Convergence of Complete Ricci-flat Manifolds

by

Jiewon Park

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Author ....
Department of Mathematics
April 17, 2020

Certified by ....
Tobias Holck Colding
Cecil and Ida Green Distinguished Professor
Thesis Supervisor

Accepted by ....
Wei Zhang
Chairman, Department Committee on Graduate Theses

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Abstract

This thesis is focused on the convergence at infinity of complete Ricci flat manifolds. In the first part of this thesis, we will give a natural way to identify between two scales, potentially arbitrarily far apart, in the case when a tangent cone at infinity has smooth cross section. The identification map is given as the gradient flow of a solution to an elliptic equation. We use an estimate of Colding-Minicozzi of a functional that measures the distance to the tangent cone. In the second part of this thesis, we prove a matrix Harnack inequality for the Laplace equation on manifolds with suitable curvature and volume growth assumptions, which is a pointwise estimate for the integrand of the aforementioned functional. This result provides an elliptic analogue of matrix Harnack inequalities for the heat equation or geometric flows.

Thesis Supervisor: Tobias Holck Colding
Title: Cecil and Ida Green Distinguished Professor

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Contents

Chapter 1

Introduction

This thesis focuses on the convergence at infinity of complete Ricci-flat manifolds of maximal volume growth. Manifolds with bounds on Ricci curvature have been widely studied in geometric analysis and other fields, as they naturally appear in the study of Ricci flow, general relativity, Kähler and Sasaki geometry, and string theory.

In order to understand the class of manifolds with bounds on Ricci curvature, it is important to study what kind of singular metric spaces can arise as limits of such manifolds (the limits are usually taken under Gromov-Hausdorff topology, as it is well-suited to the study of Ricci curvature [19]). Determining the structure of such limit spaces has been an active direction of research for decades. In particular, the *tangent cone* is an example of a limit space, which is obtained by either (1) scaling down a noncompact manifold to study its asymptotic behavior at infinity, or (2) zooming into a singularity to study a small neighborhood of it. It is a striking fact that the tangent cones are, in general, not unique; one might see different tangent cones at different scales. The uniqueness of the tangent cone is an important theme in geometric analysis. In [14], Colding-Minicozzi proved the uniqueness of the tangent cone at infinity of a complete Ricci-flat manifold with maximal volume growth, under only the assumption of the existence of a smooth tangent cone, building on Cheeger-Colding theory [5, 6]. In [29] the author gave a strengthening of Theorem 3.1.7 by showing that there is an essentially canonical way of identifying any two scales even when they are very different. The identification itself is given as the gradient flow of a solution to an elliptic equation and thus, in particular, is a diffeomorphism. One of the key ingredients is the fast decay of a functional that measures the distance to the tangent cone, which follows from the Łojasiewicz inequality in [14]. It is of independent interest to obtain a pointwise estimate for the integrand of this functional, which then yields a Harnack inequality by integrating along geodesics. In [28] the author proved a pointwise estimate under stronger assumptions on curvature. This work mirrors Hamilton's matrix inequality [20] for the heat equation, adapting the techniques to the relevant elliptic equation.

In Chapter 2, we will give a brief overview of Cheeger-Colding theory. While the full theory contains numerous important results, we will collect only some of them which are directly relevant to this thesis. Among them is the result that the existence of an almost warping function implies almost splitting. When Ricci curvature and volume are bounded from below suitably, the almost warping function is obtained by solving the Dirichlet problem on annuli. In particular, the integral estimates on the almost warping function controls the distance to the tangent cone directly; see Theorem 2.0.13.

In Chapter 3, we will introduce and prove that two arbitrary scales can be identified via a diffeomorphism. The identification is through a conformal change that brings the metric in a cylindrical form. This conformal change is designed so that the (corollary of) Łojasiewicz inequality of Colding-Minicozzi can be applied. We will briefly discuss how one might avoid the conformal change given a Hardy-Sobolev inequality.

In Chapter 4, we will discuss a sharp matrix Harnack inequality for the Laplace equation. This inequality is the pointwise bound for the integrand in the aforementioned functional which governs the convergence at infinity. This result can be seen as an elliptic analogue of Hamilton's inequality for the heat equation [20]. As a corollary, a Harnack inequality for the Green function is obtained. It is worthwhile to point out that one can also take the trace to obtain a gradient estimate for the Green function, although Colding [11] proved a stronger gradient estimate with weaker assumptions on the curvature. The proof is through a matrix maximum principle. In comparison with [20], the distance from a point r in this setting replaces the role of the time t for the heat equation.

Chapter 2

Preliminaries on convergence for Ricci curvature

Among the topologies on spaces of Riemannian manifolds, or more generally, spaces of metric spaces, the Gromov-Hausdorff topology is one of the most widely used in geometric analysis and metric geometry. In 1981, Gromov showed the compactness theorem for manifolds with Ricci curvature bounded below.

Theorem 2.0.1 ([19]). *If (M_i, g_i, p_i) is a sequence of complete pointed Riemannian manifolds such that $\mathrm{Ric}_{g_i} \geq \kappa g_i$ for some fixed $\kappa \in \mathbb{R}$, then there exists a subsequence which converges in Gromov-Hausdorff topology to a metric space.*

The singularities of X and the geometry along a convergent sequence have been actively researched for decades by Cheeger, Colding, Naber, and others [5, 6, 10, 7, 8]. While the literature on this subject is vast, in this chapter we take the opportunity to survey only a part of it that will be needed in later chapters.

The classical rigidity theorems say that if a geometric quantity such as volume or diameter attains the possible maximal value for the given curvature condition, then the metric is a certain warped product. Cheeger-Colding [5] proved quantitative versions of such rigidity theorems when the geometric quantity in question is almost maximal. In other words, they found out when the given manifold is Gromov-Hausdorff close to a warped product with an arbitrary warping function f.

We first discuss the geometry of a warped product. Define the function $\mathcal{F}$ by

$$\mathcal{F}(r) = -\int_r^b f(u)\,du. \tag{2.1}$$

Let (X, d_X) be an arbitrary metric space. The warped product $(a, b) \times_f X$ is the topological space $(a, b) \times X$ equipped with the metric d' given by

$$d'((r_1, x_1), (r_2, x_2)) = \inf \int_0^{d_X(x_1, x_2)} \left(r'(t)^2 + f(r(t))^2\right)^{1/2} dt,$$

where the infimum is taken over continuous paths $r : [0, d_X(x_1, x_2)] \to [r_1, r_2]$ with $r(0) = r_1, r(d_X(x_1, x_2)) = r_2$. Note that $d'((r_1, x_1), (r_2, x_2))$ is determined by r_1, r_2, and $d_X(x_1, x_2)$; it does not matter how x_1, x_2 are configured in X. In other words, for any $x, y \in X$, $d'(x, y)$ does not depend on the choice of the space X or the points x, y once the three numbers are given. This fact allows us to encode the model metric d' by defining the function Q by the following. Suppose that $x_1, x_2 \in X$ and $r_1 < r_2$, $r_3 < r_4$. Then there exists a function Q determined by f such that

$$d'((r_2, x_1), (r_4, x_2)) = Q(r_1, r_2, r_3, r_4, d'((r_1, x_1), (r_3, x_2))). \tag{2.2}$$

Let (M, g, p) be a complete pointed Riemannian manifold. Denote by r the distance from p, and $A_{a,b}$ the annulus $r^{-1}((a, b))$. The following theorems provide an analytic condition for almost rigidity. Namely, if there is an "almost warping" function $\tilde{\mathcal{F}}$, then the distance d induced by g on M is close to d'.

Theorem 2.0.2 ([5], Proposition 2.80, distance estimate along geodesic rays)**.** *Suppose that*

$$\mathrm{Ric}_g \geq (n-1)\Lambda, \tag{2.3}$$

for $\Lambda \in \mathbb{R}$. Let $R > 0$ and $\varepsilon > 0$. There exists $\delta = \delta(R, \varepsilon, \Lambda) > 0$ with the following *effect.*

Let $0 < a < b < R$. *Suppose that there exists* $\tilde{\mathcal{F}} : A_{a,b} \to \mathbb{R}$ *such that*

$$\operatorname{range} \tilde{\mathcal{F}} \subset \operatorname{range} \mathcal{F}, \tag{2.4}$$

$$|\nabla \mathcal{F} - \nabla \tilde{\mathcal{F}}| < \delta, \tag{2.5}$$

$$\frac{1}{\operatorname{vol}(A_{a,b})} \int_{A_{a,b}} |\nabla \mathcal{F} - \nabla \tilde{\mathcal{F}}| < \delta, \tag{2.6}$$

$$\frac{1}{\operatorname{vol}(A_{a,b})} \int_{A_{a,b}} |\mathrm{Hess}_{\tilde{\mathcal{F}}} - (H \circ \tilde{\mathcal{F}}) g| < \delta, \tag{2.7}$$

where $\mathcal{F}$ *is defined on* $A_{a,b}$ *by* $\mathcal{F}(x) = \mathcal{F}(r(x))$ *and* $H = \mathcal{F}'' \circ \mathcal{F}^{-1}$.

Let $x_1, x_2, y_1, y_2 \in B_R(p)$ *be such that*

$$r(y_1) - r(x_1) = d(x_1, y_1), \tag{2.8}$$

$$r(y_2) - r(x_2) = d(x_2, y_2). \tag{2.9}$$

Then,

$$|d(y_1, y_2) - Q\left(r(x_1), r(y_1), r(x_2), r(y_2), d(x_1, x_2)\right)| < \varepsilon. \tag{2.10}$$

If M has nonnegative Ricci curvature and large volume growth, then harmonic functions on annuli can be used to define an almost warping function.

Remark 2.0.3. *In practice,* $H \circ \tilde{\mathcal{F}}$ *in the assumption (2.7) is often replaced by a function that is more trackable, for instance* $\Delta \tilde{\mathcal{F}}/n$. *For most applications where the model warped product that* f *defines is a cone, suspension, etc.,* H *is given explicitly and is easy to work with. Then often it is possible to use the other assumptions (2.4)–(2.6) to compensate the discrepancy resulting from replacing* $H \circ \tilde{\mathcal{F}}$ *with* $\Delta \tilde{\mathcal{F}}/n$, *etc.*

Theorem 2.0.4 ([5], Section 4)**.** *Let $0 < a, b$ and $0 < \omega < 1$. Suppose that*

$$\mathrm{Ric}_g \geq 0, \tag{2.11}$$

$$\frac{\mathrm{vol}(A_{a,b})}{\mathrm{vol}(r^{-1}(a))} \geq (1-\omega)\frac{\int_a^b f^{n-1}(r)\,dr}{f^{n-1}(a)}, \tag{2.12}$$

and

$$(n-1)\frac{f'(a)}{f(a)} \geq \Delta r \quad \text{on } r^{-1}(a). \tag{2.13}$$

Then there exists $\tilde{\mathcal{F}}$ defined on $A_{a,b}$ so that the equations (2.4)–(2.7) in Theorem 2.0.2 are satisfied on $A_{a+\varepsilon,b-\varepsilon}$, where ε can be arbitrarily small if ω is sufficiently small.

$\tilde{\mathcal{F}}$ is defined using f and the solution to a Dirichlet problem on $A_{a,b}$. The integral estimate (2.7) on the Hessian is obtained by using Bochner formula,

$$\frac{1}{2}\Delta\left|\nabla\tilde{\mathcal{F}}\right|^2 = |\mathrm{Hess}_{\tilde{\mathcal{F}}}|^2 + \mathrm{Ric}\left(\nabla\tilde{\mathcal{F}}, \nabla\tilde{\mathcal{F}}\right) + g\left(\nabla\Delta\tilde{\mathcal{F}}, \nabla\tilde{\mathcal{F}}\right), \tag{2.14}$$

then multiplying by a special cutoff function ϕ, and integrating by parts to reduce to lower degree estimates. ϕ is chosen so that $|\nabla\phi|$ and $|\Delta\phi|$ are bounded above, which in turn depend on the assumption of nonnegative Ricci curvature.

Instead of solving the Dirichlet problem on annuli, one can also work on the whole manifold by considering the Green function defined on all of $M\backslash\{p\}$. This is the approach we take later in this thesis.

Definition 2.0.5. *A smooth function $G : M \times M\backslash D \to \mathbb{R}$, where D is the diagonal, is said to be a* Green function *(for the Laplacian) if it is the fundamental solution of the Laplace equation, that is,*

$$\Delta_y G(x,y) = -n(n-2)\omega_n\delta_x(y),$$

where ω_n is the volume of the unit ball in the n-dimensional Euclidean space.

The normalization is chosen so that $G = r^{2-n}$ on $\mathbb{R}^n$.

Definition 2.0.6. *A complete Riemannian manifold M is said to be nonparabolic if it possesses a positive symmetric Green function G for the Laplacian. M is said to be parabolic otherwise.*

If M has nonnegative Ricci curvature, nonparabolicity can be characterized in terms of the volume growth.

Theorem 2.0.7 ([32])**.** *If M is complete and has nonnegative Ricci curvature, and of dimension greater than 2, then M is nonparabolic if and only if*

$$\int_1^\infty \frac{t}{\text{vol}(B_p(t))}\,dt < \infty$$

for some $p \in M$.

In general, for M nonparabolic, a positive Green function G is not unique. However it is possible to choose the minimal one [24].

Now we will focus on the case of maximal volume growth. If M has nonnegative Ricci curvature, then the Bishop-Gromov inequality asserts that

$$\frac{\text{vol}(B_p(r))}{\omega_n r^n}$$

is a monotone non-increasing function of r, so the limit as $r \to \infty$ exists. M is said to have *maximal volume growth* if $\lim_{r\to\infty} \frac{\text{vol}(B_p(r))}{\omega_n r^n} > 0$. Note that if $n \geq 3$, then maximal volume growth implies nonparabolicity by Theorem 2.0.7.

We focus on the convergence at infinity of M to a cone, so we will take the warping function f to be $f(x) = |x|$. Note that if (M, g) is the Euclidean space $\mathbb{R}^n$, then $G = r^{2-n}$. It is very useful to define the function b by

$$b = G^{\frac{1}{2-n}}, \tag{2.15}$$

so that $b = r$ on $\mathbb{R}^n$.

One can take $a \to \infty$ and $b/a \to \infty$, in which case $\tilde{\mathcal{F}}$ approaches G in $\mathcal{C}^\infty(A_{a,b})$.

Theorem 2.0.8 ([5], [12]). *Let $R > 0$ and $\Omega > 1$. Suppose that M has nonnegative Ricci curvature and maximal volume growth, with*

$$\lim_{r\to\infty} \frac{\mathrm{vol}(B_p(r))}{\omega_n r^n} = V_M > 0. \tag{2.16}$$

Then there exists $R_0 > 0$ and $\delta > 0$, depending only on Ω, V_M so that whenever $R > R_0$,

$$\sup_{r\in(R,\Omega R)} \left| \frac{b}{r} - V_M^{\frac{1}{n-2}} \right| < \delta, \tag{2.17}$$

$$\frac{1}{\mathrm{vol}(A_{R,\Omega R})} \int_{A_{R,\Omega R}} \left| |\nabla b| - V_M^{\frac{1}{n-2}} \right| < \delta, \tag{2.18}$$

$$\frac{1}{\mathrm{vol}(A_{R,\Omega R})} \int_{A_{R,\Omega R}} \left| \mathrm{Hess}_{b^2} - \frac{\Delta b^2}{n} g \right| < \delta. \tag{2.19}$$

δ can be arbitrarily small if V_M is sufficiently close to 1.

Thus the conditions of Theorem 2.0.2 (or slight variations of it) are satisfied by b. Then one can estimate the Gromov-Hausdorff distance between an annulus and a discrete approximation of it by picking out points along the geodesic rays to obtain the following theorem. Here, d_{GH} refers to the the pointed Gromov-Hausdorff distance and $C(X)$ denotes the cone $(0, \infty) \times_r X$ over a metric space X.

Theorem 2.0.9 ([5]). *Let $\Omega > 1$ and $\varepsilon > 0$. Suppose that (M, g, p) is complete and has nonnegative Ricci curvature and maximal volume growth, and of dimension $n \geq 3$. Then there exists $R_0 = R_0(\varepsilon, \Omega) > 0$ and a compact metric space X so that, whenever $R > R_0$,*

$$d_{GH}\left((A_{R,\Omega R}, p), (A^{C(X)}_{R,\Omega R}, o) \right) < \varepsilon. \tag{2.20}$$

The diameter of X can be bounded from above by a factor determined by the volume growth and the decay rate of the integrals of the gradient of b and trace-free Hessian of b^2, that is, the left hand sides of equations (2.18)–(2.19).

An important corollary of Theorem 2.0.9 is the following "volume cone is metric cone" theorem. To state it, we first define the tangent cone.

Definition 2.0.10. *Let X be a metric space and $x \in X$.*

1. *A pointed metric space (Y, y) is a* tangent cone *of X at x if there exists a sequence $r_i \to 0$ so that*

$$\lim_{i\to\infty} \frac{d_{GH}\left((B_x^X(r_i), x), (B_y^Y(r_i), y)\right)}{r_i} = 0.$$

2. *A metric space Y is a* tangent cone at infinity *of X if there exists a sequence $r_i \to \infty$ so that*

$$\lim_{i\to\infty} \frac{d_{GH}\left((B_x^X(r_i), x), (B_y^Y(r_i), y)\right)}{r_i} = 0.$$

Obviously the tangent cone at infinity is trivial if X is compact.

Theorem 2.0.11 ([5], [6])**.** *1. Let $r > 0$. Suppose that (M_i, g_i, p_i) is a sequence of complete Riemannian manifolds with*

$$\mathrm{Ric}_{g_i} \geq -(n-1)\Lambda,$$

$$\mathrm{diam}(M_i, g_i) \leq L,$$

and

$$\mathrm{vol}(B_{p_i}^{M_i}(r)) \geq V.$$

Suppose that this sequence possesses the Gromov-Hausdorff limit, which we call (X, d, p). Then the tangent cone at p is a metric cone: it is isometric to a warped product $C(X) = [0, \infty) \times_r X$, where the cross section X is a compact metric space.

2. *Suppose that (M, g) is a complete Riemannian manifold with Ricci curvature bounded below and maximal volume growth. Then the tangent cone at infinity of (M, g) is a metric cone.*

In light of this theorem, one may consider the distance to the nearest cone.

Definition 2.0.12 ([11]). *Let (M, g) be a complete manifold. We define Θ_r to be the scale-free distance to the nearest cone, that is,*

$$\Theta_r = \inf \frac{d_{GH}\left((B_p^M(r), p), (B_o^{C(X)}(r), o)\right)}{r} \tag{2.21}$$

where the infimum is taken over all complete metric space X. (Θ_r is allowed to be ∞ depending on (M, g) and r.)

In particular, the distance to the tangent cone at scale r is bounded above by Θ_r, and converges to Θ_r as r tends to infinity.

The preceding results give that Θ_r can be bounded by the weighted L^2-integral of the trace-free Hessian on scale r. In fact, by carefully tracking how the quantities appearing in Theorems 2.0.2, 2.0.8, and 2.0.9 depend on each other, one can obtain a precise relationship between Θ_r and the integral of the trace-free Hessian.

Theorem 2.0.13 ([11], Theorem 4.7, Corollary 4.8). *Let $\varepsilon > 0$. Let (M, g, p) be a pointed complete manifold of dimension $n \geq 3$ with nonnegative Ricci curvature and maximal volume growth, so that*

$$\lim_{r \to \infty} \frac{\mathrm{vol}(B_p(r))}{\omega_n r^n} = V_M > 0.$$

Then there exists $C = C(\varepsilon, n, V_M)$ so that

$$\Theta_r^{2+\varepsilon} \leq C r^{-n} \int_{b \leq Cr} \left| \mathrm{Hess}_{b^2} - \frac{\Delta b^2}{n} g \right| \, d\mathrm{vol}, \tag{2.22}$$

and

$$\Theta_r^{2+\varepsilon} \leq Cr^{-n} \int_{b \leq Cr} \left| \mathrm{Hess}_{b^2} - \frac{\Delta b^2}{n} g \right|^2 d\mathrm{vol}. \tag{2.23}$$

Chapter 3

Identification of scales on a Ricci-flat manifold

3.1 Uniqueness of tangent cone for Ricci-flat manifolds

Let (M, g) be a complete, non-compact Riemannian manifold of dimension $n \geq 3$ with nonnegative Ricci curvature, with a fixed point $p \in M$. Such manifolds are not only interesting from the point of view of geometric analysis, but they also play an important role in numerous areas of mathematics and physics, including Kähler and Sasaki geometry, general relativity, and string theory.

To study the geometry of M at infinity, we consider a sequence of rescalings $(M, r_i^{-2}g, p)$ with $r_i \to \infty$. We can apply Gromov compactness theorem to obtain a subsequence that converges in the pointed Gromov-Hausdorff topology to a length space. When M has maximal volume growth, by Theorem 2.0.11, any tangent cone is a metric cone: it is isometric to a warped product $C(X) = [0, \infty) \times_r X$, where the cross section X is a compact metric space.

In general, tangent cones may depend on the choice of rescalings $\{r_i\}$; one might see different cones at different scales. There are plenty of constructions in which exactly this phenomenon happens.

Example 3.1.1. *Explicit examples are known of complete manifolds with nonnegative Ricci curvature and maximal volume growth that have non-isometric tangent cones [30, 6], constructed by finding multiply warped products with "oscillating" warping functions.*

Example 3.1.2. *Colding-Naber [16] constructed an example where the tangent cones are not only non-isomorphic, but non-homeomorphic. Their example is a sequence of 5-manifolds converging in pointed Gromov-Hausdorff topology to some (Y, p), where a tangent cone at p has cross section homeomorphic to $\mathbb{CP}^2 \sharp \mathbb{CP}^2$, and another tangent cone has cross section homeomorphic to S^4.*

Remark 3.1.3. *Even in the Ricci-flat case, Hattori [23] constructed a 4-dimensional manifold (with less than maximal volume growth) that has infinitely many non-isometric tangent cones at infinity. Hattori's example is a 4-dimensional hyperKähler manifold of type A_∞, and the isometry classes of tangent cones can be parametrized by S^1.*

Example 3.1.4. *Even in the non-unique case, by Colding-Naber [16] it is known that the tangent cones at a point vary in a Hölder continuous manner as one moves the point along a limit geodesic in the limit space, and that the set of points that have unique tangent cone is convex.*

There are works proving uniqueness of tangent cone under additional assumptions on the curvature, geometry of a tangent cone, algebraicity, etc.

Example 3.1.5. *Cheeger-Tian [9] showed that the tangent cone is unique if (M, g) is Ricci-flat and has maximal volume growth, and a tangent cone is smooth and integrable (which is an assumption on the deformation of the Einstein equation on the cross section).*

Example 3.1.6. *If one considers Kähler manifolds, Donaldson-Sun [17] proved uniqueness of tangent cone at singularities of a limit of Kähler-Einstein manifolds, by heavily using the algebraic structure of the limit space.*

In this chapter we will mainly concern ourselves with the following uniqueness result of Colding-Minicozzi [14].

Theorem 3.1.7 ([14], Theorem 0.2)**.** *Let M^n be a complete non-compact Ricci-flat manifold of maximal volume growth. If one tangent cone at infinity has smooth cross section, then the tangent cone is unique.*

The key ingredient in the proof of theorem 3.1.7 is an infinite-dimensional Łojasiewicz inequality, which implies rapid decay of Θ_r as $r \to \infty$ (where Θ_r was defined in Definition 2.0.12. We will use this rapid decay to show that there is a diffeomorphism on the manifold that identifies any two arbitrarily far apart scales in a natural way. In particular, being a smooth map, this diffeomorphism has improved regularity compared to merely Gromov-Hausdorff identifications via the tangent cone. Keeping the same notations G, b, V_M from the previous chapter, the map we consider is given by the gradient flow of b^2, which we denote by $\Phi : M \times \mathbb{R} \to M$. Hence,

$$\frac{d\Phi_t(x)}{dt} = \nabla b^2(x).$$

As a first example we examine the map Φ in the special case of the Euclidean space $(M, g, p) = (\mathbb{R}^n, g_{\text{Euc}}, 0)$. In this case $b = r$ and Φ is simply a dilation map $\Phi_t(x) = e^{2t}x$. Thus Φ_t identifies two scales by the rescaling

$$\Phi_t^* g_{\text{Euc}} = e^{4t} g_{\text{Euc}}.$$

If we perform a coordinate change to the metric $g_{\text{Euc}} = dr^2 + r^2 g_{\mathbb{S}^{n-1}}$ by $s = \log r$, so that $r^{-2} g_{\text{Euc}} = ds^2 + g_{\mathbb{S}^{n-1}}$ is now a cylindrical metric, then since $r(\Phi_t(x)) = e^{2t}x$, it follows that

$$(r \circ \Phi_t)^{-2} \Phi_t^* g_{\text{Euc}} = g_{\text{Euc}},$$

so the metric $(r \circ \Phi_t)^{-2}\Phi_t^* g_{\text{Euc}}$ is constant in t.

Our theorem generalizes this example to Ricci-flat manifolds. It identifies two scales on average after performing a conformal change to bring the metric in cylindrical form, and gives the rate of how fast they become similar. We will use the symbol $⨍$ to denote average integrals. So for instance, the notation $⨍_{b=r} f d\sigma$ refers to the average integral $\dfrac{1}{\mathcal{H}^{n-1}(\{b=r\})}\displaystyle\int_{b=r} f d\sigma$ over a level set $\{b = r\}$ where $d\sigma$ is the area measure.

Theorem 3.1.8. *Let (M^n, g) be a complete Ricci-flat manifold of dimension $n \geq 3$ and maximal volume growth. Define the family of metrics $g(t)$ by*

$$g(t) = (b \circ \Phi_t)^{-2}\Phi_t^* g. \tag{3.1}$$

Suppose that a tangent cone at infinity of M has smooth cross section. Then there exist constants $C, r_0, \beta > 0$ so that for any $r > r_0$ and $T > t > 0$,

$$⨍_{b=r} \left\{ \sup_{v \neq 0} \left| \log \frac{g(T)(v,v)}{g(t)(v,v)} \right| \right\} d\sigma \leq C t^{-\frac{\beta}{2}}. \tag{3.2}$$

In equation (3.2), the bound $Ct^{-\frac{\beta}{2}}$ is independent of T and decreases with t. Thus the scale $\Phi_t(\{b = r\})$ is identified with the scale $\Phi_T(\{b = r\})$ for any $T > t$ and the estimate becomes better if t is large. Also note that (3.2) holds in particular for $r = \exp(At)r_0$ for a constant $A > 0$, which is roughly the scale at time t.

Theorem 3.1.8 can be understood as a statement that Ricci-flat manifolds with maximal volume growth and smooth tangent cone are asymptotically conical in a weak sense. Recall that asymptotically conical manifolds are complete manifolds (M, g) such that there exists a diffeomorphism $\Psi : M\backslash K \to C\backslash B_o(R)$ where $K \subset M$ is compact and C is a metric cone with vertex o, such that $\|\Psi^* g_C - g\|_{\mathcal{C}^\infty(B_p^M(r,g))}$ tends to zero as r goes to infinity.

Remark 3.1.9. *We point out that the idea of conformally changing a given metric by a suitable factor of a positive Green function of a linear elliptic operator has appeared in other works. For instance Schoen used the Green function of the operator* $Lu = \Delta u - u$ *in his solution to the Yamabe problem [31].*

3.2 Proof of Theorem 3.1.8

The key ingredient in the proof of Theorem 3.1.8 is that the weighted L^2 integral of the trace free hessian of b^2,

$$\int_{b\leq r} b^{-n} \left|\mathrm{Hess}_{b^2} - \frac{\Delta b^2}{n} g\right|^2,$$

decays rapidly. This quantity is obviously monotone in r. Monotonicity formulae for elliptic and parabolic operators have a large number of geometric applications [11, 15, 2, 18, 1] (see also [13] for a survey).

The rapid decay of this monotone quantity follows from an infinite dimensional Łojasiewicz inequality [14].

Theorem 3.2.1 ([14])**.** *Suppose that* M *is a complete manifold with nonnegative Ricci curvature and maximal volume growth, and that a tangent cone is smooth with cross section* (N, g_N)*. Let* $\mathcal{A}$ *to be the set consisting of pairs* (g, w) *where* g *is a* $\mathcal{C}^{2,\beta}$ *metric on* N*, and* w *a* $\mathcal{C}^{2,\beta}$ *a positive function on* N*. Define the subset* $\mathcal{A}_1$ *of* $\mathcal{A}$ *to be*

$$\mathcal{A}_1 = \left\{(g, w) \;\middle|\; \int_N w \, d\mu_g = n\omega_n\right\}.$$

Define the functional $\mathcal{R} : \mathcal{A} \to \mathbb{R}$ *by*

$$\mathcal{R}(g, w) = \frac{1}{2-n}\left(\int_N w^3 \, d\mu_g - \frac{1}{n-2}\int_N R_g w \, d\mu_g\right),$$

where R_g *is the scalar curvature of* g*. Then there exists* $\alpha \in (0, 1)$ *such that*

$$\left|\mathcal{R}(g,w)-\mathcal{R}\left(V_M^{-\frac{2}{n-2}}g_N, V_M^{\frac{1}{n-2}}\right)\right|^{2-\alpha} \le |\nabla_1\mathcal{R}|^2(g,w),$$

where ∇_1 *is the restriction of* $\nabla\mathcal{R}$ *to* $\mathcal{A}_1$ *and* $(g,w)\in\mathcal{A}_1$ *is sufficiently close to* $\left(V_M^{-\frac{2}{n-2}}g_N, V_M^{\frac{1}{n-2}}\right)$. *Moreover, we have*

$$\left|\nabla_1\mathcal{R}(r^{-2}g_r,|\nabla b|)\right|^2 \le C\int_{\frac{r}{2}\le b\le\frac{3r}{2}} b^{-n}\left|\mathrm{Hess}_{b^2}-\frac{\Delta b^2}{n}g\right|^2$$

and

$$r^{1-n}\int_{b=r}|\nabla b|^3 \le \mathcal{R}(r^{-2}g_r,|\nabla b|)+C\int_{\frac{r}{2}\le b\le\frac{3r}{2}} b^{-n}\left|\mathrm{Hess}_{b^2}-\frac{\Delta b^2}{n}g\right|^2,$$

where $C>0$ *is determined by* (M,g) *and independent of* r.

By elementary methods one can deduce from Theorem 3.2.1 the following corollary, expressing the desired rapid decay.

Corollary 3.2.2 ([14], Proposition 2.25)**.** *Suppose that one tangent cone* $C(N)$ *at infinity of* M *is smooth. Then there is a constant* $C=C(\varepsilon)>0$ *such that the following is true: if* $d_{GH}(B_{2R}(p)\backslash B_R(p), B_{2R}^{C(N)}(0)\backslash B_R^{C(N)}(0))\le\varepsilon R$ *for all* $R\in[\frac{s}{100},100r]$, *then*

$$\int_{b\ge r} b^{-n}\left|\mathrm{Hess}^g_{b^2}-\frac{\Delta b^2}{n}g\right|_g^2 \le \frac{C}{\log(r/s)^{1+\beta}}. \tag{3.3}$$

Here, $\beta>0$ *is a constant depending on* (M^n,g), *but not on* ε.

The rest of this chapter is devoted to proving Theorem 3.1.8. We will first relate the change in t of the metric $g(t)$ to the weighted L^2 integral of the trace-free Hessian of b^2. The rapid decay will imply that the change in t of $g(t)$ is small. In fact, $g(t)$ and the integral estimate in Theorem 3.1.8 are designed precisely to bring this quantity into play.

We first compute the time derivative of $g(t)$.

Lemma 3.2.3. *$g'(t)$ is given by*

$$\frac{d}{dt}g(t) = 2\Phi_t^*\left[b^{-2}\left(\mathrm{Hess}_{b^2} - \frac{\Delta b^2}{n}g\right)\right]. \tag{3.4}$$

Here the Hessian and the Laplacian are taken with respect to the original metric g.

Proof. Fix any point $x \in M\backslash\{p\}$, and a tangent vector $v \in T_xM$. Then we have the following calculation, where $\mathcal{L}$ is the usual Lie derivative.

$$\begin{aligned}
&\left(\frac{d}{dt}(b\circ\Phi_t)^{-2}\Phi_t^*g\right)_x(v,v)\\
&\quad= (\mathcal{L}_{\nabla b^2}b^{-2})(\Phi_t(x))\cdot(\Phi_t^*g)_x(v,v)\\
&\qquad\quad + b\left(\Phi_t(x)\right)^{-2}\cdot(\mathcal{L}_{\nabla b^2}g)_{\Phi_t(x)}((d\Phi_t)_xv,(d\Phi_t)_xv)\\
&\quad= g(\nabla b^2,\nabla b^{-2})(\Phi_t(x))\cdot(\Phi_t^*g)_x(v,v)\\
&\qquad\quad + b\left(\Phi_t(x)\right)^{-2}\cdot 2(\mathrm{Hess}_{b^2})_{\Phi_t(x)}((d\Phi_t)_xv,(d\Phi_t)_xv)\\
&\quad= -4b(\Phi_t(x))^{-2}|\nabla b|^2(\Phi_t(x))\cdot(\Phi_t^*g)_x(v,v)\\
&\qquad\quad + b\left(\Phi_t(x)\right)^{-2}\cdot 2(\mathrm{Hess}_{b^2})_{\Phi_t(x)}((d\Phi_t)_xv,(d\Phi_t)_xv)\\
&\quad= 2b(\Phi_t(x))^{-2}\left(\Phi_t^*\left(\mathrm{Hess}_{b^2} - 2|\nabla b|^2g\right)\right)_x(v,v)\\
&\quad= 2b(\Phi_t(x))^{-2}\left(\Phi_t^*\left(\mathrm{Hess}_{b^2} - \frac{\Delta b^2}{n}g\right)\right)_x(v,v).
\end{aligned}$$

Thus the lemma is proved. □

Lemma 3.2.4. *The norm of the time derivative of $g(t)$ is controlled by the Hessian of b^2, that is,*

$$\left\|\frac{d}{dt}\,g(t)\right\|^2_{g(t)}(x) \le 4\left\|\mathrm{Hess}^g_{b^2} - \frac{\Delta b^2}{n}g\right\|^2_g(\Phi_t(x)). \tag{3.5}$$

Proof. Let $v \in T_xM$, $v \neq 0$. Then using Lemma 3.2.3 we compute

$$\begin{aligned}\frac{g'(t)(v,v)_x}{g(t)(v,v)_x} &= \frac{2\Phi_t^*\left(b^{-2}\left(\mathrm{Hess}_{b^2} - \frac{\Delta b^2}{n}g\right)\right)_x(v,v)}{(b\circ\Phi_t(x))^{-2}(\Phi_t^*g)_x(v,v)} \\ &= \frac{2\Phi_t^*\left(\mathrm{Hess}_{b^2} - \frac{\Delta b^2}{n}g\right)_x(v,v)}{(\Phi_t^*g)_x(v,v)} \\ &= 2\left(\mathrm{Hess}_{b^2} - \frac{\Delta b^2}{n}g\right)_{\Phi_t(x)}\left(\frac{d(\Phi_t)_xv}{|d(\Phi_t)_xv|_g}, \frac{d(\Phi_t)_xv}{|d(\Phi_t)_xv|_g}\right).\end{aligned}$$

Since $\left|\frac{d(\Phi_t)_xv}{|d(\Phi_t)_xv|_g}\right|_g = 1$, we have that

$$\begin{aligned}\left|\frac{g'(t)_x(v,v)}{g(t)_x(v,v)}\right|^2(x) &\leq 4\left\{\sup_{\substack{w\in T_{\Phi_t(x)}M\\ |w|=1}}\left|\left(\mathrm{Hess}_{b^2} - \frac{\Delta b^2}{n}g\right)(w,w)\right|\right\}^2 \\ &\leq 4\left\|\mathrm{Hess}_{b^2} - \frac{\Delta b^2}{n}g\right\|^2(\Phi_t(x)).\end{aligned}$$

□

Lemma 3.2.5. *For any* $0 < s < t$*, we have*

$$⨍_{b=r}\left\{\sup_{v\neq 0}\left|\log\frac{g(t)(v,v)}{g(s)(v,v)}\right|\right\}d\sigma \leq \sqrt{t-s}\left(⨍_{b=r}\int_s^t \|g'(\tau)\|^2_{g(\tau)}d\tau\, d\sigma\right)^{1/2}. \tag{3.6}$$

Proof. First note that, for any $x \in \{b = r\}$ and $v \in T_xM$, $v \neq 0$, we have

$$\left(\log\frac{g(t)(v,v)}{g(s)(v,v)}\right)(x) = \int_s^t \frac{g'(\tau)_x(v,v)}{g(\tau)_x(v,v)}\,d\tau.$$

Therefore we have

$$\begin{aligned}⨍_{b=r}\sup_{v\neq 0}\left|\log\frac{g(t)(v,v)}{g(s)(v,v)}\right|d\sigma &\leq ⨍_{b=r}\sup_{v\neq 0}\int_s^t\left|\frac{g'(\tau)(v,v)}{g(\tau)(v,v)}\right|d\tau\, d\sigma \\ &= \frac{1}{\mathcal{H}^{n-1}(\{b=r\})}\int_{b=r}\int_s^t\left(\|g'(\tau)\|^2_{g(\tau)}\right)^{\frac{1}{2}}\,d\tau\, d\sigma\end{aligned}$$

$$\leq \frac{\sqrt{t-s}}{\mathcal{H}^{n-1}(\{b=r\})^{1/2}} \left(\int_{b=r} \int_s^t \|g'(\tau)\|^2_{g(\tau)}\, d\tau\, d\sigma \right)^{1/2},$$

where we used the Cauchy-Schwarz inequality for the last step. Thus we obtain the lemma. □

In the next proposition we will bound the L^2-norm of $g'(t)$ by a weighted L^2-norm of the trace-free Hessian of b^2.

Proposition 3.2.6. *There exists $r_0 = r_0(M, g)$ such that, for all $r > r_0$, the following is true. Let $F : \{b > r_0\} \to M$ be the map that sends a point in $\{b > r_0\}$ to the unique point in the same flow line that belongs to $\{b = r\}$. Then we have that*

$$\begin{aligned} &⨍_{b=r} \int_s^t \|g'(\tau)\|^2_{g(\tau)}\, d\tau\, d\sigma \\ &\leq \frac{2r^{n-1}}{\mathcal{H}^{n-1}(\{b=r\})} \int_{\underset{s\leq\tau\leq t}{\cup} \Phi_\tau(\{b=r\})} \frac{b^{-n}}{|\nabla b| \circ F} \left\| \mathrm{Hess}^g_{b^2} - \frac{\Delta b^2}{n} g \right\|^2. \end{aligned} \tag{3.7}$$

Proof. We will first derive inequality (3.7) assuming that the set $\{b \geq r_0\}$ does not contain any critical point of b for large r_0. This assumption will be removed at the end of the proof. Let $r > r_0$. We are going to consider $\Phi : \{b = r\} \times (s, t) \to M$, $\Phi(x, \tau) = \Phi_\tau(x)$ to be a parametrization of the open set $\underset{s<\tau<t}{\cup} \Phi_\tau(\{b = r\})$, and $d\tau\, d\sigma$ as a top-degree form on this open set. Note that $d\sigma$ is the $(n-1)$-dimensional volume form on $\{b = r\}$ with respect to g, so that if $x_1, \cdots, x_{n-1}$ are coordinates on $\{b = r\}$ then $d\sigma = \sqrt{\det g|_{\{b=r\}}}dx_1 \cdots dx_{n-1}$. In particular, $d\tau\, d\sigma = \sqrt{\det g|_{\{b=r\}}}d\tau\, dx_1 \cdots dx_{n-1}$. So if $x \in \{b = r\}$, then $d\tau\, d\sigma_x$ can define a top-degree form at $\Phi_\tau(x)$ for any value of τ.

Let x denote a point in $\{b = r\}$. Then by Lemma 3.2.4, we have

$$\begin{aligned} &⨍_{b=r} \int_s^t \|g'(\tau)\|^2_{g(\tau)}(x)\, d\tau\, d\sigma_x \\ &\leq \frac{4}{\mathcal{H}^{n-1}(\{b=r\})} \int_{b=r} \int_s^t \left\| \mathrm{Hess}^g_{b^2} - \frac{\Delta b^2}{n} g \right\|^2_g (\Phi_\tau(x))\, d\tau\, d\sigma_x \end{aligned}$$

$$= \frac{4}{\mathcal{H}^{n-1}(\{b=r\})} \int_{\bigcup\limits_{s<\tau<t} \Phi_\tau(\{b=r\})} \left\| \mathrm{Hess}^g_{b^2} - \frac{\Delta b^2}{n} g \right\|_g^2 (\Phi_\tau(x)) \, \frac{d\tau \, d\sigma_x}{d\mathrm{vol}_{\Phi_\tau(x)}} \, d\mathrm{vol}_{\Phi_\tau(x)}. \tag{3.8}$$

We have to compare the form $d\tau \, d\sigma_x$ to the actual volume form $d\mathrm{vol}_{\Phi_\tau(x)}$ at $\Phi_\tau(x)$, $\tau \in (s,t)$. Recall that the Laplacian is the change of the volume element along the flow line, i.e.,

$$\frac{d}{d\tau} \left(\frac{\Phi_\tau^*(d\mathrm{vol})}{d\mathrm{vol}} \right)_x = \Delta(b^2)(\Phi_\tau(x)) \left(\frac{\Phi_\tau^*(d\mathrm{vol})}{d\mathrm{vol}} \right)_x . \tag{3.9}$$

By the previous observation we can interpret $d\tau \, d\sigma_x$ as a top-degree form either at $\Phi_\tau(x)$ or at x, and calculate that

$$\begin{aligned}
\frac{d\tau d\sigma_x}{d\mathrm{vol}_{\Phi_\tau(x)}} &= \frac{d\tau \, d\sigma_x}{d\mathrm{vol}_x} \frac{d\mathrm{vol}_x}{d\mathrm{vol}_{\Phi_\tau(x)}} \\
&= \frac{d\tau \, d\sigma_x}{\frac{1}{|\nabla b^2|} \, db^2 \, d\sigma_x} \cdot \exp\left(- \int_0^\tau \Delta b^2(\Phi_u(x)) \, du \right) \\
&= |\nabla b^2|(x) \cdot \frac{1}{\mathcal{L}_{\nabla b^2} b^2} \cdot \exp\left(- \int_0^\tau 2n |\nabla b|^2 (\Phi_u(x)) \, du \right) \\
&= |\nabla b^2|(x) \cdot \frac{1}{|\nabla b^2|^2(x)} \cdot \exp\left(- \int_0^\tau 2n |\nabla b|^2 (\Phi_u(x)) \, du \right) \\
&= \frac{1}{|\nabla b^2|(x)} \cdot \exp\left(- \int_0^\tau 2n |\nabla b|^2 (\Phi_u(x)) \, du \right).
\end{aligned} \tag{3.10}$$

On the other hand, note that the change of $\log b$ along a flow line is given by

$$\frac{d}{d\tau} \log b = \mathcal{L}_{\nabla b^2} \log b = g\left(\nabla b^2, \frac{\nabla b}{b} \right) = 2|\nabla b|^2. \tag{3.11}$$

Hence, the change of b along the flow line is

$$b(\Phi_\tau(x)) = b(x) \cdot \exp\left(\int_0^\tau 2|\nabla b|^2 (\Phi_u(x)) \, du \right). \tag{3.12}$$

Combining with (3.10), we have that

$$\frac{d\tau d\sigma_x}{d\mathrm{vol}_{\Phi_\tau(x)}} = \frac{1}{|\nabla b^2|(x)} \cdot \left(\frac{b(\Phi_\tau(x))}{b(x)}\right)^{-n} = \frac{r^{n-1}}{2|\nabla b|(x)} b(\Phi_\tau(x))^{-n}. \tag{3.13}$$

Substituting (3.13) into (3.8) yields

$$\begin{aligned}
&\fint_{b=r} \int_s^t \|g'(\tau)\|^2_{g(\tau)}(x)\, d\tau\, d\sigma_x \\
&\qquad \le \frac{2r^{n-1}}{\mathcal{H}^{n-1}(\{b=r\})} \int_{\bigcup_{s\le\tau\le t} \Phi_\tau(\{b=r\})} \frac{1}{|\nabla b|(x)} b^{-n} \left\| \mathrm{Hess}^g_{b^2} - \frac{\Delta b^2}{n} g \right\|^2,
\end{aligned} \tag{3.14}$$

which is the inequality that we wanted to prove.

Finally we argue as promised that if r is large then $\{b \ge r\}$ does not contain any critical point of b. This argument is contained in [14] but we repeat it for the convenience of the reader. We denote by $\Psi(r)$ a positive function of one variable such that $\Psi(r) \to 0$ as $r \to \infty$. $\Psi(r)$ may change from line to line.

Let $C(N)$ be a smooth tangent cone of M, which is a metric cone by Theorem 2.0.11. Denote by o the vertex of $C(N)$. Let $A > 1$ be a fixed constant. Then there exists a sequence $r_i \to \infty$ such that

$$d_{GH}\left(B_p(Ar_i), B_o^{C(N)}(Ar_i)\right) < r_i^2 \Psi(r_i), \tag{3.15}$$

where $\Psi(r) \to 0$ as $r \to \infty$. By [10], this convergence in the Gromov-Hausdorff topology is in fact a convergence in the $\mathcal{C}^\infty$ topology since M is Einstein.

In [12] the following integral gradient estimate of b was obtained,

$$\fint_{B_p(Ar)} \left| |\nabla b|^2 - V_M^{\frac{2}{n-2}} \right|^2 \le \Psi(r). \tag{3.16}$$

Since b satisfies an elliptic equation, the integral gradient estimate implies pointwise gradient bounds

$$\sup_{B_p(Ar_i)\setminus B_p(r_i/A)} \left| |\nabla b| - V_M^{\frac{1}{n-2}} \right| \leq \Psi(r_i). \tag{3.17}$$

In particular, if $\frac{r_i}{A} \leq b$ for large r_i then $|\nabla b| \neq 0$. This completes the proof. $\square$

Combining Lemma 3.2.5 and Proposition 3.2.6 gives the following proposition.

Proposition 3.2.7. *If $r > r_0$, then for any $0 < s < t$,*

$$\begin{aligned} &\fint_{b=r} \sup_{v\neq 0} \left| \log \frac{g(t)(v,v)}{g(s)(v,v)} \right| d\sigma_x \\ &\quad \leq \sqrt{t-s} \left(\frac{2r^{n-1}}{\mathcal{H}^{n-1}(\{b=r\})} \int_{\bigcup\limits_{s\leq\tau\leq t} \Phi_\tau(\{b=r\})} \frac{1}{|\nabla b|(x)} b^{-n} \left\| \mathrm{Hess}^g_{b^2} - \frac{\Delta b^2}{n} g \right\|^2 \right)^{1/2}. \end{aligned} \tag{3.18}$$

Next, we can control the area of level sets of b by the below lemma.

Lemma 3.2.8. *Suppose that M has a smooth tangent cone $C(Y)$ at infinity. Then for any $\varepsilon > 0$, if r is sufficiently large, then*

$$\left| |\nabla b| - V_M^{\frac{1}{n-2}} \right| < \varepsilon \text{ when } r \leq b. \tag{3.19}$$

Moreover, there is a constant $C = C(M, g) > 0$ such that the area of the hypersurface $\{b = r\}$ is bounded above and below,

$$\frac{r^{n-1}}{C} \leq \mathrm{Area}(\{b = r\}) \leq Cr^{n-1}. \tag{3.20}$$

Proof. The first assertion was already proved in [14], as we mentioned in Section 2 (see equation (3.17)). For the second assertion we will utilize the following facts from

[11] (Corollary 2.19, Theorem 2.12): define a function $A(r)$ by

$$A(r) = r^{1-n} \int_{b=r} |\nabla b|^3 d\sigma. \tag{3.21}$$

Then A is monotone non-increasing in r. Moreover,

$$\lim_{r\to 0} A(r) < \infty, \tag{3.22}$$

and

$$0 < \lim_{r\to\infty} A(r). \tag{3.23}$$

Now by (3.19), if r is large then there exists $C > 0$ so that

$$\frac{A(r)}{C} \leq \text{Area}(\{b = r\}) \leq CA(r). \tag{3.24}$$

The second assertion in the lemma now follows from (3.21)–(3.24) and (3.19). □

We will also utilize the following comparison result for G from [26].

Lemma 3.2.9 ([26], Subsection 1.2)**.** *Let N be a complete Riemannian manifold with maximal volume growth and nonnegative Ricci curvature of dimension $n \geq 3$. Then there exist constants C_1, C_2 with the following effect. If G is the minimal positive Green function with pole $p \in N$ and r is the distance from p, then*

$$C_1 r^{2-n} \leq G \leq C_2 r^{2-n}. \tag{3.25}$$

Now we can finish the proof of Theorem 3.1.8.

Proof of Theorem 3.1.8. Fix a small number $\varepsilon > 0$. Since $C(N)$ is the unique tangent cone by Theorem 3.1.7, there exists r_0 so that if $r > r_0$ then

$$\left| |\nabla b| - V_M^{\frac{1}{n-2}} \right| < \varepsilon \text{ when } r \leq b, \tag{3.26}$$

and

$$d_{GH}(B_{2R}(p)\backslash B_R(p), B_{2R}^{C(N)}(0)\backslash B_R^{C(N)}(0)) \leq \varepsilon R \text{ for all } R \geq \frac{r}{100}. \tag{3.27}$$

Recall the following inequality from Proposition 3.2.7, that for $0 < s < t$,

$$\begin{aligned} &⨍_{b=r} \sup_{v\neq 0} \left| \log \frac{g(t)(v,v)}{g(s)(v,v)} \right| d\sigma \\ &\quad \leq \sqrt{t-s} \left(\frac{2r^{n-1}}{\mathcal{H}^{n-1}(\{b=r\})} \int_{\underset{s\leq\tau\leq t}{\cup} \Phi_\tau(\{b=r\})} \frac{1}{|\nabla b|(x)} b^{-n} \left\| \mathrm{Hess}^g_{b^2} - \frac{\Delta b^2}{n} g \right\|^2 \right)^{1/2}. \end{aligned} \tag{3.28}$$

Since $(V_M^{\frac{1}{n-2}} - \varepsilon) \leq |\nabla b|$, it follows that there exists a positive constant C depending only on $V_M^{\frac{1}{n-2}}$ so that

$$\begin{aligned} &⨍_{b=r} \sup_{v\neq 0} \left| \log \frac{g(t)(v,v)}{g(s)(v,v)} \right| d\sigma \\ &\qquad \leq C\sqrt{t-s} \left(\frac{r^{n-1}}{\mathcal{H}^{n-1}(\{b=r\})} \int_{\underset{s\leq\tau\leq t}{\cup} \Phi_\tau(\{b=r\})} b^{-n} \left\| \mathrm{Hess}^g_{b^2} - \frac{\Delta b^2}{n} g \right\|^2 \right)^{1/2}. \end{aligned} \tag{3.29}$$

Next we look at the region of integration on the right hand side. From this point C is allowed to change line by line, as long as it is independent of r, s, and t. Recall that by Lemma 3.2.8, we have

$$\mathrm{Area}(\{b=r\}) \geq Cr^{n-1}. \tag{3.30}$$

Also note that for $x \in \{b = r\}$, by equation (3.12),

$$\begin{aligned} b(\Phi_\tau(x)) &= b(x) \exp\left(\int_0^\tau 2|\nabla b|^2(\Phi_s(x))\, ds\right) \\ &\in \left[r \cdot \exp\left(2\tau(V_M^{\frac{1}{n-2}} - \varepsilon)^2\right),\, r \cdot \exp\left(2\tau(V_M^{\frac{1}{n-2}} + \varepsilon)^2\right)\right]. \end{aligned} \tag{3.31}$$

It follows that

$$\bigcup_{s\le\tau\le t} \Phi_\tau(\{b = r\}) \subset \left\{ r \cdot \exp\left(2s(V_M^{\frac{1}{n-2}} - \varepsilon)^2\right) \le b \le r \cdot \exp\left(2t(V_M^{\frac{1}{n-2}} + \varepsilon)^2\right)\right\}. \tag{3.32}$$

Therefore, we have that

$$\begin{aligned} &\fint_{b=r} \sup_{v\neq 0} \left|\log \frac{g(t)(v,v)}{g(s)(v,v)}\right| d\sigma \\ &\le C\sqrt{t-s}\left(\int_{re^{2s\left(V_M^{\frac{1}{n-2}}-\varepsilon\right)^2} \le b \le re^{2t\left(V_M^{\frac{1}{n-2}}+\varepsilon\right)^2}} b^{-n}\left\|\mathrm{Hess}^g_{b^2} - \frac{\Delta b^2}{n} g\right\|^2\right)^{1/2} \\ &\le C\sqrt{t-s}\left(\int_{re^{2s(V_M^{\frac{1}{n-2}}-\varepsilon)^2}\le b} b^{-n}\left\|\mathrm{Hess}^g_{b^2} - \frac{\Delta b^2}{n} g\right\|^2\right)^{1/2}. \end{aligned} \tag{3.33}$$

Combining with Corollary 3.2.2, it follows that

$$\fint_{b=r} \sup_{v\neq 0} \left|\log \frac{g(t)(v,v)}{g(s)(v,v)}\right| d\sigma \le C\sqrt{t-s}\left(\frac{1}{2s\left(V_M^{\frac{1}{n-2}} - \varepsilon\right)^2 + \log(r/r_0)}\right)^{\frac{1+\beta}{2}}. \tag{3.34}$$

Taking $t = As$ with $A > 1$ in (3.34) then switching s and t, we obtain

$$\fint_{b=r} \sup_{v\neq 0} \left|\log \frac{g(At)(v,v)}{g(t)(v,v)}\right| d\sigma \le C\sqrt{(A-1)t} \cdot \left(\frac{1}{2t\left(V_M^{\frac{1}{n-2}} - \varepsilon\right)^2 + \log(r/r_0)}\right)^{\frac{1+\beta}{2}}$$

$$\leq C\sqrt{(A-1)t}\cdot\left(\frac{1}{2t\left(V_M^{\frac{1}{n-2}}-\varepsilon\right)^2}\right)^{\frac{1+\beta}{2}}$$
$$\leq C\sqrt{A-1}\cdot t^{-\frac{\beta}{2}}. \tag{3.35}$$

Iterating for $t, At, A^2t, \cdots$, we have

$$⨍_{b=r}\sup_{v\neq 0}\left|\log\frac{g(A^n t)(v,v)}{g(t)(v,v)}\right|d\sigma\leq C\sqrt{A-1}\frac{1-A^{-\frac{n\beta}{2}}}{1-A^{-\frac{\beta}{2}}}\cdot t^{-\frac{\beta}{2}}\leq\frac{C\sqrt{A-1}}{1-A^{-\frac{\beta}{2}}}t^{-\frac{\beta}{2}}. \tag{3.36}$$

Since this is true for any $n>0$ and $A>1$, we conclude that for any $T>t$,

$$⨍_{b=r}\sup_{v\neq 0}\left|\log\frac{g(T)(v,v)}{g(t)(v,v)}\right|d\sigma\leq Ct^{-\frac{\beta}{2}}. \tag{3.37}$$

This finishes the proof of the theorem. □

Remark 3.2.10. *We point out that with the aid of some functional inequalities, there is no need to consider the conformally changed metric $g(t)$. For instance, suppose that the following Hardy-Sobolev type inequality holds on M,*

$$\int_{b\leq r} b^{-n}|f-\bar{f}_r|^2\leq C\int_{b\leq r} b^{2-n}|\nabla f|^2 \qquad \text{for any } f\in\mathcal{C}^\infty(M), \lim_{r\to\infty} f=0. \tag{3.38}$$

Then one can directly work with the rescaled metrics $\tilde{g}(t)=e^{-4V_M^{\frac{2}{n-2}}t}\Phi_t^* g$ and obtain the same convergence of $\tilde{g}(t)$. The above proof carries through, with the usage of the Hardy-Sobolev inequality applied to $f=|\nabla b|^2-V_M^{\frac{2}{n-2}}$. One can conclude that Theorem 3.1.8 holds with $g(t)$ replaced with $\tilde{g}(t)$. The inequality 3.38 is to be investigated in the future. It can be shown to be true for all compactly supported f by standard integration by part techniques. Also for any f, the following modification is true,

$$\int_{b\leq r} b^{-n}|f-\bar{f}_r|^2\leq C\int_{b\leq Cr} b^{2-n}|\nabla f|^2, \tag{3.39}$$

where C is a constant depending on (M, g) but not f.

Chapter 4

Hessian estimates for the Laplace equation

4.1 Matrix Harnack inequalities

In the previous chapter we focused on the L^2-integral of trace-free Hessian of b^2. In this chapter we ask whether one can improve the integral estimate to a pointwise one. A partial motivation for this question comes from the seminal paper [25], where Li and Yau proved a sharp estimate for the gradient of the heat kernel on a complete Riemannian manifold with Ricci curvature bounded below.

Theorem 4.1.1 ([25]). *Let (M, g) be a compact Riemannian manifold and $f > 0$ a positive solution to the heat equation, $\partial_t f = \Delta f$. Suppose that M has nonnegative Ricci curvature. Then for any $t > 0$ and any vector field V on M, we have*

$$\partial_t f + \frac{n}{2t} f + 2Df(V) + f|V|^2 \geq 0. \tag{4.1}$$

Integrating inequality (4.1) along a space-time path leads to a Harnack inequality on f, which is the reason that (4.1) is sometimes referred to as a differential Harnack inequality. Later Hamilton [20] discovered a time-dependent matrix quantity that stays positive-semidefinite at all time, in the case that the manifold has nonnegative sectional curvature and parallel Ricci curvature.

Theorem 4.1.2 ([20]). *Suppose that we are in the same setting as Theorem 4.1.1, and moreover suppose that M has parallel Ricci curvature and nonnegative sectional curvature. Then for any $t > 0$ and for any vector field V_i on M, we have*

$$D_i D_j f + \frac{1}{2t} f g_{ij} + D_i f \cdot V_j + D_j f \cdot V_i + f V_i V_j \geq 0. \tag{4.2}$$

Taking the trace of this matrix inequality yields the Li-Yau gradient estimate, although the curvature assumptions are stronger for the matrix inequality.

Matrix estimates have also been developed for other situations, such as the heat equation on Kähler manifolds with nonnegative holomorphic bisectional curvature by Ni-Cao [4], Ricci flow by Hamilton [21], Kähler-Ricci flow by Ni [27], and mean curvature flow by Hamilton [22].

In the elliptic setting parallel to the aforementioned time-dependent results, Colding [11] obtained a sharp gradient estimate for the minimal positive Green function for the Laplace equation under the relatively mild assumption of nonnegative Ricci curvature.

Theorem 4.1.3 ([11]). *Let (M^n, g) be a complete non-compact Riemannian manifold of maximal volume growth and dimension $n \geq 3$. Suppose that M has nonnegative Ricci curvature. Then we have*

$$|\nabla b| \leq 1. \tag{4.3}$$

We will show that there exists a related matrix Harnack inequality. This result can be thought of as an elliptic analogue to the aforementioned parabolic matrix inequalities. We first introduce a curvature condition, necessary in carrying out the maximum principle argument.

Definition 4.1.4. *Let (M, g) be a Riemannian manifold and V a vector field on M.*

M is said to have nonnegative sectional curvature along *V if*

$$R(V, W, V, W) \geq 0$$

for any vector field W.

Theorem 4.1.5 ([28])**.** *Let (M^n, g) be a complete non-compact Riemannian manifold of maximal volume growth and dimension $n \geq 3$. Suppose that M has nonnegative sectional curvature along ∇G, and that $\nabla \mathrm{Ric} = 0$. Suppose that*

$$\mathrm{Hess}_{b^2} \leq Dg$$

on $M \backslash \{x\}$ for some $D > 0$, and that

$$\liminf_{\varepsilon \to 0} \sup_{\substack{(p,V) \\ r(p)=\varepsilon,\, V \in T_pM,\, g(V,V)=1}} \big(\mathrm{Hess}_{b^2} - Cg\big)(V, V) \leq 0.$$

where $C \geq 10$. Then

$$\mathrm{Hess}_{b^2} \leq Cg$$

holds everywhere on $M \backslash \{x\}$.

We can of course state the theorem in terms of G instead of b.

Theorem 4.1.6. *Let (M^n, g) be a complete non-compact Riemannian manifold of maximal volume growth and dimension $n \geq 3$. Suppose that M has nonnegative sectional curvature along ∇G and parallel Ricci curvature, so that $\nabla \mathrm{Ric} = 0$. Suppose that $G_{ij} + \frac{n}{2-n} \cdot \frac{G_i G_j}{G} \geq DG^{\frac{-n}{2-n}} g_{ij}$ for some $D > 0$ on $M \backslash \{x\}$, and that $G_{ij} + \frac{n}{2-n} \cdot \frac{G_i G_j}{G} \geq CG^{\frac{-n}{2-n}} g_{ij}$*

$$\limsup_{\varepsilon \to 0} \inf_{\substack{(p,V) \\ r(p)=\varepsilon,\, V \in T_pM,\, g(V,V)=1}} \left(\mathrm{Hess}_G + \frac{n}{2-n} \frac{\nabla G \otimes \nabla G}{G}\right)(V, V) \geq CG^{-\frac{n}{2-n}}$$

on a neighborhood of x for some $C \leq 10 - 5n$. Then

$$G_{ij} + \frac{n}{2-n} \cdot \frac{G_i G_j}{G} \geq C G^{\frac{-n}{2-n}} g_{ij}$$

holds everywhere on $M \backslash \{x\}$.

To motivate the above theorem, suppose for a moment that $M = \mathbb{R}^n$ where $n \geq 3$, so that $G = C(n) \cdot r^{2-n}$, where r is the distance from the origin and $C(n)$ is a dimensional constant. We observe that the first and the second order derivatives of G satisfies the following relation, which motivates a bound on the Hessian of G.

$$G_{ij} + \frac{n}{2-n} \cdot \frac{G_i G_j}{G} = (2-n) G^{\frac{-n}{2-n}} \delta_{ij}. \tag{4.4}$$

Another motivation comes from the Hessian comparison theorem for radial functions.On the Euclidean space $\mathbb{R}^n$, the Hessian of $b^2 = r^2$ satisfies

$$\mathrm{Hess}_{b^2} = 2g.$$

The Hessian comparison theorem would suggest that an inequality in the direction of $\mathrm{Hess}_{b^2} \leq Cg$ with $C \geq 2$ might be true, which is the content of Theorem 4.1.5 (although the curvature assumptions are stronger than just nonnegative Ricci curvature).

Remark 4.1.7. *The two curvature assumptions in Theorem 4.1.5 are critical in the proof, and were also imposed in [20]. However, the arguments in the proof can readily be generalized to the case where sectional curvature is bounded from below by $-K \cdot G^{-\frac{2}{2-n}}$ and the first derivative of Ricci curvature is bounded as $|\nabla_i R_{jk}| \leq L \cdot G^{-\frac{3}{2-n}}$. Then we obtain an upper bound of Hess_{b^2} in terms of n, K, L. It would be interesting to know whether a similar inequality holds under scale-invariant curvature assumptions with r instead of G, i.e. under the assumptions that the sectional curvature is bounded from below by $-K \cdot r^{-2}$ and the first derivative of Ricci curvature is bounded as*

$|\nabla_i R_{jk}| \leq L \cdot r^{-3}$.

Remark 4.1.8. *The assumption that* $\mathrm{Hess}_{b^2} \leq Cg$ *near* x *is not superfluous, and different from requiring that* $|\mathrm{Hess}_{b^2}| \leq C$ *near* x. *It implies that* $\mathrm{Hess}_{b^2}(v,w) = 0$ *whenever* $g(v,w) = 0$.

As a corollary we obtain a Harnack inequality for b. Let $y, z \in M \backslash \{x\}$ and consider a minimal geodesic segment $\overline{yz}$ parametrized by arclength s. Then the function $\frac{C}{2}s^2 - b^2$ is convex. Hence we obtain the following corollary.

Corollary 4.1.9. *Under the same assumptions as Theorem 4.1.5, let* w *be the point on a minimal geodesic* $\overline{yz}$ *such that* $d(y,w) = \lambda \cdot d(y,z)$ *and* $d(w,z) = (1-\lambda) \cdot d(y,z)$, $0 \leq \lambda \leq 1$. *Then*

$$b(w)^2 \geq (1-\lambda) \cdot b(y)^2 + \lambda \cdot b(z)^2 - \frac{C}{2}\lambda(1-\lambda) \cdot d(y,z)^2.$$

In particular, the corollary holds whether $b(y) \neq b(z)$ or not.

4.2 Proof of Theorem 4.1.5

In this section we present the proof of Theorem 4.1.5. The main tool is the maximum principle introduced by Calabi in [3], which we recall below.

Definition 4.2.1. *Let* X *be a Riemannian manifold,* $x_0 \in X$, *and* $\varphi : X \to \mathbb{R}$ *a continuous function. We say that* $\Delta\varphi \leq 0$ *at* x_0 *in barrier sense if for any* $\varepsilon > 0$, *there is a* $\mathcal{C}^2$ *function* $\psi_{x_0,\varepsilon}$ *on a neighborhood of* x_0 *such that* $\psi_{x_0,\varepsilon}(x_0) = \varphi(x_0)$, $\Delta\psi_{x_0,\varepsilon} < \varepsilon$, *and* $\psi_{x_0,\varepsilon} \geq \varphi$. *We say that* $\Delta\varphi \leq 0$ *in barrier sense if* $\Delta\varphi \leq 0$ *at* x_0 *in barrier sense for all* $x_0 \in X$.

Lemma 4.2.2 (Maximum principle for barrier subsolutions)**.** *If* $\Delta\varphi \leq 0$ *in barrier sense, then either* φ *is constant or* φ *has no weak local minimum.*

Define a tensor H as the following, motivated by the fact that it vanishes on the Euclidean space (see equation (4.4)).

$$H = \mathrm{Hess}_G + \frac{n}{2-n} \cdot \frac{\nabla G \otimes \nabla G}{G} + (n-2) \cdot G^{\frac{-n}{2-n}} g.$$

By a straightforward computation, it follows that $\mathrm{Hess}_{b^2} = -\frac{2}{n-2} G^{\frac{n}{2-n}} H + 2g$. Hence, the assumption that $\mathrm{Hess}_{b^2} \leq Dg$ is equivalent to that

$$0 \leq H + \frac{n-2}{2}(D-2) G^{\frac{-n}{2-n}} g.$$

Let $\alpha = -\frac{n}{2-n} = \frac{n}{n-2}$, so that our goal is to show that

$$0 \leq H + \frac{n-2}{2}(C-2) G^{\alpha} g.$$

For the sake of brevity we will call this tensor $\widetilde{H}$,

$$\widetilde{H} := H + \frac{n-2}{2}(C-2) G^{\alpha} g = \mathrm{Hess}_G + \frac{n}{2-n} \cdot \frac{\nabla G \otimes \nabla G}{G} + \frac{n-2}{2} C \cdot G^{\alpha} g.$$

We also define the function Λ on $M \backslash \{x\}$ to be the lowest eigenvalue of $\widetilde{H}$,

$$\Lambda(p) := \min_{V \in T_p M,\, g(V,V)=1} \widetilde{H}(V,V).$$

Then Λ is continuous, and the assumption that $\mathrm{Hess}_{b^2} \leq Dg$ implies that

$$\frac{n-2}{2}(C-D) G^{\alpha} \leq \Lambda.$$

An ingredient we will need is the following lemma, which we will prove in the next section. Let $p \in M \backslash \{x\}$ and let $\{e_i\}$ be a normal frame at p, i.e. $g_{ij}(p) = \delta_{ij}$ and $\nabla_j e_i(p) = 0$ for any i, j. Denote $\widetilde{H}_{ij} = \widetilde{H}(e_i, e_j)$.

For ease of notation, define the tensor B by

$$B := \frac{\nabla G \otimes \nabla G}{G},$$

or equivalently as $B_{ij} = \dfrac{G_i G_j}{G}$ in coordinates. It is clear that B is nonnegative-semidefinite with eigenvalues $|\nabla G|^2/G$ and 0. The following lemma follows from a straightforward computation.

Lemma 4.2.3. *The following holds at* p.

$$\begin{aligned}
&\Delta(\widetilde{H}_{ij} - \frac{n-2}{2} C \cdot G^\alpha g_{ij}) \\
&\quad = R_{ik}\widetilde{H}_{jk} + R_{jk}\widetilde{H}_{ik} - 2R_{ikjl}\widetilde{H}_{kl} - \frac{2n}{n-2} R_{ikjl}\frac{G_i G_j}{G} - \frac{2n}{(n-2)G}\widetilde{H}_{ij}^2 \\
&\qquad - \frac{n(n-2)}{2} C^2 G^{2\alpha-1} g_{ij} + \frac{4n}{n-2}\left[C \cdot G^{\alpha-1} - \frac{2|\nabla G|^2}{(n-2)^2 G^2}\right] B_{ij} \\
&\qquad + \frac{2n}{(n-2)G}\left[\widetilde{H}\left(\frac{2}{2-n}B + \frac{n-2}{2} C \cdot G^\alpha g\right) + \left(\frac{2}{2-n}B + \frac{n-2}{2} C \cdot G^\alpha g\right)\widetilde{H}\right]_{ij}.
\end{aligned}$$

Proof of Theorem 4.1.5. Note that if $\Delta\Lambda \le 0$ in barrier sense whenever $\Lambda < 0$, then the theorem would follow by the maximum principle. Indeed, in the case that Λ is constant, note that $\Lambda \ge \frac{n-2}{2}(C - D) \cdot G^\alpha$ and $G^\alpha = O(r^{-n})$, therefore $\Lambda \ge 0$. In the case that Λ is not constant, Λ takes its minimum on $\{\varepsilon \le r \le R\} \cap \{\Lambda < 0\}$ on the boundary by the maximum principle. By the same argument as in the constant case, we have that $\inf_{r=R} \Lambda \to 0$ as $R \to \infty$, and the assumption near x implies that $\liminf_{\varepsilon\to 0} \inf_{r=\varepsilon} \Lambda \ge 0$. Therefore it would suffice to establish that $\Delta\Lambda \le 0$ whenever $\Lambda < 0$.

Now suppose that $\Lambda(p) = \widetilde{H}(V, V) < 0$. Write $V = V^i e_i$ on a neighborhood of p, where each V^i is extended as a constant function. Define $\widetilde{h} = \widetilde{H}(V, V) = \widetilde{H}_{ij} V^i V^j$. We observe that $\widetilde{h}$ is an upper barrier for Λ at p. Indeed, $\widetilde{h}(p) = \Lambda(p)$ and $\widetilde{h} \ge \Lambda$ near p by definition of Λ. It only remains to show that, for any $\varepsilon > 0$, if we choose the neighborhood of p small enough then $\Delta\widetilde{h} < \varepsilon$. It is enough to show that if $\widetilde{h}(p) < 0$

then $\Delta(\widetilde{H}_{ij}V^iV^j)(p) \leq 0$, since then $\Delta\widetilde{h} < \varepsilon$ follows by continuity. Hence in what follows, all computations are at p. Note that since V^i are constant, we have that $\Delta\widetilde{h} = \Delta(\widetilde{H}_{ij})V^iV^j = (\Delta\widetilde{H}_{ij})V^iV^j$. Thus, it suffices to estimate the terms in Lemma 4.2.3.

We proceed to bound the first three terms. Without loss of generality we can assume that $\{e_i\}$ diagonalizes $\widetilde{H}$ at p and write $\widetilde{H}_{ij} = \lambda_i\delta_{ij}$. Since V is the lowest eigenvector of $\widetilde{H}$, there exists m such that $V = e_m$ with $\lambda_m = \Lambda$. Therefore (with m fixed and i, j, k, l being summed over),

$$\begin{aligned}
&\big(R_{ik}\widetilde{H}_{jk} + R_{jk}\widetilde{H}_{ik} - 2R_{ikjl}\widetilde{H}_{kl}\big)V^iV^j \\
&= R_{ik}(\widetilde{H}_{jk}V^j)V^i + R_{jk}(\widetilde{H}_{ik}V^i)V^j - 2R_{ikjl}\lambda_k\delta_{kl}V^iV^j \\
&= R_{ik}(\Lambda \cdot V^k)V^i + R_{jk}(\Lambda \cdot V^k)V^j - 2R_{ikjk}\lambda_k\delta_{im}\delta_{jm} \\
&= 2\Lambda \cdot R_{ij}\delta_{im}\delta_{jm} - 2R_{mkmk}\lambda_k \\
&= 2R_{mkmk}(\Lambda - \lambda_k) \leq 0,
\end{aligned}$$

since Λ is the lowest eigenvalue, and $R_{mkmk} \geq 0$.

The assumption on the sectional curvature implies that

$$-R_{ikjl}\frac{G_kG_l}{G}V^iV^j \leq 0.$$

It is also clear that

$$-\frac{2n}{(n-2)G}(\widetilde{H})^2_{ij}V^iV^j \leq 0.$$

For the next two of the remaining terms, we will use the sharp gradient estimate in [11] which states that $|\nabla b| \leq 1$ for nonnegative Ricci curvature. This is equivalent to $|\nabla G|^2 \leq (n-2)^2G^{\alpha+1}$. Therefore,

$$-\frac{n(n-2)}{2}C^2G^{2\alpha-1}g_{ij}V^iV^j + \frac{4n}{n-2}\left[C \cdot G^{\alpha-1} - \frac{2|\nabla G|^2}{(n-2)^2G^2}\right]B_{ij}V^iV^j$$

$$\le -\frac{n(n-2)}{2}C^2G^{2\alpha-1} + \frac{4nC\cdot G^{\alpha-1}}{n-2}B_{ij}V^iV^j$$
$$\le -\frac{n(n-2)}{2}C^2G^{2\alpha-1} + \frac{4nC\cdot G^{\alpha-2}}{n-2}|\nabla G|^2$$
$$\le -\frac{n(n-2)}{2}C^2G^{2\alpha-1} + \frac{4nC\cdot G^{\alpha-2}}{n-2}\cdot (n-2)^2G^{\alpha+1}$$
$$= -\frac{n(n-2)}{2}C(C-8)G^{2\alpha-1}.$$

For the last group of terms, we use that the top eigenvalue of B is $|\nabla G|^2/G$ and the gradient estimate $|\nabla G|^2 \le (n-2)^2G^{\alpha+1}$ to obtain that

$$\left[\widetilde{H}\left(\frac{2}{2-n}B + \frac{n-2}{2}C\cdot G^\alpha g\right) + \left(\frac{2}{2-n}B + \frac{n-2}{2}C\cdot G^\alpha g\right)\widetilde{H}\right]_{ij}V^iV^j$$
$$= \frac{2}{2-n}[\widetilde{H}B + B\widetilde{H}]_{ij}V^iV^j + (n-2)C\cdot G^\alpha \widetilde{H}_{ij}V^iV^j$$
$$\le \frac{4|\nabla G|^2}{(n-2)G}|\widetilde{h}| + (n-2)C\cdot G^\alpha\widetilde{h}$$
$$= \left[(n-2)C\cdot G^\alpha - \frac{4|\nabla G|^2}{(n-2)G}\right]\widetilde{h}$$
$$= (n-2)(C-4)\cdot G^\alpha\widetilde{h} + \frac{4}{(n-2)G}\left[(n-2)^2G^{\alpha+1} - |\nabla G|^2\right]\widetilde{h} \le 0.$$

Combining all of the above, we conclude that

$$\left(\Delta(\widetilde{H}_{ij} - \frac{n-2}{2}C\cdot G^\alpha g_{ij})\right)V^iV^j \le -\frac{n(n-2)}{2}C(C-8)G^{2\alpha-1}.$$

Routine calculation shows that $\Delta G = 0$ is equivalent to

$$\Delta(G^\alpha) = \frac{2n}{(n-2)^2}G^{\alpha-2}|\nabla G|^2.$$

(A proof of this fact is given in Section 3, Lemma 4.3.2.) Since $C \ge 10$, it follows that

$$\Delta(\widetilde{H}_{ij}V^iV^j) = \Delta\left((\widetilde{H}_{ij} - \frac{n-2}{2}C\cdot G^\alpha g_{ij} + \frac{n-2}{2}C\cdot G^\alpha g_{ij})V^iV^j\right)$$

$$
\begin{aligned}
&= \Delta\left((\widetilde{H}_{ij} - \frac{n-2}{2} C \cdot G^{\alpha} g_{ij}) V^i V^j \right) + \frac{(n-2)C}{2} \cdot \Delta G^{\alpha} \\
&= \Delta\left((\widetilde{H}_{ij} - \frac{n-2}{2} C \cdot G^{\alpha} g_{ij}) V^i V^j \right) + \frac{nC}{n-2} \cdot G^{\alpha-2} |\nabla G|^2 \\
&\leq -\frac{n(n-2)}{2} C(C-8) G^{2\alpha-1} + \frac{nC}{n-2} \cdot G^{\alpha-2} |\nabla G|^2 \\
&\leq -\frac{n(n-2)}{2} C(C-8) G^{2\alpha-1} + n(n-2)C \cdot G^{2\alpha-1} \\
&= -\frac{n(n-2)}{2} C(C-10) G^{2\alpha-1} \\
&\leq 0,
\end{aligned}
$$

where the gradient estimate for G was used for the second inequality. This establishes that $\Delta(\widetilde{H}_{ij} V^i V^j) \leq 0$. Hence Theorem 4.1.5 is proved. □

Remark 4.2.4. *In [20] it is shown that for a positive solution f of the heat equation on a closed manifold, the matrix quantity $\mathrm{Hess}_f - \frac{\nabla f \otimes \nabla f}{f} + \frac{f}{2t} g$ is positive-semidefinite for all time. One could ask whether we can introduce a cutoff function to view G as a stationary solution on an annulus in M, and obtain the same result for $\mathrm{Hess}_G - \frac{\nabla G \otimes \nabla G}{G} + \frac{G}{2t} g$, which would imply that $\mathrm{Hess}_{b^2} \leq 4g$. However this approach does not seem to work straightforwardly, since the assumption that $\partial M = \emptyset$ is essential for maximum principle argument of Hamilton.*

4.3 Derivation of Lemma 4.2.3

This section is devoted to proving Lemma 4.2.3. We compute the commutators in the case of parallel Ricci curvature.

Lemma 4.3.1. *Let $\{e_i\}$ be a normal frame at p. If M has parallel Ricci curvature, i.e. $\nabla_i R_{jk} = 0$, then for a smooth function f on M, the following identities hold at p.*

$$f_{ij} = f_{ji},$$

$$f_{ijk} - f_{ikj} = R_{jkli} f_l,$$

$$\Delta f_i - (\Delta f)_i = R_{ik} f_k,$$

$$f_{ijkl} - f_{ijlk} = R_{klmj} f_{im} + R_{klmi} f_{jm},$$

$$\Delta f_{ij} - (\Delta f)_{ij} = R_{jk} f_{ik} + R_{ik} f_{jk} - 2R_{ikjl} f_{kl},$$

where $f_{i_1 i_2 \cdots i_k}$ *refers to the derivative* $e_{i_k}(\cdots e_{i_2}(e_{i_1}(f))\cdots)$.

Proof. The first identity is simply the symmetry of the Hessian of f. For the second, we compute that

$$\begin{aligned} f_{ijk} - f_{ikj} &= e_k(g(\nabla_j \nabla f, e_i)) - e_j(g(\nabla_k \nabla f, e_i)) \\ &= g(\nabla_k \nabla_j \nabla f, e_i) + g(\nabla_j \nabla f, \nabla_k e_i) - g(\nabla_j \nabla_k \nabla f, e_i) - g(\nabla_k \nabla f, \nabla_j e_i) \\ &= g(\nabla_k \nabla_j \nabla f, e_i) - g(\nabla_j \nabla_k \nabla f, e_i) \\ &= R(e_j, e_k, \nabla f, e_i) = R_{jkli} f_l. \end{aligned}$$

In fact a similar identity holds for any 1-form S in place of df, namely,

$$(\nabla^2 S)(e_i, e_j, e_k) - (\nabla^2 S)(e_j, e_i, e_k) = S(R(e_j, e_i)e_k), \tag{4.5}$$

which can be checked in the same manner. We will use (4.5) to prove the fourth identity.

The third identity is a contraction of the one above,

$$f_{ikk} - f_{kki} = f_{kik} - f_{kki} = R_{iklk} f_l = R_{il} f_l = R_{ik} f_k.$$

The fourth identity is in fact true for any (0,2)-tensor T in the following form,

$$(\nabla^2 T)(e_l, e_k, e_i, e_j) - (\nabla^2 T)(e_k, e_l, e_i, e_j) = R_{klmj} T(e_i, e_m) + R_{klmi} T(e_m, e_j).$$

To show this, let $T = T_1 \otimes T_2$ for 1-forms T_1 and T_2, and compute using (4.5) and the normality of the coordinates, that

$$
\begin{aligned}
&(\nabla^2 T)(e_l, e_k, e_i, e_j) - (\nabla^2 T)(e_k, e_l, e_i, e_j) \\
&= \nabla^2(T_1 \otimes T_2)(e_l, e_k, e_i, e_j) - \nabla^2(T_1 \otimes T_2)(e_k, e_l, e_i, e_j) \\
&= e_l(e_k(T_1(e_i)T_2(e_j))) - e_l(T_1(\nabla_k e_i)T_2(e_j) + T_1(e_i)T_2(\nabla_k e_j)) \\
&\quad - e_k(e_l(T_1(e_i)T_2(e_j))) + e_k(T_1(\nabla_l e_i)T_2(e_j) + T_1(e_i)T_2(\nabla_l e_j)) \\
&= -[e_l(T_1(\nabla_k e_i)) - e_k(T_1(\nabla_l e_i))]T_2(e_j) - T_1(e_i)[e_l(T_2(\nabla_k e_j)) - e_k(T_2(\nabla_l e_j))] \\
&= -[\nabla^2 T_1(e_l, e_k, e_i) - \nabla^2 T_1(e_k, e_l, e_i)]T_2(e_j) - T_1(e_i)[\nabla^2 T_2(e_l, e_k, e_j) - \nabla^2 T_2(e_k, e_l, e_j)] \\
&= -T_1(R(e_k, e_l)e_i)T_2(e_j) - T_1(e_i)T_2(R(e_k, e_l)e_j) \\
&= -R_{klim}T(e_m, e_j) - R_{kljm}T(e_i, e_m) \\
&= R_{klmj}T(e_i, e_m) + R_{klmi}T(e_m, e_j).
\end{aligned}
$$

Now the fourth identity follows from taking $T = \text{Hess}_f$, and using the symmetry of the Hessian and the normality of the coordinates.

For the last identity, note that

$$
\begin{aligned}
\Delta f_{ij} = f_{ijkk} &= (f_{ikj} + R_{jkli}f_l)_k \\
&= f_{ikjk} + (\nabla_k R_{jkli})f_l + R_{jkli}f_{kl} \\
&= f_{ikkj} + R_{jkmi}f_{km} + R_{jkmk}f_{mi} + (\nabla_k R_{jkli})f_l + R_{jkli}f_{kl} \\
&= f_{kikj} - R_{ikjm}f_{km} + R_{jm}f_{im} + (\nabla_k R_{jkli})f_l - R_{ikjl}f_{kl} \\
&= (f_{kki} + R_{il}f_l)_j - 2R_{ikjl}f_{kl} + R_{jk}f_{ik} + (\nabla_k R_{jkli})f_l \\
&= (\Delta f)_{ij} + R_{il}f_{jl} + R_{jk}f_{ik} - 2R_{ikjl}f_{kl} + (\nabla_k R_{jkli})f_l.
\end{aligned}
$$

The second Bianchi identity implies that

$$
\nabla_k R_{jkli} + \nabla_l R_{jkik} + \nabla_i R_{jkkl} = \nabla_k R_{jkli} + \nabla_l R_{ji} - \nabla_i R_{jl} = 0.
$$

Since M has parallel Ricci curvature, it follows that $\nabla_k R_{jkli} = 0$. Thus we arrive at

$$\Delta f_{ij} = (\Delta f)_{ij} + R_{il} f_{jl} + R_{jk} f_{ik} - 2R_{ikjl} f_{kl}.$$

Changing k and l suitably, we have shown the lemma. □

With Lemma 4.3.1 we compute the ingredients for $\Delta \widetilde{H}_{ij}$, additionally using only the Leibniz rule.

Lemma 4.3.2. *Let $\{e_i\}$ be a normal frame at p, and suppose that M has parallel Ricci curvature. Then the following are true.*

$$\begin{aligned}
\Delta G_{ij} &= R_{jk} G_{ik} + R_{ik} G_{jk} - 2R_{ikjl} G_{kl}, \\
\Delta (G_i G_j) &= R_{ik} G_j G_k + R_{jk} G_i G_k + 2G_{ik} G_{jk}, \\
g(\nabla G, \nabla (G_i G_j)) &= G_i G_k G_{jk} + G_j G_k G_{ik}, \\
\Delta \left(\frac{G_i G_j}{G} \right) &= R_{ik} \frac{G_j G_k}{G} + R_{jk} \frac{G_i G_k}{G} \\
&\quad + \frac{2G_{ik} G_{jk}}{G} + \frac{2|\nabla G|^2 G_i G_j}{G^3} - \frac{2G_k (G_i G_{jk} + G_j G_{ik})}{G^2}, \\
\Delta G^\alpha &= \frac{2n}{(2-n)^2} G^{\alpha - 2} |\nabla G|^2.
\end{aligned}$$

Proof. The first identity is immediate from Lemma 4.3.1 and the fact that $\Delta G = 0$, and the third identity is an application of the Leibniz rule on $G_i G_j$. For the second identity,

$$\begin{aligned}
\Delta (G_i G_j) &= \Delta (G_i) G_j + G_j \Delta (G_i) + 2G_{ik} G_{jk} \\
&= [(\Delta G)_i + R_{ik} G_k] G_j + [(\Delta G)_j + R_{jk} G_k] G_i + 2G_{ik} G_{jk} \\
&= R_{ik} G_j G_k + R_{jk} G_i G_k + 2G_{ik} G_{jk}.
\end{aligned}$$

We also derive that for any β,

$$\Delta G^{\beta} = \operatorname{div}(\beta \cdots G^{\beta-1}\nabla G) = \beta(\beta-1)G^{\beta-2}|\nabla G|^2,$$

from which the last identity is immediate and it follows that $\Delta(G^{-1}) = 2G^{-3}|\nabla G|^2$. We use this and the third identity to check the fourth identity,

$$\begin{aligned}\Delta\left(\frac{G_iG_j}{G}\right) &= \frac{\Delta(G_iG_j)}{G} + \Delta(G^{-1})G_iG_j - \frac{2}{G^2}g\big(\nabla G, \nabla(G_iG_j)\big) \\ &= \frac{\Delta(G_iG_j)}{G} + \frac{2|\nabla G|^2G_iG_j}{G^3} - \frac{2G_k(G_iG_{jk}+G_jG_{ik})}{G^2}.\end{aligned}$$

□

Lemma 4.3.3. *Let* $B = \dfrac{\nabla G\otimes\nabla G}{G}$, *or equivalently in coordinates,* $B_{ij} = \dfrac{G_iG_j}{G}$ *for an orthonormal frame* $\{e_i\}$. *Then* $B^2 = \dfrac{|\nabla G|^2}{G}B$.

Proof.

$$(B^2)_{ij} = \frac{G_iG_k\cdot G_jG_k}{G^2} = \frac{|\nabla G|^2}{G}\cdot\frac{G_iG_j}{G} = \frac{|\nabla G|^2}{G}B_{ij}.$$

□

We are now ready to prove Lemma 4.2.3.

Proof of Lemma 4.2.3. By Lemma 4.3.2, we have that

$$\begin{aligned}&\Delta(\widetilde{H}_{ij} - \frac{n-2}{2}C\cdot G^{\alpha}g_{ij}) \\ &\quad= \Delta\left(G_{ij} + \frac{n}{2-n}\cdot\frac{G_iG_j}{G}\right) \\ &\quad= R_{jk}G_{ik} + R_{ik}G_{jk} - 2R_{ikjl}G_{kl} + \frac{n}{2-n}\Bigg(\frac{R_{ik}G_jG_k + R_{jk}G_iG_k}{G} \\ &\qquad\qquad + \frac{2G_{ik}G_{jk}}{G} + \frac{2|\nabla G|^2G_iG_j}{G^3} - \frac{2G_k(G_iG_{jk}+G_jG_{ik})}{G^2}\Bigg)\end{aligned}$$

$$= R_{ik}\left(G_{jk} + \frac{n}{2-n}\cdot\frac{G_jG_k}{G}\right) + R_{jk}\left(G_{ik} + \frac{n}{2-n}\cdot\frac{G_iG_k}{G}\right) - 2R_{ikjl}G_{kl} + \frac{2n}{(2-n)G}\big[\mathrm{Hess}_G - B\big]^2_{ij}.$$

Substituting the derivatives of G with expressions in $\widetilde{H}$, we obtain that

$$\begin{aligned}
&\Delta(\widetilde{H}_{ij} - \frac{n-2}{2}C\cdot G^\alpha g_{ij})\\
&= R_{ik}\left(\widetilde{H}_{jk} - \frac{n-2}{2}C\cdot G^\alpha g_{jk}\right) + R_{jk}\left(\widetilde{H}_{ik} - \frac{n-2}{2}C\cdot G^\alpha g_{ik}\right)\\
&\quad - 2R_{ikjl}\left(\widetilde{H}_{kl} - \frac{n}{2-n}\frac{G_iG_j}{G} - \frac{n-2}{2}C\cdot G^\alpha g_{kl}\right)\\
&\quad + \frac{2n}{(2-n)G}\left[\widetilde{H} - \frac{n}{2-n}B - \frac{n-2}{2}C\cdot G^\alpha g - B\right]^2_{ij}.
\end{aligned}$$

We expand the square term and rearrange as follows.

$$\begin{aligned}
&\Delta(\widetilde{H}_{ij} - \frac{n-2}{2}C\cdot G^\alpha g_{ij})\\
&= R_{ik}\widetilde{H}_{jk} + R_{jk}\widetilde{H}_{ik} - 2R_{ikjl}\widetilde{H}_{kl} - \frac{2n}{n-2}R_{ikjl}\frac{G_iG_j}{G}\\
&\quad - \frac{2n}{(n-2)G}\left[\widetilde{H} - \frac{2}{2-n}B - \frac{n-2}{2}C\cdot G^\alpha g\right]^2_{ij}\\
&= R_{ik}\widetilde{H}_{jk} + R_{jk}\widetilde{H}_{ik} - 2R_{ikjl}\widetilde{H}_{kl} - \frac{2n}{n-2}R_{ikjl}\frac{G_iG_j}{G} - \frac{2n}{(n-2)G}(\widetilde{H})^2_{ij}\\
&\quad - \frac{8n}{(n-2)^3G}(B^2)_{ij} - \frac{n(n-2)}{2}C^2G^{2\alpha-1}g_{ij} + \frac{4n}{n-2}C\cdot G^{\alpha-1}B_{ij}\\
&\quad + \frac{2n}{(n-2)G}\left[\widetilde{H}\left(\frac{2}{2-n}B + \frac{n-2}{2}C\cdot G^\alpha g\right) + \left(\frac{2}{2-n}B + \frac{n-2}{2}C\cdot G^\alpha g\right)\widetilde{H}\right]_{ij}.
\end{aligned}$$

Using Lemma 4.3.3 to replace B^2 with $\frac{|\nabla G|^2}{G}B$, it follows that

$$\begin{aligned}
&\Delta(\widetilde{H}_{ij} - \frac{n-2}{2}C\cdot G^\alpha g_{ij})\\
&= R_{ik}\widetilde{H}_{jk} + R_{jk}\widetilde{H}_{ik} - 2R_{ikjl}\widetilde{H}_{kl} - \frac{2n}{n-2}R_{ikjl}\frac{G_iG_j}{G} - \frac{2n}{(n-2)G}(\widetilde{H})^2_{ij}
\end{aligned}$$

$$
\begin{aligned}
&- \frac{8n}{(n-2)^3}\frac{|\nabla G|^2}{G^2}B_{ij} - \frac{n(n-2)}{2}C^2G^{2\alpha-1}g_{ij} + \frac{4n}{n-2}C\cdot G^{\alpha-1}B_{ij} \\
&+ \frac{2n}{(n-2)G}\left[\widetilde{H}\left(\frac{2}{2-n}B + \frac{n-2}{2}C\cdot G^{\alpha}g\right) + \left(\frac{2}{2-n}B + \frac{n-2}{2}C\cdot G^{\alpha}g\right)\widetilde{H}\right]_{ij} \\
=\ & R_{ik}\widetilde{H}_{jk} + R_{jk}\widetilde{H}_{ik} - 2R_{ikjl}\widetilde{H}_{kl} - \frac{2n}{n-2}R_{ikjl}\frac{G_iG_j}{G} - \frac{2n}{(n-2)G}(\widetilde{H})^2_{ij} \\
&- \frac{n(n-2)}{2}C^2G^{2\alpha-1}g_{ij} + \frac{4n}{n-2}\left[C\cdot G^{\alpha-1} - \frac{2|\nabla G|^2}{(n-2)^2G^2}\right]B_{ij} \\
&+ \frac{2n}{(n-2)G}\left[\widetilde{H}\left(\frac{2}{2-n}B + \frac{n-2}{2}C\cdot G^{\alpha}g\right) + \left(\frac{2}{2-n}B + \frac{n-2}{2}C\cdot G^{\alpha}g\right)\widetilde{H}\right]_{ij}.
\end{aligned}
$$

Thus we have proved Lemma 4.2.3. □

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