^{-1}(0)/\!\!/G_x \to W/\!\!/G_x$ to U' is étale, and the image $V' = (p/G_x)(U')$ is an open affine subset.

The étale map from the statement of the lemma is then obtained by the base change to $\hat{\mu}^{-1}(0)\subset W$. Its preimage in $\nu^{-1}(0)$ is exactly $\mu_x^{-1}(0)\cup\nu^{-1}(0)$. □

Note that following the proof, the desired map was induced by an étale map $U'\to V'\times_{W/\!\!/G_x}\nu^{-1}(0)/\!\!/G_x$, which also gives an étale map between the open subsets of θ_x-semistable points. This

way it extends to an étale map between the corresponding resolutions. We obtain an étale equivalence

$$(\pi, C'/\!\!/_{\theta_x} G_x, C'/\!\!/ G_x, 0) \sim_{et} (\hat{\pi}, \hat{\mu}^{-1}(0)/\!\!/_{\theta_x} G_x, \hat{\mu}^{-1}(0)/\!\!/ G_x, 0).$$

Composing it with the previous one, we get the following:

Proposition 2.7.2.17. *There exists an étale equivalence of pointed resolutions*

$$(\pi^\theta_\lambda, N_Q(\lambda,\alpha)^\theta, N_Q(\lambda,\alpha), x) \sim_{et} (\hat{\pi}, \hat{\mu}^{-1}(0)/\!\!/_{\theta_x} G_x, \hat{\mu}^{-1}(0)/\!\!/ G_x, 0).$$

Recall that by Lemma 2.7.2.6 every quadruple (A, M, ω, ζ) is equivalent to a one coming from the quiver. Let Q_x be the quiver associated to the quadruple $(A_x, W, \omega, 0)$. We also have the corresponding dimension vector $\alpha_x \in \mathbb{N}^{I_x}$, where I_x is the set of vertices of Q_x. We can then reformulate the proposition above in the following way:

Proposition 2.7.2.18. *Let Q be a quiver with a vertex set I, $\alpha \in \mathbb{N}^I$ be a dimension vector and λ a vector in $\mathbb{K}^I$. Then there exist a quiver Q_x with a vertex set I_x and a dimension vector α_x, such that*

$$(\pi^\theta_\lambda, N_Q(\lambda,\alpha)^\theta, N_Q(\lambda,\alpha), x) \sim_{et} (\pi_0^{\theta_x}, N_{Q_x}(0,\alpha_x)^{\theta_x}, N_{Q_x}(0,\alpha_x), 0)$$

Proposition says that the resolution of singularities of a Marsden-Weinstein reduction for representations of quivers étale locally at any point looks like the resolution of singularities at the central point of some other Marsden-Weinstein reduction.

$\mathbb{G}_m$-actions

Let $\lambda = 0$ and let's consider some $N_Q(0,\alpha)$. Recall that it was obtained as a Hamiltonian reduction from a symplectic vector space $\mathrm{Rep}(\overline{Q}, \alpha)$. Let's now consider a $\mathbb{G}_m$-action on $\mathrm{Rep}(\overline{Q}, \alpha)$ which is rescaling all the operators in the representation, namely $t \circ x_a = t x_a$ for any arrow $a \in \overline{Q}$. This action action commutes with the action of $\mathrm{GL}(\alpha)$ and rescales ω: we have $t \circ \omega = t^2 \omega$. Though this action does not preserve the fibers of μ, it preserves $\mu^{-1}(0)$ and so defines an action on it. Moreover, this action is contracting: the whole $\mathrm{Rep}(\overline{Q}, \alpha)$ and $\mu^{-1}(0)$ in particular are contracted to a single point 0. Since this action commutes with the action of $\mathrm{GL}(\alpha)$ it descends to a contracting action on $N_Q(0,\alpha)$. Moreover it also defines an action on $N_Q(0,\alpha)^\theta$ and turns $\pi_0^\theta \colon N_Q(0,\alpha)^\theta \to N_Q(0,\alpha)$ into a conical resolution.

Combining this with the Proposition 2.7.2.18 we get that any point $x \in N_Q(\lambda, \alpha)$ has a conical étale neighbourhood (see Definition 2.4.4.2). In other words:

Theorem 2.7.2.19. *Resolution $\pi_0^\theta \colon N_Q(0,\alpha)^\theta \to N_Q(0,\alpha)$ of singularities of Marsden-Weinstein reduction is a resolution with conical slices.*

Example: Nakajima quiver varieties

The main examples of Marsden-Weinstein reductions for representations of quivers are Nakajima quiver varieties. Here we follow the *Remarks added after 2000* in [29].

Let Q_0 be a quiver with a vertex set I. For $\underline{v}, \underline{w} \in \mathbb{N}^I$, let $M(\underline{v}, \underline{w})$ be the space

$$\mathrm{Rep}(\overline{Q_0}, \underline{v}) \oplus \bigoplus_{k \in I} \mathrm{Mat}(v_k \times w_k, \mathbb{K}) \oplus \bigoplus_{k \in I} \mathrm{Mat}(w_k \times v_k, \mathbb{K}).$$

We denote the coordinates on this space by triples (B, i, j). The natural action of $G_{\underline{v}} = \prod_{k \in I} \mathrm{GL}(v_k)$ on $M(\underline{v}, \underline{w})$ has a moment map $\mu \colon M(\underline{v}, \underline{w}) \to \oplus_{k \in I} \mathrm{Mat}(v_k, \mathbb{K})$ whose k-th component is given by the map, sending (B, i, j) to

$$\sum_{\substack{a \in Q_0 \\ h(a) = k}} B_a B_{a^*} - \sum_{\substack{a \in Q_0 \\ t(a) = k}} B_{a^*} B_a + \sum_{k \in I} i_k j_k.$$

Affine Nakajima quiver variety $\mathfrak{M}_0(\underline{v}, \underline{w})$ is defined as $\mu^{-1}(0) /\!\!/ G_{\underline{v}}$. More generally, Nakajima quiver varieties $\mathfrak{M}_\lambda^\theta(\underline{v}, \underline{w})$ are defined as $\mu^{-1}(\lambda) /\!\!/_\theta G_{\underline{v}}$.

Let's now consider a quiver Q, obtained from Q_0 by adjoining a new vertex ∞ and w_k arrows from ∞ to k for each $k \in I$. Let α be the dimension vector for Q, whose restriction to I is equal to $\underline{v}$ and such that $\alpha_\infty = 1$. Dividing the matrices in $\mathrm{Mat}(v_k \times w_k, \mathbb{K})$ into their columns and the matrices in $\mathrm{Mat}(w_k \times v_k, \mathbb{K})$ into their rows, we get an identification $M(\underline{v}, \underline{w}) \cong \mathrm{Rep}(\overline{Q}, \alpha)$. Since $\alpha_\infty = 1$, we also get that $\mathbb{P}G(\alpha) = (G_{\underline{v}} \times \mathbb{G}_m)/\mathbb{G}_m \cong G_{\underline{v}}$. It is easy to see that moment maps coincide too, and that

$$\mathfrak{M}_\lambda^\theta(\underline{v}, \underline{w}) \cong N_Q(\lambda, \alpha)^\theta$$

for all λ and θ. Assume that the map $\pi_0^\theta \colon \mathfrak{M}_0^\theta(\underline{v}, \underline{w}) \to \mathfrak{M}_0(\underline{v}, \underline{w})$ is a resolution of singularities (e.g. when the conditions of Proposition 2.7.2.11 are satisfied for α, λ and Q. Note that α is indivisible, so it is enough to check that $\alpha \in \Sigma_\lambda$). We obtain the following theorem:

Theorem 2.7.2.20. *If the map $\pi_0^\theta\colon \mathfrak{M}_0^\theta(\underline{v},\underline{w}) \to \mathfrak{M}_0(\underline{v},\underline{w})$ is a resolution of singularities, it is a resolution with conical slices.*

2.7.3 Hamiltonian reductions of vector spaces

Marsden-Weinstein reduction for representations of a quiver is a particular case of an algebraic Hamiltonian reduction of a vector space (which can be called Marsden-Weinstein reduction as well). Namely, let (V,ω) be a vector space with a symplectic form and let $G \to \mathrm{Sp}(V)$ be a symplectic representation of a reductive group G over $\mathbb{K}$. We will assume that the corresponding action is effective, namely that the map $G \to \mathrm{Sp}(V)$ is an embedding. Note that the action of G on V also induces an action of its Lie algebra $\mathfrak{g}$.

Action of G on V defines an action on the corresponding affine space, which is naturally a symplectic manifold. The symplectic G-action on V always has an algebraic moment map $\mu\colon V \to \mathfrak{g}^*$. It is given by the following formula:

$$\langle \mu(v), a\rangle = \frac{1}{2}\omega(v, a\circ v)$$

for any $v \in V$ and $a \in \mathfrak{g}$.

Let $Z(\mathfrak{g}^*) = (\mathfrak{g}^*)^G \subset \mathfrak{g}^*$ denote the subspace of the invariants of the coadjoint action. For $\lambda \in Z(\mathfrak{g}^*)$, the subvariety $\mu^{-1}(\lambda)$ is stable under the G-action. Given a character $\theta\colon G \to \mathbb{G}_m$ we can consider the GIT quotient with respect to θ:

$$\mathfrak{M}(G,V)^\theta_\lambda = \mu^{-1}(\lambda)/\!\!/_\theta G.$$

We will omit the superscript θ in the case when $\theta = 0$. For any θ we have a projective morphism $\pi^\theta_\lambda\colon \mathfrak{M}(G,V)^\theta_\lambda \to \mathfrak{M}(G,V)_\lambda$.

Darboux-type theorems

Unfortunately, the analogue of Theorem 2.7.2.1 is not true in general in this context. The important new feature is that an irreducible subrepresentation of V does not need to be isotropic. In particular, a maximal isotropic G-subrepresantation does not need to be maximal isotropic as a vector space.

Nevertheless a weaker form of Darboux theorem is still true:

Proposition 2.7.3.1. *Let (V, ω) be a symplectic representation of a reductive group G. Let $S \subset V$ be a maximal isotropic G-submodule, which is Lagrangian: $S = S^{\perp}$. Then S has an isotropic G-equivariant complement which is also Lagrangian. Moreover V is isomorphic to $S \oplus S^*$ as a symplectic representation of G.*

Proof. Since any representation of G is completely reducible, S has a G-equivariant complement C. S is Lagrangian, so $\dim V = 2 \dim S$ and $\dim C = \dim S$. Since G preserves the symplectic form on V, the natural map $C \to S^*$ induced by ω is G-equivariant. Moreover, since S is Lagrangian and ω is non-degenerate, this map is an isomorphism.

Let $f\colon S \to C^*$ be the dual map, it is an isomorphism as well. The restriction of ω to C induces a G-equivariant map $\tilde{\omega}|_C\colon C \to C^*$. Let $\theta\colon C \to S$ be equal to $(f)^{-1} \circ \tilde{\omega}|_C$. We have $\tilde{\omega}|_C = f \circ \theta$ and so $\omega(\theta(c), c') = \omega(c, c')$ for all $c, c' \in C$.

Let $D = \{c - \frac{1}{2}\theta(c) \mid c \in C\}$. Since θ was G-equivariant, this is a G-invariant subspace. Clearly $D \cap S = 0$ and $V = S \oplus D$. Then

$$\omega(c - \frac{1}{2}\theta(c), c' - \frac{1}{2}\theta(c')) = \omega(c, c') - \omega(c, \frac{1}{2}\theta(c')) - \omega(\frac{1}{2}\theta(c), c') + \omega(\frac{1}{2}\theta(c), \frac{1}{2}\theta(c')) = \\ = \omega(c, c') - \frac{1}{2}\omega(c, c') - \frac{1}{2}\omega(c, c') + 0 = 0.$$

This means that D is isotropic and has dimension $\frac{1}{2}\dim V$, so is Lagrangian. The map $\tilde{\omega}$ induces a G-equivariant isomorphism $D \xrightarrow{\sim} S^*$ and so $V = S \oplus S^*$.

□

Remark 2.7.3.2. Let $T \subset V$ be an irreducible G-submodule, such that $\omega|_T \neq 0$. Then, by irreducibility, T is symplectic and $\tilde{\omega}_T\colon T \xrightarrow{\sim} T^*$. For any self-dual irreducible representation T one can compute its Frobenius-Schur indicator $\mathrm{FS}(T)$ [25]. Since ω is skew-symmetric and G-invariant, we get that $\mathrm{FS}(T) = -1$ or in other words that T is *quaternionic*. From this it is easy to show that if all irreducible subrepresentations are *ortogonal* or *isotropic* (meaning $\mathrm{FS}(T) = 1$ or $\mathrm{FS}(T) = 0$) then any maximal isotropic subrepresentation is Lagrangian and the full statement of Theorem 2.7.2.1 holds. This way ([30], Lemma 2.1) can be considered as a statement about Frobenius-Schur indicators of self-dual representations coming from quivers.

With this weaker version of Darboux theorem we still are able to prove the analogue of Corollary 2.7.2.2. We will first need to prove a simple lemma:

Lemma 2.7.3.3. *Let $S \subset V$ be an isotropic subrepresentation. Let C be a G-equivariant complement and let W be the intersection $S^\perp \cap C$. Then the restriction of ω to W is non-degenerate and W is symplectic.*

Proof. S is isotropic, so $S \subset S^\perp$. Since C is a complement to S, $S^\perp \cong S \oplus W$. Moreover ω is non-degenerate on $V \oplus V^*$, so $(S^\perp)^\perp = S$. It follows that S is the kernel of the restriction of ω to $S^\perp$ and so the restriction of ω to W should be non-degenerate. □

Note that since ω is preserved by G, $S^\perp$ is G-invariant, and so W is a subrepresentation of G.

Proposition 2.7.3.4. *Let $S \subset V$ be an isotropic subrepresentation. Then there exists a G-equivariant complement C which is coisotropic: $C^\perp \subset C$. In this case $C = W \oplus \mathrm{Ker}\omega|_C$.*

Proof. Let's start with any G-equivariant complement C and consider $W = C \cap S^\perp$. By the lemma above, W is symplectic. It follows that $W^\perp$ is symplectic too, and $W \cap W^\perp = 0$.

Note that $\dim W = \dim V - 2\dim S$. Indeed, by the definition of C, $\dim C = \dim V - \dim S$. We have $S \subset S^\perp$ and so $S^\perp + C = V$. Since $W = C \cap S^\perp$ and $\dim S^\perp = \dim V - \dim S$, we get that:

$$\dim W = \dim C + \dim S^\perp - \dim V = \dim V - \dim S + \dim V - \dim S - \dim V = \dim V - 2\dim S.$$

From this we obtain that $\dim W^\perp = 2\dim S$. It follows that $S \subset W^\perp$ is a Lagrangian subrepresentation. Applying Proposition 2.7.3.1 to $S \subset W^\perp$, we know that there exists a G-equivariant Lagrangian complement $D \cong S^*$: $W^\perp = S \oplus D$.

Now let's take $C' = W \oplus D$. Clearly C' is a G-equivariant complement to S. Also, $(C')^\perp = W^\perp \cap D^\perp = D \subset C'$, so C' is coisotropic. Note that D is exactly $\mathrm{Ker}\omega|_{C'}$.

Finally if C is any coisotropic G-equivariant complement, by the same argument we get that $W^\perp \cap C$ is equal to $\mathrm{Ker}\omega|_C$ and so $C = W \oplus \mathrm{Ker}\omega|_C$. □

Étale local structure

We will generalise Crawley-Boevey's result to the case of the Hamiltonian reduction of the cotangent bundle of a vector space.

For any vector space V, the direct sum $V \oplus V^*$ has a natural symplectic form: $\omega((z_1, w_1), (z_2, w_2)) = \langle z_1, w_2\rangle - \langle z_2, w_1\rangle$. Action of $\mathrm{GL}(V)$ on V preserves the pairing $\langle \cdot, \cdot \rangle$, so given any linear represen-

tation $G \to \mathrm{GL}(V)$, the induced representation $G \to \mathrm{GL}(V \oplus V^*)$ is symplectic. The space $V \oplus V^*$ is identified with the total space T^*V of the cotangent bundle on V.

Let $x \in \mu^{-1}(\lambda)$ be a point with a closed G-orbit. Then, by the theorem of Matsushima [85], the stabiliser G_x is a reductive group. Let $\mathfrak{g}_x$ be the Lie algebra of G_x; group G_x acts by conjugation on $\mathfrak{g}$, preserving $\mathfrak{g}_x \subset \mathfrak{g}$. Let L be a G_x-equivariant complement to $\mathfrak{g}_x$ inside $\mathfrak{g}$: $\mathfrak{g} \cong \mathfrak{g}_x \oplus L$ as representations of G_x. Such L exists because G_x is reductive. Dually, we have a G_x-equivariant decomposition $\mathfrak{g}^* \cong \mathfrak{g}_x^* \oplus L^*$. We have a natural restriction map $r\colon \mathfrak{g}^* \to \mathfrak{g}_x^*$ and it is easy to see that $r(Z(\mathfrak{g}^*))$ is a subspace of $Z(\mathfrak{g}_x^*)$.

G_x acts on V, preserving x, moreover it preserves the subspace $\mathfrak{g} \circ x \cong \mathfrak{g}/\mathfrak{g}_x$, which is identified with the tangent space to the G-orbit of x.

Lemma 2.7.3.5. *$\mathfrak{g}/\mathfrak{g}_x \subset V \oplus V^*$ is an isotropic subspace: $\mathfrak{g}/\mathfrak{g}_x \subset (\mathfrak{g}/\mathfrak{g}_x)^\perp$*

Proof. The pairing $\langle \cdot, \cdot \rangle$ between V and V^* is being preserved by G, so $\langle a \circ z, w\rangle + \langle z, a \circ w\rangle = 0$ for any $a \in \mathfrak{g}$, $z \in V$, $w \in V^*$. Now

$$\omega(a \circ (z,w), b \circ (z,w)) = \langle a \circ z, b \circ w\rangle - \langle b \circ z, a \circ w\rangle = \\ = \langle b \circ (a \circ z), w\rangle - \langle a \circ (b \circ z), w\rangle = \mu(z,w)([b,a]) = \lambda([b,a]) = 0,$$

since $\lambda\colon \mathfrak{g}_x \to \mathbb{K}$ is Ad-invariant. □

Remark 2.7.3.6. This lemma is the main reason why we consider only symplectic representations of the form $V \oplus V^*$. In general setting, if the subspace $\mathfrak{g}/\mathfrak{g}_x$ is isotropic all arguments below are still valid and one can obtain the similar results about the étale local structure.

Let C be a G_x-equivariant coisotropic complement to $\mathfrak{g} \circ x$ inside $V \oplus V^*$. By Proposition 2.7.3.4 such C exists. We have a G_x-equivariant decomposition $V \oplus V^* \cong \mathfrak{g}/\mathfrak{g}_x \oplus C$. Let W be the intersection $C \cap (\mathfrak{g}/\mathfrak{g}_x)^\perp$. By Lemma 2.7.3.3 the restriction of ω to W is non-degenerate and so W is a symplectic representation of G_x.

Recall that we have the moment map $\mu\colon T^*V \to \mathfrak{g}^*$. Let μ_x be its composition with the restriction $r\colon \mathfrak{g}^* \to \mathfrak{g}_x^*$ and let $\hat{\mu}$ be the restriction of μ_x to W.

We define the map $\nu\colon C \to L^*$ by $\langle \nu(c), l\rangle = \frac{1}{2}(\omega(c, l \circ x) + \omega(c, l \circ c) + \omega(x, l \circ c))$. It is easy

to check that for any $a \in \mathfrak{g}_x$ we have

$$\langle \mu(x+c), a+l \rangle = \lambda(a+l) + \langle \mu_x(c), a \rangle + \langle \nu(c), l \rangle.$$

Recall that $\mathfrak{g}^* \cong \mathfrak{g}_x^* \oplus L^*$. Formula above can be viewed as the fact that the moment map $\mu\colon T^*V \to \mathfrak{g}^*$ restricted to $x + C$ decomposes as a sum of its $\mathfrak{g}_x^*$-component μ_x and the L^*-component $\nu\colon C \to L^*$.

Lemma 2.7.3.7. *The assignment $c \mapsto x + c$ induces a G_x-equivariant map $\mu_x^{-1}(0) \cap \nu^{-1}(0) \to \mu^{-1}(\lambda)$ and the induced map $(\mu_x^{-1}(0) \cap \nu^{-1}(0))/\!\!/G_x \to \mu^{-1}(\lambda)/\!\!/G$ is étale.*

Proof. Exactly the same argument as in Lemma 2.7.2.14 works here as well. □

Lemma 2.7.3.8. *The map $\nu\colon C \to L^*$ is smooth at 0, so 0 is a smooth point of $\nu^{-1}(0)$. Its tangent space at 0 is identified with W.*

Proof. Identifying the tangent spaces at 0 with C and L^* correspondingly, by the definition of ν, the differential of ν at 0 is given by

$$\langle d\nu_0(c), l \rangle = \frac{1}{2}(\omega(c, l \circ x) + \omega(x, l \circ c)).$$

Since ω is preserved by G, $\omega(x, l \circ c) = -\omega(l \circ x, c) = \omega(c, l \circ x)$. From this

$$\langle d\nu_0(c), l \rangle = \omega(c, l \circ x),$$

and so $\mathrm{Ker} d\nu_0 = C \cap (\mathfrak{g}/\mathfrak{g}_x)^\perp = W$. Since $\dim W + \dim L = \dim C$, we get that $d\nu_0$ is surjective and so ν is smooth at 0. □

Since C is isotropic, $W = C \cap (\mathfrak{g}/\mathfrak{g}_x)^\perp$ is the symplectic part of C, and $C = W \oplus \mathrm{Ker}\omega|_C$. This composition is G_x-equivariant.

Lemma 2.7.3.9. *There is a morphism $(\mu_x^{-1}(0) \cap \nu^{-1}(0))/\!\!/G_x \to \hat{\mu}^{-1}(0)/\!\!/G_x$ sending 0 to 0 and which is étale at 0.*

Proof. Let $p\colon C \to W$ be the projection along $\mathrm{Ker}\omega|_C$. Let $c = w + c' \in W \oplus \mathrm{Ker}\omega|_C$ and $a \in \mathfrak{g}_x$. By the formula for μ_x:

$$\langle \mu_x(c), a \rangle = \frac{1}{2}\omega(w + c', a \circ (w + c')) = \frac{1}{2}\omega(w, a \circ w) = \langle \hat{\mu}(w), a \rangle,$$

so $\mu_x = \hat{\mu} \circ p$. The restriction $\varphi\colon \nu^{-1}(0) \to W$ of p is étale at 0 by Lemma 2.7.3.8 and is G_x-equivariant.

Now Luna's Fundamental Lemma can be applied to φ: there exists an affine open subvariety U' of $\nu^{-1}(0)/\!\!/G_x$ containing 0 and such that the restriction of the map

$$\varphi/\!\!/G_x\colon \nu^{-1}(0)/\!\!/G_x \to W/\!\!/G_x$$

to U' is étale.

Since $\mu_x = \hat{\mu} \circ p$, we have $\varphi^{-1}(\hat{\mu}^{-1}(0)) = \mu_x^{-1}(0) \cap \nu^{-1}(0)$. So taking the pull-back of $\varphi/\!\!/G_x$ along the closed embedding $\hat{\mu}^{-1}(0)/\!\!/G_x \hookrightarrow W/\!\!/G_x$ we get a morphism $(\mu_x^{-1}(0) \cap \nu^{-1}(0))/\!\!/G_x \to W/\!\!/G_x$, which is étale at 0. □

Étale equivalences

Let's assume that a representation $G \to \mathrm{GL(V)}$, a vector $\lambda \in Z(\mathfrak{g}^*)^G$ and a character $\theta\colon G \to \mathbb{G}_m$ are chosen such that the map $\pi_\lambda^\theta\colon \mathfrak{M}(G, V \oplus V^*)_\lambda^\theta \to \mathfrak{M}(G, V \oplus V^*)_\lambda$ is a resolution of singularities.

Using Lemma 2.7.2.15 we can lift the étale maps from Lemma 2.7.3.7 and Lemma 2.7.3.9 to the level of the corresponding resolutions. We obtain the following proposition:

Proposition 2.7.3.10. *Let V be a representation of a reductive group G over $\mathbb{K}$. Let $\lambda \in Z(\mathfrak{g})^G$ and $\theta\colon G \to \mathbb{G}_m$ be such that $\pi_\lambda^\theta\colon \mathfrak{M}(G, V \oplus V^*)_\lambda^\theta \to \mathfrak{M}(G, V \oplus V^*)_\lambda$ is a resolution of singularities. Then for any point $x \in \mathfrak{M}(G, V \oplus V^*)_\lambda$ there exists a symplectic subspace W of $V \oplus V^*$, invariant under the action of the stabiliser G_x, together with an étale equivalence*

$$\left(\pi_\lambda^\theta, \mathfrak{M}(G, V \oplus V^*)_\lambda^\theta, \mathfrak{M}(G, V \oplus V^*)_\lambda, x\right) \sim_{et} \left(\pi_0^{\theta_x}, \mathfrak{M}(G_x, W)_0^{\theta_x}, \mathfrak{M}(G_x, W)_0, 0\right).$$

The contracting $\mathbb{G}_m$-action on W, which is given by $t \circ w = tw$, commutes with the action of G_x and descends to both $\mathfrak{M}(G_x, W)$ and $\mathfrak{M}(G_x, W)^{\theta_x}$, turning $\pi\colon \mathfrak{M}(G_x, W)^{\theta_x} \to \mathfrak{M}(G_x, W)$ into a conical resolution of singularities. So Proposition 2.7.3.10 implies the following theorem:

Theorem 2.7.3.11. *Let V be a representation of a reductive group G over $\mathbb{K}$. Let $\theta\colon G \to \mathbb{G}_m$ be a character such that $\pi_0^\theta\colon \mathfrak{M}(G, V \oplus V^*)_0^\theta \to \mathfrak{M}(G, V \oplus V^*)_0$ is a resolution of singularities and assume in addition that $\mathfrak{M}(G, V \oplus V^*)_0$ is normal. Then π_0^θ is a resolution with conical slices.*

Remark 2.7.3.12. When the representation V is such that the action of G for the general point of $V \oplus V^*$ is locally trivial (meaning that the stabiliser of the general point is finite), it is not hard to see that p is a smooth point of $\mu^{-1}(\lambda)$ if and only if its stabiliser is finite. Under this assumptions the moment map μ is surjective, and p is smooth if and only if $d\mu_p$ is surjective too. On the other hand, $\langle d\mu_p(v), a\rangle = \omega(v, a_p)$, so this is equivalent to $a_p \neq 0$ for any $a \in \mathfrak{g}$. In other words, the map $a \mapsto a_p$ from $\mathfrak{g}$ to $T_p(T^*V)$ should be an embedding, and consequently the stabiliser G_p should be finite. If $p \in \mu^{-1}(\lambda)^{\theta-ss}$, then this condition is equivalent to p being not only θ-semistable, but θ-stable as well. We get that $\mathfrak{M}(G, V \oplus V^*)^\theta_\lambda$ is smooth if and only if all θ-semistable points are θ-stable. Note that this is not enough though to guarantee that the map π^θ_λ is a resolution of singularities: we need also to know that it is birational. This seems to be hard to show in general.

The first potential issue is that $\mathfrak{M}(G, V \oplus V^*)^\theta_\lambda$ can happen to be empty (when the θ-semistable locus $\mu^{-1}(\lambda)^{\theta-ss} \subset \mu^{-1}(\lambda)$ is empty). One way to check that it is not, is to use hyperkähler geometry, namely a consequence of Kempf-Ness theorem (for Kempf-Ness theorem see [71], for the consequence see e.g. Lemma 3 in [28]) which states that over $\mathbb{C}$ varieties $\mu^{-1}(\theta)/\!\!/G$ and $\mu^{-1}(0)/\!\!/_\theta G$ are diffeomorphic (here some additional conditions on G, V, θ and λ should be imposed). In particular one is empty if and only if the other one is.

Even bigger potential issue is that the map π^θ_λ can happen to be not generically 1-to-1 or surjective. This is equivalent to the set of points with closed orbits being dense in $\mu^{-1}(\lambda)$, together with the existence of a θ-semistable point in each of the irreducible components of $\mu^{-1}(\lambda)$.

Finally, one also needs to check that the affine variety $\mathfrak{M}(G, V \oplus V^*)_\lambda$ is normal. It is not clear how to check this part in general too.

Remark 2.7.3.13. It can easily happen that for a fixed λ there does not exist any θ, such that $\mathfrak{M}(G, V \oplus V^*)^\theta_\lambda$ is smooth. For example, if G is semisimple then θ automatically equals to 0 and $\mathfrak{M}(G, V \oplus V^*)^\theta_\lambda = \mathfrak{M}(G, V \oplus V^*)_\lambda$, which is often very singular (especially in the case $\lambda = 0$).

Example: Marsden-Weinstein reductions for representations of quivers

Marsden-Weinstein reduction $N_Q(\lambda, \alpha)$ is a particular example of $\mathfrak{M}(G, V \oplus V^*)_\lambda$. Indeed a choice of an orientation on Q provides an identification of $\mathrm{Rep}(\overline{Q}, \alpha)$ with $\mathrm{Rep}(Q, \alpha) \oplus \mathrm{Rep}(Q, \alpha)^*$ as a representation of $\mathbb{P}G(\alpha)$. It is easy to see that after that two constructions literally coincide. So $N^\theta_Q(\lambda, \alpha) = \mathfrak{M}(\mathbb{P}G(\alpha), \mathrm{Rep}(Q, \alpha) \oplus \mathrm{Rep}(Q, \alpha)^*)^\theta_\lambda$.

In this case, Proposition 2.7.2.11 provides the sufficient conditions on Q, λ and α that guarantee

that the map π_λ^θ is a resolution of singularities.

Example: hypertoric varieties

Hypertoric varieties are a particular case of Hamiltonian reduction, namely in the case when the reductive group G is an algebraic torus. Hypertoric varieties (over $\mathbb{C}$) are hyperkähler analogues of toric varieties. Their geometry is related to complicated combinatorics of polytops, such as zonotopal tilings [5] and Stanley-Reisner rings of simplicial complexes ([56], [102]).

Let $V = \mathbb{K}^n$ be a representation of the k-dimensional algebraic torus $T^k \cong \mathbb{G}_m^k$. After a change of coordinates we can assume that the action of T^k is given by an embedding $\iota\colon T^k \to T^n$, where T^n is a standard n-dimensional torus, acting on V. We assume that the representation is faithful, or in other words, that ι is an embedding.

The quotient T^n/T^k is also an algebraic torus, namely $T^n/T^k \cong T^d$, where $d = n - k$. Let $X_*(T^n) = \mathrm{Hom}_{\mathbb{K}}(\mathbb{G}_m, T^n)$ be the cocharacter lattice. We have a non-canonical isomorphism $X_*(T^n) \cong \mathbb{Z}^n$. A short exact sequence of tori

$$1 \to T^k \to T^n \to T^d \to 1$$

induces a short exact sequence of cocharacters:

$$0 \to X_*(T^k) \to X_*(T^n) \to X_*(T^d) \to 0.$$

For each torus T we also have its character lattice $X^*(T) = \mathrm{Hom}_{\mathbb{K}}(T, \mathbb{G}_m)$. The lattice $X^*(T)$ is canonically dual to $X_*(T)$ under the pairing given by the composition of a cocharacter with a character (this composition lies in $\mathrm{Hom}_{\mathbb{K}}(\mathbb{G}_m, \mathbb{G}_m)) \cong \mathbb{Z}$). Let $\{x_1, \ldots, x_n\}$ be a basis of $X_*(T^n)$ and let $\{e_1, \ldots, e_n\}$ be the dual basis of $X^*(T^n)$. Let $\{a_1, \ldots, a_n\} \in X_*(T^d)$ be the images of $\{x_1, \ldots, x_n\}$. The map $X_*(T^n) \to X_*(T^d)$ is surjective, so $\{a_1, \ldots, a_n\}$ span $X_*(T^d)$. The set $\{a_1, \ldots, a_n\} \in X_*(T^d)$ uniquely defines the map $\iota\colon T^k \to T^n$.

The hypertoric variety $\mathfrak{HT}^\alpha$ is defined as $\mathfrak{M}(T^k, V \oplus V^*)_0^\alpha$ in the notations of Section 2.7.3. Note that α here is a character of T^k and so is an element of the lattice $X^*(T^k)$. Let $r = (r_1, \ldots, r_n) \in X_*(T^n)$ be any lift of α with respect to $\iota^*\colon X^*(T^n) \to X^*(T^k)$.

The data which we used to construct $\mathfrak{HT}_\lambda^\alpha$ can be encoded in terms of an arrangement of affine hyperplanes in $X^*(T^d)$ endowed with some additional structure. Namely a *weighted, cooriented*

hyperplane $H \subseteq X^*(T^d)$ is an affine hyperplane along with a choice of a nonzero $a \in X_*(T^d) \cong (X^*(T^d))^*$, such that $a|_H = 0$. For each $i = 1, \ldots, n$ we have a weighted cooriented hyperplane

$$H_i = \{v \in X^*(T^d) \mid \langle v, a_i \rangle + r_i = 0\}$$

with the weight and coorintation given by $a_i \in X_*(T^d)$. We will call this hyperplane arrangement by $\mathcal{A}$. A choice of a different lift of α, say r', corresponds to a simultaneous translation of all hyperplanes in $\mathcal{A}$ by the vector $r' - r \in X_*(T^d)$. This arrangement uniquely defines the map $\iota\colon T^k \to T^n$ along with the choice of α and we call by $\mathfrak{HT}(\mathcal{A})$ the corresponding hypertoric variety.

An arrangement $\mathcal{A}$ is called *simple* if for every subset of m hyperplanes in $\mathcal{A}$ their intersection has codimension m.

Remark 2.7.3.14. Here we use the convention that $\dim \emptyset = -\infty$, and so the codimension of any empty intersection is $+\infty$.

An arrangement $\mathcal{A}$ is called *unimodular* if any collection of d linearly independent weight-orientation vectors $\{a_{i_1}, \ldots, a_{i_d}\}$ spans $X_*(T^d)$ over $\mathbb{Z}$. Note that the condition of being unimodular does not depend on the choice of a character α. If the arrangement is simple and unimodular, it is called *smooth*. The following theorem explains why:

Theorem 2.7.3.15 ([21], Theorem 3.2)**.** *The variety $\mathfrak{M}(\mathcal{A})$ is smooth if and only if $\mathcal{A}$ is smooth.*

Remark 2.7.3.16. In the cited paper this fact is proven for the hyperkähler reduction, and the statement we need follows by the hyperkähler trick, which establishes the diffeomorphism of $\mathfrak{M}(\mathcal{A})_{\mathbb{C}}$ and $\mu^{-1}(\alpha)_{\mathbb{C}}/\!\!/G$ using Kempf-Ness theorem. Here the character α is also considered as an integral element of $(\mathfrak{t}^k)^*$.

Now consider the case when the character α is equal to 0. In this case the intersection of all hyperplanes from $\mathcal{A}$ is the point $0 \in X^*T^d$, so $\mathcal{A}$ is not simple unless $n = d$ and $T^k = \{1\}$. Given an arrangement $\mathcal{A} = \{H_1, \ldots, H_n\}$ with $r = 0$ (so that $H_i = \{v \in X^*(T^d) \mid \langle v, a_i \rangle = 0\}$ for some $a_i \in X_*(T^d)$), it is not hard to see, that for general $r = (r_1, \ldots, r_n) \in X_*(T^n)$, the arrangement $\widetilde{\mathcal{A}} = \{\tilde{H}_1, \ldots, \tilde{H}_n\}$, where $\tilde{H}_i = \{v \in X^*(T^d) \mid \langle v, a_i \rangle + r_i = 0\}$ is simple. In this case the arrangement $\widetilde{\mathcal{A}}$ is called a *simplification* of $\mathcal{A}$. Note that the set of weight-orientation vectors for $\widetilde{\mathcal{A}}$ and $\mathcal{A}$ is the same. In particular, $\widetilde{\mathcal{A}}$ is unimodular if and only if $\mathcal{A}$ is unimodular. Let $\alpha \in X^*(T^k)$ be the image of r under the map $\iota^*\colon X_*(T^n) \to X_*(T^d)$.

We have a projective map $\pi_{\mathcal{A}}\colon \mathfrak{M}(\widetilde{\mathcal{A}}) \to \mathfrak{M}(\mathcal{A})$ (given by the map $\pi_0^{\alpha}\colon \mathfrak{M}(T^k, V\oplus V^*)_0^{\alpha} \to \mathfrak{M}(T^k, V\oplus V^*)_0$ in the terminology of Section 2.7.3) which is known to be surjective and generically 1-to-1. If the arrangement $\mathcal{A}$ is unimodular, this is a resolution of singularities and the base is normal ([8], Section 4). From Theorem 2.7.3.11 we obtain the following:

Theorem 2.7.3.17. *The resolution of singularities $\pi_{\mathcal{A}}\colon \mathfrak{M}(\widetilde{\mathcal{A}}) \to \mathfrak{M}(\mathcal{A})$ (in the case when $\mathcal{A}$ is unimodular) is a resolution with conical slices.*

Remark 2.7.3.18. The stabiliser of any non-zero point $x \in T^*V$ in a hypertoric variety is a torus T^s of smaller dimension. Moreover, there are no self-dual representations of T^s except that of the trivial one. The vector subspace W from Proposition 2.7.3.10 can be T^s-equivariantly decomposed as $W = \mathbb{K}^m \oplus W'$, where $\mathbb{K}^m$ is the subspace of invariants and W' is the T^s-equivariant complement. It is easy to see from the definitions that $\mathfrak{M}(T^s, W)_0^{\alpha}$ can be decomposed as $\mathbb{A}^m \times \mathfrak{M}(T^s, W')_0^{\alpha}$. Since W' does not have any self-dual subresentations, by Remark 2.7.3.2 it can be T^s-equivariantly decomposed as $V_x \oplus V_x^*$. This way $\mathfrak{M}(T^s, W')_0^{\alpha}$ is also a hypertoric variety (associated with the representation V_x). Denoting by $\mathcal{A}_x$ the corresponding hyperplane arrangement, we obtain an étale equivalence

$$(\pi_{\mathcal{A}}, \mathfrak{M}(\widetilde{\mathcal{A}}), \mathfrak{M}(\mathcal{A}), x) \sim_{et} \underline{\mathbb{A}}^m \times (\pi_{\mathcal{A}_x}, \mathfrak{M}(\widetilde{\mathcal{A}}_x), \mathfrak{M}(\mathcal{A}_x), 0).$$

We point out that both in Crawley-Boevey's context and the context of hypertoric varieties, the varieties which appear in the étale equivalences provided by Proposition 2.7.3.10 are from the same class.

2.7.4 Slodowy slices

Let $\mathfrak{g}$ be any split semisimple Lie algebra over $\mathbb{K}$, let G be the adjoint group of $\mathfrak{g}$ and let $\mathcal{N} \subset \mathfrak{g}$ be the nilpotent cone. Let $\mathbb{O} \subset \mathcal{N}$ be a nonzero nilpotent adjoint orbit and let $e \in \mathbb{O}$ be a nilpotent element. By the Jacobson-Morozov theorem we can find an $\mathfrak{sl}_2$-triple (e, h, f), $h, f \in \mathfrak{g}$ associated to e, meaning that $[h, e] = 2e, [e, f] = h, [h, f] = -2f$. We fix such a triple. Let $Z(f) \subset \mathfrak{g}$ be the centraliser of f inside $\mathfrak{g}$. Then the *Slodowy slice* $\mathcal{S}_e$ to $\mathbb{O}$ at e is defined as $\mathcal{S}_e = \{e + Z(f)\} \bigcap \mathcal{N}$ (for more details see [106]). Resolution of singularities of $\mathcal{S}_e$ is given by the restriction of the Springer resolution $\pi\colon T^*\mathcal{B} \to \mathcal{N}$, where $\mathcal{B}$ is the flag variety of G. The map π is $G \times \mathbb{G}_m$-equivariant: $G \times \mathbb{G}_m$ acts on $T^*\mathcal{B}$ through the natural action of G on $\mathcal{B}$ and the contracting $\mathbb{G}_m$-action along the fibers of $T^*\mathcal{B}$, while the action on $\mathcal{N}$ is given by the adjoint action of G and the natural action

of $\mathbb{G}_m$ by dilation, which contracts $\mathcal{N}$ to 0. We denote the resolution by $\pi_e\colon \widetilde{\mathcal{S}}_e \to \mathcal{S}_e$.

Construction of the contracting $\mathbb{G}_m$-action on $\mathcal{S}_e$ is not very straightforward. The Lie algebra homomorphism $\mathfrak{sl}_2 \to \mathfrak{g}$ given by $\mathfrak{sl}_2$-triple (e,h,f) extends to a Lie group homomorphism $\tilde{\gamma}\colon \mathrm{SL}_2 \to G$. We put

$$\gamma\colon \mathbb{G}_m \to G,\ \gamma(t) = \tilde{\gamma}\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}.$$

Composition of γ with the adjoint action produces a $\mathbb{G}_m$-action on $\mathcal{N}$. Unfortunately, it does not preserve e: $(\mathrm{Ad}\gamma(t))(e) = t^2 e$. The $\mathbb{G}_m$-action on the slice is produced from this one by normalization: $\rho(t)(x) = t^2\mathrm{Ad}\gamma(t^{-1})(x)$. One can check that $\rho(t)$ preserves $Z(f)$ and contracts it to 0. So, since $\rho(t)(e+x) = e + \rho(t)x$, we get a $\mathbb{G}_m$-action on $\mathcal{S}_e$, contracting it to e. Since the map π is $G \times \mathbb{G}_m$-equivariant, this action extends to the resolution π_e, turning it into a conical resolution of singularies. Moreover, it is known to be an example of a symplectic resolution (with the symplectic form given by the restriction of the canonical symplectic form on $T^*\mathcal{B}$).

To prove that every point admits a conical neighbourhood, we note that Slodowy slice is a slice to the orbit, so in our terminology it gives an étale equivalence

$$(\pi, T^*\mathcal{B}, \mathcal{N}, e) \sim_{et} (\pi_e, \widetilde{\mathcal{S}}_e, \widetilde{\mathcal{S}}_e, e) \times \underline{\mathbb{A}}^{\dim \mathbb{O}}.$$

Moreover, for every point $e' \in \mathcal{S}_e$ it gives an equivalence

$$(\pi, T^*\mathcal{B}, \mathcal{N}, e') \sim_{et} (\pi_e, \widetilde{\mathcal{S}}_e, \mathcal{S}_e, e') \times \underline{\mathbb{A}}^{\dim \mathbb{O}}.$$

Let $\mathbb{O}' \subset \mathcal{N}$ be the orbit of e'. Then $\dim \mathbb{O}' \geq \dim \mathbb{O}$ and from the analogous decomposition for $\mathbb{O}'$ and e' we get that

$$(\pi_e, \widetilde{\mathcal{S}}_e, \mathcal{S}_e, e') \times \underline{\mathbb{A}}^{\dim \mathbb{O}} \sim_{et} (\pi, T^*\mathcal{B}, \mathcal{N}, e') \sim_{et} (\pi_{e'}, \widetilde{\mathcal{S}}_{e'}, \mathcal{S}_{e'}, e') \times \underline{\mathbb{A}}^{\dim \mathbb{O}'}.$$

Finally Lemma 2.4.5.1 provides from this an equivalence

$$(\pi_e, \widetilde{\mathcal{S}}_e, \mathcal{S}_e, e') \sim_{et} (\pi_{e'}, \widetilde{\mathcal{S}}_{e'}, \mathcal{S}_{e'}, e') \times \underline{\mathbb{A}}^{\dim \mathbb{O}' - \dim \mathbb{O}}.$$

Right hand side is a conical resolution of singularities, so we obtain the following

Theorem 2.7.4.1. *Resolution $\pi_e\colon \widetilde{\mathcal{S}}_e \to \mathcal{S}_e$ of singularities of the Slodowy slice is a resolution with conical slices.*

Chapter 3

p-adic Hodge theory for stacks

3.1 Introduction

The main goal of this chapter is to prove the Totaro's inequality for a reductive group G via p-adic Hodge theory, (more precisely the theory of prismatic cohomology [19]):

$$\dim_{\mathbb{F}_p} H^i_{\mathrm{dR}}(BG_{\mathbb{F}_p}) \geq \dim_{\mathbb{F}_p} H^i_{\mathrm{sing}}(BG(\mathbb{C}), \mathbb{F}_p).$$

In this introduction I will outline the strategy of the proof, explaining along the way the reason for introducing some of the technical tools.

The basic idea is to use the theory of prismatic cohomology that one has for smooth proper schemes. Given a functor (in our case a cohomology theory) defined on schemes, a natural way to extend it to stacks is provided by the right Kan extension. Moreover, if the cohomology theory satisfies étale descent (as it is in the case of the prismatic cohomology) we can compute its value on a stack $\mathcal{X}$ by the means of a simplicial scheme, namely a Čech object associated to some smooth cover $U \to \mathcal{X}$. If instead of G we considered an abelian variety A then we would be done, since all terms in the Čech object associated to the smooth cover $* \to BA$ by would be proper smooth schemes (given by A^n for various n) and we would then be able to deduce all the necessary properties from the case of schemes. Unfortunately BG is not proper (unless G is finite) and thus it is *impossible* to have such a simplicial resolution.

Nevertheless, if G is reductive, then BG is still *cohomologically* proper. By definition, cohomologically proper stacks do not need to be proper, but they should behave similarly when considering

cohomology of coherent sheaves (see Definition 3.4.2.12). It is rather non-clear though how this condition is manifested in the geometry of the stack (we discuss this in some detail later in Chapter 4). Under the cohomological properness assumption one can show (see Section 3.5.4) that the individual prismatic cohomology groups $H^i_{\mathbb{\Delta}}(\mathcal{X}/\mathfrak{S})$ are finitely generated $\mathfrak{S}$-modules (here $\mathfrak{S}$ is the Kisin ring $\mathbb{Z}_p[[u]]$).[1] Moreover, the prismatic cohomology compare with the de Rham cohomology $R\Gamma_{\mathrm{dR}}(\mathcal{X}_{\mathbb{F}_p})$ after the (derived) reduction modulo $(p, u-p)$; thus by semicontinuity (and the étale comparison Section 3.5.5) we get an inequality:

$$\dim_{\mathbb{F}_p} H^i_{\mathrm{dR}}(\mathcal{X}_{\mathbb{F}_p}) \geq \dim_{\mathbb{F}_p} H^i_{\text{ét}}(\widehat{\mathcal{X}}_{\mathbb{C}_p}, \mathbb{F}_p).$$

Here, $\widehat{\mathcal{X}}_{\mathbb{C}_p}$ is the Raynaud generic fiber of $\mathcal{X}$ (it is a rigid analytic stack, defined in Section 3.3.4) and unfortunately its cohomology can't a priori be related to anything like the topological space of the $\mathbb{C}$-points of $\mathcal{X}$ (as opposed to the cohomology of the *algebraic* generic fiber $\mathcal{X}_{\mathbb{C}_p}$). Now, in the case of a smooth proper scheme X the cohomology of the two generic fibers agree. Indeed, in this case, $\widehat{X}_{\mathbb{C}_p} \simeq X^{\mathrm{an}}_{\mathbb{C}_p}$, where $X^{\mathrm{an}}_{\mathbb{C}_p}$ is the analytification of the (geometric) generic fiber of X and $H^i_{\text{ét}}(X_{\mathbb{C}_p}, \mathbb{F}_p) \simeq H^i_{\text{ét}}(X_{\mathbb{C}_p}, \mathbb{F}_p)$ without any assumptions on X. Still, for a general scheme, by the universal property of $X^{\mathrm{an}}_{\mathbb{C}_p}$, there is a map $\psi_X \colon \widehat{X}_{\mathbb{C}_p} \to X^{\mathrm{an}}_{\mathbb{C}_p}$. Here is an illustration of what is going on:

Example 3.1.0.1. In the case $X = \mathbb{A}^1$ we have $\widehat{X}_{\mathbb{C}_p} = \mathbb{D}_1 = \mathrm{Sp}(\mathbb{C}_p\langle T\rangle)$ being equal to the closed disc of radius 1, while $(\mathbb{A}^1_{\mathbb{C}_p})^{\mathrm{an}}$ is an actual analytic affine line $\bigcup_{r\in\mathbb{R}_{>0}} \mathbb{D}_r$: the union of balls of all radii r centered at 0. The map $\psi_{\mathbb{A}^1} \colon \widehat{\mathbb{A}}^1_{\mathbb{C}_p} \to (\mathbb{A}^1_{\mathbb{C}_p})^{\mathrm{an}}$ is given by the embedding of the disk of radius 1 to the latter union. As we see $\psi_{\mathbb{A}^1}$ is rather far from being an isomorphism. Let's now see how the situation improves in the case of $\mathbb{P}^1$. Let's cover $\mathbb{P}^1$ by the two copies of $\mathbb{A}^1$ intersecting in $\mathbb{A}^1 \setminus 0$. The Raynaud fiber of this cover will consist of two discs of radius 1 (two semispheres) glued along the unit circle $\mathrm{Sp}(\mathbb{C}_p\langle T, T^{-1}\rangle)$ (equator), while the analytification of the generic fiber will produce two analytic affine lines (spheres without a point) glued along the analytic punctured affine line (a sphere without two points).The result of gluing gives the same thing in both cases and t the map $\psi_{\mathbb{P}^1}$ is an isomorphism. We should also warn that the étale $\mathbb{F}_p$-cohomology of $\mathbb{D}_1$ are infinite dimensional! (and thus can't be equal to the singular $\mathbb{F}_p$-cohomology of $\mathbb{A}^1(\mathbb{C}) \sim *$)

The problem with the cohomologically proper stacks is that we still have $\mathcal{X}^{\mathrm{an}}_{\mathbb{C}_p} \neq \widehat{\mathcal{X}}_{\mathbb{C}_p}$ unless

[1] For the finite-generatedness in fact a much weaker condition of Hodge-properness is enough (see Definition 3.4.2.7).

$\mathcal{X}$ is proper. Thus it seems that we did not achieve much progress after all. However, the hope could be that (may be under some additional assumptions on $\mathcal{X}$) the étale $\mathbb{F}_p$-cohomology of these analytic stacks are the same. Note that the cohomology of $\mathcal{X}^{\mathrm{an}}_{\mathbb{C}_p}$ still compares with that for $\mathcal{X}_{\mathbb{C}_p}$ even if $\mathcal{X}$ is a stack, so in the end this would indeed show that the cohomology of the algebraic and the Raynaud generic fibers are the same. Finally, we could also identify the étale cohomology of $\mathcal{X}_{\mathbb{C}_p}$ with the singular cohomology of its (appropriately defined) topological space $\mathcal{X}(\mathbb{C})$ of complex points, ultimately getting an inequality between the dimension of the de Rham cohomology with the dimension of the latter.

In particular we can try to figure this out in case of the classifying stack BG with G reductive. Restricting to this context it is a good idea to start with the simplest non-proper case $G = \mathbb{G}_m$. In topology we have a nice presentation of $B\mathbb{C}^\times \simeq BS^1$ as the infinite dimensional projective space $\mathbb{CP}^\infty = \mathrm{colim}_n \mathbb{CP}^n$. In the algebraic setting these two objects are different, and things are even worse since the étale cohomology of the Raynaud fiber is not $\mathbb{A}^1$-homotopy invariant. Still, by reducing to de Rham cohomology one can show that for $B\mathbb{G}_m$ the étale cohomology of the Raynaud fiber and the analytification of the algebraic one are the same (see Section 3.6.2). By a direct computation with Hodge cohomology one can also deduce the case of $\mathbb{A}^n/T$ with a contracting action of T (see Section 3.6.3); this then allows to cover the case of BB (with B being Borel) by considering the Čech object associated to $BT \to BB$ (and using smooth descent). Indeed, the terms of the associated Čech complex will look as $[U^n/T]$ with the diagonal adjoint action of T and thus are of the above form. The final (most nontrivial) step then is to show the statement for $[X/B]$ with X proper; given that, the isomorphism for BG is obtained by descent using the smooth cover $BB \to BG$ (with the terms of the corresponding Čech object given by $[(B\backslash G)^n/B]$ where the flag variety $B\backslash G$ is smooth and proper).

The proof of the case of $[X/B]$ (see Section 3.6.5) is exactly the place where most of the tools we introduce in this chapter are being used, most importantly derived étale sheaves on stacks (Section 3.3.4), derived local systems (Section 3.3.4) and rigid analytic stacks with étale sheaves on them (Section 3.3.4). The idea of the proof is to replace the étale cohomology of $[X/B]$ with coefficients in $\mathbb{F}_p$ by the cohomology of BB with coefficients in its pushforward under the natural morphism $[X/B] \to BB$. Then most of the techniques mentioned above is used to compare what happens with this pushforward on the algebraic and the analytic (including Raynaud generic fiber) sides. More precisely under the proper and smooth assumption we prove

the "pushforward-commutes-with-analytification" statement (see Corollary 3.3.4.38). Also we show that the cohomology sheaves of the derived pushworward of the constant sheaf on the algebraic side are local systems (Corollary 3.3.4.17), generalizing a theorem of Deligne. Since $B_{\mathbb{C}_p}$ is connected, $BB_{\mathbb{C}_p}$ is étale simply connected and does not have any non-trivial local systems. This reduces the question to the case of BB and the constant sheaf (which was covered before). Same proof works for quotients $[X/P]$ where X is proper and P is any parabolic in a reductive group.

The question of what important parts of geometry of BG were used in the proof is rather interesting (and we do not have a clear answer). For now, our conjecture with Artem (3.5.3.12) is that the isomorphism between the étale cohomology of the two generic fibers should hold if $\mathcal{X}$ satisfies formal (p-adic GAGA). We can prove this for the 1-st cohomology (the proof is not included in this thesis) and have a strategy for at least dealing with other cohomology groups at least after inverting p via some recent results in the algebraic K-theory. The formal GAGA condition also puts this conjecture in the context of *formal properness* of Halpern-Leistner and Preygel which could be considered as a good sign (see [55]).

3.1.1 Notations and conventions

- All categories are implicitly assumed to be $(\infty, 1)$ if not explicitly stated otherwise. All (co-)limits are in the sense of higher category theory. The $(\infty, 1)$-category of spaces (obtained from the category of simplicial sets with the standard Quillen model structure) will be denoted by $\mathscr{S}$. All functors (tensor product, pushforwards, e.t.c.), if not stated explicitly otherwise, are assumed to be derived.

- The category of presentable categories and continuous functors will be denoted by $\mathscr{P}r^{\mathrm{L}}$. For a commutative ring A the category of presentable R-linear categories and continuous R-linear functors will be denoted by $\mathscr{P}r^{\mathrm{L}}_A$.

- By CAlg we will denote the usual 1-category of discrete commutative rings. The category of algebras over the E_∞ operad in a symmetric monoidal category $\mathscr{C}$ will be denoted by $\mathrm{Alg}_{E_\infty}(\mathscr{C})$.

- For a commutative ring R we will denote the 1-category of (usual) R-modules by Mod_R.

- For a symmetric monoidal category $\mathscr{C}$ and an algebra object A over the commutative operad

E_∞ in $\mathscr{C}$, we will denote the category of A-modules in $\mathscr{C}$ by $\mathrm{Mod}_A(\mathscr{C})$ or just Mod_A if the underlying category $\mathscr{C}$ is clear from the context.

- For an ordinary commutative ring A we will often work with the category Mod_{HA} of modules over the corresponding Eilenberg-Maclane ring HA. This is just an $(\infty,1)$-enhancement of the usual (unbounded) derived category of complexes of A-modules. In particular Mod_{HA} admits a natural t-structure, such that the heart $\mathrm{Mod}_{HA}^{\heartsuit}$ is canonically equivalent to Mod_A.

3.2 p-adic Hodge theory for schemes: reminder

3.2.1 Fontaine's infinitesimal period ring A_{inf}

We will start by defining the Fontaine's ring $A_{\mathrm{inf}} := A_{\mathrm{inf}}(\mathcal{O}_{\mathbb{C}_p})$: all other p-adic period rings will then be defined in its terms.

Let S be a p-adically complete ring and let $\varphi\colon S/p \to S/p$ be the absolute Frobenius map. We let the *tilt* $S^\flat := (S/p)^{\mathrm{perf}}$ be the inverse limit perfection of S/p. It is a perfect ring of characteristic p.

It is not hard to see that the natural map

$$\lim_{x\mapsto x^p} S \to \lim_{x\mapsto x^p} S/p \simeq S^\flat$$

is an isomorphism of multiplicative monoids ([17], Lemma 3.2(i)). This way, any element $x \in S^\flat$ has a unique expression as a sequence $(x^{(0)}, x^{(1)}, x^{(2)}, \ldots)$ such that $(x^{(i)})^p = x^{(i-1)}$ with $x^{(i)} \in S$. Note that in this description we have $(x^{(0)}, x^{(1)}, x^{(2)}, \ldots)^{1/p} := \varphi^{-1}((x^{(0)}, x^{(1)}, x^{(2)}, \ldots)) = (x^{(1)}, x^{(2)}, x^{(3)}, \ldots)$.

Definition 3.2.1.1. Fontain's ring $A_{\mathrm{inf}}(S)$ is defined as

$$A_{\mathrm{inf}}(S) := W(S^\flat),$$

where W is the full ring of p-typical Witt vectors.

The absolute Frobenius $\varphi\colon S^\flat \xrightarrow{\sim} S^\flat$ on $S^\flat$ induces an automorphism $\varphi\colon A_{\mathrm{inf}}(S) \xrightarrow{\sim} A_{\mathrm{inf}}(S)$. Any element $x \in A_{\mathrm{inf}}(S)$ of Witt vectors has a unique p-adic expansion $x \in \sum_{i\geq 0}[x_i]\cdot p^i$ where

$[x_i]$ are the Teichmüller lifts of some elements $x_i \in S^\flat$. In terms of this expression we have $\varphi(x) = \sum_{i\geq 0}[x_i^p]\cdot p^i$. One can also define a map

$$\theta\colon A_{\mathrm{inf}}(S) \to S, \text{ given by } x = \sum_{i\geq 0}[x_i]\cdot p^i \mapsto \sum_{i\geq 0}(x_i)^{(0)}\cdot p^i \in S,$$

which turns out to be a homomorphism of rings.

For our applications, it will be mostly enough to consider the case $S = \mathcal{O}_{\mathbb{C}_p}$, where $\mathcal{O}_{\mathbb{C}_p} \subset \mathbb{C}_p$ is the ring of integers of the p-adic completion of $\overline{\mathbb{Q}}_p$. For convenience, once and for all, we put $A_{\mathrm{inf}} \coloneqq A_{\mathrm{inf}}(\mathcal{O}_C)$. In this case the p-th power map $\mathcal{O}_{\mathbb{C}_p} \xrightarrow{x\mapsto x^p} \mathcal{O}_{\mathbb{C}_p}$ is surjective, and thus so is $\theta\colon A_{\mathrm{inf}} \to \mathcal{O}_{\mathbb{C}_p}$. Moreover, $\mathcal{O}_{\mathbb{C}_p}$ is an example of a *perfectoid* ring (see [17, Definition 3.5]); in particular, the ideal $\ker(\theta)$ is principal.

We now discuss some natural elements of A_{inf}, including an explicit generator ξ for $\ker(\theta)$.

Remark 3.2.1.2. Fix a compatible choice $(1, \zeta_p, \zeta_{p^2}, \ldots)$, $\zeta_{p^{n-1}} = \zeta_{p^n}^p \in \mathcal{O}_{\mathbb{C}_p}$ of p^n-th roots of unity. It defines an element $\varepsilon \coloneqq (1, \zeta_p, \zeta_{p^2}, \ldots) \in \mathcal{O}_{\mathbb{C}_p}^\flat$ with its p-th root given by $\varepsilon^{1/p} \coloneqq (\zeta_p, \zeta_{p^2}, \zeta_{p^3}, \ldots)$. The element

$$\xi \coloneqq 1 + [\varepsilon^{1/p}] + [\varepsilon^{2/p}] + \cdots + [\varepsilon^{p-1/p}],$$

then satisfies $\theta(\xi) = 1 + \zeta_p + \zeta_p^2 + \cdots \zeta_p^{p-1} = 0$, so $\xi \in \ker(\theta)$. But by [17, Example 3.16] ξ is also distinguished (see Definition 3.2.3.8) and so it generates $\ker(\theta)$ ([17, Lemma 3.10]).

We will also denote by $\widetilde{\xi} \coloneqq \varphi(\xi) = 1 + [\varepsilon] + \cdots + [\varepsilon]^{p-1}$. Note that $\widetilde{\xi} \equiv 1 + 1 + \cdots + 1 \equiv p \mod (\xi)$. Let also $\mu \coloneqq [\varepsilon] - 1$. We have

$$\frac{\varphi(\mu)}{\mu} = \frac{[\varepsilon]^p - 1}{[\varepsilon] - 1} = 1 + [\varepsilon] + [\varepsilon]^2 + \cdots + [\varepsilon]^{p-1} = \widetilde{\xi}.$$

We will use this sort of relations later in Section 3.2.5 when looking at Breuil-Kisin-Fargues modules.

Also, for any element $\pi \in \mathfrak{m}_{\mathcal{O}_{\mathbb{C}_p}}$ in the maximal ideal and any compatible choice $\pi, \pi^{\frac{1}{p}}, \pi^{\frac{1}{p^2}}, \ldots$ of p^n-th roots of π, we get an element $\pi^\flat \coloneqq (\pi, \pi^{\frac{1}{p}}, \pi^{\frac{1}{p^2}}, \ldots) \in \mathcal{O}_{\mathbb{C}_p}^\flat$. If we localise $\mathcal{O}_{\mathbb{C}_p}^\flat$ at $\pi^\flat$, for any choice of $\pi \in \mathfrak{m}_{\mathcal{O}_{\mathbb{C}_p}}$, we obtain a field $\mathbb{C}_p^\flat \coloneqq \mathcal{O}_{\mathbb{C}_p^\flat}[\frac{1}{\pi^\flat}]$, which is the fraction field of $\mathcal{O}_{\mathbb{C}_p}^\flat$. We have a natural flat map $A_{\mathrm{inf}} \to W(\mathbb{C}_p^\flat)$.

Remark 3.2.1.3. Note that the construction $S \mapsto A_{\mathrm{inf}}(S)$ is functorial. In particular, having

a finite extension $\mathbb{Q}_p \subset K$ we get a natural G_K-action on A_{inf}. It is continuous with respect to (ξ, p)-adic topology since the ideal (ξ, p) is G_K-invariant.

3.2.2 Rational p-adic Hodge theory

In this section we briefly remind some of the main results and constructions of rational p-adic Hodge theory that we have found relevant to this thesis.

Let K be a discretely valued extension of $\mathbb{Q}_p$ with the ring of integers $\mathcal{O}_K$ and the perfect residue field k. Let's also denote by $K_0 \simeq W(k)[\frac{1}{p}] \subset K$ the maximal unramified subextension of K; let $\varphi\colon K_0 \to K_0$ be the natural Frobenius on K_0 induced by the absolute Frobenius on k. Roughly speaking, the goal of rational p-adic Hodge theory is to describe $\mathbb{Q}_p$-Galois representations of geometric nature in terms of some more accessible data of semilinear algebra.

To proceed, we first recall the definition of (some) period rings introduced by Fointaine.

Definition 3.2.2.1. Fontaine's p-adic period rings:

- (*De Rham* B_{dR}, B^+_{dR}). Probably the first important period ring (besides A_{inf}) is B^+_{dR}. It is defined as the ξ-adic completion of the ring $A_{\text{inf}}[\frac{1}{p}]$ and is a complete discrete valuation ring with residue field $\mathbb{C}_p$ and uniformizer ξ. The field B_{dR} is then defined as the localization $B^+_{\text{dR}}[\frac{1}{\xi}] \simeq \text{Frac}(B^+_{\text{dR}})$. The ideal $\ker\theta = (\xi) \subset A_{\text{inf}}$ is G_K-stable (since θ is G_K-equivariant) and this endows both B^+_{dR} and B_{dR} with a natural G_K-action. The field B_{dR} then admits a natural G_K-stable $\mathbb{Z}$-indexed decreasing filtration $F^{\geq n}B_{\text{dR}} \coloneqq \xi^n B^+_{\text{dR}} \subseteq B_{\text{dR}}$. One can show that $\text{gr}^\bullet_F B_{\text{dR}} \simeq \bigoplus_{n\in\mathbb{Z}} \mathbb{C}_p(n)$ as a G_K-representation, and, if one forgets the Galois action, there is an isomorphism $B_{\text{dR}} \simeq \mathbb{C}_p((\xi))$.

- (*Cristalline* B_{crys}, A_{crys}). The ring A_{crys} is defined as the p-adic completion of the universal PD-thickening of $\theta\colon A_{\text{inf}} \to \mathcal{O}_C$. It can also be constructed as the 0-th (derived) cristalline cohomology $H^0_{\text{crys}}(\text{Spec}(\mathcal{O}_{\mathbb{C}_p}/p)/\mathbb{Z}_p)$ [18, Theorem 8.14(3)]. More explicitely, since A_{inf} is p-torsion free and since $\ker\theta = (\xi)$, one can describe A_{crys} as the p-adic completion of the A_{inf}-subalgebra generated by $\frac{\xi^n}{n!}$ inside of $A_{\text{inf}}[\frac{1}{p}]$. The ring A_{crys} receives a natural G_K-action and a Frobenius endomorphism $\varphi\colon A_{\text{crys}} \to A_{\text{crys}}$, which is given by the Frobenius on $H^0_{\text{crys}}(\text{Spec}(\mathcal{O}_{\mathbb{C}_p}/p)/\mathbb{Z}_p)$ and agrees with (previously defined) Frobenius on A_{inf}. Then one defines B_{crys} as $A_{\text{crys}}[\frac{1}{\mu}]$. One can show that that $\varphi(\xi)$ is invertible in A_{crys} and thus, by Remark 3.2.1.2 $\varphi\colon B_{\text{crys}} \to B_{\text{crys}}$ is well-defined.

- (*The map* $B_{\mathrm{crys}} \to B_{\mathrm{dR}}$). By introducing a (quite natural) topology on B_{dR} one can show that the natural composite embedding $A_{\mathrm{inf}}[\frac{\xi^n}{n!}]_{n\geq 1} \subset A_{\mathrm{inf}}[\frac{1}{p}] \subset B_{\mathrm{dR}}$ can be extended to a natural map $j\colon A_{\mathrm{crys}} \to B_{\mathrm{dR}}$, which turns out to be an embedding [42, Section 4.1] [2]. Since $\mu \in B_{\mathrm{dR}}$ is a unit, j can also be extended to an embedding $B_{\mathrm{crys}} \subset B_{\mathrm{dR}}$. This way we can define a G_K-stable filtration $F^n B_{\mathrm{crys}} \coloneqq F^n B_{\mathrm{dR}} \cap B_{\mathrm{crys}}$. We warn that since the ideal ξA_{inf} is not stable under φ, the filtration on B_{crys} is *not* preserved by the Frobenius.

To a G_K-representation V one can associate two rational Dieudonné modules:

$$D_{\mathrm{dR}}(V) := (V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}})^{G_K} \qquad D_{\mathrm{crys}}(V) := (V \otimes_{\mathbb{Q}_p} B_{\mathrm{crys}})^{G_K}.$$

By Hensel's lemma, the map $\theta\colon B_{\mathrm{dR}} \to \mathbb{C}_p$ has a section $\overline{\mathbb{Q}}_p \dashrightarrow B_{\mathrm{dR}}$ over $\overline{\mathbb{Q}}_p \subset \mathbb{C}_p$, which is $G_{\mathbb{Q}_p}$-equivariant. In particular, we have a natural inclusion $K \hookrightarrow B_{\mathrm{dR}}$ (resp. $K_0 \hookrightarrow B_{\mathrm{crys}}$) and in fact one can show that $K \simeq (B_{\mathrm{dR}})^{G_K}$. Similarly, $K_0 \simeq (B_{\mathrm{crys}})^{G_K}$. This way $D_{\mathrm{dR}}(V)$ (reps. $D_{\mathrm{crys}}(V)$) is naturally a K- (resp. K_0-) vector space. The G_K-invariant filtration $F^{\geq n} B_{\mathrm{dR}}$ on B_{dR} induces a filtration on $D_{\mathrm{dR}}(V)$. Similarly, the filtration and the Frobenius on B_{crys} define the same type of structures on $D_{\mathrm{crys}}(V)$. Also note that the Frobenius on $D_{\mathrm{crys}}(V)$ is semi-linear with respect to the natural Frobenius on $K_0 \simeq W(k)[\frac{1}{p}]$.

There are natural maps

$$D_{\mathrm{dR}}(V) = (V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}})^{G_K} \otimes_K B_{\mathrm{dR}} \longrightarrow V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}$$
$$D_{\mathrm{crys}}(V) = (V \otimes_{\mathbb{Q}_p} B_{\mathrm{crys}})^{G_K} \otimes_K B_{\mathrm{crys}} \longrightarrow V \otimes_{\mathbb{Q}_p} B_{\mathrm{crys}}$$

induced by multiplication on B_{dR} and B_{crys} correspondingly.

Definition 3.2.2.2. Let V be a finite dimensional continuous $\mathbb{Q}_p$-representation of G_K. We call V *de Rham* if the natural map

$$D_{\mathrm{dR}}(V) \otimes_K B_{\mathrm{dR}} \to V \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}$$

[2] The image of A_{crys} in B_{dR} can be explicitely described as $\{\sum_{n\geq 0} x_n \frac{\xi^n}{n!} \mid x_n \in A_{\mathrm{inf}} \text{ and } x_n \to 0 \text{ w.r.t. } p\text{-adic toplogy.}\}$

is an isomorphism. Similarly, V is called *crystalline* if the natural map

$$D_{\mathrm{crys}}(V) \otimes_{K_0} B_{\mathrm{crys}} \to V \otimes_{\mathbb{Q}_p} B_{\mathrm{crys}}$$

is an isomorphism.

Remark 3.2.2.3. By [43, Theorem 5.2] if V is cristalline, then V is also de Rham and the natural map

$$D_{\mathrm{crys}}(V) \otimes_{K_0} K \to D_{\mathrm{dR}}(V)$$

is an isomorphism. Also, by the theory of admissible modules, V is cristalline if and only if $D_{\mathrm{crys}}(V)$ has the same dimension as V, meaning $\dim_{K_0} D_{\mathrm{crys}}(V) = \dim_{\mathbb{Q}_p} V$.

We will denote the category of crystalline representations by $\mathrm{Rep}^{\mathrm{crys}}_{G_K}$. We will just mention the existence of the category MF^{φ}_K of weakly admissible filtered φ-modules whose objects are roughly given by a K_0-vector space W with a Frobenius-linear endomorphism φ and a filtration $F^{\leq *}$ on $W \otimes_{K_0} K$, where φ and $F^{\leq *}$ are compatible in a certain way. One can show that the functor D_{crys} in fact takes values in MF^{φ}_K; moreover, its restriction to $\mathrm{Rep}^{\mathrm{crys}}_{G_K}$ is fully faithful.

Theorem 3.2.2.4 ([43, Section 5.5.2])**.** *The functor* $D_{\mathrm{crys}}\colon \mathrm{Rep}^{\mathrm{crys}}_{G_K} \to MF^{\varphi}_K$ *is symmetric monoidal and fully-faithful with the inverse on its essential image given by*

$$V_{\mathrm{crys}}(D) := \mathrm{Fil}^{\geq 0}(D \otimes_{K_0} B_{\mathrm{crys}})^{\varphi=1} := \{x \in \mathrm{Fil}^{\geq 0}(D \otimes_{K_0} B_{\mathrm{crys}}) \mid \varphi(x) = x\}.$$

In particular, we see that if V is cristalline, it is defined uniquely by its cristalline Diedonné module $D_{\mathrm{crys}}(V) \in MF^{\varphi}_K$.

The story above has a geometric counterpart, which in fact motivates all the notations for the corresponding period rings and Diedonné module functors. Namely let X be a smooth proper scheme over $\mathcal{O}_K$. Then we have:

- The p-adic rational étale cohomology of the generic fiber $H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$. This is a finite dimensional continuous $\mathbb{Q}_p$-representation of G_K;

- The algebraic de Rham cohomology of the generic fiber $H^i_{\mathrm{dR}}(X_K/K)$. This is a K-vector space equipped with the Hodge filtration.

- The rational crystalline cohomology of the special fiber $H^i_{\mathrm{crys}}(X_k/K_0) := H^i_{\mathrm{crys}}(X_k/W(k))[\frac{1}{p}]$.

This is a K_0-vector space equipped (by functoriality) with a Frobenius-linear endomorphism φ and the Hodge filtration on

$$H^i_{\mathrm{crys}}(X_k/K_0) \otimes_{K_0} K \simeq H^i_{\mathrm{dR}}(X_K/K),$$

which make it an object of MF^{φ}_K.

Then one of the main results of rational p-adic Hodge theory is the following

Theorem 3.2.2.5 ([112, Theorem 0.2], [17, Theorem 1.1])**.** *Let X be a smooth proper scheme over $\mathcal{O}_K$. Then the Galois representation $H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$ is crystalline for any $i \in \mathbb{Z}$. Moreover, there are natural isomorphism*

$$D_{\mathrm{crys}}(H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)) \simeq H^i_{\mathrm{crys}}(X_k/K_0) \qquad D_{\mathrm{dR}}(H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)) \simeq H^i_{\mathrm{dR}}(X_K/K)$$

compatible with the Frobenii and filtrations. In particular one can recover $H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$ as a Galois representation from $H^i_{\mathrm{crys}}(X_k/K_0)$ with its natural Frobenius and the Hodge filtration on $H^i_{\mathrm{dR}}(X_K/K)$ and vice versa.

3.2.3 Integral p-adic Hodge theory

We will now pass to some relevant aspects of integral p-adic Hodge theory. As one could guess already, integral p-adic Hodge theory is concerned with $\mathbb{Z}_p$-modules with a continuous G_K-action that come from geometry. Ideally, given a smooth proper scheme $X/\mathcal{O}_K$, we could hope to relate $H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_p)$, $H^i_{\mathrm{crys}}(X_k/W(k))$ and $H^i_{\mathrm{dR}}(X/\mathcal{O}_K)$ by some sort of Diedonné functors, as in Theorem 3.2.2.5. Unfortunately, the isomorphisms in Theorem 3.2.2.5 do not preserve the integral structures in general; they do so only if $ie < p-1$, for e being the ramification index of K (for the latter see e.g. [27]). Another obvious difficulty is the complete loss of information about p-torsion after we invert p.

In any case, we can at least try to understand the p-torsion free case.

Definition 3.2.3.1. *A $\mathbb{Z}_p$-lattice T* in a continuous G_K-representation is a free $\mathbb{Z}_p$-module of finite rank with an action of G_K. Let $\mathrm{Rep}^{\#}_{G_K}$ be the category of such lattices.

Since we know that the G_K-representation $H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$ is cristalline, it makes sense to restrict our attention to $\mathbb{Z}_p$-lattices T for which $V \coloneqq T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is crystalline. Let $\mathrm{Rep}^{\#,\mathrm{crys}}_{G_K} \subset \mathrm{Rep}^{\#}_{G_K}$ be the full subcategory with objects given by such lattices T. A suitable full embedding of $\mathrm{Rep}^{\#,\mathrm{crys}}_{G_K}$ into a category with objects given by certain semi-linear algebraic data was constructed by Kisin in [73]. To this end we need to introduce some more notations.

Definition 3.2.3.2. Define *the Kisin ring* $\mathfrak{S}$ to be $W(k)[[u]]$. It has a natural Frobenius morphism $\varphi_{\mathfrak{S}} \colon \mathfrak{S} \to \mathfrak{S}$ which sends u to u^p and restricts to the natural Frobenius on $W(k)$. Moreover, the choice of a uniformizer $\pi \in \mathcal{O}_K$ induces a surjective ring morphism $\widetilde{\theta} \colon \mathfrak{S} \twoheadrightarrow \mathcal{O}_K$, $u \mapsto \pi$ with the kernel being generated by the Eisenstein polynomial $E(u)$ for π.

The category of semilinear data mentioned above is then defined as follows:

Definition 3.2.3.3. *A Breuil-Kisin module* M is a finitely generated $\mathfrak{S}$-module together with an isomorphism

$$(\varphi^*_{\mathfrak{S}} M)[\tfrac{1}{E}] \xrightarrow{\sim} M[\tfrac{1}{E}].$$

The category of Breuil-Kisin modules will be denoted by $\mathrm{Mod}^{\varphi}_{\mathfrak{S}}$.

A compatible choice $\pi^\flat \coloneqq (\pi, \pi^{1/p}, \pi^{1/p^2}, \ldots) \in \mathcal{O}^\flat_{\mathbb{C}_p}$ of p^n-th roots of π, induces a natural morphism $\mathfrak{S} \to A_{\mathrm{inf}}$ by sending u to $[\pi^\flat]^p$. Let $K_\infty \coloneqq K(\pi^{1/p^\infty}) \subset \mathbb{C}_p$ be the infinite extension of K obtained by adjoining π^{1/p^n} for all $n \geq 1$. By composing $\mathfrak{S} \to A_{\mathrm{inf}}$ with a natural map $A_{\mathrm{inf}} \to W(\mathbb{C}^\flat_p)$ we obtain a homomorphism $\mathfrak{S} \to W(\mathbb{C}^\flat_p)$ which is (φ, G_{K_∞})-equivariant. Then:

Theorem 3.2.3.4 ([74, Theorem 1.2.1])**.** *There is a fully-faithful tensor (*not *exact) functor* $BK \colon \mathrm{Rep}^{\#,\mathrm{crys}}_{G_K} \to \mathrm{Mod}^{\varphi}_{\mathfrak{S}}$ *such that*

$$(BK(T)/u)[\tfrac{1}{p}] \simeq D_{\mathrm{crys}}(T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p) \qquad \textit{and} \qquad (\varphi^*_{\mathfrak{S}} BK(T)/E(u))[\tfrac{1}{p}] \simeq D_{\mathrm{dR}}(T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p).$$

Moreover $BK(T)$ *is characterized as a unique free Breuil-Kisin module such that there exists a* (φ, G_{K_∞})*-equivariant isomorphism*

$$BK(T) \otimes_{\mathfrak{S}} W(C^\flat) \simeq T \otimes_{\mathbb{Z}_p} W(C^\flat).$$

Note that $BK(T)/(u)$ gives a $W(k)$-lattice in $D_{\mathrm{crys}}(T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)$ and, similarly, $\varphi^*_{\mathfrak{S}} BK(T)/E(u)$ gives an $\mathcal{O}_K$-lattice in $D_{\mathrm{dR}}(T \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)$. Given Theorem 3.2.3.4, in view of Theorem 3.2.2.5, it

is natural to ask whether for $T = H^i(\mathfrak{X}_{\overline{K}}, \mathbb{Z}_p)$ these lattices correspond to $H^i_{\mathrm{crys}}(X_k/W(k)) \subset H^i_{\mathrm{crys}}(X_k/W(k))[\frac{1}{p}]$ and $H^i_{\mathrm{dR}}(X/\mathcal{O}_K) \subset H^i_{\mathrm{dR}}(X_K/K)$. To show this, one could try to give a geometric construction of the corresponding Breuil-Kisin module, for example as a value of some cohomology theory associated to $\mathfrak{X}$. This was first successfully achieved in [18] via topological Hochschild homology by Bhatt, Morrow and Scholze, and then reestablished via prismatic site in [19] by Bhatt and Scholze. We will follow this last approach, reminding now the notion of prismatic site and prismatic cohomology.

Definition 3.2.3.5. A *δ-ring* is a p-typical λ-ring, i.e. it is a $\mathbb{Z}_{(p)}$-algebra equipped with a map of sets $\delta\colon A \to A$ such that $\delta(0) = \delta(1) = 0$ and

$$\delta(x+y) = \delta(x) + \delta(y) + \frac{x^p + y^p - (x+y)^p}{p} \quad \text{and} \quad \delta(xy) = x^p\delta(x) + y^p\delta(y) + p\delta(x)\delta(y)$$

for all $x, y \in A$. Every δ-ring A admits a natural ring endomorphism

$$\varphi(x) := x^p + p\delta(x).$$

lifting the absolute Frobenius on $A/(p)$. Note that if A p-torsion free, a δ-structure on A is just the same as a choice of such alift φ.

Definition 3.2.3.6 ([19, Definition 3.2])**.** A *prism* is a pair (A, I), where A is a δ-ring and $I \subset A$ is an ideal defining a Cartier divisor in $\operatorname{Spec} A$, that satisfies the followong two conditions

- A is derived (p, I)-adically complete [3];
- $p \in I + \varphi(I)A$; geometrically, the divisors $V(I)$ and $V(\varphi(I))$ intersect only in characteristic p.

Definition 3.2.3.7. A prism (A, I) is called

- *bounded*, if A/I has bounded p^∞-torsion, i.e. $A[p^\infty] = A[p^n]$ for some n.
- *perfect*, if φ is an automorphism.
- *orientable*, if I is principle; the choice of a generator d is called *an orientation* of ideal I.

[3] In [19] a sligthly less restrictive condition $(p, u) \in \operatorname{rad}(A)$ is used; however all prisms that we will use will in fact be (p, u)-complete.

Definition 3.2.3.8. An element d in a δ-ring A is called *distinguished* if $\delta(d)$ is invertible.

If A is derived (p,d)-complete, then $(A,(d))$ is a prism if and only if d is distinguished ([19, Lemma 2.25]). Reversely, for any prism (A,I), pro-Zariski locally on $\operatorname{Spec} A$, one has $I=(d)$ for a distinguished element d ([19, Lemma 3.1]). Also d is automatically not a zero divisor. We refer the reader to [19, Section 3] for a thorough discussion of the notion of prisms. We will be mostly interested in the following types of prisms:

Example 3.2.3.9.

1. (*Cristalline*). Let A be a p-torsion free, p-complete ring and let $I=(p)$; δ-structure on A is then the same as a lift of Frobenius φ. Also, p is always distinguished since $\delta(p)=1-p^{p-1}\in\mathbb{Z}_p^\times$. Particular examples of such prisms include $(W(k),(p))$ and $(A_{\mathrm{crys}},(p))$.

2. (*Breuil-Kisin*). These are essentially given by Definition 3.2.3.2. Namely, given $K/\mathbb{Q}_p$ and a uniformizer $\pi\in\mathcal{O}_K$, the pair $(\mathfrak{S},(E(u)))$ is a prism, with the δ-structure on $\mathfrak{S}$ determined by $\varphi_{\mathfrak{S}}$. The element $E(u)$ is distinguished, since it is distinguished modulo u (and $\varphi_{\mathfrak{S}}((u))\subset(u)$).

3. (A_{inf}). The δ-structure on A_{inf} is the one given by Frobenius φ. An element $x=\sum_{i\geq 0}[x_i]\cdot p^i\in A_{\mathrm{inf}}$ is distinguished if and only if the coefficient $x_1\in\mathcal{O}_{\mathbb{C}_p^\flat}$ is invertible ([19, Lemma 2.33]). By [17, Lemma 3.10], $x\in\operatorname{Ker}\theta$ in fact generates it if and only if x is distinguished. In particular, $(A_{\mathrm{inf}},\ker(\theta))$ is a prism and the element ξ (3.2.1.2) is an orientation of $\ker(\theta)$, since it is distinguished ([17, Example 3.16]).

   Given $\pi^\flat=(\pi,\pi^{\frac{1}{p}},\pi^{\frac{1}{p^2}},\ldots)\in\mathcal{O}_{\mathbb{C}_p}^\flat$ for a uniformizer $\pi\in\mathcal{O}_K$ there is a natural map of prisms $\mathfrak{S}\to A_{\mathrm{inf}}$, sending u to $[\pi^\flat]$. This gives another choice of an orientation of $\ker\theta$; namely $\xi_{\pi^\flat}\coloneqq E([\pi^\flat])$ is distinguished and lies in $\ker\theta$, since $\theta(\xi_{\pi^\flat})=E(\pi)=0$.

   One can also twist everything by the Frobenius, obtaining a new prism $(A_{\mathrm{inf}},\ker(\theta\circ\varphi^{-1}))$. The elements $\widetilde{\xi}$ and $\widetilde{\xi}_{\pi^\flat}\coloneqq\varphi(\xi_{\pi^\flat})$ then give orientations of $\ker(\theta\circ\varphi^{-1})$.

We note that all of these prisms are bounded and orientable.

Definition 3.2.3.10 ([19, Definition 4.1])**.** Fix a bounded prism (A,I). A *prismatic site* $(\mathfrak{X}/A)_{\mathbb{\Delta}}$ *of a smooth p-adic formal scheme $\mathfrak{X}$ over A/I* is a category consisting of maps of bounded prisms $g\colon(A,I)\to(B,IB)$ and a map of p-adic formal schemes $f\colon\operatorname{Spf}(B/IB)\to\mathfrak{X}$ making the diagram

below commutative

$$\begin{array}{ccccc} \mathrm{Spf}(B/IB) & \xrightarrow{\quad} & & & \mathrm{Spf}(B) \\ & \xrightarrow[f]{} & \mathfrak{X} & & \downarrow{\scriptstyle g} \\ & & \downarrow & & \\ & & \mathrm{Spf}(A/I) & \longrightarrow & \mathrm{Spf}(A). \end{array}$$

Morphisms in $(\mathfrak{X}/A)_{\mathbb{\Delta}}$ are the natural ones. We endow $(\mathfrak{X}/A)_{\mathbb{\Delta}}$ with a faithfully flat topology, where a morphism

$$\big(\mathfrak{X} \leftarrow \mathrm{Spf}(C/IC) \to \mathrm{Spf}(C)\big) \longrightarrow \big(\mathfrak{X} \leftarrow \mathrm{Spf}(B/IB) \to \mathrm{Spf}(B)\big)$$

in $(\mathrm{Spf}\, X/A)_{\mathbb{\Delta}}$ is called (faithfully) flat if C is (p, IB)-completely (faithfully) flat over B.

Definition 3.2.3.11 (Prismatic cohomology)**.** Fix a prism (A, I) and let $\mathfrak{X}$ be a smooth p-adic formal A/I-scheme. *The prismatic cohomology* $R\Gamma_{\mathbb{\Delta}}(\mathfrak{X}/A)$ *of* $\mathfrak{X}$ *over* A is defined as the derived global sections of the sheaf $\big(\mathfrak{X} \leftarrow \mathrm{Spf}(B/IB) \to \mathrm{Spf}(B)\big) \mapsto B$, i.e., as the homotopy limit

$$R\Gamma_{\mathbb{\Delta}}(\mathfrak{X}/A) := \lim_{\big(\mathfrak{X} \leftarrow \mathrm{Spf}(B/IB) \to \mathrm{Spf}(B)\big) \in (\mathfrak{X}/A)_{\mathbb{\Delta}}} B.$$

Note that $R\Gamma_{\mathbb{\Delta}}(\mathfrak{X}/A)$ is naturally an E_∞-algebra over A and since all terms in the diagram above are derived (p, I)-complete, the same holds for $R\Gamma_{\mathbb{\Delta}}(R/A)$.

Note that $R\Gamma_{\mathbb{\Delta}}(\mathfrak{X}/A)$ has a natural φ_A-linear Frobenius action, which comes from the data of $\varphi_B \colon B \to B$ for any object $\big(\mathfrak{X} \leftarrow \mathrm{Spf}(B/IB) \to \mathrm{Spf}(B)\big) \in (\mathfrak{X}/A)_{\mathbb{\Delta}}$. We can consider its linearization

$$\varphi_A^* R\Gamma_{\mathbb{\Delta}}(\mathfrak{X}/A) \xrightarrow{\varphi} R\Gamma_{\mathbb{\Delta}}(\mathfrak{X}/A).$$

Also, by [19, Corollary 4.11], for any map $(A/I) \to (B/J)$ of bounded prisms one has the base change

$$R\Gamma_{\mathbb{\Delta}/A}(\mathfrak{X}/A)\widehat{\otimes}_A B \simeq R\Gamma_{\mathbb{\Delta}/B}(\mathfrak{X}_{B/J}/B)$$

With these notations we have the following:

Theorem 3.2.3.12 ([18, Theorem 11.2])**.** *Let* $\mathfrak{X}$ *be a smooth formal scheme over* $\mathcal{O}_K$*. Let's consider the Breuil-Kisin prism* $(\mathfrak{S}, (E(u)))$*, corresponding to some choice of the uniformizer* $\pi \in \mathcal{O}_K$*. Also recall the* A_{inf}*-prism* $(A_{\mathrm{inf}}, (\xi))$*. Then:*

- *If* $\mathfrak{X}$ *is proper,* $R\Gamma_{\mathbb{\Delta}}(\mathfrak{X}/\mathfrak{S})$ *is a perfect complex of* $\mathfrak{S}$*-modules.*

- *The Frobenius φ on the prismatic cohomology*

$$\varphi_{\mathfrak{S}}^* R\Gamma_{\mathbb{\Delta}}(\mathfrak{X}/\mathfrak{S})[\tfrac{1}{E}] \xrightarrow[\sim]{\varphi} R\Gamma_{\mathbb{\Delta}}(\mathfrak{X}/\mathfrak{S})[\tfrac{1}{E}]$$

  becomes an equivalence after localizing at $E(u)$.

- *(de Rham comparison) There is a canonical equivalence*

$$\varphi_{\mathfrak{S}}^* R\Gamma_{\mathbb{\Delta}}(\mathfrak{X}/\mathfrak{S}) \otimes_{\mathfrak{S}} \mathcal{O}_K \simeq R\Gamma_{\mathrm{dR}}(\mathfrak{X}/\mathcal{O}_K).$$

- *(Cristalline comparison) There is a canonical φ-equivariant equivalence*

$$\varphi_{\mathfrak{S}}^* R\Gamma_{\mathbb{\Delta}}(\mathfrak{X}/\mathfrak{S}) \otimes_{\mathfrak{S}} W(k) \simeq R\Gamma_{\mathrm{crys}}(\mathfrak{X}_k/W(k)).$$

- *(Étale comparison) There are canonical equivalences*

$$R\Gamma_{\text{ét}}(\mathfrak{X}_{\mathbb{C}_p}, \mathbb{Z}/p^n) \simeq \Big(R\Gamma_{\mathbb{\Delta}}(\mathfrak{X}_{\mathcal{O}_{\mathbb{C}_p}}/A_{\mathrm{inf}})/p^n[\tfrac{1}{\xi}]\Big)^{\varphi=1} \quad \text{and} \quad R\Gamma_{\text{ét}}(\mathfrak{X}_{\mathbb{C}_p}, \mathbb{Z}_p) \simeq \big(R\Gamma_{\mathbb{\Delta}}(\mathfrak{X}/\mathfrak{S}) \otimes_{\mathfrak{S}} W(\mathbb{C}_p^\flat)\big)^{\varphi=1},$$

  where $\mathfrak{X}_{\mathbb{C}_p}$ denotes the Raynaud's generic fiber of $\mathfrak{X}$. Consequently, for any $i \geq 0$, one has a (φ, G_{K_∞})-equivariant isomorphism

$$H^i_{\mathbb{\Delta}}(\mathfrak{X}/\mathfrak{S}) \otimes_{\mathfrak{S}} W(\mathbb{C}_p^\flat) \simeq H^i_{\text{ét}}(\mathfrak{X}_{\mathbb{C}_p}, \mathbb{Z}_p).$$

In particular, under the p-torsion freeness assumption on $H^i_{\text{ét}}(\mathfrak{X}_{\mathbb{C}_p}, \mathbb{Z}_p)$, each prismatic cohomology $H^i_{\mathbb{\Delta}}(\mathfrak{X}/\mathfrak{S})$ forms a Breuil-Kisin module, which, by the last comparison, has to be $BK(H^i_{\text{ét}}(\mathfrak{X}_{\mathbb{C}_p}, \mathbb{Z}_p))$. Also note that the integral de Rham and cristalline comparisons agree with the ones in Theorem 3.2.3.4.

Remark 3.2.3.13. Note that if $\mathfrak{X} = \widehat{X}$ came as a completion of a smooth proper scheme X over $\mathcal{O}_K$, then by [57, Theorem 3.7.2] one also has an isomorphism

$$R\Gamma_{\text{ét}}(\mathfrak{X}_{\mathbb{C}_p}, \mathbb{Z}_p) \simeq R\Gamma_{\text{ét}}(X_{\mathbb{C}_p}, \mathbb{Z}_p) \simeq {}^4 R\Gamma_{\text{ét}}(X_{\overline{\mathbb{Q}}_p}, \mathbb{Z}_p).$$

[4] Due to the invariance of the étale cohomology under a base change of the ground algebraically closed field of char 0.

Thus, Theorem 3.2.3.12 can indeed be applied in the classical algebraic setting.

For the p-torsion representations there is no analogue of Theorem 3.2.3.4: in particular, the interaction between p-torsion in $H^i_{\text{ét}}(\mathfrak{X}_{\overline{K}}, \mathbb{Z}_p)$, $H^i_{\text{crys}}(\mathfrak{X}_k/W(k))$ and $H^i_{\text{dR}}(\mathfrak{X}/\mathcal{O}_K)$ can be much more complicated (for example see [17, Section 2]). However, it follows from Theorem 3.2.3.12 (and the semicontinuity of the dimension of the fiber applied to $H^i(R\Gamma_{\mathbb{\Delta}}(\mathfrak{X}/\mathfrak{S})/p)$) that

$$\dim_k H^i_{\text{dR}}(\mathfrak{X}_k) \geq \dim_{\mathbb{F}_p} H^i_{\text{ét}}(\mathfrak{X}_{\mathbb{C}_p}, \mathbb{F}_p).$$

More generally, one can show (see [17, Theorem 1.1(ii)])) that

$$\text{length}_{W(k)}(H^i_{\text{crys}}(\mathfrak{X}_k/W(k))_{\text{tor}}/p^n) \geq \text{length}_{\mathbb{Z}_p}(H^i_{\text{ét}}(\mathfrak{X}_{\mathbb{C}_p}, \mathbb{Z}_p)_{\text{tor}}/p^n)$$

and it seems hard to say anything more precise.

3.2.4 More details on Breuil-Kisin modules

We recall the following structural result about Breuil-Kisin modules (Definition 3.2.3.3).

Proposition 3.2.4.1. *Let (M, φ_M) be a Breuil-Kisin module. Then there is a canonical exact sequence of Breuil-Kisin modules*

$$0 \longrightarrow M_{\text{tor}} \longrightarrow M \longrightarrow M_{\text{free}} \longrightarrow \overline{M} \longrightarrow 0$$

where M_{tor} is killed by a power of p, M_{free} is free over $\mathfrak{S}$, $\overline{M}$ is killed by a power of (p, E).

This gives some restrictions on what a Breuil-Kisin module can be as an $\mathfrak{S}$-module. For example, $M[\frac{1}{p}]$ should be free over $\mathfrak{S}[\frac{1}{p}]$; in particular, if M is a torsion module, it should be killed by a power of p.

The most important example of a Breuil-Kisin module is the Breuil-Kisin twist $\mathfrak{S}\{1\}$. One way do define it, is as the image of the functor BK applied to the Tate twist $\mathbb{Z}_p(1)$. One can also give a more explicit description:

Example 3.2.4.2 (Breuil-Kisin twist)**.** We fix a Breuil-Kisin prism $(\mathfrak{S}, E(u))$. As a $\mathfrak{S}$-module $\mathfrak{S}\{1\}$ is free of rank 1 and the operator $\varphi_{\mathfrak{S}\{-1\}}\colon \varphi_{\mathfrak{S}}^*\mathfrak{S}\{-1\}[\frac{1}{E}] \to \mathfrak{S}\{-1\}[\frac{1}{E}]$ is uniquely defined by $\varphi_{\mathfrak{S}\{-1\}}(1) = \frac{v}{E(u)}$, where $v = \frac{E(0)}{p} \in W(k)^\times$ is a p-adic unit (see [74, proof of the Theorem

1.2.1]). However, this explicit description can sometimes be not very useful, as we will see in Example 3.2.5.4.

There is also the following coordinate-free "geometric" description of $\mathfrak{S}\{1\}$ (see [17, Example 4.2]), which we remind for the reader's convenience. Let $E_r := E\varphi(E)\cdots\varphi^{r-1}(E)$. The $\mathfrak{S}/E_r$-module

$$(\mathfrak{S}/E_r)\{1\} := \mathbb{L}_{(\mathfrak{S}/E_r)/\mathfrak{S}}[-1] \simeq E_r\mathfrak{S}/E_r^2\mathfrak{S}$$

is free of rank 1. For $r \geq s$, the image of the natural map $(\mathfrak{S}/E_r)\{1\} \to (\mathfrak{S}/E_s)\{1\}$, induced by $\mathfrak{S}/E_r \twoheadrightarrow \mathfrak{S}/E_s$ (and given explicitely by $E_r\mathfrak{S}/E_r^2\mathfrak{S} \to E_s\mathfrak{S}/E_s^2\mathfrak{S}$), is equal to $p^{r-s}E_s\mathfrak{S}/E_s^2\mathfrak{S}$. Thus one can define a natural isomorphism $(\mathfrak{S}/E_r)\{1\} \otimes_{\mathfrak{S}/E_r} \mathfrak{S}/E_s \simeq (\mathfrak{S}/E_s)\{1\}$ by dividing this map by p^{r-s}. Then $\mathfrak{S}\{1\} := \varprojlim_r (\mathfrak{S}/E_r)\{1\}$, which is a free module of rank 1 over $\mathfrak{S} := \varprojlim_r (\mathfrak{S}/E_r)$. The natural map $E\varphi_{\mathfrak{S}}\colon \mathfrak{S}/E_r \to \mathfrak{S}/E\varphi(E_r) \simeq \mathfrak{S}/E_{r+1}$ then induces $E\varphi_{\mathfrak{S}\{1\}}\colon \mathfrak{S}\{1\} \to \mathfrak{S}\{1\}$, which defines $\varphi_{\mathfrak{S}\{1\}}$.

Remark 3.2.4.3. We warn the reader that the sequence $\{E_r\}$ does not define an element of $\mathfrak{S}\{1\}$ and that the needed correction terms look quite complicated, e.g. the image of E_r in $E\mathfrak{S}/E^2\mathfrak{S} \simeq W(k)$ for $E = u - p$ is given by $p^{r-1}\prod_{i=1}^{r-1}(p^{p^i-1} - 1)$ and not p^{r-1}.

3.2.5 Breuil-Kisin-Fargues modules

There is also an $\mathcal{O}_{\mathbb{C}_p}$-analogue of Breuil-Kisin modules introduced by Fargues:

Definition 3.2.5.1. *A Breuil-Kisin-Fargues module* M is a finitely presented A_{inf}-module equipped with an isomorphism

$$\varphi_{A_{\mathrm{inf}}}^* M[\tfrac{1}{\widetilde{\xi}}] \xrightarrow{\sim} M[\tfrac{1}{\widetilde{\xi}}]$$

and such that the localization $M[\frac{1}{p}]$ is free over $A_{\mathrm{inf}}[\frac{1}{p}]$.

Remark 3.2.5.2. We note that, differently from the Breuil-Kisin case, one needs to artificially impose the freeness condition on $M[\frac{1}{p}]$, to avoid pathological torsion examples like A_{inf}/μ, which do not exist over $\mathfrak{S}$.

There is a natural functor from Breuil-Kisin to Breuil-Kisin-Fargues modules given by base change (and thus also depending on the choice of $\pi^\flat \in \mathcal{O}_{\mathbb{C}_p^\flat}$). Indeed, the image of $E(u)$ under the map $\mathfrak{S} \hookrightarrow A_{\mathrm{inf}}$, sending u to $[\pi^\flat]$, is equal to $\widetilde{\xi}_{\pi^\flat}$. Recall also that $(\widetilde{\xi}_{\pi^\flat}) = \ker(\theta \circ \varphi^{-1}) = (\widetilde{\xi})$. This

way, we see that tensoring up with A_{inf}

$$\varphi^* M[\tfrac{1}{E}] \xrightarrow[\sim]{\varphi} M[\tfrac{1}{E}] \xrightarrow{-\otimes_{\mathfrak{S}} A_{\mathrm{inf}}} \varphi^*_{A_{\mathrm{inf}}} M_{A_{\mathrm{inf}}}[\tfrac{1}{\widetilde{\xi}}] \xrightarrow[\sim]{\varphi} M_{A_{\mathrm{inf}}}[\tfrac{1}{\widetilde{\xi}}]$$

naturally turns Breuil-Kisin modules into Breuil-Kisin-Fargues modules.

We have:

Proposition 3.2.5.3. *The map $\mathfrak{S} \to A_{\mathrm{inf}}$ is faithfully-flat ([17, Lemma 4.30]). The base change $M \otimes_{\mathfrak{S}} A_{\mathrm{inf}}$ of a Breuil-Kisin module M is naturally a Breuil-Kisin-Fargues module.*

We will now spend some time discussing what happens with the Breuil-Kisin twist after this base change.

Example 3.2.5.4 (Breuil-Kisin-Fargues twist)**.** One of the benefits of passing from $\mathfrak{S}$ to A_{inf} is that the Breuil-Kisin-Fargues twist $A_{\mathrm{inf}}\{1\} \coloneqq \mathfrak{S}\{1\} \otimes_{\mathfrak{S}} A_{\mathrm{inf}}$ can be described very explicitely.

Namely, one can use a "geometric" description similar to Example 3.2.4.2 (see [17, Example 4.24]). Let $\widetilde{\xi}_r = \varphi(\xi)\varphi^2(\xi)\cdots\varphi^r(\xi)$ and consider $A_{\mathrm{inf}}/(\widetilde{\xi}_r)$, which by [17, Lemma 3.12] is in fact isomorphic to $W_r(\mathcal{O}_{\mathbb{C}_p})$. Note that E_r from Example 3.2.4.2 generates the same ideal as $\widetilde{\xi}_r$ via the map $\mathfrak{S} \to A_{\mathrm{inf}}$. We can then consider the (p-completed) cotangent complex

$$W_r(\mathcal{O})\{1\} \coloneqq \widehat{\mathbb{L}}_{W_r(\mathcal{O})/\mathbb{Z}_p}[-1] = \widehat{\mathbb{L}}_{W_r(\mathcal{O})/A_{\mathrm{inf}}}[-1] = \widetilde{\xi}_r A_{\mathrm{inf}}/\widetilde{\xi}_r^2 A_{\mathrm{inf}}$$

and natural maps $W_r(\mathcal{O})\{1\} \simeq \widetilde{\xi}_r A_{\mathrm{inf}}/\widetilde{\xi}_r^2 A_{\mathrm{inf}} \to \widetilde{\xi}_s A_{\mathrm{inf}}/\widetilde{\xi}_s^2 A_{\mathrm{inf}} \simeq W_s(\mathcal{O})\{1\}$ for $r \geq s$. The element $\frac{\widetilde{\xi}_r}{\widetilde{\xi}_s}$ is equal to p^{r-s} times a unit modulo $\tilde{\xi}_s$ and, dividing the above maps by p^{r-s} one can form the inverse limit

$$A_{\mathrm{inf}}\{1\} \coloneqq \varprojlim_r W_r(\mathcal{O})\{1\}.$$

It will be a free $A_{\mathrm{inf}} \simeq \varprojlim_r A_{\mathrm{inf}}/(\widetilde{\xi}_r)$-module of rank 1. Note that (Example 3.2.3.9) E_r maps to $\widetilde{\xi}_{\pi^\flat,r} \coloneqq \varphi(\xi_{\pi^\flat})\cdots\varphi^r(\xi_{\pi^\flat})$ and $(\widetilde{\xi}_{\pi^\flat,r}) = (\widetilde{\xi}_r)$ since $(\xi_{\pi^\flat}) = \ker\theta = (\xi)$. Thus, by base change for the (p-completed) cotangent complex we get that $A_{\mathrm{inf}}\{1\} \coloneqq A_{\mathrm{inf}} \otimes_{\mathfrak{S}} \mathfrak{S}\{1\}$.

However, differently from $\{\widetilde{\xi}_{\pi^\flat,r}\}$, the sequence $\{\widetilde{\xi}_r\}$ now gives a well-defined element of $A_{\mathrm{inf}}\{1\}$, and, in fact, is a generator. Indeed, $\varphi(\xi) \equiv p \mod \xi$ and thus $\widetilde{\xi}_r \equiv p^{r-s}\widetilde{\xi}_s \mod (\xi_s^2)$.

One can give even more canonical description via the map $d\log\colon W_r(\mathcal{O})^\times \to \Omega^1_{W_r(\mathcal{O})/\mathbb{Z}_p}$. Namely

it induces the map

$$d\log\colon \mathbb{Z}_p(1) \to T_p(\Omega^1_{W_r(\mathcal{O})/\mathbb{Z}_p}) \simeq \widehat{\mathbb{L}}_{W_r(\mathcal{O})/\mathbb{Z}_p}[-1] = W_r(\mathcal{O})\{1\}$$

on the corresponding p-adic Tate modules. Moreover, $T_p(\Omega^1_{W_r(\mathcal{O})/\mathbb{Z}_p})$ is explicitely generated by the system $\left(\frac{d\log[\zeta_{p^s}]}{[\zeta_{p^r}]-1}\right)_{s\geq 1}$ given a trivialization $(\zeta_{p^s})_{s\geq 1} \in \mathbb{Z}_p(1)$.

This induces a map $A_{\mathrm{inf}} \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(1) \to A_{\mathrm{inf}}\{1\}$ and its image can be identified with $\mu A_{\mathrm{inf}}\{1\}$. Indeed, this follows from the fact that under the isomorphism $A_{\mathrm{inf}}(\mathcal{O}) \xrightarrow{\sim} \varprojlim_r W_r(\mathcal{O})$ the element $\mu = [\varepsilon] - 1$ goes exactly to the sequence $([\zeta_{p^r}] - 1)_{r\geq 1}$ ([17, Lemma 3.3]). Thus, as a Breuil-Kisin-Fargues module, $A_{\mathrm{inf}}\{1\}$ can be identified with $\frac{1}{\mu} \cdot A_{\mathrm{inf}}(1)$ (with the natural isomorphism $A_{\mathrm{inf}}[\frac{1}{\varphi(\mu)}][\frac{1}{\tilde{\xi}}] \simeq A_{\mathrm{inf}}[\frac{1}{\mu}][\frac{1}{\tilde{\xi}}]$ coming from $\varphi(\mu) = \widetilde{\xi} \cdot \mu$). Note that under this identification the generator $\frac{1}{\mu}$ is just multiplied by $\frac{1}{\tilde{\xi}}$.

We can also identify the Galois action on $A_{\mathrm{inf}}\{1\}$. For this we first prove the following lemma

Lemma 3.2.5.5. *The ideal $(\mu) \subset A_{\mathrm{inf}}$ is G_K-invariant.*

Proof. We have $\mu = [\varepsilon] - 1$, so $\ker(\chi_{p^\infty}) \subset G_K$ stabilizes μ and the action factors through $\mathbb{Z}_p^\times$. Let $n \in \mathbb{N}$, $(n,p) = 1$ be an integer, then $n \in \mathbb{Z}_p^\times$ and

$$n^*\mu = [\varepsilon]^n - 1 = \mu \cdot (1 + [\varepsilon] + \cdots + [\varepsilon]^{n-1}).$$

The element $\sigma_n \coloneqq (1 + [\varepsilon] + \cdots + [\varepsilon]^{n-1})$ is a unit in A_{inf} since it is so modulo ξ: $\theta(1 + [\varepsilon] + \cdots + [\varepsilon]^{n-1}) = n \in \mathcal{O}_{\mathbb{C}_p}^\times$, so $(n^*\mu) = (\mu)$. The statement for general $s \in \mathbb{Z}_p^\times$ then follows by the continuity of the action (with respect to (p,ξ)-adic topology on A_{inf}). □

Let $\sigma\colon G_K \to A_{\mathrm{inf}}^\times$ be the character obtained by composing the p-cyclotomic character χ_{p^∞} with the unique continuous homomorphism $\mathbb{Z}_p^\times \to A_{\mathrm{inf}}^\times$, sending $n \in \mathbb{Z}$ (not divisible by p) to σ_n (as above). By Lemma 3.2.5.5 the ideal $\mu \cdot A_{\mathrm{inf}}$ is G_K-invariant and is identified with A_{inf}^σ (meaning A_{inf} with he standard G_K-action twisted by σ). From the discussion in Example 3.2.5.4 we got a completely functorial identification $A_{\mathrm{inf}}\{1\} \simeq \frac{1}{\mu} \cdot A_{\mathrm{inf}}(1)$.

Lemma 3.2.5.6. *As a G_K-module $A_{\mathrm{inf}}\{1\}$ is given by $A_{\mathrm{inf}}^{\sigma^{-1}}(1)$. The action on $A_{\mathrm{inf}}\{1\}/\xi$ is identified with the standard action on $\mathcal{O}_{\mathbb{C}_p}$.*

Proof. The first assertion follows form the identification $A_{\mathrm{inf}}\{1\} \simeq \frac{1}{\mu} \cdot A_{\mathrm{inf}}(1)$ and the fact that the action on $\frac{1}{\mu} \cdot A_{\mathrm{inf}}$ is the standard action on A_{inf} twisted by σ^{-1}. The second follows form the fact that for $(n,p)=1$ we have $\sigma_n \equiv n \mod (\xi)$, thus the twist for the action on $A_{\mathrm{inf}}\{1\}/\xi$ is trivial (since it is for a dense submonoid $\mathbb{Z} \cap \mathbb{Z}_p^\times \subset \mathbb{Z}_p^\times$). □

We also record the following lemma for a future reference. Before this, note that μ is invertible in $W(\mathbb{C}_p^\flat)$ (since it is non-zero modulo p) and so we have an embedding $A_{\mathrm{inf}}[\frac{1}{\mu}] \hookrightarrow W(\mathbb{C}_p^\flat)$.

Lemma 3.2.5.7 ([17, Lemma 4.26])**.** *Let (M, φ_M) be a Breuil-Kisin-Fargues module. Let*

$$T = (M \otimes_{A_{\mathrm{inf}}} W(\mathbb{C}_p^\flat))^{\varphi=1}$$

be a $\mathbb{Z}_p$-module given by the Frobenius-invariants. Then the natural map

$$T \otimes_{\mathbb{Z}_p} W(\mathbb{C}_p^\flat) \xrightarrow{\sim} M \otimes_{A_{\mathrm{inf}}} W(\mathbb{C}_p^\flat)$$

is an isomorphism. Moreover, it descends to an isomorphism

$$T \otimes_{\mathbb{Z}_p} A_{\mathrm{inf}}[\frac{1}{\mu}] \xrightarrow{\sim} M \otimes_{A_{\mathrm{inf}}} A_{\mathrm{inf}}[\frac{1}{\mu}].$$

3.2.6 Basic properties of prismatic cohomology

In this section we also collect some basic facts about $\Gamma_{\mathbb{\Delta}}(-/\mathfrak{S})$ which follow formally from Theorem 3.2.3.12 and will be useful in this work. Our proofs here essentially reduce to the fact that $\Gamma_{\mathbb{\Delta}}(-/\mathfrak{S})$ is a one dimensional deformation of de Rham cohomology $\Gamma_{\mathrm{dR}}(-/\mathcal{O}_K)$ in a sense that

$$R\Gamma_{\mathbb{\Delta}}(-/\mathfrak{S})/\xi \simeq R\Gamma_{\mathrm{dR}}(-/\mathcal{O}_K) \qquad \text{and} \qquad R\Gamma_{\mathbb{\Delta}}(-/\mathfrak{S}) \xrightarrow{\sim} R\Gamma_{\mathbb{\Delta}}(-/\mathfrak{S})_{\widehat{\xi}},$$

where by definition $R\Gamma_{\mathbb{\Delta}}(-/\mathfrak{S})/\xi := R\Gamma_{\mathbb{\Delta}}(-/\mathfrak{S}) \otimes_{\mathfrak{S}} \mathcal{O}_K$. Using this fact and derived Nakayama's lemma it is often possible to reduce the proof of a statement about prismatic cohomology to the corresponding statement about de Rham cohomology over $\mathcal{O}_K$.

Proposition 3.2.6.1. *The presheafs*

$$R\Gamma_{\mathbb{\Delta}}(-/\mathfrak{S})\colon \mathcal{S}\mathrm{ch}^{\mathrm{sm}}_{/\mathcal{O}_K} \to D(\mathrm{Mod}_{\mathfrak{S}}) \qquad R\Gamma_{A_{\mathrm{inf}}}(-/\mathfrak{S})\colon \mathcal{S}\mathrm{ch}^{\mathrm{sm}}_{/\mathcal{O}_C} \to D(\mathrm{Mod}_{A_{\mathrm{inf}}})$$

(where $\mathcal{S}\mathrm{ch}^{\mathrm{sm}}$ denotes the category of smooth $\mathcal{O}_K$-schemes) satisfy étale hyperdescent.

Proof. We will only prove the statement for $R\Gamma_{\mathbb{\Delta}}$, the proof for $R\Gamma_{A_{\mathrm{inf}}}$ being similar. Let $p_\bullet \colon U_\bullet \to X$ be an étale hypercover. We want to prove that the induced map

$$\Gamma_{\mathbb{\Delta}}(X/\mathfrak{S}) \longrightarrow \mathrm{Tot}\, R\Gamma_{\mathbb{\Delta}}(U_\bullet/\mathfrak{S}) \tag{3.2.6.1}$$

is an equivalence. Now since both sides are ξ-adicly complete by derived Nakayama's lemma it is enough to prove (3.2.6.1) is an equivalence modulo ξ. But since $\mathfrak{S}/\xi$ is a perfect $\mathfrak{S}$-module, reduction modulo ξ commutes with limits, hence using de Rham comparison from Theorem 3.2.3.12 we obtain an equivalence

$$\left(\mathrm{Tot}\, \Gamma_{\mathbb{\Delta}}(U_\bullet/\mathfrak{S})\right)/\xi \simeq \mathrm{Tot}\left(\Gamma_{\mathrm{dR}}(U_\bullet/\mathcal{O}_K)\right),$$

under which the map (3.2.6.1)$/\xi$ identifies with the natural map $\Gamma_{\mathrm{dR}}(X/\mathcal{O}_K) \to \mathrm{Tot}\, \Gamma(U_\bullet/\mathcal{O}_K)$. Hence it is enough to prove étale hyperdescent for the de Rham cohomology over $\mathcal{O}_K$. But de Rham cohomology complete with respect to the Hodge filtration, hence it is enough to prove descent for $\Gamma(-, \Omega^i_{-/\mathcal{O}_K})$. This holds almost by definition of étale morphism, see e.g. Proposition 3.3.3.5. □

We will also prove here few properties of $\Gamma(-/\mathfrak{S})$ of topological flavor.

Proposition 3.2.6.2 (Künneth formula)**.** *Let $\mathfrak{X}, \mathfrak{Y}$ be a pair of smooth schemes over $\mathcal{O}_K$. Then there exists a natural equivalence*

$$\Gamma_{\mathbb{\Delta}}(\mathfrak{X}/\mathfrak{S}) \mathbin{\widehat{\otimes}} \Gamma_{\mathbb{\Delta}}(\mathfrak{Y}/\mathfrak{S}) \xrightarrow{\sim} \Gamma_{\mathbb{\Delta}}(\mathfrak{X} \times \mathfrak{Y}/\mathfrak{S})$$

where $\widehat{\otimes}$ is the (p, ξ)-adicly completed tensor product.

Proof. By the universal property of co-product pullback maps induce a map of E_∞-algebras

$$\Gamma_{\mathbb{\Delta}}(\mathfrak{X}/\mathfrak{S}) \otimes \Gamma_{\mathbb{\Delta}}(\mathfrak{Y}/\mathfrak{S}) \to \Gamma_{\mathbb{\Delta}}(\mathfrak{X} \times \mathfrak{Y}/\mathfrak{S})$$

which factors through $\alpha \colon \Gamma_{\mathbb{\Delta}}(\mathfrak{X}) \mathbin{\widehat{\otimes}} \Gamma_{\mathbb{\Delta}}(\mathfrak{Y}) \to \Gamma_{\mathbb{\Delta}}(\mathfrak{X} \times \mathfrak{Y})$ since the right hand side is (p, E)-adicly complete. Since both parts are E-adicly complete, it is enough to prove that α/E is an equivalence, i.e. we reduced to prove Künneth formula for de Rham cohomology. Since the map of de Rham

cohomology respects Hodge filtration, it is enough prove Künneth formula for Hodge cohomology, which follows formally by the base-change and the projection formula. □

The following proposition won't be used anywhere else, but we nevertheless decided to include it for completeness

Proposition 3.2.6.3 (Poincaré duality)**.** *Let $\mathfrak{X}$ be a smooth proper $\mathcal{O}_K$-scheme of dimension d. Then the multiplication followed by projection on the top summand*

$$\Gamma_{\mathbb{\Delta}}(\mathfrak{X}/\mathfrak{S}) \widehat{\otimes} \Gamma_{\mathbb{\Delta}}(\mathfrak{X}/\mathfrak{S}) \to \Gamma_{\mathbb{\Delta}}(\mathfrak{X}/\mathfrak{S}) \to H^d_{\mathbb{\Delta}}(\mathfrak{X}/\mathfrak{S})[d] \simeq \mathfrak{S}\{d\}[d]$$

is a perfect pairing.

Proof. First since $H^d_{\mathrm{crys}}(\mathfrak{X}_k/W(k)) \simeq W(k)$ and $H^{d+1}_{\mathrm{crys}}(\mathfrak{X}_k/W(k)) \simeq 0$ are p-torsion free we know from Theorems 3.2.3.4 and 3.2.3.12 that

$$H^d_{\mathbb{\Delta}}(\mathfrak{X}) \simeq BK\left(H^d_{\text{ét}}(\mathfrak{X}_K, \mathbb{Z}_p)\right) \simeq BK(\mathbb{Z}_p(d)) \simeq \mathfrak{S}\{d\}.$$

Next since $\Gamma_{\mathbb{\Delta}}(\mathfrak{X})$ is (p, E)-adicly complete it is enough to prove that the induced map

$$\Gamma_{\mathbb{\Delta}}(\mathfrak{X}) \to \Gamma_{\mathbb{\Delta}}(\mathfrak{X})^\vee\{d\}[d]$$

is equivalence modulo (p, E), i.e. that the de Rham cohomology of $\mathfrak{X}_k$ over k satisfy Poincaré duality. Now since the map in question respect Hodge filtration, it is enough to prove that the induced map

$$H^{p,q}(\mathfrak{X}_k) \to H^{d-p,d-q}(\mathfrak{X}_k)^\vee$$

is an isomorphism. But this is a particular case of Serre duality. □

3.3 Preliminaries on stacks

3.3.1 Artin stacks

In this subsection we recall the notion of a geometric stack and their basic properties, see also [110, Chapter 1.3], [47, Chapter 2, Section 4], or [99, Section 2]. First recall

Definition 3.3.1.1. Let $(\mathscr{C}, \tau)$ be a pair consisting of an $(\infty, 1)$-category $\mathscr{C}$ and a Grothendieck topology τ (see [82, Definition 6.2.2.1]). A presheaf $\mathcal{F}\colon \mathscr{C}^{\mathrm{op}} \to \mathscr{S}$ is called *τ-sheaf* if for every $U \in \mathscr{C}$ and a covering sieve $\alpha\colon U' \to \mathrm{Hom}_{\mathscr{C}}(-, U)$ the induced map $\mathcal{F}(\alpha)$ is an equivalence. We will denote the full subcategory of $\mathcal{PS}\mathrm{hv}(\mathscr{C}) := \mathrm{Fun}(\mathscr{C}^{\mathrm{op}}, \mathscr{S})$ by $\mathcal{S}\mathrm{hv}_{\tau}(\mathscr{C})$. By [82, Proposition 6.2.2.7] the inclusion $\mathcal{S}\mathrm{hv}_{\tau}(\mathscr{C}) \hookrightarrow \mathcal{PS}\mathrm{hv}(\mathscr{C})$ admits a left exact left adjoint functor, which we will usually denote by L_{τ}. A presheaf $\mathcal{F}$ is called *τ-hypercomplete sheaf* or *τ-hypersheaf* if for any τ-hypercover $U_{\bullet} \to U$ the induced map $\mathcal{F}(U) \to \mathrm{Tot}\, \mathcal{F}(U_{\bullet})$ is an equivalence.

Let $\mathrm{h}\mathcal{S}\mathrm{hv}_{\tau}(\mathscr{C})$ denotes the homotopy category of $\mathcal{S}\mathrm{hv}_{\tau}(\mathscr{C})$ and let $\pi_0\colon \mathcal{S}\mathrm{hv}_{\tau}(\mathscr{C}) \to \mathrm{h}\mathcal{S}\mathrm{hv}_{\tau}(\mathscr{C})$ be the natural functor. A map of sheaves $\varphi\colon \mathcal{F} \to \mathcal{G}$ is called *surjective* or *epimorphism* if the induced map $\pi_0(\varphi)$ is an epimorphism (in the 1-categorical sense).

Definition 3.3.1.2. A *geometric context* is a triple $(\mathscr{C}, \tau, \mathrm{P})$ consisting of a $(\infty, 1)$-category $\mathscr{C}$ equipped with a Grothendieck topology τ and a class of morphisms P in $\mathscr{C}$ satisfying the following assumptions:

1. Every representable presheaf is a hypercomplete τ-sheaf.
2. The class of morphisms P is closed under equivalences, compositions and pullbacks.
3. τ-coverings consist of morphisms in P.
4. P is τ-local on the source: if $f\colon X \to Y$ is a morphism in $\mathscr{C}$ such that there exists a covering $\{U_i \to X\}$ such that all compositions $U_i \to X \xrightarrow{f} Y$ are in P, then f is also in P.

Definition 3.3.1.3. Let $\mathscr{C}$ be a category category equipped with a Grothendieck topology τ. A *prestack* $\mathcal{X}$ is a just a functor $\mathcal{X}\colon \mathscr{C}^{\mathrm{op}} \to \mathscr{S}$. A *stack* is a prestack which is a sheaf with respect to τ. We will denote the category of prestacks and stacks by $\mathscr{PS}\mathrm{tk}_{\mathscr{C}}$ and $\mathscr{S}\mathrm{tk}_{\mathscr{C},\tau}$ respectively or just $\mathscr{PS}\mathrm{tk}$ and $\mathscr{S}\mathrm{tk}$ if $\mathscr{C}$ and τ are clear from the context.

Note that the previous definition did not use the class of morphisms P. The choice of P allows to define stacks which are geometric in the following sense: they can be presented as iterated quotients of representable stacks by equivalence relations lying in P. More precisely:

Definition 3.3.1.4 (Geometric stacks)**.** Let $(\mathscr{C}, \tau, \mathrm{P})$ be a geometric context. The category of *n-geometric stacks* is the full subcategory of the category of $(\mathscr{C}, \tau)$-stacks defined inductively as follows:

- ∘ A stack $\mathcal{X}$ is (-1)-*geometric stack* if it is equivalent to a coproduct of representable sheaves.
- ∘ A morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of stacks is (-1)-*representable* if for any representable sheaf S and a morphism $S \to \mathcal{Y}$ the fibered product $S \times_{\mathcal{Y}} \mathcal{X}$ is representable. It is called a P-morphism if the corresponding map $S \times_{\mathcal{Y}} \mathcal{X} \to S$ is in P for any $S \to \mathcal{Y}$.
- ⋆ Assuming the notions of an $(n-1)$-geometric stack, an $(n-1)$-representable morphism and a P-morphism are already given one can proceed further and define:
  - A stack $\mathcal{X}$ is called n-*geometric* if there exists a disjoint union of representable stacks $S = \coprod_i S_i$ and a surjective morphism of sheaves $S \to \mathcal{X}$ that is P-$(n-1)$-representable. Such a morphism is called a P-n-*atlas* of $\mathcal{X}$.
  - A morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of stacks is n-*representable* if for any representable sheaf S and a map $S \to \mathcal{Y}$ the fibered product $S \times_{\mathcal{Y}} \mathcal{X}$ is n-geometric.
  - An n-representable morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of stacks is called a P-*morphism* if for any representable sheaf S and $S \to \mathcal{Y}$ there exists a P-n-atlas $U \to S \times_{\mathcal{Y}} \mathcal{X}$ such that the composite projection $U \to S$ is a P-morphism.
- ★ A stack $\mathcal{X}$ is called *geometric* if it is n-geometric for some n. We say that it is a P-stack if the base morphism $\mathcal{Y} \to *$ is a P-morphism.

Definition 3.3.1.5. A morphism of prestacks $\mathcal{X} \to \mathcal{Y}$ is called *relatively geometric* if the pullback $S \times_{\mathcal{Y}} \mathcal{X}$ is a geometric stack for every representable S mapping to $\mathcal{Y}$.

Let $\mathrm{CAlg}_{R/}$ be the (ordinary) category of commutative R-algebras and let $\mathrm{Aff}_{/S} \coloneqq \mathrm{CAlg}_{R/}^{\mathrm{op}}$ be the category of affine schemes over $S \coloneqq \operatorname{Spec} R$. Let also $\mathscr{PS}\mathrm{tk}_R \coloneqq \mathscr{PS}\mathrm{tk}_{\mathrm{Aff}_{/S}}$ and $\mathscr{S}\mathrm{tk}_R \coloneqq \mathscr{S}\mathrm{tk}_{\mathrm{Aff}_{/S}}$. Geometric stacks on $\mathrm{Aff}_R \coloneqq \mathrm{Aff}_{/S}$ with respect to some geometric context are more commonly called *Artin stacks*. In this work we will mainly be interested in (étale, smooth)-Artin stacks in the previous notation. An immediate objection from the reader could be that classically it is more natural to consider stacks in the fppf topology. However, the following result of Toën shows that by choosing (étale, smooth)-context we in fact obtain the same category of geometric stacks:

Theorem 3.3.1.6 ([109, Théorèm 0.1])**.** *We have:*

- *Let $\mathcal{X}$ be an (étale, smooth)-Artin stack. Then it satisfies fppf descent.*

- *Let $\mathcal{X}$ be a (fppf, fppf)-Artin stack. Then $\mathcal{X}$ admits a smooth atlas.*

In other words, the full subcategories $\mathscr{S}\mathrm{tk}^{\mathrm{Art}}_{\text{ét,smooth},R}$ and $\mathscr{S}\mathrm{tk}^{\mathrm{Art}}_{\text{fppf,fppf},R}$ of $\mathscr{P}\mathscr{S}\mathrm{tk}_R$ coincide.

We will denote the category of Artin stacks in any of the two equivalent definitions by $\mathscr{S}\mathrm{tk}^{\mathrm{Art}}_R$. In Section 3.3.4 we will also consider another example of geometric stacks in the rigid analytic context.

Here are some examples of Artin stacks:

Example 3.3.1.7 (Schemes)**.** Let X be an R-scheme. If the diagonal map $\Delta\colon X \to X \times_S X$ is affine (e.g. X is separated), X is a 0-Artin stack. In general, X is still a 1-Artin stack.

Example 3.3.1.8 (Quotient stacks)**.** Let X be an S-scheme and let G be a group S-scheme acting on X. One defines *the stacky quotient* $[X/G]$ to be the sheafification of the presheaf $A \mapsto X(A)/G(A)$, where the latter means the homotopy quotient (taken in $\mathscr{S}$). In particular for $X = *$ being the base scheme we will denote $*/G$ by BG. Note that this coincides with the functor of flat-locally trivial G-torsors on Y (since both are sheafifications of the same presheaf). Moreover, if G is abelian, one can iterate the B-construction (by sheafifying $A \mapsto B^nG(A) = K(G(A), n)$) to obtain n-classifying stack B^nG for any $n \geq 0$. Note that for a scheme Y the set of maps $[Y, B^nG]$ is naturally isomorphic to $H^n_{fl}(Y, G)$. If G is smooth over R, the quotient stack X/G is 1-Artin with an atlas given by X and B^nG is n-Artin with an atlas given by the point $* = S$.

Example 3.3.1.9 (Constant stacks)**.** Let K be a homotopy type. We can define the associated *constant stack* $\underline{K}$ as the sheafification of the presheaf $A \mapsto K$. E.g. $\underline{\mathbb{S}^1} \simeq B\underline{\mathbb{Z}}$, where $\underline{\mathbb{Z}}$ is a constant group scheme associated to $\mathbb{Z}$. If $\pi_{>n}(K, x) \simeq 0$ for any point $x \in K$, the stack $\underline{K}$ is n-Artin with an atlas $\underline{\pi_0(K)}$ (one can prove this by induction on N, since the functor $K \mapsto \underline{K}$ commutes with limits). One the other hand, since for an n-Artin stack $\Omega^n(\mathcal{X}, x) \coloneqq \underbrace{x \times_X \cdots \times_X x}_{n}$ must be discrete for any R-point $x\colon \operatorname{Spec} R \to \mathcal{X}$ (e.g. by the third part of Proposition 3.3.1.14 below), we see that the condition $\pi_{>n}(K, x) \simeq 0$ is also necessary for $\underline{K}$ to be n-Artin.

Example 3.3.1.10 (Mapping stacks and G-bundles)**.** The category of pre-stacks admits an inner hom, the so called *mapping pre-stack* $\mathrm{Map}(-,-)$, given by

$$\mathrm{Map}(\mathcal{X}, \mathcal{Y})(R) \coloneqq \mathrm{Hom}_{\mathscr{P}\mathscr{S}\mathrm{tk}_{/S}}(\operatorname{Spec} R \times \mathcal{X}, \mathcal{Y}).$$

Moreover, it is easy to see that if $\mathcal{Y}$ is a stack, then so is $\mathrm{Map}(\mathcal{X}, \mathcal{Y})$ for any pre-stack $\mathcal{X}$. Mapping stack provide us an ample supply of examples. E.g. let X be a scheme and let G be a group scheme. Then one can define the *stack of G-bundles on X* by $\mathrm{Map}(X, BG)$. If X is proper this is an Artin stack ([86, a particular case of Theorem 1.1])

Having any property of a morphism in Aff_R that is local with respect to the smooth topology one can extend it to a property of a morphism of Artin stacks:

Construction 3.3.1.11. Let $\mathscr{P}$ be a class of morphisms between affine schemes closed under equivalencies, compositions and pullbacks and satisfying the following properties:

- ($\mathscr{P}$ is local on source) Let $f\colon X \to Y$ be a morphism of affine schemes and assume there exists a cover $\coprod_i U_i \twoheadrightarrow X$ such that all composite maps $U_i \to X \xrightarrow{f} Y$ are in $\mathscr{P}$. Then f is in $\mathscr{P}$.

- ($\mathscr{P}$ is local on target) Let $f\colon X \to Y$ be a morphism of affine schemes such that for some cover $\coprod_i U_i \twoheadrightarrow Y$ each pullback morphism $U_i \times_Y X \to U_i$ is in $\mathscr{P}$. Then f is in $\mathscr{P}$.

Then we say that *an n-representable morphism of stacks $f\colon \mathcal{X} \to \mathcal{Y}$ is in $\mathscr{P}$* if for any affine A mapping to $\mathcal{Y}$ there exists an atlas $\coprod_i U_i$ of $A \times_{\mathcal{Y}} \mathcal{X}$ such that all composite maps $U_i \to A$ are in $\mathscr{P}$.

See [110, Section 1.3.6] for more details about this. In particular, this way one obtains well-defined notions of *étale*, *smooth*, and *flat* morphisms of Artin stacks. If the class of morphisms is local only on target one can still introduce the following notion:

Construction 3.3.1.12. Let $\mathscr{P}$ be a class of morphisms of schemes local on target. Then the map of prestacks $f\colon \mathcal{X} \to \mathcal{Y}$ is called *schematic $\mathscr{P}$* or just $\mathscr{P}$ if for any affine scheme mapping to $\mathcal{Y}$ the pullback $S \times_{\mathcal{Y}} \mathcal{X}$ is a scheme and the natural projection $S \times_{\mathcal{Y}} \mathcal{X} \to S$ lies in $\mathscr{P}$.

Example 3.3.1.13. This way we obtain a notions of *affine*, *schematic*, *schematic projective*, and *schematic proper* morphisms of prestacks.

The following facts follow formally from the definitions and will be freely used by us in the rest of the chapter without further references:

Proposition 3.3.1.14. *1. The class of n-representable morphisms is closed under pullbacks and compositions.*

2. *A stack $\mathcal{X}$ admits an n-representable atlas $U \twoheadrightarrow \mathcal{X}$ if and only if the diagonal morphism $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is n-representable.*

3. *Any morphism from an affine scheme to an n-Artin stack is $(n-1)$-representable.*

4. *The full subcategory of n-Artin stacks is closed under finite limits.*

5. *Let $\mathscr{P}$ be a class of morphism of affine schemes as in Construction 3.3.1.11. Then the class of $\mathscr{P}$-morphism between Artin stacks is closed under equivalencies, compositions and pullbacks and is local on target.*

We will also need the following result:

Proposition 3.3.1.15 (Corollary of [109, Théorèm 0.1])**.** *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a surjective flat map. Then if $\mathcal{X}$ is smooth then so is $\mathcal{Y}$.*

Corollary 3.3.1.16. *Let G be a flat group scheme acting on a smooth scheme X. Then the quotient stack $[X/G]$ is smooth. In particular BG is smooth for any flat group scheme G.*

Quasi-compact and quasi-separated morphisms. We will also need the following classes of morphisms, which do not quite fit into the general framework of Construction 3.3.1.11:

Definition 3.3.1.17. An Artin stack $\mathcal{X}$ is called *quasi-compact* if there exists a smooth atlas $U \twoheadrightarrow \mathcal{X}$ such that U is an affine scheme (as opposed to an infinite disjoint union of affine schemes, which always exists by the definition of Artin stacks). A morphism $f\colon \mathcal{X} \to \mathcal{Y}$ is called *quasi-compact* if the pullback $\operatorname{Spec} R \times_{\mathcal{Y}} \mathcal{X}$ is quasi-compact for any $\operatorname{Spec} R$ mapping to $\mathcal{Y}$.

Definition 3.3.1.18. A stack $\mathcal{X}$ is called *quasi-separated* if the diagonal morphism $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is quasi-compact in the sense of the previous definition. A morphism of stacks $X \to \mathcal{Y}$ is called *quasi-separated* if the pullback $\operatorname{Spec} R \times_{\mathcal{Y}} \mathcal{X}$ is quasi-separated for any map $\operatorname{Spec} R \to \mathcal{Y}$.

The following properties are proved by the usual diagram chase:

Proposition 3.3.1.19. *A classes of quasi-compact and quasi-separated morphisms are closed under base change, compositions and fibered products.*

3.3.2 Quasi-coherent sheaves

To each commutative (classical) R-algebra A we can associate the unbounded derived category $D(\mathrm{Mod}_A)$ of A-modules. Given a homomorphism $f\colon A \to B$ one has a natural continuous R-linear pullback functor $f^*\colon D(\mathrm{Mod}_A) \to D(\mathrm{Mod}_B)$ defined by the derived tensor product: $f^*(M) := M \otimes_A B$. This way we obtain a functor

$$D(\mathrm{Mod}_{-})\colon \mathrm{Aff}_R^{\mathrm{op}} \to \mathscr{P}r_R^{\mathrm{L},\otimes}, \qquad \mathrm{Spec}\, A \mapsto D(\mathrm{Mod}_A)$$

from $\mathrm{Aff}_R^{\mathrm{op}}$ to the category of R-linear presentably symmetric monoidal ∞-categories and continuous functors between them. Thinking of $D(\mathrm{Mod}_A)$ as the category of quasi-coherent sheaves on $\mathrm{Spec}\, A$, given a prestack $\mathcal{Y}$, one can define the category of quasi-coherent sheaves on $\mathcal{Y}$ via the right Kan extension:

Definition 3.3.2.1. The category $\mathrm{QCoh}(\mathcal{Y})$ of quasi-coherent sheaves on $\mathcal{Y}$ is defined as the limit

$$\mathrm{QCoh}(\mathcal{Y}) := \lim_{(\mathrm{Spec}\, A \to \mathcal{Y}) \in (\mathrm{Aff}_R\, /\mathcal{Y})^{\mathrm{op}}} D(\mathrm{Mod}_A).$$

More precisely, one can extend this to a functor $\mathrm{QCoh}^*\colon \mathscr{P}\mathscr{S}\mathrm{tk}_R^{\mathrm{op}} \to \mathscr{P}r_R^{\mathrm{L}}$ ($(-)^*$ staying for the pull-back functor) as the right Kan extension of the $A \mapsto D(\mathrm{Mod}_A)$ functor along the inclusion $\mathrm{Aff}_R \subset \mathscr{P}\mathscr{S}\mathrm{tk}_R$.

In particular, for a morphism of prestacks $f\colon \mathcal{X} \to \mathcal{Y}$ we have a well-defined functor $f^*\colon \mathrm{QCoh}(\mathcal{Y}) \to \mathrm{QCoh}(\mathcal{X})$.

Remark 3.3.2.2. One can think of an object $\mathcal{F} \in \mathrm{QCoh}(\mathcal{Y})$ as a collection of complexes $\mathcal{F}_{|A} := x^*\mathcal{F} \in D(\mathrm{Mod}_A)$ associated to each A-point $x\colon \mathrm{Spec}\, A \to \mathcal{Y}$ compatible under pullbacks but only up to coherent set of higher homotopies.

The following properties follow more or less formally from the definition (see e.g. [47, Chapter 3] for more details):

Proposition 3.3.2.3. *We have:*

1. QCoh^* *is a functor from the category of prestacks to the category* $\mathscr{P}r_R^{\otimes,\mathrm{L}}$ *of presentably symmetric monoidal R-linear categories and continuous symmetric monoidal R-linear functors.*

2. *By the adjoint functor theorem there is a functor* $\mathrm{QCoh}_*\colon \mathscr{P}\mathscr{S}\mathrm{tk}_R \to \mathscr{P}r^{\mathrm{R}}_R$ *(agreeing with* QCoh^* *on objects) such that for each morphism* $f\colon \mathcal{X} \to \mathcal{Y}$ *the functor* $f_*\colon \mathrm{QCoh}(\mathcal{X}) \to \mathrm{QCoh}(\mathcal{Y})$ *is the right adjoint of* $f^*\colon \mathrm{QCoh}(\mathcal{Y}) \to \mathrm{QCoh}(\mathcal{X})$.

3. *Let* $\mathcal{X}_\bullet\colon I \to \mathscr{P}\mathscr{S}\mathrm{tk}_R$ *be a diagram of prestacks. Then the natural map*

$$\mathrm{QCoh}(\operatorname*{colim}_I \mathcal{X}_i) \longrightarrow \lim_I \mathrm{QCoh}(\mathcal{X}_i)$$

is an equivalence.

We also have the following consequence/generalization of faithfully flat descent:

Proposition 3.3.2.4 ([47, Chapter 3, Corollary 1.3.7])**.** *Let* $L\colon \mathscr{P}\mathscr{S}\mathrm{tk}_R \to \mathscr{S}\mathrm{tk}_R$ *be the hyper-sheafification functor with respect to the fpqc topology. Then for any prestack* $\mathcal{X}$ *the natural map* $\mathrm{QCoh}(L\mathcal{X}) \to \mathrm{QCoh}(\mathcal{X})$ *is an equivalence.*

Corollary 3.3.2.5. *Let* $q\colon \mathcal{U} \to \mathcal{X}$ *be an fpqc surjection and let* $\mathcal{U}_\bullet$ *be the corresponding Čech nerve. Then the natural map*

$$\mathrm{QCoh}(\mathcal{X}) \longrightarrow \operatorname{Tot} \mathrm{QCoh}(U_\bullet)$$

is an equivalence. In particular, the pull-back functor $q^*\colon \mathrm{QCoh}(\mathcal{X}) \to \mathrm{QCoh}(\mathcal{U})$ *is conservative.*

Corollary 3.3.2.6. *Let* $q\colon \mathcal{U} \to \mathcal{X}$ *be an fpqc surjection and let* $\mathcal{U}_\bullet$ *be the corresponding Čech nerve. Let* $\mathcal{F}$ *be a quasi-coherent sheaf on* $\mathcal{X}$*. Then the natural map*

$$R\Gamma(\mathcal{X}, \mathcal{F}) \longrightarrow \operatorname{Tot} R\Gamma(\mathcal{U}_\bullet, p_\bullet^*\mathcal{F})$$

is an equivalence.

Here are some basic examples of QCoh on various stacks:

Example 3.3.2.7 (Schemes)**.** Let X be an R-scheme. Then $\mathrm{QCoh}(X)$ is an $(\infty, 1)$-enhancement of the usual unbounded derived category of quasi-coherent sheaves on X.

Example 3.3.2.8 (Structure sheaf)**.** For any stack $\mathcal{Y}$ there is the *structure sheaf* $\mathcal{O}_\mathcal{Y} \in \mathrm{QCoh}(\mathcal{Y})$ defined by $(\mathcal{O}_\mathcal{Y})_{|A} \coloneqq A$. By construction, $\mathcal{O}_\mathcal{Y}$ is the monoidal unit of $\mathrm{QCoh}(\mathcal{Y})$ and is preserved under pullbacks.

Example 3.3.2.9 (Affine morphism)**.** Let $f\colon \mathcal{X} \to \mathcal{S}$ be an affine morphism of prestacks (see Example 3.3.1.13). Let $\mathcal{A} := f_*(\mathcal{O}_{\mathcal{X}})$. Since the functor f_* is right lax monoidal, f_* factors naturally through $K\colon \mathrm{QCoh}(\mathcal{X}) \to \mathrm{Mod}_{\mathcal{A}}(\mathrm{QCoh}(\mathcal{S}))$. We claim that K is an equivalence. The statement is obvious for affine $\mathcal{S}$, and the general case follows, since by the previous proposition both sides transform colimits to limits.

Example 3.3.2.10 (Constant stacks)**.** Let K be a homotopy type. Then since QCoh transforms colimits of prestacks to limits of categories, we obtain $\mathrm{QCoh}(\underline{K}) \simeq \mathrm{LocSys}(K, D(\mathrm{Mod}_R)) := \mathrm{Fun}(K, D(\mathrm{Mod}_R))$. Moreover, $R\Gamma(\underline{K}, \mathcal{O}_{\underline{K}}) \simeq C^*(K, R)$, the complex of R-cochains on K with its natural E_∞-algebra structure.

Example 3.3.2.11 (Quotient stacks)**.** Let X be an R-scheme and let G be a group scheme. Note that $[X/G]$ is equivalent to the geometric realization of the following simplicial scheme

$$\ldots \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow}\; G \times G \times X \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; G \times X \;\underset{p}{\overset{a}{\rightrightarrows}}\; X\ ,$$

where the morphisms in the diagram above are induced by the multiplication on G, the action morphisms $a\colon G \times X \to X$ and projections. Since QCoh^* transforms colimits of stacks into limits of categories, we obtain

$$\mathrm{QCoh}([X/G]) \simeq \mathrm{Tot}\left(\mathrm{QCoh}(X) \underset{p^*}{\overset{a^*}{\rightrightarrows}} \mathrm{QCoh}(G \times X) \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow}\; \mathrm{QCoh}(G \times G \times X) \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow}\; \ldots \right).$$

In particular, to give a quasi-coherent sheaf on $[X/G]$ one need to choose a quasi-coherent sheaf $\mathcal{F}$ on X, choose an equivalence $a^*\mathcal{F} \simeq p^*\mathcal{F}$, e.t.c. Hence $\mathrm{QCoh}([X/G])$ is a model for the category of G-equivariant sheaves on X.

The category $\mathrm{QCoh}(\mathcal{X})$ admits a natural t-structure. To see this, recall the following general statement about t-structures on stable categories (by a t-structure on a stable $(\infty,1)$-category $\mathscr{C}$ we mean a t-structure on the triangulated category $\mathrm{ho}(\mathscr{C})$):

Lemma 3.3.2.12. *Let $\mathscr{C}$ be a stable category. Then:*

1. *To give a t-structure on $\mathscr{C}$ is the same as to specify a full subcategory $\mathscr{C}^{\leq 0}$ closed under extensions such that the inclusion $\mathscr{C}^{\leq 0} \hookrightarrow \mathscr{C}$ admits a right adjoint functor $\tau^{\leq 0}$.*

2. *Let $\{\mathscr{C}_i\}_{i\in I}$ be a diagram of stable categories equipped with t-structures and left (reps. right) t-exact functors. Then there is a natural t-structure on $\mathscr{C} := \lim_I \mathscr{C}_i$ such that the projection functors $p_i\colon \mathscr{C} \to \mathscr{C}_i$ are left (resp. right) t-exact and $X \in \mathscr{C}^{\geq 0}$ (resp. $X \in \mathscr{C}^{\leq 0}$) if and only if $p_i(X) \in \mathscr{C}_i^{\geq 0}$ (resp. $p_i(X) \in \mathscr{C}_i^{\leq 0}$) for all $i \in I$.*

Corollary 3.3.2.13. *Let $\mathcal{X}$ be a pre-stack. There exists a natural t-structure on $\mathrm{QCoh}(\mathcal{X})$ determined by the property that $\mathcal{F} \in \mathrm{QCoh}(\mathcal{X})^{\leq 0}$ if and only if $\mathcal{F}_{|A} \in D(\mathrm{Mod}_A)^{\leq 0}$ for all $\operatorname{Spec} A \to \mathcal{X}$. Moreover, for any map $f\colon \mathcal{X} \to \mathcal{Y}$ of pre-stacks the pullback functor f^* is left t-exact and the pushforward functor f_* is right t-exact.*

If $\mathcal{X}$ is an Artin stack, this t-structure enjoys an additional nice property. First recall

Definition 3.3.2.14. A stable category $\mathscr{C}$ with a t-structure is called *left t-complete* if the natural functor

$$\mathscr{C} \longrightarrow \lim_{\leftarrow} \left(\ldots \xrightarrow{\tau^{\geq -1}} \mathscr{C}^{\geq -1} \xrightarrow{\tau^{\geq 0}} \mathscr{C}^{\geq 0} \right) =: \widehat{\mathscr{C}}$$

is an equivalence. A stable category $\mathscr{C}$ with a t-structure is called *right t-complete* if $\mathscr{C}^{op}$ is left t-complete.

Remark 3.3.2.15. The category $\widehat{\mathscr{C}}$ can be identified with a full subcategory of $\mathrm{Fun}(\mathbb{Z}^{\mathrm{op}}_{\geq 0}, \mathscr{C})$ spanned by diagrams $X(-)\colon \mathbb{Z}^{\mathrm{op}}_{\geq 0} \to \mathscr{C}$ such that $X(i) \in \mathscr{C}^{\leq -i}$ and for all $i \geq j$ the natural map $X(i) \to X(j)$ induces an equivalence on $\tau^{\geq -j}$-th truncations. Moreover, if $\mathscr{C}$ admits countable sequential limits, the natural functor $\mathscr{C} \to \widehat{\mathscr{C}}$ admits a right adjoint $X(-) \mapsto \lim_{\leftarrow} X(i)$. It follows that if $\mathscr{C}$ is left t-complete, then the unit of adjunction $X \to \lim_{\leftarrow} \tau^{\geq -i} X$ is an equivalence for all $X \in \mathscr{C}$, i.e. Postnikov's towers converge in $\mathscr{C}$.

Remark 3.3.2.16. By [83, Proposition 1.2.1.19.] if $\mathscr{C}$ admits countable infinite products such that $\mathscr{C}^{\leq 0}$ is stable under these, then $\mathscr{C}$ is left t-complete if and only if $\bigcap_{i\geq 0} \mathscr{C}^{\leq -i} \simeq 0$. This result is neat: in [90] Neeman showed that the unbounded derived category of the abelian category of finite-dimensional representations of the additive group $\mathbb{G}_a$ over a field of characteristic $p > 0$ is not left t-complete, precisely by showing that $D^{<0}(\mathrm{Rep}_{\mathbb{G}_a})$ is not closed under infinite products.

Lemma 3.3.2.17. *Let $I \to \mathscr{C}\mathrm{at}_\infty$ be a diagram of stable categories equipped with t-structures. If each term of the diagram is left (reps. right) t-complete and all translation functors are left (resp. right) t-exact, then the limit category $\mathscr{C} := \lim_I \mathscr{C}_i$ is also left (resp. right) t-complete.*

Proof. We will only prove the statement for the left complete case, the right complete case follows by passing to the opposite categories. By construction of the limit t-structure on $\mathscr{C}$ we have $\mathscr{C}^{\geq -j} \xrightarrow{\sim} \lim_I \mathscr{C}_i^{\geq -j}$ for all $j \in \mathbb{Z}$. But since all $\mathscr{C}_i$ are left t-complete, we have $\mathscr{C}_i \xrightarrow{\sim} \lim\limits_{\leftarrow j} \mathscr{C}_i^{\geq -j}$. Since limits commute with each other, we deduce that $\mathscr{C} \xrightarrow{\sim} \lim\limits_{\leftarrow j} \mathscr{C}^{\geq -j}$ as desired. □

Proposition 3.3.2.18. *Let $\mathcal{X}$ be an Artin stack. Then the natural t-structure on $\mathcal{X}$ is left and right complete (see Definition 3.3.2.14).*

Proof. Right completeness holds for arbitrary prestack by Lemma 3.3.2.17. So it is enough to prove that the natural map

$$\mathrm{QCoh}(\mathcal{X}) \longrightarrow \lim_{\leftarrow} \mathrm{QCoh}(\mathcal{X})^{\geq -i}$$

is an equivalence. Let $\mathcal{X}$ be k-Artin. We will prove the statement by induction on k. The base of induction $k = -1$ case reduces to a well known statement that Postnikov's towers converge in the derived category of modules over a ring. Let $U \twoheadrightarrow \mathcal{X}$ be a smooth atlas and denote by $U_\bullet$ the corresponding Čech nerve. By the inductive assumption the natural map

$$\mathrm{QCoh}(U_n) \longrightarrow \lim_{\leftarrow} \mathrm{QCoh}(U_n)^{\geq -i}$$

is an equivalence for all n. But by flat descent

$$\mathrm{QCoh}(\mathcal{X})^{\geq -i} \xrightarrow{\sim} \mathrm{Tot}\, \mathrm{QCoh}(U_\bullet)^{\geq -i}$$

(here we had used that the pullback functors along smooth morphisms are t-exact). We conclude, since limits commute with each other. □

We will also need the following slight generalization of the flat base change for Artin stacks. In its formulation (and proof) we will use the language of derived stacks which we have not discussed in any detail in this text. We just refer to [110] for the definition and terminology. In all our applications the derived fiber product will coincide with the non-derived one and thus the same result will apply in the non-derived setting.

Proposition 3.3.2.19 (Finite Tor-amplitude base change, [41, Corollary 1.3.17])**.** *Let $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$ be non-derived (i.e. given by Definition 3.3.1.4) Artin stacks and let $\mathcal{W}$ be the fiber product in derived stacks of a quasi-compact and quasi-separated (see Definitions 3.3.1.17 and 3.3.1.18)*

morphism $f\colon \mathcal{Y} \to \mathcal{X}$ and a morphism $p\colon \mathcal{Z} \to \mathcal{X}$ of finite Tor amplitude.

$$\begin{array}{ccc} \mathcal{W} & \xrightarrow{q} & \mathcal{Y} \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ \mathcal{Z} & \xrightarrow{p} & \mathcal{X} \end{array} \tag{3.3.2.1}$$

Then the natural transformation

$$p^* \circ f_* \longrightarrow g_* \circ q^* \tag{3.3.2.2}$$

is an equivalence.

Corollary 3.3.2.20. *In Proposition 3.3.2.19 suppose that either g or f is flat and let $\mathcal{W}'$ be the non-derived fiber product. Then $\mathcal{W}' \simeq \mathcal{W}$ and the base change transformation for $\mathcal{W}'$ is also an equivalence.*

Using this we can refine Example 3.3.2.11 in the affine case:

Proposition 3.3.2.21. *Let G be an affine flat group scheme over R and let $X := \operatorname{Spec} A$ be an affine R-scheme equipped with an action of G. Then there is a natural equivalence symmetric monoidal t-exact equivalence*

$$\mathrm{QCoh}([X/G]) \simeq \mathrm{Mod}_A(\text{co-Mod}_H(D(\mathrm{Mod}_R))).$$

Proof. Let us treat the case $[*/G] = BG$ first. Consider the fibered diagram

$$\begin{array}{ccc} G & \xrightarrow{q} & * \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle p} \\ * & \xrightarrow{p} & BG. \end{array}$$

Since the map p is surjective, the pullback functor p^* is conservative. Moreover, by flatness p^* preserves totalizations. So, by Beck's monadicity theorem $\mathrm{QCoh}(BG)$ is equivalent to the category of comodules in $D(\mathrm{Mod}_R)$ over the comonad p^*p_*. Moreover, by the Lemma 3.3.2.22 below $\text{co-Mod}_{p^*p_*}(D(\mathrm{Mod}_R)) \simeq \text{co-Mod}_H(D(\mathrm{Mod}_R))$, where $H := p^*p_*(R)$. But by the base change $H = p^*p_*(R) \simeq q_*q^*(R) \simeq \Gamma(G, \mathcal{O}_G)$.

The general case now follows from Example 3.3.2.9. □

Lemma 3.3.2.22. *Let R be a commutative ring and let Q be a colimit preserving R-linear comonad on $D(\mathrm{Mod}_R)$. Then $C := Q(R)$ has a natural R-coalgebra structure and there is a natural equivalence* $\text{co-Mod}_Q(D(\mathrm{Mod}_R)) \simeq \text{co-Mod}_C(D(\mathrm{Mod}_R))$.

Proof. For any cocomplete R-linear category $\mathscr{D}$ the evaluation at R induces an equivalence $\mathrm{Fun}^{\mathrm{L}}_R(D(\mathrm{Mod}_R), \mathscr{D})$ $\mathscr{D}$. By applying this observation to $\mathscr{D} = D(\mathrm{Mod}_R)$ we obtain a monoidal equivalence $\mathrm{End}^{\mathrm{L}}_R(D(\mathrm{Mod}_R)) \xrightarrow{\sim}$ $D(\mathrm{Mod}_R)$. It is left to note that a continuous R-linear comonad on $D(\mathrm{Mod}_R)$ is by definition a coalgebra object in $\mathrm{End}^{\mathrm{L}}_R(D(\mathrm{Mod}_R))$. □

Corollary 3.3.2.23. *Let $G = \operatorname{Spec} H$ and $X = \operatorname{Spec} A$ be as in the previous proposition. Then there is a natural equivalence*

$$\mathrm{QCoh}([X/G])^+ \simeq D^+\mathrm{Mod}_A(\text{co-Mod}_{H,R}),$$

where by $\text{co-Mod}_{H,R}$ *we denoted the abelian category of H-comodules in* Mod_R.

Proof. By the previous proposition it is enough to prove that

$$D^+(\mathrm{Mod}_A(\text{co-Mod}_{H,R})) \simeq \mathrm{Mod}_A(\text{co-Mod}_{H,D(R)})^+.$$

Note that an object $M \in \mathrm{Mod}_A(\text{co-Mod}_{H,D(R)})$ lies in the heart of the t-structure if and only if the underlying complex of R-module does. It follows, since H is flat over R, that $\mathrm{Mod}_A(\text{co-Mod}_{H,D(R)})^\heartsuit \simeq$ $\mathrm{Mod}_A(\text{co-Mod}_{H,R})$. In particular there is a natural left t-exact functor

$$K\colon D^+(\mathrm{Mod}_A(\text{co-Mod}_{H,R})) \longrightarrow \mathrm{Mod}_A(\text{co-Mod}_{H,D(R)})^+$$

inducing equivalence on the hearts. We claim that K is an equivalence.

To see this note that the forgetful functor $\text{co-Mod}_{H,D(R)} \to D(\mathrm{Mod}_R)$ has a right adjoint $M \mapsto H \otimes_R M$ and analogously for $\text{co-Mod}_{H,R}$. In particular, $\text{co-Mod}_{H,R}$ has enough injectives of the form $H \otimes_R I$, where I is an injective R-module. Next, $\text{co-Mod}_{H,D(R)}$ is symmetric monoidal with the inner hom functor $[-,-]$ such that $[M, H \otimes_R N] \simeq H \otimes_R \mathrm{Hom}_{D(\mathrm{Mod}_R)}(M, N)$ and analogously for $\text{co-Mod}_{H,R}$. It follows the forgetful functor $\mathrm{Mod}_A(\text{co-Mod}_{H,D(R)})$ admits a right adjoint $M \mapsto [A, M]$ and analogously for $\mathrm{Mod}_A(\text{co-Mod}_{A,R})$. In particular, $\mathrm{Mod}_A(\text{co-Mod}_{A,R})$ has enough

injectives of the form

$$[A, H \otimes_R I] \simeq H \otimes_R \mathrm{Hom}_{\mathrm{Mod}_R}(A, I) \simeq H \otimes_R \mathrm{Hom}_{D(\mathrm{Mod}_R)}(A, I),$$

where I is an injective R-module. Finally, since H is flat, by a series of adjunctions we have

$$\mathrm{Ext}^*_{\mathrm{Mod}_A(\text{co-Mod}_{H,D(R)})}(M, H \otimes_R \mathrm{Hom}_{D(\mathrm{Mod}_R)}(A, I)) \simeq \mathrm{Ext}^*_{\text{co-Mod}_{H,D(R)}}(M, H \otimes_R I) \simeq \mathrm{Ext}^*_{D(\mathrm{Mod}_R)}(M, I) \simeq$$
$$\simeq \mathrm{Hom}_{\mathrm{Mod}_R}(M, I) \simeq \mathrm{Hom}_{\text{co-Mod}_{H,R}}(M, H \otimes_R I) \simeq \mathrm{Hom}_{\mathrm{Mod}_A(\text{co-Mod}_{R,H})}(M, H \otimes_R \mathrm{Hom}_{\mathrm{Mod}_R}(A, I)) \simeq$$
$$\simeq \mathrm{Ext}^*_{D^+\mathrm{Mod}_A(\text{co-Mod}_{R,H})}(M, H \otimes_R \mathrm{Hom}_{\mathrm{Mod}_R}(A, I)) \simeq \mathrm{Ext}^*_{D^+\mathrm{Mod}_A(\text{co-Mod}_{R,H})}(M, H \otimes_R \mathrm{Hom}_{D(\mathrm{Mod}_R)}(A, I))$$

for any $M \in \mathrm{Mod}_A(\text{co-Mod}_{R,H})$ and injective R-module I.

Let $M, N \in D^+\mathrm{Mod}_A(\text{co-Mod}_{R,H})$. We want to prove that the induced map of R-complexes

$$K_{M,N}\colon \mathrm{Hom}_{D^+\mathrm{Mod}_A(\text{co-Mod}_{H,R})}(M, N) \longrightarrow \mathrm{Hom}_{\mathrm{Mod}_A(\text{co-Mod}_{H,D(R)})}(K(M), K(N)) \qquad (3.3.2.3)$$

is an equivalence. Since filtered colimits in Mod_R are exact the same holds for $\mathrm{Mod}_A(\text{co-Mod}_{R,H})$. In particular Whitehead towers converge in $D^+\mathrm{Mod}_A(\text{co-Mod}_{R,H})$. It follows, since K is left t-exact, that we can assume that M is discrete. By discussion above the complex N admits an increasing filtration $F^{\leq \bullet}N$ with associated graded pieces F^nN of the form $H \otimes_R \mathrm{Hom}_{D(\mathrm{Mod}_R)}(A, I_n)[-n]$ for some injective R-module I_n, $n \in \mathbb{Z}_{\geq 0}$. By the previous discussion K_{M,F^nN} is an equivalence and since both parts of (3.3.2.3) are exact, $K_{M,F^{\leq n}N}$ is also an equivalence. It is left to note that since both $\mathrm{Hom}_{D^+\mathrm{Mod}_A(\text{co-Mod}_{R,H})}(M, F^{>n}N)$ and $\mathrm{Hom}_{\mathrm{Mod}_A(\text{co-Mod}_{H,D(R)})}(K(M), K(F^{>n}N))$ are at least n coconnective, the natural maps

$$\varinjlim \mathrm{Hom}_{D^+\mathrm{Mod}_A(\text{co-Mod}_{H,R})}(M, F^{>n}N) \longrightarrow \mathrm{Hom}_{D^+\mathrm{Mod}_A(\text{co-Mod}_{H,R})}(M, N),$$
$$\varinjlim \mathrm{Hom}_{\mathrm{Mod}_A(\text{co-Mod}_{H,D(R)})}(K(M), K(F^{>n}N)) \longrightarrow \mathrm{Hom}_{\mathrm{Mod}_A(\text{co-Mod}_{H,D(R)})}(K(M), K(N))$$

are equivalences. □

Finiteness conditions

Sometimes it is useful to put some size restrictions on quasi-coherent sheaves. Classically, there are two notions of this type: perfect complexes and coherent sheaves. In this subsection we will

discuss their natural extensions to stacks.

Recall that the subcategory $D(\mathrm{Mod}_A)^{\mathrm{perf}} \subset D(\mathrm{Mod}_A)$ of perfect A-modules is defined as the minimal full subcategory containing A and closed under finite (co)limits. It can be also described as the full subcategory of dualizable objects.

Definition 3.3.2.24. A sheaf $\mathcal{F}$ on a prestack $\mathcal{Y}$ is called *perfect* if for any map $x\colon \operatorname{Spec} A \to \mathcal{Y}$ the restriction $x^*\mathcal{F}$ is a perfect A-module. We will denote the full subcategory consisting of perfect sheaves by $\mathrm{QCoh}(\mathcal{Y})^{\mathrm{perf}}$. This is an R-linear stable subcategory of $\mathrm{QCoh}(\mathcal{Y})$ closed under finite (co)limits and retracts.

Similarly to $D(\mathrm{Mod}_A)^{\mathrm{perf}}$, there is also a description of $\mathrm{QCoh}(\mathcal{Y})^{\mathrm{perf}}$ in terms of dualizability. We recall the following statement:

Lemma 3.3.2.25 ([83, Proposition 4.6.1.11])**.** *Let $\{\mathscr{C}_i\}_{i\in I}$ be a diagram of symmetric monoidal categories and symmetric monoidal functors. Then an object $X \in \lim_I \mathscr{C}_i$ is dualizable if and only if the projections of X to all $\mathscr{C}_i$ are dualizable.*

Thus, we get that $\mathcal{F} \in \mathrm{QCoh}(\mathcal{Y})$ is perfect if and only if it is dualizable. The following corollary also follows:

Corollary 3.3.2.26. *Let $\mathcal{X}_\bullet\colon I \to \mathscr{PS}\mathrm{tk}_S$ be a diagram of stacks over S. Then the natural map*

$$\mathrm{QCoh}(\operatorname*{colim}_{i\in I} \mathcal{X}_i)^{\mathrm{perf}} \longrightarrow \lim_{i\in I} \mathrm{QCoh}(\mathcal{X}_i)^{\mathrm{perf}}$$

is an equivalence.

In particular, given an fpqc-surjection $\mathcal{U} \to \mathcal{X}$ we have an equivalence $q^*\colon \mathrm{QCoh}(\mathcal{X})^{\mathrm{perf}} \xrightarrow{\sim} \operatorname{Tot} \mathrm{QCoh}(\mathcal{U}_\bullet)^{\mathrm{perf}}$. Also $\mathcal{F} \in \mathrm{QCoh}(\mathcal{X})$ is perfect if and only if $q^*\mathcal{F}$ is. We note that perfect sheaves are automatically t-bounded.

Example 3.3.2.27. Let $\mathcal{X} \coloneqq [X/G]$, where X is a smooth scheme. Then by the previous corollary a sheaf $\mathcal{F} \in \mathrm{QCoh}([X/G])$ is perfect if and only if the pullback along the quotient map $\pi\colon X \to [X/G]$ is perfect. Recall now that by Corollary 3.3.2.23 $\mathrm{QCoh}(X/G)^+$ is equivalent to the bounded below derived category of G-equivariant sheaves on X and the pullback along π correspond to forgetting the G-action. In particular, $\mathcal{F} \in \mathrm{QCoh}(BG)$ is perfect if and only if the underlying complex of the corresponding representation is perfect.

Another finiteness condition one can put on a sheaf is coherentness. For the rest of this section we assume that R is a Noetherian ring.

Definition 3.3.2.28. Let A be a Noetherian R-algebra. We will call a complex of modules $M \in D(\mathrm{Mod}_A)$ *coherent* if all but finitely many of its cohomology $H^i(M)$ vanish and $H^i(M)$ is a finitely generated A-module for any $i \in \mathbb{Z}$. We will denote the full subcategory of coherent modules by $\mathrm{Coh}(A) \subset D(\mathrm{Mod}_A)$.

It is easy to see that $\mathrm{Coh}(A)$ is closed under finite (co)limits, and retracts. Note that $D(\mathrm{Mod}_A)^{\mathrm{perf}} \subset D(\mathrm{Mod}_A)$ and if A is regular then in fact $\mathrm{Coh}(A) \simeq D(\mathrm{Mod}_A)^{\mathrm{perf}}$.

Definition 3.3.2.29. A sheaf $\mathcal{F}$ on a stack $\mathcal{Y}$ of finite type over R is called *coherent* if for all maps $y\colon \operatorname{Spec} A \to \mathcal{Y}$ the pullback $y^*\mathcal{F}$ is a coherent module over A for all finitely presented R-algebras A. We will denote the full subcategory of $\mathrm{QCoh}(\mathcal{Y})$ consisting of coherent sheaves by $\mathrm{Coh}(\mathcal{Y})$.

Remark 3.3.2.30. In fact a sheaf $\mathcal{F} \in \mathrm{QCoh}(X)$ is coherent if and only if for any smooth atlas of finite type $U \to \mathcal{X}$ the restriction $\mathcal{F}_{|U}$ lies in $\mathrm{Coh}(U)$. This follows from the fact that a module $M \in \mathrm{Mod}_A$ is finitely generated if and only $M \otimes_A B$ is finitely generated for some faithfully flat finitely generated A-algebra B. In particular a sheaf $\mathcal{F}$ is coherent if and only if all of its cohomology sheaves $\mathcal{H}^i(\mathcal{F})$ (with respect to the t-structure on $\mathrm{QCoh}(\mathcal{X})$) are coherent.

If R is regular, for smooth Artin stacks of finite type coherent means the same as perfect:

Proposition 3.3.2.31. *Assume R is regular and $\mathcal{X}$ is a smooth Artin stack of finite type over R. Then*

$$\mathrm{Coh}(\mathcal{X}) \simeq \mathrm{QCoh}^{\mathrm{perf}}(\mathcal{X}).$$

Proof. Simce $\mathcal{X}$ is of finite type there is a smooth atlas $\operatorname{Spec} A \to \mathcal{X}$. Since $\mathcal{X}$ is smooth, A is smooth over R, thus is regular. The statement then follows from $\mathrm{Coh}(A) \simeq D(\mathrm{Mod}_A)^{\mathrm{perf}}$ and the fact that both categories are local with respect to smooth topology (3.3.2.30 and 3.3.2.26). □

However if these assumptions are not satisfied, the two categories are usually different. Coherent sheaves are better behaved with respect to pushforwards along proper morphism:

Definition 3.3.2.32. A map of Artin stacks $f\colon \mathcal{X} \to \mathcal{Y}$ is called *proper schematic* if it is schematic and for any affine $\operatorname{Spec} A \to \mathcal{Y}$ the pullback $f_A\colon \operatorname{Spec} A \times_{\mathcal{Y}} \mathcal{X} \to \operatorname{Spec} A$ is a proper map of schemes.

Proposition 3.3.2.33. *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a proper map of Artin stacks of finite type. Then for any $\mathcal{F} \in \mathrm{Coh}(\mathcal{X})$ its direct image $f_*\mathcal{F}$ is coherent.*

Proof. Let $U \to \mathcal{Y}$ be a smooth atlas and let $g\colon V := U \times_{\mathcal{Y}} \mathcal{X} \to \mathcal{X}$ be the base change of f. Then by the flat base change $f_*(\mathcal{F})_{|U} \simeq g_*(\mathcal{F}_{|V})$, which is coherent by the Grothendieck's theorem for proper morphisms between schemes. □

Digression: derived tensor functors

Recall that for a scheme X one defines $\Omega^n_X := \wedge^n \Omega^1_X$, where $\wedge^n\colon \mathrm{QCoh}(X)^\heartsuit \to \mathrm{QCoh}(X)^\heartsuit$ is the usual exterior power functor. In order to define the Hodge cohomology of stacks we need to extend this construction to more general geometric objects. Unfortunately, outside of characteristic 0 case for a stack $\mathcal{X}$ one can not give the definition of $\wedge^n$ purely in terms of $\mathrm{QCoh}(\mathcal{X})$ considered as a symmetric monoidal category equipped with t-structure. Instead, following [26, Section 3] we will first define $\wedge^n_R$ as an endofunctor of $D(\mathrm{Mod}_R)$ for arbitrary commutative ring R, then show that these functors satisfy base-change, hence extend naturally to stacks.

First recall that there is a natural extension of a functor defined on the 1-category Mod_R to the connective derived category $D(\mathrm{Mod}_R)^{\leq 0}$:

Definition 3.3.2.34 (Non-abelian derived functors)**.** Let A be a commutative ring. Let us denote by $\mathrm{Mod}_A^{\mathrm{free,fg}}$ a full subcategory of Mod_A spanned by finite rank free A-modules. By [83, Section 7.2.2] for any category $\mathscr{D}$ admitting sifted colimits the restriction functor

$$\mathrm{Fun}_\Sigma(D(\mathrm{Mod}_A)^{\leq 0}, \mathscr{D}) \longrightarrow \mathrm{Fun}(\mathrm{Mod}_A^{\mathrm{free,fg}}, \mathscr{D})$$

is an equivalence with the inverse given by the left Kan extension. Here $\mathrm{Fun}_\Sigma(D(\mathrm{Mod}_A)^{\leq 0}, \mathscr{D})$ denotes the full subcategory of $\mathrm{Fun}(D(\mathrm{Mod}_A)^{\leq 0}, \mathscr{D})$ spanned by sifted colimits preserving functors. For a functor $F\colon \mathrm{Mod}_A^{\mathrm{free,fg}} \to \mathscr{D}$ we will denote its image in $\mathrm{Fun}_\Sigma(D(\mathrm{Mod}_A)^{\leq 0}, \mathscr{D})$ by $\mathrm{L}F$ and call it the *non-abelian derived functor of F*.

An extension to the full derived category $D(\mathrm{Mod}_R)$ is more involved (e.g. as explained in [67, Section 3.3], one should probably impose some assumptions on a functor to be able to extend it to the full derived category) and was first written down probably in [58, Chapitre I.4]. For technical reasons we prefer to use the language developed in [26, Section 3], which we briefly review here. First there is the following universal property of $D(\mathrm{Mod}_A)^{\mathrm{perf}}$:

Proposition 3.3.2.35 ([26, Proposition 3.8]). *Let A be a commutative ring. For any category $\mathscr{D}$ admitting sifted colimits, the restriction functor*

$$\mathrm{Fun}_{\Sigma}(D(\mathrm{Mod}_A), \mathscr{D}) \longrightarrow \mathrm{Fun}_{\sigma}(D(\mathrm{Mod}_A)^{\mathrm{perf}}, \mathscr{D})$$

is an equivalence, where $\mathrm{Fun}_{\sigma}(D(\mathrm{Mod}_A)^{\mathrm{perf}}, \mathscr{D})$ denotes the full subcategory of $\mathrm{Fun}(D(\mathrm{Mod}_A)^{\mathrm{perf}}, \mathscr{D})$ spanned by functors preserving geometric realization of m-skeletal simplicial objects for some $m \in \mathbb{Z}_{\geq 0}$. The inverse functor is given by the left Kan extension.

To use this proposition it is enough to extend our non-Abelian derived functors (defined a priory only on $D(\mathrm{Mod}_A)^{\leq 0}$) to $D(\mathrm{Mod}_A)^{\mathrm{perf}}$. To this end we will use a bit the machinery of Goodwillie's calculus of functors (see [49] and [83, Section 6.1]). Recall the following definitions:

Definition 3.3.2.36. Let T be a set. A *T-cube in a category $\mathscr{C}$* is a functor $\mathrm{P}(T) \to \mathscr{C}$, where $\mathrm{P}(T)$ denotes the poset of subsets of T. We will denote by $[n]$ the $(n+1)$-element set $\{0, 1, \ldots, n\}$ and will usually call $[n]$-cubes just n-cubes. A cube $\mathcal{X}\colon \mathrm{P}([n]) \to \mathscr{C}$ is called

- *strongly cocartesian* if it is left Kan extended from subsets of cardinality at most 1.
- *cartesian* if it is a limit diagram.

Definition 3.3.2.37. Let $\mathscr{C}$ be a category admitting finite colimits and let $\mathscr{D}$ be a category admitting finite limits. A functor $F\colon \mathscr{C} \to \mathscr{D}$ is called *n-excisive* if for any strongly cocartesian n-cube $\mathcal{X}$ the cube $F(\mathcal{X})$ is cartesian, i.e. if the natural map

$$F(\mathcal{X}_{\varnothing}) \longrightarrow \lim_{S \in \mathrm{P}([n]) \setminus \varnothing} F(\mathcal{X}_S)$$

is an equivalence.

Example 3.3.2.38. If $\mathscr{C}$ admits the final object $*$, then the functor $F\colon \mathscr{C} \to \mathscr{D}$ is 0-excisive if and only if it is constant with the value $F(*)$. If the category $\mathscr{D}$ is stable (so that a square in $\mathscr{D}$ is Cartesian if and only if it is coCartesian), then the functor $F\colon \mathscr{C} \to \mathscr{D}$ is 1-excisive if and only if it preserves finite (co-)limits.

Theorem 3.3.2.39 ([83, Theorem 6.1.1.10]). *Let $\mathscr{C}$ be a category admitting finite colimits and a final object and let $\mathscr{D}$ be a category admitting finite limits and countable sequential colimits and such*

that those commute with each other. Let us denote by $\mathrm{Exc}^n(\mathscr{C},\mathscr{D})$ *the full subcategory of* $\mathrm{Fun}(\mathscr{C},\mathscr{D})$ *spanned by* n*-excisive functors. Then the inclusion functor* $i\colon \mathrm{Exc}^n(\mathscr{C},\mathscr{D}) \hookrightarrow \mathrm{Fun}(\mathscr{C},\mathscr{D})$ *admits a left adjoint* P_n*, which moreover preserves finite limits.*

The relevance of this notion to our problem is explained by the following proposition:

Proposition 3.3.2.40 ([26, Theorem 3.35])**.** *The restriction functor*

$$\mathrm{Exc}^n(D(\mathrm{Mod}_A)^{\mathrm{perf},\leq 0},\mathscr{D}) \longrightarrow \mathrm{Exc}^n(D(\mathrm{Mod}_A)^{\mathrm{perf}},\mathscr{D})$$

is an equivalence.

I.e. any n-excisive functor defined on $D(\mathrm{Mod}_A)^{\mathrm{perf},\leq 0}$ admits an essentially unique n-excisive extension to $D(\mathrm{Mod}_A)^{\mathrm{perf}}$. To be able to apply this proposition we need to know that $\wedge^n\colon D(\mathrm{Mod}_A)^{\mathrm{perf},\leq 0} \to D(\mathrm{Mod}_A)$ is n-excisive. To this end recall the following definition:

Definition 3.3.2.41. We say that a functor $F\colon \mathscr{C}\to\mathscr{D}$ *is of degree* 0 if it is constant. F is called *degree* n if the difference functor $D_XF(Y) := \mathrm{fib}\big(F(X\oplus Y)\to F(Y)\big)$ is a degree $n-1$ functor.

Example 3.3.2.42. Classically, $\mathrm{Sym}^n, \wedge^n$ and Γ^n are degree n-functors.

Then one has:

Proposition 3.3.2.43 ([61, Proposition 5.10], [26, Proposition 3.34])**.** *Let* $F\colon \mathrm{Mod}_A\to\mathscr{D}$ *be a degree* n *functor. Then the derived functor* $\mathrm{L}F\colon D^{\leq 0}(\mathcal{A})\to\mathscr{D}$ *is* n*-excisive.*

Finally, we have the following useful observation:

Proposition 3.3.2.44. *Let* $F\colon \mathscr{C}\to\mathscr{D}$ *be a filtered colimit preserving functor and assume that* $\mathscr{D}$ *is stable and cocomplete. Then for each* $n\in\mathbb{Z}_{\geq 0}$ *the functor* P_nF *(see 3.3.2.39) preserves sifted colimits.*

Proof. Consider the fiber sequence $D_nF\to P_nF\to P_{n-1}F$. By induction it is enough to prove that D_nF preserves sifted colimits. By [83, Theorem 6.1.4.7] there is a functor $\mathrm{cr}_n F\colon \mathscr{C}^{\times n}\to\mathscr{D}$ which is 1-excisive separately in each argument and such that $D_nF(X)\simeq \big(\mathrm{cr}_n F(X,X,\dots X)\big)_{h\Sigma_n}$. Since by assumption F preserves filtered colimits, by construction the same holds for $\mathrm{cr}_n F$. It follows that $\mathrm{cr}_n F$ preserves all colimits separately in each argument. Since by definition of a sifted diagram I the diagonal functor $I\to I^n$ is cofinal, the functor $X\mapsto \mathrm{cr}_n F(X,X,\dots,X)$ preserves sifted colimits. Since colimits commute, the same holds for D_nF. □

Combing all this one can perform the following construction:

Construction 3.3.2.45. Let $F\colon \mathrm{Mod}_A^{\mathrm{free,fg}} \to \mathscr{D}$ be a degree n functor. By Proposition 3.3.2.43 the derived functor $\mathrm{L}F_{|D(\mathrm{Mod}_A)^{\mathrm{perf},\leq 0}}\colon D(\mathrm{Mod}_A)^{\mathrm{perf},\leq 0} \to \mathscr{D}$ is n-excisive. By Proposition 3.3.2.40 it extends in a canonical way to an n-excisive functor $\mathrm{L}F'\colon D(\mathrm{Mod}_A)^{\mathrm{perf}} \to \mathscr{D}$. Moreover, by Proposition 3.3.2.44 $\mathrm{L}F'$ preserves geometric realizations. Taking the left Kan extension, we obtain a unique (by Proposition 3.3.2.35) sifted colimit preserving functor $\mathrm{RL}F\colon D(\mathrm{Mod}_A) \to \mathscr{D}$. Note that since $\mathrm{RL}F$ preserves sifted colimits, its restriction to $D(\mathrm{Mod}_A)^{\leq 0}$ is equivalent to $\mathrm{L}F$.

Notation 3.3.2.46. For a commutative ring A we will denote by $\mathrm{Sym}^n_A, \wedge^n_A, \Gamma^n_A\colon D(\mathrm{Mod}_A) \to D(\mathrm{Mod}_A)$ the functors obtained by the previous construction from the classical symmetric, exterior and divided power functors respectively. If the ring A is clear from the context we will usually omit it from notations and write just Sym^n, $\wedge^n$ and Γ^n.

Next we explain how to extend this functors to the category of quasi-coherent sheaves on stacks.

Proposition 3.3.2.47. *Let $A \to B$ be a homomorphism of commutative rings and let $X \in D(\mathrm{Mod}_A)$ be an A-module. Then the natural map*

$$(\wedge^n_A X) \otimes_A B \longrightarrow \wedge^n_B (X \otimes_A B)$$

is an equivalence.

Proof. First assume that X is connective. Then $X \simeq |X_\bullet|$ for some simplicial diagram $X_\bullet$ of discrete free A-modules. Since both parts commute with sifted colimits, we reduced the statement to the case of free modules, which is well known. The general case follows from equivalences of Proposition 3.3.2.40 and Proposition 3.3.2.35. □

Passing to the right Kan extensions we obtain:

Corollary 3.3.2.48. *For each $n \in \mathbb{Z}_{\geq 0}$ there exist natural endotransformations Sym^n, $\wedge^n$ and Γ^n of the functor*

$$\mathrm{QCoh}^*\colon \mathscr{P}\mathscr{S}\mathrm{tk}^{\mathrm{op}} \longrightarrow \mathrm{Pr}^{\Sigma}$$

(where Pr^{Σ} denotes the $(\infty,1)$-category of presentable categories and sifted colimit preserving functors) determined by the property that their restriction to $\mathrm{Aff}^{\mathrm{op}}$ is equivalent to the derived tensor functors from Notation 3.3.2.46.

Finally, we will need some standard results about derived tensor functor, which we recall here.

Proposition 3.3.2.49 (Decalage equivalence)**.** *Let $\mathcal{X}$ be a prestack and let $\mathcal{F}$ be a quasi-coherent sheaf on $\mathcal{X}$. For each $n \in \mathbb{Z}_{\geq 0}$ there are canonical equivalences:*

$$\mathrm{Sym}^n(\mathcal{F}[1]) \simeq \wedge^n(\mathcal{F})[n] \qquad \textit{and} \qquad \wedge^n(\mathcal{F}[1]) \simeq \Gamma^n(\mathcal{F})[n].$$

Proof. The statement is local, so we can assume that $\mathcal{X} \simeq \mathrm{Spec}\, A$ for some commutative ring A. Then the result for connective $\mathcal{F}$ is proved in [84, Proposition 25.2.4.2]. The general case follows from equivalences of Proposition 3.3.2.40 and Proposition 3.3.2.35. □

We also have the following finiteness result:

Proposition 3.3.2.50. *Let $\mathcal{X}$ be a prestack and let $\mathcal{E}$ be a perfect sheaf on $\mathcal{X}$. Then for all $n \in \mathbb{Z}_{\geq 0}$ the quasi-coherent sheaves $\mathrm{Sym}^n(\mathcal{E})$, $\wedge^n(\mathcal{E})$ and $\Gamma^n(\mathcal{E})$ are also perfect.*

Proof. By the decalage equivalence it is enough to prove the proposition only for Sym^n. Moreover, since the statement is local, we can assume that $\mathcal{X} \simeq \mathrm{Spec}\, A$ for some commutative ring A. Let E be perfect k-connective module for some $k \in \mathbb{Z}$. We will prove the statement by induction on n and k. The base of induction $k = 0$ was treated in [84, Proposition 25.2.5.3]. Since by construction Sym^n is n-excisive we have an equivalence

$$\mathrm{Sym}^n(E) \simeq \lim_{S \in \mathrm{P}([n]) \setminus \varnothing} \mathrm{Sym}^n(S \star E),$$

where $S \star E \simeq (E[1])^{\oplus |S|-1}$. Since $E[1]$ is $(k+1)$-connective and since Sym^n_A of a direct sum can be expressed as a product of Sym^i for $i \leq n$, we conclude by inductive assumption. □

3.3.3 Cotangent complex

In this section we will recall the definition and some basic properties of the cotangent complex $\mathbb{L}_{\mathcal{Y}}$ of a prestack $\mathcal{Y}$. To give a definition we first note that there exists a natural extension of $\mathcal{Y}$ to a functor on simplicial R algebras $\mathrm{CAlg}_{R/}^{\Delta^{\mathrm{op}}}$ (with its natural $(\infty,1)$-structure) given by

$$\mathcal{Y}(A_\bullet) := (\mathrm{Lan}_{i_\Delta} \mathcal{Y})(A_\bullet), \tag{3.3.3.1}$$

where $\mathrm{Lan}_{i_\Delta}\mathcal{Y}$ is the left Kan extension of $\mathcal{Y}$ along the embedding of commutative R-algebras into simplicial commutative R-algebras.

Remark 3.3.3.1. The definition above is natural from the following perspective: if we denote by $i_{\mathrm{red}}\colon \mathrm{CAlg}^{\mathrm{red}}_{R/} \hookrightarrow \mathrm{CAlg}_{R/}$ the natural embedding of the category of reduced R-algebras, then it is easy to see that for any reduced R-scheme $Y \in \mathcal{S}\mathrm{ch}_{/S}$ and any (not necessary reduced) R-algebra A we have

$$Y(A) \simeq (\mathrm{Lan}_{i_{\mathrm{red}}} Y_{|\,\mathrm{CAlg}^{\mathrm{red}}_{/R}})(A).$$

Now, one can think about simplicial algebra $A_\bullet$ as a derived nilpotent thickening of the usual algebra $\pi_0(A_\bullet)$ (some elements of $A_\bullet$ are so nilpotent, that they live in higher homotopy degrees). Hence (3.3.3.1) generalizes the usual embedding of the full subcategory of reduced schemes into the category of all schemes.

Remark 3.3.3.2. Outside of characteristic 0 there are two different notions of cotangent complex: one controls deformations as a simplicial commutative algebra, the other deformations as E_∞-algebras. The usual polynomial algebra $R[x]$ is not free over R when considered as an E_∞-algebra, thus its E_∞-cotangent complex will in general be quite different from the usual module of Kahler differentials. For us this behavior is undesirable since we would at least like the cotangent complex of a smooth classical algebra A to be given by the A-module of differential 1-forms $\Omega^1_{A/R}$. Essentially by construction one doesn't have this problem in the simplicial context, hence we stick to this case.

Let A be an R-algebra and let $M \in D(\mathrm{Mod}_A)^{\leq 0}$ be a connective complex. Then one can consider the *trivial square-zero extension* $A \oplus M$ *of* A *by* M which is given as follows: by Dold-Kan correspondence we have a simplicial module $M_\bullet \in \mathrm{Mod}_A^{\Delta^{op}}$ corresponding to M and then can apply the usual square-zero extension degreewise. With this notations we can formulate the defining property of $\mathbb{L}_{\_}$:

Definition 3.3.3.3. Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a morphism of prestacks. The defining property of the *relative cotangent complex* $\mathbb{L}_f = \mathbb{L}_{\mathcal{X}/\mathcal{Y}} \in \mathrm{QCoh}(\mathcal{X})$ is that for any R algebra A[5], a point $x \in \mathcal{X}(A)$ and a module $M \in D(\mathrm{Mod}_A)^{\leq 0}$ the space of morphisms $\mathrm{Hom}_{D(\mathrm{Mod}_A)}(x^*\mathbb{L}_{\mathcal{X}/\mathcal{Y}}, M)$ is equivalent to

[5]One could also ask the analogues representability property for all simplicial R-algebras $A_\bullet$. For non-derived stacks there is no difference.

the space of lifts

$$\begin{array}{ccc} \operatorname{Spec} A & \xrightarrow{x} & \mathcal{X} \\ \downarrow & \nearrow & \downarrow f \\ \operatorname{Spec} A \oplus M & \xrightarrow[f\circ x\circ q]{} & \mathcal{Y} \end{array}$$

where $q\colon \operatorname{Spec} A \oplus M \to \operatorname{Spec} A$ is the projection morphism. In the special case $\mathcal{Y} = S$ we will call $\mathbb{L}_{\mathcal{X}/S}$ just *cotangent complex of* $\mathcal{X}$ and will just write $\mathbb{L}_{\mathcal{Y}}$ if the base S is clear from the context.

The relative cotangent complex $\mathbb{L}_{\mathcal{X}/\mathcal{Y}}$ does not need to exist in general; however, it does if $f\colon \mathcal{X} \to \mathcal{Y}$ is n-representable for some n (and in many other cases).

We will now recall some basic properties of the cotangent complexes that are going to be relevant later. The following proposition follows formally from the defining property of $\mathbb{L}_{\_}$:

Proposition 3.3.3.4. *In the above notation we have:*

1. *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a morphism of prestacks such that both $\mathcal{X}$ and $\mathcal{Y}$ admit cotangent complexes over S. Then the relative cotangent complex $\mathbb{L}_f$ exists and is canonically equivalent to the cofiber of the natural map $f^*\mathbb{L}_{\mathcal{Y}} \to \mathbb{L}_{\mathcal{X}}$.*

2. *(Base change for $\mathbb{L}_{\_}$) Let*

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{q} & \mathcal{Z} \\ \downarrow & & \downarrow \\ \mathcal{Y} & \xrightarrow{p} & \mathcal{W} \end{array}$$

   be a pullback square of prestacks such that $\mathbb{L}_{\mathcal{Z}/\mathcal{W}}$ and $\mathbb{L}_{\mathcal{X}/\mathcal{Y}}$ exist and lower horizontal or right vertical arrow is flat. Then there exists a canonical equivalence $q^*\mathbb{L}_{\mathcal{Z}/\mathcal{W}} \simeq \mathbb{L}_{\mathcal{X}/\mathcal{Y}}$.

3. *Let $\mathcal{X}_\bullet\colon I \to \mathscr{S}\mathrm{tk}_{/S}$ be a diagram of pre-stacks admitting cotangent complex and flat maps between them and let $\mathcal{X} \coloneqq \lim_I \mathcal{X}_i$. Then $\mathbb{L}_{\mathcal{X}}$ exists and, if $p_i\colon \mathcal{X} \to \mathcal{X}_i$ is the natural projection, the induced map $\operatorname{colim}_I p_i^*\mathbb{L}_{\mathcal{X}_i} \to \mathbb{L}_{\mathcal{X}}$ is an equivalence.*

It also satisfies the étale hyperdescent:

Proposition 3.3.3.5. *Let $p_\bullet\colon \mathcal{U}_\bullet \to \mathcal{X}$ be an étale hypercover with $\mathcal{X}$ admitting a cotangent complex. Then each $\mathcal{U}_i$ also admits a cotangent complex and the natural map*

$$\mathbb{L}_{\mathcal{X}} \longrightarrow \operatorname{Tot} p_{\bullet *}(\mathbb{L}_{\mathcal{U}_\bullet})$$

is an equivalence.

Proof. Since all maps $p_n\colon \mathcal{U}_n \to \mathcal{X}$ are étale we have canonical equivalences $p_n^*\mathbb{L}_{\mathcal{X}} \xrightarrow{\sim} \mathbb{L}_{\mathcal{U}_n}$ (in particular, the latter exists). It follows that

$$\operatorname{Tot} p_{\bullet *}(\mathbb{L}_{\mathcal{U}_\bullet}) \simeq \operatorname{Tot} p_{\bullet *}p_\bullet^*\mathbb{L}_{\mathcal{X}} \simeq \mathbb{L}_{\mathcal{X}},$$

where the last equivalence is provided by the flat descent for quasi-coherent sheaves. □

If $\mathcal{Y} = Y$ is a classical smooth scheme then $\mathbb{L}_{\mathcal{Y}/S}$ is given by the sheaf $\Omega^1_{Y/S} \in \mathrm{QCoh}(Y)^\heartsuit$ of S-relative differential 1-forms. Note that in this case $\Omega^1_{Y/S}$ is locally free and so is a perfect complex. Similar statement is also true for n-Artin stacks

Proposition 3.3.3.6. *Let $\mathcal{Y}$ be a smooth n-Artin stack over S. Then $\mathbb{L}_{\mathcal{Y}}$ is perfect and of* Tor-*amplitude* $[0, n]$. *In particular $\mathcal{H}^i(\mathbb{L}_{\mathcal{Y}}) \simeq 0$ for $i \notin [0; n]$.*

Proof. Let $\mathcal{Y}$ be n-Artin. We will prove the statement by induction on n. If $n = 0$, (i.e. $\mathcal{Y}$ is a smooth scheme) since Zariski locally every smooth scheme admits an étale map into affine space we can assume $\mathcal{Y} \simeq \mathbb{A}^n_S$. In this case it is not hard to see from the universal property that $\mathbb{L}_{\mathbb{A}^n_S/S} \simeq \mathcal{O}^{\oplus n}_{\mathbb{A}^n_S}$, hence is flat (which is equivalent to be of Tor-amplitude $[0; 0]$) and dualizable.

Assume now the statement is proved for $n' < n$. Let $p\colon U \twoheadrightarrow \mathcal{Y}$ be a smooth atlas. Since the functor p^* is conservative, t-exact and detects dualizable objects, it is enough to prove the statement for $p^*\mathbb{L}_{\mathcal{Y}}$ instead of $\mathbb{L}_{\mathcal{Y}}$. We have a co-fiber sequence

$$p^*\mathbb{L}_{\mathcal{Y}} \longrightarrow \mathbb{L}_U \longrightarrow \mathbb{L}_p$$

Since dualizable objects are closed under finite limit and since a fiber of $[0; n-1]$ Tor-amplitude objects has Tor-amplitude at most $[0; n]$, it is enough to prove $\mathbb{L}_p$ is perfect and has Tor-amplitude $[0; n-1]$. For this end consider the fibered square

$$\begin{array}{ccc} U \times_{\mathcal{Y}} U & \xrightarrow{q} & U \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle p} \\ U & \xrightarrow{p} & \mathcal{Y} \end{array}$$

Again since q is smooth and surjective is enough to prove $q^*\mathbb{L}_p$ has desired properties. But by

base-change for the cotangent complex $q^*\mathbb{L}_p \simeq \mathbb{L}_q$. We conclude by induction, since the map q is smooth and $(n-1)$-representable. □

Example 3.3.3.7. Let G be a flat group scheme over R and consider the classifying stack BG from Example 3.3.1.8. It is not hard to see that the diagonal map $BG \to BG \times BG$ classifies the action of $G \times G$ on G given by $(g_1, g_2) \cdot h = g_1 h g_2^{-1}$. Consider the following pullback diagram

$$\begin{array}{ccc} G_{\mathrm{Ad}_G} & \xrightarrow{p} & BG \\ {\scriptstyle e}\uparrow\downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \Delta} \\ BG & \xrightarrow{\Delta} & BG \times BG, \end{array}$$

where G_{Ad_G} stands for the quotient of G by the adjoint action and e is a section of p induced by the unit morphism $* \to G$. It follows by base change that $\mathbb{L}_{G_{\mathrm{Ad}_G}/BG} \simeq p^*\mathbb{L}_{BG/BG\times BG} \simeq p^*\mathbb{L}_{\mathbb{B}G}[1]$. On the other hand, by definition $e^*\mathbb{L}_{G_{\mathrm{Ad}_G}/BG} \simeq \mathfrak{g}^\vee$ with the adjoint action on the right hand side. It follows that $\mathfrak{g}^\vee \simeq e^*\mathbb{L}_{G_{\mathrm{Ad}_G}/BG} \simeq e^*p^*\mathbb{L}_{BG} \simeq \mathbb{L}_{BG}[1]$, hence $\mathbb{L}_{BG} \simeq \mathfrak{g}^\vee[-1]$.

Example 3.3.3.8. Let X be an R-scheme and let G be a flat group R-scheme acting on X. Note that the structure map $X \to *$ induces a map $p\colon X/G \to BG$ and that there is a fibered square

$$\begin{array}{ccc} X & \xrightarrow{q} & X/G \\ \downarrow & & \downarrow{\scriptstyle p} \\ * & \longrightarrow & BG. \end{array}$$

It follow, there is a fiber sequence $p^*\mathbb{L}_{BG} \to \mathbb{L}_{X/G} \to \mathbb{L}_p$. Moreover, by base change for cotangent complex, $q^*\mathbb{L}_p \simeq \mathbb{L}_X$, hence $\mathbb{L}_p$ is just $\mathbb{L}_X$ with its natural G-equivariant structure. One can check that the natural map $\mathbb{L}_X \to p^*\mathbb{L}_{BG}[1] \simeq \mathfrak{g}^\vee$ is just the natural coaction of $\mathfrak{g}^\vee$ on $\mathbb{L}_X$. In particular for a smooth X the cotangent complex $\mathbb{L}_{X/G}$ is equivalent to the two-term complex $\Omega^1_X \to \mathfrak{g}^\vee$ of G-equivariant sheaves on X.

We will finish this section by proving flat descent for the cotangent complex. The proof is essentially due to Bhatt (see [15, Corollary 2.7, Remark 2.8] or [18, Section 3]):

Proposition 3.3.3.9 (Flat descent for the cotangent complex)**.** *Let $p\colon \mathcal{U} \to \mathcal{X}$ be a surjective flat morphism between Artin stacks and denote by $p_\bullet\colon \mathcal{U}_\bullet \to \mathcal{X}$ the corresponding Čech nerve. Then the natural map*

$$\mathbb{L}_{\mathcal{X}} \longrightarrow \mathrm{Tot}\, p_{\bullet *}(\mathbb{L}_{\mathcal{U}_\bullet})$$

is an equivalence.

Proof. For every $n \in \mathbb{Z}_{\geq 0}$ we have a co-fiber sequence

$$p_n^* \mathbb{L}_{\mathcal{X}} \longrightarrow \mathbb{L}_{\mathcal{U}_n} \longrightarrow \mathbb{L}_{\mathcal{U}_n/\mathcal{X}}$$

in $\mathrm{QCoh}(\mathcal{U}_n)$. Applying $p_{\bullet *}$ we obtain a co-fiber sequence of co-simplicial objects

$$p_{\bullet *} p_\bullet^* \mathbb{L}_{\mathcal{X}} \longrightarrow p_{\bullet *} \mathbb{L}_{\mathcal{U}_\bullet} \longrightarrow p_{\bullet *} \mathbb{L}_{\mathcal{U}_\bullet/\mathcal{X}}$$

in $\mathrm{QCoh}(\mathcal{X})$ which induces a co-fiber sequence of totalizations

$$\mathrm{Tot}\, p_{\bullet *} p_\bullet^* \mathbb{L}_{\mathcal{X}/k} \longrightarrow \mathrm{Tot}\, p_{\bullet *} \mathbb{L}_{\mathcal{U}_\bullet/k} \longrightarrow \mathrm{Tot}\, p_{\bullet *} \mathbb{L}_{\mathcal{U}_\bullet/\mathcal{X}}\ .$$

By flat descent for QCoh [84, Corollary D.6.3.4] the induced map $\mathbb{L}_{\mathcal{X}} \to \mathrm{Tot}\, p_{\bullet *} p_\bullet^* \mathbb{L}_{\mathcal{X}}$ is an equivalence. So it is enough to prove that $\mathrm{Tot}\, p_{\bullet *} \mathbb{L}_{\mathcal{U}_\bullet/\mathcal{X}} \simeq 0$. Moreover, since the map p is faithfully flat, it is enough to prove that the pullback $p^* \mathrm{Tot}\, p_{\bullet *} \mathbb{L}_{\mathcal{U}_\bullet/\mathcal{X}}$ of the totalization above is null-homotopic. Now, by base change for the relative cotangent complex (second part of Proposition 3.3.3.4) and for quasi-coherent sheaves (Proposition 3.3.2.19) we have

$$p^* p_{\bullet *} \mathbb{L}_{\mathcal{U}_\bullet/\mathcal{X}} \simeq q_{\bullet *} \mathbb{L}_{\mathcal{U}_\bullet \times_{\mathcal{X}} \mathcal{U}/\mathcal{U}},$$

where $q_\bullet \colon \mathcal{U}_\bullet \times_{\mathcal{X}} \mathcal{U} \to \mathcal{U}$ is the pullback $q_\bullet := p_\bullet \times_{\mathcal{X}} \mathcal{U}$ of the Čech nerve $p_\bullet \colon \mathcal{U}_\bullet \to \mathcal{X}$ on $\mathcal{U}$ along p. But since $\mathcal{U}_\bullet \times_{\mathcal{X}} \mathcal{U} \to \mathcal{U}$ is a split simplicial object, the same holds for $\mathbb{L}_{\mathcal{U}_\bullet \times_{\mathcal{X}} \mathcal{U}/\mathcal{U}}$ (since split simplicial objects are stable under any functor) and so by Remark A.1.1.2 the natural map

$$0 \simeq \mathbb{L}_{\mathcal{U}/\mathcal{U}} \longrightarrow \mathrm{Tot}\, q_{\bullet *} \mathbb{L}_{\mathcal{U}_\bullet \times_{\mathcal{X}} \mathcal{U}/\mathcal{U}}$$

is an equivalence. Finally, since by flat descent $\mathrm{QCoh}(\mathcal{X})$ is comonadic over $\mathrm{QCoh}(\mathcal{U})$, the pullback functor p^* commutes with totalizations of p^*-split co-simplicial objects, hence

$$p^* \mathrm{Tot}\, p_{\bullet *} \mathbb{L}_{\mathcal{U}_\bullet/\mathcal{X}} \simeq \mathrm{Tot}\, q_{\bullet *} \mathbb{L}_{\mathcal{U}_\bullet \times_{\mathcal{X}} \mathcal{U}/\mathcal{U}} \simeq 0. \quad \square$$

3.3.4 Étale sheaves and local systems on stacks

Étale sheaves

Let Λ be a commutative ring such that $\mathrm{char}(\Lambda)$ is invertible in R. In this section we will develop some rudiments of the theory of étale sheaves on Artin stacks and will show that many familiar results about sheaves on schemes remain valid in this generality. We refer an interested reader to [80] for a more exhaustive treatment of the formalism of six operations for étale sheaves on Artin stacks. We won't need the full power of six operations and in any case the results of loc.cit. are not quite adapted to our situation, so we decided to develop the theory independently from scratch, taking the known results for schemes as an input.

For an affine R-scheme T let $\mathcal{S}\mathrm{hv}_{\text{ét}}(T,\Lambda) \in \mathscr{P}r^{\mathrm{L}}_{\underline{\Lambda}}$ be the $(\infty,1)$-category of sheaves of $\underline{\Lambda}$-modules on the small étale site $\text{ét}/T$. Given a map $f\colon T_1 \to T_2$ we have the natural pull-back functor $f^{-1}\colon \mathcal{S}\mathrm{hv}_{\text{ét}}(T_2,\Lambda) \to \mathcal{S}\mathrm{hv}_{\text{ét}}(T_1,\Lambda)$.

Construction 3.3.4.1. Let $\mathcal{X}$ be a prestack over R. We define the category of Λ-*étale sheaves on* $\mathcal{X}$ as

$$\mathcal{S}\mathrm{hv}_{\text{ét}}(\mathcal{X},\Lambda) \coloneqq \lim_{(T\to\mathcal{X})\in(\mathrm{Aff}_{R/\mathcal{X}}/\mathcal{X})^{\mathrm{op}}} \mathcal{S}\mathrm{hv}_{\text{ét}}(T,\Lambda),$$

where the limit is taken in $\mathscr{P}r^{\mathrm{L}}_{\Lambda}$. For a sheaf $\mathcal{F} \in \mathcal{S}\mathrm{hv}_{\text{ét}}(\mathcal{X},\Lambda)$ and a map $x\colon T \to \mathcal{X}$ we will sometimes denote the canonical projection $x^{-1}\mathcal{F} \in \mathcal{S}\mathrm{hv}_{\text{ét}}(T,\Lambda)$ by $\mathcal{F}_{|T}$.

Similarly to quasi-coherent sheaves, we can extend this to a functor $\mathcal{S}\mathrm{hv}_{\text{ét}}(-,\Lambda)^{-1}\colon \mathscr{P}\mathscr{S}\mathrm{tk}_R^{\mathrm{op}} \to \mathscr{P}r^{\mathrm{L}}_{\Lambda}$ as the right Kan extension of the functor $T \mapsto \mathcal{S}\mathrm{hv}_{\text{ét}}(T,\Lambda)$ along the inclusion $\mathrm{Aff}_R \subset \mathscr{P}\mathscr{S}\mathrm{tk}_R$.

The functor $\mathcal{S}\mathrm{hv}_{\text{ét}}(-,\Lambda)$ enjoys the following properties:

- By the adjoint functor theorem there is a functor

$$\mathcal{S}\mathrm{hv}_{\text{ét}}(-,\Lambda)_*\colon \mathscr{P}\mathscr{S}\mathrm{tk}_R \to \mathrm{Pr}$$

  agreeing with $\mathcal{S}\mathrm{hv}_{\text{ét}}(-,\Lambda)^{-1}$ on objects and such that for each morphism $f\colon \mathcal{X} \to \mathcal{Y}$ the functor $f_*\colon \mathcal{S}\mathrm{hv}_{\text{ét}}(\mathcal{X}) \to \mathcal{S}\mathrm{hv}_{\text{ét}}(\mathcal{Y})$ is the right adjoint of $f^{-1}\colon \mathrm{QCoh}(\mathcal{Y}) \to \mathrm{QCoh}(\mathcal{X})$.

- Let $\mathcal{X}_\bullet \colon I \to \mathscr{P}\mathscr{S}\mathrm{tk}_R$ be a diagram of prestacks. Then the natural map

$$\mathcal{S}\mathrm{hv}_{\text{ét}}(\operatorname*{colim}_I \mathcal{X}_i) \longrightarrow \lim_I \mathcal{S}\mathrm{hv}_{\text{ét}}(\mathcal{X}_i)$$

  is an equivalence.

- Since the functor $\mathcal{S}\mathrm{hv}_{\text{ét}}(-,\Lambda)^{-1}_{|\mathrm{Aff}_R^{\mathrm{op}}} \colon \mathrm{Aff}_R^{\mathrm{op}} \to \mathscr{P}r^{\mathrm{L}}_\Lambda$ is (essentially by definition) a hypersheaf in the étale topology, so is the functor $\mathcal{S}\mathrm{hv}_{\text{ét}}(-,\Lambda)$. In particular if $\varepsilon \colon \mathcal{X} \to L_{\text{ét}}\mathcal{X}$ is the hypersheafification map, the induced functor ε^{-1} is an equivalence.

Combining the properties above, we deduce the following descent statement for $\mathcal{S}\mathrm{hv}_{\text{ét}}(-,\Lambda)$:

Proposition 3.3.4.2. *Let $\mathcal{U} \to \mathcal{X}$ be an étale surjection of prestacks over R with the corresponding Čech object $\mathcal{U}_\bullet$, then the natural map*

$$\mathcal{S}\mathrm{hv}_{\text{ét}}(\mathcal{X},\Lambda) \longrightarrow \mathrm{Tot}\, \mathcal{S}\mathrm{hv}_{\text{ét}}(\mathcal{U}_\bullet,\Lambda)$$

is an equivalence.

Proof. Since the natural map $|\mathcal{U}_\bullet| \to \mathcal{X}$ induces equivalence after sheafification, we have

$$\mathcal{S}\mathrm{hv}_{\text{ét}}(\mathcal{X},\Lambda) \simeq \mathcal{S}\mathrm{hv}_{\text{ét}}(L_{\text{ét}}\mathcal{X},\Lambda) \simeq \mathcal{S}\mathrm{hv}_{\text{ét}}(L_{\text{ét}}|\mathcal{U}_\bullet|,\Lambda) \simeq \mathcal{S}\mathrm{hv}_{\text{ét}}(|\mathcal{U}_\bullet|,\Lambda) \simeq \mathrm{Tot}\, \mathcal{S}\mathrm{hv}_{\text{ét}}(\mathcal{U}_\bullet,\Lambda). \quad \square$$

Remark 3.3.4.3. Let $p_\bullet \colon \mathcal{U}_\bullet \to \mathcal{X}$ be as above. The inverse functor to the equivalence above is equal to the adjoint of the latter and thus sends a compatible system of sheaves $\mathcal{F}^\bullet \in \mathcal{S}\mathrm{hv}_{\text{ét}}(\mathcal{U}_\bullet,\Lambda)$ to the limit $\mathrm{Tot}\, p_{\bullet *}(\mathcal{F}^\bullet)$.

Here are some applications of Proposition 3.3.4.2:

Example 3.3.4.4 (Smooth descent of étale sheaves on schemes)**.** Let $p \colon U \to X$ be a smooth surjective (in the usual sense) map of schemes. Since a smooth morphism admits a section étale locally, it follows that p is a surjection in the étale topology. By discussion above we then have

$$\mathcal{S}\mathrm{hv}_{\text{ét}}(X,\Lambda) \xrightarrow{\sim} \mathrm{Tot}\, \mathcal{S}\mathrm{hv}_{\text{ét}}(U_\bullet,\Lambda)$$

where $U_\bullet$ is the Čech nerve of p.

Example 3.3.4.5. More generally, let $p\colon \mathcal{U} \to \mathcal{X}$ be a surjective (in étale topology), e.g. smooth, schematic map of Artin stacks and let $\mathcal{U}_\bullet$ be the corresponding Čech nerve. Then the natural map

$$\mathcal{S}\mathrm{hv}_{\text{ét}}(\mathcal{X}, \Lambda) \longrightarrow \mathrm{Tot}\, \mathcal{S}\mathrm{hv}_{\text{ét}}(\mathcal{U}_\bullet, \Lambda)$$

is an equivalence. In particular the pullback functor p^{-1} is conservative.

Similarly to what we had for quasi-coherent sheaves, by Lemma 3.3.2.17 the category $\mathcal{S}\mathrm{hv}_{\text{ét}}(-, \Lambda)$ has a natural t-structure. Since all translation maps in the diagram defining $\mathcal{S}\mathrm{hv}_{\text{ét}}(\mathcal{X}, \Lambda)$ are t-exact, the t-structure we obtain is particularly nice.

Proposition 3.3.4.6. *We have:*

1. *Let $\mathcal{X}$ be a pre-stack. Then the category $\mathcal{S}\mathrm{hv}_{\text{ét}}(\mathcal{X}, \Lambda)$ admits a t-structure such that $\mathcal{F} \in \mathcal{S}\mathrm{hv}_{\text{ét}}(\mathcal{X}, \Lambda)^{\leq 0}$ if and only if for each map $S \in \mathrm{Aff}_{R/\mathcal{X}}$ the restriction $\mathcal{F}_{|S}$ lands to $\mathcal{S}\mathrm{hv}_{\text{ét}}(S, \Lambda)^{\leq 0}$.*
2. *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a morphism of pre-stacks. Then the pullback functors f^{-1} is t-exact and f_* being right adjoint is automatically left t-exact.*

Proof. This follows from t-exactness of $f^{-1}\colon \mathcal{S}\mathrm{hv}_{\text{ét}}(T_2, \Lambda) \to \mathcal{S}\mathrm{hv}_{\text{ét}}(T_1, \Lambda)$ for any morphisms of schemes $T_1 \to T_2$, the definition of $\mathcal{S}\mathrm{hv}_{\text{ét}}(\mathcal{X}, \Lambda)$ and Lemma 3.3.2.12. □

Proposition 3.3.4.7. *Let $\mathcal{X}$ be a prestack over R. Then the category $\mathcal{S}\mathrm{hv}_{\text{ét}}(\mathcal{X}, \Lambda)$ is right t-complete.*

Proof. The t-structure on the category of sheaves $\mathcal{S}\mathrm{hv}_{\text{ét}}(T, \Lambda)$ is such that $\mathcal{S}\mathrm{hv}_{\text{ét}}(T, \Lambda)^{\geq 0}$ is stable under filtered colimits (since totalization preserves $D(\mathrm{Mod}_A)^{\geq 0}$). Thus $\mathcal{S}\mathrm{hv}_{\text{ét}}(T, \Lambda)$ is right t-complete by the dual statement to [83, Proposition 1.2.1.19.]. The rest follows form passing to the limit by 3.3.2.17. □

We now pass to a discussion of the pushforward f_*. Though formal nonsense allowed us to define it, we do not yet know how to compute it. In the case of schemes we have the following result in this direction:

Theorem 3.3.4.8 (Smooth base change)**.** *Let*

$$\begin{array}{ccc} X_2 & \xrightarrow{q} & X_1 \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ Y_2 & \xrightarrow{p} & Y_1 \end{array}$$

be a pullback square of schemes such that p is smooth and f is quasi-compact and quasi-separated. Then the natural map of functors $p^{-1} \circ f_* \to g_* \circ q^{-1}$ renders the diagram below commutative

$$\begin{array}{ccc} \mathcal{S}\mathrm{hv}_{\text{ét}}^{>-\infty}(X_2, \Lambda) & \xleftarrow{q^{-1}} & \mathcal{S}\mathrm{hv}_{\text{ét}}^{>-\infty}(X_1, \Lambda) \\ \downarrow{\scriptstyle g_*} & & \downarrow{\scriptstyle f_*} \\ \mathcal{S}\mathrm{hv}_{\text{ét}}^{>-\infty}(Y_2, \Lambda) & \xleftarrow{p^{-1}} & \mathcal{S}\mathrm{hv}_{\text{ét}}^{>-\infty}(Y_1, \Lambda). \end{array}$$

The rest of this section will be devoted to the proof of the analogous statement for Artin stacks. As usual we will more or less formally reduce the statement to the case of schemes, details are provided below.

Recall the following notion [83, Section 4.7.4]:

Definition 3.3.4.9. A commutative diagram of categories

$$\begin{array}{ccc} \mathscr{C}_2 & \xleftarrow{q} & \mathscr{C}_1 \\ \uparrow{\scriptstyle F_2} & & \uparrow{\scriptstyle F_1} \\ \mathscr{D}_2 & \xleftarrow{p} & \mathscr{D}_1 \end{array}$$

is called *right adjointable* if both F_1, F_2 admit right adjoints G_1, G_2 and the natural map

$$p \circ G_1 \longrightarrow G_2 \circ F_2 \circ p \circ G_1 \simeq G_2 \circ q \circ F_1 \circ G_1 \longrightarrow G_2 \circ q$$

is an equivalence.

Lemma 3.3.4.10 ([83, Proposition 4.7.4.19])**.** *Let $\mathscr{C}_\bullet, \mathscr{D}_\bullet \colon I \to \mathrm{Pr}^L$ be a pair of diagrams of presentable categories and continuous functors indexed by I. Let $F_\bullet \colon \mathscr{D}_\bullet \to \mathscr{C}_\bullet$ be a natural transformation such that for each morphism $i \to j$ in I the square*

$$\begin{array}{ccc} \mathscr{C}_j & \longleftarrow & \mathscr{C}_i \\ \uparrow{\scriptstyle F_j} & & \uparrow{\scriptstyle F_i} \\ \mathscr{D}_j & \longleftarrow & \mathscr{D}_i \end{array}$$

is right adjointable. Then for each $i \in I$ the square

$$\begin{array}{ccc} \mathscr{C}_i & \longleftarrow & \lim \mathscr{C}_i \\ \uparrow{\scriptstyle F_i} & & \uparrow{\scriptstyle \lim F_i} \\ \mathscr{D}_i & \longleftarrow & \lim \mathscr{D}_i \end{array}$$

is also right adjointable.

The following lemma is formal:

Lemma 3.3.4.11. *Let*

$$\begin{array}{ccccc} \mathscr{C}_3 & \longleftarrow & \mathscr{C}_2 & \longleftarrow & \mathscr{C}_1 \\ \uparrow & & \uparrow & & \uparrow \\ \mathscr{D}_3 & \xleftarrow{\pi} & \mathscr{D}_2 & \longleftarrow & \mathscr{D}_1 \end{array}$$

be a commutative diagram of categories. Then

- *If left and right hand squares are right adjointable, then the big square is also right adjointable.*
- *If big and left hand squares are right adjointable and the functor π is conservative, then the right hand square is also right adjointable.*

Finally we will need the following auxiliary result about interaction of totalizations of cosimplicial and pullbacks

Proposition 3.3.4.12. *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a map between R-prestacks. Then the functor $f^{-1}\colon \mathcal{S}\mathrm{hv}_{\text{ét}}(\mathcal{Y}, \Lambda) \to \mathcal{S}\mathrm{hv}_{\text{ét}}(\mathcal{X}, \Lambda)$ preserves totalizations of uniformly bounded below sheaves.*

Proof. Follows easily from Corollary A.1.1.7 and the fact that the t-structure on $\mathcal{S}\mathrm{hv}_{\text{ét}}(\mathcal{X}, \Lambda)$ is right t-complete by Proposition 3.3.4.7. □

We are now ready to prove

Proposition 3.3.4.13 (Smooth base change for Artin stacks)**.** *Let*

$$\begin{array}{ccc} \mathcal{X}_2 & \xrightarrow{q} & \mathcal{X}_1 \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ \mathcal{Y}_2 & \xrightarrow{p} & \mathcal{Y}_1 \end{array}$$

be a pullback square of Artin stacks such that f is a quasi-compact quasi-separated (see Definitions 3.3.1.17 and 3.3.1.18) and p is a smooth morphism. Then the following diagram is right adjointable

$$\begin{array}{ccc} \mathcal{S}\mathrm{hv}_{\text{ét}}^{>-\infty}(\mathcal{X}_2,\Lambda) & \xleftarrow{q^{-1}} & \mathcal{S}\mathrm{hv}_{\text{ét}}^{>-\infty}(\mathcal{X}_1,\Lambda) \\ \uparrow{\scriptstyle g^{-1}} & & \uparrow{\scriptstyle f^{-1}} \\ \mathcal{S}\mathrm{hv}_{\text{ét}}^{>-\infty}(\mathcal{Y}_2,\Lambda) & \xleftarrow{p^{-1}} & \mathcal{S}\mathrm{hv}_{\text{ét}}^{>-\infty}(\mathcal{Y}_1,\Lambda). \end{array}$$

Proof. Let n be such that all $\mathcal{X}_i, \mathcal{Y}_i$ are at most n-Artin stacks and assume the statement is already proved for $(n-1)$-Artin stacks (the case $n = 0$ being Theorem 3.3.4.8). We will prove the proposition by induction on n and a series of reductions.

Step 1. First note that without loss of generality we can assume that $\mathcal{Y}_2$ is a disjoint union of affines. Indeed, let $\pi\colon U \twoheadrightarrow \mathcal{Y}_2$ be a smooth atlas. Then by assumption the big and left hand squares in the diagram below are right adjointable

$$\begin{array}{ccccc} \mathcal{S}\mathrm{hv}_{\text{ét}}^{>-\infty}(U\times_{\mathcal{Y}_2}\mathcal{X}_2,\Lambda) & \longleftarrow & \mathcal{S}\mathrm{hv}_{\text{ét}}^{>-\infty}(\mathcal{X}_2,\Lambda) & \xleftarrow{q^{-1}} & \mathcal{S}\mathrm{hv}_{\text{ét}}^{>-\infty}(\mathcal{X}_1,\Lambda) \\ \uparrow & & \uparrow{\scriptstyle g^{-1}} & & \uparrow{\scriptstyle f^{-1}} \\ \mathcal{S}\mathrm{hv}_{\text{ét}}^{>-\infty}(U,\Lambda) & \xleftarrow{\pi^{-1}} & \mathcal{S}\mathrm{hv}_{\text{ét}}^{>-\infty}(\mathcal{Y}_2,\Lambda) & \xleftarrow{p^{-1}} & \mathcal{S}\mathrm{hv}_{\text{ét}}^{>-\infty}(\mathcal{Y}_1,\Lambda) \end{array}$$

Since the map π^{-1} is conservative, by the Lemma 3.3.4.11 the right hand square is also right adjointable.

Step 2. Assume $\mathcal{Y}_1 = Y_1$ is a union of affines. By the first step we can assume $\mathcal{Y}_2 = Y_2$ is also a union of affines. Let $U \twoheadrightarrow \mathcal{X}_1$ be an atlas and set $V := \mathcal{X}_2 \times_{\mathcal{X}_1} U$. Let $U_\bullet, V_\bullet$ be the corresponding Čech nerves and $f_i\colon U_i \to \mathcal{X}_1 \xrightarrow{f} Y_1$, $g_i\colon V_i \to \mathcal{X}_2 \xrightarrow{g} Y_2$ and $q_i\colon V_i \to U_i$ be the canonical maps

$$\begin{array}{ccccc} V_i & \xrightarrow{q_i} & U_i & & \\ \downarrow{\scriptstyle g_i} & & \downarrow{\scriptstyle f_i} & & \\ \mathcal{X}_2 & \xrightarrow{q} & \mathcal{X}_1 & & \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} & & \\ Y_2 & \xrightarrow{p} & Y_1 & & \end{array}$$

(in particular $q_{-1} = q$). By descent

$$\mathcal{S}\mathrm{hv}_{\text{ét}}^{>-\infty}(\mathcal{X}_1,\Lambda) \simeq \mathrm{Tot}\,\mathcal{S}\mathrm{hv}_{\text{ét}}^{>-\infty}(U_\bullet,\Lambda) \qquad \mathcal{S}\mathrm{hv}_{\text{ét}}^{>-\infty}(\mathcal{X}_2,\Lambda) \simeq \mathrm{Tot}\,\mathcal{S}\mathrm{hv}_{\text{ét}}^{>-\infty}(V_\bullet,\Lambda)$$

and $q^{-1} \simeq \mathrm{Tot}(q_i^{-1})$. Let $\mathcal{F}$ be an object of $\mathcal{S}\mathrm{hv}_{\text{ét}}^{\geq -\infty}(\mathcal{X}_1, \Lambda)$. Then by descent and the fact that f_* and g_* (being right adjoint) commute with limits, we have

$$p^{-1} f_* \mathcal{F} \simeq p^{-1} \mathrm{Tot}(f_{\bullet *} \mathcal{F}^\bullet) \quad \text{and} \quad g_*(q^{-1}\mathcal{F}) \simeq \mathrm{Tot}(g_{\bullet *}(q^{-1}\mathcal{F})^\bullet) \simeq \mathrm{Tot}(g_{\bullet *} q_\bullet^{-1} \mathcal{F}^\bullet)$$

Now since $\mathcal{X}_1$ is n-Artin stack, all elements in the Čech nerves $U_\bullet, V_\bullet$ are $(n-1)$-Artin. It follows by induction

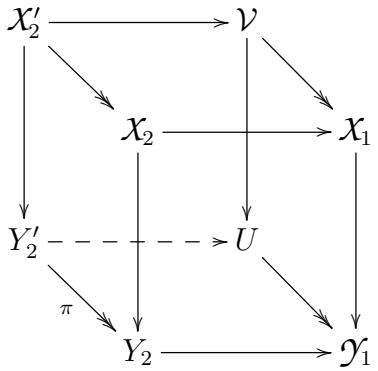
$$\mathrm{Tot}(g_{\bullet *} q_\bullet^{-1} \mathcal{F}^\bullet) \simeq \mathrm{Tot}(p^{-1} f_{\bullet *} \mathcal{F}^\bullet)$$

Hence it is enough to prove that the natural map

$$p^{-1} \mathrm{Tot}(f_{\bullet *} \mathcal{F}^\bullet) \to \mathrm{Tot}(p^{-1} f_{\bullet *} \mathcal{F}^\bullet)$$

is an equivalence. But since $\mathcal{F}$ is bounded from below the same is true for all $f_{\bullet *}\mathcal{F}^\bullet$, hence we conclude by Proposition 3.3.4.12.

Step 3. We claim that it is enough to prove the statement for $Y_2 \to \mathcal{Y}_1$ being surjective (i.e. an atlas). Indeed, let $U \twoheadrightarrow \mathcal{Y}_1$ be an atlas. By definition of surjection there exists an étale cover $\pi \colon Y_2' \twoheadrightarrow Y_2$ such that the composite map $Y_2' \twoheadrightarrow Y_2 \to \mathcal{Y}_2$ factors through U. Consider the following diagram where all vertical squares are pullbacks



We know base change for the left and right back vertical faces by assumption and for the back vertical face by the previous step. It follows big and left hand squares in the diagram below

$$\begin{array}{ccccc}
\mathcal{S}\mathrm{hv}_{\text{ét}}^{\geq -\infty}(\mathcal{X}_2', \Lambda) & \longleftarrow & \mathcal{S}\mathrm{hv}_{\text{ét}}^{\geq -\infty}(\mathcal{X}_2, \Lambda) & \xleftarrow{q^{-1}} & \mathcal{S}\mathrm{hv}_{\text{ét}}^{\geq -\infty}(\mathcal{X}_1, \Lambda) \\
\uparrow & & \uparrow{\scriptstyle g^{-1}} & & \uparrow{\scriptstyle f^{-1}} \\
\mathcal{S}\mathrm{hv}_{\text{ét}}^{\geq -\infty}(Y_2', \Lambda) & \xleftarrow{\pi^{-1}} & \mathcal{S}\mathrm{hv}_{\text{ét}}^{\geq -\infty}(Y_2, \Lambda) & \xleftarrow{p^{-1}} & \mathcal{S}\mathrm{hv}_{\text{ét}}^{\geq -\infty}(\mathcal{Y}_1, \Lambda)
\end{array}$$

are right adjointable. We conclude by Lemma 3.3.4.11 again.

Step 4. By the first and third steps we can assume $\mathcal{Y}_2 = Y_2$ is a union of affines and the map $Y_2 \to \mathcal{Y}_1$ is surjective. Let $\mathcal{Y}_1$ be k-Artin. We will finish the proof by a second induction on k, the base of induction $k = 0$ is precisely the second step.

Set $U := Y_2$, $V := \mathcal{X}_2$ and let us denote by $U_\bullet$ and $V_\bullet$ Čech nerves of $U \twoheadrightarrow \mathcal{Y}_1$ and $V \twoheadrightarrow \mathcal{X}_1$ respectively. Also let us denote by $p_i \colon U_i \to \mathcal{Y}_1$, $q_i \colon V_i \to \mathcal{X}_1$ and $g_i \colon V_i \to U_i$ the canonical maps

$$\begin{array}{ccccc} & & \overset{q_i}{\frown} & & \\ V_i & \longrightarrow & V_0 = \mathcal{X}_2 & \underset{q_0=q}{\rightrightarrows} & \mathcal{X}_1 \\ {\scriptstyle g_i}\downarrow & & {\scriptstyle g_0=g}\downarrow & & \downarrow{\scriptstyle f} \\ U_i & \longrightarrow & U_0 = Y_2 & \underset{p_i}{\overset{p_0=p}{\rightrightarrows}} & \mathcal{Y}_1 \end{array}$$

By descent

$$\mathcal{S}\mathrm{hv}^{>-\infty}(\mathcal{Y}_1, \Lambda) \simeq \mathrm{Tot}\, \mathcal{S}\mathrm{hv}^{>-\infty}(U_\bullet, \Lambda) \qquad \mathcal{S}\mathrm{hv}^{>-\infty}(\mathcal{X}_1, \Lambda) \simeq \mathrm{Tot}\, \mathcal{S}\mathrm{hv}^{>-\infty}(V_\bullet, \Lambda)$$

and f^{-1} is equivalent to $\lim g_i^{-1}$. Now since the pullback functor preserves fibered products, we have $V_\bullet \simeq U_\bullet \times_{Y_2} \mathcal{X}_1$ and in particular all squares

$$\begin{array}{ccc} V_{i+1} & \longrightarrow & V_i \\ {\scriptstyle g_{i+1}}\downarrow & & \downarrow{\scriptstyle g_i} \\ U_{i+1} & \longrightarrow & U_i \end{array}$$

are pullbacks. Note that since $\mathcal{Y}_1$ is k-Artin all terms U_i are at most $(k-1)$-Artin stacks. Hence by induction all squares

$$\begin{array}{ccc} \mathcal{S}\mathrm{hv}^{>-\infty}_{\text{ét}}(V_{i+1}, \Lambda) & \longleftarrow & \mathcal{S}\mathrm{hv}^{>-\infty}_{\text{ét}}(V_i, \Lambda) \\ \uparrow{\scriptstyle g_{i+1}^{-1}} & & \uparrow{\scriptstyle g_i^{-1}} \\ \mathcal{S}\mathrm{hv}^{>-\infty}_{\text{ét}}(U_{i+1}, \Lambda) & \longleftarrow & \mathcal{S}\mathrm{hv}^{>-\infty}_{\text{ét}}(U_i, \Lambda) \end{array}$$

are right adjointable. Since $f_* \simeq \lim g_{i*}$ by Lemma 3.3.4.10 we have that

$$p^{-1} \circ f_* \simeq p_0^{-1} \circ \lim g_i \simeq (f_0)_* \circ q_0^{-1} = g_* \circ q^{-1}. \quad \square$$

Local systems

Let's consider a map of schemes $f\colon X \to Y$ and an etale sheaf $\mathcal{F}$ on X, and let's say we are interested in computing the global sections $R\Gamma_{\text{ét}}(X, \mathcal{F})$. Then it be can helpful to look at $R\Gamma_{\text{ét}}(Y, f_*\mathcal{F})$ instead. A particularly nice situation is if $\mathcal{F}$ is a local system and f is smooth and proper: then the individual cohomology sheaves $R^i f_*\mathcal{F}$ are also local systems and we get a filtration on $R\Gamma_{\text{ét}}(X, \mathcal{F})$ with the associated graded given by $\oplus_i R\Gamma_{\text{ét}}(Y, R^i f_* F)[-i]$. If, moreover, Y was simply-connected then all terms in the latter sum will be given by the global sections of the constant sheaf with some multiplicities. We would like to generalize this picture to the context of Artin stacks.

Let R be the base ring and let Λ be a *field* such that $\mathrm{char}(\Lambda)$ is invertible on R. The goal of this section is to define the category $\mathscr{L}_{\text{ét}}(\mathcal{X}, \Lambda)$ of étale Λ-local systems on an Artin stack $\mathcal{X}$ over R with the following properties:

- For $\mathcal{X}$ being an ordinary scheme the category $\mathscr{L}_{\text{ét}}(\mathcal{X}, \Lambda)$ coincides with the derived variant of étale local systems on $\mathcal{X}$ (see Definition 3.3.4.14).
- The category $\mathscr{L}_{\text{ét}}(\mathcal{X}, \Lambda)$ admits a natural t-structure.
- For any map $f\colon \mathcal{X} \to \mathcal{Y}$ there is a well defined t-exact functor $f^{-1}\colon \mathscr{L}_{\text{ét}}(\mathcal{Y}, \Lambda) \to \mathscr{L}_{\text{ét}}(\mathcal{X}, \Lambda)$.
- Most importanly, if f is smooth and proper, f^{-1} admits a right adjoint f_* (which is automatically right t-exact).

In particular, for a smooth proper map $f\colon \mathcal{X} \to \mathcal{Y}$ we want the complex $f_*\mathcal{F} \in \mathscr{L}_{\text{ét}}(\mathcal{Y}, \Lambda)^{>-\infty}$ (for $\mathcal{F} \in \mathscr{L}_{\text{ét}}(\mathcal{X}, \Lambda)^{>-\infty}$) to admit an exhaustive filtration with the associated graded given by shifts of objects in the heart $\mathscr{L}_{\text{ét}}(\mathcal{Y}, \Lambda)^{\heartsuit}$.

We proceed to the construction. First, for a scheme X we put:

Definition 3.3.4.14. The category $\mathscr{L}_{\text{ét}}(X, \Lambda)$ of étale Λ-local systems on X is defined as the full subcategory of $\mathcal{S}\mathrm{hv}_{\text{ét}}(X, \Lambda)$ on (complexes of) sheaves $\mathcal{L}$ such that each cohomology sheaf $\mathcal{H}^i(\mathcal{L})$ is an ordinary étale local system (i.e. an étale sheaf locally isomorphic to $\underline{\Lambda}^{\oplus k}$ for some k).

Note that $\mathscr{L}_{\text{ét}}(X, \Lambda) \subset \mathcal{S}\mathrm{hv}_{\text{ét}}(X, \Lambda)$ is stable only under *finite* (co)limits. Also, $f^{-1}\colon \mathscr{L}_{\text{ét}}(X_2, \Lambda) \to \mathscr{L}_{\text{ét}}(X_1, \Lambda)$ does not have a right adjoint unless $f\colon X_1 \to X_2$ is smooth and proper.

Using results of the previous subsection it is straightforward to extend this definition to Artin stacks.

Definition 3.3.4.15. Let $\mathcal{X}$ be an Artin stack. The subcategory $\mathscr{L}_{\text{ét}}(X,\Lambda) \subset \mathcal{S}\text{hv}_{\text{ét}}(\mathcal{X},\Lambda)$ of *étale* Λ*-local systems* is defined as the full subcategory spanned by objects $\mathcal{L} \in \mathcal{S}\text{hv}_{\text{ét}}(\mathcal{X},\Lambda)$ with the property that for any map $p\colon T \to \mathcal{X}$ from an affine scheme T, the restriction $p^{-1}(\mathcal{L})$ is an étale Λ-local system on X.

The category of local systems $\mathscr{L}_{\text{ét}}(\mathcal{X},\Lambda)$ receives a natural t-structure defined uniquely by

$$\mathscr{L}_{\text{ét}}(\mathcal{X},\Lambda)^{\geq 0} \coloneqq \mathcal{S}\text{hv}_{\text{ét}}(\mathcal{X},\Lambda)^{\geq 0} \cap \mathscr{L}_{\text{ét}}(\mathcal{X},\Lambda).$$

One can characterize $\mathscr{L}_{\text{ét}}(\mathcal{X},\Lambda)$ and $\mathscr{L}_{\text{ét}}(\mathcal{X},\Lambda)^{\geq 0}$ in terms of a surjective cover:

Lemma 3.3.4.16. *Let $p\colon \mathcal{U} \to \mathcal{X}$ be a surjective morphism. Then*

1. *$\mathcal{F} \in \mathcal{S}\text{hv}_{\text{ét}}(\mathcal{X},\Lambda)$ is a local system if and only if $p^{-1}(\mathcal{F})$ is.*
2. *Étale local system $\mathcal{L}$ lies in $\mathscr{L}_{\text{ét}}(\mathcal{X},\Lambda)^{\geq 0}$ (resp. $\mathscr{L}_{\text{ét}}(\mathcal{X},\Lambda)^{\leq 0}$) if and only if $p^{-1}(\mathcal{L}) \in \mathscr{L}_{\text{ét}}(\mathcal{U},\Lambda)^{\geq 0}$ (resp. $p^{-1}(\mathcal{L}) \in \mathscr{L}_{\text{ét}}(\mathcal{U},\Lambda)^{\leq 0}$).*

Proof. We will only prove the first part, the proof of the second statement being similar. Let $X \to \mathcal{X}$ be a map, where X is a disjoint union of affines. Then, since $\mathcal{U} \twoheadrightarrow \mathcal{X}$ is surjective, for some étale cover $V \to X$ there exists a dashed arrow making the diagram below commutative

$$\begin{array}{ccc} V & \dashrightarrow & \mathcal{U} \\ \downarrow & & \downarrow \\ X & \longrightarrow & \mathcal{X} \end{array}$$

Then $\mathcal{F}_{|V}$ is a local system because it is a local system already restricted on U. On the other hand since the property of being a local system on a scheme can be checked étale locally, we conclude that $\mathcal{F}_{|X}$ is a local system. □

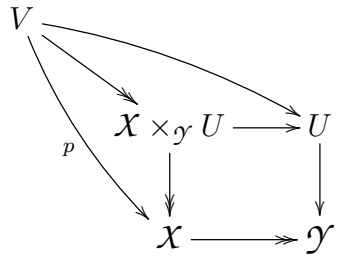
This lemma is enough to show that f^{-1} on étale sheaves sends local systems to local systems. It is also enough to show that it admits an adjoint f_* provided f is smooth and proper. Recall the definition (3.3.2.32) of a proper morphism between Artin stacks

Corollary 3.3.4.17. *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a morphism of Artin stacks. Then:*

1. *The pullback functor $f^{-1}\colon \mathcal{S}\text{hv}_{\text{ét}}(\mathcal{Y},\Lambda) \to \mathcal{S}\text{hv}_{\text{ét}}(\mathcal{X},\Lambda)$ restricts to a t-exact functor $f^{-1}\colon \mathcal{L}_{\text{ét}}(\mathcal{Y},\Lambda) \to \mathcal{L}_{\text{ét}}(\mathcal{X},\Lambda)$.*

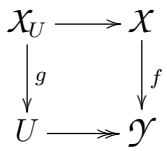
2. *If f is smooth and proper, the functor $f_*\colon \mathcal{L}_{\text{ét}}^{\geq -\infty}(\mathcal{X},\Lambda) \to \mathcal{L}_{\text{ét}}^{\geq -\infty}(\mathcal{Y},\Lambda)$ restricts to a right adjoint of f^{-1}.*

Proof. 1. It is enough to prove that $f^{-1}\colon \mathcal{S}\mathrm{hv}_{\text{ét}}(\mathcal{Y},\Lambda) \to \mathcal{S}\mathrm{hv}_{\text{ét}}(\mathcal{X},\Lambda)$ restricts to a functor between the full subcategories of local systems. To this end let $U \twoheadrightarrow \mathcal{Y}$ be a smooth atlas and consider the following commutative diagram



where $V \twoheadrightarrow \mathcal{X} \times_{\mathcal{Y}} U$ is a smooth cover by a union of affine schemes, which exists since a fibered product of Artin stacks is again an Artin stack. Now for any $\mathcal{L} \in \mathscr{L}_{\text{ét}}(\mathcal{Y},\Lambda)$ we have that $\mathcal{L}_{|U}$ is a local system and hence $(f^{-1}\mathcal{L})_{|V} \simeq p^{-1}\mathcal{L}_{|U}$ is a local system as well. We conclude by Lemma 3.3.4.16.

2. Again it is enough to prove that f_* preserves the full subcategory of bounded below local systems. Let k be such that $\mathcal{X},\mathcal{Y}$ are at most k-Artin. We will prove the statement by induction on k, the base of induction $k=0$ being the theorem of Deligne. Assume the statement is proved for $k' < k$. Let us take a smooth atlas $U \twoheadrightarrow \mathcal{Y}$. By Lemma 3.3.4.16 it is enough to prove that the restriction $(f_*\mathcal{L})_{|U}$ is a local system. Consider the fiber square



By the base change we have $(f_*\mathcal{L})_{|U} \simeq g_*(\mathcal{L}_{|V})$. By the equivalence above it is enough to prove $g_*\mathcal{M}$ is a local system for $\mathcal{M} := \mathcal{L}_{|V}$. Let $V \twoheadrightarrow \mathcal{X}_U$ be an atlas, $V_\bullet$ be the corresponding Čech nerve, and $g_i\colon V_i \to \mathcal{X}_U \xrightarrow{g} U$ be the canonical morphisms (note that all of V_i are at most $(k-1)$-Artin stacks). Then by descent we have $g_*(\mathcal{M}) \simeq \mathrm{Tot}(g_{\bullet *}\mathcal{M}^\bullet)$. By induction assumption all $g_{i*}\mathcal{M}^i$ are local systems. By shifting if necessary we can assume $\mathcal{M} \in \mathscr{L}_{\text{ét}}(\mathcal{X}_U,\Lambda)^{\geq 0}$ and thus so are all the terms in the cosimplicial diagram $g_{\bullet *}(\mathcal{M}^\bullet)$. Hence by Proposition 3.3.4.12 we have

$$\mathcal{H}^n(g_*\mathcal{M}) \simeq \mathcal{H}^n(\mathrm{Tot}^{\leq n}(g_{\bullet *}\mathcal{M}^\bullet))$$

. But since $\mathrm{Tot}^{\leq n}$ is a finite limit all $\mathrm{Tot}^{\leq n}(g_{\bullet *}\mathcal{M}^{\bullet})$ are local systems. It follows that all cohomology of $g_*\mathcal{M}$ are classical local systems, and so $g_*\mathcal{M}$ is a local system as desired. □

From Proposition 3.3.4.2 and Lemma 3.3.4.16 we also obtain the following corollary:

Corollary 3.3.4.18. *Let $\mathcal{U}_\bullet \twoheadrightarrow \mathcal{X}$ be an étale hypercover. Then the natural functor $\mathscr{L}_{\text{ét}}(\mathcal{X}, \Lambda) \to \mathrm{Tot}\, \mathscr{L}_{\text{ét}}(\mathcal{U}_\bullet, \Lambda)$ is an equivalence respecting t-structures on both sides.*

The following result describes the heart of the t-structure on $\mathcal{S}\mathrm{hv}_{\text{ét}}(-, \Lambda)$ in the simply-connected case. From now on we assume $R = C$ is an algebraically closed field.

Definition 3.3.4.19. A prestack $p\colon \mathcal{X} \to \mathrm{Spec}\, C$ is called *étale connected* if the natural map

$$\mathrm{Mod}_\Lambda^{\text{f.g.}} \simeq \mathscr{L}_{\text{ét}}^{\heartsuit}(\mathrm{Spec}\, C, \Lambda) \xrightarrow{p^{-1}} \mathscr{L}_{\text{ét}}(\mathcal{X}, \Lambda)^{\heartsuit}$$

is fully-faithful. A prestack $\mathcal{X}$ is called *étale simply connected* if the functor above is an equivalence.

Example 3.3.4.20. A scheme X over C is étale connected if and only if it is connected in Zariski topology and it is étale simply connected if and only if $\pi_1^{\text{ét}}(X, x) \simeq *$ for any (equivalently for all) geometric point x in X.

Proposition 3.3.4.21. *Let $\mathcal{U}_\bullet \to \mathcal{X}$ be an étale hypercover and let $p\colon \mathcal{X} \to \mathrm{Spec}\, C$ be a structure map.*

1. *If all terms of $\mathcal{U}_\bullet$ are étale connected, then the functor of a constant local system*

$$\mathrm{Mod}_\Lambda^{\heartsuit} \simeq \mathcal{S}\mathrm{hv}_{\text{ét}}^{\heartsuit}(\mathrm{Spec}\, C, \Lambda) \xrightarrow{p^{-1}} \mathscr{L}_{\text{ét}}(\mathcal{X}, \Lambda)^{\heartsuit}$$

   is fully-faithful, i.e. $\mathcal{X}$ is étale connected as well.

2. *If furthermore $\mathcal{U}_0$ is étale simply connected, then p^{-1} is an equivalence, i.e. $\mathcal{X}$ is étale simply connected.*

Proof. 1. Let $M_1, M_2 \in \mathcal{S}\mathrm{hv}_{\text{ét}}(\mathrm{Spec}\, R, \Lambda)$. Then by descent

$$\mathrm{Hom}_{\mathscr{L}_{\text{ét}}(\mathcal{X}, \Lambda)}(p^{-1}M_1, p^{-1}M_2) \simeq \mathrm{Tot}\, \mathrm{Hom}_{\mathscr{L}_{\text{ét}}(\mathcal{U}_\bullet, \Lambda)}(p_\bullet^{-1}M_1, p_\bullet^{-1}M_2).$$

But since all $\mathcal{U}_n$ are étale connected we have

$$\mathrm{Hom}_{\mathscr{L}_{\text{ét}}(\mathcal{U}_\bullet,\Lambda)}(p_n^{-1}M_1, p_n^{-1}M_2) \simeq \mathrm{Hom}_{\mathscr{L}_{\text{ét}}(\mathrm{Spec}\, R,\Lambda)}(M_1, M_2)$$

and all translation maps are necessary equivalencies. We conclude, since the projection from a limit over a co-simplicial diagram with all translation maps being equivalencies to any of its terms is an equivalence.

2. By the previous part we know that the pullback $p^{-1}\colon \mathrm{Mod}_\Lambda^\heartsuit \to \mathscr{L}(\mathcal{X},\Lambda)^\heartsuit$ is fully-faithful, hence it is enough to prove essential surjectivity. For this end let $\mathcal{L}$ be a local system on $\mathcal{X}$. Since $\mathcal{U}_0$ is simply connected the restriction $\mathcal{L}_{|\mathcal{U}_0}$ is constant, hence so are all $\mathcal{L}_{|\mathcal{U}_n}$. Now since all $\mathcal{U}_n$ are connected, for any pair of constant sheaves $\underline{M}_1, \underline{M}_2$ we have

$$\mathrm{Hom}_{\mathscr{L}_{\text{ét}}(\mathcal{U}_n,\Lambda)}(\underline{M}_1, \underline{M}_2) \simeq \mathrm{Hom}_\Lambda(M_1, M_2).$$

Hence the diagram $p_\bullet^{-1}\mathcal{L}$ is a diagram of constant sheaves. We conclude by descent. □

Example 3.3.4.22. Let G be a connected C-group scheme. Then the Čech nerve of $\mathrm{Spec}\, C \to BG$ satisfies conditions of the second part of the proposition above, hence BG is étale simply connected.

Example 3.3.4.23. Let A be a connected abelian C-group scheme. Then by induction all $B^nA, n \geq 1$ are étale simply connected.

Rigid analytic étale sheaves

For the rest of this subsection we fix the following notation:

Notation 3.3.4.24. Let K be a complete non-Archimedean field with the ring of integers $\mathcal{O}_K := \{f \in K \mid |f| \leq 1\}$. We fix a pseudo-uniformizer $\pi \in \mathcal{O}_K$, i.e. an element with the property $0 < |\pi| < 1$.

In this subsection for a K-rigid analytic stack $\mathcal{X}$ (see Definition 3.3.4.28 below) we construct the corresponding category of étale sheaves on $\mathcal{X}$ following a path similar to Section 3.3.4. The resulting category $\mathcal{S}\mathrm{hv}_{\text{ét}}(\mathcal{X},\Lambda)$ behaves very similarly to what we had in the algebraic setting. If $\mathcal{X} \simeq (\widehat{\mathcal{X}_{\mathcal{O}_K}})_K$ comes as the Raynaud generic fiber of an Artin $\mathcal{O}_K$-stack $\mathcal{X}_{\mathcal{O}_K}$, we also discuss the relationship of $\mathcal{S}\mathrm{hv}_{\text{ét}}(\mathcal{X},\Lambda)$ with the category $\mathcal{S}\mathrm{hv}_{\text{ét}}(\mathcal{X}_K,\Lambda)$ of étale sheaves on the algebraic generic fiber $\mathcal{X}_K := \mathcal{X}_{\mathcal{O}_K} \times_{\mathrm{Spec}\,\mathcal{O}_K} \mathrm{Spec}\, K$. First recall the following definition:

Definition 3.3.4.25. A *Tate algebra* $K\langle x_1, \dots, x_n\rangle$ over K is the subring of the ring of formal power series $K[[x_1, \dots, x_n]]$ consisting of those $f = \sum_I f_I x^I$ such that $\lim_{I\to\infty} |f_I| \to 0$. The ring $K\langle x_1, \dots, x_n\rangle$ has the natural *Gauss norm* given by $\|f\| \coloneqq \max_I |f_I|$, making it a Banach algebra. *An affinoid K-algebra* A by definition is a Banach algebra that is a quotient of some $K\langle x_1, \dots, x_n\rangle$ by a closed ideal I. A *morphism of affinoid algebras* is a homomorphism that is bounded.

The Tate algebra $K\langle x_1, \dots, x_n\rangle$ can be thought of as the algebra of analytic functions on the closed polydisk $\mathbb{D}_1^n \subset K^n$ and affinoid algebras are supposed to be rings of analytic functions on closed subvarieties in those. We denote the opposite category of affinoid K-algebras by Afd_K; this is the category of *affinoid spaces over* K. We refer to [9] for the foundations of rigid analytic geometry which is built up starting from this notion. We denote the category of rigid analytic spaces over K by Rig_K. We have an embedding $\mathrm{Afd}_K \to \mathrm{Rig}_K$, given by $A \mapsto \mathrm{Sp}(A)$. We put $\widehat{\mathbb{A}}_K^n \coloneqq \mathrm{Sp}\, K\langle x_1, \dots, x_n\rangle$.

One has a notion of an étale map and étale cover (the latter being a little subtle and using a concept of admissibility to ensure some quasicompactness properties) of affinoid spaces.

Definition 3.3.4.26 ([57, Definition 2.1.1])**.** A morphism of rigid spaces $f\colon X \to Y$ is called *étale* if for any point $x \in X$ the induced map of local rings $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is flat and unramified. An étale morphism $f\colon X \to Y$ is called *strongly surjective* if for any quasi-compact admissible open subset V of Y there exists a finite collection of admissible open subsets $U_1, \dots, U_n$ of X such that $V = f(\bigcup_i U_i)$. For a rigid analytic space we define a *rigid analytic étale site* $\text{ét}^{\mathrm{an}}_{/X}$ to be the category of rigid analytic spaces étale over X equipped with the Grothendieck topology generated by strongly surjective étale morphisms.

There is also a notion of a smooth map:

Definition 3.3.4.27. The map $f\colon X \to Y$ of affinoids is called *smooth* if for any point $x \in X$ there is a neighbourhood $V \subset X$ such that $f_{|V}$ can be factored as

$$\begin{array}{ccc} V & \xrightarrow{g} & Y \times \widehat{\mathbb{A}}_K^n \\ & \searrow & \downarrow p \\ & & Y \end{array}$$

with g étale and p the natural projection.

Obviously, étale maps are smooth and we get a well-defined geometric context ($\mathrm{Afd}_K^{\mathrm{op}}$, étale, smooth). Following [99] we introduce rigid analytic (pre)stacks as follows:

Definition 3.3.4.28. Let K be a complete non-Archimedean field. A category of *locally finite type K-analytic prestacks* $\mathscr{PS}\mathrm{tk}_K^{\mathrm{an,lft}}$ is defined as a category of presheaves on the category of affinoid K-analytic spaces Afd_K:

$$\mathscr{PS}\mathrm{tk}_K^{\mathrm{an,lft}} := \mathrm{Fun}(\mathrm{Afd}_K^{\mathrm{op}}, \mathscr{S}).$$

The category $Stk_K^{\mathrm{an,lft}}$ of *locally finite type K-rigid analytic Artin stacks* is defined as the full subcategory of $\mathscr{PS}\mathrm{tk}_K^{\mathrm{an,lft}}$ consisting of geometric stacks (see Definition 3.3.1.3) with respect to the context ($\mathrm{Afd}_K^{\mathrm{op}}$, étale, smooth).

As promised, now we define the category of étale sheaves on a rigid-analytic stack. Given an affinoid $T \in \mathrm{Afd}_K$, for a commutative ring Λ we will denote the corresponding $(\infty,1)$-sheaf category for $\text{ét}_{/T}^{\mathrm{an}}$ by $\mathcal{S}\mathrm{hv}_{\text{ét}}(T,\Lambda)$. Given a map $f\colon T_1 \to T_2$ we have the pull-back functor $f^{-1}\colon \mathcal{S}\mathrm{hv}_{\text{ét}}(T_2,\Lambda) \to \mathcal{S}\mathrm{hv}_{\text{ét}}(T_1,\Lambda)$.

Construction 3.3.4.29. Let Λ be a ring. We define the functor of rigid analytic étale sheaves as the right Kan extension of the functor

$$\mathcal{S}\mathrm{hv}_{\text{ét}}^{-1}(-,\Lambda)\colon \mathrm{Afd}_K^{\mathrm{op}} \longrightarrow \mathscr{P}r_\Lambda^{\mathrm{L}}$$

along the Yoneda embedding $\mathrm{Afd}_K^{\mathrm{op}} \hookrightarrow (\mathscr{PS}\mathrm{tk}_K^{\mathrm{an,lft}})^{\mathrm{op}}$.

The proof of the following properties is similar to that for the étale sheaves in the algebraic context (see Section 3.3.4):

Proposition 3.3.4.30. *In the notation above we have:*

1. *For any rigid analytic prestack $\mathcal{X}$ over K the category $\mathcal{S}\mathrm{hv}_{\text{ét}}(\mathcal{X},\Lambda)$ is presentable and Λ-linear.*

2. *The assignment $\mathcal{X} \mapsto \mathcal{S}\mathrm{hv}_{\text{ét}}(\mathcal{X},\Lambda)$ lifts to a functor*

$$\mathcal{S}\mathrm{hv}_{\text{ét}}(-,\Lambda)^{-1}\colon (\mathscr{PS}\mathrm{tk}_K^{\mathrm{an,lft}})^{\mathrm{op}} \longrightarrow \mathrm{Pr}_\Lambda^{\mathrm{L}}$$

and by adjoint functor theorem there is also a functor

$$\mathcal{S}\mathrm{hv}_{\text{ét}}(-,\Lambda)_*\colon\ \mathscr{P}\mathscr{S}\mathrm{tk}_K^{\mathrm{an,lft}}\ \longrightarrow\ \mathscr{P}r_\Lambda^{\mathrm{R}}.$$

For a morphism $f\colon \mathcal{X}\to\mathcal{Y}$ of prestacks we will denote the induced pullback and pushforward functors by f^{-1} and f_* respectively.

3. *The functor $\mathcal{S}\mathrm{hv}_{\text{ét}}(-,\Lambda)^{-1}$, being a right Kan extension, transforms colimits of prestacks into limits of categories.*

4. *Since the functor $\mathcal{S}\mathrm{hv}_{\text{ét}}(-,\Lambda)^{-1}_{|\mathrm{Afd}_K^{\mathrm{op}}}\colon \mathrm{Afd}_K^{\mathrm{op}}\to \mathrm{Pr}_\Lambda^{\mathrm{L}}$ is (by definition) a hypersheaf in the étale topology, so is the functor $\mathcal{S}\mathrm{hv}_{\text{ét}}(-,\Lambda)$. In particular if $\varepsilon\colon \mathcal{X}\to L_{\text{ét}}\mathcal{X}$ is the étale sheafification map, the induced functor ε^{-1} is an equivalence.*

5. *For each prestack $\mathcal{X}$ the category $\mathcal{S}\mathrm{hv}_{\text{ét}}(\mathcal{X},\Lambda)$ admits a natural t-structure determined by the property that a sheaf $\mathcal{F}$ belongs to $\mathcal{S}\mathrm{hv}_{\text{ét}}^{\geq 0}(\mathcal{X},\Lambda)$ if and only if its restriction $\mathcal{F}_S$ to any K-affinoid S mapping to X lands to $\mathcal{S}\mathrm{hv}_{\text{ét}}^{\geq 0}(\widehat{S},\Lambda)$.*

6. *For a morphism of stacks $f\colon \mathcal{X}\to\mathcal{Y}$ the pullback functor f^{-1} is t-exacts and the pushforward functor f_* is left t-exact.*

We now turn to studying the relation between algebraic and rigid analytic étale sheaves. First recall that to a finite type K-scheme X one can attach a rigid analytic space $X^{\mathrm{an}}\in \mathrm{Rig}_K$, a so-called *analitification of* X, characterized by the property that to give a map of rigid analytic spaces $Y\to X^{\mathrm{an}}\in\mathrm{Rig}_K$ is the same as to give a map of locally ringed G-spaces $Y\to X$ (see [24, Section 5.4] for more details). On the other hand, if $\mathfrak{X}$ is a flat finite type formal $\mathcal{O}_K$-scheme, one can attach to $\mathfrak{X}$ the so-called *Raynaud generic fiber* $\mathfrak{X}_K$ *of* $\mathfrak{X}$. Explicitly, if $\mathfrak{X}\simeq \mathrm{Spf}\,\widehat{A}$ for some π-adically complete finitely generated flat $\mathcal{O}_K$-algebra $\widehat{A}$, then $\widehat{A}\otimes_{\mathcal{O}_K}K$ is an affinoid K-algebra and $\mathfrak{X}_K$ is just $\mathrm{Sp}(\widehat{A}\otimes_{\mathcal{O}_K}K)$. We note that both construction commute with finite limits.

Thus we see that there are two natural ways to associate a rigid analytic space to a finite type flat $\mathcal{O}_K$-scheme X: one can take an algebraic generic fiber X_K of X and then consider its analytification $X_K^{\mathrm{an}}\coloneqq (X_K)^{\mathrm{an}}$, or one can take a π-adic completion $\widehat{X}$ of X and then consider its Raynaud generic fiber $\widehat{X}\coloneqq(\widehat{X})_K$. By the universal property of X_K^{an} these two constructions are related to each other:

Proposition 3.3.4.31. *For a finite type flat $\mathcal{O}_K$-scheme S there is a natural map of locally ringed G-spaces*

$$\widehat{S}_K \xrightarrow{\psi_S} S_K^{\mathrm{an}} \xrightarrow{\varphi_S} S_K.$$

Moreover, the pullbacks along these maps induce maps of the corresponding étale sites. In particular there is a natural pullback functor on the categories of étale sheaves:

$$\lambda_S^{-1} \colon \mathcal{S}\mathrm{hv}_{\text{ét}}(S_K, \Lambda) \longrightarrow \mathcal{S}\mathrm{hv}_{\text{ét}}(\widehat{S}_K, \Lambda).$$

If S is proper then φ_S is an isomorphism. Also, note that by a non-archimedean analogue of Artin comparison ([57, Theorem 3.8.1]) φ_S^{-1} does not change cohomology (at least when applied to constructible sheaves). On the opposite, ψ_S^{-1} changes cohomology a lot (unless S was proper).

Now we are going to generalize this picture to stacks. We note that the Raynaud generic fiber construction $\operatorname{Spec} A \mapsto \operatorname{Sp}(\widehat{A} \otimes_{\mathcal{O}_K} K)$ sends étale maps/covers to étale maps/covers and thus extends to a well-defined functor between the big sites

$$\widehat{-}_K \colon \text{ét}(\mathrm{Aff}^{\mathrm{f.g.}}_{\mathcal{O}_K}) \to \text{ét}(\mathrm{Afd}_K).$$

This functor can be left Kan extended to a functor

$$\widehat{-}_K \colon \mathscr{P}\mathscr{S}\mathrm{tk}^{\mathrm{lft}}_{\mathcal{O}_K} \to \mathscr{P}\mathscr{S}\mathrm{tk}^{\mathrm{lft,an}}_K$$

which then descends to the corresponding categories of sheaves

$$\widehat{-}_K \colon \mathscr{S}\mathrm{tk}^{\mathrm{lft}}_{\mathcal{O}_K} \to \mathscr{S}\mathrm{tk}^{\mathrm{lft,an}}_K.$$

Moreover, since smooth maps are also preserved under the Raynaud generic fiber functor, geometric stacks go to geometric stacks.

Situation with the functor $(-)^{\mathrm{an}}_K$ is slightly more complicated. The problem is that $(\operatorname{Spec} A)^{\mathrm{an}}_K \in \mathrm{Rig}_K$ is usually not an affinoid space, e.g. in the case $A = \mathcal{O}_K[x]$ we get the rigid affine line $(\mathbb{A}^1_K)^{\mathrm{an}}$ which is not quasi-compact. However, the construction $\operatorname{Spec} A \to (\operatorname{Spec} A)^{\mathrm{an}}_K$ still sends étale covers/maps on the algebraic side to étale covers/maps on the rigid analytic side. Note that we have a natural functor $\mathrm{Rig}_K \to \mathscr{S}\mathrm{tk}^{\mathrm{lft,an}}_K$ (e.g. by left Kan extending the Yoneda embedding $\mathrm{Afd}_K \hookrightarrow$

$\mathscr{S}\mathrm{tk}_K^{\mathrm{lft,an}}$ via the embedding $\mathrm{Afd}_K \hookrightarrow \mathrm{Rig}_K$). Moreover, it once again sends étale maps/covers to étale maps/surjections. Left Kan exteding the resulting functor $\mathrm{Aff}_{\mathcal{O}_K} \to \mathscr{S}\mathrm{tk}_K^{\mathrm{lft,an}}$ we obtain a well-defined functor

$$(-)_K^{\mathrm{an}} \colon \mathscr{S}\mathrm{tk}_{\mathcal{O}_K}^{\mathrm{lft}} \to \mathscr{S}\mathrm{tk}_K^{\mathrm{lft,an}}.$$

We will need the following technical lemma:

Lemma 3.3.4.32. *Let $\mathcal{X}$ be a finitely presentable flat Artin stack over $\mathcal{O}_K$. Then the natural maps*

$$\mathcal{S}\mathrm{hv}_{\text{ét}}(\mathcal{X}_K, \Lambda) \longrightarrow \lim_{S \in \mathrm{Aff}^{\mathrm{ft,flat,op}}_{\mathcal{O}_K/\mathcal{X}}} \mathcal{S}\mathrm{hv}_{\text{ét}}(S_K, \Lambda) \qquad \mathcal{S}\mathrm{hv}_{\text{ét}}(\widehat{\mathcal{X}}_K, \Lambda) \longrightarrow \lim_{S \in \mathrm{Aff}^{\mathrm{ft,flat,op}}_{\mathcal{O}_K/\mathcal{X}}} \mathcal{S}\mathrm{hv}_{\text{ét}}(\widehat{S}_K, \Lambda)$$

$$\mathcal{S}\mathrm{hv}_{\text{ét}}(\mathcal{X}_K^{\mathrm{an}}, \Lambda) \longrightarrow \lim_{S \in \mathrm{Aff}^{\mathrm{ft,flat,op}}_{\mathcal{O}_K/\mathcal{X}}} \mathcal{S}\mathrm{hv}_{\text{ét}}(S_K^{\mathrm{an}}, \Lambda)$$

are equivalences.

Proof. Let $\mathcal{X}$ be n-Artin. We will prove the statement by induction on n, the base of induction $n = -1$ being obvious since the category $\mathrm{Aff}_{\mathcal{O}_K/\mathcal{X}}$ has a final object in this case. To make an inductive step note that in each case both parts satisfy étale descent. Hence, if $U \twoheadrightarrow \mathcal{X}$ is a smooth atlas of $\mathcal{X}$, the statement follows by induction from the corresponding statement for the elements of the Čech nerve $U_\bullet$ and descent. □

We are now ready to define functors λ^{-1}, φ^{-1} and ψ^{-1} for arbitrary geometric stacks:

Construction 3.3.4.33. The map of étale sites from Proposition 3.3.4.31 induces a natural transformation

$$\lambda_-^{-1} \colon \mathcal{S}\mathrm{hv}_{\text{ét}}(-_K, \Lambda) \longrightarrow \mathcal{S}\mathrm{hv}_{\text{ét}}(\widehat{-}_K, \Lambda)$$

of functors from $\mathrm{Aff}^{\mathrm{fl,op}}_{\mathcal{O}_K}$ to $\mathrm{Pr}^{\mathrm{L}}_{\Lambda}$. Passing to the right Kan extensions and using the equivalences from Lemma 3.3.4.32, for a finitely presentable flat Artin $\mathcal{O}_K$-stack $\mathcal{X}$ we obtain a functorial natural transformation

$$\lambda_{\mathcal{X}}^{-1} \colon \mathcal{S}\mathrm{hv}_{\text{ét}}(\mathcal{X}_K, \Lambda) \longrightarrow \mathcal{S}\mathrm{hv}_{\text{ét}}(\widehat{\mathcal{X}}_K, \Lambda).$$

We will also denote the right adjoint of $\lambda_{\mathcal{X}}^{-1}$ by $\lambda_{\mathcal{X}*}$. Similarly, we can also define pairs of adjoint functors

$$\mathcal{S}\mathrm{hv}_{\text{ét}}(\mathcal{X}_K, \Lambda) \underset{\varphi_{X*}}{\overset{\varphi_X^{-1}}{\rightleftarrows}} \mathcal{S}\mathrm{hv}_{\text{ét}}(\mathcal{X}_K^{\mathrm{an}}, \Lambda) \qquad \text{and} \qquad \mathcal{S}\mathrm{hv}_{\text{ét}}(\mathcal{X}_K^{\mathrm{an}}, \Lambda) \underset{\psi_{X*}}{\overset{\psi_X^{-1}}{\rightleftarrows}} \mathcal{S}\mathrm{hv}_{\text{ét}}(\widehat{\mathcal{X}}_K, \Lambda).$$

If $f\colon \mathcal{X} \to \mathcal{Y}$ is a map of finitely presentable flat Artin $\mathcal{O}_K$-stacks we have a natural equivalence $\lambda_{\mathcal{X}}^{-1} \circ f_K^{-1} \simeq \widehat{f}_K^{-1} \circ \lambda_{\mathcal{Y}}$. Our goal now will be to show that the corresponding square is not only commutative, but right adjoinable when restricted to the category of local systems. We will do this in two steps (basically decomposing λ as $\varphi \circ \psi$).

Proposition 3.3.4.34. *Let $f_K\colon \mathcal{X}_K \to \mathcal{Y}_K$ be a schematic smooth proper morphism of finitely presentable Artin K-stacks. Then the diagram*

$$\begin{array}{ccc} \mathcal{S}\mathrm{hv}_{\text{ét}}^{>-\infty}(\mathcal{X}_K^{\mathrm{an}}, \Lambda) & \xleftarrow{\varphi_{\mathcal{X}}^{-1}} & \mathcal{L}_{\text{ét}}^{>-\infty}(\mathcal{X}_K, \Lambda) \\ \uparrow{\scriptstyle (f_K^{\mathrm{an}})^{-1}} & & \uparrow{\scriptstyle f_K^{-1}} \\ \mathcal{S}\mathrm{hv}_{\text{ét}}^{>-\infty}(\mathcal{Y}_K^{\mathrm{an}}, \Lambda) & \xleftarrow{\varphi_{\mathcal{Y}}^{-1}} & \mathcal{L}_{\text{ét}}^{>-\infty}(\mathcal{Y}_K, \Lambda). \end{array}$$

is right adjointable (see Definition 3.3.4.9).

Remark 3.3.4.35. In fact, replacing local systems in the diagram above by a suitable version of constructible sheaves, one can allow arbitrary schematic morphism f.

Proof. Let's first assume that both X and Y are schemes. In this case the statement follows from the theorem of Huber [57, Theorem 3.8.1].

Now assume that both $\mathcal{X}$ and $\mathcal{Y}$ are n-Artin. We will prove the statement by induction on n. Let $U \twoheadrightarrow \mathcal{Y}$ be an atlas of $\mathcal{Y}$ and set $V \coloneqq U \times_{\mathcal{Y}} \mathcal{X}$ (which is a scheme since f is schematic). By étale descent and definition of t-structure on $\mathcal{S}\mathrm{hv}_{\text{ét}}(-, \Lambda)$ for analytic spaces we have

$$\mathcal{S}\mathrm{hv}_{\text{ét}}^{>-\infty}(\mathcal{X}_K^{\mathrm{an}}, \Lambda) \xrightarrow{\sim} \operatorname{Tot} \mathcal{S}\mathrm{hv}_{\text{ét,ad}}^{>-\infty}(V_{\bullet,K}^{\mathrm{an}}, \Lambda) \qquad \text{and} \qquad \mathcal{S}\mathrm{hv}_{\text{ét}}^{>-\infty}(\mathcal{Y}_K^{\mathrm{an}}, \Lambda) \xrightarrow{\sim} \operatorname{Tot} \mathcal{S}\mathrm{hv}_{\text{ét,ad}}^{>-\infty}(U_{\bullet,K}^{\mathrm{an}}, \Lambda),$$

where $U_\bullet$ and $V_\bullet$ are Čech nerves of $U \twoheadrightarrow \mathcal{Y}$ and $V \twoheadrightarrow \mathcal{X}$ respectively. Similar equalities for $\mathcal{X}_K$ and $\mathcal{Y}_K$ were shown in Proposition 3.3.4.2. By induction each square

$$\begin{array}{ccc} V_{i+1} & \longrightarrow & V_i \\ \downarrow & & \downarrow \\ U_{i+1} & \longrightarrow & U_i \end{array}$$

is right adjointable. The induction step is then completed by the fact that right adjointable squares are closed under limits [83, Corollary 4.7.4.18.]. □

To deal with ψ we will use proper base change for adic morphisms (in particular, properness assumption on f will be essential). To use it we will need the following lemma.

Lemma 3.3.4.36. *Let $f\colon X \to Y$ be a proper morphism of finite type flat $\mathcal{O}_K$-schemes. Then the commutative square*

$$\begin{array}{ccc} \widehat{X}_K & \xrightarrow{\psi_X} & X_K^{\mathrm{an}} \\ \downarrow{\scriptstyle \widehat{f}_K} & & \downarrow{\scriptstyle f_K^{\mathrm{an}}} \\ \widehat{Y}_K & \xrightarrow{\psi_Y} & Y_K^{\mathrm{an}} \end{array}$$

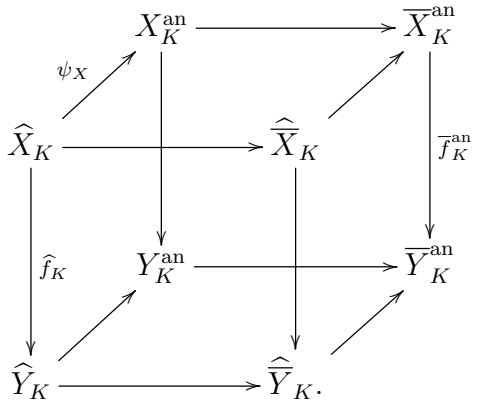
is a pullback square of rigid analytic spaces.

Proof. By Nagata's compactification theorem there exist compactifications $\overline{X}, \overline{Y}$ of X and Y respectively. By replacing $\overline{X}$ with the closure of the graph $\Gamma_f \subset \overline{X} \times \overline{Y}$ we can choose $\overline{X}$ such that there is a commutative square

$$\begin{array}{ccc} X & \longrightarrow & \overline{X} \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \overline{f}} \\ Y & \longrightarrow & \overline{Y} \end{array}$$

with $\overline{f}$ being proper. It is also easy to see that this commutative square is fibered (since f was already proper and thus taking closure of Γ_f does not affect fibers over $Y \subset \overline{Y}$).

Consider now a commutative diagram of rigid spaces



The back and front faces of this diagram are pullback: this follows from the fact that both the Raynaud generic fiber and the analytification commute with fibered products. Moreover, since $\overline{X}$ and $\overline{Y}$ are proper, we have $\widehat{\overline{X}}_K \simeq \overline{X}_K^{\mathrm{an}}$ and $\widehat{\overline{Y}}_K \simeq \overline{Y}_K^{\mathrm{an}}$. Hence in the right face horizontal maps

are isomorphisms and in particular it is fibered. It follows that in the commutative diagram

$$\begin{array}{ccccc} \widehat{X}_K & \longrightarrow & X_K^{\mathrm{an}} & \longrightarrow & \overline{X}_K^{\mathrm{an}} \\ \downarrow{\scriptstyle \widehat{f}_K} & & \downarrow{\scriptstyle f_K} & & \downarrow{\scriptstyle \overline{f}_K} \\ \widehat{Y}_K & \longrightarrow & Y_K^{\mathrm{an}} & \longrightarrow & \overline{Y}_K^{\mathrm{an}} \end{array}$$

big and right hand squares are fibered. It follows that the left hand square is also fibered. □

Proposition 3.3.4.37. *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a schematic proper morphism of finitely presentable flat Artin $\mathcal{O}_K$-stacks. Then the diagram*

$$\begin{array}{ccc} \mathcal{S}\mathrm{hv}_{\text{ét}}^{>-\infty}(\widehat{\mathcal{X}}_K, \Lambda) & \xleftarrow{\psi_{\mathcal{X}}^{-1}} & \mathcal{S}\mathrm{hv}_{\text{ét}}^{>-\infty}(\mathcal{X}_K^{\mathrm{an}}, \Lambda) \\ \uparrow{\scriptstyle \widehat{f}_K^{-1}} & & \uparrow{\scriptstyle f_K^{-1}} \\ \mathcal{S}\mathrm{hv}_{\text{ét}}^{>-\infty}(\widehat{\mathcal{Y}}_K, \Lambda) & \xleftarrow{\psi_{\mathcal{Y}}^{-1}} & \mathcal{S}\mathrm{hv}_{\text{ét}}^{>-\infty}(\mathcal{Y}_K^{\mathrm{an}}, \Lambda). \end{array}$$

is right adjointable (see Definition 3.3.4.9).

Proof. First assume that both X and Y are schemes. By the lemma above the commutative square

$$\begin{array}{ccc} \widehat{X}_K & \xrightarrow{\psi_X} & X_K^{\mathrm{an}} \\ \downarrow{\scriptstyle \widehat{f}_K} & & \downarrow{\scriptstyle f_K^{\mathrm{an}}} \\ \widehat{Y}_K^{\mathrm{an}} & \xrightarrow{\psi_Y} & Y_K \end{array}$$

is a fibered square of adic spaces. Hence the result in this case follows from the proper base change for adic spaces [57, Theorem 4.4.1]. Note that ψ_Y is an open embedding, thus $\dim.\mathrm{tr}(\psi_Y) = 0$ and the mentioned base change result can indeed be applied.

Now assume that both $\mathcal{X}$ and $\mathcal{Y}$ are n-Artin. We argue as before, by induction on n. Let $U \twoheadrightarrow \mathcal{Y}$ be an atlas of $\mathcal{Y}$ and set $V \coloneqq U \times_{\mathcal{Y}} \mathcal{X}$ (which is a scheme). By étale descent we have

$$\mathcal{S}\mathrm{hv}_{\text{ét}}^{>-\infty}(\widehat{\mathcal{X}}_K, \Lambda) \xrightarrow{\sim} \mathrm{Tot}\, \mathcal{S}\mathrm{hv}_{\text{ét}}^{>-\infty}(\widehat{V}_{\bullet,K}, \Lambda) \qquad \text{and} \qquad \mathcal{S}\mathrm{hv}_{\text{ét}}^{>-\infty}(\widehat{\mathcal{Y}}_K, \Lambda) \xrightarrow{\sim} \mathrm{Tot}\, \mathcal{S}\mathrm{hv}_{\text{ét,ad}}^{>-\infty}(\widehat{U}_{\bullet,K}, \Lambda),$$

where $U_\bullet$ and $V_\bullet$ are Čech nerves of $U \twoheadrightarrow \mathcal{Y}$ and $V \twoheadrightarrow \mathcal{X}$ respectively. Similar equalities are true

for $\mathcal{X}_K^{\mathrm{an}}$ and $\mathcal{Y}_K^{\mathrm{an}}$ (see the proof of 3.3.4.34). By induction each square

$$\begin{array}{ccc} V_{i+1} & \longrightarrow & V_i \\ \downarrow & & \downarrow \\ U_{i+1} & \longrightarrow & U_i \end{array}$$

is right adjointable and we are done by passing to the limit. □

Corollary 3.3.4.38. *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a smooth proper morphism of finitely presentable flat Artin $\mathcal{O}_K$-stacks. Then the diagram*

$$\begin{array}{ccc} \mathcal{S}\mathrm{hv}_{\text{ét}}^{>-\infty}(\widehat{\mathcal{X}}_K, \Lambda) & \xleftarrow{\lambda_{\mathcal{X}}^{-1}} & \mathcal{L}_{\text{ét}}^{>-\infty}(\mathcal{X}_K, \Lambda) \\ \uparrow{\scriptstyle \widehat{f}_K^{-1}} & & \uparrow{\scriptstyle f_K^{-1}} \\ \mathcal{S}\mathrm{hv}_{\text{ét}}^{>-\infty}(\widehat{\mathcal{Y}}_K, \Lambda) & \xleftarrow{\lambda_{\mathcal{Y}}^{-1}} & \mathcal{L}_{\text{ét}}^{>-\infty}(\mathcal{Y}_K, \Lambda). \end{array}$$

is right adjointable (see Definition 3.3.4.9).

Proof. Follows by composing squares in Propositions 3.3.4.34 and 3.3.4.37. □

3.4 Hodge-proper stacks

In this section we give definition of Hodge-proper stacks (which will also be used crucially in Chapter 4).

3.4.1 Hodge and de Rham cohomology

In this section we set up the Hodge and de Rham cohomology for n-Artin stacks and prove some technical results needed in subsequent sections. For the rest of this section fix a base ring R. We refer the reader to [110] and Section 3.3 for an introduction to the theory of Artin stacks and cotangent complexes.

Definition 3.4.1.1 (Hodge cohomology)**.** Let $\mathcal{X}$ be an Artin stack over R. Define *Hodge cohomology* $R\Gamma_{\mathrm{H}}(\mathcal{X})$ *of* $\mathcal{X}$ to be

$$R\Gamma_{\mathrm{H}}(\mathcal{X}) \coloneqq \bigoplus_{p \geq 0} R\Gamma\left(\mathcal{X}, \wedge^p \mathbb{L}_{\mathcal{X}/R}[-p]\right),$$

where $\mathbb{L}_{\mathcal{X}/R}$ is the cotangent complex of $\mathcal{X}$ over R and $\wedge^p\mathbb{L}_{\mathcal{X}/R}$ is its p-th derived exterior power (see [58, Chapitre I.4] or [26, Section 3]). For a fixed $n \in \mathbb{Z}$ we will also denote

$$H^n_{\mathrm{H}}(\mathcal{X}) \coloneqq H^n R\Gamma_{\mathrm{H}}(\mathcal{X}) \simeq \bigoplus_{p+q=n} H^{p,q}(\mathcal{X}), \quad \text{where} \quad H^{p,q}(\mathcal{X}) \coloneqq H^q\left(\mathcal{X}, \wedge^p\mathbb{L}_{\mathcal{X}/R}\right).$$

Notation 3.4.1.2. Let $S \coloneqq \operatorname{Spec} A$ be an affine smooth R-algebra. The algebraic de Rham complex of S over R

$$A \xrightarrow{d} \Omega^1_{A/R} \xrightarrow{d} \Omega^2_{A/R} \xrightarrow{d} \dots$$

will be denoted by $\Omega^\bullet_{S/R,\mathrm{dR}}$. We define $R\Gamma_{\mathrm{dR}}(S) \coloneqq \Omega^\bullet_{S/R,\mathrm{dR}} \in D(\mathrm{Mod}_R)$.

Definition 3.4.1.3 (de Rham cohomology)**.** Let $\mathcal{X}$ be a smooth Artin stack over R. Define the *(Hodge-completed) de Rham cohomology* $R\Gamma_{\mathrm{dR}}(\mathcal{X})$ *of* $\mathcal{X}$ to be

$$R\Gamma_{\mathrm{dR}}(\mathcal{X}) \coloneqq \lim_{Y \in \mathrm{Aff}^{\mathrm{sm,op}}_{/\mathcal{X}}} R\Gamma_{\mathrm{dR}}(S),$$

where $\mathrm{Aff}^{\mathrm{sm}}_{/\mathcal{X}}$ is the full subcategory of stacks over $\mathcal{X}$ consisting of affine schemes that are smooth over $\mathcal{X}$. We will also denote $H^n R\Gamma_{\mathrm{dR}}(\mathcal{X})$ by $H^n_{\mathrm{dR}}(\mathcal{X})$.

In fact the Hodge cohomology complex admits a description similar to our definition of the de Rham cohomology:

Proposition 3.4.1.4. *For any $p \in \mathbb{Z}_{\geq 0}$ the natural map*

$$R\Gamma(\mathcal{X}, \wedge^p\mathbb{L}_{\mathcal{X}/R}) \to \lim_{S \in \mathrm{Aff}^{\mathrm{sm,op}}_{/\mathcal{X}}} R\Gamma(S, \wedge^p\mathbb{L}_{S/R}) \tag{3.4.1.1}$$

is an equivalence.

Proof. Since the relative cotangent complex of an étale morphism vanishes, the cotangent complex satisfies étale descent. Moreover, since étale locally any smooth morphism admits a section, the cotangent complex satisfies smooth descent as well. It follows that both sides of (3.4.1.1) satisfy smooth descent. Since n-Artin stacks are by definition iterated smooth quotients of schemes, by induction n we reduce to the case when $\mathcal{X}$ is a smooth affine scheme, where the assertion of the proposition is true, since $S \in \mathrm{Aff}^{\mathrm{sm}}_{/\mathcal{X}}$ has a final object given by $\mathcal{X}$. □

Corollary 3.4.1.5. *Let $\mathcal{X}$ be a smooth Artin stack. Then*

1. *There exists a complete (decreasing)* Hodge filtration $F^\bullet R\Gamma_{\mathrm{dR}}(\mathcal{X})$ *such that* $\operatorname{gr} F^\bullet R\Gamma_{\mathrm{dR}}(\mathcal{X}) \simeq R\Gamma_{\mathrm{H}}(\mathcal{X})$.

2. *There exists a strongly convergent spectral sequence* $E_1^{p,q} = H^q(\mathcal{X}, \wedge^p \mathbb{L}_{\mathcal{X}}) \Rightarrow H_{\mathrm{dR}}^{p+q}(\mathcal{X})$.

Proof. Note that since $\mathcal{X}$ is smooth, all schemes $S \in \mathrm{Aff}^{\mathrm{sm}}_{/\mathcal{X}}$ are smooth. Since the Hodge filtration on $R\Gamma_{\mathrm{dR}}(S)$ is complete, the same holds for $R\Gamma_{\mathrm{dR}}(\mathcal{X})$, since complete filtered complexes are closed under limits. Moreover, by construction we have

$$\operatorname{gr} F^\bullet R\Gamma_{\mathrm{dR}}(\mathcal{X}) \simeq \lim_{S \in \mathrm{Aff}^{\mathrm{sm,op}}_{/\mathcal{X}}} \operatorname{gr} F^\bullet R\Gamma_{\mathrm{dR}}(S) \simeq \lim_{S \in \mathrm{Aff}^{\mathrm{sm,op}}_{/\mathcal{X}}} R\Gamma_{\mathrm{H}}(S) \simeq R\Gamma_{\mathrm{H}}(\mathcal{X}),$$

where the last equivalence follows from the previous proposition. This filtration induces a spectral sequence with $E_1^{p,q}$ as stated. To prove it is strongly convergent, note that by smoothness of $\mathcal{X}$, for each n the induced filtration on $H^n_{\mathrm{dR}}(\mathcal{X})$ is finite. □

The following simple observation will be quite useful in what follows:

Remark 3.4.1.6. Let $\mathcal{X}$ be a smooth Artin stack. Then the cotangent complex $\mathbb{L}_{\mathcal{X}/R}$ (and its exterior powers) is concentrated in nonnegative cohomological degrees (with respect to the natural t-structure on $\mathrm{QCoh}(\mathcal{X})$). Since the global section functor $R\Gamma$ is left t-exact, it follows that the natural map $R\Gamma_{\mathrm{dR}}(\mathcal{X}) \to R\Gamma_{\mathrm{dR}}(\mathcal{X})/F^p R\Gamma_{\mathrm{dR}}(\mathcal{X})$ induces an isomorphism on $H^{<p}$.

Finally, we will need the following

Proposition 3.4.1.7 (Base-change)**.** *Let $\mathcal{X}$ be a smooth Artin stack over R and let $R \to R'$ be a ring homomorphism of finite Tor-amplitude. Then for $\mathcal{X}' := \mathcal{X} \otimes_R R'$ the natural map $R\Gamma_{\mathrm{dR}}(\mathcal{X}/R) \otimes_R R' \to R\Gamma_{\mathrm{dR}}(\mathcal{X}'/R')$ is a filtered equivalence. In particular, for each $p \in \mathbb{Z}_{\geq 0}$ the natural map $R\Gamma(\mathcal{X}, \wedge^p \mathbb{L}_{\mathcal{X}/R}) \otimes_R R' \to R\Gamma(\mathcal{X}', \wedge^p L_{\mathcal{X}'/R'})$ is an equivalence.*

Proof. By the smoothness assumption on $\mathcal{X}$ the fiber product $\mathcal{X} \otimes_R R'$ coincides with the derived fiber product. It follows by [110, Lemma 1.4.1.16 (2)] that $\mathbb{L}_{\mathcal{X}/R} \otimes_R R' \simeq \mathbb{L}_{\mathcal{X}'/R'}$. By the base change for QCoh (see [47, Chapter 3., Proposition 2.2.2 (b)]) we deduce that the natural map $R\Gamma_{\mathrm{H}}(\mathcal{X}/R) \otimes_R R' \to R\Gamma_{\mathrm{H}}(\mathcal{X}'/R')$ is an equivalence.

Next, note that the condition on the morphism $R \to R'$ guarantees that the natural map $R\Gamma_{\mathrm{dR}}(\mathcal{X}) \otimes_R R' \to \lim_{\longleftarrow p}((R\Gamma_{\mathrm{dR}}(\mathcal{X})/F^p R\Gamma_{\mathrm{dR}}(\mathcal{X})) \otimes_R R')$ is an equivalence. Since both sides are

complete with respect to the Hodge filtration, and, since by the above the induced map on the associated graded pieces

$$R\Gamma_{\mathrm{H}}(\mathcal{X}) \otimes_R R' \simeq \operatorname{gr} R\Gamma_{\mathrm{dR}}(\mathcal{X}/R) \otimes_R R' \to \operatorname{gr} R\Gamma_{\mathrm{dR}}(\mathcal{X}'/R') \simeq R\Gamma_{\mathrm{H}}(\mathcal{X}'/R')$$

is an equivalence, we deduce that the base-change map for de Rham cohomology is an equivalence as well. □

3.4.2 Hodge-proper stacks

Fix a Noetherian base ring R. In this section we will introduce a reasonable (at least from the point of view of Hodge-to-de Rham degeneration) generalization of the notion of properness for stacks.

Definition 3.4.2.1. A complex of R-modules X is called *nearly coherent*[6] if it is cohomologically bounded below and for any $i \in \mathbb{Z}$ the cohomology module $H^i(X)$ is finitely generated over R. We will denote the full subcategory of $D(\mathrm{Mod}_R)$ consisting of nearly coherent R-modules by $\mathrm{Coh}^+(R)$.

Remark 3.4.2.2. We use the term *nearly* coherent for objects of $\mathrm{Coh}^+(R)$ to distinguish them from coherent complexes, which in our convention are necessarily bounded (both from above and below).

We have the following basic properties of $\mathrm{Coh}^+(R)$:

Proposition 3.4.2.3. *Let R be a Noetherian ring. Then:*

1. *The category $\mathrm{Coh}^+(R)$ is closed under finite (co-)limits and retracts. In particular $\mathrm{Coh}^+(R)$ is a stable subcategory of $D(\mathrm{Mod}_R)$.*

2. *For each $n \in \mathbb{Z}$ the category $\mathrm{Coh}^{\geq n}(R) \coloneqq \mathrm{Coh}^+(R) \cap D(\mathrm{Mod}_R)^{\geq n}$ is closed under totalizations.*

Proof. 1. This follows from the fact that for a Noetherian R the abelian category of finitely generated R-modules is closed under (co)kernels, extensions and direct summands.

[6] In the previous version of this text we called such complexes *almost coherent*. We decided to change the notation to avoid possible clashes with almost mathematics.

2. Let $X^\bullet$ be a co-simplicial object of $\mathrm{Coh}^{\geq n}(R)$. By shifting if necessary, we can assume that all X^i are coconnective. Since coconnective modules are closed under limits, $\mathrm{Tot}(X^\bullet) \in D(\mathrm{Mod}_R)^{\geq 0}$; hence it is enough to prove that $H^i \mathrm{Tot}(X^\bullet)$ is finitely generated R-module for all $i \in \mathbb{Z}_{\geq 0}$. Since all X^i are connective, the natural map $\mathrm{Tot}(X^\bullet) \to \mathrm{Tot}^{\leq k}(X^\bullet)$ induces an isomorphism on $H^{\leq k}$. But since $\mathrm{Tot}^{\leq k}$ is a finite limit, each $H^i \mathrm{Tot}^{\leq k}(X^\bullet)$ is a finitely generated R-module. □

Example 3.4.2.4. Let G be a topological group, such that the underlying space of G is a finite CW-complex (e.g. G might be just a finite discreet group). Then the co-chains complex $C^*(BG, R)$ is a nearly coherent R-module. To see this note that $BG \simeq |B_\bullet G|$, where $B_n G := G^{\times n}$, and hence $C^*(BG, R) \simeq \mathrm{Tot}\, C^*(G^{\times \bullet}, R)$. Now, by assumption on G, for all $n \in \mathbb{Z}_{\geq 0}$ the complexes $C^*(G^n, R)$ are perfect (and hence nearly coherent), so we conclude by the second part of Proposition 3.4.2.3.

The property of being nearly coherent may be checked flat locally:

Proposition 3.4.2.5. *Let $R \to R'$ be a faithfully flat map. Then a module $X \in D(\mathrm{Mod}_R)$ is nearly coherent if and only if $H^i(X \otimes_R R')$ is a finitely presentable R'-module for all $i \in \mathbb{Z}$ and $H^{\ll 0}(X \otimes_R R') \simeq 0$.*

Remark 3.4.2.6. We stress that R' is not assumed to be Noetherian. If R' is Noetherian, the proposition says that X is nearly coherent R-module if and only if $M \otimes_R R'$ is nearly coherent R'-module.

Proof. By flatness of R' over R we have $H^i(X \otimes_R R') \simeq H^i(X) \otimes_R R'$. Hence it is enough to prove that a (classical) R-module M is finitely presentable if and only if $M \otimes_R R'$ is finitely presentable over R'. This is proved e.g. in [7, Tag 03C4, Lemma 10.82.2(2)]. □

After this technical digression we are ready to introduce the notion of a Hodge-proper stack:

Definition 3.4.2.7 (Hodge-proper stacks)**.** A smooth Artin stack $\mathcal{X}$ over R is called *Hodge-proper* if for every $p \in \mathbb{Z}_{\geq 0}$ the complex $R\Gamma(\mathcal{X}, \wedge^p \mathbb{L}_{\mathcal{X}/R})$ is nearly coherent.

For us the most important implication of Hodge-properness is that the de Rham cohomology is nearly coherent:

Proposition 3.4.2.8. *Let $\mathcal{X}$ be a smooth Hodge-proper Artin stack. Then $R\Gamma_{\mathrm{dR}}(\mathcal{X})$ is nearly coherent.*

Proof. By smoothness $R\Gamma_{\mathrm{dR}}(\mathcal{X})$ is bounded below by 0, hence it is enough to prove that for each $n \in \mathbb{Z}_{\geq 0}$ the cohomology module $H^n_{\mathrm{dR}}(X)$ is finitely generated over R. By Remark 3.4.1.6 the natural map $R\Gamma_{\mathrm{dR}}(\mathcal{X}) \to R\Gamma_{\mathrm{dR}}(\mathcal{X})/F^{n+1}R\Gamma_{\mathrm{dR}}(\mathcal{X})$ induces an isomorphism on $H^{\leq n}$. We conclude, since $R\Gamma_{\mathrm{dR}}(\mathcal{X})/F^{n+1}R\Gamma_{\mathrm{dR}}(\mathcal{X})$, being a finite extension of nearly coherent complexes $R\Gamma(\mathcal{X}, \wedge^i \mathbb{L}_{\mathcal{X}/R}[-i])$, $0 \leq i \leq n$, is nearly coherent. □

In the next section we will see some rather complicated examples of Hodge-proper stacks Section 4.4. As for now we will just prove some general statements about those. First of all, it is easy to see that the property to be Hodge-proper is fppf-local:

Proposition 3.4.2.9. *Let $R \to R'$ be faithfully flat map of Noetherian rings. Then if $\mathcal{Y}' := \mathcal{Y} \times_{\operatorname{Spec} R} \operatorname{Spec} R'$ is Hodge-proper over R', then $\mathcal{Y}$ is Hodge-proper over R.*

Proof. Let $q\colon \mathcal{Y}' \to \mathcal{Y}$ be the canonical projection. Then by the flat base change Proposition 3.3.2.19 we have

$$R\Gamma(\mathcal{Y}, \mathcal{F}) \otimes_R R' \simeq R\Gamma(\mathcal{Y}', q^*\mathcal{F})$$

for any quasi-coherent sheaf $\mathcal{F} \in \mathrm{QCoh}(\mathcal{Y})$. In particular the complex $R\Gamma(\mathcal{Y}, \wedge^i \mathbb{L}_{\mathcal{Y}/R}) \otimes_R R' \simeq R\Gamma(\mathcal{Y}', \wedge^i \mathbb{L}_{\mathcal{Y}'/R'})$ is nearly coherent by assumption on $\mathcal{Y}'$. We conclude by Proposition 3.4.2.5. □

We also have:

Proposition 3.4.2.10. *Let $\pi\colon \mathcal{U} \twoheadrightarrow \mathcal{X}$ be a smooth surjective map of smooth Artin stacks and assume that all terms of the corresponding Čech nerve $\mathcal{U}_n$ are Hodge-proper. Then $\mathcal{X}$ is Hodge-proper.*

Proof. By the smooth descent Proposition 3.3.3.9 we have

$$R\Gamma(\mathcal{X}, \wedge^i \mathbb{L}_{\mathcal{X}}) \simeq \operatorname{Tot} R\Gamma(\mathcal{U}_\bullet, \wedge^i \mathbb{L}_{\mathcal{U}_\bullet}).$$

By assumption all complexes $R\Gamma(\mathcal{U}_n, \wedge^i \mathbb{L}_{\mathcal{U}_n})$ are nearly coherent and cohomologically bounded below by 0 (by smoothness). So $R\Gamma(\mathcal{X}, \wedge^i \mathbb{L}_{\mathcal{X}})$ is also nearly coherent by Proposition 3.4.2.3. □

In fact the following notion of properness is more reasonable, see Section 4.3.2 for more details.

Definition 3.4.2.11. Let $\mathcal{X}$ be a finitely presentable Artin stack. A sheaf $\mathcal{F} \in \mathrm{QCoh}(\mathcal{X})$ is called *nearly coherent* if $\mathcal{H}^{\ll 0}(\mathcal{F}) \simeq 0$ and $\mathcal{H}^i(\mathcal{F})$ is coherent for all $i \in \mathbb{Z}$. We will denote the full subcategory of $\mathrm{QCoh}(\mathcal{X})$ consisting of nearly coherent sheaves by $\mathrm{Coh}^+(\mathcal{X})$.

Definition 3.4.2.12. A morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of finitely presentable Artin stacks is called *cohomologically proper* if the induced functor $f_*\colon \mathrm{QCoh}(\mathcal{X}) \to \mathrm{QCoh}(\mathcal{Y})$ sends $\mathrm{Coh}^+(\mathcal{X})$ to $\mathrm{Coh}^+(\mathcal{Y})$. A finitely presentable Artin stack $\mathcal{X}$ is called *cohomologically proper* if the structure morphism $\mathcal{X} \to \operatorname{Spec} R$ is cohomologically proper.

Remark 3.4.2.13. By the left exactness of f_* it is enough to prove that $f_*(\mathrm{Coh}(\mathcal{X})^\heartsuit) \subset \mathrm{Coh}^+(\mathcal{Y})$.

The following properties of cohomologically proper morphisms are formal:

Proposition 3.4.2.14. *In the notations above:*

1. *The class of cohomologically proper morphisms is closed under compositions and flat base changes.*

2. *The property of being a cohomologically proper morphism is fppf-local on the target: the map $f\colon \mathcal{X} \to \mathcal{Y}$ is cohomologically proper if and only if there exists an fppf surjection $\mathcal{U} \twoheadrightarrow \mathcal{Y}$ such that the base-change $f \times_{\mathcal{Y}} \mathcal{U}$ is cohomologically proper.*

3. *Let $\mathcal{U}_\bullet \twoheadrightarrow \mathcal{X}$ be a flat hypercover such that all $\mathcal{U}_n$ are cohomologically proper. Then $\mathcal{X}$ is cohomologically proper.*

The previous notion is stronger than the Hodge-properness:

Proposition 3.4.2.15. *Let $\mathcal{X}$ be a smooth cohomologically proper Artin stack. Then $\mathcal{X}$ is Hodge-proper.*

Proof. By Proposition 3.3.3.6 and Proposition 3.3.2.50 all exterior powers of the cotangent complex of $\mathcal{X}$ are perfect, hence coherent. So by assumption on $\mathcal{X}$ all complexes $R\Gamma(\mathcal{X}, \wedge^i \mathbb{L}_{\mathcal{X}})$ are nearly coherent. □

3.4.3 Derived Nakayama lemma for Coh^+

Let $\xi \in R$ be a non-zero divisor such that R is ξ-adically complete. Recall that a complex $X \in D(\mathrm{Mod}_R)$ is called (derived) ξ-adically complete if the natural map $X \to \varprojlim X/\xi^k$ is an equivalence. Later we will need the following variant of (derived) Nakayama lemma for nearly coherent sheaves.

Proposition 3.4.3.1. *Let R be a Noetherian ring and let $\xi \in R$ be a non-zero divisor. Then a ξ-adicly complete complex $X \in D(\mathrm{Mod}_R)$ is nearly coherent if and only if $X/\xi := X \otimes_R R/\xi$ is nearly coherent as an R/ξ module.*

The proof consists of a series of lemmas. We start with the following connectivity estimate:

Proposition 3.4.3.2. *Let X be a ξ-adically complete R-module. Then*

1. *X is n-connective if and only if X/ξ is n-connective.*

2. *If X/ξ is n-coconnective, then so is X.*

Proof. We will only prove the first part, the proof of the second one being similar but slightly easier. Since a tensor product of an n-connective and an m-connective modules is $(n+m)$-connective we have that if X is n-connective then so is $X/\xi \simeq X \otimes R/\xi$. Conversely assume X/ξ is n-connective for some $n \in \mathbb{Z}$. Shifting if necessary we can assume $n = 0$. Now using a co-fiber sequence $X/\xi \to X/\xi^k \to X/\xi^{k-1}$ we find $H^{>0}(X/\xi^k) \simeq 0$ for all $k \geq 1$. Since $X \simeq \varprojlim X/\xi^k$ for any i we have the Milnor exact sequence

$$0 \longrightarrow \varprojlim{}^1 H^{i-1}(X/\xi^k) \longrightarrow H^i(X) \longrightarrow \varprojlim H^i(X/\xi^k) \longrightarrow 0.$$

Hence to prove $H^{>0}(X) \simeq 0$ it is enough to prove that the $\varprojlim^1$-term on the left vanishes. For $i > 1$ this is obvious since $H^{i-1}(X/\xi^k) \simeq 0$ for all k. For $i = 1$ notice that since $H^{-1}(X/\xi) = 0$, the maps $H^0(X/\xi^k) \to H^0(X/\xi^{k-1})$ are surjective for any k, hence $\varprojlim^1 H^0(X/\xi^k) \simeq 0$. □

Note that if X and Y are ξ-adically complete, then so is the cofiber Y/X of any map $X \to Y$.

Lemma 3.4.3.3. *Let X, Y be connective ξ-complete R-modules and let $\varphi\colon X \to Y$ be a map inducing surjection on H^0 modulo ξ. Then $H^0(\varphi)\colon H^0(X) \to H^0(Y)$ is also surjective.*

Proof. It is enough to prove that $H^0(Y/X) \simeq 0$. But by Proposition 3.4.3.2 this is equivalent to $(Y/X)/\xi \simeq (Y/\xi)/(X/\xi)$ being connected, which is equivalent to the assumption that $\pi_0(\varphi/\xi)$ is surjective. □

Lemma 3.4.3.4. *In notations above let X be a discrete (derived) ξ-adically complete bounded R-module. Then X is coherent if and only if X/ξ is coherent as an $R/(\xi)$-module.*

Proof. If $X \simeq H^0(X)[0]$ is finitely generated, then so does X/ξ. For the other implication it is enough to prove that $H^0(X)$ is finitely generated as an R-module. Let $\varphi\colon R^{\oplus n} \to X$ be any map which induces a surjection on $H^0(X/\xi) \simeq H^0(X)/\xi$ (such a map exists by assumption on X/ξ). By the previous lemma the induced map $H^0(\varphi)\colon R^{\oplus n} \to H^0(X)$ is also surjective. Thus $\square$

Proof of Proposition 3.4.3.1. By Proposition 3.4.3.2 X is bounded below. By shifting if necessary, we can assume that $H^{<0}(X) \simeq 0$. Next we claim that $H^0(X)[0]/\xi$ is coherent. Note that since ξ is a non-zero divisor, $H^0(X)[0]/\xi$ is concentrated in cohomological degrees $[-1; 0]$, so it is enough to prove that $H^{-1}(H^0(X)[0]/\xi)$ and $H^0(H^0(X)[0]/\xi)$ are finitely generated $R/(\xi)$-modules. Consider a long exact sequence of cohomology induced by the cofiber sequence $H^0(X)[0]/\xi \to X/\xi \to X^{>0}/\xi$:

$$0 \longrightarrow H^{-1}(H^0(X)[0]/\xi) \longrightarrow H^{-1}(X/\xi) \longrightarrow H^{-1}(X^{>0}/\xi) \simeq 0$$
$$\longrightarrow H^0(H^0(X)[0]/\xi) \longrightarrow H^0(X/\xi) \longrightarrow \cdots .$$

So $H^{-1}(H^0(X)[0]/\xi) \simeq H^{-1}(X/\xi)$ and $H^0(H^0(X)[0]/\xi)$ is a submodule of a finitely generated $R/(\xi)$-module $H^0(X/\xi)$. Hence both are finitely generated.

By the previous lemma we deduce that $H^0(X)$ is finitely generated. Moreover, if we denote $X' := X^{>0}$, we see that X'/ξ being a cofiber of a map of nearly coherent $R/(\xi)$-modules, is nearly coherent. By repeating the argument for X' instead of X we find that $H^1(X)$ is finitely generated e.t.c. $\square$

We also record the following:

Lemma 3.4.3.5. *Let's assume R is ξ-complete and Noetherian. Then any $\mathcal{F} \in \mathrm{Coh}^+(R)$ is (derived) ξ-adically complete.*

Proof. We need to show that the map $\mathcal{F} \to \varprojlim \mathcal{F}/\xi^k$ is an equivalence. Any truncation $\tau^{\leq n}F$ is coherent, each of its cohomology is finitely generated and thus is classically ξ-adically complete. In particular $\tau^{\leq n}\mathcal{F}$ is derived ξ-adically complete for any n. The statement then follows by observing that the fiber of

$$\tau^{>n}\mathcal{F} \to \varprojlim_k \tau^{>n}\mathcal{F}/\xi^k$$

is at most $(n-1)$-coconnected and thus becomes zero as $n \to \infty$. $\square$

3.4.4 Example: quotients by reductive group action

In this section we will consider the most important for this work examples of Hodge-proper stack: global quotient $[X/G]$ of a smooth proper scheme X by reductive group scheme G action. The fact that $[X/G]$ is Hodge-proper in this case could also be derived from the proof of Theorem 4.4.1.3 or Theorem 4.4.2.10, but for completeness here we give another argument which does not rely on hard cohomology vanishing results from [46] (but we have to assume that X is proper instead of more general proper over affine version of [76, Theorem 2.2.13]).

For the rest of this section let R be a Noetherian ring and let X be a smooth proper R-scheme. Let G be a reductive group R-scheme acting on X. By Proposition 3.4.2.15 it is enough to prove that $[X/G]$ is cohomologically proper.

Remark 3.4.4.1. The statement is more or less obvious if R is a $\mathbb{Q}$-algebra: in this case G is linearly reductive and one recovers the underlying complex of $R\Gamma([X/G], \mathcal{F}) \simeq R\Gamma(X, \mathcal{F})^G$ as a direct summand of $R\Gamma(X, \mathcal{F})$. However if R is just an algebra over $\mathbb{Z}$ this is no more true, but one can still go around reducing to the case of algebraic torus which is linearly reductive.

The proof will occupy the rest of this section (also see Proposition 3.4.4.8 for a slightly more general result). First note that by Proposition 3.4.2.14 and the following assertion it is enough to consider the case of split G:

Proposition 3.4.4.2 (Raynaud)**.** *Let G be a reductive group scheme over R. Then there exists an étale cover $R \to R'$ such that $G_{R'}$ is split.*

The proof consists of three steps: first we will treat the case of split torus T (which is easy, since T is linearly reductive), then we deduce the case of Borel subgroup $B \subseteq G$ and then a general parabolic subgroup $P \subseteq G$.

Torus. Let T be a split algebraic torus $T \simeq \mathbb{G}_m^d$ over R. Let $X^*(T) \simeq \mathbb{Z}^d$ denote the character lattice. Then the category $\mathrm{QCoh}(BT)^{\mathrm{perf}}$ can be identified with the subcategory of the derived category $\mathrm{Rep}_T(D(\mathrm{Mod}_R))$ on complexes of T-representations which are perfect as R-modules. Moreover, the category $\mathrm{Rep}_T^{\heartsuit}$ is equivalent to the category of $X^*(T)$-graded R-modules.

Let now $V \simeq R^n$ be a representation of T. As any representation of T it is necessarily a direct sum of weight submodules, say $V = V_{\alpha_1} \oplus \cdots \oplus V_{\alpha_k}$. We can consider the corresponding affine space $\mathbb{A}^n = \operatorname{Spec} \operatorname{Sym}_R(V^\vee)$, where $V^\vee := \operatorname{Hom}_R(V, R)$ is the dual representation of T. There is a natural

action of T on $\mathbb{A}^n$ and we consider the corresponding quotient stack $[\mathbb{A}^n/T]$. Note that the action of T on $\mathbb{A}^n$ corresponds to the natural $X^*(T)$-grading on $\mathrm{Sym}_R(V^\vee)$ and that $\mathrm{QCoh}([\mathbb{A}^n/T])^{\mathrm{perf}}$ can be identified with the full subcategory of the derived category of $X^*(T)$-graded $\mathrm{Sym}_R(V^\vee)$-modules consisting of objects which are perfect as $\mathrm{Sym}_R(V^\vee)$-modules. The global sections are again given by the grading 0 part of the corresponding complex.

Proposition 3.4.4.3. *Let $\alpha_1, \dots, \alpha_k$ be the weights of V. Assume that $\alpha_i \neq 0$ for all i and that the commutative monoid $\mathbb{N} \cdot \alpha_1 + \dots + \mathbb{N} \cdot \alpha_k$ does not contain a copy of $\mathbb{Z}$ (in other words if $x \in \mathbb{N} \cdot \alpha_1 + \dots + \mathbb{N} \cdot \alpha_k$ then $-x \notin \mathbb{N} \cdot \alpha_1 + \dots + \mathbb{N} \cdot \alpha_k$). Then the corresponding quotient stack $[\mathbb{A}^n/T]$ is cohomologically proper.*

Proof. Any object of $\mathrm{QCoh}([\mathbb{A}^n/T])^{\mathrm{perf}}$ can be represented by a finite length complex with terms being $X^*(T)$-graded finitely generated modules over $\mathrm{Sym}_R(V^\vee)$. But the condition on the weights on V guarantees that for any finitely generated $X^*(T)$-graded $\mathrm{Sym}_R(V^\vee)$ module the submodule of invariants (or equivalently the grading 0 part) is finitely generated over R (because it is for $\mathrm{Sym}_R(V^\vee)$ as well as its finite direct sums with itself). Thus the corresponding module of global sections lies in $D(\mathrm{Mod}_R)^{\mathrm{perf}} \subset \mathrm{Coh}^+(R)$. □

Remark 3.4.4.4. Given a split reductive group G with a maximal torus T an example of such $\mathbb{A}^n$ is given by the maximal unipotent subgroup U of some Borel subgroup $B \subset G$ such that B contains T. Torus T acts on U by conjugation, moreover U can be identified with the affine space $\mathbb{A}^n$ with coordinates x_α corresponding to positive roots $\alpha \in \Delta$ with the action given by the negative of the corresponding weight: $t \circ x_\alpha = \alpha(t)^{-1} \cdot x$. Thus we are in the context of Proposition 3.4.2.14 with V given by $\mathfrak{u} := \mathrm{Lie}(U)$: indeed, the set of positive roots (which are exactly the weights of $\mathfrak{u}$) clearly satisfies the conditions of the proposition. Note that the same is true for the diagonal (adjoint) action of T on U^n.

Corollary 3.4.4.5. *For any $n \in \mathbb{Z}_{\geq 1}$ the iterated classifying stack B^nT is cohomologically proper.*

Proof. For $n = 1$ this is a special case of Proposition 3.4.4.3. The general case follows by induction using Proposition 3.4.2.10. □

Borel subgroup. Let $B \subset G$ be a Borel subgroup containing a split maximal torus T.

Proposition 3.4.4.6. *The classifying stack BB is cohomologically proper.*

Proof. We use the cover $\pi\colon BT \to BB$ and the fact that $B \simeq T \ltimes U$. The corresponding Čech simplicial object $\check{C}(\pi)_\bullet$ of $\pi\colon BT \to BB$ is given by the following: $\check{C}(\pi)_n$ is identified with $[T\backslash \underbrace{B \times_T B \times_T \cdots \times_T B}_{n}/T]$ with the coboundary maps given by the associated projections. On the other hand there is an isomorphism

$$[T\backslash \underbrace{B \times_T B \times_T \cdots \times_T B}_{n}/T] \simeq [(\underbrace{U \times U \times \cdots \times U}_{n})/T]$$

with the T-action on $U \times \cdots \times U$ given by the simultaneous adjoint action on all factors $t \circ (u_1, \ldots, u_n) = (tu_1t^{-1}, \ldots, tu_nt^{-1})$. The corresponding map sends

$$[(b_1, \ldots, b_n)] \in [T\backslash \underbrace{B \times_T B \times_T \cdots \times_T B}_{n}/T] \mapsto ([b_1], \ldots, [b_n]) \in [(\underbrace{U \times U \times \cdots \times U}_{n})/T],$$

where each $[b_i]$ is a class in $[T\backslash B] \simeq U$. It is easy to see by induction on n that the map defined this way is an isomorphism. Remark 3.4.4.4 tells that the quotient stack $(U \times U \times \cdots \times U)/T$ is cohomologically proper and so $\check{C}(\pi)_n$ is cohomologically proper for all n. Finally, the cover $\pi\colon BT \to BB$ is smooth, so Proposition 3.4.2.14 can be applied. It follows that BB is cohomologically proper. □

Corollary 3.4.4.7. *Let X be a smooth proper R-scheme equipped with an action of B. Then $[X/B]$ is cohomologically proper.*

Proof. By assumption on X the structure map $[X/B] \to BB$ is proper, and hence is cohomologically proper. Since the composition of cohomologically proper maps is cohomologically proper, we can conclude by the previous proposition. □

Parabolic subgroup. Finally, we can pass to the case of BP, where $P \subset G$ is any parabolic subgroup.

Proposition 3.4.4.8. *Let $P \subset G$ be a parabolic subgroup. Then BP is cohomologically proper.*

Proof. We use the cover $\pi : BB \to BP$. The terms $\check{C}(\pi)_n$ of the corresponding Čech simplicial object are given by

$$\check{C}(\pi)_n \simeq [B\backslash \underbrace{P \times_B \cdots \times_B P}_{n}/B].$$

The scheme $(P \times_B \cdots \times_B P)/B$ is an iterated fibration with fibers given by the variety P/B which is the flag variety for the Levi subgroup $L \subset P$ and so is smooth and proper. It follows by Corollary 3.4.4.7 that $\check{C}(\pi)_n$ is cohomologically proper for all n. By Proposition 3.4.2.10 BP is cohomologically proper as well. □

Similarly to the proof of Corollary 3.4.4.7 one deduces:

Corollary 3.4.4.9. *Let X be a smooth proper R-scheme and let P be a parabolic subgroup in a reductive R-group scheme. Then $[X/P]$ is cohomologically proper.*

3.5 p-adic cohomology theories for stacks

3.5.1 Completed Hodge and de Rham cohomology

In the p-adic context we will also need a completed variant of the de Rham cohomology. Let $\xi \in R$ be an element such that R is ξ-adically complete.

Definition 3.5.1.1. Let $\mathcal{X}/R$ be a smooth Artin stack. *The ξ-completed Hodge and de Rham cohomology* are defined as $\widehat{R\Gamma}_{\mathrm{H}}(\mathcal{X}) \coloneqq \varprojlim R\Gamma_{\mathrm{H}}(\mathcal{X})/\xi^k$ and $\widehat{R\Gamma}_{\mathrm{dR}}(\mathcal{X}) \coloneqq \varprojlim R\Gamma_{\mathrm{dR}}(\mathcal{X})/\xi^k$ correspondingly.

Remark 3.5.1.2. Similarly to Definition 3.4.1.3, $\widehat{R\Gamma}_{\mathrm{dR}}(\mathcal{X})$ has a description as a limit

$$\widehat{R\Gamma}_{\mathrm{dR}}(\mathcal{X}) \coloneqq \lim_{Y \in \mathrm{Aff}^{\mathrm{sm,op}}_{/\mathcal{X}}} R\Gamma_{\mathrm{dR}}(\widehat{Y}),$$

where $\widehat{Y}$ is a formal $\operatorname{Spf} R$-scheme obtained by ξ-adically completing Y. Indeed, the de Rham complex of $\widehat{Y}$ is obtained from the de Rham complex of Y by ξ-adically completing termwise. The terms in the latter are ξ-torsion free by smoothness and so this exactly gives the derived ξ-completion.

Remark 3.5.1.3. Similarly to Section 3.3.4 one can consider the formal stack $\widehat{\mathcal{X}}$ given by the formal ξ-adic completion of $\mathcal{X}$. Then one can show from Remark 3.5.1.2 that $\widehat{R\Gamma}_{\mathrm{dR}}(\mathcal{X}) \simeq R\Gamma_{\mathrm{dR}}(\widehat{\mathcal{X}})$. We will sometimes use the latter notation as well for the beauty of the formulas.

Lemma 3.5.1.4. *Let R be Noetherian (and ξ-adically complete) and let $\mathcal{X}$ be a smooth Hodge-proper stack over R. Then*

$$\widehat{R\Gamma}_{\mathrm{dR}}(\mathcal{X}) \simeq R\Gamma_{\mathrm{dR}}(\mathcal{X}).$$

Proof. This follows from Lemma 3.4.3.5. □

3.5.2 Crystalline cohomology

Crystalline cohomology for stacks were previously thoroughly studied by Olsson in [93], where he used them (among other things) to prove the Fontaine's C_{st}-conjecture. In this section we compare his construction and compare it with the right Kan extension of the crystalline cohomology from schemes. Since in applications we will be interested only in computing the crystalline cohomology of k-stacks, to simplify notations we restrict just to cristalline cohomology over $W(k)$.

Definition 3.5.2.1 ([93, Section 1.3])**.** Let $\mathrm{Aff}^{\mathrm{pd}}_{/W(k)}$ denote the category consisting of pairs $(U \hookrightarrow T, \delta)$, where $U \hookrightarrow T$ is a closed embedding of an affine k-scheme U into an affine $W(k)$-schemes T and δ is a pd-structure on the ideal of U in T compatible with the canonical pd-structure on $(p) \subset W(k)$. Note that there is a natural forgetful functor $\mathrm{Aff}^{\mathrm{pd}}_{/W(k)} \to \mathrm{Aff}_k$ sending $(U \hookrightarrow T, \delta)$ to U.

For a stack $\mathcal{X}$ over k we define *lisse-crystalline site* $\mathrm{Crys}(\mathcal{X}/W(k))$ of $\mathcal{X}$ over $W(k)$ as a fibered product $\mathrm{Aff}^{\mathrm{pd}}_{/W(k)} \times_{\mathrm{Aff}_k} \mathrm{Aff}^{\mathrm{sm}}_{/\mathcal{X}}$. Unwinding the definition, one sees that an object of $\mathrm{Crys}(\mathcal{X}/W(k))$ is a triple $(U \to \mathcal{X}, U \hookrightarrow T, \delta)$, where $U \to \mathcal{X}$ is a smooth map and T is a pd-thickening of U over $W(k)$. There is a sheaf of rings $\mathcal{O}_{\mathcal{X}_{\mathrm{crys}}}$ on $\mathrm{Crys}(\mathcal{X}/W(k))$ sending (U, T, δ) to $\Gamma(T, \mathcal{O}_T)$. Define

$$R\Gamma(\mathcal{X}_{\mathrm{crys}}, \mathcal{O}_{\mathcal{X}_{\mathrm{crys}}}) := \lim_{(U,T,\delta) \in \mathrm{Crys}(\mathcal{X}/W(k))^{\mathrm{op}}} \Gamma(T, \mathcal{O}_T).$$

Remark 3.5.2.2. In his definition Olsson allows U and T to be arbitrary (i.e. not necessary affine) schemes. This definition gives the same answer by Zariski descent.

One can also define crystalline cohomology via right Kan extension:

Definition 3.5.2.3. Let $\mathcal{X}$ be a smooth Artin stack over k. *Crystalline cohomology* $R\Gamma_{\mathrm{crys}}(\mathcal{X}/W(k))$ *of* $\mathcal{X}$ are defined as

$$R\Gamma_{\mathrm{crys}}(\mathcal{X}/W(k)) := \lim_{U \in (\mathrm{Aff}^{\mathrm{sm}}_{/\mathcal{X}})^{\mathrm{op}}} R\Gamma_{\mathrm{crys}}(U/W(k)),$$

where the limit is taken over the diagram of affine schemes smooth over $\mathcal{X}$.

In fact definitions 3.5.2.1 and 3.5.2.3 produce the same answer:

Proposition 3.5.2.4. *Let $\mathcal{X}$ be a smooth Artin stack over k. Then there is a natural equivalence*

$$R\Gamma(\mathcal{X}_{\mathrm{crys}}, \mathcal{O}_{\mathcal{X}_{\mathrm{crys}}}) \simeq R\Gamma_{\mathrm{crys}}(\mathcal{X}/W(k)).$$

Proof. Let $\pi\colon \mathrm{Crys}(\mathcal{X}/W(k)) \to \mathrm{Aff}^{\mathrm{sm}}_{/\mathcal{X}}$ be the natural projection. By general properties of Kan extensions we can compute Olsson's crystalline cohomology $R\Gamma(\mathcal{X}_{\mathrm{crys}}, \mathcal{O}_{\mathcal{X}_{\mathrm{crys}}})$ by first right Kan extending the functor $\mathcal{O}_{\mathcal{X}_{\mathrm{crys}}}$ along π and then by computing the limit of $\mathrm{Ran}_\pi \mathcal{O}_{\mathcal{X}_{\mathrm{crys}}}$ over $\mathrm{Aff}^{\mathrm{sm}}_{/\mathcal{X}}$. For $U \in \mathrm{Aff}^{\mathrm{sm}}_{/\mathcal{X}}$ we have

$$(\mathrm{Ran}_\pi \mathcal{O}_{\mathcal{X}_{\mathrm{crys}}})(U) \simeq \lim_{(V,T,\delta)\in(\pi/U)^{\mathrm{op}}} \Gamma(T, \mathcal{O}_T).$$

Note that an object in π/U is a pair $(V \to U, V \hookrightarrow T)$, where $V \to U$ is arbitrary morphism such that the composition $V \to U \to \mathcal{X}$ is smooth and T is a pd-thickening of U.

There is a full subcategory $\mathrm{Crys}'(U/W(k))$ in π/U consisting of pairs $(V \to U, T)$ where $V \to U$ is an identity morphism. We claim that $\mathrm{Crys}'(U/W(k))$ is cofinal in π/U. By Quillen's Theorem A (see e.g. [82, Theorem 4.1.3.1]) it is enough to prove that for a fixed $P := (V \to U, T)$ the comma category $P/\mathrm{Crys}'(U/W(k))$ is non-empty and weakly contractible. Now, $P/\mathrm{Crys}'(U/W(k))$ consists of pd-thickenings T' of U such that the diagram

$$\begin{array}{ccc} V & \longrightarrow & U \\ \downarrow & & \downarrow \\ T & \longrightarrow & T' \end{array}$$

is commutative. But by [10, Lemma 5.11] the pushout $U \coprod_V T$ admits a pd-structure, so $P/\mathrm{Crys}'(U/W(k))$ is non-empty. To see that it is contractible, note that $P/\mathrm{Crys}'(U/W(k))$ has products given by pd-envelope of U diagonally embedded in $T' \times T''$.

It follows

$$(\mathrm{Ran}_\pi \mathcal{O}_{\mathcal{X},\mathrm{crys}})(U) \simeq \lim_{(T,\delta)\in\mathrm{Crys}'(U/W(k))^{\mathrm{op}}} \Gamma(T, \mathcal{O}_T) \simeq R\Gamma_{\mathrm{crys}}(U/W(k))$$

and

$$R\Gamma(\mathcal{X}_{\mathrm{crys}}, \mathcal{O}_{\mathcal{X}_{\mathrm{crys}}}) = \lim_{U\in(\mathrm{Aff}^{\mathrm{sm}}_{/\mathcal{X}})^{\mathrm{op}}} (\mathrm{Ran}_\pi \mathcal{O}_{\mathcal{X}_{\mathrm{crys}}})(U) \simeq \lim_{U\in(\mathrm{Aff}^{\mathrm{sm}}_{/\mathcal{X}})^{\mathrm{op}}} R\Gamma_{\mathrm{crys}}(U/W(k)) = R\Gamma_{\mathrm{crys}}(\mathcal{X}/W(k)),$$

finishing the proof. □

3.5.3 Étale cohomology of algebraic and Raynaud generic fibers

In this subsection we define two versions of étale $\mathbb{Z}_p$-cohomology of the generic fiber of a stack $\mathcal{Y}$ over $\mathcal{O}_K$: an algebraic one and an adic one. As we will see in Proposition 3.5.5.3, the generic fiber of the prismatic cohomology only gives the comparison with the latter. For the rest of this section let $\mathbb{C}_p$ be a complete non-archimedean algebraically closed field extension of $\mathbb{Q}_p$ with the ring of integers $\mathcal{O}_{\mathbb{C}_p}$. Let Λ be a torsion ring.

Algebraic generic fiber. We first introduce a simpler version of algebraic étale cohomology:

Definition 3.5.3.1. Fix an algebraically closed field F. We define a functor of *(algebraic) étale cohomology*

$$R\Gamma_{\text{ét}}(-,\Lambda)\colon \mathscr{P}\mathscr{S}\mathrm{tk}_F^{\mathrm{op}} \longrightarrow D(\mathrm{Mod}_\Lambda)$$

as a right Kan extension of the functor of the usual étale cohomology

$$R\Gamma_{\text{ét}}(-,\Lambda)\colon \mathrm{Aff}_F^{\mathrm{op}} \longrightarrow D(\mathrm{Mod}_\Lambda)$$

along the Yoneda embedding $\mathrm{Aff}_F^{\mathrm{op}} \hookrightarrow \mathscr{P}\mathscr{S}\mathrm{tk}_F^{\mathrm{op}}$. Note that by construction this functor satisfies étale descent. Finally, for an F-stack $\mathcal{X}$ and $\mathbb{Z}_p$-coefficients we give a somewhat ad hoc definition $R\Gamma_{\text{ét}}(\mathcal{X},\mathbb{Z}_p) := \varprojlim R\Gamma_{\text{ét}}(\mathcal{X},\mathbb{Z}/p^n)$.

Remark 3.5.3.2. Let $\underline{\Lambda} \in \mathcal{S}\mathrm{hv}_{\text{ét}}(\mathcal{X},\Lambda)$ denote the constant sheaf. Then there is a natural equivalence

$$R\Gamma_{\text{ét}}(\mathcal{X},\underline{\Lambda}) := \mathrm{End}_{\mathcal{S}\mathrm{hv}_{\text{ét}}(\mathcal{X},\underline{\Lambda})}(\underline{\Lambda}) \xrightarrow{\sim} R\Gamma_{\text{ét}}(\mathcal{X},\Lambda).$$

To see this note that by definition both sides are the same for affine schemes and satisfy étale descent.

As in the case of schemes, for F being a field of complex numbers $\mathbb{C}$, the étale cohomology of a stack can be identified with the singular cohomology of the underlying homotopy type of a stack:

Construction 3.5.3.3. Define the *underlying homotopy type of a prestack functor*

$$-(\mathbb{C})\colon \mathscr{P}\mathscr{S}\mathrm{tk}_{\mathbb{C}} \to \mathscr{S}$$

as the left Kan extension of the functor

$$\operatorname{Spec} A \mapsto (\operatorname{Spec} A)(\mathbb{C})$$

(where we consider the right hand side as an underlying homotopy type of the topological space $(\operatorname{Spec} A)(\mathbb{C})$ with its natural complex topology) along the Yoneda embedding $\mathrm{Aff}_{\mathbb{C}} \hookrightarrow \mathscr{P}\mathscr{S}\mathrm{tk}_{\mathbb{C}}$. Since for a smooth morphism $U \to V$ the induced map $U(\mathbb{C}) \to V(\mathbb{C})$ is a fibration, we see that $(-)(\mathbb{C})$ satisfy smooth descent. In particular, if we denote by $L_{\text{ét}}$ the étale sheafification functor $\mathscr{P}\mathscr{S}\mathrm{tk}_{\mathbb{C}} \to \mathscr{S}\mathrm{tk}_{\mathbb{C},\text{ét}}$, we see that the $-(\mathbb{C})\colon \mathscr{P}\mathscr{S}\mathrm{tk}_{\mathbb{C}} \to \mathscr{S}$ factors through a functor $\mathscr{S}\mathrm{tk}_{\mathbb{C},\text{ét}} \to \mathscr{S}$, which we will also denote by $-(\mathbb{C})$.

Example 3.5.3.4. By construction the functor $-(\mathbb{C})\colon \mathscr{S}\mathrm{tk}_{\mathbb{C},\text{ét}} \to \mathscr{S}$ commutes with colimits. In particular, for an algebraic group G over $\mathbb{C}$ we have that $(BG)(\mathbb{C})$ is naturally equivalent to the topological classifying space of the topological group $G(\mathbb{C})$. More generally, for a $\mathbb{C}$-scheme X equipped with an action of G there is a natural equivalence $(X/G)(\mathbb{C}) \simeq X(\mathbb{C})_{hG(\mathbb{C})}$, where $-_{hG(\mathbb{C})}\colon \mathscr{S}^{BG(\mathbb{C})} \to \mathscr{S}$ is the Borel's homotopy quotient by $G(\mathbb{C})$ functor.

Proposition 3.5.3.5 (Artin's comparison)**.** *Let $\mathcal{X}$ be an Artin finite type $\mathbb{C}$-stack and let Λ be a finite abelian group. Then there is a natural equivalence*

$$R\Gamma_{\text{ét}}(\mathcal{X}, \Lambda) \simeq C^*(\mathcal{X}(\mathbb{C}), \Lambda),$$

where $C^*(-,\Lambda)\colon \mathscr{S}^{\mathrm{op}} \to D(\mathrm{Mod}_\Lambda)$ is the singular Λ-cochains functor.

Proof. Both sides satisfy étale descent, hence it is enough to prove the equivalence for $\mathcal{X}$ being an affine finite type $\mathbb{C}$-scheme. This is a classical theorem of Artin, see [6, Exposé XVI, Theorem 4.1]. □

Adic étale cohomology. In general the étale comparison from Theorem 3.2.3.12 relates $\mathbb{A}$-cohomology not with étale cohomology of a scheme, but with étale cohomology of its Raynaud generic fiber. To give an appropriate definition of adic étale cohomology of stacks we first introduce the following notation:

Notation 3.5.3.6. Let $U = \operatorname{Spec} A$ be an affine finite type flat scheme over $\mathcal{O}_C$. Consider the corresponding formal scheme $\widehat{U} = \operatorname{Spf} \widehat{A}$, where $\widehat{A} = \varprojlim A/p^n$ is the p-adic completion. To $\widehat{U}$ we

can associate a *Raynaud generic fiber* $\widehat{U}_C$ *of* $\operatorname{Spf}\widehat{A}$

$$\widehat{U}_C := \operatorname{Spa}(\widehat{A},\widehat{A}) \times_{\operatorname{Spa}(\mathcal{O}_C,\mathcal{O}_C)} \operatorname{Spa}(C,\mathcal{O}_C).$$

This is a rigid analytic C-space.

Definition 3.5.3.7. We define the functor of *étale cohomology of Raynaud generic fiber* or just *adic* Λ*-étale cohomology*

$$R\Gamma_{\text{ét}}(\widehat{-}_C,\Lambda)\colon \mathscr{P}\mathscr{S}\mathrm{tk}^{\mathrm{op}}_{\mathcal{O}_C} \longrightarrow D(\mathrm{Mod}_\Lambda)$$

as the right Kan extension along the Yoneda embedding $\mathrm{Aff}^{\mathrm{op}}_{\mathcal{O}_C} \hookrightarrow \mathscr{P}\mathscr{S}\mathrm{tk}^{\mathrm{op}}_{\mathcal{O}_C}$ of the functor, which sends an affine $\mathcal{O}_C$-scheme U to the complex of adic étale cohomology of the adic generic fiber $\widehat{U}_C$. I.e., for a prestack $\mathcal{X}$ we have:

$$R\Gamma_{\text{ét}}(\widehat{\mathcal{X}}_C,\Lambda) = \lim_{U\in \mathrm{Aff}^{\mathrm{op}}_{/\mathcal{X}}} R\Gamma_{\text{ét}}(\widehat{U}_C,\Lambda).$$

Again, we define $R\Gamma_{\text{ét}}(\widehat{\mathcal{X}}_C,\mathbb{Z}_p) := \varprojlim R\Gamma_{\text{ét}}(\widehat{\mathcal{X}}_C,\mathbb{Z}/p^n)$.

Remark 3.5.3.8. It is not hard to see from smooth descent that $R\Gamma_{\text{ét}}(\widehat{\mathcal{X}}_C,\Lambda)$ indeed agrees with the étale cohomology of the stack $\widehat{\mathcal{X}}_C$ defined in Section 3.3.4.

There is a natural map comparing these two versions of étale cohomology:

Construction 3.5.3.9. Let $\mathcal{X}$ be a flat finitely presentable Artin $\mathcal{O}_C$-stack. Recall that by Construction 3.3.4.33 there is a natural pair of adjoint functors

$$\lambda_{\mathcal{X}}^{-1}\colon \mathcal{S}\mathrm{hv}_{\text{ét}}(\mathcal{X}_C,\Lambda) \rightleftarrows \mathcal{S}\mathrm{hv}_{\text{ét}}(\widehat{\mathcal{X}}_C,\Lambda)\colon \lambda_{\mathcal{X}*}.$$

Applying the global section functor to the unit of adjunction $\underline{\Lambda} \to \lambda_{\mathcal{X}*}\underline{\Lambda}$ we obtain a natural comparison map

$$\Upsilon_{\mathcal{X}}\colon R\Gamma_{\text{ét}}(\mathcal{X}_C,\Lambda) \longrightarrow R\Gamma_{\text{ét}}(\widehat{\mathcal{X}}_C,\Lambda).$$

Example 3.5.3.10. In the case when $\mathcal{X} = X$ is a smooth proper scheme, $\Upsilon_{\mathcal{X}}$ is an equivalence (since the left hand side agrees with $R\Gamma_{\text{ét}}(X_C^{\mathrm{an}},\Lambda)$ and for X proper $X_C^{\mathrm{an}} \simeq \widehat{X}_C$).

Example 3.5.3.11. In the case X is not proper this map is very from being an isomorphism: in the example of $X = \mathbb{A}^1$ all higher étale cohomology of $\mathbb{A}^1_C$ vanish, but the first $\mathbb{F}_p$-cohomology of $\widehat{\mathbb{A}}^1_C$ (which is a p-adic disk) is infinite-dimensional over $\mathbb{F}_p$.

Conjecture 3.5.3.12. Let $\mathcal{X}$ be a smooth formally proper Artin stack (in the sense of [55, Definition 1.1.3], roughly meaning that p-adic formal GAGA holds $\mathrm{Coh}(\mathcal{X}) \simeq \mathrm{Coh}(\widehat{\mathcal{X}})$) over $\mathcal{O}_K$. Then the map

$$\Upsilon_{\mathcal{X}} \colon R\Gamma_{\text{ét}}(\mathcal{X}_C, \Lambda) \longrightarrow R\Gamma_{\text{ét}}(\widehat{\mathcal{X}}_C, \Lambda)$$

is an equivalence.

Remark 3.5.3.13. For now, the largest class of stacks for which we can check that $\Upsilon_{\mathcal{X}}$ is an equivalence are the quotient stacks $[X/G]$ under the assumption that X is proper and G is reductive (see Theorem 3.6.5.2). The state of the art though is that the formal properness of $[X/G]$ over a mixed characteristic base is not known, but this (along with more complicated examples) should come as a consequence of the upcoming work of Halpern-Leistner [52].

3.5.4 Prismatic cohomology

Definition 3.5.4.1. As usual we define the functor of *prismatic cohomology* as the right Kan extension of the functor

$$R\Gamma_{\mathbb{\Delta}}(-/\mathfrak{S}) \colon \mathrm{Aff}^{\mathrm{op}}_{\mathcal{O}_K} \longrightarrow D(\mathrm{Mod}_{\mathfrak{S}})$$

along the Yoneda embedding $\mathrm{Aff}^{\mathrm{op}}_{\mathcal{O}_K} \hookrightarrow \mathscr{PS}\mathrm{tk}^{\mathrm{op}}_{\mathcal{O}_K}$. Note that by extension from the subcategory of schemes there is a canonical natural transformation

$$\varphi \colon \varphi^*_{\mathbb{\Delta}} R\Gamma_{\mathbb{\Delta}}(-/\mathfrak{S}) \longrightarrow R\Gamma_{\mathbb{\Delta}}(-/\mathfrak{S}).$$

Remark 3.5.4.2. The following properties follow formally from the case of schemes:

- Since complete objects are closed under limits, for any $\mathcal{X} \in \mathscr{PS}\mathrm{tk}_{\mathcal{O}_K}$ the complex $R\Gamma_{\mathbb{\Delta}}(\mathcal{X}/\mathfrak{S})$ is (p,u)-adicly complete.
- y construction and Proposition 3.2.6.1 the functor

$$R\Gamma_{\mathbb{\Delta}}(-/\mathfrak{S}) \colon \mathscr{PS}\mathrm{tk}^{\mathrm{op}}_{\mathcal{O}_K} \longrightarrow D(\mathrm{Mod}_{\mathfrak{S}})$$

satisfy étale (equivalently smooth) descent.

We will establish various comparison and finiteness results in the next subsection. For now we will only prove that the natural Frobenius on $R\Gamma_{\mathbb{\Delta}}(-/\mathfrak{S})$ is an isomorphism if we invert E:

Proposition 3.5.4.3. *Let $\mathcal{X}$ be a smooth Artin stack over $\mathcal{O}_K$. Then the induced map*

$$\varphi[\tfrac{1}{E}]\colon \varphi_{\mathfrak{S}}^{*} R\Gamma_{\mathbb{\Delta}}(\mathcal{X}/\mathfrak{S})[\tfrac{1}{E}] \longrightarrow R\Gamma_{\mathbb{\Delta}}(\mathcal{X}/\mathfrak{S})[\tfrac{1}{E}]$$

is an equivalence.

Proof. Let $\mathcal{X}$ be n-Artin. We will prove the statement by induction on n. The base of induction $n = 0$ is [19, Theorem 1.8 (6)]. Let $U \twoheadrightarrow \mathcal{X}$ be a smooth atlas and let $U_\bullet$ be the corresponding Čech nerve. By induction assumption the map $\varphi[\frac{1}{E}]$ is an equivalence for all $R\Gamma_{\mathbb{\Delta}}(U_i/\mathfrak{S})$. Since by descent for prismatic cohomology the natural map

$$R\Gamma_{\mathbb{\Delta}}(\mathcal{X}/\mathfrak{S}) \longrightarrow \operatorname{Tot} R\Gamma_{\mathbb{\Delta}}(U_\bullet/\mathfrak{S})$$

is an equivalence, it is enough to prove that the natural map

$$\varphi_{\mathfrak{S}}^{*}\left(\operatorname{Tot} R\Gamma_{\mathbb{\Delta}}(U_\bullet/\mathfrak{S})\right)[\tfrac{1}{E}] \longrightarrow \operatorname{Tot}\left(R\Gamma_{\mathbb{\Delta}}(U_\bullet/\mathfrak{S})[\tfrac{1}{E}]\right)$$

is an equivalence. This holds by Corollary A.1.1.7, since the localization map $\mathfrak{S} \to \mathfrak{S}[\frac{1}{E}]$ is flat (and so is $\varphi_{\mathfrak{S}}$). □

Corollary 3.5.4.4. *Let $\mathcal{X}$ be a smooth Hodge-proper Artin stack over $\mathcal{O}_K$. Then $H^i_{\mathbb{\Delta}}(\mathcal{X}/\mathfrak{S})$ is a Breuil-Kisin module for all $i \in \mathbb{Z}_{\geq 0}$.*

Proof. From the de Rham comparison (Proposition 3.5.5.1) below and Hodge-properness it follows that $\varphi_{\mathfrak{S}}^{*} R\Gamma_{\mathbb{\Delta}}(\mathcal{X}/\mathfrak{S})/E \in \mathrm{Coh}^{+}(\mathcal{O}_K)$ (see Proposition 3.4.2.8). Then, by Proposition 3.4.3.1 we get that $\varphi_{\mathfrak{S}}^{*} R\Gamma_{\mathbb{\Delta}}(\mathcal{X}/\mathfrak{S}) \in \mathrm{Coh}^{+}(\mathfrak{S})$ and, since $\varphi_{\mathfrak{S}}^{*}$ is faithfully flat, also that $R\Gamma_{\mathbb{\Delta}}(\mathcal{X}/\mathfrak{S}) \in \mathrm{Coh}^{+}(\mathfrak{S})$. Thus $H^i_{\mathbb{\Delta}}(\mathcal{X}/\mathfrak{S})$ is finitely generated over $\mathfrak{S}$ for all $i \in \mathbb{Z}_{\geq 0}$. The natural map $\varphi[\frac{1}{E}] \curvearrowright H^i_{\mathbb{\Delta}}(\mathcal{X}/\mathfrak{S})[\frac{1}{E}]$ is an isomorphism by 3.5.4.3. □

3.5.5 Comparisons and finiteness results

De Rham and crystalline comparisons

The de Rham and crystalline comparisons follow formally from the case of schemes (if we use 3.5.2.3 as a definition of the crystalline cohomology of stacks):

Proposition 3.5.5.1. *Let $\mathcal{X}$ be a smooth Artin stack over $\mathcal{O}_K$. Then*

- *(De Rham comparison) There is a canonical equivalence*

$$R\Gamma_{\mathbb{\Delta}}(\mathcal{X}/\mathfrak{S}) \otimes_{\mathfrak{S},\varphi} \mathcal{O}_K \simeq \widehat{R\Gamma}_{\mathrm{dR}}(\mathcal{X}/\mathcal{O}_K).$$

  If $\mathcal{X}$ is Hodge-proper, the latter is isomorphic to $R\Gamma_{\mathrm{dR}}(\mathcal{X}/\mathcal{O}_K)$ (3.5.1.4.)

- *(Crystalline comparison) There is a canonical equivalence*

$$R\Gamma_{\mathbb{\Delta}}(\mathcal{X}/\mathfrak{S}) \otimes_{\mathfrak{S},\varphi} W(k) \simeq R\Gamma_{\mathrm{crys}}(\mathcal{X}_k/W(k)).$$

Proof. Since the Frobenius homomorphism $\varphi_{\mathfrak{S}}\colon \mathfrak{S} \to \mathfrak{S}$ is flat and since $\mathcal{O}_K$ and $W(k)$ are perfect over $\mathfrak{S}$, the tensor product functors $- \otimes_{\mathfrak{S},\varphi} \mathcal{O}_K$ and $- \otimes_{\mathfrak{S},\varphi} W(k)$ commute with totalizations of coconnected modules. So by descent the result follows from the corresponding comparisons for schemes [19, Theorem 1.8]. □

In Hodge-proper case we can deduce finiteness of various cohomology theories from the de Rham comparison:

Proposition 3.5.5.2. *Let $\mathcal{X}$ be a smooth Hodge-proper Artin stack. Then $R\Gamma_{\mathbb{\Delta}}(\mathcal{X}/\mathfrak{S})$ and $R\Gamma_{\mathrm{crys}}(\mathcal{X}_k/W(k))$ are nearly coherent over $\mathfrak{S}$ and $W(k)$ respectively.*

Proof. Since $W(k)$ is of finite Tor-amplitude over $\mathfrak{S}$, the result for $R\Gamma_{\mathrm{crys}}(\mathcal{X}_k/W(k))$ follows formally from the case for $R\Gamma_{\mathbb{\Delta}}(\mathcal{X}/\mathfrak{S})$ by the crystalline comparison. Since by Remark 3.5.4.2 the complex $\varphi^* R\Gamma_{\mathbb{\Delta}}(\mathcal{X}/\mathfrak{S})$ is E-adicly complete and $\varphi^* R\Gamma_{\mathbb{\Delta}}(\mathcal{X}/\mathfrak{S})/E \simeq R\Gamma_{\mathrm{dR}}(\mathcal{X}/\mathcal{O}_K)$ is nearly coherent over $\mathcal{O}_K$ (see Proposition 3.4.2.8) by Proposition 3.4.3.1 we deduce that $\varphi^* \Gamma_{\mathbb{\Delta}}(\mathcal{X}/\mathfrak{S})$ is nearly coherent over $\mathfrak{S}$. Since the Frobenius homomorphism $\varphi_{\mathfrak{S}}\colon \mathfrak{S} \to \mathfrak{S}$ is flat, we conclude by Proposition 3.4.2.5. □

Étale comparison

In this section we prove the étale comparison. However, as we have seen in Section 3.5.3, in the context of stacks there are two natural options for what one can call the étale cohomology of the generic fiber of $\mathcal{X}$: the étale cohomology of either algebraic or Raynaud generic fiber of $\mathcal{X}$. For a general Artin stack the prismatic cohomology compare only with the latter:

Proposition 3.5.5.3. *Let $\mathcal{X}$ be a smooth Artin stack over $\mathcal{O}_K$. Then there are canonical equivalences*

$$R\Gamma_{\text{ét}}(\widehat{\mathcal{X}}_{\mathbb{C}_p}, \mathbb{Z}/p^n) \simeq \left(R\Gamma_{\mathbb{\Delta}}(\mathcal{X}_{\mathcal{O}_{\mathbb{C}_p}}/A_{\text{inf}})/p^n[\tfrac{1}{\xi}]\right)^{\varphi=1} \quad \text{and} \quad R\Gamma_{\text{ét}}(\widehat{\mathcal{X}}_{\mathbb{C}_p}, \mathbb{Z}_p) \simeq \left(R\Gamma_{\mathbb{\Delta}}(\mathcal{X}/\mathfrak{S}) \otimes_{\mathfrak{S}} W(\mathbb{C}_p^\flat)\right)^{\varphi=1}.$$

Proof. For the first equivalence note that both parts satisfy étale (equivalently smooth) descent. Hence the result follows from the affine case [19, Theorem 1.8(4)]. The second equivalence follows from the first one and the fact that the p-adic completion of $A_{\text{inf}}[1/\xi]$ is isomorphic to $W(\mathbb{C}_p^\flat)$. □

Corollary 3.5.5.4. *Let $\mathcal{X}$ be a smooth Artin stack over $\mathcal{O}_K$. Then there is a canonical equivalence*

$$R\Gamma_{\mathbb{\Delta}}(\mathcal{X}/\mathfrak{S}) \otimes_{\mathfrak{S}} W(\mathbb{C}_p^\flat) \simeq R\Gamma_{\text{ét}}(\widehat{\mathcal{X}}_{\mathbb{C}_p}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} W(\mathbb{C}_p^\flat).$$

Proof. By the second equivalence from the previous proposition it is enough to prove the following more general statement: for a nearly coherent complex X of $W(\mathbb{C}_p^\flat)$-modules the natural map

$$X^{\varphi=1} \otimes_{\mathbb{Z}_p} W(\mathbb{C}_p^\flat) \longrightarrow X$$

is an equivalence. Note that both parts commute with colimits, hence it is enough to prove the statement for $X^{\leq n}$. But since $W(\mathbb{C}_p^\flat)$ is regular, coherent $W(\mathbb{C}_p^\flat)$-modules are perfect, so the result follows from [16, Lemma 8.5]. □

Finally, we can deduce finiteness of étale cohomology under the Hodge-properness assumption:

Corollary 3.5.5.5. *Let $\mathcal{X}$ be a smooth Hodge-proper Artin stack over $\mathcal{O}_K$. Then $R\Gamma_{\text{ét}}(\widehat{\mathcal{X}}_{\mathbb{C}_p}, \mathbb{Z}_p)$ is a nearly coherent complex of $\mathbb{Z}_p$-modules.*

Proof. By the étale comparison and Proposition 3.5.5.2 the complex $R\Gamma_{\text{ét}}(\widehat{\mathcal{X}}_{\mathbb{C}_p}, \mathbb{Z}_p) \otimes W(\mathbb{C}_p^\flat)$ is nearly coherent. Since the natural map $\mathbb{Z}_p \to W(\mathbb{C}_p^\flat)$ is faithfully flat, we conclude by Proposition 3.4.2.5. □

In particular, each individual cohomology $H^n_{\text{ét}}(\widehat{\mathcal{X}}_{\mathbb{C}_p}, \mathbb{Z}_p)$ is a finitely generated $\mathbb{Z}_p$-module.

3.5.6 Inequality on dimensions

Let's restrict to the unramified context $\mathcal{O}_K = W(k)$. Note that (since the Tor-dimension of $\mathfrak{S}$ is finite, given by 2) for a fixed i and $n \gg 0$ we have $H^i(\tau^{\leq n} R\Gamma_{\mathbb{\Delta}}(\mathcal{X}/\mathfrak{S})/\xi) \simeq H^i(R\Gamma_{\mathbb{\Delta}}(\mathcal{X}/\mathfrak{S})/\xi)$. Since for $\mathcal{X}/W(k)$ Hodge-proper stack $\tau^{\leq n} R\Gamma_{\mathbb{\Delta}}(\mathcal{X}/\mathfrak{S})$ is coherent for any n, by semicontinuity of the dimensions of stalks (and the comparisons) we get

Theorem 3.5.6.1. *Let $\mathcal{X}$ be a smooth Hodge-proper stack over $W(k)$. Then for any $i \geq 0$*

$$\dim_k H^i_{\text{dR}}(\mathcal{X}_k/k) \geq \dim_{\mathbb{F}_p} H^i_{\text{ét}}(\widehat{\mathcal{X}}_{\mathbb{C}_p}, \mathbb{F}_p).$$

3.6 Comparison between étale cohomology of generic fibers of $[X/G]$

Let $C = \mathbb{C}_p$. In this section, we show that the two versions of étale cohomology of the generic fiber agree for $\mathcal{Y} \coloneqq [X/G]$, with X proper and G reductive. Namely we show that $R\Gamma_{\text{ét}}(\widehat{\mathcal{Y}}_C, \mathbb{Z}_p) \simeq R\Gamma_{\text{ét}}(\mathcal{Y}_C, \mathbb{Z}_p)$ where $\mathcal{Y}_C$ is the fiber of $\mathcal{Y}$ over $\operatorname{Spec} C \to \operatorname{Spec} \mathcal{O}_K$. While for a proper scheme X we have an equality of rigid spaces $X^{\text{an}}_K \simeq \widehat{X}_K$, for stacks that are (only) cohomologically proper, this is no more true; nevertheless, at least in some cases, one could still expect at least the cohomology to be the same. Note that $R\Gamma_{\text{ét}}(\widehat{\mathcal{Y}}_C, \mathbb{Z}_p)$ depends exclusively on the underlying formal stack $\widehat{\mathcal{Y}}$. On the other hand, $R\Gamma_{\text{ét}}(\mathcal{Y}_C, \mathbb{Z}_p)$ is purely algebraic and using Artin comparison (3.5.3.5) can be identified with the cohomology of the corresponding analytic stack, which in turn is equal to cohomology of a certain space, very familiar in some cases (3.5.3.4).

For example proving this for BG where G is reductive leads to the inequality

$$\dim_{\mathbb{F}_p} H^i_{\text{dR}}(BG_{\mathbb{F}_p}) \geq \dim_{\mathbb{F}_p} H^i_{\text{sing}}(BG(\mathbb{C}), \mathbb{F}_p),$$

where $G(\mathbb{C})$ is the topological group of $\mathbb{C}$-points of G and $BG(\mathbb{C})$ is its corresponding classifying space. This was observed through an explicit computation by Totaro in [111] for a large list of groups; he conjectured that the inequality should hold in general and possibly follow from the

recent results in integral p-adic Hodge theory. One of the things we do is show that his hopes were true.

3.6.1 De Rham cohomology of $B\mathbb{G}_m$

Recall that in Section 3.5.1 for any Artin stack $\mathcal{X}$ over any base ring R we have defined the Hodge cohomology complex $R\Gamma_{\mathrm{H}}(\mathcal{X}/R)$ and the de Rham cohomology complex $R\Gamma_{\mathrm{dR}}(\mathcal{X}/R)$. In this section we compute them for $X = B\mathbb{G}_m$ over $\mathbb{Z}$. We also show that pullback under the natural map $\mathbb{P}^\infty \to B\mathbb{G}_m$ induces an equivalence on de Rham cohomology.

The cotangent complex of $B\mathbb{G}_m$ is given by $\mathcal{O}_{B\mathbb{G}_m/\mathbb{Z}}[-1]$ and more generally $\wedge^i \mathbb{L}_{B\mathbb{G}_m/\mathbb{Z}} \simeq \mathcal{O}_{B\mathbb{G}_m/\mathbb{Z}}[-i]$. Moreover, under the equivalence $\mathrm{QCoh}(B\mathbb{G}_m) \simeq D(\mathrm{Rep}_{\mathbb{G}_m})$, the algebra $\bigoplus_{i=0}^\infty \wedge^i(\mathbb{L}_{B\mathbb{G}_m/\mathbb{Z}})[-i]$ corresponds just to $\bigoplus_{i=0}^\infty \mathbb{Z}(0)[-2i]$, where $\mathbb{Z}(0)$ is the trivial rank 1 representation of $\mathbb{G}_m$, and the algebra structure is given by the natural isomorphisms $\mathbb{Z}(0)[-2i] \otimes \mathbb{Z}(0)[-2j] \simeq \mathbb{Z}(0)[-2(i+j)]$. This graded algebra is equivalent to the polynomial ring $\mathbb{Z}(0)[x]$, where x has cohomological degree 2. We also have $R\Gamma(\mathbb{G}_m, \mathbb{Z}(0)) \simeq \mathbb{Z}(0)^{\mathbb{G}_m} \simeq \mathbb{Z}$, since the functor of $\mathbb{G}_m$-invariants is t-exact over arbitrary base. We get

Lemma 3.6.1.1. *The algebra of Hodge cohomology $H^\bullet_{\mathrm{H}}(B\mathbb{G}_m/\mathbb{Z})$ of $B\mathbb{G}_m$ over $\mathbb{Z}$ is isomorphic to $\mathbb{Z}[x]$, $\deg(x) = 2$.*

Corollary 3.6.1.2. *The Hodge-de Rham spectral sequence for $B\mathbb{G}_m$ over $\mathbb{Z}$ degenerates and $H^\bullet_{\mathrm{dR}}(B\mathbb{G}_m/\mathbb{Z}) \simeq \mathbb{Z}[x]$, $\deg(x) = 2$.*

Proof. The first statement holds for every linearly reductive group G, since $E_1^{p,q} \simeq 0$ for $p \neq q$ by semi-simplicity of the category of G-representations. So $\mathrm{gr}^\bullet H^\bullet_{\mathrm{dR}}(B\mathbb{G}_m/\mathbb{Z}) \simeq H^\bullet_{\mathrm{H}}(B\mathbb{G}_m/\mathbb{Z}) \simeq \mathbb{Z}[x]$. By choosing a homogeneous lift of x we obtain a filtered map $\mathbb{Z}[x] \to H^\bullet_{\mathrm{dR}}(B\mathbb{G}_m/\mathbb{Z})$, which induces an isomorphism on the associated graded pieces, hence it is an isomorphism itself. □

Remark 3.6.1.3. Since $B\mathbb{G}_m$ is smooth over $\mathbb{Z}$, by base change the same result holds for any R:

$$H^\bullet_{\mathrm{dR}}(B\mathbb{G}_m/R) \simeq H^\bullet_{\mathrm{H}}(B\mathbb{G}_m/R) \simeq R[x],\ \deg(x) = 2.$$

Let $\mathbb{P}^n$ be the projective space over $\operatorname{Spec}\mathbb{Z}$. Then by the standard computation of the Hodge

cohomology of $\mathbb{P}^n$ (e.g. see Lemma B.1.2) we have

$$H^\bullet_{\mathrm{H}}(\mathbb{P}^n/\mathbb{Z}) \simeq \mathbb{Z}[y]/y^{n+1},\ \deg(y) = 2.$$

The Hodge-de Rham spectral sequence degenerates since the cohomology of the graded pieces $R\Gamma(\mathbb{P}^n/\mathbb{Z}, \Omega^i_{\mathbb{P}^n}[-i])$ are concentrated in even total degrees and so we have the same answer for the de Rham cohomology

$$H^\bullet_{\mathrm{dR}}(\mathbb{P}^n/\mathbb{Z}) \simeq \mathbb{Z}[y]/y^{n+1}.$$

Note that any subspace inclusion $i\colon \mathbb{P}^k \hookrightarrow \mathbb{P}^n$ induces a map $i^*\colon R\Gamma_{\mathrm{dR}}(\mathbb{P}^n/\mathbb{Z}) \to R\Gamma_{\mathrm{dR}}(\mathbb{P}^n/\mathbb{Z})$, which on the cohomology can be identified with the quotient by $y^{k+1}\colon \mathbb{Z}[y]/(y^{n+1}) \twoheadrightarrow \mathbb{Z}[y]/(y^{k+1})$.

The Serre's line bundle $\mathcal{O}_{\mathbb{P}^n}(1)$ on $\mathbb{P}^n$ is classified by a map $\mathcal{O}_{\mathbb{P}^n}(1)\colon \mathbb{P}^n \to B\mathbb{G}_m$ and we can consider the corresponding pull-back map $\mathcal{O}_{\mathbb{P}^n}(1)^*\colon R\Gamma_{\mathrm{dR}}(B\mathbb{G}_m/\mathbb{Z}) \to R\Gamma_{\mathrm{dR}}(\mathbb{P}^n/\mathbb{Z})$ on the de Rham cohomology.

Proposition 3.6.1.4. *Under the identifications $H^\bullet_{\mathrm{dR}}(B\mathbb{G}_m/\mathbb{Z}) \simeq \mathbb{Z}[x]$ and $H^\bullet_{\mathrm{dR}}(\mathbb{P}^n/\mathbb{Z}) \simeq \mathbb{Z}[y]/y^{n+1}$ we have*

$$\mathcal{O}_{\mathbb{P}^n}(1)^*(x) = \pm y \qquad \text{and} \qquad H^2_{\mathrm{dR}}(B\mathbb{G}_m/\mathbb{Z}) \simeq H^2_{\mathrm{dR}}(\mathbb{P}^n/\mathbb{Z}).$$

As a consequence, the fiber of the pullback map $\mathcal{O}_{\mathbb{P}^n}(1)^*\colon R\Gamma_{\mathrm{dR}}(B\mathbb{G}_m/\mathbb{Z}) \to R\Gamma_{\mathrm{dR}}(\mathbb{P}^n/\mathbb{Z})$ is $(2n+1)$-coconnected.

Proof. Consider the map of presheaves $d\log\colon \mathcal{O}^\times_- \to \Omega^1_-$ on $\mathrm{Aff}^{\mathrm{op,sm}}_{/B\mathbb{G}_m}$ given on each $X \to B\mathbb{G}_m$ by the map $d\log\colon \mathcal{O}^\times_X \to \Omega^1_X$ that sends an invertible function $f \in \mathcal{O}^\times_X(X)$ to the differential form $d\log(f) := df/f$. Passing to the limit over $\mathrm{Aff}^{\mathrm{op,sm}}_{/B\mathbb{G}_m}$ and applying H^1, we obtain a map

$$d\log\colon \mathrm{Pic}(B\mathbb{G}_m) \longrightarrow H^1(B\mathbb{G}_m, \Omega^1_-) \overset{(*)}{\simeq} H^1(B\mathbb{G}_m, \mathbb{L}_{B\mathbb{G}_m/\mathbb{Z}}) \simeq H^2_{\mathrm{dR}}(B\mathbb{G}_m) \simeq \mathbb{Z},$$

where $\mathrm{Pic}(\mathcal{X}) := \pi_0 \mathrm{Hom}_{\mathscr{S}\mathrm{tk}}(\mathcal{X}, B\mathbb{G}_m)$ and the equivalence $(*)$ follows from the smooth descent for cotangent complex (Proposition 3.4.1.4). On the other hand

$$\mathrm{Pic}(B\mathbb{G}_m) = \pi_0 \mathrm{Hom}_{\mathscr{S}\mathrm{tk}}(B\mathbb{G}_m, B\mathbb{G}_m) \simeq \mathrm{Hom}_{\mathrm{Grp}}(\mathbb{G}_m, \mathbb{G}_m) \simeq \mathbb{Z}.$$

So the map $d\log\colon \mathrm{Pic}(B\mathbb{G}_m) \to H^2_{\mathrm{dR}}(B\mathbb{G}_m)$ is some map from $\mathbb{Z}$ to $\mathbb{Z}$ and we need to show it is an isomorphism.

For this end consider the pull-back to $\mathbb{P}^n$ of the transformation $d\log\colon R\Gamma_{\text{ét}}(-,\mathcal{O}^\times)\to\Omega^1$ under $\mathcal{O}_{\mathbb{P}^n}(1)\colon \mathbb{P}^n\to B\mathbb{G}_m$. This gives a commutative square:

$$\begin{array}{ccc} \mathrm{Pic}(B\mathbb{G}_m/\mathbb{Z}) & \xrightarrow{\mathcal{O}(1)^*} & \mathrm{Pic}(\mathbb{P}^n/\mathbb{Z}) \\ \downarrow{\scriptstyle d\log} & & \downarrow{\scriptstyle d\log} \\ H^2_{\mathrm{dR}}(B\mathbb{G}_m/\mathbb{Z}) & \xrightarrow{\mathcal{O}(1)^*} & H^2_{\mathrm{dR}}(\mathbb{P}^n/\mathbb{Z}). \end{array}$$

Since the line bundle $\mathcal{O}_{\mathbb{P}^n}(1)$ generates $\mathrm{Pic}(\mathbb{P}^n/\mathbb{Z})$ the top horizontal arrow is an isomorphism. Since all groups in the diagram are isomorphic to $\mathbb{Z}$, if we show that the right vertical arrow is an isomorphism it will follow that all remaining arrows are isomorphisms as well. Consider an embedding of a line $i\colon \mathbb{P}^1\hookrightarrow\mathbb{P}^n$, then i^* induces isomorphisms on both the Picard groups and H^2_{dR} and so it is enough to check that $d\log$ is an isomorphism for $\mathbb{P}^1$. This is easy (see Lemma B.1.2). We get that a generator x in $H^2_{\mathrm{dR}}(B\mathbb{G}_m/\mathbb{Z})$ is getting mapped to a generator of $H^2_{\mathrm{dR}}(\mathbb{P}^n/\mathbb{Z})$, two options given by $\pm y$.

The pullback $\mathcal{O}_{\mathbb{P}^n}(1)^*\colon H^\bullet_{\mathrm{dR}}(B\mathbb{G}_m)\to H^\bullet_{\mathrm{dR}}(\mathbb{P}^n)$ is a homomorphism of algebras and since $\mathcal{O}_{\mathbb{P}^n}(1)^*(x)=\pm y$ it can be identified with the projection $\mathbb{Z}[y]\twoheadrightarrow\mathbb{Z}[y]/(y^{n+1})$ and we have $H^\bullet(\mathrm{fib}(\mathcal{O}_{\mathbb{P}^n}(1)^*))\simeq y^{n+1}\cdot\mathbb{Z}[y]$ which lives in cohomological degrees $\geq 2n+2$, so the fiber is $(2n+1)$-coconnected. □

Now consider an infinite system of subspaces $\mathbb{P}^0\overset{i_0}{\hookrightarrow}\mathbb{P}^1\overset{i_1}{\hookrightarrow}\cdots\overset{i_{n-1}}{\hookrightarrow}\mathbb{P}^n\overset{i_n}{\hookrightarrow}\cdots$. The collection of line bundles $\mathcal{O}_{\mathbb{P}^n}(1)$ on $\mathbb{P}^n$ produces a compatible system of maps $\mathcal{O}_{\mathbb{P}^n}(1)\colon\mathbb{P}^n\to B\mathbb{G}_m$. Let $\mathcal{O}_{\mathbb{P}^\infty}(1)^*\colon R\Gamma_{\mathrm{dR}}(\mathbb{B}G_m/\mathbb{Z})\to\varprojlim R\Gamma_{\mathrm{dR}}(\mathbb{P}^n/\mathbb{Z})$ be the natural map to the limit.

Corollary 3.6.1.5. *The pullback map*

$$\mathcal{O}_{\mathbb{P}^\infty}(1)^*\colon R\Gamma_{\mathrm{dR}}(B\mathbb{G}_m/\mathbb{Z})\longrightarrow\varprojlim R\Gamma_{\mathrm{dR}}(\mathbb{P}^n/\mathbb{Z})$$

is an equivalence.

Proof. By Proposition 3.6.1.4, the coconnectivity of the fiber of the map $\mathcal{O}_{\mathbb{P}^n}(1)^*$ tends to ∞ when $n\to\infty$. □

3.6.2 Prismatic and étale cohomology of $B\mathbb{G}_m$

Fix a Breuil-Kisin prism $(\mathfrak{S},(E))$ with $\mathfrak{S}/E \simeq \mathcal{O}_K$. Again we consider a system $\mathbb{P}^0 \overset{i_0}{\hookrightarrow} \mathbb{P}^1 \overset{i_1}{\hookrightarrow} \cdots \overset{i_{n-1}}{\hookrightarrow} \mathbb{P}^n \overset{i_n}{\hookrightarrow} \cdots$, but now over $\mathcal{O}_K$. We have compatible maps $\mathcal{O}_{\mathbb{P}^n}(1)\colon \mathbb{P}^n \to B\mathbb{G}_m$ which produce a map $\mathcal{O}_{\mathbb{P}^\infty}(1)^*\colon R\Gamma_{\mathbb{\Delta}}(B\mathbb{G}_m/\mathfrak{S}) \to \varprojlim R\Gamma_{\mathbb{\Delta}}(\mathbb{P}^n/\mathfrak{S})$. To identify the right hand side recall that the étale cohomology of $\mathbb{P}^n_C$ are given by

$$H^*_{\text{ét}}(\mathbb{P}^n_C,\mathbb{Z}_p) \simeq \bigoplus_{i=0}^{n} \mathbb{Z}_p(-i)[-2i].$$

Recall that by [74, Theorem 1.2.1] the functor $\mathrm{BK}\colon \mathrm{Rep}^{\#,\mathrm{crys}}_{G_K} \to \mathrm{Mod}^{\varphi}_{\mathfrak{S}}$ is a symmetric monoidal equivalence and $\mathrm{BK}(\mathbb{Q}_p(n)) \simeq \mathfrak{S}\{n\}$. Moreover, since the crystalline cohomology of $\mathbb{P}^n$ are torsion free, by [17, Theorem 14.6(iii)] there is a natural isomorphism of Breuil-Kisin modules $H^i_{\mathbb{\Delta}}(\mathbb{P}^n/\mathfrak{S}) \simeq \mathrm{BK}(H^i_{\text{ét}}(\mathbb{P}^n_K,\mathbb{Q}_p))$. So

$$H^*_{\mathbb{\Delta}}(\mathbb{P}^n/\mathfrak{S}) \simeq \mathrm{BK}(H^*_{\text{ét}}(\mathbb{P}^n_{\mathbb{C}_p},\mathbb{Z}_p)) \simeq \bigoplus_{i=0}^{n} \mathrm{BK}(\mathbb{Z}_p(-i))[-2i] \simeq \bigoplus_{i=0}^{n} \mathfrak{S}\{-i\}[-2i]$$

as a ring object in the category of graded Breuil-Kisin modules. Here the ring structure on the right hand side is induced by isomorphisms $\mathfrak{S}\{-i\} \otimes_{\mathfrak{S}} \mathfrak{S}\{-j\} \simeq \mathfrak{S}\{-i-j\}$ if $i+j \leq n$ and zero otherwise. In particular the underlying $\mathfrak{S}$-module is just $\mathfrak{S}[x]/(x^{n+1})$.

Lemma 3.6.2.1. *The map $\mathcal{O}_{\mathbb{P}^\infty}(1)^* : R\Gamma_{\mathbb{\Delta}}(B\mathbb{G}_m/\mathfrak{S}) \to \varprojlim R\Gamma_{\mathbb{\Delta}}(\mathbb{P}^n/\mathfrak{S})$ is an equivalence.*

Proof. We will use the de Rham comparison and the derived Nakayama. Namely, since $\varphi_{\mathfrak{S}}$ is faithfully flat, it is enough to prove that

$$\varphi_{\mathfrak{S}}(\mathcal{O}_{\mathbb{P}^\infty}(1)^*)\colon \varphi^*_{\mathfrak{S}} R\Gamma_{\mathbb{\Delta}}(B\mathbb{G}_m/\mathfrak{S}) \longrightarrow \varphi^*_{\mathfrak{S}}(\varprojlim R\Gamma_{\mathbb{\Delta}}(\mathbb{P}^n/\mathfrak{S}))$$

is an equivalence. Moreover, since both sides are E-adicly complete, it is enough to check this modulo $E \in \mathfrak{S}$. Coconnectivity of the natural projections $\varprojlim R\Gamma_{\mathbb{\Delta}}(\mathbb{P}^n/\mathfrak{S}) \to R\Gamma_{\mathbb{\Delta}}(\mathbb{P}^n/\mathfrak{S})$ tends to infinity, so, by flatness of $\varphi_{\mathfrak{S}}$, the map

$$\varphi^*_{\mathfrak{S}}(\varprojlim R\Gamma_{\mathbb{\Delta}}(\mathbb{P}^n/\mathfrak{S})) \longrightarrow \varprojlim \varphi^*_{\mathfrak{S}} R\Gamma_{\mathbb{\Delta}}(\mathbb{P}^n/\mathfrak{S}).$$

is an equivalence. Moreover, since $\mathcal{O}_K$ is a perfect $\mathfrak{S}$-module, the natural map

$$(\varprojlim \varphi_{\mathfrak{S}}^* R\Gamma_{\blacktriangle}(\mathbb{P}^n/\mathfrak{S})) \otimes_{\mathfrak{S}} \mathcal{O}_K \longrightarrow \varprojlim \left(\varphi_{\mathfrak{S}}^* R\Gamma_{\blacktriangle}(\mathbb{P}^n/\mathfrak{S}) \otimes_{\mathfrak{S}} \mathcal{O}_K\right)$$

is also an equivalence. So it is enough to check that

$$\varphi_{\mathfrak{S}}(\mathcal{O}_{\mathbb{P}^\infty}(1)^*)/\xi \colon \varphi_{\mathfrak{S}}^*(R\Gamma_{\blacktriangle}(B\mathbb{G}_m/\mathfrak{S})) \otimes_{\mathfrak{S}} \mathcal{O}_K \longrightarrow \varprojlim \left(\varphi_{\mathfrak{S}}^* R\Gamma_{\blacktriangle}(\mathbb{P}^n/\mathfrak{S}) \otimes_{\mathfrak{S}} \mathcal{O}_K\right)$$

is an equivalence. But by the de Rham comparison (Proposition 3.5.5.1) this map identifies with

$$\mathcal{O}_{\mathbb{P}^\infty}(1)^* \colon R\Gamma_{\mathrm{dR}}(B\mathbb{G}_m/\mathcal{O}_K) \longrightarrow \varprojlim R\Gamma_{\mathrm{dR}}(\mathbb{P}^n/\mathcal{O}_K),$$

which is an equivalence by Corollary 3.6.1.5 and base change. □

Remark 3.6.2.2. In particular we get that

$$H^*_{\blacktriangle}(B\mathbb{G}_m/\mathfrak{S}) \simeq \mathrm{Sym}(\mathfrak{S}\{-1\}[-2]) \simeq \bigoplus_{i\geq 0} \mathfrak{S}\{-i\}[-2i]$$

as a ring object in the category of graded Breuil-Kisin modules (with the underlying $\mathfrak{S}$-algebra isomorphic to $\mathfrak{S}[x]$).

Corollary 3.6.2.3. *The pullback map on the étale cohomology of the Raynaud generic fibers*

$$\mathcal{O}_{\mathbb{P}^\infty}(1)^* \colon R\Gamma_{\text{ét}}((\widehat{B\mathbb{G}_m})_C, \mathbb{Z}_p) \longrightarrow \varprojlim R\Gamma_{\text{ét}}((\widehat{\mathbb{P}^n})_C, \mathbb{Z}_p)$$

is an equivalence.

Proof. By the étale comparison Proposition 3.5.5.3 we have

$$R\Gamma_{\text{ét}}((\widehat{B\mathbb{G}_m})_C, \mathbb{Z}_p) \simeq \left(R\Gamma_{\blacktriangle}(B\mathbb{G}_m/\mathfrak{S}) \otimes_{\mathfrak{S}} W(C^\flat)\right)^{\varphi=1} \quad \text{and} \quad R\Gamma_{\text{ét}}((\widehat{\mathbb{P}^n})_C, \mathbb{Z}_p) \simeq \left(R\Gamma_{\blacktriangle}(\mathbb{P}^n/\mathfrak{S}) \otimes_{\mathfrak{S}} W(C^\flat)\right)^{\varphi=1}$$

Moreover, by flatness of $W(C^\flat)$ over $\mathfrak{S}$ the natural map

$$\left((\varprojlim R\Gamma_{\blacktriangle}(\mathbb{P}^n/\mathfrak{S})) \otimes_{\mathfrak{S}} W(C^\flat)\right)^{\varphi=1} \longrightarrow \varprojlim \left(R\Gamma_{\blacktriangle}(\mathbb{P}^n/\mathfrak{S}) \otimes_{\mathfrak{S}} W(C^\flat)\right)^{\varphi=1}$$

is an equivalence. So we conclude by the previous lemma. □

The map $\mathcal{O}_{\mathbb{P}^\infty}(1)^*\colon R\Gamma_{\text{ét}}((B\mathbb{G}_m)_C, \mathbb{Z}_p) \to \varprojlim R\Gamma_{\text{ét}}((\mathbb{P}^n)_C, \mathbb{Z}_p)$ is an equivalence by the $\mathbb{A}^1$-homotopy invariance of the (algebraic) étale cohomology in Γ:

Lemma 3.6.2.4. *The map $\mathcal{O}_{\mathbb{P}^\infty}(1)^*\colon R\Gamma_{\text{ét}}((B\mathbb{G}_m)_C, \mathbb{Z}_p) \to \varprojlim R\Gamma_{\text{ét}}((\mathbb{P}^n)_C, \mathbb{Z}_p)$ is an equivalence.*

Proof. First note that for each n we have an open embedding $j_n\colon \mathbb{P}^n \simeq [(\mathbb{A}^{n+1} \setminus 0)/\mathbb{G}_m] \to [\mathbb{A}^{n+1}/\mathbb{G}_m]$ and a projection $p_n\colon [\mathbb{A}^{n+1}/\mathbb{G}_m] \to [\mathrm{pt}/\mathbb{G}_m] = B\mathbb{G}_m$. The map $\mathcal{O}_{\mathbb{P}^n}(1)\colon \mathbb{P}^n \to \mathbb{B}\mathbb{G}_m$ factors as $p_n \circ j_n$. We also have a section $s_n\colon B\mathbb{G}_m \simeq [\{0\}/\mathbb{G}_m] \hookrightarrow [\mathbb{A}^{n+1}/\mathbb{G}_m]$ of p_n. Since (algebraic) étale $\mathbb{Z}_p$-cohomology over C are $\mathbb{A}^1$-homotopy invariant, the maps

$$p_n^*\colon R\Gamma_{\text{ét}}((B\mathbb{G}_m)_C, \mathbb{Z}_p) \leftrightarrows R\Gamma_{\text{ét}}([\mathbb{A}^n/\mathbb{G}_m]_C, \mathbb{Z}_p) \ :s_n^*$$

are inverse equivalences. In particular, if we choose a sequence of linear embeddings (which are then automatically $\mathbb{G}_m$-equivariant) $\mathbb{A}^1 \overset{i_0}{\hookrightarrow} \mathbb{A}^2 \overset{i_1}{\hookrightarrow} \mathbb{A}^3 \overset{i_2}{\hookrightarrow} \dots$ the natural map $p_\infty^*\colon R\Gamma_{\text{ét}}((B\mathbb{G}_m)_C, \mathbb{Z}_p) \to \varprojlim R\Gamma_{\text{ét}}([\mathbb{A}^n/\mathbb{G}_m]_C, \mathbb{Z}_p)$ is an equivalence. So it is enough to prove,for each n, that the fiber of the map $j_n^*\colon R\Gamma_{\text{ét}}([\mathbb{A}^{n+1}/\mathbb{G}_m]_C, \mathbb{Z}_p) \to R\Gamma_{\text{ét}}(\mathbb{P}^n_C, \mathbb{Z}_p)$ induced by the open embedding $j_n\colon \mathbb{P}^n \simeq (\mathbb{A}^n \setminus \{0\})/\mathbb{G}_m \to [\mathbb{A}^n/\mathbb{G}_m]$ is n-coconnected.

To see this we will use Čech objects for covers $u_1\colon \mathbb{A}^{n+1} \to [\mathbb{A}^{n+1}/\mathbb{G}_m]$ and $u_2\colon \mathbb{A}^{n+1} \setminus \{0\} \to \mathbb{P}^n$. The open embedding $j_n\colon \mathbb{A}^{n+1} \setminus \{0\} \to \mathbb{A}^{n+1}$ is $\mathbb{G}_m$-equivariant, inducing exactly $j_n\colon \mathbb{P}^n \to [\mathbb{A}^{n+1}/\mathbb{G}_m]$ on the quotients, and the corresponding Čech objects naturally map to each other:

$$\begin{array}{ccc}
\dots & \longrightarrow & \dots \\
\downarrow & & \downarrow \\
\mathbb{G}_m^k \times \mathbb{A}^{n+1} \setminus \{0\} & \xrightarrow{j_{n,k}} & \mathbb{G}_m^k \times \mathbb{A}^{n+1} \\
\downarrow & & \downarrow \\
\dots & \longrightarrow & \dots \\
\downarrow & & \downarrow \\
\mathbb{G}_m \times \mathbb{A}^{n+1} \setminus \{0\} & \xrightarrow{j_{n,1}} & \mathbb{G}_m \times \mathbb{A}^{n+1} \\
\downarrow & & \downarrow \\
\mathbb{A}^{n+1} \setminus \{0\} & \xrightarrow{j_n = j_{n,0}} & \mathbb{A}^{n+1}.
\end{array}$$

For each $j_{n,k}$ we have the map $\underline{\mathbb{Z}}_p \to (j_{n,k})_* j_{n,k}^* \underline{\mathbb{Z}}_p \simeq (j_{n,k})_* \mathbb{Z}_p$ of (pro)-étale sheaves on $(\mathbb{G}_m^k \times \mathbb{A}^{n+1})_C$ and the fiber is identified with $(i_{j,n})_* i_{j,n}^! \underline{\mathbb{Z}}_p$ (where $i_{j,n}$ is the embedding $\mathbb{G}_m^k \times \{0\} \hookrightarrow \mathbb{G}_m^k \times \mathbb{A}^{n+1}$) which is equivalent to $(i_{j,n})_* \underline{\mathbb{Z}}_p[n+1]$ by the local Poincare duality. It follows then that the corresponding map on global sections $j_{n,k}^*\colon R\Gamma_{\text{ét}}((\mathbb{G}_m^k \times \mathbb{A}^{n+1})_C, \mathbb{Z}_p) \to R\Gamma_{\text{ét}}((\mathbb{G}_m^k \times \mathbb{A}^{n+1} \setminus \{0\})_C, \mathbb{Z}_p)$

has n-coconnected fiber. Finally, the fiber of the map $j_n^*\colon R\Gamma_{\text{ét}}([\mathbb{A}^{n+1}/\mathbb{G}_m]_C, \mathbb{Z}_p) \to R\Gamma_{\text{ét}}(\mathbb{P}^n_C, \mathbb{Z}_p)$ is given by the totalization of fibers of $j_{n,k}^*\colon R\Gamma_{\text{ét}}((\mathbb{G}_m^k \times \mathbb{A}^{n+1})_C, \mathbb{Z}_p) \to R\Gamma_{\text{ét}}((\mathbb{G}_m^k \times \mathbb{A}^{n+1} \setminus \{0\})_C, \mathbb{Z}_p)$, each of them is n-coconnected, so the totalization is as well. □

We finish this subsection by showing that the natural map from algebraic to adic étale $\mathbb{Z}_p$-cohomology of $B\mathbb{G}_m$ is an equivalence:

Proposition 3.6.2.5. *The map $\Upsilon_{B\mathbb{G}_m}\colon R\Gamma_{\text{ét}}((B\mathbb{G}_m)_C, \mathbb{Z}_p) \to R\Gamma_{\text{ét}}((\widehat{B\mathbb{G}_m})_C, \mathbb{Z}_p)$ from Construction 3.5.3.9 is an equivalence.*

Proof. By Corollary 3.6.2.3 and Lemma 3.6.2.4 we have natural equivalences

$$R\Gamma_{\text{ét}}((\widehat{B\mathbb{G}_m})_C, \mathbb{Z}_p) \simeq \lim_{\leftarrow} R\Gamma_{\text{ét}}((\widehat{\mathbb{P}^n})_C, \mathbb{Z}_p) \qquad \text{and} \qquad R\Gamma_{\text{ét}}((B\mathbb{G}_m)_C, \mathbb{Z}_p) \simeq \lim_{\leftarrow} R\Gamma_{\text{ét}}((\mathbb{P}^n)_C, \mathbb{Z}_p)$$

under which the map $\Upsilon_{B\mathbb{G}_m}$ is identified with the limit $\lim_{\leftarrow} \Upsilon_{\mathbb{P}^n}$. But each $\Upsilon_{\mathbb{P}^n}$ is an equivalence (since $\mathbb{P}^n$ is proper), so we are done. □

Remark 3.6.2.6. The same argument can be applied to a split torus $T \simeq \mathbb{G}_m^k$ of higher dimension. In particular there are natural isomorphisms of graded rings:

$$H^*_{\mathbb{\Delta}}(BT/\mathfrak{S}) \simeq \mathrm{Sym}_{\mathrm{Mod}^{\varphi}_{\mathfrak{S}}}(T_p(T)^\vee \otimes_{\mathbb{Z}_p} \mathfrak{S}\{-1\}[-2]) \simeq \mathfrak{S}[x_1, \dots, x_k],\ \deg(x_i) = 2,$$

$$R\Gamma_{\text{ét}}(BT_C, \mathbb{Z}_p) \xrightarrow{\sim} R\Gamma_{\text{ét}}(\widehat{BT}_C, \mathbb{Z}_p) \simeq \mathrm{Sym}_{\mathrm{Rep}(G_K)}(T_p(T)^\vee[-2]) \simeq \mathbb{Z}_p[x_1, \dots, x_k],\ \deg(x_i) = 2,$$

where $T_p(T)^\vee \simeq \mathbb{Z}_p(-1)^{\oplus k}$ is the $\mathbb{Z}_p$-dual of the Tate module $T_p(T)$ of T.

3.6.3 The case of $[\mathbb{A}^n/T]$

Let $T \simeq \mathbb{G}_m^r$ be a split algebraic torus over $\mathcal{O}_K$ and let $\chi : T \to \mathbb{G}_m$ be a character. Consider a T-representation V which is free as an $\mathcal{O}_K$-module. V is a sum of characters $V \simeq \chi_1 \oplus \cdots \oplus \chi_n$ and for the rest of this subsection we make the following assumption:

Assumption 3.6.3.1. We assume that $V^T = 0$ and that the commutative monoid $R(V) = \mathbb{N}\cdot\chi_1 + \cdots + \mathbb{N}\cdot\chi_n \subset X^*(T)$ spanned by $\chi_1, \dots, \chi_n$ does not contain a copy of $\mathbb{Z}$ inside. Note that under this condition for any non-zero $\chi \in R(V)$ we have $0 \notin \chi + R(V)$.

Let $A = \mathrm{Sym}_{\mathcal{O}_K}(V^\vee)$ and $\mathbb{A}^n := \operatorname{Spec} A$ be the affine space corresponding to V. Surely, this $\mathbb{A}^n$ has a natural T-action. Let's pick coordinates $x_1, \ldots, x_n \in V^* \subset A$ which agree with the direct sum decomposition $V \simeq \chi_1 \oplus \cdots \oplus \chi_n$. As usual the action of T makes A a $X^*(T)$-graded algebra which we denote by $A_\bullet$. Note that the non-zero graded parts in A are given by $-R(V) \subset X^*(T)$. Recall that the category $\mathrm{QCoh}^+([\mathbb{A}^n/T])$ is identified with the bounded below derived category of graded $A_\bullet$-modules. Recall also that $\mathrm{QCoh}(BT)$ is identified with the derived category of $X^*(T)$-graded $\mathcal{O}_K$-modules (with respect to the trivial grading on $\mathcal{O}_K$). In this terms the functor of global sections $R\Gamma$ on both categories is given by taking the degree 0 part of the corresponding complex with respect to $X^*(T)$-grading.

Given a character $\chi \in X^*(T)$, for a T-module M we will denote by $M(\chi)$ the corresponding shift of the grading, $M(\chi)_\mu := M_{\mu-\chi}$.

Lemma 3.6.3.2. *Let $p\colon [\mathbb{A}^n/T] \to BT$ be the natural projection. Then the pullback map*

$$p^*\colon R\Gamma_{\mathbb{\Delta}}(BT/\mathfrak{S}) \longrightarrow R\Gamma_{\mathbb{\Delta}}([\mathbb{A}^n/T]/\mathfrak{S})$$

is an equivalence.

Proof. By the usual argument it is enough to prove the analogous statement for Hodge cohomology or equivalently for the global sections of the wedge powers of the cotangent complex. Let $\mathfrak{t} := \mathrm{Lie}(T)$ be the Lie algebra of T. As a complex of graded modules the cotangent complex $\mathbb{L}_{[\mathbb{A}^n/T]}$ (relative to $\mathcal{O}_K$) is equivalent to the 2-term complex $\Omega^1_{\mathbb{A}^n} \xrightarrow{a^*} \mathfrak{t}^\vee \otimes_{\mathcal{O}_K} \mathcal{O}_{\mathbb{A}^n}$ of cohomological amplitude $[0,1]$, where a^* is the coaction of $\mathfrak{t}^\vee$ on $\Omega^1_{\mathbb{A}^n}$. We have a T-equivariant isomorphism

$$\Omega^1_{\mathbb{A}^n} \simeq \bigoplus_{i=1}^n A_\bullet \cdot dx_i \simeq \bigoplus_{i=1}^n A_\bullet(-\chi_i)$$

and so the cotangent complex is equivalent to a 2-term complex $(\bigoplus_{i=1}^n A_\bullet(-\chi_i)) \to A_\bullet \otimes_{\mathcal{O}_K} \mathfrak{t}^\vee$. Similarly $\wedge^i \mathbb{L}_{[\mathbb{A}^n/T]}$ is equivalent to the corresponding wedge power $\wedge^i_A \left((\bigoplus_{i=1}^n A_\bullet(-\chi_i)) \xrightarrow{p} A_\bullet \right)$.

On the other hand, the cotangent complex $\mathbb{L}_{BT}$ is equivalent to $\mathfrak{t}^\vee[-1]$. The pullback functor p^* is given by tensoring up with A, so $p^*\mathbb{L}_{BT}$ is equivalent to $(A_\bullet \otimes_{\mathcal{O}_K} \mathfrak{t}^\vee)[-1]$. The map $R\Gamma(BT, \mathbb{L}_{BT}) \to R\Gamma([\mathbb{A}^n/T], p^*\mathbb{L}_{BT})$ is then given by the restriction of the embedding

$$\mathfrak{t}^\vee[-1] \to A_\bullet \otimes_{\mathcal{O}_K} \mathfrak{t}^\vee[-1]$$

to the degree 0 part. Since $A_0 \simeq \mathcal{O}_K$ we see that it is an equivalence.

Finally, the map $p^*\mathbb{L}_{BT} \to \mathbb{L}_{[\mathbb{A}^n/T]}$ is given by sending $p^*\mathbb{L}_{BT} \simeq A_\bullet \otimes_{\mathcal{O}_K} \mathfrak{t}^\vee[-1]$ to the similar term in $\mathbb{L}_{[\mathbb{A}^n/T]} \simeq (\bigoplus_{i=1}^n A_\bullet(-\chi_i)) \to A_\bullet \otimes_{\mathcal{O}_K} \mathfrak{t}^\vee$:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & A_\bullet \otimes_{\mathcal{O}_K} \mathfrak{t}^\vee & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \sim} & & \\ \cdots & \longrightarrow & (\bigoplus_{i=1}^n A_\bullet(-\chi_i)) & \longrightarrow & A_\bullet \otimes_{\mathcal{O}_K} \mathfrak{t}^\vee & \longrightarrow & \cdots . \end{array}$$

From the assumptions on V we know that the degree 0 component of $\bigoplus_{i=1}^n A_\bullet(-\chi_i)$ is 0. So this map is an equivalence when restricted to the part of degree 0. Thus $p^*\colon R\Gamma(BT, \mathbb{L}_{BT}) \to R\Gamma([\mathbb{A}^n/T], \mathbb{L}_{[\mathbb{A}^n/T]})$ is an equivalence. Similar argument works for $\wedge^i \mathbb{L}_{[\mathbb{A}^n/T]}$. □

From the étale comparison (Proposition 3.5.5.3) we deduce:

Corollary 3.6.3.3. *Under the assumptions from 3.6.3.1 on V the pullback map*

$$p^*\colon R\Gamma_{\text{ét}}(\widehat{BT}_C, \mathbb{Z}_p) \longrightarrow R\Gamma_{\text{ét}}(\widehat{[\mathbb{A}^1/T]}_C, \mathbb{Z}_p)$$

is an equivalence.

By the $\mathbb{A}^1$-homotopy invariance of the algebraic étale cohomology over characteristic zero fields we also have:

Lemma 3.6.3.4. *For any T-representation V the map*

$$p^*\colon R\Gamma_{\text{ét}}(BT_C, \mathbb{Z}_p) \longrightarrow R\Gamma_{\text{ét}}([\mathbb{A}^n/T]_C, \mathbb{Z}_p)$$

is an equivalence.

Proposition 3.6.3.5. *Under the assumptions on V from 3.6.3.1 the comparison map*

$$\Upsilon_{[\mathbb{A}^n/T]}\colon R\Gamma_{\text{ét}}([\mathbb{A}^n/T]_C, \mathbb{Z}_p) \longrightarrow R\Gamma_{\text{ét}}(\widehat{[\mathbb{A}^n/T]}_C, \mathbb{Z}_p)$$

is an equivalence.

Proof. This follows from the existence of the commutative square

$$\begin{array}{ccc} R\Gamma_{\text{ét}}(BT_C, \mathbb{Z}_p) & \xrightarrow[\sim]{\Upsilon_{BT}} & R\Gamma_{\text{ét}}(\widehat{BT}_C, \mathbb{Z}_p) \\ \sim\downarrow{\scriptstyle p^*} & & \sim\downarrow{\scriptstyle p^*} \\ R\Gamma_{\text{ét}}([\mathbb{A}^n/T]_C, \mathbb{Z}_p) & \xrightarrow{\Upsilon_{[\mathbb{A}^n/T]}} & R\Gamma_{\text{ét}}(\widehat{[\mathbb{A}^n/T]}_C, \mathbb{Z}_p), \end{array}$$

where we already know that all arrows except $\Upsilon_{[\mathbb{A}^n/T]}$ are equivalences by Proposition 3.6.2.5, Corollary 3.6.3.3 and Lemma 3.6.3.4. □

3.6.4 The case of BB

Let G be a quasi-split reductive group over $\mathcal{O}_K$ and let $B \subset G$ be a Borel subgroup. In this subsection using the results of the previous two subsections we prove that the comparison map from Construction 3.5.3.9 is an equivalence for the classifying stack BB.

Let $T \subset B$ be the maximal torus and consider the natural cover $\pi\colon BT \to BB$. Then the n-term $\check{C}(\pi)_n$ of the corresponding Čech diagram is equivalent to

$$[T\backslash \underbrace{B \times_T B \times_T \cdots \times_T B}_{n}/T] \simeq [(\underbrace{U \times U \times \cdots \times U}_{n})/T]$$

with the T-action on $U \times \cdots \times U$ given by the simultaneous adjoint action on all factors $t \circ (u_1, \ldots, u_n) = (tu_1t^{-1}, \ldots, tu_nt^{-1})$. The corresponding map sends

$$[(b_1, \ldots, b_n)] \in [T\backslash B \times_T B \times_T \cdots \times_T B/T] \mapsto ([b_1], \ldots, [b_n]) \in [(U \times U \times \cdots \times U)/T],$$

where each $[b_i]$ is a class in $[T\backslash B] \simeq U$. Thus the terms of the Čech object associated to π are given by U^n/T.

Proposition 3.6.4.1. *The pullback map $\pi^*\colon R\Gamma_{\mathbb{\Delta}}(BB/\mathfrak{S}) \to R\Gamma_{\mathbb{\Delta}}(BT/\mathfrak{S})$ is an equivalence.*

Proof. By the smooth descent (Remark 3.5.4.2) we know that

$$R\Gamma_{\mathbb{\Delta}}(BB/\mathfrak{S}) \simeq \operatorname{Tot} R\Gamma_{\mathbb{\Delta}}((U^\bullet/T)/\mathfrak{S}).$$

On the other hand, as a scheme U is isomorphic to an affine space $\mathfrak{u}\operatorname{Spec}(\operatorname{Sym}(\mathfrak{u}^\vee))$ with coordinates x_α corresponding to positive roots $\alpha \in \Delta$ with the action given by the negative of the corresponding

weight: $t \circ x_\alpha = \alpha(t)^{-1} \cdot x$. Representation $\mathfrak{u}$ satisfies assumptions from 3.6.3.1 and more generally the same holds for each U^n with the diagonal T-action. Hence by Lemma 3.6.3.2 all pullback maps $R\Gamma_{\mathbb{\Delta}}(BT/\mathfrak{S}) \to R\Gamma_{\mathbb{\Delta}}((U^\bullet/T)/\mathfrak{S})$ are equivalences, so the result follows. □

From this we deduce that Υ_{BB} is an equivalence:

Proposition 3.6.4.2. *Let G be a split reductive group, $T \subset B \subset G$ be a maximal torus and a Borel subgroup correspondingly. Then we have a diagram of quasi-isomorphisms*

$$\begin{array}{ccc} R\Gamma_{\text{ét}}(BB_C, \mathbb{Z}_p) & \xrightarrow[\sim]{\Upsilon_{BB}} & R\Gamma_{\text{ét}}(\widehat{BB}_C, \mathbb{Z}_p) \\ {\scriptstyle \pi^*}\downarrow{\scriptstyle \sim} & & {\scriptstyle \pi^*}\downarrow{\scriptstyle \sim} \\ R\Gamma_{\text{ét}}(BT_C, \mathbb{Z}_p) & \xrightarrow[\sim]{\Upsilon_{BT}} & R\Gamma_{\text{ét}}(\widehat{BT}_C, \mathbb{Z}_p). \end{array}$$

In particular the comparison map $\Upsilon_{BB}\colon R\Gamma_{\text{ét}}(BB_C, \mathbb{Z}_p) \to R\Gamma_{\text{ét}}(\widehat{BB}_C, \mathbb{Z}_p)$ is an equivalence.

Proof. The top horizontal arrow is an equivalence by Remark 3.6.2.6 and the right vertical arrow is an equivalence by Proposition 3.6.4.1 below and the étale comparison (Proposition 3.5.5.3). So it is enough to prove that the left vertical map is an equivalence.

To see this, note that since $B \simeq T \rtimes U$, we have an isomorphism of schemes $B \simeq T \times U$. Moreover, as a scheme U is isomorphic to an affine space. So from homotopy $\mathbb{A}^1$-invariance we get that the pullback map

$$(\pi \times \ldots \times \pi)^*\colon R\Gamma_{\text{ét}}(B_C \times \ldots \times B_C, \mathbb{Z}_p) \longrightarrow R\Gamma_{\text{ét}}(T_C \times \ldots \times T_C, \mathbb{Z}_p)$$

is an equivalence. The statement for BT_C and BB_C now follows from considering the simplicial schemes $(BT)_\bullet$ and $(BB)_\bullet$ and identifying $R\Gamma_{\text{ét}}(BT_C, \mathbb{Z}_p)$ and $R\Gamma_{\text{ét}}(BB_C, \mathbb{Z}_p)$ with the corresponding totalizations. □

3.6.5 The general case

In this subsection using the results of Section 3.3.4 and of the previous subsections we will deduce that for a smooth proper scheme X over $\mathcal{O}_K$ and a parabolic subgroup P of a reductive group G acting on X the comparison map 3.5.3.9 between two versions of étale cohomology of $[X/P]$ is an equivalence. We first establish the following particular case:

Proposition 3.6.5.1. *Let $\mathcal{X} \to BB$ be a smooth proper map. Then the comparison map for $\mathcal{X}$*

$$\Upsilon_{\mathcal{X}} \colon R\Gamma_{\text{ét}}(\mathcal{X}_C, \mathbb{Z}_p) \longrightarrow R\Gamma_{\text{ét}}(\widehat{\mathcal{X}}_C, \mathbb{Z}_p)$$

is an equivalence.

Proof. Since by construction both parts are p-adically derived complete it is enough to prove that the natural map

$$\Upsilon_{\mathcal{X},\mathbb{F}_p} \colon R\Gamma_{\text{ét}}(\mathcal{X}_C, \mathbb{F}_p) \longrightarrow R\Gamma_{\text{ét}}(\widehat{\mathcal{X}}_C, \mathbb{F}_p)$$

is an equivalence. Let $\pi \colon \mathcal{X} \to BB$ be the structure map. Recall that we defined the comparison map $\Upsilon_{\mathcal{X},\mathbb{F}_p}$ as a map induced by applying the global sections functor $R\Gamma_{\text{ét}}(BB, -) \colon \mathcal{S}\text{hv}_{\text{ét}}(\mathcal{X}, \mathbb{F}_p) \to D(\text{Mod}_{\mathbb{F}_p})$ to the unit $\underline{\mathbb{F}_p} \to \lambda_{\mathcal{X}*}\lambda_{\mathcal{X}}^{-1}\underline{\mathbb{F}_p}$ of adjunction $\lambda_{\mathcal{X}}^{-1} \dashv \lambda_{\mathcal{X}*}$. Recall also that in Corollary 3.3.4.38 we proved that the diagram

$$\begin{array}{ccc} \mathcal{S}\text{hv}_{\text{ét}}^{>-\infty}(\widehat{\mathcal{X}}_C, \mathbb{F}_p) & \xleftarrow{\lambda_{\mathcal{X}}^{-1}} & \mathcal{S}\text{hv}_{\text{ét}}^{>-\infty}(\mathcal{X}_C, \mathbb{F}_p) \\ \downarrow{\scriptstyle (\widehat{\pi}_C)_*} & & \downarrow{\scriptstyle \pi_*} \\ \mathcal{S}\text{hv}_{\text{ét}}^{>-\infty}(\widehat{BB}_C, \mathbb{F}_p) & \xleftarrow{\lambda_{BB}^{-1}} & \mathcal{S}\text{hv}_{\text{ét}}^{>-\infty}(BB_C, \mathbb{F}_p) \end{array}$$

is commutative. By a diagram chasing and commutativity of the diagram above, it is not hard to see that $\Upsilon_{\mathcal{X},\mathbb{F}_p}$ is equivalent to the map obtained by applying the global section functor $R\Gamma_{\text{ét}}(BB, -)$ to the unit of the adjunction $\pi_*\underline{\mathbb{F}_p} \to \lambda_{BB*}\lambda_{BB}^{-1}(\pi_*\underline{\mathbb{F}_p})$.

Now, since the pushforward functor π_* is right t-exact, the associated graded of the Whitehead filtration of the étale sheaf $\pi_*\underline{\mathbb{F}_p}$ is $\bigoplus_{i=0}^{\infty} \mathcal{H}^i(\pi_*\underline{\mathbb{F}_p})[-i]$, where by Corollary 3.3.4.17 all $\mathcal{H}^i(\pi_*\underline{\mathbb{F}_p}) \in \mathcal{L}_{\text{ét}}^{\heartsuit}(BB, \mathbb{F}_p)$. But by Proposition 3.3.4.21 BB is étale simply connected, so each $\mathcal{H}^i(\pi_*\underline{\mathbb{F}_p})$ is isomorphic to a finite direct sums of the constant sheaves $\underline{\mathbb{F}_p}$. Since by Proposition 3.6.4.2 the comparison map is an equivalence for $\underline{\mathbb{F}_p}$, the same holds by exactness of $R\Gamma_{\text{ét}}(BB, -)$ for $\tau^{\leq n}\pi_*\underline{\mathbb{F}_p}$ for all $n \geq 0$. The statement for $\pi_*\underline{\mathbb{F}_p}$ follows, since the natural map $\tau^{\leq n}\pi_*\underline{\mathbb{F}_p} \to \pi_*\underline{\mathbb{F}_p}$ is n-coconnected and the global sections functor $R\Gamma_{\text{ét}}(BB, -)$ is right t-exact. □

Now we can deduce the main result of this section:

Theorem 3.6.5.2. *Let G be a split reductive $\mathcal{O}_K$-group scheme and let P be a parabolic subgroup of G. Let X be a smooth proper $\mathcal{O}_K$-scheme equipped with an action of P. Then the comparison*

map

$$R\Gamma_{\text{ét}}([X/P]_C, \mathbb{Z}_p) \longrightarrow R\Gamma_{\text{ét}}(\widehat{[X/P]}_C, \mathbb{Z}_p)$$

is an equivalence.

Proof. Let $B \subseteq P$ be a Borel subgroup. Then the natural cover $[X/B] \to [X/P]$ is smooth and proper, in particular all elements of the corresponding Čech nerve are smooth and proper over BB. Since both parts of the comparison in question satisfy smooth descent, the result follows by the previous proposition. □

Corollary 3.6.5.3. *Let k be perfect and let G be a reductive $W(k)$-group scheme with $P \subset G$ being a parabolic subgroup. Let X be a smooth proper $W(k)$-scheme equipped with an action of P. Then for all $i \geq 0$*

$$\dim_k H^i_{\text{dR}}([X_k/G_k]) \geq \dim_{\mathbb{F}_p} H^i_{\text{sing}}(X(\mathbb{C})_{hG(\mathbb{C})}, \mathbb{F}_p).$$

Remark 3.6.5.4. Note that (since the de Rham cohomology over k and $\overline{k}$ agree) both sides of the inequality depend only on base change of G to $\mathcal{O}_{\mathbb{C}_p}$, so indeed no splitness assumption is necessary (by passing to $W(k')$ for some extension k' of k).

3.7 Other applications

3.7.1 Differential 1-forms on conical resolutions

From the proof of Theorem 4.4.2.10 (and Example 4.4.2.14(2)) one can see that the quotient $[X/\mathbb{G}_m]$ of a conical resolution X over $\mathbb{Z}_p$ is cohomologically proper. Then, from Theorem 3.5.6.1 we get

$$\dim_{\mathbb{F}_p} H^1_{\text{dR}}([X/\mathbb{G}_m]_{\mathbb{F}_p}) \geq \dim_{\mathbb{F}_p} H^1_{\text{ét}}(\widehat{[X/\mathbb{G}_m]}_{\mathbb{C}_p}, \mathbb{F}_p).$$

Assuming Conjecture 3.5.3.12 (note that $[X/\mathbb{G}_m]$ satisfies formal GAGA due to the semiorthogonal decomposition (4.4.2.1)) we also get an inequality

$$\dim_{\mathbb{F}_p} H^1_{\text{dR}}([X/\mathbb{G}_m]_{\mathbb{F}_p}) \geq \dim_{\mathbb{F}_p} H^1_{\text{sing}}(X(\mathbb{C})_{h\mathbb{C}^\times}, \mathbb{F}_p).$$

Thus we would get the statement of Conjecture 2.5.2.3. It is not hard to see from the Lerray-Serre spectral sequence for $X(\mathbb{C})_{h\mathbb{C}^\times} \to B\mathbb{C}^\times$ that the dimension on the right is bounded below by the

p-torsion in $H^1_{\text{sing}}(X(\mathbb{C}),\mathbb{Z})$. Thus, returning to the case of a conical resolution (namely, under the assumption $H^1(X,\mathcal{O}_X)=0$), a non-trivial p-torsion in $H^1_{\text{sing}}(X(\mathbb{C}),\mathbb{Z})$ implies the existence of a non-zero $\mathbb{G}_m$-invariant differential 1-form on $X_{\mathbb{F}_p}$.

3.7.2 De Rham and prismatic cohomology of BG

In this section we recall some of the results of [111] about the de Rham cohomology of BG for large enough p and deduce the analogous statements for the prismatic cohomology.

Let G be a split reductive group. First we will recall the definition of a torsion prime:

Definition 3.7.2.1. Let $\overline{k}$ be an algebraically closed field and consider the flag variety $(G/B)_{\overline{k}}$ over $\overline{k}$. There is a natural map $X^*(T)\to \mathrm{CH}^1((G/B)_{\overline{k}})$ associating to a character λ the divisor of a rational section of the line bundle $\mathcal{O}(\lambda)$. Let $N:=\dim G/B$ be the dimension of the flag variety. The map above induces a map from the N-th symmetric power $\mathrm{Sym}^N_{\mathbb{Z}}(X^*(T))\to \mathrm{CH}^N((G/B)_{\overline{k}})$ which then can be composed with the degree map for the 0-cycles $\mathrm{CH}^N((G/B)_{\overline{k}})\to\mathbb{Z}$ which in the case of G/B is an isomorphism. A prime p is called a *torsion prime* for G if the image of the composition $S^N_{\mathbb{Z}}(X^*(T))\to\mathbb{Z}$ is 0 modulo p.

Borel showed that p is a torsion prime if and only if the cohomology $H^*_{\text{sing}}(B(G(\mathbb{C})),\mathbb{Z})$ has p-torsion (here $G(\mathbb{C})$ is the corresponding topological group of $\mathbb{C}$-points). Totaro gave a description of torsion (or rather non-torsion) primes in terms of the classifying stack of the reduction of G modulo p:

Theorem 3.7.2.2 ([111, Therorem 10.1])**.** *Let G be a reductive group over a field k of characteristic $p\geq 0$. Then $H^{>0}(G,\mathcal{O}(\mathfrak{g}))\simeq 0$ if and only if p is not a torsion prime for G.*

In particular any *good prime* is non-torsion prime since by ([60, Proof of Theorem 2.2] and [39, II.4.22]) $\mathcal{O}(\mathfrak{g})$ has a good filtration and so $H^{>0}(G,\mathcal{O}(\mathfrak{g}))\simeq 0$. Explicitly, p is good if $p\neq 2$ if G has a simple factor not of type A_n, $p\neq 2,3$ if G has a simple factor of an exceptional type and $p\neq 2,3,5$ if G has an E_8 factor. More generally for G is simply connected, p is non-torsion if $p\neq 2$ if G has a simple factor not of type A_n or C_n, $p\neq 2,3$ if G has a simple factor of type F_4, E_6, E_7, or E_8, and $p\neq 2,3,5$ if G has a simple factor E_8.

Theorem 3.7.2.2 then can be used to compute the de Rham cohomology of BG locally at any non-torsion prime:

Theorem 3.7.2.3 ([111, Theorem 10.2]). *Let G be a split reductive group over $\mathbb{Z}$ and let p be a non-torsion prime for G. Then the de Rham cohomology $H^*_{\mathrm{dR}}(BG/\mathbb{Z})$, localized at p, is a polynomial ring on generators of degrees equal to 2 times the fundamental degrees of G. These graded rings are isomorphic to the cohomology of the topological classifying space $B(G(\mathbb{C}))$ with $\mathbb{Z}_{(p)}$-coefficients.*

Remark 3.7.2.4. Below (see Corollary 3.7.2.7) we show that after the p-adic completion this theorem can be reformulated in terms of the prismatic cohomology of BG over $\mathbb{Z}_p$. Namely, under the same assumption on p, prismatic cohomology groups $H^*_{\mathbb{\Delta}}(BG/\mathfrak{S})$ are free $\mathfrak{S}$-modules, which specialize to $H^*_{\mathrm{dR}}(BG/\mathbb{Z}_p)$ and $H^*_{\mathrm{sing}}(B(G(\mathbb{C})),\mathbb{Z}_p)$ at de Rham and étale specialization points of $\operatorname{Spec}\mathfrak{S}$. This allows us to construct the prismatic characteristic classes that form a bridge connecting topological mod p characteristic classes and de Rham characteristic classes in characteristic p.

We recall that the *fundamental degrees* of a reductive group G over a field k are the the degrees of the generators of the polynomial ring $\operatorname{Sym}(X^*(T)\otimes_{\mathbb{Z}}\mathbb{Q})^W$ of invariants under the Weyl group W. For split reductive group this ring does not depend on k and if $\operatorname{char} k = 0$ the corresponding degrees can be identified with the degrees of the generators of the polynomial ring $\mathcal{O}(\mathfrak{g})^G$ of invariant functions on the Lie algebra. Here is the list of fundamental degrees of the simple groups:

$$
\begin{aligned}
A_n &\quad 2,3,\dots,n+1,\\
B_n &\quad 2,4,6,\dots,2n,\\
C_n &\quad 2,4,6,\dots,2n,\\
D_n &\quad 2,4,6,\dots,2n-2,n,\\
G_2 &\quad 2,6,\\
F_4 &\quad 2,6,8,12,\\
E_6 &\quad 2,5,6,8,9,12,\\
E_7 &\quad 2,6,8,10,12,14,18,\\
E_8 &\quad 2,8,12,14,18,20,24,30.
\end{aligned}
$$

Let now $\mathcal{O}_K = \mathbb{Z}_p$, the corresponding Kisin ring $\mathfrak{S}$ is $\mathbb{Z}_p[[u]]$. We use Theorem 3.7.2.3 to compute the prismatic cohomology $H^*_{\mathbb{\Delta}}(BG/\mathfrak{S})$ when p is non-singular:

Theorem 3.7.2.5. *Let G be a split reductive group over $\mathbb{Z}$ and let p be a non-torsion prime for G. Then the prismatic cohomology $H^*_{\mathbb{\Delta}}(BG/\mathfrak{S})$ is a free graded polynomial algebra over $\mathfrak{S}$ on generators of degrees equal to 2 times the fundamental degrees of G.*

Proof. By the de Rham comparison and the universal coefficient formula we have short exact sequences

$$0 \longrightarrow \mathrm{Tor}^1_{\mathfrak{S}}(H^{2i+1}_{\mathbb{\Delta}}(BG/\mathfrak{S}), \mathfrak{S}/(u-p)) \longrightarrow H^{2i}_{\mathrm{dR}}(BG/\mathbb{Z}_p) \longrightarrow H^{2i}_{\mathbb{\Delta}}(BG/\mathfrak{S}) \otimes_{\mathfrak{S}} \mathfrak{S}/(u-p) \longrightarrow 0,$$

$$0 \longrightarrow \mathrm{Tor}^1_{\mathfrak{S}}(H^{2i+2}_{\mathbb{\Delta}}(BG/\mathfrak{S}), \mathfrak{S}/(u-p)) \longrightarrow H^{2i+1}_{\mathrm{dR}}(BG/\mathbb{Z}_p) \longrightarrow H^{2i+1}_{\mathbb{\Delta}}(BG/\mathfrak{S}) \otimes_{\mathfrak{S}} \mathfrak{S}/(u-p) \longrightarrow 0.$$

By Theorem 3.7.2.3 the de Rham cohomology of BG over $\mathbb{Z}_p$ is a polynomial ring with generators of even degree, in particular $H^{2i+1}_{\mathrm{dR}}(BG/\mathbb{Z}_p) \simeq 0$ for all i and so $H^{2i+1}_{\mathbb{\Delta}}(BG/\mathfrak{S}) \otimes_{\mathfrak{S}} \mathfrak{S}/(u-p) \simeq 0$ as well. The group $\mathrm{Tor}^1_{\mathfrak{S}}(H^{2i+1}_{\mathbb{\Delta}}(BG/\mathfrak{S}), \mathfrak{S}/(u-p))$ is isomorphic to the $(u-p)$-torsion in $H^{2i+1}_{\mathbb{\Delta}}(BG/\mathfrak{S})$. However, since $H^{2i+1}_{\mathbb{\Delta}}(BG/\mathfrak{S})$ is finitely generated, $H^{2i+1}_{\mathbb{\Delta}}(BG/\mathfrak{S})[u-p] \not\simeq 0$ would imply $H^{2i+1}_{\mathbb{\Delta}}(BG/\mathfrak{S}) \otimes_{\mathfrak{S}} \mathfrak{S}/(u-p) \not\simeq 0$ — a contradiction. So $\mathrm{Tor}^1_{\mathfrak{S}}(H^{2i+1}_{\mathbb{\Delta}}(BG/\mathfrak{S}), \mathfrak{S}/(u-p)) \simeq 0$ and

$$H^{2i}_{\mathrm{dR}}(BG/\mathbb{Z}_p) \xrightarrow{\sim} H^{2i}_{\mathbb{\Delta}}(BG/\mathfrak{S}) \otimes_{\mathfrak{S}} \mathfrak{S}/(u-p).$$

We also have

$$H^{2i+1}_{\mathrm{dR}}(BG/\mathbb{Z}_p) \xrightarrow{\sim} H^{2i+1}_{\mathbb{\Delta}}(BG/\mathfrak{S}) \otimes_{\mathfrak{S}} \mathfrak{S}/(u-p)$$

since both groups vanish. All together this provides an isomorphism of graded $\mathbb{Z}_p$-algebras

$$H^*_{\mathbb{\Delta}}(BG/\mathfrak{S}) \otimes_{\mathfrak{S}} \mathfrak{S}/(u) \xrightarrow{\sim} H^*_{\mathrm{dR}}(BG/\mathbb{Z}_p).$$

It also follows that the $(u-p)$-torsion $H^i_{\mathbb{\Delta}}(BG/\mathfrak{S})[u-p]$ vanish for all i.

Let $x_{e_1}, \ldots, x_{e_n}$ be the generators of $H^*_{\mathrm{dR}}(BG/\mathbb{Z}_p)$ of degree $2e_i$, where e_i are the fundamental degrees and let $\tilde{x}_{e_1}, \ldots, \tilde{x}_{e_n}$ be any lifts under the projection modulo $(u-p)$

$$H^*_{\mathbb{\Delta}}(BG/\mathfrak{S}) \longrightarrow H^*_{\mathbb{\Delta}}(BG/\mathfrak{S}) \otimes_{\mathfrak{S}} \mathfrak{S}/(u-p) \simeq H^*_{\mathrm{dR}}(BG/\mathbb{Z}_p).$$

The choice of lifts gives a map from the polynomial ring

$$\tilde{f} \colon \mathfrak{S}[y_{e_1}, \ldots, y_{e_n}] \to H^*_{\mathbb{\Delta}}(BG/\mathfrak{S}),$$

which is an isomorphism modulo $(u-p)$. The graded components on both sides are finitely generated $\mathfrak{S}$-modules (so are $(u-p)$-adically complete) and are $u-p$-torsion free, so Nakayama lemma can be applied. Thus $\tilde{f}$ is an isomorphism and we are done. □

In most cases one can also describe $H^*_{\blacktriangle}(BG/\mathfrak{S})$ by looking at its image in $H^*_{\blacktriangle}(BT/\mathfrak{S})$ under the natural map $q\colon BT \to BG$:

Proposition 3.7.2.6. *Let G be a split reductive group and let p be a non-torsion prime or $p \neq 2$ if G has a simple factor of type C_n. Then $q^*\colon H^*_{\blacktriangle}(BG/\mathfrak{S}) \to H^*_{\blacktriangle}(BT/\mathfrak{S})$ maps isomorphically onto the subring of W-invariants:*

$$H^*_{\blacktriangle}(BG/\mathfrak{S}) \xrightarrow{\sim} H^*_{\blacktriangle}(BT/\mathfrak{S})^W.$$

Proof. Since cohomology groups $H^*_{\blacktriangle}(BG/\mathfrak{S})$ are free as an $\mathfrak{S}$-module we have an embedding

$$H^*_{\blacktriangle}(BG/\mathfrak{S}) \hookrightarrow H^*_{\blacktriangle}(BG/\mathfrak{S}) \otimes_{\mathfrak{S}} W(\mathbb{C}_p^\flat) \simeq H^*_{\text{ét}}(BG_{\mathbb{C}_p}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} W(\mathbb{C}_p^\flat),$$

where we use Theorem 3.6.5.2 and Corollary 3.5.5.4 for the isomorphism on the right. Considering the same diagram for T we obtain a commutative square

$$\begin{array}{ccc} H^*_{\blacktriangle}(BG/\mathfrak{S}) & \longrightarrow & H^*_{\text{ét}}(BG_{\mathbb{C}_p}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} W(\mathbb{C}_p^\flat) \\ \downarrow & & \downarrow \\ H^*_{\blacktriangle}(BT/\mathfrak{S}) & \longrightarrow & H^*_{\text{ét}}(BT_{\mathbb{C}_p}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} W(\mathbb{C}_p^\flat), \end{array}$$

where the bottom horizontal map is W-equivariant. The W-action on $H^*_{\text{ét}}(BT_{\mathbb{C}_p}, \mathbb{Z}_p)$ is induced by the adjoint action of elements of $G_{\mathbb{C}_p}$ lying in the connected component of unit element and so this action is trivial restricted to the image of $q^*\colon H^*_{\text{ét}}(BG_{\mathbb{C}_p}, \mathbb{Z}_p) \to H^*_{\text{ét}}(BT_{\mathbb{C}_p}, \mathbb{Z}_p)$. It follows that the image of $q^*\colon H^*_{\blacktriangle}(BG/\mathfrak{S}) \to H^*_{\blacktriangle}(BT/\mathfrak{S})$ lands in W-invariants $H^*_{\blacktriangle}(BT/\mathfrak{S})^W$.

It remains to check that

$$q^*\colon H^*_{\blacktriangle}(BG/\mathfrak{S}) \longrightarrow H^*_{\blacktriangle}(BT/\mathfrak{S})^W$$

is an isomorphism. By Remark 3.6.2.6 we can identify the right hand side with the ring of polynomial functions $\mathcal{O}(\mathfrak{t}_{\mathfrak{S}})$ on $\mathfrak{t}_{\mathfrak{S}}$, moreover this identification is W-equivariant. So $H^*_{\blacktriangle}(BT/\mathfrak{S})^W \simeq \mathcal{O}(\mathfrak{t}_{\mathfrak{S}})^W$. By Thèoréme of [37] under a milder condition on p the ring $\mathcal{O}(\mathfrak{t}_{\mathfrak{S}})^W$ is a polynomial ring in variables of degrees equal to the fundamental degrees of G, in particular each graded component is free as an $\mathfrak{S}$-module. So by Nakayama it is enough to check that the map q^* is an isomorphism

modulo the ideal $(p, u-p) \subset \mathfrak{S}$ which is checked during the proof of Theorem 10.2 in [111]. $\square$

Corollary 3.7.2.7. *Under the same assumption on G and p as in the previous proposition let $e_1, \ldots, e_n$ be the fundamental degrees. Then*

$$H^*_{\mathbb{\Delta}}(BG/\mathfrak{S}) \simeq \mathrm{Sym}_{\mathfrak{S}}(\mathfrak{S}\{-e_1\}[-2e_1] \oplus \cdots \oplus \mathfrak{S}\{-e_n\}[-2e_n])$$

as a graded ring in the category of Breuil-Kisin modules. Consequently,

$$H^*_{\mathbb{\Delta}}(BG/A_{\mathrm{inf}}) \simeq \mathrm{Sym}_{A_{\mathrm{inf}}}(A_{\mathrm{inf}}\{-e_1\}[-2e_1] \oplus \cdots \oplus A_{\mathrm{inf}}\{-e_n\}[-2e_n])$$

as a graded ring in the category of Breuil-Kisin-Fargues modules.

Chapter 4

Hodge-to-de Rham degeneration for stacks

4.1 Introduction

4.1.1 Deligne-Illusie method for schemes

Let X be a smooth scheme over $\mathbb{C}$ and let $X(\mathbb{C})$ be the topological space of its complex points. Grothendieck has shown that there is a formula for the singular cohomology of $X(\mathbb{C})$ in purely algebraic terms, namely

$$H^n_{\mathrm{sing}}(X(\mathbb{C}),\mathbb{C}) \simeq H^n_{\mathrm{dR}}(X/\mathbb{C}),$$

where the de Rham cohomology $H^n_{\mathrm{dR}}(X/\mathbb{C})$ is defined as the n-th hypercohomology of the algebraic de Rham complex of X. If, moreover, X is projective, using Hodge theory one obtains the Hodge decomposition

$$H^n_{\mathrm{sing}}(X(\mathbb{C}),\mathbb{C}) \simeq \bigoplus_{p+q=n} H^q(X,\Omega^p_X).$$

Unfortunately, it is only possible to get such a decomposition utilizing some transcendental methods (like Hodge theory). However, for X proper, using just algebraic geometry we still obtain a functorial filtration $F^\bullet H^n_{\mathrm{dR}}(X/\mathbb{C})$ whose associated graded is given by the sum above. Namely, the de Rham complex has a natural cellular (also called "stupid") filtration $\Omega^{\geq p}_{X,\mathrm{dR}}$ given by subcomplexes

$$\Omega^{\geq p}_{X,\mathrm{dR}} \coloneqq \ldots \longrightarrow 0 \longrightarrow \Omega^p_X \xrightarrow{d} \Omega^{p+1}_X \xrightarrow{d} \ldots \xrightarrow{d} \Omega^{\dim X}_X \ .$$

This filtration induces a filtration on the complex of global sections $R\Gamma_{\mathrm{dR}}(X/\mathbb{C}) \coloneqq R\Gamma(X,\Omega^\bullet_{X,\mathrm{dR}})$

whose associated graded pieces are $R\Gamma(X, \Omega_X^p[-p])$. As a consequence one gets the so-called *Hodge-to-de Rham spectral sequence*

$$E_1^{p,q} = H^q(X, \Omega_X^p) \;\Rightarrow\; H_{\mathrm{dR}}^{p+q}(X/\mathbb{C}).$$

As was shown by Deligne and Illusie [36] there is a purely algebraic proof of the degeneration of the spectral sequence above, thus the induced filtration $F^\bullet H_{\mathrm{dR}}^n(X/\mathbb{C})$ on the de Rham cohomology has the associated graded

$$\mathrm{gr}_F^\bullet H_{\mathrm{dR}}^n(X/\mathbb{C}) \simeq \bigoplus_{p+q=n} H^{p,q}(X), \quad \text{where} \quad H^{p,q}(X) := H^q(X, \Omega_X^p).$$

The strategy of Deligne-Illusie is to reduce the statement in characteristic 0 to an analogous question in big enough positive characteristic. Let k be a perfect field of characteristic p and let Y be a smooth scheme over k. Then we have:

Theorem 4.1.1.1 (Cartier)**.** *Let $Y^{(1)}$ denote the Frobenius twist of Y and let $\varphi_Y \colon Y \to Y^{(1)}$ be the relative Frobenius morphism. Then there exists a unique isomorphism of sheaves of $\mathcal{O}_{Y^{(1)}}$-algebras on $Y^{(1)}_{\mathrm{Zar}}$*

$$C_Y^{-1} \colon \bigoplus_i \Omega^i_{Y^{(1)}} \to \bigoplus_i \mathcal{H}^i(\varphi_{Y*}\Omega^\bullet_{Y,\mathrm{dR}}),$$

determined by the property that for any local section f of $\mathcal{O}_Y$

$$C_Y^{-1}(df) = \text{“}df^p/p\text{”} := f^{p-1}df.$$

The map C_Y^{-1} is called the inverse Cartier isomorphism.

This way we see that the Postnikov (also called “canonical”) filtration on $\varphi_{Y*}\Omega^\bullet_{Y,\mathrm{dR}}$ induces another filtration on $R\Gamma(Y, \Omega^\bullet_{Y,\mathrm{dR}}) \simeq R\Gamma(Y^{(1)}, \varphi_{Y*}\Omega^\bullet_{Y,\mathrm{dR}})$ whose associated graded pieces are $R\Gamma(Y^{(1)}, \Omega^p_{Y^{(1)}}[-p])$. Taking the spectral sequence induced by this filtration we obtain the so-called *conjugate spectral sequence*

$$E_2^{p,q} = H^p(Y^{(1)}, \Omega^q_{Y^{(1)}}) \Rightarrow H_{\mathrm{dR}}^{p+q}(Y/k).$$

Note that for any spectral sequence the E_∞-page is always a subfactor of the E_r-page ($r \geq 0$), hence $\dim_k E_\infty^{*,*} \leq \dim_k E_r^{*,*}$. If all vector spaces $E_*^{*,*}$ are finite-dimensional, equality holds if and only if all differentials starting from the r-th page vanish. It follows that for Y proper, the

conjugate spectral sequence degenerates if and only if $\dim_k H^n_{\mathrm{dR}}(Y) = \sum_{p+q=n} \dim_k H^{p,q}(Y^{(1)})$. Since $\dim_k H^{p,q}(Y^{(1)}) = \dim_k H^{p,q}(Y)$ this happens if and only if the Hodge-to-de Rham spectral sequence degenerates as well.

The differentials in the conjugate spectral sequence are induced by the connecting homomorphisms for the Postnikov filtration on $\varphi_{Y_*}\Omega^\bullet_{Y,\mathrm{dR}}$. In particular, if $\varphi_{Y_*}\Omega^\bullet_{Y,\mathrm{dR}}$ is formal (i.e. quasi-isomorphic to the sum of its cohomology), then the conjugate spectral sequence degenerates. While in general this is hard to guarantee, the formality of the truncation $\tau^{\leq p-1}\varphi_{Y_*}\Omega^\bullet_{Y,\mathrm{dR}}$ turns out to be equivalent to the existence of a lift to the second Witt vectors $W_2(k)$:

Theorem 4.1.1.2 (Deligne-Illusie)**.** *A smooth scheme Y over k admits a lift to $W_2(k)$ if and only if there exists an equivalence*

$$\bigoplus_{i=0}^{p-1} \Omega^i_{Y^{(1)}}[-i] \xrightarrow{\sim} \tau^{\leq p-1}\varphi_{Y*}\Omega^\bullet_{Y,\mathrm{dR}}$$

inducing the inverse Cartier isomorphism C_Y^{-1} on $\mathcal{H}^*$. In particular, if Y admits a lift to $W_2(k)$ and $\dim Y < p$, then the complex $\varphi_{Y*}\Omega^\bullet_{Y,\mathrm{dR}}$ is formal, and hence the Hodge-to-de Rham spectral sequence degenerates at the first page.

The proof of the degeneration in characteristic 0 is then accomplished by choosing a smooth proper model (the so-called *spreading*) X_R of X over some finitely generated $\mathbb{Z}$-subalgebra R of F. Enlarging R if needed, one can assume that the R-modules $H^q(X_R, \Omega^p_{X_R})$ and $H^n_{\mathrm{dR}}(X_R/R)$ are free of finite rank. Picking a closed point of residue characteristic $p > \dim X$ one reduces to Theorem 4.1.1.2.

4.1.2 Generalization to stacks

In this work we extend the results of Deligne-Illusie to the case of Artin stacks. For a smooth proper Deligne-Mumford stack one can proceed with the original arguments (see e.g. [104, Corollary 1.7]), but they do not seem to work for a general smooth Artin stack (see Remark 4.1.2.3). Instead we use another approach relying on quasi-syntomic descent for the derived de Rham cohomology.

As in the case of schemes, to establish Hodge-to-de Rham degeneration, we need to impose some properness assumptions. However, the standard notion of a proper stack is too restrictive for our purposes. For example, the quotient stack $[X/G]$ of a proper scheme X by an action of a linear

algebraic group G is proper if and only if the stabilizers of all points of X are finite group schemes. On the other hand, as we will see in Section 4.4.1, the Hodge-to-de Rham spectral sequence for $[X/G]$ with reductive G always degenerates.

This suggests that we should look for a more general notion of properness:

Definition 4.1.2.1. Let R be a Noetherian ring. A smooth Artin stack $\mathcal{X}$ over R is called *Hodge-proper* if $H^q(\mathcal{X}, \wedge^p \mathbb{L}_{\mathcal{X}/R})$ is a finitely generated R-module for all p and q, where $\mathbb{L}_{\mathcal{X}/R}$ is the cotangent complex of $\mathcal{X}$ over R.

The complex $R\Gamma(\mathcal{X}, \wedge^p \mathbb{L}_{\mathcal{X}/R})$ is a natural analogue of $R\Gamma(X, \Omega^p_X)$ and, similarly to the scheme case, the de Rham cohomology complex $R\Gamma_{\mathrm{dR}}(\mathcal{X}/R)$ has a natural (Hodge) filtration whose associated graded pieces are $R\Gamma(\mathcal{X}, \wedge^p \mathbb{L}_{\mathcal{X}/R}[-p])$; see Section 3.4.1 for more details. In this way one obtains a spectral sequence

$$E_1^{p,q} = H^q(\mathcal{X}, \wedge^p \mathbb{L}_{\mathcal{X}/R}) \ \Rightarrow \ H^{p+q}_{\mathrm{dR}}(\mathcal{X}/R).$$

In the case $R = F$ is a field this spectral sequence degenerates if and only if

$$\dim_F H^n_{\mathrm{dR}}(\mathcal{X}/F) = \sum_{p+q=n} \dim_F H^q(\mathcal{X}, \wedge^p \mathbb{L}_{\mathcal{X}/F}). \tag{4.1.2.1}$$

Remark 4.1.2.2. By smooth descent for the cotangent complex, $R\Gamma(\mathcal{X}, \wedge^p \mathbb{L}_{\mathcal{X}/F})$ produces the same answer as the definition of the Hodge cohomology via the lisse-étale site of $\mathcal{X}$ (see Proposition 3.4.1.4).

We will now explain the strategy of our proof of the equality (4.1.2.1) above. The first step is to extend Theorem 4.1.1.2 to the setting of stacks:

Theorem (4.2.1.18)**.** *Let $\mathcal{Y}$ be a smooth Artin stack over a perfect field k of characteristic p admitting a smooth lift to the ring of the second Witt vectors $W_2(k)$. Then there is a canonical equivalence*

$$R\Gamma(\mathcal{Y}, \tau^{\leq p-1}\Omega^\bullet_{\mathcal{Y},\mathrm{dR}}) \simeq R\Gamma\left(\mathcal{Y}^{(1)}, \bigoplus_{i=0}^{p-1} \wedge^i \mathbb{L}_{\mathcal{Y}^{(1)}}[-i]\right).$$

In particular for $n \leq p-1$ we have $H^n_{\mathrm{dR}}(\mathcal{Y}) \simeq H^n_{\mathrm{H}}(\mathcal{Y}^{(1)})$.

Since the de Rham cohomology for Artin stacks are defined as the right Kan extension from

smooth affine schemes (Definition 3.4.1.3) one can more or less formally deduce the theorem above from the following very functorial form of Deligne-Illusie splitting for affine schemes:

Theorem (4.2.1.16). *Let* $\mathrm{Aff}^{\mathrm{sm}}_{/W_2(k)}$ *be the category of smooth affine schemes over* $W_2(k)$. *Then there is a natural* k*-linear equivalence of functors*

$$\bigoplus_{i=0}^{p-1} \Omega^i_{-} \colon B \mapsto \bigoplus_{i=0}^{p-1} \Omega^i_{(B^{(1)}/p)/k}[-i] \qquad \textit{and} \qquad \tau^{\leq p-1}\Omega^{\bullet}_{-,\mathrm{dR}} \colon B \mapsto \tau^{\leq p-1}\Omega^{\bullet}_{(B/p)/k,\mathrm{dR}}$$

from $\mathrm{Aff}^{\mathrm{sm,op}}_{/W_2(k)}$ *to the* ∞*-category* $D(\mathrm{Mod}_k)$ *which induces the Cartier isomorphism on the level of the individual cohomology functors.*

The splitting in Theorem 4.1.1.2 is already functorial with respect to liftings to $W_2(k)$, but only on the level of the underlying homotopy category and not the ∞-category of complexes $D(\mathrm{Mod}_k)$ itself. To get this higher functoriality we follow [44, Section II] using a more convenient language of [18].

The idea is to extend the de Rham (and the crystalline) cohomology functor to a larger category of quasisyntomic algebras (Definition 4.2.1.1). This category, endowed with the quasisyntomic topology, has a basis consisting of quasi-regular semiperfectoid $W_n(k)$-algebras (Definition 4.2.1.2), on which the values of $R\Gamma_{\mathrm{dR}}$ (and $R\Gamma_{\mathrm{crys}}$) become ordinary rings. Additionally, the Frobenius morphism, the Hodge filtration and the conjugate filtration can be described explicitly. This way, using quasi-syntomic descent, the question reduces to a certain computation in commutative algebra.

More concretely, for a quasi-regular semiperfect k-algebra S one can prove that $R\Gamma_{\mathbb{L}\mathrm{crys}}(S/W_n(k)) \simeq \mathbb{A}_{\mathrm{crys}}(S)/p^n$, where $\mathbb{A}_{\mathrm{crys}}(S)$ is the divided power envelope of the kernel of the natural surjection $W((S)^{\flat}) \twoheadrightarrow S$ (see Construction 4.2.1.10). Under this identification the Hodge filtration on $R\Gamma_{\mathrm{crys}}(-/k) \simeq R\Gamma_{\mathrm{dR}}(-/k)$ corresponds to the filtration by the divided powers of the pd-ideal $I \lhd \mathbb{A}_{\mathrm{crys}}(S)/p$. The conjugate filtration $\mathrm{Fil}^{\mathrm{conj}}_{*}$ admits an explicit description as well (see Definition 4.2.1.12). Given a lifting $\widetilde{S}$ of S to $W_2(k)$ there is a natural morphism $\theta \colon \mathbb{A}_{\mathrm{crys}}(S)/p^2 \to \widetilde{S}$. The image of $K \coloneqq \ker\theta$ under the first divided Frobenius map φ_1 then provides a splitting of $\mathrm{Fil}^{\mathrm{conj}}_1$ into $\mathrm{Fil}^{\mathrm{conj}}_0 \oplus \mathrm{Fil}^{\mathrm{conj}}_1 / \mathrm{Fil}^{\mathrm{conj}}_0 \simeq S^{\flat}/I \oplus I/I^2$ (Proposition 4.2.1.17). By multiplicativity this extends to the splitting of $\mathrm{Fil}^{\mathrm{conj}}_{p-1}$ whose descent to smooth schemes gives Theorem 4.2.1.16.

Remark 4.1.2.3. The original approach of Deligne-Illusie (at least applied literally) does not seem to work for a general Artin stack; the key result of [36] is the equivalence of two gerbes on the

étale site of $Y^{(1)}/k$ for a smooth k-scheme Y: the one of splittings of $\tau^{\leq 1}\varphi_{Y*}\Omega^{\bullet}_{X,\mathrm{dR}}$ in $\mathrm{QCoh}(Y^{(1)})$ and the one of liftings of $Y^{(1)}$ to $W_2(k)$. A general smooth Artin stack $\mathcal{Y}$ can be covered by an affine scheme only *smooth locally*, so one needs to replace the étale site of $\mathcal{Y}$ by the smooth one. But both the space of splittings of $\tau^{\leq 1}\varphi_{Y*}\Omega^{\bullet}_{Y,\mathrm{dR}}$ and the space of liftings to $W_2(k)$ are not even presheaves there. Nevertheless, it would be still interesting to have an explicit description of the space of liftings to $W_2(k)$ for an arbitrary smooth n-Artin stack $\mathcal{Y}$. We do not discuss this question here.

Spreadings. Let now $\mathcal{X}$ be a smooth Hodge-proper stack over a field F of characteristic 0. If there exits $\mathbb{Z}$-subalgebra $R \subset F$, which is finitely generated over $\mathbb{Z}$[1], and a Hodge-proper stack $\mathcal{X}_R$ over R such that $\mathcal{X}_R \otimes_R F \simeq \mathcal{X}$ (a *Hodge-proper spreading of* $\mathcal{X}$), then one can deduce the equality (4.1.2.1) for the n-th cohomology from Theorem 4.2.1.18 by taking a suitable closed point $\operatorname{Spec} k \hookrightarrow \operatorname{Spec} R$ of characteristic $p > n$ and considering the fiber $\mathcal{X}_k$. This way we obtain

Theorem (4.2.2.2)**.** *Let $\mathcal{X}$ be a smooth Hodge-properly spreadable Artin stack over a field F of characteristic zero. Then the Hodge-to-de Rham spectral sequence for $\mathcal{X}$ degenerates at the first page. In particular for each $n \geq 0$ there exists a (non canonical) isomorphism*

$$H^n_{\mathrm{dR}}(\mathcal{X}) \simeq \bigoplus_{p+q=n} H^{p,q}(\mathcal{X}).$$

We must warn the reader that smooth Hodge-properly spreadable stacks do not enjoy many of the nice properties that smooth proper schemes have, in particular the natural mixed Hodge structure on the singular n-th cohomology is not necessarily pure (see Remark 4.3.3.17). The main motivation for the definition is that it is the most general class of stacks for which the Deligne-Illusie method can be applied. This, however, does not exceed all examples of the Hodge-to-de Rham degeneration (see Remark 4.3.3.14).

In order to address the question of the existence of a Hodge-proper spreading we first extend the standard spreading out results for finitely presentable schemes to the case of Artin stacks:

Theorem (4.3.1.12 and 4.3.3.2)**.** *Let $\{S_i\}$ be a filtered diagram of affine schemes with limit S. For a class of morphism $\mathcal{P}$ = proper, smooth, flat, surjective, or any other class of norphisms that*

[1]More generally, in Definition 4.2.2.1 we also allow subrings $R \subset F$ that are localizations of finitely generated $\mathbb{Z}$-algebras under the assumption that the image of $\operatorname{Spec} R$ in $\operatorname{Spec} Z$ is open, but it is fine to assume that R is finitely generated throughout the introduction.

satisfies the conditions of Definition 4.3.1.8. For an affine scheme T, let $\mathscr{S}\mathrm{tk}_{/T}^{n\text{-Art,fp},\mathcal{P}}$ denote the category of finitely presentable n-Artin stacks over T and morphisms in $\mathcal{P}$ between them. Then the natural functor

$$\varinjlim_i \mathscr{S}\mathrm{tk}_{/S_i}^{n\text{-Art,fp},\mathcal{P}} \longrightarrow \mathscr{S}\mathrm{tk}_{/S}^{n\text{-Art,fp},\mathcal{P}}$$

(induced by base-change) is an equivalence.

As a corollary we deduce that any smooth n-Artin stack $\mathcal{X}$ over F admits a smooth spreading $\mathcal{X}_R$ over some finitely generated $\mathbb{Z}$-algebra $R \subset F$ and that any two such spreadings become equivalent after enlarging R. Since all smooth proper stacks are Hodge-proper (see Proposition 4.3.2.10), we immediately deduce the Hodge-to-de Rham degeneration in this case. Note that this includes smooth proper Deligne-Mumford stacks as a special case.

However, Hodge-proper spreadings need not to exist in general: one can show that the classifying stack BG is Hodge-proper for any finite-type group scheme G over F (see Proposition 4.3.3.6) but it is not necessarily Hodge-properly spreadable. Indeed, the classifying stack $B\mathbb{G}_a$ of the additive group has nontrivial Hodge cohomology but is de Rham contractible (i.e. has the de Rham cohomology of a point), so the Hodge-to-de Rham spectral sequence is clearly nondegenerate. By Theorem 4.2.2.2 it follows that it is not Hodge-properly spreadable and this forces the Hodge cohomology of $B\mathbb{G}_{a,\mathbb{Z}}$ to have infinitely generated p-torsion for a dense set of primes p, which one can also see from the explicit description (see Example 4.3.3.7). This illustrates the general phenomenon: the non-degeneracy of the Hodge-to-de Rham spectral sequence in characteristic 0 is always reflected arithmetically, namely the Hodge cohomology of any spreading *has to be* infinitely generated over the base.

In the main case of our interest, namely the quotient stacks $\mathcal{X} = [X/G]$, we exhibit some sufficient conditions for Hodge-proper spreadability purely in terms of the geometry of X, G and the action $G \curvearrowright X$. In this case the spreadability is not easy to show, especially if we can't spread G to a linearly reductive group (which is only possible if G is a torus or an extension of a finite group by one). Nevertheless, using certain cohomological finiteness results from [46] we prove

Theorem (4.4.1.3)**.** *Let F be an algebraically closed field of characteristic 0. Let X be a smooth scheme and let Y be a finite-type quasi-separated scheme over F, both endowed with an action of a reductive group G. Assume that*

1. *There is a proper G-equivariant map $\pi\colon X \to Y$.*

2. *The G-action on Y is locally linear (Definition 4.4.1.1).*

3. *The categorical quotient $Y//G$ is proper.*

Then the quotient stack $[X/G]$ is Hodge-properly spreadable[2].

Theorem 4.4.1.3 applies to some natural examples of X with a G-action, in particular, equivariant "proper-over-affine" varieties (see Example 4.4.1.5) and the GIT quotients, whose coarse moduli space is proper (see Example 4.4.1.6).

We also prove a variant of Theorem 4.4.1.3 where we drop the reductivity assumption on G but impose an additional Bialynicki-Birula (BB)-completeness assumption on the action when restricted to a subgroup $h\colon \mathbb{G}_m \to G$. Moreover, the extra structure given by the map π is replaced by the internal condition on the properness of $h(\mathbb{G}_m)$-fixed points; see Theorem 4.4.2.10 for details. Also, using the results of the upcoming work [52], one can show that global quotients of KN-complete varieties and, more generally, Θ-stratified stacks with certain properness assumptions on the strata are Hodge-properly spreadable; see Section 4.4.3.

As an application, for any Hodge-properly spreadable quotient stack $[X/G]$, we deduce an equivariant Hodge-to-de Rham degeneration:

Corollary (4.2.3.2)**.** Let X be a smooth scheme over $\mathbb{C}$ endowed with an action of an algebraic group G such that the quotient stack $[X/G]$ is Hodge-properly spreadable. Then there is a (non-canonical) decomposition

$$H^n_{G(\mathbb{C})}(X(\mathbb{C}),\mathbb{C}) \simeq \bigoplus_{p+q=n} H^q([X/G], \wedge^p \mathbb{L}_{[X/G]}).$$

Finally, it turns out that Theorem 4.2.2.2 can be applied even in the case of some non-proper schemes, as we discuss in some detail in Section 4.3.3.

4.1.3 Relation to previous work and further directions

Our definition of Hodge-proper stacks is partially motivated by the work [55] by Halpern-Leistner and Preygel, where several generalized notions of properness for stacks are studied. In Questions 1.3.2 and 1.3.3 of loc.cit. authors ask if any formally proper stack (Definition 1.1.3 of loc.cit.)

[2] In fact we prove a stronger statement, namely that $[X/G]$ is cohomologically properly spreadable, see Definitions 4.3.2.1 and 4.3.3.1.

admits a formally proper spreading and if the Hodge-to-de Rham spectral sequence degenerates for a formally proper stack over a field of characteristic 0. It follows from our work that the first statement implies the second; however, the method of Section 4.3.1 does not help to show the existence of a formally proper spreading. In fact, for the degeneration, only the existence of a Hodge-proper spreading would suffice, but this still seems hard to show (see Question 4.4.3.6 in the very end of the thesis).

The splitting of the $(p-1)$-st truncation of the de Rham complex for a smooth tame 1-Artin stack over a perfect field k of characteristic p was established (among other things) in [104]. The key observation in [104] is that a smooth tame stack admits a smooth lift together with a lift of Frobenius étale-locally on its coarse moduli space, which enables to follow the original argument of Deligne-Illusie. Our proof is different and works for an arbitrary smooth n-Artin stack.

It is worth to mention that there is still no example of a smooth liftable scheme X in characteristic p whose Hodge-to-de Rham spectral sequence does not degenerate (recall that the Deligne-Illusie method gives such a degeneration only for $i+j<p$). Motivated by the recent examples of non-degeneration for the HKR-filtration constructed in [4] one could first look for such a counterexample in the world of stacks. The de Rham cohomology of various classifying stacks were considered recently in great detail in [111]; however, in all considered examples the Hodge-to-de Rham spectral sequence did degenerate.

The equivariant Hodge-to-de Rham degeneration for a reductive group G acting on a scheme X, under the Kempf-Ness-completeness assumption was treated (among other things) in [108] by completely different methods. We discuss how to prove the degeneration in a (slightly) more general case via Theorem 4.2.2.2 in Section 4.4.3; for this we need some semiorthogonal decomposition results that are going to appear in the upcoming work of Halpern-Leistner [52]. The same strategy applies to Θ-stratified stacks with properness (or, more generally, cohomologically proper spreadability) assumptions on the fixed points of the strata (see Remark 4.4.3.3).

Another approach to the equivariant Hodge theory was introduced in [54], where the authors deduce (among other things) the noncommutative Hodge-to-de Rham degeneration for $\mathrm{QCoh}([X/G])^{\mathrm{perf}}$ (under the KN-completeness assumption) and some purely non-commutative examples (like categories of matrix factorizations) by exploiting methods of non-commutative geometry. Note that the result of Kaledin (see [65] and [68]) does not apply in this situation, since the DG-category $\mathrm{QCoh}([X/G])^{\mathrm{perf}}$ is usually not smooth. It is natural to ask whether the commutative

degeneration implies the noncommutative one in this case. This is not immediately clear, since the relation between the Hochschild/periodic cyclic homology of the category of perfect complexes and the Hodge/de Rham cohomology for Artin stacks is more subtle than in the case of schemes.

4.1.4 Plan of the chapter

Section 4.2 is devoted to a proof of the degeneration of the Hodge-to-de Rham spectral sequence for Hodge-properly spreadable stacks. For the review of Hodge and de Rham cohomology of stacks and the definition of Hodge-proper stacks see Subsections 3.4.1 and 3.4.2. In Section 4.2.1 we prove (a truncated version of) the Hodge-to-de Rham degeneration in positive characteristic for Hodge-proper stacks that admit a lift to $W_2(k)$. Then, in Section 4.2.2 we prove the Hodge-to-de Rham degeneration in characteristic 0 for stacks that are Hodge-properly spreadable. As a corollary, in Section 4.2.3, in the case of a quotient stack, we also deduce a (non-canonical) Hodge decomposition for the corresponding equivariant singular cohomology.

In Section 4.3 we study the spreadability of Hodge proper stacks. In Subsection 4.3.1 we extend the standard spreading out results for finitely presented schemes and their morphisms to the case of Artin stacks (see Theorem 4.3.1.12). In 4.3.2 we introduce a more convenient class of cohomologically proper stacks which includes all Hodge-proper ones. In Section 4.3.3 we give first examples of spreadable Hodge-proper stacks: in Section 4.3.3 we cover the case of smooth proper stacks, in Section 4.3.3 we discuss for which algebraic groups G the classifying stack BG is Hodge-properly spreadable. Then, in Section 4.3.3 we discuss the case schemes.

In Section 4.4 we concentrate on the spreadability of quotient stacks. In Section 4.4.1 we discuss the case of global quotients by reductive groups whose coarse moduli space is proper. In Section 4.4.2 we prove Hodge-proper spreadability of $[X/\mathbb{G}_m]$ under the condition that the associated Bialynicki-Birula stratification is full and the fixed points subscheme is proper; then, in Section 4.4.2 we use the $\mathbb{G}_m$-case to prove spreadability for another class of global quotients, including some quotients by non-reductive groups. Finally, in Section 4.4.3 we also discuss how to show the spreadability of Kempf-Ness complete varieties by a similar method.

Notations and conventions.

1. We will freely use the language of higher categories, modeled e.g. by quasi-categories of [82]. If not explicitly stated otherwise all categories are assumed to be $(\infty, 1)$ and all (co-)limts are

homotopy ones. The $(\infty,1)$-category of Kan complexes will be denoted by $\mathscr{S}$ and we will call it *the category of spaces*. By $\mathrm{Lan}_i F$ and $\mathrm{Ran}_i F$ we will denote left and right Kan extensions of a functor F along i (see e.g. [82, Definition 4.3.2.2] for more details).

2. For a commutative ring R by $D(\mathrm{Mod}_R)$ we will denote the canonical $(\infty,1)$-enhancement of the triangulated unbounded derived category of the abelian category of A-modules Mod_A. All tensor product, pullback and pushforward functors are implicitly derived.

3. Our conventions on (higher) Artin stacks are the standard ones: see e.g. [110, Section 1.3.3.] or [47, Chapter 2.4] (in the definition we use fppf-topology). We stress that we work with classical (as opposed to derived) higher Artin stacks, i.e. they are defined on the category of *ordinary* commutative rings. We will denote the category of n-Artin stacks over a base scheme S by $\mathscr{S}\mathrm{tk}^{n\text{-Art}}_{/S}$.

4. For a stack $\mathcal{X}$ we will denote by $\mathrm{QCoh}(\mathcal{X})$ the category of *quasi-coherent sheaves on* $\mathcal{X}$ defined as the limit $\lim_{\mathrm{Spec}\, A\to\mathcal{X}} D(\mathrm{Mod}_A)$ over all affine schemes $\mathrm{Spec}\, A$ mapping to $\mathcal{X}$ (see [47, Chapter 3.1] for more details). Note that $\mathrm{QCoh}(\mathcal{X})$ admits a natural t-structure such that $\mathcal{F}\in\mathrm{QCoh}(\mathcal{X})^{\leq 0}$ if and only if $x^*(\mathcal{F})\in D(\mathrm{Mod}_A)^{\leq 0}$ for any A-point $x\in\mathcal{X}(A)$. Moreover, by [47, Chapter 3, Corollary 1.5.7] if $\mathcal{X}$ is Artin stack, then $\mathrm{QCoh}(\mathcal{X})$ is left- and right-complete (i.e. Postnikov's and Whitehead's towers converge) and the truncation functors commute with filtered colimits.

5. For an affine group scheme G over a ring R, given a representation M (i.e. a comodule over the corresponding Hopf algebra $R[G]$) we denote by $R\Gamma(G,M)\in D(\mathrm{Mod}_A)$ the *rational cohomology* complex of G, namely the derived functor of G-invariants $M\mapsto M^G$. By flat descent, for G flat over R, the abelian category $\mathrm{Rep}(G):=\mathrm{Rep}_G(\mathrm{Vect}_F)^{\heartsuit}$ is identified with $\mathrm{QCoh}(BG)^{\heartsuit}$ and $R\Gamma(G,M)\simeq R\Gamma(BG,M)$.

4.2 Degeneration of the Hodge-to-de Rham spectral sequence

4.2.1 Hodge-to-de Rham degeneration in positive characteristic

Let $\mathcal{Y}$ be a Hodge-proper Artin stack over a perfect field k of characteristic p admitting a smooth lift to the ring of the second Witt vectors $W_2(k)$. In this section we will prove that the Hodge-to-de Rham spectral sequence $H^j(\mathcal{Y},\wedge^i\mathbb{L}_{\mathcal{Y}/k})\Rightarrow H^{i+j}_{\mathrm{dR}}(\mathcal{Y}/k)$ degenerates at the first page for

$i+j<p$. Our strategy is to interpret both Hodge and de Rham cohomology in terms of crystalline cohomology and then, following Fontaine-Messing [44] (and Bhatt-Morrow-Scholze [18]), use (quasi-)syntomic descent for the crystalline cohomology to get a very functorial form of the Deligne-Illusie splitting.

We denote by $\sigma\colon k \xrightarrow{x\mapsto x^p} k$ the absolute Frobenius morphism of k. We denote by the same letter σ the induced automorphisms $W(k)\to W(k)$ and $W_n(k)\to W_n(k)$ for any $n\in\mathbb{N}$. For a $W(k)$-algebra (e.g. a $W_s(k)$-algebra for some s) A we denote by $A^{(1)}\coloneqq A\otimes_{W(k),\sigma}W(k)$ its Frobenius twist and by $A^{(-1)}\coloneqq A\otimes_{W(k),\sigma^{-1}}W(k)$ its Frobenius untwist. For each $n\in\mathbb{Z}$ we have the relative Frobenius map $\varphi\colon A^{(n)}\to A^{(n-1)}$.

Definition 4.2.1.1. A morphism $A\to B$ of $W_n(k)$-algebras is called *quasisyntomic* if it is flat and $\mathbb{L}_{B/A}$ has cohomological Tor amplitude $[-1,0]$. A morphism $A\to B$ is *a quasisyntomic cover* if it is quasisyntomic and faithfully flat. We will denote by QSyn_n the site consisting of quasisyntomic $W_n(k)$-algebras with the topology generated by quasisyntomic covers.

The notion of a quasisyntomic morphism is a generalization of more classical notion of a *syntomic* morphism: a flat map $A\to B$ that is locally a complete intersection in smooth one. Syntomic morphisms include smooth morphisms, and (in the case we are over k) the relative Frobenius morphism $\varphi\colon A^{(1)}\to A$. The advantage of quasisyntomic morphisms is that they also include some natural non-finite-type maps, most importantly the direct limit perfection $A\to A_{\mathrm{perf}}\coloneqq\varinjlim_{\varphi,n\geq 0}A^{(-n)}$ and its tensor powers $A\to A_{\mathrm{perf}}\otimes_A\ldots\otimes_A A_{\mathrm{perf}}$ for a smooth k-algebra A. Using standard properties of the cotangent complex it is not hard to show that quasisyntomic morphisms are stable under composition and pushouts along arbitrary morphisms of algebras (and same for quasisyntomic covers). We refer to Section 4 of [18] for more details.

Recall that an $\mathbb{F}_p$-algebra S is called semiperfect if $\varphi\colon S\to S$ is surjective.

Definition 4.2.1.2. A k-algebra S is called *quasiregular semiperfect* if S is quasisyntomic and the relative Frobenius homomorphism $\varphi\colon S^{(1)}\to S$ is surjective. We call a $W_n(k)$-algebra $\widetilde{S}$ *quasiregular semiperfectoid* if it is flat over $W_n(k)$ and $\widetilde{S}/p$ is quasiregular semiperfect. We will denote by $\mathrm{QRSPerf}_n$ the site consisting of quasiregular semiperfect $W_n(k)$-algebras with the topology generated by faithfully flat covers.

For any k-algebra S, $H^0(\mathbb{L}_{S/k})$ is identified with the Kahler differentials $\Omega^1_{S/k}$. Since $d(x^p)=0$, we get that $H^0(\mathbb{L}_{S/k})=0$ for S semiperfect, and that $\mathbb{L}_{S/k}$ is concentrated in a single cohomologi-

cal degree -1 for S quasiregular semiperfectoid. The same is true for $\mathbb{L}_{\widetilde{S}/W_n(k)}$ for a quasiregular semiperfectoid $W_n(k)$-algebra $\widetilde{S}$. Moreover, any flat map $\widetilde{S}_1 \to \widetilde{S}_2$ between quasiregular semiperfectoids over $W_n(k)$ is quasisyntomic. This gives a map of sites $\mathrm{QRSPerf}_n \to \mathrm{QSyn}_n$.

In fact quasiregular semiperfect algebras form a basis of topology in QSyn_n. This leads to an equivalence between the corresponding categories of sheaves:

Proposition 4.2.1.3. *The restriction along the natural embedding $u\colon \mathrm{QRSPerf}_n \to \mathrm{QSyn}_n$ induces an equivalence*

$$\mathcal{S}\mathrm{hv}(\mathrm{QSyn}_n, \mathscr{C}) \xrightarrow{u^{-1}}_{\sim} \mathcal{S}\mathrm{hv}(\mathrm{QRSPerf}_n, \mathscr{C})$$

of the categories of sheaves with values in any presentable ∞-category $\mathscr{C}$.

Proof. Similar to [18, Proposition 4.31]. □

Remark 4.2.1.4. For a sheaf $\mathcal{F}$ on $\mathrm{QRSPerf}_n$ we will denote its image under the inverse equivalence in Proposition 4.2.1.3 by $\mathcal{F}$ as well.

Example 4.2.1.5. Let B be a smooth algebra over $W_n(k)$. By smoothness there is an étale map $P \to B$ from the polynomial algebra $P = P_d \coloneqq W_n(k)[x_1, \dots, x_d]$ for some d. Let $P_{\mathrm{perf}} = W_n(k)[x_1^{1/p^\infty}, \dots, x_d^{1/p^\infty}]$ and let $B_{\mathrm{perf}} \coloneqq B \otimes_P P_{\mathrm{perf}}$; it is a quasiregular[3] perfectoid $W_n(k)$-algebra and the natural map $B \to B_{\mathrm{perf}}$ is a quasisyntomic cover. Moreover all terms $(B_{\mathrm{perf}} \otimes_B \dots \otimes_B B_{\mathrm{perf}})_n$ in the corresponding Cech object are also quasiregular semiperfectoids. Given any sheaf $\mathcal{F}$ on $\mathrm{QRSPerf}_n$ its value on $B \in \mathrm{QSyn}_n$ (via Proposition 4.2.1.3) can be computed as "the unfolding":

$$R\Gamma_{\mathrm{QSyn}_n}(B, \mathcal{F}) \xrightarrow{\sim} \mathrm{Tot}\left(\mathcal{F}(B_{\mathrm{perf}}) \rightrightarrows \mathcal{F}(B_{\mathrm{perf}} \otimes_B B_{\mathrm{perf}}) \substack{\rightarrow\\\rightarrow\\\rightarrow} \mathcal{F}(B_{\mathrm{perf}} \otimes_B B_{\mathrm{perf}} \otimes_B B_{\mathrm{perf}}) \substack{\rightarrow\\\rightarrow\\\rightarrow\\\rightarrow} \cdots \right).$$

For a ring R let $\mathrm{Poly}_R \subset \mathrm{CAlg}_{R/}$ denote the full subcategory of finitely generated *polynomial R-algebras*. Recall that one of the ways to define the cotangent complex $\mathbb{L}_{A/R}$ for an R-algebra A is to consider the left Kan extension of the functor $B \mapsto \Omega^1_{B/R}$ from the category of polynomial R-algebras, namely

$$\mathbb{L}_{A/R} \simeq \operatorname*{colim}_{\mathrm{Poly}_R/A} \Omega^1_{B/R}.$$

One can extend de Rham and crystalline cohomology functors in a similar way:

Construction 4.2.1.6. Let k be a perfect field.

[3] In fact it is even *quasismooth*, $\mathbb{L}_{B/W_n(k)} = 0$.

- The *derived de Rham cohomology* functor

$$R\Gamma_{\mathbb{L}\mathrm{dR}}(-/W_n(k))\colon \mathrm{CAlg}_{W_n(k)/} \to D(\mathrm{Mod}_{W_n(k)})$$

  is defined as the left Kan extension of the functor $B \mapsto \Omega^{\bullet}_{B/W_n(k),\mathrm{dR}}$ on $\mathrm{Poly}_{W_n(k)}$.

- The *derived crystalline cohomology* functor

$$R\Gamma_{\mathbb{L}\mathrm{crys}}(-/W(k))\colon \mathrm{CAlg}_{W_n(k)/} \to D(\mathrm{Mod}_{W(k)})$$

  is defined as the (derived) p-adic completion of the left Kan extension of the functor $B \mapsto R\Gamma_{\mathrm{crys}}((B/p)/W(k))$ on $\mathrm{Poly}_{W_n(k)}$.

Similarly, we can extend the functors $B \mapsto \Omega^{\leq m}_{B/W_n(k),\mathrm{dR}}$ and $B \mapsto \tau^{\leq m}\Omega^{\bullet}_{B/W_n(k),\mathrm{dR}}$ to get filtered objects $(R\Gamma_{\mathbb{L}\mathrm{dR}}(-/W_n(k)), F^*_{\mathrm{H}})$ (Hodge filtration) and $(R\Gamma_{\mathbb{L}\mathrm{dR}}(-/W_n(k)), \mathrm{Fil}^{\mathrm{conj}}_*)$ (conjugate filtration). The derived de Rham cohomology is complete with respect to its conjugate filtration since colimits commute. For $R\Gamma_{\mathbb{L}\mathrm{dR}}(-/k)$ the Cartier isomorphism identifies the corresponding associated graded with $\bigoplus_{r\geq 0} \wedge^r \mathbb{L}_{B^{(1)}/k}[-r]$. Thus $R\Gamma_{\mathbb{L}\mathrm{dR}}(-/k)$ satisfies flat descent (by flat descent for the cotangent complex [18, Theorem 3.1]); for $R\Gamma_{\mathbb{L}\mathrm{dR}}(-/W_n(k))$ and $R\Gamma_{\mathbb{L}\mathrm{crys}}((-/p)/W(k))$ the same holds since they are p-adically complete and their reduction mod p is $R\Gamma_{\mathbb{L}\mathrm{dR}}(-/k)$. In particular all these functors define sheaves on QSyn_n.

Remark 4.2.1.7. Let B be a smooth $W_n(k)$-algebra. We claim that the natural morphism $R\Gamma_{\mathbb{L}\mathrm{dR}}(B) \to R\Gamma_{\mathrm{dR}}(B)$ is an equivalence. To see this it is enough to prove that the induced map on the associated graded of the conjugate filtration is an equivalence. But, since B is smooth, $\Lambda^i \mathbb{L}_{B^{(1)}/W_n(k)} \xrightarrow{\sim} \Omega^i_{B^{(1)}/W_n(k)}$.

Remark 4.2.1.8. For any $W_n(k)$-algebra B the complexes $R\Gamma_{\mathbb{L}\mathrm{crys}}(B/W(k)) \otimes^{\mathbb{L}}_{W(k)} W_n(k)$ and $R\Gamma_{\mathbb{L}\mathrm{dR}}(B/W_n(k))$ are canonically equivalent. Indeed, by construction both functors commute with geometric realizations, hence it is enough to prove the statement for B being a smooth $W_n(k)$-algebra. In this case this is a basic result in the crystalline cohomology theory, see e.g. [10, Corollary 7.4].

Remark 4.2.1.9. Since the absolute Frobenius $\sigma\colon k \to k$ is an automorphism, the cotangent complex $\mathbb{L}_{k/\mathbb{F}_p}$ (and all its wedge powers) vanishes. It follows that $R\Gamma_{\mathrm{dR}}(k/F_p) \simeq k$. Given

any k-algebra B we have a natural morphism of E_∞-algebras $R\Gamma_{\mathrm{dR}}(k/\mathbb{F}_p) \to R\Gamma_{\mathrm{dR}}(B/\mathbb{F}_p)$. This endows $R\Gamma_{\mathrm{dR}}(B/\mathbb{F}_p)$ with a natural k-linear structure. Similarly, for any k-algebra A the complex $R\Gamma_{\mathbb{L}\mathrm{crys}}(A/\mathbb{Z}_p)$ has a natural $W(k)$-linear structure. Moreover, the natural morphism

$$R\Gamma_{\mathbb{L}\mathrm{crys}}(B/\mathbb{Z}_p) \to R\Gamma_{\mathbb{L}\mathrm{crys}}(B/W(k)) \tag{4.2.1.1}$$

is $W(k)$-linear. We claim that (4.2.1.1) is an equivalence. Since both sides are p-adically complete it is enough to show that it is an equivalence mod p, where we get an analogous map, but for the derived de Rham cohomology of the reduction B/p. On the associated graded of the conjugate filtration $\mathrm{Fil}^{\mathrm{conj}}_*$ the induced map is an equivalence, since in the transitivity triangle

$$\mathbb{L}_{k/\mathbb{F}_p} \otimes_k B \to \mathbb{L}_{B/\mathbb{F}_p} \to \mathbb{L}_{B/k}$$

the term $\mathbb{L}_{k/\mathbb{F}_p}$ is equivalent to 0. Thus (4.2.1.1) is an equivalence.

Recall that the cotangent complex $\mathbb{L}_{\widetilde{S}/W_n(k)}$ of $\widetilde{S} \in \mathrm{QRSPerf}_n$ is supported in cohomological degree -1, thus $\bigoplus_r \wedge^r \mathbb{L}_{\widetilde{S}/W_n(k)}[-r]$ is supported in cohomological degree 0. The same holds for $R\Gamma_{\mathbb{L}\mathrm{dR}}(\widetilde{S}/W_n(k))$; in other words, it is a classical commutative ring. It has a description in terms of one of the Fontaine's period rings A_{crys}:

Construction 4.2.1.10. Let S be a semiperfect k-algebra and let $S^\flat$ be the inverse limit perfection $S^\flat := \varprojlim_{\varphi, n\geq 0} S^{(n)}$. We have a natural map $S^\flat \to S$ which is surjective. The ring $\mathbb{A}_{\mathrm{crys}}(S)$ is defined as the p-adic completion of the divided power envelope of the kernel of the natural composite surjection $\theta_{1,S}\colon W(S^\flat) \twoheadrightarrow S^\flat \twoheadrightarrow S$ (where the divided power structure agrees with the one on $W(k)$). Note that $\mathbb{A}_{\mathrm{crys}}(S)/p$ is identified with the PD-completion $D_I^{PD}(S^\flat)$ along the ideal $I \subset S^\flat$ defined as the kernel of the natural map $S^\flat \twoheadrightarrow S$.

Theorem 8.14(3) of [18] (together with Remark 4.2.1.9) identifies $R\Gamma_{\mathbb{L}\mathrm{crys}}(S/W(k))$ with $\mathbb{A}_{\mathrm{crys}}(S)$. The ring $\mathbb{A}_{\mathrm{crys}}(S)$ comes with a natural ring morphism $\varphi\colon \mathbb{A}_{\mathrm{crys}}(S)^{(1)} \to \mathbb{A}_{\mathrm{crys}}(S)$ induced by the relative Frobenius $\varphi\colon S^{(1)} \to S$. It is identified with the natural Frobenius $\varphi\colon R\Gamma_{\mathbb{L}\mathrm{crys}}(S/W(k))^{(1)} \to R\Gamma_{\mathbb{L}\mathrm{crys}}(S/W(k))$ on the crystalline cohomology. For each n we define a presheaf of rings $\mathbb{A}_{\mathrm{crys}}$ on $\mathrm{QRSPerf}_n$ by sending $\widetilde{S} \in \mathrm{QRSPerf}_n$ to $\mathbb{A}_{\mathrm{crys}}(\widetilde{S}/p)$. By the above identification it is in fact a sheaf. Note that by the universal property of the PD-envelope there is a natural map[4] $\theta_{n,\widetilde{S}} : \mathbb{A}_{\mathrm{crys}}(\widetilde{S}/p) \to \widetilde{S}$ which quotients through $\mathbb{A}_{\mathrm{crys}}(\widetilde{S}/p)/p^n$.

[4]Here we endow $(p) \subset \widetilde{S}$ with the standard PD-structure, given by $p^{[k]} := p^k/k!$.

The following two filtrations on $\mathbb{A}_{\text{crys}}/p$ correspond to the Hodge and the conjugate filtrations:

Definition 4.2.1.11. Let S be a quasiregular semiperfect k-algebra and let I be the ideal of the natural projection $S^\flat \twoheadrightarrow S$. The descending *Hodge filtration* on $\mathbb{A}_{\text{crys}}(S)/p \simeq D^{PD}_I(S^\flat)$ is defined as the filtration by the divided powers of I: $\mathbb{A}_{\text{crys}}(S)/p \simeq I^0 \supset I \supset I^{[2]} \supset I^{[3]} \supset \cdots$. This filtration is functorial in S and thus defines a filtration by presheaves $\mathbb{I}^0 \supset \mathbb{I} \supset \mathbb{I}^{[2]} \supset \mathbb{I}^{[3]} \supset \cdots$ on the sheaf $\mathbb{A}_{\text{crys}}/(p)$ on QRSPerf_n for any n. Via Proposition 8.12 of [18] it is identified with the Hodge filtration on $R\Gamma_{\mathbb{L}\text{dR}}(S/k) \simeq \mathbb{A}_{\text{crys}}(S)/p$ and thus is in fact a filtration by sheaves.

Definition 4.2.1.12. The ascending *conjugate filtration* $\text{Fil}^{\text{conj}}_*$ on $\mathbb{A}_{\text{crys}}(S)/p \simeq D^{PD}_{S^\flat}(I)$ is defined by taking F^{conj}_r to be the $S^\flat$-submodule generated by the elements of the form $s_1^{[l_1]}s_2^{[l_2]}\dots s_m^{[l_m]}$ with $s_i \in I$ and $\sum_{i=1}^m l_i < (r+1)p$. This construction is functorial in S and determines an (ascending) filtration $\text{Fil}^{\text{conj}}_*$ on the sheaf $\mathbb{A}_{\text{crys}}/p$ on QRSPerf_n for any n. By Proposition 8.12 of [18] it is identified with the conjugate filtration on $R\Gamma_{\mathbb{L}\text{dR}}(S/k)$ and thus is also a filtration by sheaves.

Note that both filtrations are multiplicative and exhaustive.

The following is an analogue of the inverse Cartier isomorphism (see Theorem 4.1.1.1) between $(\mathbb{A}_{\text{crys}}/p, \mathbb{I}^{[*]})$ and $(\mathbb{A}_{\text{crys}}/p, \text{Fil}^{\text{conj}}_*)$:

Proposition 4.2.1.13 ([18], Proposition 8.11)**.** *Let S be a semiperfect k-algebra. There is a well-defined surjective homomorphism of $S^\flat$-algebras $\kappa_*\colon \Gamma^*_{S^\flat}(I/I^2)^{(1)} \otimes_{(S^\flat)^{(1)}} S^\flat \to \text{gr}^{\text{conj}}_*(\mathbb{A}_{\text{crys}}(S)/p)$[5]. If S is quasiregular κ_* is an isomorphism.*

Proof. The map is defined as follows: for $s_i \in I$

$$\kappa_{k_1+\cdots k_m} : (s_1^{[k_1]}\cdots s_m^{[k_m]})^{(1)} \otimes 1 \mapsto \prod_{i=1}^m \left(\frac{(pk_i)!}{p^{k_i}k_i!}\right) s_1^{[pk_1]}\cdots s_m^{[pk_m]}.$$

We have $(s_1s_2)^{[pk]} = p!(s_1^k)^{[p]}s_2^{[pk]} = 0$ and $(s_1s_2)^{[l]} \in \text{Fil}^{\text{conj}}_0$ for any $l < p$. This shows that for $s \in I^2$, $s^{[l]} \in \text{Fil}^{\text{conj}}_0$ for all l and so the map is well-defined. Elements $\{s_1^{[pk_1]}\cdots s_m^{[pk_m]}\}_{k_1+\cdots k_m<r+1}$ in fact generate $\text{Fil}^{\text{conj}}_r$ over $S^\flat$. Since the integer $\prod_{i=1}^m \left(\frac{(pk_i)!}{p^{k_i}k_i!}\right)$ is a p-adic unit the map κ_* is surjective. The fact that κ_* is an isomorphism for S quasiregular semiperfect is a part of Proposition 8.12 of [18]. □

Remark 4.2.1.14. In particular we get an isomorphism $\kappa_*\colon \Gamma^*_{S^\flat}(\mathbb{I}/\mathbb{I}^2) \xrightarrow{\sim} \text{gr}^{\text{conj}}_*(\mathbb{A}_{\text{crys}}/p)$ of sheaves on $\text{QRSPerf}_{\text{n}}$.

[5] Where Γ^* denotes the free commutative divided power algebra.

Now we descend everything back to the quasisyntomic site QSyn_n. We record what the sheaves defined above give when computed on a smooth $W_n(k)$-algebra B.

Proposition 4.2.1.15. *Let B be a smooth $W_n(k)$-algebra considered as an object of QSyn_n. Then:*

1. *For any $0 \leq s \leq n$ there is a natural equivalence of E_∞-algebras $R\Gamma_{\mathrm{QSyn}_n}(B, \mathbb{A}_{\mathrm{crys}}/p^s) \simeq \Omega^\bullet_{(B/p^s)/W_s(k),\mathrm{dR}}$.*

2. *For any $r \in \mathbb{Z}_{\geq 0}$ there is a natural equivalence $R\Gamma_{\mathrm{QSyn}_n}(B, \mathbb{I}^{[r]}) \simeq \Omega^{\geq r}_{(B/p)/k,\mathrm{dR}}$, where $\Omega^{\geq r}_{B/p,\mathrm{dR}/k}$ is the r-th term of the Hodge filtration.*

   In particular, $R\Gamma_{\mathrm{QSyn}_n}(B, \mathbb{I}^{[r]}/\mathbb{I}^{[r+1]}) \simeq \Omega^r_{(B/p)/k}[-r]$.

3. *For any $r \in \mathbb{Z}_{\geq 0}$ there is a natural equivalence $R\Gamma_{\mathrm{QSyn}_n}(B, \mathrm{Fil}^{\mathrm{conj}}_r) \simeq \tau^{\leq r}\Omega^\bullet_{(B/p)/k,\mathrm{dR}}$.*

4. *The natural map $\Gamma^r_{S^\flat}(\mathbb{I}/\mathbb{I}^2)^{(1)} \to \mathbb{I}^{[r]}/\mathbb{I}^{[r+1]}$ induces an equivalence*

$$R\Gamma_{\mathrm{QSyn}_n}(B, \Gamma^r_{S^\flat}(\mathbb{I}/\mathbb{I}^2))^{(1)} \simeq R\Gamma_{\mathrm{QSyn}_n}(B, \mathbb{I}^{[r]}/\mathbb{I}^{[r+1]})$$

   for any $r \geq 0$.

5. *The isomorphism $\kappa_*\colon \Gamma^*_{S^\flat}(\mathbb{I}/\mathbb{I}^2)^{(1)} \xrightarrow{\sim} \mathrm{gr}^{\mathrm{conj}}_*(\mathbb{A}_{\mathrm{crys}}/p)$ from Proposition 4.2.1.13 induces the inverse Cartier isomorphism*

$$\bigoplus_{r=0}^{\infty} \Omega^r_{(B^{(1)}/p)/k} \xrightarrow[\sim]{C^{-1}} \bigoplus_{r=0}^{\infty} H^r\left(\Omega^\bullet_{(B/p)/k,\mathrm{dR}}\right)$$

   via the above equivalencies.

Proof. Parts $1, 2, 3$ follow from Proposition 8.12 and Theorem 8.14(3) of [18] and flat descent for the derived crystalline cohomology (using Remarks 4.2.1.7, 4.2.1.8 and 4.2.1.9).

4. We use the notations of Example 4.2.1.5. We have

$$R\Gamma_{\mathrm{QSyn}_n}(B, \mathcal{F}) \xrightarrow{\sim} \mathrm{Tot}\left(\mathcal{F}(B_{\mathrm{perf}}) \rightrightarrows \mathcal{F}(B_{\mathrm{perf}} \otimes_B B_{\mathrm{perf}}) \;\substack{\rightarrow\\\rightarrow\\\rightarrow}\; \mathcal{F}(B_{\mathrm{perf}} \otimes_B B_{\mathrm{perf}} \otimes_B B_{\mathrm{perf}}) \;\substack{\rightarrow\\\rightarrow\\\rightarrow\\\rightarrow}\; \cdots\right)$$

for any quasisintomic sheaf $\mathcal{F}$. Moreover all terms $(B_{\mathrm{perf}} \otimes_B \ldots \otimes_B B_{\mathrm{perf}})_n$ are in fact regular semiperfect, meaning $I \subset S^\flat$ is generated by a regular sequence. Thus for them $\Gamma^r_{S^\flat}(I/I^2) \xrightarrow{\sim} I^{[r]}/I^{[r+1]}$ and so $R\Gamma_{\mathrm{QSyn}_n}(B, \Gamma^r_{S^\flat}(\mathbb{I}/\mathbb{I}^2)) \simeq R\Gamma_{\mathrm{QSyn}_n}(B, \mathbb{I}^{[r]}/\mathbb{I}^{[r+1]})$.

5. The inverse Cartier isomorphism C^{-1} is uniquely defined by the property that it is multiplicative, $C^{-1}(f) = f^p$ and $C^{-1}(df) = f^{p-1}df$ for any $f \in B$. The map κ_* is multiplicative, κ_0 is by definition given by Frobenius, so it remains to check the third assertion. By functoriality (considering the map $k[x] \xrightarrow{x \mapsto f} B$) it is enough to check this in the case $B = k[x]$. Recall that the map $f \mapsto C^{-1}(df)$ is given more functorially as $f \mapsto \beta(f^p)$ where $\beta\colon H^0_{\mathrm{dR}}(k[x]) \to H^1_{\mathrm{dR}}(k[x])$ is the Bockstein operator associated to the de Rham complex of the lifting $\widetilde{B} \coloneqq W_2(k)[x]$ (see e.g. Proposition 4 of [45]). Thus it is enough to check that κ_1 agrees with Bockstein (associated to $R\Gamma_{\mathrm{QSyn}_2}(\widetilde{B}, \mathbb{A}_{\mathrm{crys}}/p^2) \simeq \Omega^{\bullet}_{\widetilde{B}/W_2(k),\mathrm{dR}}$) in the same way. Let $\widetilde{B}_{\mathrm{perf}} \coloneqq W_2(k)[x^{1/p^\infty}]$, then $R\Gamma_{\mathrm{QSyn}_2}(\widetilde{B}, \mathbb{A}_{\mathrm{crys}}/p^2)$ can be computed by the "unfolding", given in this case by

$$W_2\left(k[x^{1/p^\infty}]\right) \xrightarrow{d} W_2\left(k[y_1^{1/p^\infty}, y_2^{1/p^\infty}]^{\wedge}_{(y_1-y_2)}\right)^{PD}_{\mathrm{Ker}\theta_1} \to \dots$$

By part 1 we know that the cohomology is non-trivial only in degree 0 and 1. We have an analogous complex for $R\Gamma_{\mathrm{QSyn}_2}(\widetilde{B}, \mathbb{A}_{\mathrm{crys}}/p)$:

$$k[x^{1/p^\infty}] \to k[y_1^{1/p^\infty}, y_2^{1/p^\infty}]^{PD}_{(y_1-y_2)} \to \dots$$

The first arrow in both is given by $x \mapsto y_1 - y_2$. For $\mathbb{A}_{\mathrm{crys}}/p$ its kernel is $k[x^p]$. We see that the class of $\beta(x^p) \in H^1_{\mathrm{dR}}(k[x])$ under the identification $R\Gamma_{\mathrm{QSyn}_2}(\widetilde{B}, \mathbb{A}_{\mathrm{crys}}/p) \simeq \Omega^{\bullet}_{B/k,\mathrm{dR}}$ can be computed using the complex for $\mathbb{A}_{crys}/p^2$ as

$$\frac{d([x]^p)}{p} \in k[y_1^{1/p^\infty}, y_2^{1/p^\infty}]^{PD}_{(y_1-y_2)} \simeq W_2\left(k[y_1^{1/p^\infty}, y_2^{1/p^\infty}]^{\wedge}_{(y_1-y_2)}\right)^{PD}_{\mathrm{Ker}\theta_1} \otimes_{W_2(k)} k$$

since the Teichmuller lift $[x^p] \in W_2(k[x^{1/p^\infty}])$ is indeed a lifting of $x^p \in k[x^{1/p^\infty}]$. We have $d([x]^p) = [y_1 - y_2]^p = p! \cdot [y_1 - y_2]^{[p]}$ and so $\dfrac{d([x]^p)}{p} = (p-1)! \cdot (y_1 - y_2)^{[p]}$.

The complex $R\Gamma_{\mathrm{QSyn}_2}(\widetilde{B}, \mathbb{I}/\mathbb{I}^2)$ is computed by

$$0 \to k[y_1^{1/p^\infty}, y_2^{1/p^\infty}] \cdot (y_1 - y_2) \;/\; (y_1 - y_2)^2 \to \dots$$

while $R\Gamma_{\mathrm{QSyn}_2}(\widetilde{B}, \mathrm{Fil}_1^{\mathrm{conj}} \,/\, \mathrm{Fil}_0^{\mathrm{conj}})$ is computed by

$$0 \to k[y_1^{1/p^\infty}, y_2^{1/p^\infty}] \cdot (y_1 - y_2)^{[p]} \to \dots$$

By a direct computation one can see that $1 \cdot (y_1 - y_2)$ generates $H^1_{\mathrm{QSyn}_2}(\widetilde{B}, \mathbb{I}/\mathbb{I}^2)$, and that it is sent exactly to $(p-1)! \cdot (y_1 - y_2)^{[p]}$ under the map κ_1 from Proposition 4.2.1.13. On the other hand $y_1 - y_2$ is equal to $d(x) \in k[y_1^{1/p^\infty}, y_2^{1/p^\infty}]^{PD}_{(y_1 - y_2)}$ in the complex for $\mathbb{A}_{crys}/p$ and so corresponds to the differential form dx under the quasiisomorphism $R\Gamma_{\mathrm{QSyn}_2}(\widetilde{B}, \mathrm{Fil}_1^{\mathrm{conj}} / \mathrm{Fil}_0^{\mathrm{conj}}) \simeq \Omega^1_{B/k}[-1]$. Thus κ_1 agrees with C^{-1}.

□

Next we prove the following enhancement of the classical Deligne-Illusie theorem:

Theorem 4.2.1.16. *Let $\mathrm{Aff}^{\mathrm{sm}}_{/W_2(k)}$ be the category of smooth affine schemes over $W_2(k)$. Then there is a natural k-linear equivalence of functors*

$$\bigoplus_{i=0}^{p-1} \Omega^i_{-^{(1)}} \colon B \mapsto \bigoplus_{i=0}^{p-1} \Omega^i_{(B^{(1)}/p)/k}[-i] \quad \text{and} \quad \tau^{\leq p-1}\Omega^\bullet_{-,\mathrm{dR}} \colon B \mapsto \tau^{\leq p-1}\Omega^\bullet_{(B/p)/k,\mathrm{dR}}$$

from $\mathrm{Aff}^{\mathrm{sm,op}}_{/W_2(k)}$ to $D(\mathrm{Mod}_k)$ which induces the Cartier isomorphism on the level of the individual cohomology functors.

By Proposition 4.2.1.3 and Proposition 4.2.1.15 to deduce the statement of the theorem it is enough to prove the following:

Proposition 4.2.1.17. *There is a natural isomorphism $f \colon \bigoplus_{r=0}^{p-1} \Gamma^r_{S^\flat}(\mathbb{I}/\mathbb{I}^2) \simeq \mathrm{Fil}^{\mathrm{conj}}_{p-1}$ of sheaves of abelian groups on $\mathrm{QRSPerf}_2$ such that it agrees with $\kappa_{\leq p-1} \colon \Gamma^{\leq p-1}_{S^\flat}(\mathbb{I}/\mathbb{I}^2) \xrightarrow{\sim} \mathrm{gr}^{\mathrm{conj}}_{\leq p-1}(\mathbb{A}_{\mathrm{crys}}/p)$ after passing to the associated graded.*

Proof. Given $\widetilde{S} \in \mathrm{QRSPerf}_2$ we denote by S the reduction of $\widetilde{S}$ modulo p. As before we denote the kernel of the natural map $S^\flat \to S$ by I. Note that $\Gamma^i_{S^\flat}(I/I^2) \simeq \mathrm{Sym}^i_{S^\flat}(I/I^2)$ for $i \leq p-1$ and so, extending the map by multiplicativity, it is enough to construct a splitting $f \colon S^\flat/I \oplus I/I^2 \xrightarrow{\sim} \mathrm{Fil}^{\mathrm{conj}}_1$. Recall that we have a natural endomorphism $\varphi \colon \mathbb{A}_{\mathrm{crys}}(S) \to \mathbb{A}_{\mathrm{crys}}(S)$. We consider the Nygaard filtration (see Definition 8.9 of [18])

$$\mathcal{N}^{\geq i}\mathbb{A}_{\mathrm{crys}}(S) \coloneqq \{x \in \mathbb{A}_{\mathrm{crys}}(S) \mid \varphi(x) \in p^i \mathbb{A}_{\mathrm{crys}}(S)\}.$$

In fact we will be interested only in the first two of its associated graded terms. We will construct f by using the divided Frobenii, defined as follows. By Theorem 8.15(1) of [18] $\mathbb{A}_{\mathrm{crys}}(S)$ is p-torsion

free and so for each $i \geq 0$ one has a well defined map

$$\varphi_i := \varphi/p^i \colon \mathcal{N}^i \mathbb{A}_{\mathrm{crys}}(S) \to \mathbb{A}_{\mathrm{crys}}(S)/p$$

from the i-th graded piece $\mathcal{N}^i \mathbb{A}_{\mathrm{crys}}(S) := \mathcal{N}^i \mathbb{A}_{\mathrm{crys}}(S)/\mathcal{N}^{i+1}\mathbb{A}_{\mathrm{crys}}(S)$ of the Nygaard filtration.

It is clear that $p \cdot \mathbb{A}_{\mathrm{crys}}(S) \subset \mathcal{N}^{\geq 1}\mathbb{A}_{\mathrm{crys}}(S)$; moreover, by Theorem 8.14(4) of [18], $\mathcal{N}^{\geq 1}\mathbb{A}_{\mathrm{crys}}(S) \bmod p \cdot \mathbb{A}_{\mathrm{crys}}$ is given by $I \subset \mathbb{A}_{\mathrm{crys}}(S)/p$. Thus $\mathcal{N}^0 := \mathcal{N}^{\geq 0}/\mathcal{N}^{\geq 1} \simeq S^\flat/I$ and φ_0 induces an isomorphism $S^\flat/I \xrightarrow{\sim} \mathrm{Fil}_0^{\mathrm{conj}}$ (since $\kappa_0 = \varphi$, this follows from Proposition 4.2.1.13). We then also have a map $\varphi_1 \colon \mathcal{N}^1\mathbb{A}_{\mathrm{crys}}(S) \to \mathbb{A}_{\mathrm{crys}}(S)/p$, which, by Theorem 8.14(2) of [18], is an isomorphism onto $\mathrm{Fil}_1^{\mathrm{conj}}$. Multiplication by p induces a natural map $\mathcal{N}^0\mathbb{A}_{\mathrm{crys}}(S) \to \mathcal{N}^1\mathbb{A}_{\mathrm{crys}}(S)$ which after composing with φ_1 is identified with the embedding $\mathrm{Fil}_0^{\mathrm{conj}} \subset \mathrm{Fil}_1^{\mathrm{conj}}$. In fact, by flatness of $\mathbb{A}_{\mathrm{crys}}(S)$, we have $\mathcal{N}^{\geq 1}\mathbb{A}_{\mathrm{crys}}(S) \cap p \cdot \mathbb{A}_{\mathrm{crys}}(S) \simeq \mathcal{N}^{\geq 0}\mathbb{A}_{\mathrm{crys}}(S)$ and so $\mathrm{Fil}_0^{\mathrm{conj}}$ (under the isomorphism given by φ_1) is identified exactly with the subspace of those elements in $\mathcal{N}^1\mathbb{A}_{\mathrm{crys}}(S) \simeq \mathrm{Fil}_1^{\mathrm{conj}}$ that lift to elements of $\mathcal{N}^{\geq 1}\mathbb{A}_{\mathrm{crys}}(S)$ divisible by p.

We now use the lifting $\widetilde{S}$ of S to construct the splitting of $\mathrm{Fil}_1^{\mathrm{conj}}$. Recall that we have a map $\theta_{2,\widetilde{S}} \colon \mathbb{A}_{\mathrm{crys}}(S)/p^2 \twoheadrightarrow \widetilde{S}$ and let $K := \ker \theta_{2,\widetilde{S}}$. Since both $\widetilde{S}$ and $\mathbb{A}_{\mathrm{crys}}(S)/p^2$ are flat over $W_2(k)$, we get that K is also flat over $W_2(k)$ and that $K/pK \simeq I$:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & \mathbb{A}_{\mathrm{crys}}(S)/p^2 & \longrightarrow & \widetilde{S} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & I & \longrightarrow & \mathbb{A}_{\mathrm{crys}}(S)/p & \longrightarrow & S & \longrightarrow & 0. \end{array}$$

The splitting is then given by applying φ_1 to K. Namely, since $\varphi(I) = 0 \in \mathbb{A}_{\mathrm{crys}}(S)/p$ it follows that $\varphi(K) \subset p \cdot \mathbb{A}_{\mathrm{crys}}(S)/p^2$ and $K \subset \mathcal{N}^{\geq 1}\mathbb{A}_{\mathrm{crys}}(S) \bmod p^2\mathbb{A}_{\mathrm{crys}}(S)$. The natural projection from K to $\mathcal{N}^1\mathbb{A}_{\mathrm{crys}}(S)$ contains $p \cdot K + \mathcal{N}^{\geq 2}\mathbb{A}_{\mathrm{crys}}(S) \bmod p^2\mathbb{A}_{\mathrm{crys}}(S)$ in its kernel. Since $K/pK = I$ and the image of $\mathcal{N}^{\geq 2}\mathbb{A}_{\mathrm{crys}}$ modulo p is given by I^2 (e.g. by Theorem 8.14(4) of [18]), we get that φ_1 (applied to K) gives a well-defined map $f \colon I/I^2 \to \mathrm{Fil}_1^{\mathrm{conj}}$. Moreover $K \cap (p \cdot \mathbb{A}_{\mathrm{crys}}(S)/p^2) \subset p \cdot K$, since K is flat over $W_2(k)$, and so the image of f does not intersect with $\mathrm{Fil}_0^{\mathrm{conj}}$.

It remains to check that the constructed $f \colon I/I^2 \to \mathrm{Fil}_1^{\mathrm{conj}}$ coincides with κ_1 after the projection to $\mathrm{Fil}_1^{\mathrm{conj}}/\mathrm{Fil}_0^{\mathrm{conj}}$. Given $s \in I$ let $\widetilde{s} = [s] + p \cdot s' \in K \subset \mathbb{A}_{\mathrm{crys}}/p^2$ be a lifting of s to an element of

K. Then

$$\varphi(\widetilde{s}) = \varphi([s]) + p \cdot \varphi(s') = (p-1)! \cdot p \cdot [s]^{[p]} + p \cdot \varphi(s') \;\Rightarrow\; f(s) = (p-1)! \cdot s^{[p]} + \varphi(s').$$

By the discussion above (see also Theorem 8.14(2) in [18]) $\varphi(s') \in \mathrm{Fil}_0^{\mathrm{conj}}$ and $f(s) = (p-1)! \cdot s^{[p]}$ modulo $\mathrm{Fil}_0^{\mathrm{conj}}$.

Since the above splitting is clearly functorial in $\widetilde{S}$ we get the statement of the proposition. □

As a corollary we deduce

Theorem 4.2.1.18. *Let $\mathcal{Y}$ be a smooth Artin stack over a perfect field k of characteristic p admitting a smooth lift to the ring of the second Witt vectors $W_2(k)$. Then there is a canonical equivalence*

$$R\Gamma(\mathcal{Y}, \tau^{\leq p-1}\Omega^{\bullet}_{\mathcal{Y},\mathrm{dR}}) \simeq R\Gamma\left(\mathcal{Y}^{(1)}, \bigoplus_{i=0}^{p-1} \wedge^i \mathbb{L}_{\mathcal{Y}^{(1)}}[-i]\right).$$

In particular for $n \leq p-1$ we have $H^n_{\mathrm{dR}}(\mathcal{Y}) \simeq H^n_{\mathrm{H}}(\mathcal{Y}^{(1)})$.

Proof. Let $\pi\colon \mathscr{S}\mathrm{tk}^{n\text{-Art,sm}}_{/W_2(k)} \to \mathscr{S}\mathrm{tk}^{n\text{-Art,sm}}_{k}$ be the reduction functor, $\widetilde{\mathcal{Y}} \mapsto \widetilde{\mathcal{Y}} \otimes_{W_2(k)} k$. By Theorem 4.2.1.16 it is enough to prove that the natural map (existing by the universal property of the right Kan extensions)

$$R\Gamma_{\mathrm{dR}}(-/k) \circ \pi \to \mathrm{Ran}_{i_2}(R\Gamma_{\mathrm{dR}}(-/k) \circ \pi_{|\,\mathrm{Aff}^{\mathrm{sm}}_{/W_2(k)}}) \tag{4.2.1.2}$$

(where i_2 denotes the inclusion functor $\mathrm{Aff}^{\mathrm{sm}}_{/W_2(k)} \hookrightarrow \mathscr{S}\mathrm{tk}^{n\text{-Art,sm}}_{/W_2(k)}$) is an equivalence. Since both sides of (4.2.1.2) satisfy smooth descent, by induction on n we reduce the statement to the case of smooth affine schemes over $W_2(k)$, where (4.2.1.2) is evidently an equivalence. □

Corollary 4.2.1.19. *Let $\mathcal{Y}$ be a smooth Hodge-proper stack over a perfect field k of characteristic p admitting a smooth lift to $W_2(k)$. Then the Hodge-to-de Rham spectral sequence $H^j(\mathcal{Y}, \wedge^i \mathbb{L}_{\mathcal{Y}/k}) \Rightarrow H^{i+j}_{\mathrm{dR}}(\mathcal{Y}/k)$ degenerates at the first page for $i + j < p$.*

Proof. This follows from Theorem 4.2.1.18 and the equality of dimensions $\dim_k H^n_{\mathrm{H}}(\mathcal{Y}) = \dim_k H^n_{\mathrm{H}}(\mathcal{Y}^{(1)})$. □

4.2.2 Degeneration in characteristic zero

To reduce the statement in characteristic 0 to results of the previous section we introduce the following notion:

Definition 4.2.2.1. A smooth Hodge-proper Artin stack $\mathcal{X}$ over a field F of characteristic 0 is called *Hodge-properly spreadable* if there exists a $\mathbb{Z}$-subalgebra $R \subset F$ and an Artin stack $\mathcal{X}_R$ over $\operatorname{Spec} R$ such that

- R is a localization of a finitely generated $\mathbb{Z}$-algebra such that the image of $\operatorname{Spec} R$ in $\operatorname{Spec} \mathbb{Z}$ is open.
- $\mathcal{X}_R$ is smooth over R and $\mathcal{X} \otimes_R F \coloneqq \mathcal{X}_R \times_{\operatorname{Spec} R} \operatorname{Spec} F \simeq \mathcal{X}$.
- $\mathcal{X}_R$ is Hodge-proper over R, namely $R\Gamma(\mathcal{X}_R, \wedge^p \mathbb{L}_{X_R/R})$ is nearly coherent over R for any $p \geq 0$.

We defer a thorough discussion of spreadability of stacks till the next section. We only stress here again, that (unlike in the case of schemes) Hodge-proper spreadings do not exist in general (see Example 4.3.3.7).

Now we will deduce the promised Hodge-to-de Rham degeneration in characteristic 0:

Theorem 4.2.2.2. *Let $\mathcal{X}$ be a smooth Hodge-properly spreadable Artin stack over a field F of characteristic zero. Then the Hodge-to-de Rham spectral sequence for $\mathcal{X}$ degenerates at the first page. In particular for each $n \geq 0$ there exists a (non-canonical) isomorphism*

$$H^n_{\mathrm{dR}}(\mathcal{X}) \simeq \bigoplus_{p+q=n} H^{p,q}(\mathcal{X}).$$

Proof. For the rest of the proof fix $n \in \mathbb{Z}_{\geq 0}$. By Hodge-properness of $\mathcal{X}$ it is enough to prove

$$\dim_F H^n_{\mathrm{dR}}(\mathcal{X}) = \dim_F H^n_{\mathrm{H}}(\mathcal{X}).$$

Let R and $\mathcal{X}_R$ be as in Definition 4.2.2.1. Note that by the assumption on $\mathcal{X}_R$ and Proposition 3.4.2.8 both $H^n_{\mathrm{dR}}(\mathcal{X}_R)$ and $H^n_{\mathrm{H}}(\mathcal{X}_R)$ are finitely generated R-modules. Localizing R if necessary, we can assume that R is regular connected of Krull dimension d, and that the i-th cohomology

groups $H^i_{\mathrm{dR}}(\mathcal{X}_R)$ and $H^i_{\mathrm{H}}(\mathcal{X}_R)$ are free R-modules of finite rank for $i = n, n+1, \ldots, n+d$.[6] Note that for any point $s\colon \operatorname{Spec} k \to \operatorname{Spec} R$ the map $R \to k$ can be factored as a composition of a flat map $R \to R_s$ (where R_s is a local ring of s) and a map $R_s \to k$ of finite Tor-amplitude (by regularity assumption k is perfect as an R_s-module). Hence by Proposition 3.4.1.7 we have

$$R\Gamma_{\mathrm{dR}}(\mathcal{X}_k/k) \simeq R\Gamma_{\mathrm{dR}}(\mathcal{X}_R/R) \otimes_R k \qquad \text{and} \qquad R\Gamma_{\mathrm{H}}(\mathcal{X}_k/k) \simeq R\Gamma_{\mathrm{H}}(\mathcal{X}_R/R) \otimes_R k,$$

where $\mathcal{X}_k \coloneqq \mathcal{X}_R \otimes_R k$. Since the $n, (n+1), \ldots, (n+d)$-th cohomology groups are free as R-modules and since the Tor-amplitude of k over R is bounded by d, we get $H^n_{\mathrm{dR}}(\mathcal{X}_k/k) \simeq H^n_{\mathrm{dR}}(\mathcal{X}_R/R) \otimes_R k$, so

$$\dim_F H^n_{\mathrm{dR}}(\mathcal{X}) = \operatorname{rank}_R H^n_{\mathrm{dR}}(\mathcal{X}_R) = \dim_k H^n_{\mathrm{dR}}(\mathcal{X}_k)$$

and analogously for the Hodge cohomology. In particular, to prove that $\dim_F H^n_{\mathrm{dR}}(\mathcal{X}/F) = \dim_F H^n_{\mathrm{H}}(\mathcal{X}/F)$ it is enough to show that $\dim_k H^n_{\mathrm{dR}}(\mathcal{X}_k/k) = \dim_k H^n_{\mathrm{H}}(\mathcal{X}_k/k)$ for some point $s\colon \operatorname{Spec} k \to \operatorname{Spec} R$.

To do so, note that by Lemma 4.2.2.3 below, there exists a closed point $s \simeq \operatorname{Spec} k \in \operatorname{Spec} R$ of characteristic greater then n, such that the map $R \to k \to k^{\mathrm{perf}}$ factors through the second Witt vectors $W_2(k^{\mathrm{perf}})$. Since the base change $\mathcal{X}_{W_2(k^{\mathrm{perf}})} \coloneqq \mathcal{X}_R \times_R W_2(k^{\mathrm{perf}})$ is smooth and Hodge-proper over $W_2(k^{\mathrm{perf}})$, by Theorem 4.2.1.18 we have

$$\dim_{k^{\mathrm{perf}}} H^n_{\mathrm{dR}}(\mathcal{X}_{k^{\mathrm{perf}}}/k^{\mathrm{perf}}) = \dim_{k^{\mathrm{perf}}} H^n_{\mathrm{H}}(\mathcal{X}_{k^{\mathrm{perf}}}/k^{\mathrm{perf}}).$$

Finally, by base change (applied to $k \to k(k^{\mathrm{perf}})$ we get

$$\dim_k H^n_{\mathrm{dR}}(\mathcal{X}_k/k) = \dim_k H^n_{\mathrm{H}}(\mathcal{X}_k/k)$$

as desired. □

Lemma 4.2.2.3. *Let R be a regular localization of a finitely generated $\mathbb{Z}$-algebra such that the image of $\operatorname{Spec} R$ in $\operatorname{Spec} \mathbb{Z}$ is open. Then there exists a closed point $s \simeq \operatorname{Spec} k \in \operatorname{Spec} R$ of arbitrary large residue characteristic, such that the composite map $R \to k \hookrightarrow k^{\mathrm{perf}}$ factors through the ring $W_2(k^{\mathrm{perf}})$ of the second Witt vectors.*

[6] Note that there does not necessarily exist a localization $R[s^{-1}]$ such that *for all* i the $R[s^{-1}]$-modules $H^i_{\mathrm{dR}}(\mathcal{X}_R)[s^{-1}]$ (or $H^i_{\mathrm{H}}(\mathcal{X}_R)[s^{-1}]$) are free, since there are infinitely many of them.

Proof. By assumption R is regular over $\mathbb{Z}[1/N]$ for some N. Hence by Popescu's theorem ([7, Tag 07GB]) R is a filtered co-limit of smooth $\mathbb{Z}[1/N]$-algebras. In particular, the cotangent complex $\mathbb{L}_{R[1/N]/\mathbb{Z}[1/N]}$ is concentrated in degree zero and is flat over $R[1/N]$. Let us take some closed point $s \simeq \operatorname{Spec} k \in \operatorname{Spec} R[1/N]$ of large enough residue characteristic. By the basic deformation theory the obstruction to lift a map $R[1/N] \to k^{\mathrm{perf}}$ to $R[1/N] \to W_2(k^{\mathrm{perf}}) \to k^{\mathrm{perf}}$ lies in $\operatorname{Ext}^1_{k^{\mathrm{perf}}}(\mathbb{L}_{R[1/N]/\mathbb{Z}[1/N]} \otimes_{R[1/N]} k^{\mathrm{perf}}, k^{\mathrm{perf}})$. But the latter group vanishes, since by flatness of $\mathbb{L}_{R[1/N]/\mathbb{Z}[1/N]}$, the restriction $\mathbb{L}_{R[1/N]/\mathbb{Z}[1/N]} \otimes_{R[1/N]} k^{\mathrm{perf}}$ is a complex of k^{perf}-vector spaces concentrated in degree 0. □

4.2.3 Equivariant Hodge decomposition

In this section we apply Theorem 4.2.2.2 to obtain a (non-canonical) Hodge decomposition for the equivariant singular cohomology of an algebraic variety X with a G action, under the assumption that the corresponding quotient stack $[X/G]$ is Hodge-properly spreadable.

Let X be a homotopy type with an action of a topological group H (i.e. an $(\infty,1)$-functor $X_\bullet \colon BH \to \mathscr{S}$). Recall that the H-equivariant cohomology $C^*_H(X,\Lambda)$ of X with coefficients in a ring Λ are defined as

$$C^*_H(X,\Lambda) \coloneqq C^*(X_{hH},\Lambda),$$

where X_{hH} is the homotopy quotient of X by H (i.e. a colimit of the corresponding functor $X_\bullet$, or, more classically, $(X \times EH)/H$). If X is a smooth algebraic variety over a field $F \subseteq \mathbb{C}$ equipped with an action of a smooth algebraic group G, then the de Rham cohomology of $[X/G]$ gives a model for the $G(\mathbb{C})$-equivariant singular cohomology of $X(\mathbb{C})$:

Proposition 4.2.3.1. *Let X and G be as above. Then there is a canonical equivalence*

$$C^*_{G(\mathbb{C})}(X(\mathbb{C}),\mathbb{C}) \simeq R\Gamma_{\mathrm{dR}}([X/G]/F) \otimes_F \mathbb{C}.$$

Proof. By definition we have

$$\left| \dots \substack{\rightarrow\\\rightarrow\\\rightarrow} G \times G \times X \substack{\rightarrow\\\rightarrow} G \times X \rightarrow X \right| \simeq [X/G],$$
$$\left| \dots \substack{\rightarrow\\\rightarrow\\\rightarrow} G(\mathbb{C}) \times G(\mathbb{C}) \times X(\mathbb{C}) \substack{\rightarrow\\\rightarrow} G(\mathbb{C}) \times X(\mathbb{C}) \rightarrow X(\mathbb{C}) \right| \simeq X(\mathbb{C})_{hG(\mathbb{C})}.$$

Since the functor of cochains $C^*(-,\mathbb{C})$ sends colimits of homotopy types to limits of complexes

and by smooth descent for $R\Gamma_{\mathrm{dR}}(-/F)\otimes_F \mathbb{C}$, the result follows from the analogous comparison between algebraic de Rham and Betti cohomology for ordinary smooth schemes $X\times G^n$. □

Corollary 4.2.3.2 (Equivariant Hodge decomposition)**.** *Let X be a scheme over $\mathbb{C}$ with an action of a an algebraic group G. Assume that $[X/G]$ is Hodge-properly spreadable (e.g. X and G satisfy the conditions of Theorem 4.4.1.3 or 4.4.2.10). Then for all $n\in\mathbb{Z}_{\geq 0}$ there exists an isomorphism*

$$H^n_{G(\mathbb{C})}(X(\mathbb{C}),\mathbb{C})\simeq \bigoplus_{p+q=n} H^q([X/G],\wedge^p\mathbb{L}_{[X/G]}).$$

Example 4.2.3.3. Let $X=\operatorname{Spec}\mathbb{C}$. Then $\wedge^n\mathbb{L}_{BG}\simeq \operatorname{Sym}^n(\mathfrak{g}^\vee)[-n]$ and we get a standard isomorphism

$$H^n_{G(\mathbb{C})}(\mathrm{pt},\mathbb{C})\simeq\begin{cases}\operatorname{Sym}^k(\mathfrak{g}^\vee)^G & \text{if } n=2k,\\ 0 & \text{if } n=2k+1.\end{cases}$$

In particular,

$$H^\bullet_{G(\mathbb{C})}(\mathrm{pt},\mathbb{C})\simeq \operatorname{Sym}(\mathfrak{g}^\vee)^G,\quad \text{where}\quad \deg(\mathfrak{g}^\vee)=2.$$

Example 4.2.3.4. As another example one can take a conical resolution $\pi\colon X\to\operatorname{Spec}A$ (see the second example of 4.4.2.14). Following Example 4.4.2.14, the quotient stack $[X/\mathbb{G}_m]$ is Hodge-properly spreadable and we get a decomposition for $H^\bullet_{\mathbb{C}^\times}(X(\mathbb{C}),\mathbb{C})$ as in Corollary 4.2.3.2. In particular, the first equivariant cohomology is isomorphic to the non-equivariant first cohomology (since $\tau_{\leq 1}B\mathbb{C}^\times\simeq\mathrm{pt}$) and we get a decomposition

$$H^1(X(\mathbb{C}),\mathbb{C})\simeq H^0([X/\mathbb{G}_m],\mathbb{L}_{[X/\mathbb{G}_m]})\oplus H^1([X/\mathbb{G}_m],\mathcal{O}_{[X/\mathbb{G}_m]}).$$

We have $\mathbb{L}_{[X/\mathbb{G}_m]}\simeq\Omega^1_X\xrightarrow{a^*}\mathcal{O}_X$ as a complex of $\mathbb{G}_m$-equivariant sheaves on X, where a^* is the map dual to the derivative of the action $\mathrm{Lie}(\mathbb{G}_m)\otimes_{\mathbb{C}}\mathcal{O}_X\to\mathbb{T}_X$ (where $\mathbb{T}_X$ denotes the tangent bundle). Then $\mathbb{H}^0(X,\Omega^1_X\xrightarrow{a^*}\mathcal{O}_X)\simeq\ker\left(H^0(X,\Omega^1_X)\xrightarrow{a^*}H^0(X,\mathcal{O}_X)\right)$, which is identified with the invariants of the Lie algebra action, which also identifies with the group invariants $H^0(X,\Omega^1_X)^{\mathbb{G}_m}$. Finally we get

$$H^0([X/\mathbb{G}]m,\mathbb{L}_{[X/\mathbb{G}_m]})\simeq H^0(X,\Omega^1_X\xrightarrow{a^*}\mathcal{O}_X)^{\mathbb{G}_m}\simeq H^0(X,\Omega^1_X)^{\mathbb{G}_m}$$

as well. The second summand is just $H^1(X,\mathcal{O}_X)^{\mathbb{G}_m}$. Thus for any conical resolution we get a

formula

$$H^1(X(\mathbb{C}),\mathbb{C}) \simeq H^0(X,\Omega^1_X)^{\mathbb{G}_m} \oplus H^1(X,\mathcal{O}_X)^{\mathbb{G}_m}.$$

This is a partial generalization of results of Section 6 in [77] to the case when $R^1\pi_*\mathcal{O}_X$ is not necessarily 0.

4.3 Spreadings

To apply Theorem 4.2.2.2 we need to find a good model of our stack over a finitely generated $\mathbb{Z}$-algebra, namely a Hodge-proper spreading. However, as we will see, such a spreading does not necessarily exist in general.

In Section 4.3.1 we first prove a general result about the existence of spreadings for some more natural classes of morphisms between Artin stacks (like smooth, flat, etc). Then some examples of Hodge-properly spreadable and nonspreadable stacks are given in Section 4.3.3.

4.3.1 Spreadable classes

Definition 4.3.1.1. Let $\mathcal{P}$ be a class of morphisms of schemes (e.g. $\mathcal{P}$ = smooth, flat or proper morphisms) containing all isomorphisms and closed under compositions. For a scheme S, define $\mathcal{S}\mathrm{ch}^{\mathrm{fp},\mathcal{P}}_{/S}$ to be the (non-full) subcategory of schemes over S consisting of finitely-presentable S-schemes and morphisms between them that belong to $\mathcal{P}$.

Theorem 4.3.1.2 ([38, IV(3), Theorems 8.10.5, 11.2.6] and [38, IV(4), Theorem 17.7.8])**.** *Let $\{S_i\}$ be a filtered diagram of affine schemes with limit S and let $\mathcal{P}$ be one of the following classes of morphisms: isomorphisms, surjections, closed embeddings, flat, smooth or proper morphisms*[7]*. Then the natural functor*

$$\varinjlim_i \mathcal{S}\mathrm{ch}^{\mathrm{fp},\mathcal{P}}_{/S_i} \to \mathcal{S}\mathrm{ch}^{\mathrm{fp},\mathcal{P}}_{/S}$$

(induced by the base change $\mathcal{S}\mathrm{ch}^{\mathrm{fp},\mathcal{P}}_{/S_i} \ni X \mapsto X \times_{S_i} S$) is an equivalence.

We will say that a scheme X is a *$\mathcal{P}$-scheme over S* ($\mathcal{P}$-scheme/S) if X is an S-scheme and the structure morphism $X \to S$ is in $\mathcal{P}$.

[7] The list is not even nearly complete. See [95, Appendix C.1] for a much more exhaustive list of classes of morphisms and their properties with precise references.

Corollary 4.3.1.3. *Let $\{S_i\}_{i\in I}, S$ and $\mathcal{P}$ be as above. Then if X is a finitely presentable $\mathcal{P}$-scheme/S, then there exists $i \in I$ and a finitely presentable $\mathcal{P}$-scheme X_i over S_i, such that $X \simeq X_i \times_{S_i} S$.*

Proof. Let $\pi\colon X \to S$ be the structure morphism. By the previous theorem and the description of objects in filtered colimits of categories (see e.g. [103]) there exists a finitely presented scheme $\pi_j\colon X_j \to S_j$ such that $\pi_j \times_{S_j} S = \pi$. A morphism in a filtered colimit of categories is a filtered co-limit of morphisms, hence

$$\mathrm{Hom}_{\mathcal{S}\mathrm{ch}^{\mathrm{fp},\mathcal{P}}_{/S}}(X, S) \simeq \varinjlim_k \mathrm{Hom}_{\mathcal{S}\mathrm{ch}^{\mathrm{fp},\mathcal{P}}_{/S_i}}(X_i \times_{S_j} S_i, S_i).$$

Since the left hand side is non-empty by assumption, the right hand side also must be nonempty for some i, i.e. there exists $i \in I$ such that $\pi_i\colon X_i \to S_i$ is in $\mathcal{P}$. □

Our goal in this section is to extend Theorem 4.3.1.2 to the setting of Artin stacks. First we recall how "finitely presentable" is defined in Artin setting:

Definition 4.3.1.4 (Finitely presentable Artin stacks)**.** A (-1)-Artin stack $\mathcal{X}$ over R is called *finitely presentable*, if $\mathcal{X} \simeq \operatorname{Spec} A$ and A is a finitely presentable R-algebra. Then, an n-Artin stack $\mathcal{X}$ over R is called finitely presentable if there exists a smooth atlas $U \twoheadrightarrow \mathcal{X}$ such that U is a finitely presentable affine scheme and $U \times_{\mathcal{X}} U$ is a finitely presentable $(n-1)$-Artin R-stack. We will denote the category of finitely presentable n-Artin stacks by $\mathscr{S}\mathrm{tk}^{n\text{-Art,fp}}$.

Our general strategy for proving results about spreadability is to inductively reduce to the case of finitely presentable schemes. For this we will need a description of an n-Artin stack as a quotient of a hypercover by that depends only on a finite amount of maps between them. This is provided by the work of Pridham [100].

Construction 4.3.1.5. Let $X_\bullet\colon \Delta^{\mathrm{op}} \to \mathscr{C}$ be a simplicial object in a category $\mathscr{C}$ admitting finite limits. Define $X(-)\colon \mathrm{SSet}^{\mathrm{fin,op}} \to \mathscr{C}$ to be the right Kan extension of $X_\bullet$ along the inclusion of Δ^{op} into the opposite $\mathrm{SSet}^{\mathrm{fin,op}}$ of the category of finite simplicial sets (meaning simplicial sets with only finitely many non-degenerate simplices). More concretely, for a finite simplicial set K

$$X(K) \simeq \varprojlim_{\Delta^n \in \Delta_{/S}} X(\Delta^n).$$

In particular, we denote $M_n(X_\bullet) \coloneqq X(\partial\Delta^n)$ and call it the *n-th matching object of $X_\bullet$*.

Definition 4.3.1.6. Let $\mathbb{H}$ be an ∞-topos. An augmented simplicial object $X_\bullet \to X_{-1}$ is called a *hypercover of* X_{-1} if for any $n \in \mathbb{Z}_{\geq 0}$ the natural map $X_n \to M_n(X_\bullet)$ is an effective epimorphism (M_n is computed in the category $\mathbb{H}_{/X_{-1}}$). A hypercover $X_\bullet$ is called *n-coskeletal* if additionally for each $m > n$ the natural map $X_m \to M_m(X_\bullet)$ is an equivalence (equivalently $X_\bullet$ coincides with the right Kan extension of its restriction to $\Delta_{\leq n}$).

With this notation, n-Artin stacks can be thought of as some special $(n-1)$-coskeletal hepercovers:

Theorem 4.3.1.7 ([100, Theorem 4.15])**.** *Let $\mathcal{X}$ be a finitely presented n-Artin stack over S. Then there exists an $(n-1)$-coskeletal hypercover $X_\bullet$ of $\mathcal{X}$ such that all X_k are finitely presentable affine schemes and for all m, k, with $0 \leq m \leq k$, the maps $X_k \to X(\Lambda^k_m)$ are smooth surjections. Conversely, given $X_\bullet$ as above, its geometric realization $|X_\bullet|$ is a finitely presentable n-Artin stack.*

For convenience we introduce the following notation:

Definition 4.3.1.8 (Spreadable class)**.** A class of morphism $\mathcal{P}$ between Artin stacks is called *spreadable* if

- $\mathcal{P}$ is closed under arbitrary base changes, compositions and contains all equivalences.
- (Locality on source and target) Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a morphism of finitely presentable Artin stacks. Then f lies in $\mathcal{P}$ if and only if there exist smooth finitely presentable affine atlases $U \twoheadrightarrow \mathcal{Y}$ and $V \twoheadrightarrow U \times_{\mathcal{Y}} \mathcal{X}$ such that the composite map $V \to U \times_{\mathcal{Y}} \mathcal{X} \to U$ is in $\mathcal{P}$.
- (Affine spreadability) Let $\{S_i\}$ be a filtered diagram of affine schemes with the limit S. Let $f\colon X \to Y$ be a morphism in $\mathcal{P}$ between affine finitely presentable S-schemes. Then for some i there exits a map $f_i\colon X_i \to Y_i$ in $\mathcal{P}$ of affine finitely presentable S_i-schemes, such that $f \simeq f_i \times_{S_i} S$.

Example 4.3.1.9. If $\mathcal{P}$ and $\mathcal{Q}$ is a pair of spreadable classes, then $\mathcal{P} \cap \mathcal{Q}$ and $\mathcal{P} \cup \mathcal{Q}$ are also spreadable. There exists the smallest spreadable class (consisting only of equivalences) and the largest one (consisting of all finitely presentable morphisms).

Example 4.3.1.10. Since surjective, smooth and flat morphisms of Artin stacks are by definition local on the source and the target for the flat topology, by Theorem 4.3.1.2 we get that these classes are spreadable.

Definition 4.3.1.11. Let $\mathcal{P}$ be a spreadable class and let S be a scheme. Let us denote by $\mathscr{S}\mathrm{tk}_{/S}^{n\text{-Art},\mathrm{fp},\mathcal{P}}$ the subcategory of the category of finitely presentable n-Artin stacks over S and morphisms from $\mathcal{P}$ between them.

We are now ready to prove the main technical result of this section (which also explains our choice of notation for Definition 4.3.1.8):

Theorem 4.3.1.12. *Let $\{S_i\}$ be a filtered diagram of affine schemes with limit S. Let $\mathcal{P}$ be a spreadable class. Then the natural functor*

$$\varinjlim_i \mathscr{S}\mathrm{tk}_{/S_i}^{n\text{-Art},\mathrm{fp},\mathcal{P}} \longrightarrow \mathscr{S}\mathrm{tk}_{/S}^{n\text{-Art},\mathrm{fp},\mathcal{P}}$$

(induced by the base-change $\mathscr{S}\mathrm{tk}_{/S_i}^{n\text{-Art},\mathrm{fp},\mathcal{P}} \ni \mathcal{X}_i \mapsto \mathcal{X}_i \times_{S_i} S$) is an equivalence.

Proof. We will prove the statement by induction on n. The base of the induction $n = -1$, i.e. the case of affine schemes, holds by the definition of a spreadable class. To make the induction step, we first prove the statement for $\mathcal{P} =$ "all (finitely presented) morphisms" (using the induction assumption for smooth surjective morphisms) and then deduce the statement for a general spreadable class $\mathcal{P}$.

Essential surjectivity for $\mathcal{P} =$ "all". Since all n-Artin stacks are $(n+1)$-truncated, the Yoneda embedding $\mathscr{S}\mathrm{tk}^{n\text{-Art}} \hookrightarrow \mathrm{Fun}(\mathrm{CAlg}, \mathscr{S})$ factors through a full subcategory $\mathrm{Fun}(\mathrm{CAlg}, \mathscr{S}_{\leq n+1}) =: \mathscr{P}\mathscr{S}\mathrm{tk}_{\leq n+1}$. Let now $\mathcal{X}$ be a finitely presented n-Artin S-stack and let $X_\bullet$ be a simplicial diagram of finitely presented affine S-schemes, so that $|X_\bullet| \simeq \mathcal{X}$ (as in Theorem 4.3.1.7). Since for any simplicial diagram $A_\bullet$ in any $(n+1,1)$-category the natural map $|A_\bullet|_{\leq n+2} \to |A_\bullet|$ is an equivalence, we see that $\mathcal{X} \simeq |X_\bullet|_{\leq n+2}$ in $\mathscr{P}\mathscr{S}\mathrm{tk}_{\leq n+1}$. But $X_{\bullet|\Delta^{\mathrm{op}}_{\leq n+2}}$ is a finite diagram of finitely presented affine schemes, hence there exists S_i and a diagram $X_{\bullet|\Delta^{\mathrm{op}}_{\leq n+2},S_i}$ such that $X_{\bullet|\Delta^{\mathrm{op}}_{\leq n+2}} \simeq X_{\bullet|\Delta^{\mathrm{op}}_{\leq n+2},S_i} \times_{S_i} S$. We set $\mathcal{X}_{S_i} := |X_{\leq n+2,S_i}|$. By applying the inductive assumption with $\mathcal{P} =$ "smooth surjective", we can assume that all maps $X_{k,S_j} \to X_{\bullet,S_j}(\Lambda^k_m)$ are smooth and surjective for some S_j; hence by Theorem 4.3.1.7 $\mathcal{X}_{S_j}$ is a finitely presented n-Artin spreading of $\mathcal{X}$.

Fully-faithfulness for $\mathcal{P} =$ "all". Let $\mathcal{X}_i, \mathcal{Y}_i$ be a pair of n-Artin stacks of finite presentation over

S_i. We then have

$$\varinjlim_j \mathrm{Hom}_{\mathscr{P}\mathscr{S}\mathrm{tk}_{/S_j}}(\mathcal{X}_i \times_{S_i} S_j, \mathcal{Y}_i \times_{S_i} S_j) \simeq \varinjlim_j \mathrm{Hom}_{\mathscr{P}\mathscr{S}\mathrm{tk}_{/S_i}}(\mathcal{X}_i \times_{S_i} S_j, \mathcal{Y}_i) \simeq \varinjlim_j \mathrm{Hom}_{\mathscr{P}\mathscr{S}\mathrm{tk}_{\leq n+1/S_i}}(\mathcal{X}_i \times_{S_i} S_j, \mathcal{Y}_i), \tag{4.3.1.1}$$

where the second equivalence follows from the fact that filtered co-limits commute with π_*, hence preserve $(n+1)$-truncated spaces. Let now $X_\bullet \to \mathcal{X}$ be as in Theorem 4.3.1.7. Then

$$(4.3.1.1) \ldots \simeq \varinjlim_j \mathrm{Hom}_{\mathscr{P}\mathscr{S}\mathrm{tk}_{\leq n+1/S_i}}(|X_\bullet \times_{S_i} S_j|_{\leq n+2}, \mathcal{Y}_i) \simeq \mathrm{Tot}_{\leq n+2} \varinjlim_j \mathrm{Hom}_{\mathscr{P}\mathscr{S}\mathrm{tk}_{\leq n+1/S_i}}(X_\bullet \times_{S_i} S_j, \mathcal{Y}_i) \simeq$$

$$\simeq \mathrm{Tot}_{\leq n+2} \varinjlim_j \mathrm{Hom}_{\mathscr{S}\mathrm{tk}_{/S_i}}(X_\bullet \times_{S_i} S_j, \mathcal{Y}_i),$$

where the second equivalence follows from the fact that, since $\Delta_{\leq n+2}$ is a finite diagram, limits along $\Delta_{\leq n+2}$ commute with filtered co-limits. Similarly, one shows that

$$\mathrm{Hom}_{\mathscr{S}\mathrm{tk}_{/S}}(\mathcal{X}_i \times_{S_i} S, \mathcal{Y}_i \times_{S_i} S) \simeq \mathrm{Tot}_{\leq n+2} \mathrm{Hom}_{\mathscr{S}\mathrm{tk}_{/S_i}}(X_\bullet \times_{S_i} S, \mathcal{Y}_i).$$

Finally, since $\mathcal{Y}_i$ is finitely presentable, by [47, Chapter 2, Proposition 4.5.2]

$$\varinjlim_j \mathrm{Hom}_{\mathscr{S}\mathrm{tk}_{/S_i}}(X_\bullet \times_{S_i} S_j, \mathcal{Y}_i) \simeq \mathrm{Hom}_{\mathscr{S}\mathrm{tk}_{/S_i}}(X_\bullet \times_{S_i} S, \mathcal{Y}_i).$$

<u>*General* $\mathcal{P}$</u>. Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a morphism in a spreadable class $\mathcal{P}$ over S. It is enough to prove that there exists i and a map between finitely presentable n-Artin S_i-stacks $f_i\colon \mathcal{X}_i \to \mathcal{Y}_i$ such that $f_i \times_{S_i} S \simeq f$ and $f_i \in \mathcal{P}$. Choose affine finitely presentable atlases $U \twoheadrightarrow \mathcal{Y}$ and $V \twoheadrightarrow U \times_{\mathcal{Y}} \mathcal{X}$. The induced map $g\colon V \to U$ belongs to $\mathcal{P}$, so by the previous part and definition of spredable classes, the diagram

$$\begin{array}{ccc} V & \twoheadrightarrow & \mathcal{X} \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ U & \twoheadrightarrow & \mathcal{Y} \end{array}$$

can be spread out to some S_i, such that g_{S_i} belongs to $\mathcal{P}$. It follows by the definition of spreadable class, that f_{S_i} is also in $\mathcal{P}$. □

A stack $\mathcal{X}$ is called an *n-Artin $\mathcal{P}$-stack over S* if the structure morphism $\pi\colon \mathcal{X} \to S$ exhibits

$\mathcal{X}$ as an n-Artin stack and π is in $\mathcal{P}$.

Corollary 4.3.1.13 (Existence of spreading in a predefined class)**.** *Let $\{S_i\}_{i\in I}$ be a filtered diagram of affine schemes, $S := \varprojlim S_i$ and $\mathcal{P}$ be a spreadable class. Then if $\mathcal{X}$ is a finitely presentable n-Artin $\mathcal{P}$-stack over S, then there exists $i \in I$ and a finitely presentable n-Artin $\mathcal{P}$-stack $\mathcal{X}_i$ over S_i, such that $\mathcal{X} \simeq \mathcal{X}_i \times_{S_i} S$.*

Proof. One can apply Theorem 4.3.1.12 to the structure morphism $\mathcal{X} \to S$ and argue as in the proof of Corollary 4.3.1.3. □

4.3.2 Cohomologically proper stacks

In most examples for which we are able to construct a Hodge-proper spreading, the spreading in fact satisfies a stronger property, namely it is *cohomologically proper*. This property enjoys many natural properties that Hodge-properness does not: e.g. it translates along proper maps and a cohomologically proper scheme is necessarily proper. To introduce it we first need to extend Definition 3.4.2.1 to all locally Noetherian Artin stacks:

Definition 4.3.2.1. An Artin stack is called *locally Noetherian* if it admits an atlas $\coprod_i U_i$, where all U_i are Noetherian affine schemes. For a locally Noetherian Artin stack $\mathcal{X}$ we will denote by $\mathrm{Coh}(\mathcal{X})$ (resp. $\mathrm{Coh}^+(X)$) the full subcategory of $\mathrm{QCoh}(\mathcal{X})$ consisting of sheaves $\mathcal{F}$ such that the restriction of $\mathcal{F}$ to some (equivalently to any) locally Noetherian atlas has bounded (resp. bounded below) coherent cohomology sheaves.

Definition 4.3.2.2. A morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of locally Noetherian Artin stacks is called *cohomologically proper* if the induced functor $f_*\colon \mathrm{QCoh}(\mathcal{X}) \to \mathrm{QCoh}(\mathcal{Y})$ preserves the full subcategory of nearly coherent sheaves. A locally Noetherian Artin stack $\mathcal{X}$ over a Noetherian ring R is called *cohomologically proper* if the structure morphism $\mathcal{X} \to \mathrm{Spec}\, R$ is cohomologically proper.

Remark 4.3.2.3. By the left exactness of f_* it is enough to prove that $f_*(\mathrm{Coh}(\mathcal{X})^\heartsuit) \subset \mathrm{Coh}^+(\mathcal{Y})$.

The following properties of cohomologically proper morphisms are formal:

Proposition 4.3.2.4. *In the notations above we have:*

1. *The class of cohomologically proper morphism is closed under composition and flat base change.*

2. *The property of being a cohomologically proper morphism is fpqc-local on the target: the map $f\colon \mathcal{X} \to \mathcal{Y}$ is cohomologically proper if and only if there exists an fpqc surjection $\mathcal{U} \twoheadrightarrow \mathcal{Y}$ such that the base-change $f \times_{\mathcal{Y}} \mathcal{U}$ is cohomologically proper.*

If R is regular, the cohomological properness is stronger then the Hodge-properness:

Proposition 4.3.2.5. *Let $\mathcal{X}$ be a smooth cohomologically proper Artin stack over a regular Noetherian ring R. Then $\mathcal{X}$ is Hodge-proper over R.*

Proof. Since $\mathcal{X}$ is smooth over a regular Noetherian ring R, the category of coherent sheaves on $\mathcal{X}$ coincides with the category of perfect complexes. So by assumption, it is enough to prove that $\wedge^i \mathbb{L}_{\mathcal{X}/R}$ is perfect for all $i \geq 0$. By smoothness, the cotangent complex $\mathbb{L}_{\mathcal{X}/R}$ is perfect and concentrated in non-positive cohomological degrees. It follows that $\mathbb{L}_{\mathcal{X}/R}$ admits a finite filtration with the associated graded pieces being negative shifts of vector bundles. Hence by induction it is enough to prove that if E is a quasi-coherent sheaf on $\mathcal{X}$ such that $\wedge^{\leq i} E$ is perfect, then $\wedge^i(E[-1])$ is also perfect. But by construction (see [26, Theorem 3.35]) the functor $\wedge^i$ is i-excisive, so $\wedge^i(E[-1])$ is a finite limit of sheaves of the form $\wedge^i(E^{\oplus n}), n \leq i$, hence is perfect. □

Moreover, all proper morphisms are cohomologically proper. To show this, let's first recall the notion of a proper morphisms between higher stacks (following [99, Section 4]):

Definition 4.3.2.6. A 0-representable morphism $\mathcal{X} \to \mathcal{Y}$ is called *proper* if for any affine scheme S mapping to $\mathcal{Y}$, the pullback $\mathcal{X} \times_{\mathcal{Y}} S$ is a proper S-scheme. Next, assuming that the notion of a proper $(n-1)$-representable morphism is already defined, an n-representable morphism $f\colon \mathcal{X} \to \mathcal{Y}$ is called *proper* if

- f is *separated*, i.e. the diagonal map $\mathcal{X} \to \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ (which is $(n-1)$-representable) is proper.
- For any affine scheme S mapping to $\mathcal{Y}$ the pullback $\mathcal{X}_S := \mathcal{X} \times_{\mathcal{Y}} S$ admits a surjective S-morphism $P \twoheadrightarrow \mathcal{X}_S$ such that P is a proper S-scheme.

Remark 4.3.2.7. Since the property of a morphism of schemes to be proper is flat local on the target, it is enough in the definition above to check the second condition only for some atlas of $\mathcal{Y}$.

Remark 4.3.2.8. A potentially more familiar definition of a (classical) proper algebraic stack $p : \mathcal{X} \to S$ is that p should be separated, finite type and universally closed. We note that such stacks over S are proper 1-Artin stacks in the definition above. Indeed, by [92, Theorem 1.1] in this case there exists a proper surjective map $U \to \mathcal{X}$ from a proper scheme U.

From the standard results about proper morphisms of schemes and representable morphisms of stacks one formally deduces:

Proposition 4.3.2.9. *With the notations above:*

1. *Proper morphism are closed under base change.*

2. *The property of being a proper morphism is flat local on the target.*

3. *Proper morphisms are also closed under compositions.*

The fact that proper morphisms are cohomologically proper was proved in [99, Theorem 5.13], but in slightly different context. Their proof essentially follows the argument of [78, Theorem 15.6] in the case of classical proper stacks. For the reader's convenience we sketch the argument here:

Proposition 4.3.2.10. *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a proper morphism between locally Noetherian Artin stacks. Then f is cohomologically proper.*

Sketch of the proof. The question is local on the target, hence we can assume that $\mathcal{Y} = Y$ is an affine Noetherian scheme. Moreover, by localizing further if necessary, we can assume that there exists a surjective map $\pi\colon P \twoheadrightarrow \mathcal{X}$ such that P is a proper scheme over Y. Let us also assume that $\mathcal{X}$ is n-Artin for some $n \geq 0$ and let us prove the statement by induction on n. The statement for the $n = 0$ is the fundamental result about the direct image of a coherent sheaf under a proper morphism of schemes [38, III(1), Chapter III, Theorem 3.2.1].

By Remark 4.3.2.3 it is enough to prove that $f_*(\mathrm{Coh}^{\heartsuit}(\mathcal{X})) \subset \mathrm{Coh}^+(Y)$. Let $\mathcal{F} \in \mathrm{Coh}^{\heartsuit}(\mathcal{X})$. Since $\mathcal{X}$ is proper over a Noetherian base, it is Noetherian itself. It follows that there exists a finite filtration (by power of nil-radical of $\mathcal{O}_{\mathcal{X}}$) of $\mathcal{F}$ with the associated graded pieces coming from $\mathcal{X}^{\mathrm{red}}$. Since $\mathrm{Coh}^+(Y)$ is closed under finite extensions, it follows that we can assume that both $\mathcal{X}$ and Y are reduced.

Let us denote $\mathrm{Tot}\, \pi_{\bullet,*}(\mathcal{H}^0(\pi_{\bullet}^*\mathcal{F}))$ by $\mathcal{F}'$, where $\pi_{\bullet}\colon P_{\bullet} \to \mathcal{X}$ is the Čech nerve of the map $P \to \mathcal{X}$. By the higher "generic flatness" [99, Theorem 8.3] there exists an open dense substack $\mathcal{U}$ of $\mathcal{X}$ such that the induced map $P_{\mathcal{U}} := P \times_{\mathcal{X}} \mathcal{U} \to \mathcal{U}$ is flat. In particular, $\pi_{\mathcal{U},n}^*(\mathcal{F}) \simeq \mathcal{H}^0(\pi_{\mathcal{U},n}^*(\mathcal{F}))$ for all $n \in \mathbb{Z}_{\geq 0}$. It follows by flat descent that the natural map $\mathcal{F} \to \mathcal{F}'$ becomes an equivalence when restricted to $\mathcal{U}$. By Noetherian induction we can assume that $f_*(\mathrm{fib}(\mathcal{F} \to \mathcal{F}'))$ lies in $\mathrm{Coh}^+(Y)$. So to prove that $\pi_*(\mathcal{F})$ lies in $\mathrm{Coh}^+(Y)$ it is enough to show that $\pi_*(\mathcal{F}') \in \mathrm{Coh}^+(Y)$. On the other hand, all elements P_n of the Čech nerve are $(n-1)$-Artin proper stacks over Y and all sheaves

$\mathcal{H}^0(p_n^*(\mathcal{F}))$ are coherent. Since the global section functors $f_{n,*}\colon \mathrm{QCoh}(P_n) \to \mathrm{QCoh}(Y)$ are right t-exact, it follows by induction and Proposition 3.4.2.3 that the totalization

$$f_*(\mathcal{F}') \simeq \mathrm{Tot}\, f_{\bullet,*}(\mathcal{H}^0(p_n^*\mathcal{F}))$$

lies in $\mathrm{Coh}^+(Y)$. □

Corollary 4.3.2.11. *Let $\mathcal{X}$ be a smooth proper Artin stack over a regular Noetherian ring R. Then $\mathcal{X}$ is Hodge-proper.*

Proof. Follows immediately from the previous proposition and Proposition 4.3.2.5. □

Finally, we record the following observation, which allows to construct new examples of cohomologically proper stacks in inductive way.

Proposition 4.3.2.12. *Let $\pi_\bullet\colon \mathcal{U}_\bullet \to \mathcal{X}$ be a flat hypercover such that all $\mathcal{U}_n$ are cohomologically proper. Then $\mathcal{X}$ is cohomologically proper.*

Proof. Let $\mathcal{F}$ be a coherent sheaf on $\mathcal{X}$. By shifting if necessary we can assume that $\mathcal{H}^{<0}(\mathcal{F}) \simeq 0$. By the flat descent

$$R\Gamma(\mathcal{X}, \mathcal{F}) \simeq \mathrm{Tot}\, R\Gamma(\mathcal{U}_\bullet, \pi_\bullet^*\mathcal{F}).$$

Since the global section functors $R\Gamma(\mathcal{U}_n, -)$ are right t-exact and by assumptions on $\mathcal{U}_n$ the diagram $R\Gamma(\mathcal{U}_\bullet, \pi_\bullet^*\mathcal{F})$ consists of coconective nearly coherent complexes. By Proposition 3.4.2.3 the complex $R\Gamma(\mathcal{X}, \mathcal{F})$ is also nearly coherent. □

4.3.3 Examples of Hodge-properly spreadable stacks

In this subsection we begin to study which Hodge-proper stacks in characteristic 0 admit a Hodge-proper spreading over some finitely generated $\mathbb{Z}$-algebra. We will make extensive use of Theorem 4.3.1.12 in the following situation: let F be an algebraically closed field of characteristic 0, then $\mathrm{Spec}\, F \simeq \varprojlim R$ where $R \subset F$ runs through subrings of F that are regular and finitely generated over $\mathbb{Z}$. In particular we have an equivalence

$$\varinjlim_{R \subset F} \mathscr{S}\mathrm{tk}_{/R}^{n\text{-Art,fp},\mathcal{P}} \xrightarrow{\sim} \mathscr{S}\mathrm{tk}_{/F}^{n\text{-Art,fp},\mathcal{P}}$$

for any spreadable class $\mathcal{P}$. Another option is to also allow localizations of finitely generated $\mathbb{Z}$-algebras with the property that the image of $\operatorname{Spec} R$ in $\operatorname{Spec} \mathbb{Z}$ is open (as in Definition 4.2.2.1). In Section 4.3.3 we will see that this actually makes a difference. In what follows F will always denote an algebraically closed field of characteristic 0 and R will be a finitely generated regular $\mathbb{Z}$-subalgebra (or, in Section 4.3.3, also a localization of one) of F. We also freely use the standard spreading out results for schemes (Theorem 4.3.1.2) and their easy consequences (like spreading out group schemes, group actions, group homomorphisms, closed subgroups, etc.) without any additional reference.

We start with the Hodge-proper spreadability for proper Artin stacks, which is deduced from the spreadability of proper morphisms (Theorem 4.3.1.12). This is done in Section 4.3.3. Then we discuss in great detail the question of Hodge-proper spreadability of BG in Section 4.3.3; the case of more general quotient stacks is postponed till Section 4.4. Finally, in Section 4.3.3 we try to grasp the scope of potential applications of Theorem 4.2.2.2 concentrating on the case of schemes: in fact a particular set of examples given by semiabelian surfaces.

As was mentioned, often, along with Hodge-proper spreadability, we are able to prove a somewhat stronger statement, saying that the stacks we consider admit a cohomologically proper spreading. For this it is convenient to introduce the following variant of Definition 4.3.2.2:

Definition 4.3.3.1. A morphism $f\colon \mathcal{X} \to \mathcal{Y}$ of Artin stacks over a field F of characteristic 0 is called *cohomologically properly spreadable* if there exists a finitely generated $\mathbb{Z}$-algebra $R \subset F$[8] and a morphism $f_R\colon \mathcal{X}_R \to \mathcal{Y}_R$ over $\operatorname{Spec} R$, such that

- $f_R \otimes_R F := f_R \times_R F \simeq f$.
- $f_R\colon \mathcal{X}_R \to Y_R$ is cohomologically proper (see Definition 4.3.2.2).

In the case $\mathcal{Y} = \operatorname{Spec} F$ we will call $\mathcal{X}$ *cohomologically properly spreadable*. By Proposition 4.3.2.5 any such $\mathcal{X}$ is also Hodge-properly spreadable.

Proper stacks

In this subsection we show that all proper stacks are cohomologically (and in particular Hodge-)properly spreadable. By Proposition 4.3.2.10 it is just enough to show that proper morphisms spread out.

[8] Or a suitable localization of one, as in Definition 4.2.2.1.

Following the convention of Section 4.3.1, for an affine scheme S we denote by $\mathscr{S}\mathrm{tk}_{/S}^{n\text{-Art,fp,pr}} \subset \mathscr{S}\mathrm{tk}_{/S}^{n\text{-Art,fp}}$ the subcategory consisting of finitely presented n-Artin S-stacks and with morphisms given by proper maps (see Definition 4.3.2.6).

The results of Section 4.3.1 allow to deduce the spreadability of proper morphisms from the analogous statement for classical schemes:

Proposition 4.3.3.2. *Let $\{S_i\}$ be a filtered diagram of affine schemes with a limit S. Then the natural functor*

$$\varinjlim_i \mathscr{S}\mathrm{tk}_{/S_i}^{n\text{-Art,fp,pr}} \longrightarrow \mathscr{S}\mathrm{tk}_{/S}^{n\text{-Art,fp,pr}}$$

is an equivalence.

Proof. By Theorem 4.3.1.12 it is enough to prove that for a proper morphism $f\colon \mathcal{X} \to \mathcal{Y}$ there exists a proper morphism $f_i\colon \mathcal{X}_i \to \mathcal{Y}_i$ such that $f_i \times_{S_i} S \simeq f$. Assume that f is n-representable. We will prove the statement by induction on n.

For $n = 0$ let $U \twoheadrightarrow \mathcal{Y}$ be an affine finitely presentable atlas. Then by assumption $\mathcal{X}_U := \mathcal{X} \times_{\mathcal{Y}} U$ is a scheme proper over U. By Theorem 4.3.1.12 we can spread the commutative square

$$\begin{array}{ccc} \mathcal{X}_U & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow{\scriptstyle f} \\ U & \twoheadrightarrow & \mathcal{Y} \end{array}$$

to some S_i. By spreadability of equivalences, smooth surjective morphisms of stacks and proper morphisms of schemes we can assume, taking base change to some S_j if necessary, that the natural map $\mathcal{X}_{U,j} \to U_j \times_{\mathcal{Y}_j} \mathcal{X}_j$ is an equivalence. We can also assume that $U_j \to \mathcal{Y}_j$ is a smooth atlas and that $\mathcal{X}_{U,j}$ is proper scheme over U_j. Since the property of a map of schemes to be proper is flat local on target, it follows that for any T mapping to $\mathcal{Y}_j$ the pullback $T \times_{\mathcal{Y}_j} \mathcal{X}_j$ is a proper T-scheme. So the map $f_j\colon \mathcal{X}_j \to \mathcal{Y}_j$ is proper.

Finally, assume that the statement for $(n-1)$-representable morphism is already proved. Let $U \twoheadrightarrow \mathcal{Y}$ be a smooth finitely presentable atlas and let $P \twoheadrightarrow \mathcal{X}_U$ be a surjection from a proper U-scheme P. Then by the induction assumption we can find a spreading $f_i\colon \mathcal{X}_i \to \mathcal{Y}_i$ such that f_i is separated. By taking base change to some S_j we can assume that P_j is proper over U_j and that the map $P_j \to \mathcal{X}_{U_j}$ is surjective. □

Corollary 4.3.3.3. *Let $\mathcal{X}$ be a smooth proper stack over a field F of characteristic 0. Then $\mathcal{X}$ is Hodge-proper and Hodge-properly spreadable.*

Proof. By Proposition 4.3.3.2 (and Corollary 4.3.1.13) applied to $\mathcal{X} \to \operatorname{Spec} F$ we get a smooth proper spreading $\mathcal{X}_R \to \operatorname{Spec} R$. Then $\mathcal{X}_R$ is Hodge-proper by Propositions 4.3.2.5 and 4.3.2.10. □

We will also use the following corollary:

Corollary 4.3.3.4. *Let $f\colon \mathcal{X} \to \mathcal{Y}$ be a proper map of finitely presentable Artin stacks over a field F of characteristic 0 such that $\mathcal{Y}$ is cohomologically properly spreadable (see Definition 4.3.3.1). Then $\mathcal{X}$ is also cohomologically properly spreadable.*

Proof. Let $f_R\colon \mathcal{X}_R \to \mathcal{Y}_R$ be some proper spreading of f. Since any two spreadings become equivalent after some finite localization of R, we can assume that $\mathcal{Y}_R$ is cohomologically proper over R. Then we conclude by Proposition 4.3.2.10 and the first part of Proposition 4.3.2.4. □

Remark 4.3.3.5. More generally, a composition of two cohomologically properly spreadable morphisms is again cohomologically properly spreadable.

Classifying stacks

Let's first understand for which groups G in characteristic 0 the classifying stack BG is Hodge-proper. The answer is easy: more or less for all G. In fact BG is even cohomologically proper:

Proposition 4.3.3.6. *Let G be a finite type group scheme over an algebraically closed field F of characteristic 0. Then BG is cohomologically (and, in particular, Hodge-)proper.*

Proof. In fact we will show a more precise statement, namely that for any coherent sheaf $\mathcal{F}$ on BG, $R\Gamma(BG, \mathcal{F})$ lies in $\operatorname{Coh}(F)$ if G is linear and in $\operatorname{Coh}^+(F)$ if G is general.

Note that, since we are in characteristic 0, G is smooth and thus so is the natural map $\operatorname{Spec} F \to BG$. In particular, the natural t-structure on $\operatorname{Coh}(BG)$ coincides with the usual t-structure on $\operatorname{Coh}(F)$ after taking pullback $\operatorname{Coh}(BG) \to \operatorname{Coh}(F)$ (aka the forgetful functor in terms of representations). It is enough to show the statement for $\mathcal{F} \in \operatorname{Coh}(BG)^\heartsuit$ (Remark 4.3.2.3). Note that such $\mathcal{F}$ is the same thing as a finite-dimensional algebraic representation of G over F.

By Chevalley's structure theorem there is an exact sequence $1 \to L \to G \to A \to 1$ where L is a linear algebraic group and A is proper. Then for L we have another short exact sequence

$$1 \to U \to L \to H \to 1,$$

where U is the unipotent radical of L and $H \simeq L/U$ is reductive.

Let $j\colon BU \to BL$, $f\colon BL \to BH$, $i\colon BL \to BG$ and $p\colon BG \to BA$ be the corresponding maps between classifying stacks. We will prove the statement step by step, starting from the unipotent case.

Case 1. $G = U$ is unipotent. We assume $\mathcal{F} \in \mathrm{Coh}(BU)^{\heartsuit}$. Since the characteristic of F is 0 and U is unipotent, $R\Gamma(BU, \mathcal{F})$ can be computed as the cohomology of the Lie algebra $\mathfrak{u}$. Explicitly, this is given by the Chevalley complex:

$$0 \to \mathcal{F} \to \mathcal{F} \otimes \mathfrak{u}^* \to \mathcal{F} \otimes \wedge^2 \mathfrak{u}^* \to \ldots \mathcal{F} \otimes \wedge^{\dim U} \mathfrak{u}^* \to 0.$$

Since $\mathcal{F}$ is finite dimensional this complex is clearly perfect.

Case 2. $G = H$ is reductive. This follows from the fact that the abelian category $\mathrm{Rep}(H)$ is semi-simple (since $\mathrm{char}(F) = 0$). Namely for $\mathcal{F} \in \mathrm{Coh}(BH)^{\heartsuit}$, the complex $R\Gamma(BH, \mathcal{F})$ is equal to the H-invariants $\mathcal{F}^H$ (in cohomological degree 0). Since $\mathcal{F}$ is finite-dimensional we get $R\Gamma(BH, \mathcal{F}) \in \mathrm{Coh}(F)$.

Case 3. $G = A$ is proper. Let $\mathcal{F} \in \mathrm{Coh}(BA)$. We can compute $R\Gamma(BA, \mathcal{F})$ using the smooth $q\colon \operatorname{Spec} F \to BA$. Let $p_n\colon A^n \to BA$ be the map from the n-th term of the associated Čech simplicial object. We get a cosimplicial object

$$[n] \mapsto R\Gamma(A^n, p_n^* \mathcal{F}),$$

in Mod_F, and

$$R\Gamma(BA, \mathcal{F}) \simeq \operatorname{Tot} R\Gamma(A^\bullet, p_\bullet^* \mathcal{F}).$$

However, each term $R\Gamma(A^n, p_n^* \mathcal{F})$ lies in $\mathrm{Coh}(F)$ (since A^n is proper) and has cohomology only in non-negative degrees. By Proposition 3.4.2.3 it follows that $R\Gamma(BA, \mathcal{F})$ lies in $\mathrm{Coh}^+(F)$.

Case 4. $G = L$ is linear. We assume $\mathcal{F} \in \mathrm{Coh}(BU)^{\heartsuit}$ and consider $f_* \mathcal{F} \in \mathrm{QCoh}(BH)$ (for $f\colon BL \to BH$). We claim that $f_* \mathcal{F} \in \mathrm{Coh}(BH)$. It is enough to check that after taking pull-back to the smooth cover $q\colon \operatorname{Spec} F \to BH$. We have a fibered square

$$\begin{array}{ccc} BU & \xrightarrow{j} & BL \\ \downarrow & & \downarrow f \\ \operatorname{Spec} F & \xrightarrow{q} & BH \end{array}$$

and by base change we have $q^*f_*\mathcal{F} \simeq R\Gamma(BU, j^*\mathcal{F})$. The map j is flat, so $j^*\mathcal{F}$ is coherent and thus $R\Gamma(BU, j^*\mathcal{F}) \in \mathrm{Coh}(F)$ by Case 1. It follows that $f_*\mathcal{F} \in \mathrm{Coh}(BH)$. But then $R\Gamma(BL, \mathcal{F}) \simeq R\Gamma(BH, f_*\mathcal{F})$ and we are done by Case 2. At this point we have the statement for G linear.

Case 5. G is general. The argument in Case 4 works here as well, replacing U with L and H with A. Namely $p_*\mathcal{F} \in \mathrm{Coh}(BA)$ and then by Case 3

$$R\Gamma(BL, \mathcal{F}) \simeq R\Gamma(BH, f_*\mathcal{F})$$

. □

Even though BG is Hodge-proper practically for any G, there are definitely some algebraic groups G for which BG is not Hodge-properly spreadable. Indeed, consider $G = \mathbb{G}_a$. If $B\mathbb{G}_a$ were spreadable, then by Corollary 4.2.3.2 we would get a decomposition

$$H^n_{\mathrm{dR}}(B\mathbb{G}_a) \simeq \bigoplus_{p+q=n} H^q(B\mathbb{G}_a, \wedge^p \mathbb{L}_{B\mathbb{G}_a}).$$

However, this is impossible. Indeed, the left hand side vanishes for $n > 0$ by the $\mathbb{A}^1$-homotopy invariance of the de Rham cohomology in characteristic 0. On the other hand $\wedge^p \mathbb{L}_{B\mathbb{G}_a} \simeq \mathcal{O}_{B\mathbb{G}_a}[-p]$ and $H^i(B\mathbb{G}_a, \mathcal{O}_{B\mathbb{G}_a})$ is non-zero for $i = 0, 1$. In particular, the right hand side is non-zero for all n, contradiction.

Note that by Theorem 4.2.2.2 it follows that the Hodge cohomology of any spreading of $B\mathbb{G}_a$ *has to be* infinitely generated, which is also confirmed by a direct computation:

Example 4.3.3.7. Let $\mathcal{X} = B\mathbb{G}_a$ and let $\mathcal{X}_R$ be a Hodge-proper spreading of $\mathcal{X}$. Then, since $(B\mathbb{G}_a)_R$ is a spreading of $B\mathbb{G}_a$, by Theorem 4.3.1.12, after enlarging R, we can actually assume $\mathcal{X}_R \simeq (B\mathbb{G}_a)_R$. The cohomology of $\mathcal{O}_{B\mathbb{G}_a}$ over $\mathbb{Z}$ is given by[9]

$$H^\bullet(B\mathbb{G}_a, \mathcal{O}_{B\mathbb{G}_a}) \simeq \mathbb{Z}[v_1]/v_1^2 \otimes_{\mathbb{Z}} \mathrm{Sym}^\bullet\left(\bigoplus_p \mathbb{F}_p\langle v_p, v_{p^2}, v_{p^3}, \ldots\rangle\right),$$

where the sum is taken over all primes p, and $\mathbb{F}_p\langle v_p, v_{p^2}, v_{p^3}, \ldots\rangle$ denotes the free vector space spanned by $v_p, v_{p^2}, v_{p^3}, \ldots$, element v_1 has degree 1 and all other v_i have degree 2. Since $R \subset F$ is torsion-free, by base change we also get that

[9]The indexing of the generators is not arbitrary and corresponds to their $\mathbb{G}_m$-weights, via the natural action of $\mathbb{G}_m$ on $\mathbb{G}_a$ by rescaling, we will discuss that in more detail in Example 4.3.3.9.

$$H^\bullet(\mathcal{X}_R, \mathcal{O}_{\mathcal{X}_R}) \simeq R[v_1]/v_1^2 \otimes_R \mathrm{Sym}^\bullet\left(\bigoplus_p R/p\langle v_p, v_{p^2}, v_{p^3}, \ldots\rangle\right).$$

Since R is a finitely generated $\mathbb{Z}$-algebra, for p big enough we have $R/p \neq 0$ and thus $H^i(\mathcal{X}_R, \mathcal{O}_{\mathcal{X}_R})$ is not finitely generated over R for $i \geq 2$. A similar thing happens for any unipotent group U, namely p-torsion in cohomology of BU is gigantic for almost all p.

So it is natural to ask when BG is Hodge-properly spreadable. We provide a list of examples:

- G is proper (=an extesion of a finite group by an abelian variety). Then BG is a proper stack and this is covered by Corollary 4.3.3.3;
- G is reductive. This is a particular case of Theorem 4.4.1.3;
- $G{=}P \subset H$ is some parabolic subgroup of some reductive group H. This is a particular case of Theorem 4.4.2.10. Alternatively, it follows from the previous point and Corollary 4.3.3.4 (using that $BP \to BH$ is proper).

Remark 4.3.3.8. By an argument similar to Proposition 4.3.3.6 it is also possible to show the spreadability of BG for an extension of an abelian variety by a parabolic subgroup of some reductive group.

The fact that BP is spreadable can look a little surprising and we would like to illustrate what happens by the simplest non-trivial example, a Borel subgroup $B \subset \mathrm{SL}_2$:

Example 4.3.3.9. Let $G = B \subset \mathrm{SL}_2$ be the standard Borel subgroup of SL_2, namely

$$B = \left\{ \begin{pmatrix} t & s \\ 0 & t^{-1} \end{pmatrix} \right\} \subset \mathrm{SL}_2\,.$$

Then $B \simeq \mathbb{G}_a \rtimes \mathbb{G}_m$ with $\mathbb{G}_m = \operatorname{Spec} \mathbb{Z}[t, t^{-1}]$ acting on $\mathbb{G}_a = \operatorname{Spec} \mathbb{Z}[x]$ by multiplication of x by t^2. Consider the natural map $p\colon BB \to B\mathbb{G}_m$ and take $p_*(\mathcal{O}_{BB})$. We have a fiber square

$$\begin{array}{ccc} B\mathbb{G}_a & \xrightarrow{j} & BB \\ \downarrow & & \downarrow \\ \mathrm{pt} & \xrightarrow{q} & B\mathbb{G}_m. \end{array}$$

We have $j^*\mathcal{O}_{BB} \simeq \mathcal{O}_{B\mathbb{G}_a}$ and by base change the underlying complex $q^*p_*\mathcal{O}_{BB}$ is equal to $R\Gamma(B\mathbb{G}_a, \mathcal{O}_{B\mathbb{G}_a})$. It follows that

$$R\Gamma(BB, \mathcal{O}_{BB}) \simeq R\Gamma(B\mathbb{G}_m, p_*\mathcal{O}_{BB}) \simeq R\Gamma(B\mathbb{G}_a, \mathcal{O}_{B\mathbb{G}_a})^{\mathbb{G}_m},$$

since $\mathbb{G}_m$-invariants is an exact functor. So it left to understand the $\mathbb{G}_m$-action on $R\Gamma(B\mathbb{G}_a, \mathcal{O}_{B\mathbb{G}_a})$.

The description in Example 4.3.3.7 can be made more functorial. Namely let $V \simeq \mathbb{Z}^n$ be a lattice and let $\mathbb{V} \simeq \mathbb{G}_a \otimes_{\mathbb{Z}} V$ be the corresponding vector group scheme. Let $V_{\mathbb{F}_p} \simeq V \otimes_{\mathbb{Z}} \mathbb{F}_p$ be the reduction and let $V_{\mathbb{F}_p}^{(i)}$ denote the i-th Frobenius twist. Then

$$H^\bullet(B\mathbb{V}, \mathcal{O}_{B\mathbb{V}}) \simeq \wedge^\bullet_{\mathbb{Z}} V \otimes_{\mathbb{Z}} \mathrm{Sym}^\bullet\left(\bigoplus_p V_{\mathbb{F}_p}^{(1)} \oplus V_{\mathbb{F}_p}^{(2)} \oplus V_{\mathbb{F}_p}^{(3)} \oplus \cdots\right),$$

as a representation of the algebraic group $GL(V)$. Here V is in degree 1 and all $V_{\mathbb{F}_p}^{(i)}$ are in degree 2. Applying this to our case and using the notation of Example 4.3.3.7 for the description of $H^\bullet(B\mathbb{G}_a, \mathcal{O}_{B\mathbb{G}_a})$ we see that $\mathbb{G}_m$ acts on v_1 with the weight 2 and on v_{p^k} with the weight $2p^k$. Since $H^\bullet(B\mathbb{G}_a, \mathcal{O}_{B\mathbb{G}_a})$ is freely generated by v_i's it follows that the $\mathbb{G}_m$-invariants are given by $H^0(B\mathbb{G}_a, \mathcal{O}_{B\mathbb{G}_a}) \simeq \mathbb{Z}$. Consequently $R\Gamma(BB, \mathcal{O}_{BB}) = \mathbb{Z}$.

Summarizing, we see that even though the cohomology of $B\mathbb{G}_a$ is enormous, the $\mathbb{G}_m$-action contracts it to something finitely generated and nice, ultimately making BB Hodge-proper over $\mathbb{Z}$.

Non-proper schemes

Considering schemes, it is natural to ask whether a Hodge-properly spreadable scheme is necessarily proper. On the other extreme, one can ask whether any schematic example of the Hodge-to-de Rham degeneration is Hodge-properly spreadable. Below we consider an example of a semiabelian surface X given by an extension of an elliptic curve by $\mathbb{G}_m$; as we will see, an appropriate choice of extension gives counterexamples to both statements above.

Example 4.3.3.10. Let E be an elliptic curve over a field k. Let K/k be a (not necessarily algebraic) field extension and let $\mathcal{L} \in \mathrm{Pic}^0(E)(K) \simeq E(K)$ be a degree 0 line bundle on E_K. Let X be the total space of the associated $\mathbb{G}_m$-torsor. The K-scheme X is clearly smooth and non-proper; moreover, by [105, VII.3.16], X is in fact an algebraic group, more concretely a semiabelian surface.

Lemma 4.3.3.11. *X is Hodge-proper over K if and only if $\mathcal{L}$ is non-torsion. If* $\mathrm{char}\, K = 0$ *this is also equivalent to the degeneration of the Hodge-to-de Rham spectral sequence for X.*

Proof. Note that $\Omega^1_X \simeq \mathcal{O}_X^{\oplus 2}$, since X is a group. Denote the natural projection $X \to E_K$ by π. We find

$$R\pi_*\mathcal{O}_X \simeq \pi_*\mathcal{O}_X \simeq \bigoplus_{n\in\mathbb{Z}} \mathcal{L}^n.$$

Next, since the degree of $\mathcal{L}$ is zero and $\mathcal{L} \neq \mathcal{O}_X$, we have $R\Gamma(E_K, \mathcal{L}) \simeq 0$. If $\mathcal{L}$ is non-torsion, the same holds for $\mathcal{L}^n$, for all $n \neq 0$. So

$$R\Gamma(X, \Omega^2_X) \simeq R\Gamma(X, \mathcal{O}_X) \simeq R\Gamma(E_K, \mathcal{O}_{E_K}) \simeq K \oplus K[-1], \quad R\Gamma(X, \Omega^1_X) \simeq R\Gamma(X, \mathcal{O}_X^{\oplus 2}) \simeq K^{\oplus 2} \oplus K[-1]^{\oplus 2}.$$

If, on the other hand, $\mathcal{L}$ is torsion, $\pi_*\mathcal{O}_X$ has infinitely many copies of $\mathcal{O}_{E_K}$ as direct summands and so $H^0(X, \mathcal{O}_X)$ is infinite-dimensional.

For the second assertion it is enough to consider the case $K = \mathbb{C}$, where we can compare the de Rham cohomology with the singular one. Since the degree of $\mathcal{L}$ is 0, $\mathcal{L}$ is topologically trivial, and X is homotopy equivalent to $(\mathbb{S}^1)^{\times 3}$; comparing the dimensions we see that the Hodge-to-de Rham spectral sequence degenerates at the first page. □

Remark 4.3.3.12. Note that if K is finite or, more generally, is isomorphic to $\overline{\mathbb{F}_p}$, then $\mathrm{Pic}^0(E)(K)$ is a torsion abelian group. Thus, in this case X is never Hodge-proper.

Now let E be an elliptic curve over $\mathbb{Q}$. Let's consider $\mathcal{L}_\mathbb{C} \in \mathrm{Pic}^0(E)(\mathbb{C})$; the corresponding semiabelian variety $X_\mathbb{C}$ is a variety over $\mathbb{C}$.

Proposition 4.3.3.13. *$X_\mathbb{C}$ is Hodge-properly spreadable if and only if $\mathcal{L}_\mathbb{C} \in \mathrm{Pic}^0(E)(\mathbb{C}) \backslash \mathrm{Pic}^0(E)(\overline{\mathbb{Q}})$.*

Proof. First, let $\mathcal{L}_\mathbb{C} \in E(\overline{\mathbb{Q}})$. Let $K \subset \overline{\mathbb{Q}}$ be the field of definition of $\mathcal{L}_\mathbb{C}$; then $\mathcal{L}_\mathbb{C}$ and $X_\mathbb{C}$ are defined over K and we will denote the corresponding line bundle and the variety over K by $\mathcal{L}$ and X. Let $\mathcal{O}_K \subset K$ be the ring of integers. Consider the filtered system $\{\mathcal{O}_K[1/n]\}_{n\in\mathbb{N}}$ of subrings of K; we have $K = \mathrm{colim}_n \mathcal{O}_K[1/n]$. By Corollary 4.3.1.3 X has a smooth spreading X_R over $R = \mathcal{O}_K[1/n]$ for n big enough and, localizing R further if necessary, we can assume that X_R is the total space of a line bundle $\mathcal{L}_R$ over an elliptic curve E_R (with E_R and $\mathcal{L}_R$ being spreadings of E_K and $\mathcal{L}$). Note that all points of $\mathrm{Spec}\, R$ of positive characteristic are closed and have finite residue fields. Localizing R further we can assume that base change for Hodge and de Rham cohomology holds with respect to all closed points of R. Let $x\colon \mathrm{Spec}\, \mathbb{F}_q \hookrightarrow \mathrm{Spec}\, R$ be some closed point. By Remark 4.3.3.12 the reduction $\mathcal{L}_{\mathbb{F}_q}$ is torsion and thus $X_{\mathbb{F}_q}$ is not Hodge-proper. It follows that neither was X_R. Since the system $\{\mathcal{O}_K[1/n]\}_{n\in\mathbb{N}}$ is cofinal we get that X is not Hodge-properly

spreadable. It follows that $X_{\mathbb{C}}$ is also not Hodge-properly spreadable; indeed, given any Hodge-proper spreading X_R over some localization $R \subset \mathbb{C}$ of some finitely generated $\mathbb{Z}$-algebra A we can replace R with $R \cdot \mathcal{O}_K$, which after a suitable localization of R can be assumed to be flat over both $\mathcal{O}_K$ and R. Moreover, (by Remark 4.3.3.12) after a further localization we can assume that X_R came as a base change from $\mathcal{O}_K[1/n]$ to R of some spreading $X_{\mathcal{O}_K[1/n]}$. By flat base change, comparing the Hodge cohomology of X_R, $X_{R\cdot\mathcal{O}_K}$ and $X_{\mathcal{O}_K[1/n]} \times_{\mathcal{O}_K[1/n]} R$, we get that X_R was not Hodge-proper (since $X_{\mathcal{O}_K[1/n]}$ is not) and so $X_{\mathbb{C}}$ is not Hodge-properly spreadable.

It remains to deal with the transcendent case $\mathcal{L}_{\mathbb{C}} \notin \mathrm{Pic}^0(E)(\overline{\mathbb{Q}})$. Let's consider the universal line bundle $\mathcal{P}$ on $E \times \mathrm{Pic}^0(E)$. Since $\mathcal{L}$ is not a $\overline{\mathbb{Q}}$-point, the corresponding map $\operatorname{Spec}\mathbb{C} \to \mathrm{Pic}^0(E)$ quotients through the generic point $\operatorname{Spec}\mathbb{Q}(E) \subset E \simeq \mathrm{Pic}^0(E)$ and thus both $\mathcal{L}_{\mathbb{C}}$ and $X_{\mathbb{C}}$ are defined over $\mathbb{Q}(E)$. We denote the corresponding bundle and $\mathbb{Q}(E)$-scheme by $\mathcal{L}$ and X. Let $y^2 = x^3+ax+b$ be the equation of (the affine part of) E. Let $B = \mathbb{Z}[1/n][x,y]/(y^2-x^3-ax-b) \subset \mathbb{Q}(E)$ where n is big enough to be divisible by the denominators of both a and b, and so that B is smooth over $\operatorname{Spec}\mathbb{Z}$. Note that E has a smooth proper model $E_{\mathbb{Z}[1/n]}$ over $\mathbb{Z}[1/n]$ given by the projective closure of $\operatorname{Spec} B$. Now let $R = B[S^{-1}] \subset \mathbb{Q}(E)$ be the localization of B with respect to the set S of elements $s \in B$ that are non-zero modulo all primes $p \in \mathbb{Z}$ provided $(p,n) = 1$.[10] The reduction $R/(p)$ is equal to the field of fractions of $B/(p)$ which is nothing but $\mathbb{F}_p(E_{\mathbb{F}_p})$. Identifying $\mathrm{Pic}^0(E)$ with E, we obtain spreadings $\mathcal{L}_R$ (over E_R) and X_R of $\mathcal{L}$ and X (considered as a line bundle on $E_{\mathbb{Q}(E)}$ and a scheme over $\mathbb{Q}(E)$ correspondingly). Localizing R we can assume that X_R is a group scheme and thus it is enough to show that the cohomology of $R\Gamma(X_R, \mathcal{O}_{X_R})$ is finitely generated over R. We have

$$R\pi_*\mathcal{O}_{X_R} \simeq \pi_*\mathcal{O}_{X_R} \simeq \bigoplus_{n\in\mathbb{Z}} \mathcal{L}_R^n.$$

Each $\mathcal{L}_R^n$ is a coherent sheaf on E_R and $R\Gamma(E_R, \mathcal{L}_R^n) \in \mathrm{Mod}_R^{\mathrm{coh}}$. We claim that it is zero if $n \neq 0$; note that R is regular, thus $R\Gamma(E_R, \mathcal{L}_R^n)$ is perfect and so it is enough to check this modulo all primes $p \in \mathbb{Z}$. But by the construction the reduction $\mathcal{L}_{R/p}$ is the restriction of the universal line bundle on $E_{\mathbb{F}_p} \times_{\mathbb{F}_p} \mathrm{Pic}^0(E_{\mathbb{F}_p})$ to $E_{\mathbb{F}_p} \times_{\mathbb{F}_p} \operatorname{Spec}\mathbb{F}_p(E_{\mathbb{F}_p})$; in particular it is non-torsion. Thus $R\Gamma(E_R, \mathcal{L}_R^n) \simeq 0$, and so

$$R\Gamma(X_R, \mathcal{O}_{X_R}) \simeq R\Gamma(E_R, \pi_*\mathcal{O}_{X_R}) \simeq R\Gamma(E_R, \mathcal{O}_{E_R})$$

[10] More explicitely, one can see that it is enough to invert functions $y^n - 1$ for $n \geq 1$.

is coherent. □

Remark 4.3.3.14. Due to Proposition 4.3.3.13 and Lemma 4.3.3.11, a semiabelian surface $X_{\mathbb{C}}$, for the choice of a non-torsion line bundle $\mathcal{L}_{\mathbb{C}} \in \mathrm{Pic}^0(E)(\overline{\mathbb{Q}})$, gives an example of a scheme which is not Hodge-properly spreadable, but the Hodge-to-de Rham spectral sequence degenerates. On the other hand, taking $\mathcal{L}_{\mathbb{C}} \notin \mathrm{Pic}^0(E)(\overline{\mathbb{Q}})$ gives an example of a Hodge-properly spreadable scheme which is not proper.

Remark 4.3.3.15. Note that even in the case $\mathcal{L}_{\mathbb{C}} \notin \mathrm{Pic}^0(E)(\overline{\mathbb{Q}})$, the scheme $X_{\mathbb{C}}$ does not have a Hodge-proper spreading over a finitely generated $\mathbb{Z}$-algebra R (because then $\operatorname{Spec} R$ necessarily has points with finite residue fields over which $\mathcal{L}$ becomes torsion). Thus it really makes a difference to allow arbitrary localizations of the latter in Definition 4.2.2.1.

Remark 4.3.3.16. The ring R used in the proof of Proposition 4.3.3.13 is a slight generalization of a ring, that could be called "quantum integers/rationals". Recall that the "quantum integer n" polynomial $[n]_q \in \mathbb{Z}[q]$ is defined as $[n]_q \coloneqq 1 + q + \ldots + q^{n-1}$; we then can consider $\mathbb{Q}_{\mathbf{q}} \coloneqq \mathbb{Z}[q][[n]_q^{-1}]_{n\in\mathbb{N}}$. The ring $\mathbb{Q}_{\mathbf{q}}$ is a principle ideal domain of Krull dimension 1 whose reduction modulo a prime p is given by

$$\mathbb{Q}_{\mathbf{q}} \otimes_{\mathbb{Z}} \mathbb{F}_p \simeq \mathbb{F}_p[q][[n]_q^{-1}]_{n\in\mathbb{N}} \simeq \mathbb{F}_p(q),$$

the field of rational functions over $\mathbb{F}_p$.

Remark 4.3.3.17. Topologically, $X_{\mathbb{C}}(\mathbb{C}) \simeq \mathbb{C}^\times \times (\mathbb{S}^1)^2$, since $\mathcal{L}_{\mathbb{C}}$ has degree 0 and is topologically trivial. The space $H^1_{\mathrm{sing}}(X_{\mathbb{C}}(\mathbb{C}), \mathbb{C}) \simeq \mathbb{C}^3$ is odd dimensional and thus the corresponding mixed Hodge structure can't be pure of weight 1. In particular this shows that the mixed Hodge structure[11] on the n-th singular cohomology of a Hodge-properly spreadable stack is not necessarily pure of weight n.

4.4 Hodge-proper spreadability of quotient stacks

In this section we study in more detail the case of quotient stacks, providing several new sets of examples of cohomologically proper spreadable stacks. Here the proofs of spreadability are much more involved; the following two important representation-theoretic results will be used:

[11] Appropriately defined, say via Section 8.3 of [35].

Theorem 4.4.0.1 (Theorem 9 and Proposition 57 of [46]). *Let $G_{\mathbb{Z}}$ be a split reductive group over $\mathbb{Z}$ and R be a finitely generated algebra over $\mathbb{Z}$. Let A be a finitely generated R-algebra endowed with a (rational) action of G_R. Then the algebra A^{G_R} of G_R-invariants is finitely generated over R and $H^n(G_R, A)$ is a finitely generated A^{G_R}-module for any $n \geq 0$.*

Theorem 4.4.0.2 (Kempf's theorem: Proposition II.4.5 of [60]). *Let $G_{\mathbb{Z}}$ be a split connected reductive group over $\mathbb{Z}$ and let $B_{\mathbb{Z}} \subset G_{\mathbb{Z}}$ be a Borel subgroup. Let $(G/B)_{\mathbb{Z}}$ be the corresponding flag variety. Then $R\Gamma((G/B)_{\mathbb{Z}}, \mathcal{O}_{(G/B)_{\mathbb{Z}}}) \simeq \mathbb{Z}$.*

As we will see, these two theorems, together with the semiorthogonal decompositions of derived categories constructed by Halpern-Leistner ([53], [52]) allow to prove in a lot of cases that the stack is cohomologically properly spreadable.

We stick to the notations of Section 4.3.3, in particular F will denote an algebraically closed field of characteristic 0 and $R \subset F$ will be assumed to be a finitely generated $\mathbb{Z}$-subalgebra which is regular.

Here is a plan of the section. In Section 4.4.1 we show that a quotient by reductive group is cohomologically properly spreadable (Theorem 4.4.1.3), provided its coarse moduli space is proper and the action is locally linear. The key result is Proposition 4.4.1.2, where we use 4.4.0.1 to show that under the latter assumption the natural morphism $q\colon [Y/G] \to Y//G$ is cohomologically properly spreadable. Particular examples then include a proper-over-affine scheme with an action of a reductive group (Example 4.4.1.5) and quotients coming from GIT (Example 4.4.1.6). In Section 4.4.2 we look at some other set of examples (4.4.2.10), given by quotients that come from BB-complete $\mathbb{G}_m$-actions (Definition 4.4.2.2). Theorem 4.4.2.10 also allows some quotients by non-reductive groups, and 4.4.0.2 is used to pass from the quotient by a Borel subgroup to the quotient by the whole group (4.4.2.13). In Section 4.4.3 we also give a recipe of reestablishing the degeneration results of [108] in the KN-complete case using Theorem 4.2.2.2; here, however, we need to assume some results which are going to appear in the upcoming work of Halpern-Leistner [52].

4.4.1 Global quotients by reductive groups

In [108] Teleman proved the Hodge-to-de Rham degeneration for the quotient of a smooth scheme X by a Kempf-Ness (KN) complete action of a reductive group. In this section we establish spreadability for certain class of global quotients, which includes the semistable (single KN-stratum) case

$X = X^{\mathrm{ss}}(\mathcal{L})$ (4.4.1.6) and another standard KN-complete example given by equivariant "projective-over-affine" variety (4.4.1.5). Moreover, the "projectivity" condition in the latter is replaced by the "proper" one almost for free.

Let Y be a quasi-separated finite type scheme over F and let G be a reductive group acting on Y.

Definition 4.4.1.1. The action of G on Y is called *locally linear* if there exists a G-invariant affine cover of Y.

In this case there exists the categorical quotient $Y//G$; in other words, the quotient stack $[Y/G]$ has a coarse moduli $q\colon [Y/G] \to Y//G$ and $Y//G$ is representable by a scheme. More explicitely, if $\{U_i\}_{i\in I}$, with $U_i \coloneqq \operatorname{Spec} A_i$, is the G-invariant affine cover of X, the categorical quotient $X//G$ is glued out of $\{U_i//G\}_{i\in I}$, with $U_i//G \coloneqq \operatorname{Spec} A_i^G$, with the natural gluing maps unduced by the gluing maps for $\{U_i\}_{i\in I}$. Note that if G is a torus and Y is normal, the action is automatically locally linear by the result of Sumihiro (Corollary 2 of [107]).

Proposition 4.4.1.2. *Let Y be a quasi-separated finite type scheme over F with a locally linear action of a reductive group G. Then the natural morphism $q\colon [Y/G] \to Y//G$ is cohomologically properly spreadable.*

Proof. The group G is split and has a Chevalley model $G_{\mathbb{Z}}$ over $\mathbb{Z}$; this defines a split reductive spreading of G over any R, namely just put $G_R \coloneqq G_{\mathbb{Z}} \otimes_{\mathbb{Z}} R$. We can also spread Y to a quasi-separated scheme Y_R. Moreover, picking a G-invariant affine cover $\{U_i\}_{i\in I}$ as above and a spreading $A_{i,R}$ of each A_i, enlarging R, we can assume that the affine schemes $U_{i,R} \coloneqq \operatorname{Spec} A_{i,R}$ have a G_R-action and give a G_R-equivariant affine cover of Y_R. Let $Y_R//G_R$ be the categorical quotient, namely the scheme obtained by gluing the spectra $\operatorname{Spec} A_{i,R}^{G_R}$ of the invariants in the same way that was described above. Note that by Theorem 4.4.0.1 the scheme $Y_R//G_R$ is of finite type and so it is a valid spreading of $Y//G$ (since $A_{i,R}^{G_R} \otimes_R F \simeq A_i$ for all $i \in I$). We have a natural map $q_R\colon [Y_R/G_R] \to Y_R//G_R$ given locally (on $Y_R//G_R$) by $q_{i,R}\colon [\operatorname{Spec} A_{i,R}/G_R] \to \operatorname{Spec} A_{i,R}^{G_R}$. The map q_R is a spreading of q, thus it is enough to show that q_R is cohomologically proper. This can be checked locally, so it is enough to show that for any finitely generated R-algebra A the map $q_R\colon [\operatorname{Spec} A/G_R] \to \operatorname{Spec} A^{G_R}$ is cohomologically proper.

Let $\mathcal{F} \in \operatorname{Coh}([\operatorname{Spec} A/G_R])^{\heartsuit}$ and let M be the corresponding G_R-equivariant finitely generated A-module. Since $\operatorname{Spec} A$ is affine, the module $q_{R*}\mathcal{F}$ has a simple description: it is just given by

$R\Gamma(G_R, M)$ considered as a complex of modules over A^{G_R}. The complex $R\Gamma(G_R, M)$ lies in $\mathrm{Mod}^{\geq 0}_{A^{G_R}}$ and its cohomology are finitely generated by Theorem 4.4.0.1. Thus $R\Gamma(G_R, M) \in \mathrm{Coh}^+(A^{G_R})$ and q_R is cohomologically proper. □

Note that we did note assume that Y was smooth. This was on purpose: the actual cohomologically properly spreadable examples are given by the following theorem.

Theorem 4.4.1.3. *Let X be a smooth scheme and Y be a finite-type quasi-separated scheme over F, both endowed with an action of a reductive group G. Assume that*

1. *There is a proper G-equivariant map $\pi\colon X \to Y$;*
2. *The G-action on Y is locally linear;*
3. *The categorical quotient $Y//G$ is proper.*

Then the quotient stack $[X/G]$ is cohomologically properly spreadable.

Proof. Consider the map $q\colon [Y/G] \to Y//G$; by Proposition 4.4.1.2 it is cohomologically properly spreadable. The map $\pi\colon [X/G] \to [Y/G]$ is proper, thus $[X/G]$ is cohomologically properly spreadable by Corollary 4.3.3.4. □

Remark 4.4.1.4. More generally one can replace $[X/G]$ by any $\mathcal{X}$ with a cohomologically properly spreadable morphism $\pi\colon \mathcal{X} \to [Y/G]$.

We discuss some applications of Theorem 4.4.1.3:

Example 4.4.1.5. Let X be a smooth proper-over-affine scheme X with $\dim H^0(X, \mathcal{O}_X)^G < \infty$. By definition, this means that there is a proper G-equivariant map $\pi\colon X \to \operatorname{Spec} A$. By replacing $\operatorname{Spec} A$ with the image of π we can assume that π is surjective. Then A^G embedds in $H^0(X, \mathcal{O}_X)^G$ and thus is finite-dimensional; equivalently, $(\operatorname{Spec} A)//G$ is finite, and in particular proper. Applying 4.4.1.3 to $Y = \operatorname{Spec} A$ we get that $[X/G]$ is cohomologically properly spreadable. Also note that we were able to relax the "projectivity" assumption on the map π to the "proper" one.

Example 4.4.1.6. Let X have an ample G-equivariant line bundle $\mathcal{L}$, and let's assume that $X = X^{\mathrm{ss}} := X^{\mathrm{ss}}(\mathcal{L})$. Basically by definition, the action on $X^{\mathrm{ss}}(\mathcal{L})$ is locally linear (see e.g. the

proof of [33, Theorem 1.10]). Assume further that $\dim_F H^0(X, \mathcal{O}_X)^G < \infty$. Then the scheme

$$X//G \simeq \operatorname{Proj}\Big(\bigoplus_{n\geq 0} H^0(X, \mathcal{L}^{\otimes n})^G\Big)$$

is projective over $\operatorname{Spec} H^0(X, \mathcal{O}_X)^G$ and hence also projective over F. Thus $[X/G]$ is cohomologically properly spreadable by Theorem 4.4.1.3 applied to $Y = X$.

4.4.2 Global quotients coming from BB-complete $\mathbb{G}_m$-actions

In this section we prove another result (Theorem 4.4.2.10) concerning the cohomologically proper spreadability of quotient stacks, which also allows quotients by groups that are not necessarily reductive. Another benefit of Theorem 4.4.2.10 (compared, say, to Theorem 4.4.1.3) is that the condition on X (and the G-action) is internal: no additional structure, such as a map to another scheme Y, is involved.

Varieties with a $\mathbb{G}_m$-action and Bialynicki-Birula stratification

Let X be a smooth scheme over F with an action $a\colon \mathbb{G}_m \curvearrowright X$. By [107] such an action is always locally linear: X has a $\mathbb{G}_m$-invariant affine cover $\{U_i = \operatorname{Spec} A_i\}_{i\in I}$. A $\mathbb{G}_m$-action on a given affine scheme $\operatorname{Spec} A$ induces a $\mathbb{Z}$-grading A^* on A; we denote by $I^+ \subset A^*$ the ideal generated by $A^{>0}$ and by $I^\pm \subset A^*$ the ideal generated by $A^{>0}$ and $A^{<0}$.

Here are some examples:

Example 4.4.2.1.

- $\mathrm{pt} \coloneqq \operatorname{Spec} F$ with the trivial $\mathbb{G}_m$-action; here $A = A^0 \simeq F$ and $I^+ = I^\pm = 0$.
- $\mathbb{A}^1 \coloneqq \operatorname{Spec} F[x]$ endowed with the standard action by dilation, $s \mapsto ts$ for $s \in \mathbb{A}^1$; in this case $\deg x = -1$, so $I^+ = 0$ and $I^\pm = (x) \subset F[x]$.

There are natural $\mathbb{G}_m$-equivariant maps $\mathrm{pt} \underset{p}{\overset{i_0}{\rightleftarrows}} \mathbb{A}^1$ given by the projection and the embedding of $0 \in \mathbb{A}^1(F)$. We also have a (non-equivariant) map $i_1\colon \mathrm{pt} \to \mathbb{A}^1$ given by the embedding of $1 \in \mathbb{A}^1(F)$.

To a smooth scheme X endowed with a $\mathbb{G}_m$-action one can associate the following objects:

- *The fixed points* $X^0 \coloneqq \operatorname{Maps}(\mathrm{pt}, X)^{\mathbb{G}_m}$; its functor of points is given by $X^0(S) = \operatorname{Maps}(S, X)^{\mathbb{G}_m}$, meaning the $\mathbb{G}_m$-equivariant maps from S to X, where the action on S is trivial. There is

a natural map $\iota\colon X^0 \to X$ sending a map $f \in X^0$ to its evaluation $f(\mathrm{pt})$. The map ι is a closed embedding ([40, Proposition 1.2.2]); the affine cover $\{U_i\}_{i\in I}$ defines an affine cover $\{U_i^0\}_{i\in I}$ of X^0 with $U_i^0 \coloneqq \mathrm{Spec}(A_i/I_i^{\pm})$ (glued along U_{ij}^0). There is a natural $\mathbb{G}_m$-action on X^0, which is trivial.

- *The attractor* $X^+ \coloneqq \mathrm{Map}(\mathbb{A}^1, X)^{\mathbb{G}_m}$; here the functor of points is given by $X^+(S) = \mathrm{Maps}(S\times\mathbb{A}^1, X)^{\mathbb{G}_m}$, where the $\mathbb{G}_m$-action on $S\times\mathbb{A}^1$ is diagonal. By [40], Corollary 1.4.3 this functor is indeed represented by a scheme. There are two natural $\mathbb{G}_m$-actions on $\mathrm{Map}(\mathbb{A}^1, X)$, one coming from the $\mathbb{G}_m$-action on $\mathbb{A}^1$, the other coming from the action on X; their restrictions to $\mathrm{Map}(\mathbb{A}^1, X)^{\mathbb{G}_m}$ coincide and thus define the same $\mathbb{G}_m$-action on X^+

  There are natural $\mathbb{G}_m$-equivariant maps $X^0 \underset{\pi}{\overset{\sigma}{\rightleftarrows}} X^+ \xrightarrow{j} X$ induced by i_0, i_1 and p. Namely,

  - $\sigma(f) \in X^+$ is given by pre-composing $f\colon \mathrm{pt} \to X$ with $p\colon \mathbb{A}^1 \to \mathrm{pt}$; the map $\sigma\colon X^0 \to X^+$ is a closed embedding.
  - $\pi(f) \in X^0$ is given by the evaluation of $f\colon \mathbb{A}^1 \to X$ at $0 \in \mathbb{A}^1$; by [20] $\pi\colon X^+ \to X^0$ is a Zariski locally trivial fibration with the fiber given by an affine space.
  - $j(f) \in X \simeq \mathrm{Map}(\mathrm{pt}, X)$ is given by the evaluation of $f\colon \mathbb{A}^1 \to X$ at $1 \in \mathbb{A}^1$; the map $j\colon X^+ \to X$, however, is only a locally closed embedding.

  Similarly to X^0, the affine cover $\{U_i\}_{i\in I}$ defines an affine cover $\{U_i^+\}_{i\in I}$ of X^+ with $U_i^+ \coloneqq \mathrm{Spec}(A_i/I_i^+)$.

Also, by [20] both X^0 and X^+ are smooth.

Let $\pi_0(X^0) = \pi_0(X^+)$ be the set of connected components of X^0 (equivalently, X^+). For a given $c \in \pi_0(X^0)$ we denote by $Z_c \subset X^0$ the corresponding connected component of X^0, and by $S_c \coloneqq \pi^{-1}(S_c) \subset X^+$ the corresponding connected component of X^+; we call $\{S_c\}_{c\in\pi_0(X^0)}$ *the Bialynicki-Birula (BB) strata.*

Definition 4.4.2.2. The $\mathbb{G}_m$-action $a\colon \mathbb{G}_m \curvearrowright X$ is called *BB-complete* if the map $j\colon X^+ \to X$ is a surjection on the underlying topological spaces $|j|\colon |X^+| \twoheadrightarrow |X|$.

Equivalently, $j\colon X^+ \to X$ gives a *full* stratification of X with individual strata being locally closed. In this case, the limit of $a(t)\circ x$ for $t \to 0$ exists for any point $x \in X$; in particular, the Bialynicki-Birula stratification $\{S_c\} \in \pi_0(X^0)$ gives a full stratification of X. We will fix some

ordering on $\pi_0(X^0)$ with the only condition that $c' > c$ if $\dim S_{c'} < \dim S_c$; in this case a stratum S_c is necessarily closed in $X_{\leq c} := X \setminus \cup_{c'>c} S_{c'}$.

Remark 4.4.2.3. Since both X^+ and X are of finite type and F is algebraically closed, $|j|\colon |X^+| \to |X|$ is surjective if and only if the corresponding map $j(F)\colon X^+(F) \to X(F)$ between the F-points is.

BB-complete quotients by $\mathbb{G}_m$

Spreading BB-stratification. Let X be a smooth scheme over F with a $\mathbb{G}_m$-action and let $\{U_i\}_{i\in I}$, $U_i \simeq \operatorname{Spec} A_i$ be a $\mathbb{G}_m$-invariant affine cover as above. Then (by Corollary 4.3.1.3) we can spread X to a smooth R-scheme X_R endowed with an action of $\mathbb{G}_{m,R} := \operatorname{Spec} R[t, t^{-1}]$. We can also spread the cover $\{U_i\}_{i\in I}$ to a $\mathbb{G}_m$-invariant affine cover $\{U_{i,R}\}_{i\in I}$, $U_{i,R} \simeq \operatorname{Spec} A_{i,R}$ over some regular finitely generated $\mathbb{Z}$-algebra $R \subset F$. Each algebra $A_{i,R}$ is $\mathbb{Z}$-graded and we can consider closed subschemes $U^+_{i,R} := \operatorname{Spec}(A_{i,R}/I^+_{i,R})$ and $U^0_{i,R} := \operatorname{Spec}(A_{i,R}/I^{\pm}_{i,R})$, as well as schemes X^+_R and X^0_R, obtained as their gluings along $U^+_{ij,R}$ and $U^0_{ij,R}$ (defined analogously). We have $(X^+_R) \times_R F \simeq X^+$ and $(X^0_R) \times_R F \simeq X^0$; moreover, (e.g. by [55, Example 1.2.3]) the functors $\operatorname{Map}(\operatorname{Spec} R, X_R)^{\mathbb{G}_{m,R}}$ and $\operatorname{Map}(\mathbb{A}^1, X_R)^{\mathbb{G}_{m,R}}$ are represented by schemes locally of finite presentation and so (at least after enlarging R) we can assume that they are equal to X^0_R and X^+_R (since it is true after base changing to F).

Similarly, we have maps $X^0_R \underset{\pi_R}{\overset{\sigma_R}{\rightleftarrows}} X^+_R \xrightarrow{j_R} X_R$, and after enlarging R further, by Corollary 4.3.1.3 we can assume that σ_R (resp. j_R) is a (resp. locally) closed embedding. Picking an open cover $\{V_i\}_{i\in I}$ on which the affine fibration $\pi\colon X^+ \to X^0$ is trivial, $\pi^{-1}(V_i) \simeq V_i \times \mathbb{A}^d$, we can spread it out; enlarging R we can assume that $\{V_{i,R}\}_{i\in I}$ cover X^0_R, $\pi_R^{-1}(V_{i,R})$ cover X^+_R and $\pi_R^{-1}(V_{i,R}) \simeq V_{i,R} \times_R \mathbb{A}^d_R$. Furthermore, we can assume that X^0_R, and consequently X^+_R, are smooth over R.

After enlarging R further, the set $\pi_0(X^0_R)$ can be identified with $\pi_0(X^0)$; the connected component $Z_{c,R}$ for $c \in \pi_0(X^0_R) \simeq \pi_0(X^0)$ then is a spreading of $Z_c \subset X^0$. Similarly, $S_{c,R} := \pi_R^{-1}(Z_{c,R})$ is a connected component of X^+_R and a spreading of S_c.

If the $\mathbb{G}_m$-action on X is BB-complete, we can assume the same about the $\mathbb{G}_{m,R}$-action on X_R; indeed, enlarging R we can assume that the map $X^+_R \to X_R$ is a surjection. In particular, we can assume that the spreading $\{S_{c,R}\}$ of the stratification $\{S_c\}$ of X gives a full stratification of X_R by locally closed smooth subschemes.

Semiorthogonal decomposition of $\mathrm{Coh}(\mathcal{X}_R)$. We now discuss certain semiorthogonal decompositions of the category $\mathrm{Coh}(\mathcal{X}_R)$ given by the theory of so-called magic windows developed by Halpern-Leistner in [53]. Let X be a smooth scheme over F with a $\mathbb{G}_m$-action and let X_R be a spreading of X as constructed above. Let $S_R := S_{c,R} \subset X_R$ be a closed stratum among $\{S_{c',R}\}$. Let $Z_R := Z_{c,R}$ for the same c; the scheme Z_R is called the *centrum* of S_R. The subschemes S_R and Z_R enjoy the following nice properties:

Proposition 4.4.2.4. • *Both S_R and Z_R are smooth over R;*

- *There is a $\mathbb{G}_m$-equivariant map $\pi_R\colon S_R \simeq S_R^+ \to S_R^0 \simeq Z_R$, which is a Zariski locally trivial fibration with an affine space $\mathbb{A}_R^d$ (for some d) as a fiber;*
- *The conormal bundle $N^\vee_{S_R} X_R$ has positive weights when restricted to the fixed locus $Z_R \subset S_R$.*

Proof. Since Z_R and S_R are connected components of X_R^0 and X_R^+ correspondingly, it is enough to show that the above properties hold for X_R^0 and X_R^+ in place of Z_R and S_R. The first two properties are included in our construction of the spreading. The last property can be checked locally, thus we can assume $X_R \simeq \operatorname{Spec} A_R$, $S_R \simeq \operatorname{Spec} A_R/I_R^+$ and $Z_R \simeq \operatorname{Spec} A_R/I_R^\pm$. The normal bundle $N^\vee_{X_R^+} X_R$ is then given by $I_R^+/(I_R^+)^2$ as a A_R/I_R^+-module and its restriction to X_R^0 is given by $I_R^+/(I_R^+ \cdot I_R^\pm)$, considered as a $A_R/I_R^\pm$-module). Since by definition I_R^+ is generated by elements of strictly positive weight, the weights of $I_R^+/(I_R^+ \cdot I_R^\pm)$ are also strictly positive. □

Remark 4.4.2.5. In other words, using the terminology of [53], $S_R \subset X_R$ is a smooth KN-stratum satisfying properties (A) and $(L+)$ (in the case $G = \mathbb{G}_{m,R}$).

Let $\mathcal{X}_R := [X_R/\mathbb{G}_{m,R}]$, $\mathcal{S}_R := [S_R/\mathbb{G}_{m,R}]$ and $\mathcal{Z}_R := [Z_R/\mathbb{G}_{m,R}]$. Also let $U_R := X_R \setminus S_R$ be the complement and let $\mathcal{U}_R := [U_R/\mathbb{G}_{m,R}]$. Let $\mathrm{Coh}(\mathcal{X}_R)$ be the (bounded derived) category of coherent sheaves on $\mathcal{X}_R$ and let $\mathrm{Coh}_{\mathcal{S}_R}(\mathcal{X}_R) \subset \mathrm{Coh}(\mathcal{X}_R)$ be the full subcategory of sheaves whose pull-back to the flat cover $X_R \twoheadrightarrow \mathcal{X}_R$ is set-theoretically supported on S_R. The action of $\mathbb{G}_{m,R}$ on Z_R is trivial and so $\mathcal{Z}_R \simeq Z_R \times_R B\mathbb{G}_{m,R}$. Thus the heart $\mathrm{Coh}(\mathcal{Z}_R)^\heartsuit$ of the category of coherent sheaves on $\mathcal{Z}_R$ is identified with the category of $\mathbb{Z}$-graded objects in $\mathrm{Coh}(Z_R)^\heartsuit$. For a given w we denote by $\mathrm{Coh}(\mathcal{Z}_R)_{<w} \subset \mathrm{Coh}(\mathcal{Z}_R)$ the full subcategory spanned by objects $\mathcal{F} \in \mathrm{Coh}(\mathcal{Z}_R)$ whose cohomology sheaves $\mathcal{H}^i(\mathcal{F}) \in \mathrm{Coh}(\mathcal{Z}_R)^\heartsuit$ have grading less than w. Similarly, by $\mathrm{Coh}(\mathcal{Z}_R)_{\geq w} \subset \mathrm{Coh}(\mathcal{Z}_R)$ we denote the full subcategory spanned by objects whose cohomology sheaves have grading greater or equal than w.

Let $i_R\colon \mathcal{U}_R \to \mathcal{X}_R$ and $j_R\colon \mathcal{S}_R \to \mathcal{X}_R$ be the natural embeddings. Then, given $w \in \mathbb{Z}$, by [53], we have a semiorthogonal decomposition

$$\mathrm{Coh}(\mathcal{X}_R) = \langle\, \mathrm{Coh}_{\mathcal{S}_R}(\mathcal{X}_R)_{<w}, \mathrm{G}_w, \mathrm{Coh}_{\mathcal{S}_R}(\mathcal{X}_R)_{\geq w} \,\rangle, \tag{4.4.2.1}$$

where

- $\mathrm{Coh}_{\mathcal{S}_R}(\mathcal{X}_R)_{\geq w} \coloneqq \{\mathcal{F} \in \mathrm{Coh}_{\mathcal{S}_R}(\mathcal{X}_R) \mid \sigma_R^*\mathcal{F} \in \mathrm{Coh}(\mathcal{Z}_R)_{\geq w}\} \subset \mathrm{Coh}_{\mathcal{S}_R}(\mathcal{X}_R)$,
- $\mathrm{Coh}_{\mathcal{S}_R}(\mathcal{X}_R)_{<w} \coloneqq \{\mathcal{F} \in \mathrm{Coh}_{\mathcal{S}_R}(\mathcal{X}_R) \mid \sigma_R^* j_R^! \mathcal{F} \in \mathrm{Coh}(\mathcal{Z}_R)_{<w}\} \subset \mathrm{Coh}_{\mathcal{S}_R}(\mathcal{X}_R)$

and $\mathrm{G}_w \subset \mathrm{Coh}(\mathcal{X}_R)$ is a certain (full) subcategory such that the functor $i_R^* : \mathrm{Coh}(\mathcal{X}_R) \to \mathrm{Coh}(\mathcal{U}_R)$ restricted to G_w

$$i_R^*|_{\mathrm{G}_w}\colon \mathrm{G}_w \xrightarrow{\sim} \mathrm{Coh}(\mathcal{U}_R)$$

is an equivalence. Note that even though [53] assumes the base to be a field of characteristic 0, in the case $G = \mathbb{G}_m$ the proofs of Section 3 of *loc.cit.* work verbatim as long as $\mathcal{S}_R \subset \mathcal{X}_R$ is a KN-stratum over R that satisfies the conclusions of Proposition 4.4.2.4.

Proposition 4.4.2.6. *$\mathcal{X}_R$ is cohomologically proper if and only if both $\mathcal{U}_R$ and $\mathcal{S}_R$ are.*

Proof. Fix some $w \in \mathbb{Z}$.

"$\Rightarrow$" $\mathcal{S}_R$ is cohomologically proper, since for $\mathcal{F} \in \mathrm{Coh}(\mathcal{S}_R)$ we have $j_{R*}\mathcal{F} \in \mathrm{Coh}(\mathcal{X}_R)$ and $R\Gamma(\mathcal{S}_R, F) \simeq R\Gamma(\mathcal{X}_R, j_{R*}\mathcal{F})$. Since $\mathcal{X}_R$ is smooth, we have $\mathrm{Coh}(\mathcal{X}_R) \simeq \mathrm{Perf}(\mathcal{X}_R)$ and so any object of $\mathrm{Coh}(\mathcal{X}_R)$ is dualizable; in particular, for any $V_1, V_2 \in \mathrm{Coh}(\mathcal{X}_R)$ we have

$$\mathrm{Hom}_{\mathrm{Coh}(\mathcal{X}_R)}(V_1, V_2) \simeq R\Gamma(\mathcal{X}_R, \mathscr{H}om(V_1, V_2)),$$

where $\mathscr{H}om(V_1, V_2) \in \mathrm{Coh}(\mathcal{X}_R)$. Thus $\mathrm{Hom}_{\mathrm{Coh}(\mathcal{X}_R)}(V_1, V_2) \in \mathrm{Coh}^+(R)$, or, in other words, $\mathrm{Coh}\,(\mathcal{X}_R)$ is a *nearly proper* R-linear DG-category. Now, taking $E \in \mathrm{Coh}(\mathcal{U}_R)$, we get

$$R\Gamma(\mathcal{U}_R, E) \simeq \mathrm{Hom}_{\mathrm{Coh}(\mathcal{U}_R)}(\mathcal{O}_{\mathcal{U}_R}, E) \simeq \mathrm{Hom}_{\mathrm{Coh}(\mathcal{X}_R)}((i_R^*|_{\mathrm{G}_w})^{-1}\mathcal{O}_{\mathcal{U}_R}, (i_R^*|_{\mathrm{G}_w})^{-1}E) \in \mathrm{Coh}^+(R),$$

via the equivalence $(i_R^*|_{\mathrm{G}_w})^{-1}\colon \mathrm{Coh}(\mathcal{U}_R) \xrightarrow{\sim} \mathrm{G}_w \subset \mathrm{Coh}(\mathcal{X}_R)$.

"$\Leftarrow$" It is enough to show that the DG-category $\mathrm{Coh}(\mathcal{X}_R)$ is nearly proper. Let's first show that the subcategory $\mathrm{Coh}_{\mathcal{S}_R}(\mathcal{X}_R) \subset \mathrm{Coh}(\mathcal{X}_R)$ is nearly proper. Every object of $\mathrm{Coh}_{\mathcal{S}_R}(\mathcal{X}_R)$ has a finite

filtration with graded pieces of the form $j_{R*}\mathcal{F}$, where $\mathcal{F} \in \mathrm{Coh}(\mathcal{S}_R)$. Thus it is enough to show that $\mathrm{Hom}(j_{R*}\mathcal{F}_1, j_{R*}\mathcal{F}_2) \in \mathrm{Coh}^+(R)$ for any $\mathcal{F}_1, \mathcal{F}_2 \in \mathrm{Coh}(\mathcal{S}_R)$. Since $\mathcal{S}_R$ and $\mathcal{X}_R$ are smooth we have

$$\mathrm{Hom}_{\mathrm{Coh}(\mathcal{S}_R)}(j_{R*}\mathcal{F}_1, j_{R*}\mathcal{F}_2) \simeq R\Gamma(\mathcal{S}_R, \mathscr{H}om(j_R^* j_{R*}\mathcal{F}_1, \mathcal{F}_2))$$

with $\mathscr{H}om(j_R^* j_{R*}\mathcal{F}_1, \mathcal{F}_2) \in \mathrm{Coh}(\mathcal{S}_R)$; then $R\Gamma(\mathcal{S}_R, \mathscr{H}om(j_R^* j_{R*}\mathcal{F}_1, \mathcal{F}_2)) \in \mathrm{Coh}^+(R)$ since $\mathcal{S}_R$ is cohomologically proper.

More generally, given $\mathcal{F} \in \mathrm{Coh}(\mathcal{S}_R)$ and $V \in \mathrm{Coh}(\mathcal{X}_R)$ we have

$$\mathrm{Hom}_{\mathrm{Coh}(\mathcal{X}_R)}(V, j_{R*}\mathcal{F}) \simeq \mathrm{Hom}_{\mathrm{Coh}(\mathcal{S}_R)}(j_R^* V, \mathcal{F}) \in \mathrm{Coh}^+(R),$$
$$\mathrm{Hom}_{\mathrm{Coh}(\mathcal{X}_R)}(j_{R*}\mathcal{F}, V) \simeq \mathrm{Hom}_{\mathrm{Coh}(\mathcal{S}_R)}(\mathcal{F}, j_R^! V) \in \mathrm{Coh}^+(R),$$

where $j_R^! V \in \mathrm{Coh}(\mathcal{S}_R)$, since both $\mathcal{X}_R$ and $\mathcal{S}_R$ are smooth. It follows that $\mathrm{Hom}_{\mathrm{Coh}(\mathcal{X}_R)}(V, E)$ and $\mathrm{Hom}_{\mathrm{Coh}(\mathcal{X}_R)}(E, V)$ both lie in $\mathrm{Coh}^+(R)$ if $E \in \mathrm{Coh}_{\mathcal{S}_R}(\mathcal{X}_R)$ and V is any coherent sheaf on $\mathcal{X}_R$.

Now, due to the semiorthogonal decomposition in (4.4.2.1) any $V \in \mathrm{Coh}(\mathcal{X}_R)$ has a finite (in fact three-step) filtration with the associated graded pieces lying either in $\mathrm{Coh}_{\mathcal{S}_R}(\mathcal{X}_R)$ or G_w. By the above discussion, hom-complex $\mathrm{Hom}_{\mathrm{Coh}(\mathcal{X}_R)}(E, -)$ for any object E in $\mathrm{Coh}_{\mathcal{S}_R}(\mathcal{X}_R)$ is always nearly coherent. The category $\mathrm{G}_w \simeq \mathrm{Coh}(\mathcal{U}_R)$ is nearly proper since $\mathcal{U}_R$ is cohomologically proper. Taking such filtrations for a given pair $\mathcal{F}_1, \mathcal{F}_2 \in \mathrm{Coh}(\mathcal{X}_R)$ and using the exactness of Hom in each variable we get that $\mathrm{Hom}_{\mathrm{Coh}(\mathcal{X}_R)}(\mathcal{F}_1, \mathcal{F}_2) \in \mathrm{Coh}^+(R)$. □

The $\mathbb{G}_{m,R}$-action on X_R^0 is trivial; thus $\mathcal{X}_R^0$ is isomorphic to $X_R^0 \times_R B\mathbb{G}_{m,R}$. Let $p\colon \mathcal{X}_R^0 \to X_R^0$ be the projection.

Lemma 4.4.2.7. *The morphism $p \circ \pi_R\colon \mathcal{X}_R^+ \to X_R^0$ is cohomologically proper.*

Proof. The statement is local on X_R^0. Let $\{U_{i,R}^0\}_{i\in I}$, $U_{i,R}^0 = \operatorname{Spec} A_{i,R}^0$ be an affine cover of X_R^0 such that the affine bundle given by $\pi_R\colon X_R^+ \to X_R^0$ is trivial. Let $U_{i,R}^+ \coloneqq \pi_R^{-1}(U_{i,R}^0)$. It is enough to show that the morphism $\mathcal{U}_{i,R}^+ \coloneqq [U_{i,R}^+/\mathbb{G}_{m,R}] \to U_{i,R}^0$ induced by the composition of π_R and p is cohomologically proper. We have $U_{i,R}^+ \simeq U_{i,R}^0 \times_R \mathbb{A}_R^d$ where $\mathbb{G}_{m,R}$ acts with negative weights on $\mathbb{A}_R^d$. Let $A_{i,R}^+$ be the ring of functions on $U_{i,R}^+$; it is naturally $\mathbb{Z}$-graded and $A_{i,R}^+ \simeq A_{i,R}^0[x_1, \dots, x_d]$ where x_i's can be chosen to be homogeneous of strictly negative degree.

The functor $(p \circ \pi_R)_*$ is t-exact, since π_R is affine and $\mathbb{G}_{m,R}$ is linearly reductive. We have an equivalence between the abelian category $\mathrm{Coh}(\mathcal{U}_{i,R}^+)^\heartsuit$ and the category of finitely generated

graded $A^+_{i,R}$-modules. Via this equivalence, $(p \circ \pi_R)_*$ sends a graded module M^* to the $A^0_{i,R}$-module M^0. Since the degrees of x_i's are strictly negative, it is straightforward to see that if a $A^0_{i,R}[x_1, \ldots, x_d]$-module M is finitely generated, the $A^0_{i,R}$-module M^0 is finitely generated as well, and thus corresponds to a coherent sheaf on $U^0_{i,R}$. □

Proposition 4.4.2.8. *Let X be smooth scheme over F and with a BB-complete $\mathbb{G}_m$-action. Let X_R be a spreading as above. Then the following are equivalent:*

1. *$\mathcal{X}_R$ is cohomologically proper.*

2. *$\mathcal{X}^+_R$ is cohomologically proper.*

3. *X^0_R is proper.*

Proof. $1 \Leftrightarrow 2$. Note that $\mathcal{X}^+_R \simeq \cup_{c \in \pi_0(X^0)} \mathcal{S}_{c,R}$. Let's fix an ordering on $\pi_0(X^0)$ such that each $S_{c,R}$ is closed in $X_{\leq c,R} := X_R \setminus \cup_{c'>c} S_{c',R}$. Since $\{S_{c,R}\}$ form a full finite stratification of X_R, applying Proposition 4.4.2.6 we are done by induction on c.

$2 \Rightarrow 3$. The morphism $\sigma_R \colon \mathcal{X}^0_R \to \mathcal{X}^+_R$ is a closed embedding (in particular proper) and thus is cohomologically proper. It follows that $\mathcal{X}^0_R$ is cohomologically proper. On the other hand $\mathcal{X}^0_R \simeq X^0_R \times_R B\mathbb{G}_{m,R}$ and so X^0_R is cohomologically proper as well. Indeed, $p_* \mathcal{O}_{\mathcal{X}^0_R} \simeq \mathcal{O}_{X^0_R}$ and given $\mathcal{F} \in \mathrm{Coh}(X^0_R)$ we have $R\Gamma(\mathcal{X}^0_R, p^*\mathcal{F}) \simeq R\Gamma(X^0_R, \mathcal{F})$ by the projection formula; since $p^*\mathcal{F} \in \mathrm{Coh}(\mathcal{X}^0_R)$ we get $R\Gamma(X^0_R, \mathcal{F}) \in \mathrm{Coh}^+(R)$. Being a cohomologically proper R-scheme, X^0_R is forced to be proper (Corollary 3.8 of [51]).

$2 \Leftarrow 3$: The structure morphism $\mathcal{X}^+_R \to \operatorname{Spec} R$ factors as the compostion of $\mathcal{X}^+_R \xrightarrow{p \circ \pi_R} X^0_R$ and $X^0_R \to \operatorname{Spec} R$. The first map is cohomologically proper by Lemma 4.4.2.7, the second — since X^0_R is proper. □

Corollary 4.4.2.9. *Let X be a smooth scheme over F with an action of $\mathbb{G}_m$. Assume that the action is BB-complete and that the scheme of $\mathbb{G}_m$-fixed points X^0 is proper. Then $\mathcal{X}$ is cohomologically properly spreadable.*

Proof. Let X_R be a spreading as above, then X^0_R is a spreading of X^0 and thus, after enlarging R, we can assume that X^0_R is proper. Then $\mathcal{X}_R$ is cohomologically proper by Proposition 4.4.2.8. □

Quotients by G that are BB-complete with respect to a subgroup

In Corollary 4.4.2.9 we gave some sufficient conditions for the quotient stack $[X/\mathbb{G}_m]$ to be cohomologically spreadable. As we will see soon, Kempf's theorem allows to generalize this to a quotient by an arbitrary linear group G; however, a certain extra weight-positivity assumption with respect to a 1-parameter subgroup $h\colon \mathbb{G}_m \to G$ is necessary.

Let G be a linear algebraic group and let $B \subset G$ be a Borel subgroup[12]. Let $U \subset B$ be the unipotent radical of B and let $T \subset B$ be a maximal torus. We have a short exact sequence $1 \to U \to B \to T \to 1$. Let $X^*(T) \coloneqq \mathrm{Hom}(T, \mathbb{G}_m)$ and $X_*(T) \coloneqq \mathrm{Hom}(\mathbb{G}_m, T)$ be the character and cocharacter lattices of T. One has $X_*(T) \simeq X^*(T)^\vee$. Given a T-representation V and a character $\lambda \in X^*(T)$ we denote by $V_\lambda \subset V$ the subspace of V of weight λ. The adjoint action of T on U induces an action on the Lie algebra $\mathfrak{u}$ of U and we denote by $\Phi^+ \subset X^*(T)$ the set of weights of $\mathfrak{u}$ with respect to this action.

Theorem 4.4.2.10. *Let X be a smooth scheme over F endowed with an action of a linear algebraic group G. Let $B \subset G$ be a Borel subgroup and let $T \subset B$ be a maximal torus. Let $\Phi^+ \subset X^*(T)$ be the set of T-weights of the Lie algebra $\mathfrak{u}$ of the unipotent radical $U \subset B$ with respect to the adjoint action of T on U.*

Assume that there is a subgroup $h\colon \mathbb{G}_m \to T$, $h \in X_*(T)$, such that

1. *$h(\Phi^+) > 0$;*
2. *The $\mathbb{G}_m$-action induced by h is BB-complete;*
3. *The $h(\mathbb{G}_m)$-fixed points X^0 are proper.*

Then the quotient stack $[X/G]$ is cohomologically properly spreadable.

Proof. Let's first assume that G is connected. Note that B is isomorphic to a semidirect product $T \ltimes U$ and can be spread out to a semidirect product $T_R \ltimes U_R$ of a split torus T_R and a unipotent group U_R over a finitely generated $\mathbb{Z}$-algebra $R \subset F$. Since T_R is split $X^*(T_R) \simeq X^*(T)$. In particular we have a cocharacter $h\colon \mathbb{G}_{m,R} \to T_R$. The subgroup $B \subset G$ can be spread out to a closed subgroup $B_R \subset G_R$. Let U_G be the unipotent radical of G. Then G/U_G is reductive and can be spread out to a split reductive group $(G/U_G)_R$. We then also have a spreading $p_R\colon G_R \to (G/U_G)_R$ of the projection $p\colon G \to G/U_G$ and the kernel $(U_G)_R \coloneqq \mathrm{Ker}(p_G)$ is a spreading of U_G and thus

[12]Recall that a subgroup $B \subset G$ is called Borel if it is a maximal Zariski-closed solvable subgroup of G.

can be assumed to be unipotent. Since U_G is a closed subgroup of B, we can assume that $(U_G)_R$ is a closed subgroup of B_R. The image of B_R under p_R is a spreading of $B/U_G \subseteq G/U_G$ and thus can be assumed to be a Borel subgroup of the split reductive group $(G/U)_R$. Note that with all these assumptions $G_R/B_R \simeq (G/U_G)_R/p_R(B_R)$.

We can also spread X with the action $a\colon G \curvearrowright X$ to a smooth scheme X_R over R with an action $a_R\colon G_R \curvearrowright X_R$. Note that by [107] the restriction of the action to $\mathbb{G}_m$ (via h) is locally linear; consider a $\mathbb{G}_m$-invariant open cover $\{U_i\}_{i\in I}$ of X, $U_i = \operatorname{Spec} A_i$. We have spreadings $A_{i,R}$ of A_i with an action of $\mathbb{G}_{m,R}$; localizing R if necessary, we can assume that $U_{i,R} := \operatorname{Spec} A_{i,R}$ cover X_R and that the restriction of a_R to $\mathbb{G}_{m,R}$ via h is locally linear. Localizing R, we can assume that the $\mathbb{G}_{m,R}$-fixed points X_R^0 are proper, the action is BB-complete and X_R is such that the conditions of 4.4.2.8 are satisfied. Thus we have that $[X_R/h(\mathbb{G}_{m,R})]$ is cohomologically proper. It is enough to show that $[X_R/G_R]$ is cohomologically proper.

We split the argument into a sequence of lemmas:

Lemma 4.4.2.11. *Let X_R be as above. Then*

$[X_R/h(\mathbb{G}_{m,R}))]$ *is cohomologically proper over R* $\Rightarrow$ $[X_R/T_R]$ *is cohomologically proper over R.*

Proof. Let $p\colon X_R \to [X_R/T_R]$ and $q\colon [X/h(\mathbb{G}_{m,R})] \to [X_R/T_R]$ be the natural smooth covers. Then, given $\mathcal{F} \in \operatorname{Coh}([X_R/T_R])$, we have $R\Gamma([X_R/T_R], \mathcal{F}) \simeq R\Gamma(X_R, p^*\mathcal{F})^{T_R}$. But

$$R\Gamma(X_R, p^*\mathcal{F})^{T_R} \simeq \left(R\Gamma(X_R, p^*\mathcal{F})^{h(\mathbb{G}_{m,R})}\right)^{T/h(\mathbb{G}_{m,R})} \simeq R\Gamma([X_R/h(\mathbb{G}_{m,R})], s^*\mathcal{F})^{T/h(\mathbb{G}_{m,R})};$$

since $s^*\mathcal{F} \in \operatorname{Coh}([X_R/h(\mathbb{G}_{m,R})])$ we have $R\Gamma([X_R/h(\mathbb{G}_{m,R})], s^*\mathcal{F}) \in \operatorname{Coh}^+(R)$. Recall that the coherence of a complex of R-modules is equivalent to being t-bounded and having finitely generated cohomology. The group scheme $T/h(\mathbb{G}_{m,R})$ is a torus and the functor of $T_R/h(\mathbb{G}_{m,R})$-invariants is t-exact. Thus $R\Gamma([X_R/h(\mathbb{G}_{m,R})], s^*\mathcal{F})^{T/h(\mathbb{G}_{m,R})}$ is also bounded and has finitely generated cohomology, so is coherent. □

Lemma 4.4.2.12. *Let X_R be as above. Then*

$[X_R/h(\mathbb{G}_{m,R}))]$ *is cohomologically proper over R* $\Rightarrow$ $[X_R/B_R]$ *is cohomologically proper over R.*

Proof. Consider the natural smooth cover $q\colon [X_R/T_R] \to [X_R/B_R]$ induced by the embedding

$T_R \to B_R$. Since $B_R \simeq T_R \ltimes U_R$, the n-th term of the Čech complex associated to q is given by

$$[X_R \overset{T_R}{\times} \underbrace{B_R \overset{T_R}{\times} B_R \overset{T_R}{\times} \cdots \overset{T_R}{\times} B_R}_{n}/T_R] \simeq [(X_R \times \underbrace{U_R \times U_R \times \cdots \times U_R}_{n})/T_R]^{13},$$

where the action of T_R on $X_R \times U_R \times U_R \times \cdots \times U_R$ is given by the product of the action on X_R and the adjoint action on each copy of U_R.

Note that the underlying scheme of U_R can be T_R-equivariantly identified with its Lie algebra $\mathfrak{u}_R$ (see II.1.7 in [60]); this way functions on U_R (resp U_R^n) are identified with $\mathrm{Sym}_R(\mathfrak{u}_R^*)$ (resp. $\mathrm{Sym}_R(\mathfrak{u}_R^*)^{\otimes n}$). Since $h(\Phi^+) > 0$ we get that the $\mathbb{G}_{m,R}$-weights on non-constant homogeneous functions on U_R^n are strictly negative. It follows that $U_R^n \simeq (U_R^n)^+$ and $(U_R^n)^0 \simeq \operatorname{Spec} R$. We have $(X_R \times U_R^n)^+ \simeq X_R^+ \times U_R^n$, so the map $(X_R \times U_R^n)^+ \to X_R \times U_R^n$ is surjective and thus the $\mathbb{G}_{m,R}$-action on $X_R \times U_R^n$ is BB-complete for every n. Also, $(X_R \times U_R^n)^0 \simeq X_R^0$ is proper. Finally, since U_R^n is isomorphic to the affine space, the Bialynicki-Birula strata still satisfy the conditions in 4.4.2.4. Consequently, Proposition 4.4.2.8 applies to $X_R \times U_R^n$ for all n; we get that $[(X_R \times U_R^n)/h(\mathbb{G}_{m,R})]$ is cohomologically proper. By Lemma 4.4.2.11 it follows that $[(X_R \times U_R^n)/T_R]$ is cohomologically proper for all n. By Proposition 4.3.2.12 we get that $[X_R/B_R]$ is cohomologically proper. □

To pass from B_R to G_R we use the Kempf's theorem (4.4.0.2):

Lemma 4.4.2.13. *Let X_R be as above. Then*

$$[X_R/B_R] \text{ is cohomologically proper over } R \quad \Rightarrow \quad [X_R/G_R] \text{ is cohomologically proper over } R.$$

Proof. Let $j\colon BB_R \to BG_R$ be the natural morphism. Then by the projection formula

$$R\Gamma(BB_R, j^*\mathcal{F}) \simeq R\Gamma(BG_R, j_*j^*\mathcal{F}) \simeq R\Gamma(BG_R, \mathcal{F} \otimes j_*\mathcal{O}_{BB_R}).$$

By base change, the underlying complex of R-modules of $j_*\mathcal{O}_{BB_R}$ is equivalent to $R\Gamma(G_R/B_R, \mathcal{O}_{G_R/B_R})$. But $G_R/B_R \simeq (G/U_G)_R/(p_R(B_R))$, where $p_R(B_R) \subset (G/U_G)_R$ is a Borel subgroup and thus $R\Gamma(G_R/B_R, \mathcal{O}_{G_R/B_R}) \simeq R$ by Theorem 4.4.0.2. Consequently, $R\Gamma(BG_R, \mathcal{F}) \simeq R\Gamma(BB_R, j^*\mathcal{F})$ for

[13] The isomorphism is given by the formula $[(x, b_1, \ldots, b_n)] \mapsto [(x, u_1, \ldots, u_n)]$, where $b_i = t_i \cdot u_i \in T_R \ltimes U_R$.

any sheaf $\mathcal{F} \in \mathrm{QCoh}(BG_R)$. We now apply this as follows: there is a fibered square

$$\begin{array}{ccc} [X_R/B_R] & \xrightarrow{f} & [X_R/G_R] \\ \downarrow & & \downarrow \\ BB_R & \xrightarrow{j} & BG_R \end{array}$$

and, given a coherent sheaf $\mathcal{F} \in \mathrm{Coh}([Y_R/G_R])$, its pull-back $f^*\mathcal{F} \in \mathrm{Coh}([Y_R/B_R])$ is also coherent. Applying base change and the above isomorphism we get that $R\Gamma([Y_R/G_R], \mathcal{F}) \simeq R\Gamma([Y_R/B_R], f^*\mathcal{F})$; in particular, one complex is nearly coherent if and only if the other one is. The statement of the lemma follows. □

It remains to cover the case of a disconnected G. We can write $[X/G] \simeq [[X/G^0]/\pi_0(G)]$, where G^0 is the connected component of $e \in G$ and $\pi_0(G)$ is the finite group of components. The homomorphism $p\colon G \to \pi_0(G) \simeq G/G^0$ can be spread out to $p_R\colon G_R \to \pi_0(G)_R$ where G_R is some spreading out of G and $\pi_0(G)_R$ is the constant group R-scheme associated to $\pi_0(G)$. Moreover the kernel G^0_R of p_R is a spreading of G^0.

We have just shown that the quotient stack $[X_R/G^0_R]$ is cohomologically proper over R. We also have $[X_R/G_R] \simeq [[X_R/G^0_R]/\pi_0(G)_R]$. It follows that for any $\mathcal{F} \in \mathrm{Coh}([X/G_R])$

$$R\Gamma([X_R/G_R], \mathcal{F}) \simeq R\Gamma\left(B\pi_0(G)_R, R\Gamma([X_R/G^0_R], \mathcal{F})\right).$$

Replacing R with $R[1/|\pi_0(G)|]$ we can assume that $|\pi_0(G)|$ is invertible in R, and so the functor of $\pi_0(G)$-invariants is t-exact. Then we get $H^q([X_R/G_R], F) \simeq H^q([X_R/G^0_R], \mathcal{F})^{\pi_0(G)}$. In particular, $R\Gamma([X_R/G_R], \mathcal{F})$ is t-bounded and its cohomology are finitely generated over R, so $R\Gamma([X_R/G_R], \mathcal{F}) \in \mathrm{Coh}^+(R)$. □

We end this subsection by giving some examples to which Theorem 4.4.2.10 does apply:

Example 4.4.2.14. 1. X is proper. In this case by the valuative criterion of properness every $\mathbb{G}_m$-orbit of an F-point has a limit as $t \to 0$, so the map $X^+ \to X$ is surjective on F-points. It follows that any $\mathbb{G}_m$-action on X is BB-complete. Moreover $X^0 \subset X$ is a closed subscheme and so is proper. Thus, the only condition to check is on G: namely there should exist $h \in X_*(T)$ such that $h(\Phi^+) > 0$ (since all Borel subgroups of G are conjugate to each other this does not depend on the choice of B). Here is a list of linear algebraic groups G which satisfy this:

- G reductive. Then one can take $h \in X_*(T)$ given by any dominant coweight. This case is also covered by Theorem 4.4.1.3;
- $G = P \subset H$ is a parabolic subgroup of a reductive group H. Any h that is dominant with respect to some Borel subgroup $B \subset P$ applies;
- More or less tautologically, any G with a 1-dimensional subtorus $\mathbb{G}_m \subset G$ such that the adjoint action of $\mathbb{G}_m$ on the the Lie algebra $\mathfrak{u}_G$ of the unipotent radical $U_G \subset G$ has strictly positive weights and such that the projection of $\mathbb{G}_m$ to G/U_G gives a regular coweight (meaning its centralizer is given by a maximal torus). Then one picks B as the preimage of a Borel subgroup of G/U_G, with respect to which the $\mathbb{G}_m$ above gives a dominant coweight, under the projection $G \twoheadrightarrow G/U_G$ and take h given by any lifting $\mathbb{G}_m \to B$. As a non-parabolic example of such G one can take any semidirect product $\mathbb{G}_m \ltimes U$ where U is unipotent and $\mathbb{G}_m$ acts on $\mathfrak{u}$ with strictly positive weights.

2. There are also natural examples that are more in the spirit of Theorem 4.4.1.3. Namely, let $\pi\colon X \to Y$ be a proper G-equivariant morphism where Y is not necessarily smooth. Then, given a cocharacter $h\colon \mathbb{G}_m \to G$ that satisfies $h(\Phi^+) > 0$ for some $B \subset G$, we have that if $Y^+ \twoheadrightarrow Y$ is a surjection and Y^0 is proper, X satisfies the conditions of Theorem 4.4.2.10. Indeed the induced map $X^0 \to Y^0$ is proper and so X^0 is proper. Also, given any point $x \in X(F)$, the image $\pi(\mathbb{G}_m \cdot x)$ of its orbit is the orbit $\mathbb{G}_m \cdot \pi(x)$. Since $Y^+ \twoheadrightarrow Y$ is a surjection, the limit $\lim_{t\to 0} t \circ \pi(x)$ exists. This gives a diagram

$$\begin{array}{ccc} \mathbb{G}_m & \xrightarrow{\cdot\circ x} & X \\ \downarrow & \nearrow & \downarrow \\ \mathbb{A}^1 & \xrightarrow{\cdot\circ\pi(x)} & Y \end{array}$$

and, due to the valuative criterion of properness, the lifting $\mathbb{A}^1 \dashrightarrow X$, this way producing the limit of $t \circ x$ as $t \to 0$. We get that $X^+ \to X$ is a surjection on F-points and that the $\mathbb{G}_m$-action given by h is BB-complete.

This applies, in particular, to the case when $Y^0 \simeq \operatorname{Spec} F$ and $G = \mathbb{G}_m$ (where we can assume $h = \mathrm{id}\colon \mathbb{G}_m \to \mathbb{G}_m$). In this case we basically arrive at the definition of a conical resolution (see e.g. [77]). Namely, we have $Y \simeq Y^+ \to \operatorname{Spec} F$ is affine, so $Y \simeq \operatorname{Spec} A$; the induced $\mathbb{Z}$-grading on A is such that $A \simeq A^{\leq 0}$ and $A^0 \simeq F$[14]. The map $\pi\colon X \to \operatorname{Spec} A$ is proper, X is smooth and the

[14]Note the change of sign in the grading compared to [77]. In the case of a commutative group action there are

$\mathbb{G}_m$-action on X agrees with the grading on A. The geometry of such X is the following: it is not proper itself, but it has a proper $\mathbb{G}_m$-equivariant map to $\operatorname{Spec} A$ so that the $\mathbb{G}_m$-action contracts it to the central fiber $\pi^{-1}(Y^0)$ which is proper over F. Note that even if X is smooth, $\pi^{-1}(Y^0)$ can be singular (for example in the case of the minimal resolution of the A_n-singularity for $n > 2$).

4.4.3 Relation to the work of Teleman

In [108] Teleman showed the degeneration of the Hodge-to-de Rham spectral sequence for the quotient stacks $[X/G]$ with G reductive under the condition that the action on X is KN-complete (see Section 1 of *loc.cit.* for the definition of a KN-complete action and Section 7 for the proof of degeneration). In this section we comment on how one could deduce his results (in a slightly more general form) from Theorem 4.2.2.2. Let us briefly recall what a KN-stratification is: first of all, it is a stratification

$$X = X^{\mathrm{ss}} \cup \bigcup_i S_i$$

by locally closed G-invariant subschemes. For each i there exists a one-parameter subgroup $\lambda_i \colon \mathbb{G}_m \to G$; let $L_i \subset G$ be the centralizer of $\lambda_i(\mathbb{G}_m)$. Then there exists an open variety $Z_i \subset X^{\lambda_i(\mathbb{G}_m)}$ and the KN-stratum S_i is defined by the G-span $G \cdot Y_i$ of

$$Y_i \coloneqq \{x \in X | \lim_{t\to 0} \lambda_i(t) \cdot x \in Z_i\}.$$

Z_i is called the center of S_i. Also the natural action map $G \times Y_i \to X$ should induce an isomorphism $S_i \simeq G \overset{P_i}{\times} Y_i$, where $P_i \subset G$ is the (parabolic) subgroup of elements $p \in G$ for which the limit of $\lambda_i(t) p \lambda_i(t)^{-1}$ as $t \to 0$ exists. In the case when a KN-stratification comes from GIT the i-th center $Z_i \subset X^{\lambda_i(\mathbb{G}_m)}$ is the semistable locus of the action of $L_i' \coloneqq L_i/\lambda_i(\mathbb{G}_m)$ on $X^{\lambda_i(\mathbb{G}_m)}$; note that in this case the L_i-action on Z_i is automatically locally linear and thus Theorem 4.4.1.3 applies to $\mathcal{Z}_i \coloneqq [Z_i/L_i]$. We will assume further on that we are in this setting.

Definition 4.4.3.1. A KN-stratification is called *complete* if the categorical quotients $X^{\mathrm{ss}}//G$ and $Z_i//L_i'$ are projective. It is called *locally linear* if the action of L_i on Z_i (including the action of G on X^{ss}) is locally linear for all i.

Similarly to how we have spread out the Bialynicki-Birula stratification in Section 4.4.2, it

two natural left actions on the space of functions on Y, induced either by the action of g or g^{-1} on Y. This is exactly the difference we are facing here.

is not hard to see that one can spread X, λ_i, P_i and the Kempf-Ness stratification, preserving the properties $(S1)$-$(S3)$, (A) and $(L+)$ (of Section 2 of [53]). Then, similarly to Section 4.4.2, one can try to reduce the question of cohomological properness of the spreading $[X_R/G_R]$ to the cohomological properness of the individual KN-strata. However, in this more general situation, the proofs of [53] need some modification to work over non-characteristic 0 base. Nevertheless, the statement still holds in this generality: we were informed by Halpern-Leistner that the needed result over any locally Noetherian base (in a more general context of a Θ-stratification) is going to appear in his upcoming work [52].

Let $\mathcal{X} \coloneqq [X/G]$, $\mathcal{X}^{\mathrm{ss}} \coloneqq [X^{\mathrm{ss}}/G]$ and $\mathcal{S}_i \coloneqq [S_i/G]$. Let $\mathcal{X}_R \coloneqq [X_R/G]$, $\mathcal{X}_R^{\mathrm{ss}} \coloneqq [X_R^{\mathrm{ss}}/G_R]$ and $\mathcal{S}_{i,R} \coloneqq [S_{i,R}/G_R]$ be their smooth spreadings, such that $\{\mathcal{S}_{i,R}\}$ form a Kempf-Ness stratification over R in the sense of Section 2 of [53].

Claim 4.4.3.2. *Pick a closed KN-stratum $\mathcal{S}_R \subset \mathcal{X}_R$. Then one has a semiorthogonal decomposition*

$$\mathrm{Coh}(\mathcal{X}_R) = \langle\, \mathrm{Coh}_{\mathcal{S}_R}(\mathcal{X}_R)_{<w}, \mathrm{G}_w, \mathrm{Coh}_{\mathcal{S}_R}(\mathcal{X}_R)_{\geq w} \,\rangle \tag{4.4.3.1}$$

where $\mathrm{Coh}_{\mathcal{S}_R}(\mathcal{X}_R)_{\geq w} \subset \mathrm{Coh}_{\mathcal{S}_R}(\mathcal{X}_R)$ and $\mathrm{Coh}_{\mathcal{S}_R}(\mathcal{X}_R)_{<w} \subset \mathrm{Coh}_{\mathcal{S}_R}(\mathcal{X}_R)$ are certain full subcategories defined similarly to (4.4.3.1) (see Definition 2.8 of [53]), $\mathrm{G}_w \subset \mathrm{Coh}(\mathcal{X}_R)$ is a certain (full) subcategory such that the functor $i_R^*\colon \mathrm{Coh}(\mathcal{X}_R) \to \mathrm{Coh}(X_R^{\mathrm{ss}})$ restricted to G_w

$$i_R^*|_{\mathrm{G}_w} : \mathrm{G}_w \xrightarrow{\sim} \mathrm{Coh}(\mathcal{X}_R^{\mathrm{ss}})$$

is an equivalence.

Assuming Claim 4.4.3.2, similarly to the arguments in Propositions 4.4.2.6 and 4.4.2.8, one can reduce the cohomological properness of $\mathcal{X}_R$ to the cohomological properness of the semistable part $\mathcal{X}_R^{\mathrm{ss}}$ (which, under the completeness and local linearity assumption, is covered by Theorem 4.4.1.3) and the individual strata $\mathcal{S}_i$. For a given stratum $\mathcal{S}_R$ one can use the infinite semiorthogonal decomposition

$$\mathrm{Coh}(\mathcal{S}_R) = \langle \ldots, \mathrm{Coh}(\mathcal{Z}_R)_w, \mathrm{Coh}(\mathcal{Z}_R)_{w+1}, \ldots \rangle,$$

where $\mathrm{Coh}(\mathcal{Z}_R) \simeq \bigoplus_{w\in\mathbb{Z}} \mathrm{Coh}(\mathcal{Z}_R)_w$ is the decomposition corresponding to the weigths of the 1-parameter subgroup $\lambda\colon \mathbb{G}_{m,R} \to G_R$ that acts trivially on $\mathcal{Z}_R$. The inclusion $\mathrm{Coh}(\mathcal{Z}_R)_w \subset \mathrm{Coh}(\mathcal{S}_R)$ is just given by π^* for the natural map $\pi\colon \mathcal{S}_R \to \mathcal{Z}_R$ induced by the map $Y_R \simeq Y_R^+ \to Y_R^0 \simeq Z_R$

via the $\mathbb{G}_{m,R}$-action provided by λ. By a similar argument to Lemma 4.4.2.7 one can show that $\mathrm{Hom}(\pi^*\mathcal{F}, \pi^*\mathcal{G}) \in \mathrm{Coh}^+(R)$, provided $\mathcal{F}, \mathcal{G} \in \mathrm{Coh}(\mathcal{Z}_R)$ and $\mathcal{Z}_R$ is cohomologically proper (which, under the local linearity assumption, is again covered by Theorem 4.4.1.3). Thus, due to the semiorthogonal decomposition above, it follows that $\mathrm{Coh}(\mathcal{S}_R)$ is a nearly proper R-linear stable ∞-category, and that, consequently, $\mathcal{X}_R$ is cohomologically proper.

Remark 4.4.3.3. More generally, we expect that following a similar strategy (and the upcoming work [52]) one can show that any Θ-stratified stack, for which the "fixed points" of the Θ-strata are cohomologically properly spreadable, is also cohomologically properly spreadable. This would provide a generalization of the Hodge-to-de Rham degeneration result to stacks that are not necessarily of the form $[X/G]$.

Remark 4.4.3.4. In [54] the non-commutative Hodge-to-de Rham degeneration was proved for a sligthly more general definition of a KN-complete stratification: namely one does not need to assume that the L_i-action on Z_i is locally linear, only that there exists a good quotient $q\colon [Z_i/L_i] \to Z_i//L_i$. In this case the strata are not necessarily covered by Theorem 4.4.1.3 (and the above strategy) but we still hope that they could be cohomologically properly spreadable. More generally, it would be interesting to answer the following question:

Question 4.4.3.5. Let $q\colon \mathcal{Y} \to Y$ be a good moduli space (e.g. see the Definition in Section 1.2 of [1]) and assume that Y is a proper algebraic space. Is it true that $\mathcal{Y}$ is cohomologically properly spreadable?

The notion of a good moduli space does note spread out well unless the stabilizers are *nice*, i.e. extension of a finite group by a torus. Thus we think it would be very interesting to understand if the property of having a good proper moduli space in characteristic 0 implies any cohomological properness for its mixed characteristic spreading (like it happens in the case of BG for a reductive G).

Motivated by Questions 1.3.2 and 1.3.3 of [55], one can also ask the following:

Question 4.4.3.6. Let $\mathcal{X}$ be a formally proper stack (in the sense of Definition 1.1.3 of [55]) over F. Is $\mathcal{X}$ cohomologically properly spreadable?

Appendix A

Appendix A

A.1 (Co)-simplicial objects and descent

In this section we will collect standard notions

A.1.1 (Co-)simplicial objects

We will use the following (standard) notations

Notation A.1.1.1. Let $X_\bullet\colon \Delta^{op} \to \mathscr{C}$ be a simplicial object. We will denote $\operatorname{colim}_{\Delta^{op}} X_\bullet$ by $|X_\bullet|$ and will call it *geometric realization of* $X_\bullet$. Analogously, if $Y^\bullet\colon \Delta \to \mathscr{C}$ is a co-simplicial object then we will denote $\lim_\Delta Y^\bullet$ by $\operatorname{Tot} Y^\bullet$ and will call it *totalization of* $Y^\bullet$.

Recall the definition of a split simplicial object [83, Section 4.7.3].

Remark A.1.1.2. Split simplicial object $X_\bullet \to X_{-1}$ is always a colimit diagram, i.e. the canonical arrow $|X_\bullet| \to X_{-1}$ is an equivalence. One of the most important properties of split simplicial objects is that they are preserved by *any* functor, in particular for any functor f we have $|f(X_\bullet)| \xrightarrow{\sim} f(X_{-1})$.

Example A.1.1.3. Let C be an ordinary additive category. Then under Dold-Kan equivalence $C^{\Delta^{op}} \simeq \operatorname{Comp}^{\geq 0}(C)$ split simplicial objects in C correspond to split complexes of objects of C.

Example A.1.1.4. Let $f\colon B \to A$ be a morphism. Then Čech nerve $\check{C}(f)_\bullet$ is an augmented simplicial object of $\mathscr{C}$. If the morphism f admits a section, then $\check{C}(f)_\bullet$ admits a splitting. In particular the canonical map

$$|\check{C}(f)_\bullet| \to A$$

is an equivalence.

The following result about commutation of left t-exact functors and totalization of bounded below co-simplicial object will be useful

Proposition A.1.1.5. *Let $\mathscr{C}$ be a stable category with a t-structure. Let $X^\bullet$ be a co-simplicial object of $\mathscr{C}$ such that each term X^i lies in $\mathscr{C}^{\geq 0}$. Then the natural map $\mathrm{Tot}(X^\bullet) \to \mathrm{Tot}^{\leq n}(X^\bullet)$ induces equivalence on $\tau^{\leq n}$.*

Proof. The fiber $\mathrm{Tot}^{n+1}(X^\bullet)$ of $\mathrm{Tot}^{\leq n+1}(X^\bullet) \to \mathrm{Tot}^{\leq n}(X^\bullet)$ is a retract of $\Omega^{n+1}X^{n+1}$. Since $X^{n+1} \in \mathscr{C}^{\geq 0}$ we find that $\mathrm{Tot}^{n+1}(X^\bullet) \in \mathscr{C}^{>n}$. It follows the fiber F_n of the natural map $\mathrm{Tot}(X^\bullet) \to \mathrm{Tot}^{\leq n}(X^\bullet)$ admits a decreasing filtration with associated graded pieces $\mathrm{Tot}^i(X^\bullet), i > n$ from $\mathscr{C}^{>n}$. Since $\mathscr{C}^{>n}$ is closed under extensions and limits, $F_n \in \mathscr{C}^{>n}$, hence vanish after applying $\tau^{\leq n}$. □

Definition A.1.1.6. Let $\mathscr{C}$ be a category with a t-structure. A co-simplicial object $X^\bullet \in \mathscr{C}^\Delta$ is called *uniformly bounded below* if there exists $N \in \mathbb{N}$ such that $\Sigma^N X^i \in \mathscr{C}^{\geq 0}$ for all i.

Corollary A.1.1.7. *Let $f\colon \mathscr{C} \to \mathscr{D}$ be a left t-exact functor between stable categories equipped with t-structures. Then if $\mathscr{D}$ is right t-complete, the natural map $f(\mathrm{Tot}\, X^\bullet) \to \mathrm{Tot}\, f(X^\bullet)$ is an equivalence for any uniformly bounded below co-simplicial object $X^\bullet$ of $\mathscr{C}$.*

Proof. By shifting if necessary we can assume that all X^i lie in $\mathscr{C}^{\geq 0}$. Then by the previous proposition the natural map $\mathrm{Tot}(X^\bullet) \to \mathrm{Tot}^{\leq n}(X^\bullet)$ is n-coconnective. Since f is left t-exact, the same holds for $f(\mathrm{Tot}(X^\bullet)) \to f(\mathrm{Tot}^{\leq n}(X^\bullet)) \simeq \mathrm{Tot}^{\leq n}(f(X^\bullet))$. It follows that the limit map $f(\mathrm{Tot}(X^\bullet)) \to \mathrm{Tot}(f(X^\bullet))$ is ∞-coconnective and hence is an equivalence by assumption on $\mathscr{D}$. □

Appendix B

Appendix B

B.1 Hodge cohomology of $\mathbb{P}^n$ over $\mathbb{Z}$

In this section we compute the Hodge cohomology ring of $\mathbb{P}^n$ over the ring of integers $\mathbb{Z}$.

Lemma B.1.1.

$$H^i(\mathbb{P}^n, \Omega^j_{\mathbb{P}^n/\mathbb{Z}}) \simeq \begin{cases} \mathbb{Z} & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Proof. This follows from the Euler exact sequence:

$$0 \to \Omega^1_{\mathbb{P}^n/\mathbb{Z}} \to \mathcal{O}_{\mathbb{P}^n}(-1)^{\oplus n+1} \to \mathcal{O}_{\mathbb{P}^n} \to 0.$$

One has $H^i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(-1)) \simeq 0$ for all i and so $H^i(\mathbb{P}^n, \Omega^j_{\mathbb{P}^n/\mathbb{Z}}) \simeq H^{i-1}(\mathbb{P}^n, \Omega^{j-1}_{\mathbb{P}^n/\mathbb{Z}}) \simeq \ldots \simeq H^{i-j}(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n/\mathbb{Z}})$. It remains to note that

$$H^i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n/\mathbb{Z}}) = \begin{cases} \mathbb{Z} & \text{if } i = 0, \\ 0 & \text{if } i \neq 0. \end{cases}$$

□

From the lemma we get that

$$H^*_{\mathrm{H}}(\mathbb{P}^n/\mathbb{Z}) \simeq \bigoplus_{i+j=\bullet} H^i(\mathbb{P}^n, \Omega^j_{\mathbb{P}^n/\mathbb{Z}}) \simeq \bigoplus_{i=0}^{n} \mathbb{Z}[-2i].$$

This gives the computation of Hodge cohomology as a (graded) abelian group, but we would also

like to compute them as a (graded) algebra.

Lemma B.1.2. *One has an isomorphism of graded algebras* $H^*_{\mathrm{H}}(\mathbb{P}^n/\mathbb{Z}) \simeq \mathbb{Z}[y]/(y^{n+1})$, *where* $\deg(y) = 2$.

Proof. Since we have $H^\bullet_{\mathrm{H}}(\mathbb{P}^n/\mathbb{Z}) \simeq \bigoplus_{i=0}^n \mathbb{Z}[-2i]$ it will be enough to find a class $y \in H^2_{\mathrm{H}}(\mathbb{P}^n/\mathbb{Z})$ such that its n-th power $y^n \in H^{2n}_{\mathrm{H}}(\mathbb{P}^n/\mathbb{Z}) \simeq \mathbb{Z}$ is a generator.

For the computation we will use the reduced Čech complex associated to the standard Zariski cover of $\mathbb{P}^n$ given by the open subsets $U_i = \{(z_0 : \ldots : z_n) \in \mathbb{P}^n \mid z_i \neq 0\} \simeq \mathbb{A}^n$, $i = 0, \ldots, n$. We also have the intersections $U_{i_1,\ldots,i_k} = U_{i_1} \cap \ldots \cap U_{i_k}$ and the reduced Čech complex $C^{red}(U_\bullet, \mathcal{F})$ is given by

$$0 \to \oplus_i \mathcal{F}(U_i) \to \oplus_{i_1<i_2} \mathcal{F}(U_{i_1,i_2}) \to \cdots \to \mathcal{F}(U_{0,1,\ldots,n}) \to 0$$

where the differentials are given by the signed sums of the restriction maps. We have $U_i \simeq \mathbb{Z}[\frac{z_0}{z_i}, \ldots, \frac{z_n}{z_i}]$ and more generally $U_{i_1,\ldots,i_k} \simeq \mathbb{Z}[\frac{z_0}{z_{i_1}}, \ldots, \frac{z_n}{z_{i_1}}, \frac{z_0}{z_{i_2}}, \ldots, \frac{z_n}{z_{i_2}}, \ldots, \frac{z_n}{z_{i_k}}]$. Given an (ordered) subset $\{i_1, \ldots, i_k\} \subset \{0, \ldots, n\}$ we put $s_1, \ldots, s_n$ to be the functions $\frac{z_0}{z_{i_1}}, \ldots, \frac{z_n}{z_{i_1}}$ (excluding $\frac{z_{i_1}}{z_{i_1}} = 1$), then $U_{i_1,\ldots,i_k} \simeq \mathbb{Z}[s_1, \ldots, s_n, s_{i_2}^{-1}, \ldots, s_{i_k}^{-1}]$.

First, we consider the case $\mathcal{F} = \Omega^n$. We have a torus $T = \mathbb{G}_m^{n+1}$ acting on $\mathbb{P}^n$ rescaling the coordinates, the cover $U_\bullet$ is preserved by the action of T and the restriction maps are T-equivariant. Given $U_{i_1,\ldots,i_k}$ the coordinate functions s_k are homogeneous: if $t = (t_0, \ldots, t_n) \in T$, then $t \circ s_k = \frac{t_k}{t_{i_1}} s_k$, and the corresponding characters are linearly independent. Each $\Omega^n(U_{i_1,\ldots,i_k})$ has an eigenbasis given by monomial forms $s^\alpha ds_1 \wedge \cdots \wedge ds_n$ where the multi-index α takes values in $\mathbb{Z}$ for indices $i_1, \ldots, i_k$ and in $\mathbb{N}$ for the others. The only way such a form is T-invariant is when $s^\alpha = s_1^{-1} \cdots s_n^{-1}$ which we have only for $U_{0,1,\ldots,n}$. The form $s_1^{-1} \cdots s_n^{-1} ds_1 \wedge \cdots \wedge ds_n$ can be rewritten as $d\log s_1 \wedge \cdots \wedge d\log s_n$. The complex $C^{red}(U_\bullet, \mathcal{F})$ splits as a direct sum over the characters of T. In particular we get a map

$$\begin{array}{ccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots & \longrightarrow & \mathbb{Z} \cdot d\log s_1 \wedge \cdots \wedge d\log s_n \longrightarrow 0 \\
 & & \downarrow \sim & & \downarrow \sim & & \downarrow \sim & & \downarrow \sim \\
0 & \longrightarrow & \oplus_i \Omega^n(U_i)^T & \longrightarrow & \oplus_{i_1<i_2} \Omega^n(U_{i_1,i_2})^T & \longrightarrow & \cdots & \longrightarrow & \Omega^n(U_{0,1,\ldots,n})^T \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \oplus_i \Omega^n(U_i) & \longrightarrow & \oplus_{i_1<i_2} \Omega^n(U_{i_1,i_2}) & \longrightarrow & \cdots & \longrightarrow & \Omega^n(U_{0,1,\ldots,n}) \longrightarrow 0
\end{array}$$

which is an embedding of a direct summand on cohomology. Since we know $H^n(\mathbb{P}^n, \Omega^n) = \mathbb{Z}$ this

map needs to be an isomorphism. Thus the class of the form $d\log s_1 \wedge \cdots \wedge d\log s_n$ on $U_{0,1,\ldots,n}$ is a generator for $H^n(\mathbb{P}^n, \Omega^n)$.

Now let's deal with $\mathcal{F} = \Omega^1$. Our candidate for $y \in H^1(\mathbb{P}^1, \Omega^1)$ will be the first Chern class c_1 of $\mathcal{O}_{\mathbb{P}^n}(1)$. The standard Čech cocycle representing $[\mathcal{O}_{\mathbb{P}^n}(1)] \in H^1(\mathbb{P}^n, \mathcal{O}^\times_{\mathbb{P}^n})$ is given by the choice of functions $\frac{z_{i_1}}{z_{i_2}}$ on $U_{i_1 i_2}$, the class $c_1(\mathcal{O}_{\mathbb{P}^n}(1)) \in H^1(\Omega^1)$ is then obtained by applied the $d\log$ map. So let's choose y to be class given by the Čech cocycle $(d\log(\frac{z_{i_1}}{z_{i_2}}))$ on $U_{i_1 i_2}$. Its n-th power map $H^1(\mathbb{P}^n, \Omega^1) \to H^n(\mathbb{P}^n, \Omega^n)$ can be computed on the level of Čech cocycles as $(\omega_{i_1 i_2})_{i_1 i_2} \mapsto \omega_{01} \wedge \omega_{12} \wedge \cdots \wedge \omega_{n-1n}$, so we get that

$$\begin{aligned} y^n &= \left[d\log \frac{z_0}{z_1} \wedge d\log \frac{z_1}{z_2} \wedge \cdots \wedge d\log \frac{z_{n-1}}{z_n}\right] = \left[d\log s_1 \wedge d\log \frac{s_2}{s_1} \wedge d\log \frac{s_3}{s_2} \wedge \cdots \wedge d\log \frac{s_n}{s_{n-1}}\right] = \\ &= [d\log s_1 \wedge (d\log s_2 - d\log s_1) \wedge (d\log s_3 - d\log s_2) \wedge \cdots \wedge (d\log s_n - d\log s_{n-1})] = \\ &= [d\log s_1 \wedge \cdots \wedge d\log s_n] \end{aligned}$$

This finishes the proof. □

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