Stable Characters for Symmetric Groups and Wreath Products

by

Christopher Ryba

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Abstract

Given a Hopf algebra $\mathcal{R}$, the Grothendieck group of $\mathcal{C} = \mathcal{R}$-mod inherits the structure of a ring. We define a ring $\mathcal{G}_{\infty}^{\mathbb{Z}}(\mathcal{C})$, which is "the $n \to \infty$ limit" of the Grothendieck rings of modules for the wreath products $\mathcal{R} \wr S_n$; it is the Grothendieck group of a certain wreath product Deligne category. The construction yields a basis of $\mathcal{G}_{\infty}^{\mathbb{Z}}(\mathcal{C})$ corresponding to irreducible objects. The structure constants of this basis are stable tensor product multiplicities for the wreath products $\mathcal{R} \wr S_n$.

We generalise $\mathcal{G}_{\infty}^{\mathbb{Z}}(\mathcal{C})$, allowing an arbitrary ring to be substituted for the Grothendieck ring of $\mathcal{C}$. Aside from being a Hopf algebra, $\mathcal{G}_{\infty}^{\mathbb{Z}}(\mathcal{C})$ is the algebra of distributions on a certain affine group scheme.

In the special case where $\mathcal{C}$ is the category of vector spaces (over $\mathbb{C}$, say), $\mathcal{G}_{\infty}^{\mathbb{Z}}(\mathcal{C})$ is the ring of symmetric functions. The basis obtained by our construction is the family of stable Specht polynomials, which is closely related to the problem of calculating restriction multiplicities from $GL_n(\mathbb{C})$ to S_n. We categorify the stable Specht polynomials by producing a resolution of irreducible representations of S_n by modules restricted from $GL_n(\mathbb{C})$.

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Contents

Chapter 1

Introduction

Perhaps the most important open question in the characteristic zero representation theory of the symmetric group is the *Kronecker problem*. Let $\mathcal{S}^\lambda$ be the irreducible representation of the symmetric group S_n indexed by a partition λ of size n. The Kronecker coefficients $k_{\mu,\nu}^{\lambda}$ are the tensor product multiplicities of the irreducible representations of symmetric groups:

$$k_{\mu,\nu}^{\lambda} = \dim(\hom_{\mathbb{C}S_n}(\mathcal{S}^\lambda, \mathcal{S}^\mu \otimes \mathcal{S}^\nu)).$$

The Kronecker problem asks for a manifestly positive combinatorial formula for the $k_{\mu,\nu}^{\lambda}$. Although a solution to the problem remains out of reach over a century after it was posed, considerable progress has been made.

For a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$, let $\lambda[n] = (n - |\lambda|, \lambda_1, \lambda_2, \ldots, \lambda_r)$, which is a partition of size n provided that $n \geq |\lambda| + \lambda_1$. A remarkable result of Murnaghan [20] states that for any three partitions, λ, μ, ν, the limit

$$\tilde{k}_{\mu,\nu}^{\lambda} = \lim_{n\to\infty} k_{\mu[n],\nu[n]}^{\lambda[n]}$$

exists and is finite. The limiting values $\tilde{k}_{\mu,\nu}^{\lambda}$ are called the *reduced Kronecker coefficients*. The existence of the reduced Kronecker coefficients is a "stability phenomenon" closely related to a number of topics of active research, such as Deligne categories and FI-modules.

The character values of $\mathcal{S}^{\lambda[n]}$ (for fixed λ and varying n) were shown by Specht [30] to be polynomial in a certain sense. Recently, Orellana and Zabrocki [21] (and independently, Assaf and Speyer [2]) introduced a basis $s^\dagger_\lambda$ of the ring symmetric functions manifesting this polynomiality. The $s^\dagger_\lambda$ may be thought of as "stable characters" of symmetric groups, making the ring of symmetric functions a "stable Grothendieck ring".

In this thesis, we construct a theory of stable characters and stable Grothendieck rings for the wreath products $\mathcal{R} \wr S_n$, where $\mathcal{R}$ is a Hopf algebra, or more generally a tensor category. A key feature is that if $\mathcal{R}$ is not cocommutative, the stable Grothendieck ring may not be commutative (and so cannot be isomorphic to the ring of symmetric functions, which is commutative). Let $\mathcal{C} = \mathcal{R} - \text{mod}$. We construct a stable Grothendieck ring $\mathcal{G}^{\mathbb{Z}}_\infty(\mathcal{C})$ and determine its structure. In particular, we show that as $\mathbb{Q}$-algebras,

$$\mathbb{Q} \otimes_{\mathbb{Z}} \mathcal{G}^{\mathbb{Z}}_\infty(\mathcal{C}) = \bigotimes_{i=1}^{\infty} U(\mathcal{G}(\mathcal{C})),$$

where $U(\mathcal{G}(\mathcal{C}))$ is the universal enveloping algebra of the rational-coefficient Grothendieck ring of $\mathcal{C}$, viewed as a Lie algebra. It turns out that $\mathcal{G}^{\mathbb{Z}}_\infty(\mathcal{C})$ has the structure of a Hopf algebra, and is isomorphic to the Hopf algebra of distributions supported at the identity of a particular affine group scheme.

There is a $\mathbb{Z}$-basis $X_{\vec{\lambda}}$ of $\mathcal{G}^{\mathbb{Z}}_\infty(\mathcal{C})$ generalising the $s^\dagger_\lambda$ basis of the ring of symmetric functions. We give a generating function expressing $X_{\vec{\lambda}}$ in terms of certain generators.

One of the original motivations for considering the $s^\dagger_\lambda$ was to understand restriction multiplicities along the embedding of S_n into $GL_n(\mathbb{C})$ as permutation matrices; expressing Schur functions in terms of $s^\dagger_\lambda$ amounts to solving this problem, at least for n sufficiently large. We produce a resolution of $\mathbb{S}^{\lambda[n]}$ by representations restricted from $GL_n(\mathbb{C})$, categorifying the $s^\dagger_\lambda$.

The thesis is organised as follows. In Chapter 2, we review the background material on partitions, symmetric functions, and homological algebra, which also serves to establish notation. We then discuss related topics (Deligne categories, $\text{Rep}(\mathfrak{S})$, properties of $s^\dagger_\lambda$) in Chapter 3. This is not logically required for the remainder of the thesis and may be skipped.

In Chapter 4, we construct the stable Grothendieck ring $\mathcal{G}_\infty^{\mathbb{Z}}(\mathcal{C})$, and study it over $\mathbb{Q}$. We then analyse the properties of $\mathcal{G}_\infty^{\mathbb{Z}}$ as a $\mathbb{Z}$-algebra in Chapter 5 and conclude with a categorification of the symmetric functions $s_\lambda^\dagger$ using Lie algebra (co)homology in Chapter 6. The last three chapters may be read independently.

Chapter 2

Background

In this chapter we review the background material used in the remainder of the thesis.

2.1 Partitions and Symmetric Functions

We will make considerable use of partition combinatorics, which we review briefly. All the material that we will need can be found in the first chapter of [18].

We say that $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ is a partition of n if $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_l$ are nonnegative integers summing to n, and we call the λ_i the parts of the partition. Two partitions that differ only by the number of trailing zeroes are considered equivalent (in particular, $\lambda_i = 0$ when $i > l$). The set of all partitions is denoted $\mathcal{P}$. The expression (n, λ) is an abbreviation for $(n, \lambda_1, \lambda_2, \ldots, \lambda_l)$, which is also a partition provided $n \geq \lambda_1$. We write $\lambda \vdash n$ to mean that λ is a partition of n. An alternative way of expressing λ is $(1^{m_1} 2^{m_2} \cdots r^{m_r})$, where m_i is the number of j such that $\lambda_j = i$; in case it is unclear which partition we are considering, we write $m_i(\lambda)$ for the number of parts of λ equal to i. The size of λ is $|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_l = 1m_1 + 2m_2 + \cdots + rm_r$ (where $r = \lambda_1$ is the largest part of λ), $|\lambda|$ is the unique integer n such that $\lambda \vdash n$. The length $l(\lambda)$ is the number of nonzero parts of λ, so we have $l(\lambda) = m_1 + m_2 + \cdots + m_r$. If $\lambda^{(j)}$ are partitions, we write $\cup_j \lambda^{(j)}$ for the partition μ obtained by merging all the partitions $\lambda^{(j)}$ together, so $m_i(\mu) = \sum_j m_i(\lambda^{(j)})$. We write $\varepsilon_\lambda = (-1)^{|\lambda|-l(\lambda)}$, and $z_\lambda = (m_1!)1^{m_1}(m_2!)2^{m_2}\cdots(m_n!)n^{m_n}$.

Recall that the ring of symmetric functions, Λ, is defined as a graded inverse limit of the rings of invariants $\mathbb{Z}[x_1, x_2, \ldots, x_n]^{S_n}$, where the symmetric group acts by permuting the variables. It is freely generated as a polynomial algebra by the elementary symmetric functions e_i, and also by the complete symmetric functions h_i, so $\Lambda = \mathbb{Z}[e_1, e_2, \ldots] = \mathbb{Z}[h_1, h_2, \ldots]$. There are also power-sum symmetric functions p_n which do not generate Λ over $\mathbb{Z}$, but do satisfy $\Lambda_\mathbb{Q} = \Lambda \otimes_\mathbb{Z} \mathbb{Q} = \mathbb{Q}[p_1, p_2, \ldots]$. If we define the formal power series $E(t) = \sum_{n=0}^\infty e_n t^n$, $H(t) = \sum_{n=0}^\infty h_n t^n$ (here $e_0 = h_0 = 1$), and $P(t) = \sum_{n=0}^\infty p_{n+1} t^n$, then we have the relations $H(t)E(-t) = 1$, and $\frac{E'(t)}{E(t)} = P(-t)$. For a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$, we write $e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_l}$, and similarly we define h_λ and p_λ (so the e_λ and h_λ form $\mathbb{Z}$-bases of Λ). Another important family of symmetric functions are the Schur functions s_λ (indexed by $\lambda \in \mathcal{P}$), which form a $\mathbb{Z}$-basis of Λ.

The irreducible representations of the symmetric group S_n in characteristic zero are indexed by partitions $\lambda \vdash n$. They are called Specht modules and are denoted by $\mathcal{S}^\lambda$. Since the conjugacy classes of S_n are also parametrised by partitions of n via cycle type, we may write χ^λ_μ for the value of the character of $\mathcal{S}^\lambda$ on an element of cycle type μ. This allows us to express the Schur function s_λ in terms of power-sum symmetric function as follows:

$$s_\lambda = \sum_{\mu \vdash |\lambda|} \frac{\chi^\lambda_\mu p_\mu}{z_\mu}.$$

Since $h_n = s_{(n)}$ and $e_n = s_{(1^n)}$, we have:

$$h_n = \sum_{\lambda \vdash n} \frac{p_\lambda}{z_\lambda}, \qquad e_n = \sum_{\lambda \vdash n} \frac{\varepsilon_\lambda p_\lambda}{z_\lambda}.$$

There is a nondegenerate bilinear form $\langle -, - \rangle$ on Λ for which the Schur functions are orthonormal. It satisfies $\langle p_\lambda, p_\mu \rangle = \delta_{\lambda,\mu} z_\lambda$, where $\delta_{\lambda,\mu}$ is the Kronecker delta.

One can think of elements of $\Lambda \otimes \Lambda$ as symmetric functions that are symmetric in two sets of variables separately. We write $f(\mathbf{x})$ to indicate that f is a symmetric function in the set of variables $\{x_i\}$ (we will suppress the index set of the variables), or $f(\mathbf{x}, \mathbf{y})$ to indicate that

f is a symmetric function in the set of variables $\{x_i\} \cup \{y_j\}$. Similarly, we write $f(\mathbf{xy})$ when the variable set is $\{x_i y_j\}$, for example, $p_n(\mathbf{x}, \mathbf{y}) = p_n(\mathbf{x}) + p_n(\mathbf{y})$ and $p_n(\mathbf{xy}) = p_n(\mathbf{x})p_n(\mathbf{y})$. With this in mind, we have the Cauchy identity:

$$\prod_{i,j} \frac{1}{1 - x_i y_j} = \sum_{\lambda \in \mathcal{P}} s_\lambda(\mathbf{x}) s_\lambda(\mathbf{y}) = \sum_{\mu \in \mathcal{P}} \frac{p_\mu(\mathbf{x}) p_\mu(\mathbf{y})}{z_\mu}$$

We also note that $s_\lambda(\mathbf{xy}) = \sum_{\mu,\nu \in \mathcal{P}} k^\lambda_{\mu,\nu} s_\mu(\mathbf{x}) s_\nu(\mathbf{y})$, where $k^\lambda_{\mu,\nu}$ are the Kronecker coefficients, defined for $|\lambda| = |\mu| = |\nu|$ as multiplicities in tensor products of Specht modules and taken to be zero otherwise:

$$\mathcal{S}^\mu \otimes \mathcal{S}^\nu = \bigoplus_\lambda \left(\mathcal{S}^\lambda\right)^{\oplus k^\lambda_{\mu,\nu}}$$

On the other hand, $s_\lambda(\mathbf{x}, \mathbf{y}) = \sum_{\mu,\nu} c^\lambda_{\mu,\nu} s_\mu(\mathbf{x}) s_\nu(\mathbf{y})$, where $c^\lambda_{\mu,\nu}$ are the Littlewood-Richardson coefficients, which also satisfy the property that $s_\mu s_\nu = \sum_\lambda c^\lambda_{\mu,\nu} s_\lambda$. The Littlewood-Richardson coefficient $c^\lambda_{\mu,\nu}$ is taken to be zero if $|\mu| + |\nu| \neq |\lambda|$.

We will consider symmetric functions with many variable sets, some of which may be repeated. To indicate that a certain variable set occurs multiple times, we write $f(\mathbf{x}, \mathbf{x}, \ldots, \mathbf{x}) = f(\mathbf{x}^{\oplus n})$, where n is the number of occurrences of $\mathbf{x}$ in the left hand side. More generally, if we have a family of variable sets $\mathbf{x}^{(i)}$, then we write $f(\mathbf{x}^{(i_1)}, \mathbf{x}^{(i_2)}, \ldots) = f(\bigoplus_i \mathbf{x}^{(i)})$.

Definition 2.1.1. *Suppose that λ is a partition, and n is an integer such that $n - |\lambda| \geq \lambda_1$. We write $\lambda[n] = (n - |\lambda|, \lambda)$ for the partition obtained by adding a part at the beginning of λ such that the total size is n. (If $n - |\lambda| < \lambda_1$, we leave $\lambda[n]$ undefined for now, although we will revisit this in Chapter 6.)*

The Kronecker coefficients famously satisfy the following stability property [20].

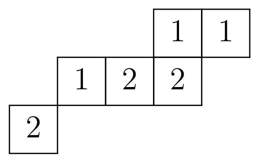
Lemma 2.1.2. *The sequence of Kronecker coefficients $k^{\lambda[n]}_{\mu[n],\nu[n]}$ eventually becomes constant, and the stable limit, called the reduced Kronecker coefficient, is denoted $\tilde{k}^\lambda_{\mu,\nu}$.*

The following result shows that in special cases, more can be said [9].

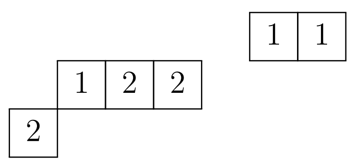
Lemma 2.1.3. *If $|\mu| + |\nu| = |\lambda|$, then the reduced Kronecker coefficient agrees with the Littlewood-Richardson coefficient: $\tilde{k}^\lambda_{\mu,\nu} = c^\lambda_{\mu,\nu}$. If $|\mu| + |\nu| < |\lambda|$, then $\tilde{k}^\lambda_{\mu,\nu} = 0$.*

Proposition 2.1.4. *Suppose that for a partition λ we write λ^{*m} for the partition obtained by adding m to λ_1. Then if λ, μ, ν are partitions, then the sequence of Littlewood-Richardson coefficients $c^{\lambda}_{\mu,\nu}, c^{\lambda^{*1}}_{\mu^{*1},\nu}, c^{\lambda^{*2}}_{\mu^{*2},\nu}, \ldots$ is eventually constant.*

Proof. Using the Littlewood-Richardson rule, it suffices to count the number of skew tableaux of shape λ^{*m}/μ^{*m} and weight ν satisfying the lattice word condition. The diagrams are related for successive m by shifting the first row. We illustrate this with an example. Suppose $\lambda = (5, 4, 1)$, $\mu = (3, 1)$ and $\nu = (3, 3)$. An example of a skew-tableau of shape λ/μ and weight ν is



But when $\lambda^{*2} = (7, 4, 1)$, $\mu^{*2} = (5, 1)$ and $\nu = (3, 3)$ an example of a skew-tableau of shape λ^{*2}/μ^{*2} and weight ν is



Increasing m in λ^{*m}/μ^{*m} shifts the top row of the diagram to the right. As soon as m is large enough that the first row of the skew diagram of shape λ^{*m}/μ^{*m} is disconnected from the rest of the diagram then the operation of further shifting the top row to the right leads to an obvious bijection of skew tableaux. The disconnected condition guarantees that the tableau property is unaffected by shifting the top row. Since it also preserves the lattice words associated to the tableaux, the lattice word property is also preserved by this row-shifting bijection. Counting the number of such tableaux gives the Littlewood-Richardson coefficient $c^{\lambda^{*m}}_{\mu^{*m},\nu}$, which gives the result. □

We recall some facts about representations of general linear groups in characteristic zero, as well the Koszul and Chevalley-Eilenberg complexes. These will be important in Chapter 6.

For a vector space V over k, the irreducible polynomial representations of the general linear group $GL(V)$ are given by Schur functors $\mathbb{S}^\lambda(V)$ for $l(\lambda) \leq \dim(V)$. Here,

$$\mathbb{S}^\lambda(V) = \left(V^{\otimes n} \otimes \mathcal{S}^\lambda\right)^{S_n},$$

where the action of S_n on $V^{\otimes n}$ is by permutation of tensor factors. In particular, $\mathbb{S}^{(n)}(V)$ and $\mathbb{S}^{(1^n)}(V)$ are the n-th symmetric and exterior powers of V respectively. The character of $\mathbb{S}^\lambda(V)$ is the Schur function s_λ in the following sense. Fix a matrix $g \in GL(V)$, whose eigenvalues are $\{x_i\}$, then the trace of g acting on $\mathbb{S}^\lambda(V)$ is equal to $s_\lambda(\mathbf{x})$. One formulation of Schur-Weyl duality asserts that the functor

$$SW_n(-) = \left(V^{\otimes n} \otimes -\right)^{S_n}$$

is an equivalence of categories between degree n polynomial representations of $GL(V)$ and the of representations of S_n, provided that $\dim(V) \geq n$.

Given a Lie algebra $\mathfrak{g}$, the Lie algebra homology with coefficients in a $\mathfrak{g}$-module M is defined as

$$H_i(\mathfrak{g}, M) = \mathrm{Tor}_i^{U(\mathfrak{g})}(k, M),$$

where $U(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$, and k is the trivial $\mathfrak{g}$-module. The Lie algebra homology may be computed with the Chevalley-Eilenberg complex, whose r-th term is

$$\bigwedge^r(\mathfrak{g}) \otimes M.$$

We omit the differential because we will not need it. Details can be found in [33]. There is a similar complex computing Lie algebra cohomology

$$H^i(\mathfrak{g}, M) = \mathrm{Ext}^i_{U(\mathfrak{g})}(k, M),$$

whose chain groups are

$$\hom_k\left(\bigwedge^i(\mathfrak{g}), M\right).$$

Let $A = \bigoplus_{i\geq 0} A_i$ be a $\mathbb{Z}_{\geq 0}$-graded algebra such that $A_0 = k$, and each A_i is finite dimensional over k. Let $A^+ = \oplus_{i>0} A_i$ be the positively graded part of A. Then $k = A/A^+$ is a graded A-module. We say that A is Koszul if the induced ("internal") grading on the Ext algebra

$$\mathrm{Ext}^\bullet_A(k,k)$$

coincides with the homological grading.

A Koszul algebra A is necessarily a quadratic algebra, i.e. it is generated in degree one with relations in degree two:

$$A = T(A_1)/(R),$$

where $R \subseteq A_1 \otimes A_1$. We may define the Koszul dual algebra $A^!$ to be

$$A^! = T(A_1^*)/(R^\perp),$$

where $R^\perp \subseteq A_1^* \otimes A_1^*$ is the set of elements orthogonal to R under the natural pairing. Note that $A^!$ is also a graded algebra and $A^{!!} = A$.

Koszul algebras admit a free resolution of k (as a graded A-module) called the Koszul complex, whose r-th chain module is

$$A \otimes (A^!_r)^* = \mathrm{Hom}_k(A^!_r, A).$$

When we view the elements as linear functions from $A^!_r$ to A, the differential is given by

$$d(f) = \sum_i x_i f(- \cdot x_i^*),$$

where $\{x_i\}$ is a basis of A_1, and $\{x_i^*\}$ is the dual basis of A_1^*. The tensor product of Koszul algebras is again Koszul, and the associated Koszul complex is the tensor product of the individual Koszul complexes.

Two particular examples will be useful for us. The first is that $\mathrm{Sym}(V)$ and $\bigwedge(V^*)$ are

Koszul dual, yielding the Koszul complex

$$\cdots \to \mathrm{Sym}(V) \otimes \bigwedge^2(V) \to \mathrm{Sym}(V) \otimes V \to \mathrm{Sym}(V) \otimes k \to 0,$$

where the differential is given by $d = \sum_i v_i \otimes v_i^*$, where v_i is a basis of V (acting by multiplication on $\mathrm{Sym}(V)$), and v_i^* is the dual basis of V^* (acting on $\bigwedge(V)$ by contraction). The second example is the tensor algebra $T(V)$, and the algebra $k \oplus V^*$, where k has degree zero and V^* has degree one. This gives the complex

$$0 \to T(V) \otimes V \to T(V) \to 0,$$

where the map in question is simply multiplication (viewing V as the degree one graded part of $T(V)$). Finally, we remark that both of these complexes are $GL(V)$-equivariant and also they are well defined even if V is infinite dimensional (although the corresponding algebras are not Koszul). This is because the complexes are the direct limit of the corresponding complexes for finite dimensional subspaces of V, and direct limits preserve exactness.

Chapter 3

Related Literature

In this chapter, we survey literature closely related to this thesis. It is not logically required for the remainder of this thesis, and may be skipped.

3.1 Deligne Categories

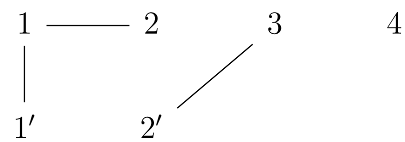
In this section we work over a commutative ground ring R with a distinguished element $t \in R$. The two main cases of interest are $R = \mathbb{Q}$ with $t \in \mathbb{Z}_{\geq 0}$ and $R = \mathbb{Z}[t]$, where the distinguished element is the variable t. We omit most of the proofs; they can be found in [7], which serves as an excellent introduction.

A (m, n)-*partition diagram* is a set partition of the set

$$W_{m,n} = \{1, 2, \dots, m, 1', 2', \dots, n'\}.$$

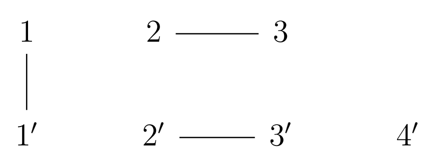
We write $\mathrm{Par}_{m,n}$ for the set of these. We depict such a partition diagram by means of a graph whose vertices are labelled by the set $W_{m,n}$ and whose connected components are the parts of the set partition. Although there are many choices of graphs with the same connected components, they are all equivalent for our purposes. For convenience we arrange the vertices into two rows, the first consisting of the unprimed vertices and the second consisting of the primed vertices.

Example 3.1.1. *The diagram below represents the set partition* $\{\{1,2,1'\},\{3,2'\},\{4\}\} \in Par_{4,2}$.



The *pre-Deligne category* $\underline{\mathrm{Rep}}_0(S_t)$ is defined as follows. The objects $[n]$ are indexed by $n \in \mathbb{Z}_{\geq 0}$. The morphisms $\mathrm{Hom}([m],[n])$ are R-linear combinations of (m,n)-partition diagrams. Composition of morphisms is R-linear and the composition of a (m,n)-partition diagram D_1 with a (k,m)-partition diagram D_2 is obtained by the following process. Consider the labelled graphs associated to the two partition diagrams; identify the vertices $1,2,\ldots,m$ in the first partition diagram with the elements $1',2',\ldots,m'$ in the second partition diagram, retaining all edges. This yields the graph of a (k,n)-partition diagram D_3 (containing some of the intermediate vertices which were identified) together with r connected components consisting only of vertices which were identified. We define $D_1 \circ D_2 = t^r D_3$.

Example 3.1.2. *We demonstrate composition of morphisms in the pre-Deligne category by composing the partition* $\{\{1,2,1'\},\{3,2'\},\{4\}\} \in \mathrm{Par}_{4,2}$ *from Example 3.1.1 with the partition* $\{\{1,1'\},\{2,3\},\{2',3'\},\{4'\}\} \in \mathrm{Par}_{3,4}$, *which corresponds to the following diagram:*



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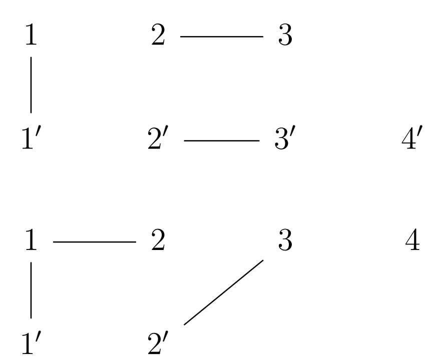
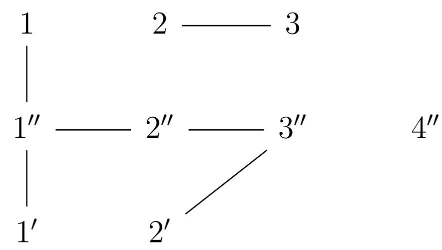
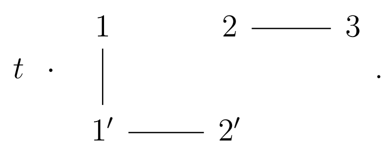
We now show the two partitions so that the vertices to be merged are vertically adjacent to each other.



Merging the appropriate vertices gives the following intermediate result, where we label the merged vertices with double primes.



Noting that we have one connected component that is not connected to either $1, 2, 3$ *or* $1', 2'$ *(namely,* $\{4''\}$*), we obtain a scalar factor of* t^1 *when we pass to the induced diagram on* $\{1, 2, 3, 1', 2'\}$*. Our final result is*



There is a symmetric monoidal structure on $\underline{\mathrm{Rep}}_0(S_t)$ given by $[n] \otimes [m] = [n+m]$ on objects, with the action on morphisms given by concatenation (and relabelling) as follows. If $D_1 \in \mathrm{Par}_{m_1,n_1}$ and $D_2 \in \mathrm{Par}_{m_2,n_2}$, then $D_1 \otimes D_2 \in \mathrm{Par}_{m_1+m_2,n_1+n_2}$ is the diagram where vertices with labels in $\{1, 2, \ldots, m_1, 1', 2', \ldots, n_1'\}$ have the same connections as D_1, while vertices with labels in

$$\{m_1 + 1, m_1 + 2, \ldots, m_1 + m_2, (n_1 + 1)', (n_1 + 2)', \ldots, (n_1 + n_2)'\}$$

have the same connections as D_2. Here, the vertex set is identified with that of D_2 by subtracting m_1 from vertex labels without primes, and n_1 from vertex labels with primes. No edges are included between the vertices corresponding to D_1 and the vertices corresponding to D_2.

In fact, $\underline{\mathrm{Rep}}_0(S_t)$ is a rigid symmetric monoidal category, but we will not discuss this here. We direct the interested reader to Section 2 of [7].

The *Deligne category* $\underline{\mathrm{Rep}}(S_t)$ is the Karoubian envelope of the pre-Deligne category $\underline{\mathrm{Rep}}_0(S_t)$. It inherits the structure of a rigid symmetric monoidal category.

Suppose that the distinguished element $t \in R$ is set to be $d \in \mathbb{Z}_{\geq 0}$. There is a symmetric monoidal functor F_d from $\underline{\mathrm{Rep}}(S_d)$ (the Deligne category with $t = d$) to $RS_d - \mathrm{mod}$, the category of finitely generated representations of the symmetric group S_d over R. This functor is defined first for $\underline{\mathrm{Rep}}_0(S_t)$, and then we pass to the Karoubian envelope. Let V be the permutation representation of S_d on R^d. Then, $F_d([m]) = V^{\otimes m}$. To define the action of F_n on morphisms, let $v_1, v_2, \ldots, v_n$ be the standard basis of V. Then a diagram $D \in \mathrm{Par}_{p,q}$ acts on the pure tensor $v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_p}$ to give the sum of all $v_{i_{1'}} \otimes v_{i_{2'}} \otimes \cdots \otimes v_{i_{q'}}$ such that if two vertices x, y (possibly with or without primes) are in the same component of D, then $i_x = i_y$.

Taking the R-span of $\mathrm{Par}_{n,n}$ for a fixed $n \in \mathbb{Z}_{\geq 0}$, we obtain an associative algebra (depending on the parameter t) using this multiplication. These algebras are called the partition algebras, and are often denoted $\mathrm{Par}_n(t)$. By the above, we have a map $\mathrm{Par}_n(d) \to \mathrm{End}_{S_d}(V^{\otimes n})$. This has been used to study an analogue of Schur-Weyl duality between symmetric group algebras and partition algebras. It allows for the study of the representations of one algebra in terms of the other. This has been the subject of much work in recent years; for example, [3] and [4].

Suppose that we take R to be a field of characteristic zero. Then the indecomposable objects X_μ are indexed by the set of all partitions μ. Further, $F_d(X_\mu)$ is equal to $S^{(d-|\mu|,\mu)}$ if $d - |\mu| \geq \mu_1$ (so that this defines a valid partition), and $F_d(X_\mu) = 0$ otherwise.

So, the objects X_μ (which we have not defined here) can be thought of as "interpolating" the irreducible representations $\mathcal{S}^{\mu[n]}$ of S_n as n varies. This can be used to show that the structure constants for the monoidal product are the reduced Kroencker coefficients (see [1]). However, Deligne categories see much more categorical structure than just this: when $R = \mathbb{C}$ (or any field of characteristic zero), F_d is essentially surjective and full.

There are several variations of this construction that "interpolate" other representation categories. In particular, $\underline{\mathrm{Rep}}(GL_t), \underline{\mathrm{Rep}}(O_t), \underline{\mathrm{Rep}}(Sp_t)$ in the case of classical groups. Another example is the case of wreath products (with a fixed tensor category). More details can be found in [11] and [19]

3.2 The Category $\mathrm{Rep}(\mathfrak{S})$

Let S_∞ be the group of permutations of $\mathbb{Z}_{>0}$ that fix all but finitely many elements. We let $V = k^\infty$ be the permutation representation of S_∞. An algebraic representation of S_∞ is defined to be a subquotient of a direct sum of tensor powers of V. We let $\mathrm{Rep}(\mathfrak{S})$ be the category of algebraic representations. We follow Section 6 of [27].

In certain respects, $\mathrm{Rep}(\mathfrak{S})$ is quite different from the representation categories of finite symmetric groups. For example, although S_∞ has a well-defined sign representation, it is not algebraic, so it is not an object of $\mathrm{Rep}(\mathfrak{S})$. Note that the trivial representation of S_∞ is a quotient of V, by mapping each element of V to its sum of coordinates in k. However, the trivial representation is not a subrepresentation of V; any invariant vector would have equal coordinates, but all but finitely many entries of a vector in V must be zero. Thus, $\mathrm{Rep}(\mathfrak{S})$ is not semisimple.

The simple objects of $\mathrm{Rep}(\mathfrak{S})$ may be described in a way that is consistent with finite symmetric groups. Let $V = k^n$ be the permutation representation of S_n. For $1 \leq i \leq n$, we have a map $t_i : V^{\otimes r} \to V^{\otimes(r-1)}$ that is the quotient map $V \to k$ on the i-th tensor factor. Note that componentwise multiplication $s : V \otimes V \to V$ respects permutation of coordinates, and hence is a homomorphism. For $1 \leq i < j \leq n$ we also have maps $s_{i,j} : V^{\otimes r} \to V^{\otimes(r-1)}$ which are s applied to the i-th and j-th tensor factors. Let $T^n_{[r]}$ be the intersection of the kernels of all the t_i and $s_{i,j}$. One can check that $T^n_{[r]}$ is preserved by the action of S_r on $V^{\otimes r}$ by permuting tensor factors. For a partition $\mu \vdash r$, let

$$V^n_\lambda = \hom_{kS_r}(\mathcal{S}^\mu, T^n_{[r]}).$$

This is a representation of S_n, which is isomorphic to $\mathcal{S}^{\lambda[n]}$ if $n \geq |\lambda| + \lambda_1$ (i.e. $\lambda[n]$ is a partition), and is zero otherwise. The simple objects of $\mathrm{Rep}(\mathfrak{S})$ are precisely the V^n_λ when

$n = \infty$. It turns out that the objects $V^{\otimes r}$ are injective in $\mathrm{Rep}(\mathfrak{S})$; these split as a direct sum of Schur functors applied to V, so the $\mathbb{S}^\lambda(V)$ are injective objects. However, they are not typically indecomposable.

Let H_d be the subgroup of S_∞ that fixes $\{1, 2, \ldots, d\}$ (it is itself isomorphic to S_∞). Because H_d commutes with S_d (viewed as the subgroup of S_∞ permuting $\{1, 2, \ldots, d\}$), taking H_d-invariants of a representation of S_∞ yields a representation of S_d. We therefore have a functor

$$\Gamma_d : \mathrm{Rep}(\mathfrak{S}) \to \mathrm{Rep}(S_d)$$

defined by $M \mapsto M^{H_d}$. This is called a *specialisation functor*. We have

$$\Gamma_d(V_\lambda^\infty) = V_\lambda^d.$$

A remarkable fact is that Γ_d is a left-exact tensor functor. We may therefore speak of its derived functors. It turns out that for any fixed λ, $R^i\Gamma_d(V_\lambda^\infty)$ is either zero for all i, or is nonzero in precisely one degree i, where its value is an irreducible representation of S_d. The precise rule requires some setup to state, so we omit it here (although it can be found in Subsection 6.4 of [27], and we will perform our own computation in Chapter 6).

3.3 Stable Specht polynomials

The symmetric group S_n may be viewed as the subgroup of the general linear group $GL_n(k)$ consisting of permutation matrices. We may therefore consider the restriction to S_n of irreducible $GL_n(k)$ representations. Let us write $[M]$ for the image of a module in the Grothendieck ring of kS_n-modules. The restriction multiplicities a_μ^λ (from $GL_n(k)$ to S_n) are defined via

$$[\mathrm{Res}_{S_n}^{GL_n}(\mathbb{S}^\lambda(\mathbb{C}^n))] = \sum_{\mu \vdash n} a_\mu^\lambda [\mathcal{S}^\mu].$$

Whilst a positive combinatorial formula for the restriction multiplicities is not currently known, there is an expression using plethysm of symmetric functions (see [18] Chapter 1

Section 8 for background about plethysm). Let $f[g]$ denote the plethysm of a symmetric function f with another symmetric function g. Then,

$$a_\mu^\lambda = \langle s_\lambda, s_\mu[1 + h_1 + h_2 + \cdots]\rangle,$$

see [13] or Exercise 7.74 of [31]. We will need to consider the Lyndon symmetric function,

$$L_n = \frac{1}{n}\sum_{d|n} \mu(d) p_d^{n/d},$$

where $\mu(d)$ is the Möbius function and p_d is the d-th power-sum symmetric function. In fact, L_n is the $GL(V)$ character of the degree n components of the free Lie algebra on V (see the first proof of Theorem 8.1 of [22], which proves this to deduce a related result). For convenience we define the total Lyndon symmetric function $L = L_1 + L_2 + \cdots$; this is the character of the (whole) free Lie algebra on V.

Instead of asking for the restriction coefficients a_μ^λ, we may ask the inverse question: how can one express $[\mathcal{S}^\mu]$ in terms of $[\mathrm{Res}_{S_n}^{GL_n(k)}(\mathbb{S}^\lambda(\mathbb{C}^n))]$? This question was recently answered by Assaf and Speyer in [2]. Assaf and Speyer showed

$$[\mathcal{S}^{\mu[n]}] = \sum_\lambda b_\lambda^\mu [\mathbb{S}^\nu(\mathbb{C}^n)],$$

where

$$b_\lambda^\mu = (-1)^{|\mu|-|\lambda|} \sum_{\mu/\nu \text{ vert. strip}} \langle s_{\nu'}, s_{\lambda'}[L]\rangle.$$

The notation μ/ν vert. strip means that the diagram of μ may be obtained from the diagram of ν by adding boxes, no two in the same row, and primes indicate dual partitions.

Consider the symmetric functions

$$s_\lambda^\dagger = \sum_\mu b_\lambda^\mu s_\lambda,$$

which Assaf and Speyer call the stable Specht polynomials (Orellana and Zabrocki call these the irreducible character basis). Evaluating a Schur function at the eigenvalues of an element $g \in GL_n(k)$ gives the trace of that element acting on the corresponding Schur functor. If we choose g to be a permutation matrix, and evaluate $s^\dagger_\lambda$ at its eigenvalues, we will get the character of $\mathcal{S}^{\lambda[n]}$.

It was shown by Specht [30] (see also Chapter 1, Section 7, Examples 13 and 14 of [18]) that there exist polynomials (called *character polynomials*),

$$X^\lambda \in \mathbb{Q}[a_1, a_2, \ldots],$$

with the following property. Suppose that $\mu = (1^{a_1}2^{a_2}\cdots)$ is a partition of size $n \geq |\lambda| + \lambda_1$, so that $\lambda[n]$ is defined. Then the character of $\mathcal{S}^{\lambda[n]}$ on an element of cycle type μ is given by evaluating X^λ:

$$\chi^{\lambda[n]}_\mu = X^\lambda(a_1, a_2, \ldots).$$

It is not difficult to deduce that the structure constants are reduced Kronecker coefficients:

$$X^\mu X^\nu = \sum_\lambda \tilde{k}^\lambda_{\mu,\nu} X^\lambda.$$

The relation to the s_λ is as follows. Observe that a cycle of size r contributes all r-th roots of unity to the multiset of eigenvalues of a permutation matrix. If ζ is a primitive r-th root of unity, the power-sum symmetric function p_s evaluated at r-th roots of unity is

$$p_s(\zeta^0, \zeta^1, \ldots, \zeta^{r-1}) = \sum_{i=0}^{s-1} \zeta^{ir} = \begin{cases} r & \text{if } r \text{ divides } s \\ 0 & \text{otherwise} \end{cases}.$$

Extending this to the whole multiset of eigenvalues amounts to summing this over each of the cycles, so p_s takes the value $\sum_{r|s} r a_r$. Performing Möbius inversion, we may express a_r

(the number of r-cycles) as a function of the eigevalues of the permutation matrix:

$$a_r = \frac{1}{r} \sum_{s|r} \mu(r/s) p_s.$$

It is precisely this substitution which relates character polynomials and stable Specht polynomials.

Chapter 4

Stable Grothendieck Rings of Wreath Product Categories

4.1 Introduction

In this chapter, we consider the Grothendieck groups of the categories of modules over the wreath products $\mathcal{R} \wr S_n$, where $\mathcal{R}$ is a Hopf algebra, and S_n is the symmetric group on n symbols. The Hopf algebra structure gives rise to a multiplication on the Grothendieck groups, so we may speak of Grothendieck rings. We study the ring structure in the "limit" $n \to \infty$.

Using Mackey theory, we describe the multiplication on the Grothendieck rings in terms of data associated to $\mathcal{R}$ and S_n. We show that this multiplication exhibits a certain stability property which allow us to define a "limiting Grothendieck ring", $\mathcal{G}^{\mathbb{Z}}_{\infty}(\mathcal{C})$ (here $\mathcal{C} = \mathcal{R}-\mathrm{mod}$). This ring is the Grothendieck ring of the wreath product Deligne categories $S_t(\mathcal{C})$ introduced in [19] and considered in [14]. When $\mathcal{C}$ is the category of finite-dimensional vector spaces over the field k (characteristic zero and algebraically closed), we recover the original Deligne category $\underline{\mathrm{Rep}}(S_t)$.

Our first main result is Theorem 4.7.8, which gives the structure of $\mathcal{G}^{\mathbb{Q}}_{\infty}(\mathcal{C}) = \mathbb{Q} \otimes_{\mathbb{Z}} \mathcal{G}^{\mathbb{Z}}_{\infty}(\mathcal{C})$. If $\mathcal{G}(\mathcal{C})_i$ denotes a copy of the Grothendieck group of $\mathcal{C}$ with rational coefficients, we have a

Lie algebra structure coming from the associative multiplication. This allows us to take the universal enveloping algebra $U(\mathcal{G}(\mathcal{C})_i)$. Then:

$$\mathcal{G}_\infty^{\mathbb{Q}}(\mathcal{C}) = \bigotimes_{i=1}^{\infty} U(\mathcal{G}(C)_i)$$

This generalises the fact (due to Deligne) that the Grothendieck ring of $\underline{\text{Rep}}(S_t)$ is the free polynomial algebra on certain elements (see [8]); in [14] these elements were generalised to *basic hooks*. It was proved in [14] that the basic hooks generate the Grothendieck ring of $S_t(\mathcal{C})$, although for arbitrary $\mathcal{C}$ they do not commute (so the Grothendieck ring is not a free polynomial algebra).

Our second main result is about the Grothendieck ring of wreath product Deligne categories. Knowing that $\mathcal{G}_\infty^{\mathbb{Q}}(\mathcal{C})$ is isomorphic to the Grothendieck ring of $S_t(\mathcal{C})$, there is a natural basis of $\mathcal{G}_\infty^{\mathbb{Q}}(\mathcal{C})$, $X_{\vec{\lambda}}$, coming from the images of the indecomposable objects of $S_t(\mathcal{C})$. In Theorem 4.8.10 we give a generating function that describes the $X_{\vec{\lambda}}$ basis of $\mathcal{G}_\infty^{\mathbb{Q}}(\mathcal{C})$ in the presentation in Theorem 4.7.8. Finally, we discuss implications of these results for the asymptotic representation theory of wreath products and symmetric groups, and explain that all our results actually hold when $\mathcal{C}$ is a tensor category, without the need for a Hopf algebra $\mathcal{R}$.

The outline of the chapter is as follows. In Section 2 we briefly discuss our setup. Then in Section 3, we discuss wreath products and their irreducible representations. In Section 4 we apply Mackey theory to show tensor products of irreducible representations of wreath products decompose in ways controlled by double coset representatives of Young subgroups of symmetric groups, which we discuss in Section 5. We see that the double coset representatives exhibit certain stability properties that allow us to define a "limiting Grothendieck ring" $\mathcal{G}_\infty^{\mathbb{Z}}(\mathcal{C})$ in Section 6, and we establish the structure of $\mathcal{G}_\infty^{\mathbb{Q}}(\mathcal{C})$ in Section 7. In Section 8 we use partition combinatorics to determine explicitly how certain basis elements of the ring are expressed in terms of the the basic hooks. Finally, we discuss some applications to asymptotic representation theory of wreath products in Section 9. In Section 10 we explain how all these results generalise to the setting where $\mathcal{C}$ is a tensor category.

4.2 Preliminaries

Throughout this chapter, we work with a fixed algebraically closed field of characteristic zero, k, and a category $\mathcal{C} = \mathcal{R} - mod$ of finite-dimensional modules for $\mathcal{R}$, a fixed Hopf algebra over k. We work with $\mathcal{C}$ rather than $\mathcal{R}$ to stress that our results only depend on the module category. In fact, all our results generalise to the setting where $\mathcal{C}$ is a tensor category; we prove things in a suitable level of generality, for example we do not make use of dual modules constructed using the antipode of $\mathcal{R}$, which do not exist in a general tensor category.

It is clear that if $\mathbf{1}$ is the trivial $\mathcal{R}$-module, then $\mathrm{End}_{\mathcal{C}}(\mathbf{1}) = k$. Also, the tensor product in $\mathcal{C}$ is exact in both arguments and bilinear with respect to direct sums. The Grothendieck group, $\mathcal{G}(\mathcal{C})$, has a basis (as a free abelian group) consisting of isomorphism classes of simple $\mathcal{R}$-modules. The exactness of the tensor product implies that it respects the relations of the Grothendieck group and therefore descends to a bilinear distributive multiplication on $\mathcal{G}(\mathcal{C})$. Finally, the image of $\mathbf{1}$ in $\mathcal{G}(\mathcal{C})$ is a multiplicative identity. Thus, $\mathcal{G}(\mathcal{C})$ inherits the structure of a ring. This will be the main setting in which we work.

4.3 Wreath Products

We outline some features of wreath products that are important for us.

4.3.1 Construction of $\mathcal{W}_n(\mathcal{C})$ and Restriction/Induction

Definition 4.3.1. *Let $\mathcal{R} \wr S_n$ be the algebra isomorphic to $\mathcal{R}^{\otimes n} \otimes kS_n$ as a vector space, with multiplication defined as follows. Suppose that a_i and b_i, $i \in \{1, 2, \ldots, n\}$ are elements of $\mathcal{R}$, whilst σ and ρ are elements of S_n. Then,*

$$\begin{aligned} & ((a_1 \otimes a_2 \otimes \cdots \otimes a_n) \otimes \sigma)((b_1 \otimes b_2 \otimes \cdots \otimes b_n) \otimes \rho) \\ = \; & (a_1 b_{\sigma^{-1}(1)} \otimes a_2 b_{\sigma^{-1}(2)} \otimes \cdots \otimes a_n b_{\sigma^{-1}(n)}) \otimes (\sigma\rho). \end{aligned}$$

It is well known that this algebra naturally inherits the structure of a Hopf algebra from the Hopf algebra structure on $\mathcal{R}$ in the following way. The maps $\mathcal{R} \to \mathcal{R} \wr S_n$ obtained by embedding $\mathcal{R}$ into the r-th tensor factor of $\mathcal{R}^{\otimes n} \otimes kS_n$ ($r = 1, 2, \ldots, n$) are maps of Hopf algebras, and similarly, the embedding $kS_n \to \mathcal{R}^{\otimes n} \otimes kS_n$ by mapping into the final tensor factor is a map of Hopf algebras. The images of these $n+1$ maps generate $\mathcal{R} \wr S_n$, and therefore determine the comultiplication. The *wreath product category* $\mathcal{W}_n(\mathcal{C})$ is the category $(\mathcal{R} \wr S_n) - mod$. We suppress $\mathcal{R}$ in the notation because $\mathcal{W}_n(\mathcal{C})$ can be constructed from $\mathcal{C} = \mathcal{R} - \text{mod}$ alone (see the final section for details).

In our situation, it will be important to consider actions of subgroups of S_n. In the above, we may form the Hopf subalgebra $\mathcal{R} \wr G = \mathcal{R}^{\otimes n} \otimes kG$ for any subgroup G of S_n. If H is a subgroup of G, we have a restriction functor $\text{Res}_H^G : \mathcal{R} \wr G - \text{mod} \to \mathcal{R} \wr H - \text{mod}$. Additionally there is an induction functor $\text{Ind}_H^G : \mathcal{R} \wr H - \text{mod} \to \mathcal{R} \wr G - \text{mod}$ which is both right adjoint and left adjoint to Res_H^G. The induction functor may be written as a sum over coset representatives of H in G as follows:

$$\text{Ind}_H^G(M) = \bigoplus_{g \in G/H} gM.$$

In the above formula, gM denotes an object isomorphic to M as a vector space, and the action of G is that of an induced representation. Explicitly, to see how $x \in G$ acts on gM, note that $xg = g'h$ for unique coset representative $g' \in G/H$ and $h \in H$. Then, x takes gM to $g'M$, whilst acting by the usual action of $h \in H$. Because induction and restriction are exact functors, they define homomorphisms between the Grothendieck groups of $\mathcal{R} \wr G$ and $\mathcal{R} \wr H$. We will be interested in the case where $G = S_n$ and H is a subgroup of the following type.

Definition 4.3.2. *If $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ is a composition of n (that is, a finite sequence of non-negative integers summing to n), then $S_\alpha = \prod_{i=1}^k S_{\alpha_i}$ is the* Young subgroup *of S_n corresponding to the composition α (i.e. $Sym(\{1, 2, \ldots, \alpha_1\}) \times Sym(\{\alpha_1 + 1, \ldots, \alpha_1 + \alpha_2\}) \times \cdots \times Sym(\{n - \alpha_k + 1, \ldots, n\})$). We refer to the factors S_{α_i} as the component groups of S_α.*

Note that for $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$, we may identify $\mathcal{R} \wr S_\alpha = \mathcal{R}^{\otimes n} \otimes kS_\alpha$ with $\bigotimes_i (\mathcal{R}^{\otimes \alpha_i} \otimes kS_{\alpha_i}) = \bigotimes_i (\mathcal{R} \wr S_{\alpha_i})$.

4.3.2 Description of Simple Objects in $\mathcal{W}_n(\mathcal{C})$

Definition 4.3.3. *Suppose that M and N are modules over algebras A and B respectively. Then we write $M \boxtimes N$ for $M \otimes N$ viewed as an $A \otimes B$-module. We also write $M^{\boxtimes n}$ for $M^{\otimes n}$ viewed as an $A^{\otimes n}$-module.*

In the sequel we will often consider objects of the following form.

Definition 4.3.4. *Let $M \in \mathcal{C}$, and V be a finite-dimensional representation of S_n over k. We define the following object of $\mathcal{W}_n(\mathcal{C})$:*

$$M^{\boxtimes n} \otimes V.$$

This has a $\mathcal{R}^{\otimes n}$-action by acting on the first tensor factor. An element of S_n acts by permuting the factors of $M^{\boxtimes n}$ in the obvious way, as well as acting on V.

We now introduce some standard properties of the $M^{\boxtimes n} \otimes V$. The proofs of most of these statements are well known, and therefore omitted.

Proposition 4.3.5. *Let V_1, V_2 be finite-dimensional representations of S_n, and let M, M' be objects of $\mathcal{C}$. We have the following:*

1.

$$M^{\boxtimes n} \otimes (V_1 \oplus V_2) \cong (M^{\boxtimes n} \otimes V_1) \oplus (M^{\boxtimes n} \otimes V_2),$$

2.

$$(M^{\boxtimes n} \otimes V_1) \otimes (M'^{\boxtimes n} \otimes V_2) \cong (M \otimes M')^{\boxtimes n} \otimes (V_1 \otimes V_2).$$

When considering Mackey theory, it will also be necessary to understand the behaviour of $M^{\boxtimes n} \otimes V$ under induction and restriction.

Proposition 4.3.6. *We have the following identities.*

1. *Suppose that M is an object of $\mathcal{C}$, V_1 is a finite-dimensional representation of S_{n_1}, and V_2 is a finite-dimensional representation of S_{n_2}. Then,*

$$\mathrm{Ind}_{S_{n_1}\times S_{n_2}}^{S_{n_1+n_2}}((M^{\boxtimes n_1}\otimes V_1)\boxtimes(M^{\boxtimes n_2}\otimes V_2)) = M^{\boxtimes(n_1+n_2)}\otimes \mathrm{Ind}_{S_{n_1}\times S_{n_2}}^{S_{n_1+n_2}}(V_1\boxtimes V_2).$$

   Note that the induction on the left relates to modules for wreath products, while the induction on the right relates to modules for symmetric groups.

2. *Suppose that M is an object of $\mathcal{C}$, and V is a finite-dimensional representation of S_n such that $\mathrm{Res}_{S_{n_1}\times S_{n_2}}^{S_n}(V) = \bigoplus_i V_1^{(i)}\boxtimes V_2^{(i)}$ (where $V_j^{(i)}$ is a representation of S_{n_j} for $j=1,2$). Then,*

$$\mathrm{Res}_{S_{n_1}\times S_{n_2}}^{S_n}(M^{\boxtimes n}\otimes V) = \bigoplus_i (M^{\boxtimes n_1}\otimes V_1^{(i)})\boxtimes(M^{\boxtimes n_2}\otimes V_2^{(i)}).$$

Definition 4.3.7. *Let $I(\mathcal{C})$ be the set of isomorphism classes of simple objects of $\mathcal{C}$. Let $\mathcal{G}(\mathcal{W}_n(\mathcal{C}))$ denote the Grothendieck group of $\mathcal{W}_n(\mathcal{C})$, and for an object R of $\mathcal{W}_n(\mathcal{C})$, let $[R]$ denote the image of R in $\mathcal{G}(\mathcal{W}_n(\mathcal{C}))$. As in [14], let:*

$$\mathcal{P}_n^{\mathcal{C}} = \{\vec{\lambda} : I(\mathcal{C}) \to \mathcal{P} \mid \sum_{U\in I(\mathcal{C})} |\vec{\lambda}(U)| = n\}$$

Thus, $\mathcal{P}_n^{\mathcal{C}}$ is the set of multipartitions of n whose constituent partitions are indexed by isomorphism classes of simple objects in $\mathcal{C}$. We will indicate multipartitions (elements of $\mathcal{P}_n^{\mathcal{C}}$ for some n) with arrows (e.g. $\vec{\lambda}, \vec{\mu}, \vec{\nu}$), while ordinary partitions (elements of $\mathcal{P}$) will not have arrows (e.g. $\lambda, \mu, \nu, \rho, \sigma, \tau$).

Eventually, we will pass to the Grothendieck group of $\mathcal{W}_n(\mathcal{C})$, and we will wish to understand the composition factors of $M^{\boxtimes n}\otimes V$. The following proposition will allow us to calculate the composition factors that we will be interested in. The proof is routine, and we omit it.

Proposition 4.3.8. *Suppose N is a subobject of M in $\mathcal{W}_n(\mathcal{C})$. If $\mathbf{1}_G$ denotes the trivial representation of a group G, we have the following equality in $\mathcal{G}(\mathcal{W}_n(\mathcal{C}))$:*

$$[M^{\boxtimes n} \otimes \mathbf{1}_{S_n}] = \sum_{r=0}^{n} [\mathrm{Ind}_{S_r \times S_{n-r}}^{S_n} \left((N^{\boxtimes r} \otimes \mathbf{1}_{S_r}) \boxtimes ((M/N)^{\boxtimes (n-r)} \otimes \mathbf{1}_{S_{n-r}}) \right)].$$

We now describe the simple objects in the category $\mathcal{W}_n(\mathcal{C})$. The set $\mathcal{P}_n^{\mathcal{C}}$ gives an index set for the isomorphism classes of simple objects of $\mathcal{W}_n(\mathcal{C})$.

Definition 4.3.9. *Let $\vec{\lambda} \in \mathcal{P}_n^{\mathcal{C}}$, and $K = \prod_{U \in I(\mathcal{C})} S_{|\vec{\lambda}(U)|}$, a Young subgroup of S_n. We define $R_{\vec{\lambda}}$, an object of $\mathcal{W}_n(\mathcal{C})$:*

$$R_{\vec{\lambda}} = \mathrm{Ind}_K^{S_n} \left(\boxtimes_{U \in I(\mathcal{C})} \left(U^{\boxtimes |\vec{\lambda}(U)|} \otimes \mathcal{S}^{\vec{\lambda}(U)} \right) \right)$$

As before, $\mathcal{S}^{\mu}$ denotes the Specht module associated to a partition μ.

The $R_{\vec{\lambda}}$ are the (pairwise non-isomorphic) simple objects of $\mathcal{W}_n(\mathcal{C})$. In [19], this is shown in the context of indecomposable objects of an additive category, but the proof in our setting is analogous. We will use Mackey theory to calculate tensor products of the $R_{\vec{\lambda}}$, and hence the multiplicative structure of the Grothendieck ring $\mathcal{G}(\mathcal{W}_n(\mathcal{C}))$.

4.4 Mackey Theory

We now consider how to take tensor products of induced objects, which we will later apply to $R_{\vec{\lambda}}$.

Definition 4.4.1. *Suppose that H and K are subgroups of a group G. A (H, K)*-double coset *in G is an orbit of the group action of $H \times K$ on G given by $(h, k) \cdot g = hgk^{-1}$. A* double coset representative *is any element of a double coset, and we write $H\backslash G/K$ for a set of (H, K)-double coset representatives in G (a set containing one representative from each (H, K)-double coset in G).*

The following results are completely analogous to the corresponding versions for representations of finite groups, including the proofs, which we omit (see for example, [35]).

Proposition 4.4.2. *Let G be a finite group with subgroups H and K. Suppose that M is a K-equivariant object of a k-linear abelian category with a G-action. We have the following formula for the composition of induction and restriction:*

$$\mathrm{Res}_H^G(\mathrm{Ind}_K^G(M)) \cong \bigoplus_{s\in H\backslash G/K} \mathrm{Ind}_{H\cap sKs^{-1}}^H(\mathrm{Res}_{H\cap sKs^{-1}}^{sKs^{-1}}(sM)).$$

Lemma 4.4.3. *If H is a subgroup of the finite group G, and M, N are objects of a monoidal k-linear abelian category such that M is H-equivariant and N is G-equivariant, then we have $\mathrm{Ind}_H^G(M)\otimes N \cong \mathrm{Ind}_H^G(M\otimes \mathrm{Res}_H^G(N))$. Similarly, $N\otimes \mathrm{Ind}_H^G(M)\cong \mathrm{Ind}_H^G(\mathrm{Res}_H^G(N)\otimes M)$.*

Proposition 4.4.4. *Suppose that H and K are subgroups of the finite group G. If M is an H-equivariant object of a monoidal k-linear abelian category with an action of G, and N is a K-equivariant object, we have the following.*

$$\mathrm{Ind}_H^G(M)\otimes \mathrm{Ind}_K^G(N) \cong \bigoplus_{s\in H\backslash G/K} \mathrm{Ind}_{H\cap sKs^{-1}}^G(\mathrm{Res}_{H\cap sKs^{-1}}^H(M)\otimes \mathrm{Res}_{H\cap sKs^{-1}}^{sKs^{-1}}(sN))$$

Proof. We compute directly:

$$\begin{aligned}
\mathrm{Ind}_H^G(M)\otimes \mathrm{Ind}_K^G(N) &\cong \mathrm{Ind}_H^G(M\otimes \mathrm{Res}_H^G(\mathrm{Ind}_K^G(N)))\\
&\cong \mathrm{Ind}_H^G(M\otimes \bigoplus_{s\in H\backslash G/K} \mathrm{Ind}_{H\cap sKs^{-1}}^H(\mathrm{Res}_{H\cap sKs^{-1}}^{sKs^{-1}}(sN)))\\
&\cong \bigoplus_{s\in H\backslash G/K} \mathrm{Ind}_H^G(M\otimes \mathrm{Ind}_{H\cap sKs^{-1}}^H(\mathrm{Res}_{H\cap sKs^{-1}}^{sKs^{-1}}(sN)))\\
&\cong \bigoplus_{s\in H\backslash G/K} \mathrm{Ind}_H^G(\mathrm{Ind}_{H\cap sKs^{-1}}^H(\mathrm{Res}_{H\cap sKs^{-1}}^H(M)\otimes \mathrm{Res}_{H\cap sKs^{-1}}^{sKs^{-1}}(sN)))\\
&\cong \bigoplus_{s\in H\backslash G/K} \mathrm{Ind}_{H\cap sKs^{-1}}^G(\mathrm{Res}_{H\cap sKs^{-1}}^H(M)\otimes \mathrm{Res}_{H\cap sKs^{-1}}^{sKs^{-1}}(sN))
\end{aligned}$$

Here we have used the transitivity of induction, namely that $\mathrm{Ind}_H^G\circ\mathrm{Ind}_{H\cap sKs^{-1}}^H = \mathrm{Ind}_{H\cap sKs^{-1}}^G$

(the proof of this is again analogous to the proof for representations of finite groups). □

In our setting, H and K will be Young subgroups of S_n (recall that the simple objects of the wreath product category are obtained by applying induction functors from Young subgroups). So if we are to use the previous proposition to decompose tensor products of simple objects, it will be important to understand double coset representatives of Young subgroups of S_n.

4.5 Double Cosets of Young Subgroups

We now prove some facts about minimal length double coset representatives of Young subgroups of symmetric groups. Let $\sigma \in S_n$ be considered as a bijective function from the set $\{1, 2, \ldots, n\}$ to itself. If $\mu = (\mu_1, \mu_2, \ldots)$ and $\nu = (\nu_1, \nu_2, \ldots)$ are compositions of n and $S_\mu = \prod_i S_{\mu_i}, S_\nu = \prod_i S_{\nu_i}$ are the associated Young subgroups of S_n, we seek to describe the (S_μ, S_ν)-double cosets of S_n. We write A_i for the subset of $\{1, 2, \ldots, n\}$ that is permuted by S_{μ_i} (considered as a subgroup of S_μ), so that $A_1 = \{1, 2, \ldots, \mu_1\}$, $A_2 = \{\mu_1 + 1, \mu_1 + 2, \ldots, \mu_1 + \mu_2\}$, and so on. Similarly we define B_i to be the subsets of $\{1, 2, \ldots, n\}$ permuted by the S_{ν_i}.

Definition 4.5.1. *Say that $\sigma \in S_n$ is* fully ordered *if (in the notation defined above), for each B_i and A_j, the restrictions $\sigma|_{B_i}$ and $\sigma^{-1}|_{A_j}$ are monotone increasing functions.*

Remark 4.5.2. *The property of being fully ordered will turn out to be equivalent to being a minimal length (S_μ, S_ν)-double coset representative. However, it will be more convenient to work with the above definition.*

Lemma 4.5.3. *The numbers $C_{i,j}(\sigma) = |\{x \in B_i | \sigma(x) \in A_j\}|$ are double coset invariants. Moreover, σ_1 and σ_2 are in the same double coset if and only if $C_{i,j}(\sigma_1) = C_{i,j}(\sigma_2)$ for all i, j. Additionally, each double coset has a unique fully ordered element.*

Proof. The $C_{i,j}(\sigma)$ are constant on each double coset since both the left (S_μ) and right (S_ν) actions preserve each A_j and B_i. We show that every element can be acted on by the left by

S_μ and on the right by S_ν to obtain a fully ordered element. Then we show that a totally ordered element is determined by the $C_{i,j}(\sigma)$. This implies that if two elements of S_n have the same $C_{i,j}(\sigma)$, then they have the same fully ordered element in their double coset, implying that they are in the same coset.

Given $\sigma \in S_n$, we may act on it on the right by elements of the S_{ν_i} (and hence an element of S_ν) to reorder the elements of B_i in order of increasing image under σ. This gives a new $\sigma \in S_n$ such that if $x < y$ are elements of B_i, then $\sigma(x) < \sigma(y)$. This means that the restriction of σ to each B_i is a monotone increasing function. We may also act on the left by the S_{μ_i} to sort the preimages of each A_j; thus we may assume if $x \in B_a$ and $y \in B_b$ with $a < b$, such that $\sigma(x)$ and $\sigma(y)$ are in A_i, then $\sigma(x) < \sigma(y)$. Note that this process preserves the property that σ is monotone increasing when restricted to the B_i. Thus the result of these actions is a fully ordered element.

Next we inductively construct a fully ordered σ from prescribed $C_{i,j}$. For a collection of natural numbers $C'_{i,j}$, there is a $\sigma \in S_n$ such that $C_{i,j}(\sigma) = C'_{i,j}$ if and only if the following two conditions hold. For each j, $\sum_i C'_{i,j} = |B_j|$ and for each i, $\sum_j C'_{i,j} = |A_i|$. The first $C_{1,1}(\sigma)$ elements of B_1 must map to the the first $C_{1,1}(\sigma)$ elements of A_1 (in a monotone increasing way, hence uniquely). Then, the next $C_{1,2}(\sigma)$ elements of B_1 map to the first $C_{1,2}(\sigma)$ elements of A_2, and so on. This means that the image of B_1 is determined uniquely. Then, the first $C_{2,1}(\sigma)$ elements of B_2 map to the next $C_{2,1}(\sigma)$ elements of A_1, and so on (again without choice). Repeating for all i and j, we obtain a fully ordered element σ and each step of the construction was forced, so the fully ordered element is unique. $\square$

Remark 4.5.4. *Noting that the length of $\sigma \in S_n$ is equal to the number of inversions (that is, pairs (i, j) with $1 \le i < j \le n$ such that $\sigma(j) < \sigma(i)$), the property of being fully ordered is the same as being a minimal length double coset representative. Note that the number of inversions is bounded below by $\sum_{i_1<i_2} \sum_{j_1<j_2} |C_{i_1,j_2}(\sigma)||C_{i_2,j_1}(\sigma)|$, and a fully ordered σ attains this bound.*

For convenience, in this section we require that for a composition α, the factors in the Young subgroup $S_\alpha = \prod_i S_{\alpha_i}$ are ordered in increasing order from left to right, i.e. $S_{\alpha_k} \times$

$\cdots \times S_{\alpha_1}$. For example, $S_{(3,2,1)} = S_1 \times S_2 \times S_3$. We will be interested in the operation of increasing the largest part of a partition by 1 (hence passing from partitions of n to partitions of $n+1$), so this will affect the final component group of S_α according to this convention.

Definition 4.5.5. *If α is a composition of n, write α^* for composition of $n+1$ obtained by adding 1 to the first part of α.*

Correspondingly, we discuss (S_μ, S_ν)-double coset representatives under the operation of adding 1 to the largest parts of the partitions μ and ν. If f is a bijection from the set $\{1, 2, \ldots, n\}$ to itself satisfying the fully ordered property, it continues to satisfy the fully ordered property as a function on $\{1, 2, \ldots, n+1\}$ when we define $f(n+1) = n+1$ (note that this corresponds to the inclusion $S_n \hookrightarrow S_{n+1}$ by considering elements fixing $n+1$).

Proposition 4.5.6. *Let μ, ν be partitions of n. After sufficiently many repeated applications of the operation $(\mu, \nu, n) \mapsto (\mu^*, \nu^*, n+1)$, the number of (S_μ, S_ν)-double cosets in S_n stabilises. Moreover, one can choose representatives which are identified for different n (sufficiently large) via the usual inclusions of symmetric groups.*

Proof. Observe that if the first parts of μ^* and ν^* each exceed $n/2$, then for any $\sigma \in S_n$, $C_{(\mu^*)_1, (\nu^*)_1}(\sigma) \geq 1$ by the pigeonhole principle. This means that for a fully ordered double coset representative σ, $\sigma(n+1) = n+1$. In particular, each double coset representative is obtained from a double coset representative of (S_μ, S_ν) under the inclusion of S_n in S_{n+1}. □

Remark 4.5.7. *Ordering the multiplicative factors in the definition of Young subgroup from smallest to largest allows us to take the inclusions $S_n \hookrightarrow S_{n+1}$ obtained by extending functions on $\{1, 2, \ldots, n\}$ by requiring them to fix $n+1$. If we did not do this, we would have to use a nonstandard inclusion. The Young subgroups related by different orderings of their component groups are conjugate in S_n, so induced objects coming from the two embeddings are related by a twist which will be irrelevant for our purposes.*

4.6 The Limiting Grothendieck Ring

We work towards understanding the tensor product, with the aim of constructing a "stable" Grothendieck ring.

4.6.1 Tensor Products of Irreducible Modules

Recall that $\mathcal{G}(\mathcal{W}_n(\mathcal{C}))$ naturally inherits the structure of a ring, and has a basis given by $[R_{\vec{\lambda}}]$ for $\vec{\lambda} \in \mathcal{P}_n^{\mathcal{C}}$.

Example 4.6.1. *If $\mathcal{C}$ is the category of finite-dimensional vector spaces over k (e.g. if $\mathcal{R} = k$), $\mathcal{G}(\mathcal{W}_n(\mathcal{C}))$ is the representation ring of S_n over k.*

Given $\vec{\mu}, \vec{\nu} \in \mathcal{P}_n^{\mathcal{C}}$, we wish to describe $[R_{\vec{\mu}}][R_{\vec{\nu}}] = [R_{\vec{\mu}} \otimes R_{\vec{\nu}}]$ as a linear combination of $[R_{\vec{\lambda}}]$. For this task, we use the categorical Mackey theory results. We write $H_{\vec{\lambda}} = \prod_{U \in I(\mathcal{C})} S_{|\vec{\lambda}(U)|}$ for the subgroup of S_n from which $R_{\vec{\lambda}}$ is induced. Note that $H_{\vec{\lambda}}$ is itself a Young subgroup of S_n.

Lemma 4.6.2. *Using Proposition 4.4.4, we have the following:*

$$\begin{aligned}
R_{\vec{\mu}} \otimes R_{\vec{\nu}} &\cong \operatorname{Ind}_{H_{\vec{\mu}}}^{S_n} \left(\boxtimes_{U \in I(\mathcal{C})} \left(U^{\boxtimes|\vec{\mu}(U)|} \otimes \mathcal{S}^{\vec{\mu}(U)}\right)\right) \otimes \operatorname{Ind}_{H_{\vec{\nu}}}^{S_n} \left(\boxtimes_{U \in I(\mathcal{C})} \left(U^{\boxtimes|\vec{\nu}(U)|} \otimes \mathcal{S}^{\vec{\nu}(U)}\right)\right) \\
&\cong \bigoplus_{t \in H_{\vec{\mu}} \backslash S_n / H_{\vec{\nu}}} \operatorname{Ind}_{H_{\vec{\mu}} \cap t H_{\vec{\nu}} t^{-1}}^{S_n} \Big(\operatorname{Res}_{H_{\vec{\mu}} \cap t H_{\vec{\nu}} t^{-1}}^{H_{\vec{\mu}}} \left(\boxtimes_{U \in I(\mathcal{C})} \left(U^{\boxtimes|\vec{\mu}(U)|} \otimes \mathcal{S}^{\vec{\mu}(U)}\right)\right) \\
&\qquad \otimes \operatorname{Res}_{H_{\vec{\mu}} \cap t H_{\vec{\nu}} t^{-1}}^{t H_{\vec{\nu}} t^{-1}} \left(t \boxtimes_{U \in I(\mathcal{C})} \left(U^{\boxtimes|\vec{\nu}(U)|} \otimes \mathcal{S}^{\vec{\nu}(U)}\right)\right)\Big).
\end{aligned}$$

Remark 4.6.3. *Observe that $H_{\vec{\mu}} \cap t H_{\vec{\nu}} t^{-1}$ is a proper subgroup of at least one of $H_{\vec{\mu}}$ and $t H_{\vec{\nu}} t^{-1}$ unless these two are equal. Later on, this observation will imply the vanishing of certain restrictions of virtual representations. We also note that by the fully ordered property of t, $H_{\vec{\mu}} \cap t H_{\vec{\nu}} t^{-1}$ is a Young subgroup of S_n; it independently permutes contiguous subsets of $\{1, 2, \ldots, n\}$.*

We now exploit the stability property of double cosets of Young subgroups to define a "limiting Grothendieck group".

Definition 4.6.4. *Let* $\mathbf{1}$ *be the unit object of* $\mathcal{C}$. *If* $\vec{\lambda} \in \mathcal{P}_n^{\mathcal{C}}$, *write* $\vec{\lambda}^*$ *for the element of* $\mathcal{P}_{n+1}^{\mathcal{C}}$ *obtained by adding 1 to the largest part of* $\vec{\lambda}(\mathbf{1})$. *We denote the result of applying the operation* $\vec{\lambda} \mapsto \vec{\lambda}^*$ *successively* k *times by* $\vec{\lambda}^{*k}$.

Specifically, we will consider products $[R_{\vec{\mu}^{*k}}][R_{\vec{\nu}^{*k}}]$ for $\vec{\mu}, \vec{\nu} \in \mathcal{P}_n^{\mathcal{C}}$ (for some n) as $k \to \infty$. We first introduce notation to conveniently describe the limit.

Definition 4.6.5. *If* $\vec{\lambda} \in \mathcal{P}_n^{\mathcal{C}}$, *we define the multipartition* $\vec{\lambda}' \in \mathcal{P}_{n-\vec{\lambda}(\mathbf{1})_1}^{\mathcal{C}}$ *by removing the largest part of the partition* $\vec{\lambda}(\mathbf{1})$ *(if this was the empty partition, it remains the empty partition). Also, if* $\vec{\lambda} \in \mathcal{P}_m^{\mathcal{C}}$, *write* $\vec{\lambda}[n]$ *for the element of* $\mathcal{P}_n^{\mathcal{C}}$ *obtained by appending a part of size* $n-m$ *to the start of the partition* $\vec{\lambda}(\mathbf{1})$; *this is only defined if* $n-m \geq \vec{\lambda}(\mathbf{1})_1$. *Explicitly, for all* U *different from* $\mathbf{1}$, $\vec{\lambda}'(U) = \vec{\lambda}[n](U) = \vec{\lambda}(U)$, *and* $\vec{\lambda}'(\mathbf{1}) = \vec{\lambda}(\mathbf{1}) \backslash (\vec{\lambda}(\mathbf{1})_1)$, *whilst for* $n-m \geq \vec{\lambda}(\mathbf{1})_1$, $\vec{\lambda}[n](\mathbf{1}) = (n-m, \vec{\lambda}(\mathbf{1}))$. *We leave the operation undefined if the inequality does not hold. Finally, we define the following set which will index a basis of the limiting Grothendieck ring:*

$$\mathcal{P}^{\mathcal{C}} = \bigcup_{n \geq 0} \mathcal{P}_n^{\mathcal{C}} = \{\lambda : I(\mathcal{C}) \to \mathcal{P} \mid \sum_{U \in I(C)} |\lambda(U)| < \infty\}.$$

The above operations satisfy some trivial properties.

Lemma 4.6.6. *The operations* $\vec{\lambda} \mapsto \vec{\lambda}'$ *and* $\vec{\lambda} \mapsto \vec{\lambda}[n]$ *satisfy the following relations.*

1. $\vec{\lambda}' = \vec{\lambda}^{*\prime}$.

2. *If* $\vec{\lambda} \in \mathcal{P}_n^{\mathcal{C}}$, *then* $\vec{\lambda}'[n] = \vec{\lambda}$.

3. *If* $\vec{\lambda} \in \mathcal{P}_m^{\mathcal{C}}$ *and* $(n-m) \geq \vec{\lambda}(\mathbf{1})_1$, *then* $\vec{\lambda}[n]' = \vec{\lambda}$. *In particular, this holds for* n *sufficiently large.*

4. $\bigcup_{n \in \mathbb{Z}_{\geq 0}} \{\vec{\lambda}' \mid \vec{\lambda} \in \mathcal{P}_n^{\mathcal{C}}\} = \mathcal{P}^{\mathcal{C}}$.

Definition 4.6.7. *Given* $\vec{\mu}, \vec{\nu} \in \mathcal{P}^{\mathcal{C}}$, *we may write (in* $\mathcal{G}(\mathcal{W}_n(\mathcal{C}))$ *for* n *sufficiently large)*

$$[R_{\vec{\mu}[n]}][R_{\vec{\nu}[n]}] = \sum_{\vec{\lambda} \in \mathcal{P}^{\mathcal{C}}} k_{\vec{\mu},\vec{\nu}}^{\vec{\lambda}}(n)[R_{\vec{\lambda}[n]}]$$

In the above sum, we only consider terms for which $\overrightarrow{\lambda}[n]$ is well defined. We define the numbers $k^{\overrightarrow{\lambda}}_{\overrightarrow{\mu},\overrightarrow{\nu}}(n) \in \mathbb{Z}_{\geq 0}$ via the preceding equation, and note that for fixed $\overrightarrow{\mu}, \overrightarrow{\nu}, \overrightarrow{\lambda}$ it is defined for all sufficiently large n.

We use Mackey theory to show that the $k^{\overrightarrow{\lambda}}_{\overrightarrow{\mu},\overrightarrow{\nu}}(n)$ become constant as $n \to \infty$. Specifically, Lemma 4.6.2 describes a partial tensor product decomposition and Proposition 4.5.6 implies that the index set of the sum stabilises. So, it suffices to show that for any fixed t in the set of double coset representatives, the corresponding summand also stabilises:

$$\mathrm{Ind}^{S_n}_{H_{\overrightarrow{\mu}[n]} \cap tH_{\overrightarrow{\nu}[n]}t^{-1}} \Big(\mathrm{Res}^{H_{\overrightarrow{\mu}[n]}}_{H_{\overrightarrow{\mu}[n]} \cap tH_{\overrightarrow{\nu}[n]}t^{-1}} \left(\boxtimes_{U \in I(\mathcal{C})} \left(U^{\boxtimes|\overrightarrow{\mu}[n](U)|} \otimes \mathcal{S}^{\overrightarrow{\mu}[n](U)} \right)\right) \\ \otimes \mathrm{Res}^{tH_{\overrightarrow{\nu}[n]}t^{-1}}_{H_{\overrightarrow{\mu}[n]} \cap tH_{\overrightarrow{\nu}[n]}t^{-1}} \left(t \boxtimes_{U \in I(\mathcal{C})} \left(U^{\boxtimes|\overrightarrow{\nu}[n](U)|} \otimes \mathcal{S}^{\overrightarrow{\nu}[n](U)} \right)\right) \Big).$$

To demonstrate stability, we first show that the restrictions in the above expression stabilise in a particular sense.

Lemma 4.6.8. *Let n be sufficiently large, depending on $\overrightarrow{\mu}$, $\overrightarrow{\nu}$, and t, where t is a fully ordered $(H_{\overrightarrow{\mu}[n]}, H_{\overrightarrow{\nu}[n]})$-double coset representative. There exists $g \in S_n$ (identified for different n via usual inclusions of symmetric groups) such that $g(H_{\overrightarrow{\mu}[n]} \cap tH_{\overrightarrow{\nu}[n]}t^{-1})g^{-1} = S_{\sigma[n]}$, where $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_l)$ is some partition. Additionally, the restriction*

$$\mathrm{Res}^{H_{\overrightarrow{\mu}[n]}}_{H_{\overrightarrow{\mu}[n]} \cap tH_{\overrightarrow{\nu}[n]}t^{-1}} \left(\boxtimes_{U \in I(\mathcal{C})} \left(U^{\boxtimes|\overrightarrow{\mu}[n](U)|} \otimes \mathcal{S}^{\overrightarrow{\mu}[n](U)} \right)\right)$$

is equal to a finite direct sum of expressions of the following form, where the multiplicity of each term does not vary with n, provided n is sufficiently large:

$$\left(\boxtimes_{i=1}^{l} \left(U_i^{\boxtimes|\tau^{(i)}|} \otimes \mathcal{S}^{\tau^{(i)}} \right)\right) \boxtimes \left(\mathbf{1}^{\boxtimes(n-|\sigma|)} \otimes \mathcal{S}^{(n-|\tau^{(0)}|-|\sigma|, \tau^{(0)})} \right).$$

Here the U_i are not necessarily distinct, and each $\tau^{(i)}$ is a partition of σ_i ($\tau^{(0)}$ is arbitrary, but only finitely many cases appear).

Proof. The subgroup $H_{\overrightarrow{\mu}[n]} \cap tH_{\overrightarrow{\nu}[n]}t^{-1}$ is a Young subgroup of S_n by Remark 4.6.3, which we may conjugate to reorder the component groups in order of increasing size. Explicitly, we have $g \in S_n$ such that $g(H_{\vec{\mu}[n]} \cap tH_{\vec{\nu}[n]}t^{-1})g^{-1} = S_\alpha$ for some partition $\alpha = (\sigma_0, \sigma_1, \ldots, \sigma_l)$. We now show that we may take $\sigma_1, \sigma_2, \ldots, \sigma_l$ to be constant with respect to n, and hence $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_l)$ is the desired partition from the statement of the lemma (and $\sigma_0 = n - |\sigma|$).

By Proposition 4.5.6, we may assume the double coset representative t is preserved under the inclusions $S_n \hookrightarrow S_{n+1} \hookrightarrow \cdots$. Hence, t fixes i for i larger than some fixed constant m_0 depending on t. If $m_1 = |\vec{\mu}| - |\vec{\mu}(\mathbf{1})_1|$, then by construction $H_{\vec{\mu}}$ contains the symmetric group on $\{m_1+1, m_1+2, \ldots, n\}$ (this is a subgroup of the component group whose representation $U^{\boxtimes|\tau|} \otimes \mathcal{S}^\tau$ has $U = \mathbf{1}$). Similarly if $m_2 = |\vec{\nu}| - |\vec{\nu}(\mathbf{1})_1|$, then $H_{\vec{\nu}}$ contains the symmetric group on $\{m_2+1, m_2+2, \ldots, n\}$. Now, we let $M = \max(m_0, m_1, m_2)$, and we observe that the symmetric group on $\{M+1, M+2, \ldots, n\}$ is contained in $H_{\vec{\mu}} \cap tH_{\vec{\nu}}t^{-1}$ (it is contained in each of $H_{\vec{\mu}}$ and $H_{\vec{\nu}}$ and commutes with t). Thus we may choose g in the previous paragraph to fix $M+1, M+2, \ldots, n$ by making the component group permuting the orbit of n appear as the last factor in the construction of $S_{(\sigma_0, \sigma_1, \ldots, \sigma_l)}$ (for n sufficiently large, this is consistent with our convention of ordering of component groups in a Young subgroup of a symmetric group). By similar reasoning, the other component groups of $H_{\vec{\mu}} \cap tH_{\vec{\nu}}t^{-1}$ are stable when passing from n to $n+1$, meaning that $H_{\vec{\mu}} \cap tH_{\vec{\nu}}t^{-1}$ decomposes as a product of a fixed number of symmetric groups S_{σ_i} (where σ_i are constant with respect to n) and $S_{n-\sum_i \sigma_i}$. Since g fixes all i greater than M, it is compatible with the inclusions $S_n \hookrightarrow S_{n+1}$.

Next we discuss stability of the restriction. The restriction of an external product is the same as the external product of restrictions:

$$\begin{aligned} &\mathrm{Res}^{H_{\vec{\mu}[n]}}_{H_{\vec{\mu}[n]} \cap tH_{\vec{\nu}[n]}t^{-1}} \left(\boxtimes_{U \in I(\mathcal{C})} \left(U^{\boxtimes|\vec{\mu}[n](U)|} \otimes \mathcal{S}^{\vec{\mu}[n](U)} \right) \right) \\ \cong\ & \boxtimes_{U \in I(\mathcal{C})} \mathrm{Res}^{S_{|\vec{\mu}[n](U)|}}_{S_{|\vec{\mu}[n](U)|} \cap tH_{\vec{\nu}[n]}t^{-1}} \left(U^{\boxtimes|\vec{\mu}[n](U)|} \otimes \mathcal{S}^{\vec{\mu}[n](U)} \right). \end{aligned}$$

Because the intersection $H_{\vec{\mu}[n]} \cap tH_{\vec{\nu}[n]}t^{-1}$ is a Young subgroup of S_n, it follows that for each U, $S_{|\vec{\mu}[n](U)|} \cap tH_{\vec{\nu}[n]}t^{-1}$, is a Young subgroup of $S_{|\vec{\mu}[n](U)|}$ and the product of these across $U \in I(\mathcal{C})$ will give $H_{\vec{\mu}[n]} \cap tH_{\vec{\nu}[n]}t^{-1}$. Because $H_{\vec{\mu}[n]} \cap tH_{\vec{\nu}[n]}t^{-1}$ is conjugate to $S_{(n-|\sigma|, \sigma)}$ by reordering of component groups, we may write $S_{|\vec{\mu}[n](U)|} \cap tH_{\vec{\nu}[n]}t^{-1} = \prod_{j \in I_U} S_{\sigma_j}$, where the I_U are disjoint subsets of $\{0, 1, \ldots, l\}$ indexed by $U \in I(\mathcal{C})$ that partition $\{0, 1, \ldots, l\}$. In particular, when $U \neq \mathbf{1}$, $\prod_{j\in I_U} S_{\sigma_j}$ is independent of n (the $U = \mathbf{1}$ term contains $S_{\sigma_0} = S_{n-|\sigma|}$ which does depend on n).

For a function $f : I_U \to \mathcal{P}$, let $c_f \in \mathbb{Z}_{\geq 0}$ be defined by restricting representations of symmetric groups:

$$\mathrm{Res}^{S_{|\vec{\mu}[n](U)|}}_{\prod_{i\in I_U} S_{\sigma_i}} \left(\mathcal{S}^{\vec{\mu}[n](U)}\right) = \bigoplus_{f: I_U\to\mathcal{P}, |f(i)|=\sigma_i} \left(\boxtimes_{i\in I_U} \mathcal{S}^{f(i)}\right)^{\oplus c_f}.$$

After suitably applying Proposition 4.3.6 to decompose such a restriction, we get the following:

$$\mathrm{Res}^{S_{|\vec{\mu}[n](U)|}}_{S_{|\vec{\mu}[n](U)|}\cap tH_{\vec{\nu}[n]}t^{-1}} \left(U^{\boxtimes|\vec{\mu}[n](U)|} \otimes \mathcal{S}^{\vec{\mu}[n](U)}\right) = \bigoplus_{f: I_U\to\mathcal{P}, |f(i)|=\mu_i} \left(\boxtimes_{i\in I_U} \left(U^{\boxtimes|f(i)|} \otimes \mathcal{S}^{f(i)}\right)\right)^{\oplus c_f}.$$

This makes it clear that the only dependence on n enters through the term corresponding to $\sigma_0 = n - |\sigma|$ in the $U = \mathbf{1}$ case (the cases for other U do not involve $S_{n-|\sigma|}$ and hence do not depend on n). Let $k = \sum_{i\in I_{\mathbf{1}}\setminus\{0\}} \sigma_i$. In that case, we observe that restriction from $S_{|\vec{\mu}[n](\mathbf{1})|}$ to $(\prod_{i\in I_{\mathbf{1}}\setminus\{0\}} S_{\sigma_i}) \times S_{n-|\sigma|}$ is the same as first restricting to $S_k \times S_{n-|\sigma|}$ and then restricting the first factor to $\prod_{i\in I_{\mathbf{1}}\setminus\{0\}} S_{\sigma_i}$ where the latter operation will be independent of n, similarly to the case $U \neq \mathbf{1}$. Thus it is enough to show that the operation of restricting to $S_k \times S_{n-|\sigma|}$ is stable in the sense described by the statement of the lemma.

To understand the restriction, we fix an integer partition ρ. We must demonstrate the stability of the following expression (understood as a sum of terms of the form $\mathcal{S}^\eta \boxtimes \mathcal{S}^{(n-k-|\tau|,\tau)}$ for $\eta \vdash k$):

$$\mathrm{Res}^{S_{n-|\mu|+k}}_{S_k\times S_{n-|\mu|}}(\mathcal{S}^{(n-k-|\rho|,\rho)}).$$

The restriction multiplicities are given by Littlewood-Richardson coefficients, and the stability condition is immediately implied by Proposition 2.1.4. □

Remark 4.6.9. *The analogous stability statement for*

$$\mathrm{Res}^{tH_{\vec{\nu}[n]}t^{-1}}_{H_{\vec{\mu}[n]}\cap tH_{\vec{\nu}[n]}t^{-1}} \left(t \boxtimes_{U\in I(\mathcal{C})} \left(U^{\boxtimes|\vec{\nu}[n](U)|} \otimes \mathcal{S}^{\vec{\nu}[n](U)}\right)\right)$$

is proved similarly.

Finally, we are able to prove stability of the coefficients $k^{\vec{\lambda}}_{\vec{\mu},\vec{\nu}}(n)$.

Theorem 4.6.10. *For any choice of* $\vec{\mu}$, $\vec{\nu}$, $\vec{\lambda}$, $\lim_{n\to\infty} k^{\vec{\lambda}}_{\vec{\mu},\vec{\nu}}(n)$ *exists and is a non-negative integer.*

Proof. We use Lemma 4.6.8, again reducing to the case of a fixed double coset representative t (of which there are finitely many, and they are stable with respect to n). We must demonstrate stability of

$$\begin{aligned}\mathrm{Ind}^{S_n}_{H_{\vec{\mu}[n]}\cap tH_{\vec{\nu}[n]}t^{-1}} \Big(&\mathrm{Res}^{H_{\vec{\mu}[n]}}_{H_{\vec{\mu}[n]}\cap tH_{\vec{\nu}[n]}t^{-1}} \left(\boxtimes_{U\in I(\mathcal{C})} \left(U^{\boxtimes|\vec{\mu}[n](U)|} \otimes \mathcal{S}^{\vec{\mu}[n](U)}\right)\right)\\ &\otimes \mathrm{Res}^{tH_{\vec{\nu}[n]}t^{-1}}_{H_{\vec{\mu}[n]}\cap tH_{\vec{\nu}[n]}t^{-1}} \left(t\boxtimes_{U\in I(\mathcal{C})} \left(U^{\boxtimes|\vec{\nu}[n](U)|} \otimes \mathcal{S}^{\vec{\nu}[n](U)}\right)\right)\Big).\end{aligned}$$

The restrictions give a finite number of stable summands (by Lemma 4.6.8), so it suffices to show that products of summands exhibit the relevant stabilisation property. We write:

$$\begin{aligned}&\mathrm{Ind}^{S_n}_{H_{\vec{\mu}[n]}\cap tH_{\vec{\nu}[n]}t^{-1}} \Big(\mathrm{Res}^{H_{\vec{\mu}[n]}}_{H_{\vec{\mu}[n]}\cap tH_{\vec{\nu}[n]}t^{-1}} \left(\boxtimes_{U\in I(\mathcal{C})} \left(U^{\boxtimes|\vec{\mu}[n](U)|} \otimes \mathcal{S}^{\vec{\mu}[n](U)}\right)\right)\\ &\qquad\otimes \mathrm{Res}^{tH_{\vec{\nu}[n]}t^{-1}}_{H_{\vec{\mu}[n]}\cap tH_{\vec{\nu}[n]}t^{-1}} \left(t\boxtimes_{U\in I(\mathcal{C})} \left(U^{\boxtimes|\vec{\nu}[n](U)|} \otimes \mathcal{S}^{\vec{\nu}[n](U)}\right)\right)\Big)\\ =\ &\bigoplus \mathrm{Ind}^{S_n}_{S_\sigma\times S_{n-|\sigma|}} \Big(\Big(\boxtimes_{i=1}^{l}\Big(U_i^{\boxtimes|\tau^{(i)}|}\otimes\mathcal{S}^{\tau^{(i)}}\Big)\Big)\boxtimes\Big(\mathbf{1}^{\boxtimes(n-|\sigma|)}\otimes\mathcal{S}^{(n-|\tau^{(0)}|-|\sigma|,\tau^{(0)})}\Big)\\ &\qquad\otimes\Big(\boxtimes_{i=1}^{l}\Big(V_i^{\boxtimes|\rho^{(i)}|}\otimes\mathcal{S}^{\rho^{(i)}}\Big)\Big)\boxtimes\Big(\mathbf{1}^{\boxtimes(n-|\sigma|)}\otimes\mathcal{S}^{(n-|\rho^{(0)}|-|\sigma|,\rho^{(0)})}\Big)\Big)\\ =\ &\bigoplus \mathrm{Ind}^{S_n}_{S_\sigma\times S_{n-|\sigma|}} \Big(\Big(\boxtimes_{i=1}^{l}\Big((U_i\otimes V_i)^{|\tau^{(i)}|}\otimes\mathcal{S}^{\tau^{(i)}}\otimes\mathcal{S}^{\rho^{(i)}}\Big)\Big)\\ &\qquad\boxtimes\Big(\mathbf{1}^{\boxtimes(n-|\sigma|)}\otimes\mathcal{S}^{(n-|\tau^{(0)}|-|\sigma|,\tau^{(0)})}\otimes\mathcal{S}^{(n-|\rho^{(0)}|-|\sigma|,\rho^{(0)})}\Big)\Big).\end{aligned}$$

Here the first equality used the statement of the preceding lemma (hence the implied sum is finite and independent of n); $\tau^{(i)}$ and $\rho^{(i)}$ are the partitions coming from the statement of the lemma. The second equality used Proposition 4.3.5. Each $U_i \otimes V_i$ decomposes into a linear combination of simple $[U]$ when we pass to the Grothendieck group. Proposition 4.3.8 can be used to replace the summand with a sum of similar terms where the $U_i\otimes V_i$ are replaced with $[U]$ for some $U \in I(\mathcal{C})$ and the resulting quantity is independent of n. The result that the term $\mathbf{1}^{\boxtimes(n-|\sigma|)} \otimes \left(\mathcal{S}^{(n-|\tau^{(0)}|-|\sigma|,\tau^{(0)})} \otimes \mathcal{S}^{(n-|\rho^{(0)}|-|\sigma|,\rho^{(0)})}\right)$ admits a stable limit in terms of $\mathbf{1}^{\boxtimes(n-|\sigma|)} \otimes \mathcal{S}^{(n-|\lambda|-|\sigma|,\lambda)}$ is equivalent to the stability of Kronecker coefficients. Then, taking the exterior tensor product with a finite number of fixed $\mathbf{1}^{\boxtimes|\tau^{(i)}|} \otimes \mathcal{S}^{\tau^{(i)}}$ (coming from the finite terms in the product) and inducing to a larger symmetric group:

$$\mathrm{Ind}_{S_{n-|\sigma|}\times\prod_i S_{|\tau^{(i)}|}}^{S_n} \left(\left(\mathbf{1}^{\boxtimes(n-|\sigma|)} \otimes \mathcal{S}^{(n-|\lambda|-|\sigma|,\lambda)} \right) \boxtimes \left(\boxtimes_{i=1}^{l} \mathbf{1}^{\boxtimes|\tau^{(i)}|} \otimes \mathcal{S}^{\tau^{(i)}} \right) \right)$$

also has a stable limit because the multiplicities are described by Littlewood-Richardson coefficients which we already know have suitable stability properties as per Proposition 2.1.4. We obtain a linear combination of $[R_{\vec{\lambda}[n]}]$ in the Grothendieck group. The coefficients are finite because they are limits of eventually constant sequences of integers. □

4.6.2 Definition and Basic Properties of the Limiting Grothendieck Ring

We come to the definition of the main object of this paper.

Definition 4.6.11. *Let $\mathcal{G}_\infty^{\mathbb{Z}}(\mathcal{C})$ be the free $\mathbb{Z}$-module having basis $X_{\vec{\lambda}}$ for $\vec{\lambda} \in \mathcal{P}^{\mathcal{C}}$ and a multiplication defined by*

$$X_{\vec{\mu}} X_{\vec{\nu}} = \sum_{\vec{\lambda}\in\mathcal{P}^{\mathcal{C}}} \left(\lim_{n\to\infty} c_{\vec{\mu},\vec{\nu}}^{\vec{\lambda}}(n) \right) X_{\vec{\lambda}},$$

and let $\mathcal{G}_\infty^{\mathbb{Q}}(\mathcal{C}) = \mathbb{Q} \otimes_{\mathbb{Z}} \mathcal{G}_\infty^{\mathbb{Z}}(\mathcal{C})$. We will show that this is multiplication is associative and unital, making these into associative algebras.

Remark 4.6.12. *We will focus on $\mathcal{G}_\infty^{\mathbb{Q}}(\mathcal{C})$, as it contains certain elements of interest to us that do not lie in $\mathcal{G}_\infty^{\mathbb{Z}}(\mathcal{C})$. The integral form will be studied in Chapter 5.*

We also introduce a collection of elements that will be important.

Definition 4.6.13. *A* basic hook *is an element $\vec{\lambda} \in \mathcal{P}^{\mathcal{C}}$ such that $\vec{\lambda}(U) = (1^n)$ for some $U \in I(\mathcal{C})$, and $\vec{\lambda}(V)$ is the empty partition for all V different from U. By abuse of terminology* *we also refer to* $X_{\vec{\lambda}} \in \mathcal{G}_{\infty}^{\mathbb{Q}}(\mathcal{C})$ *as a basic hook whenever the indexing multipartition* $\vec{\lambda}$ *is a basic hook; we also denote* $X_{\vec{\lambda}}$ *as* $\overline{e}_n(U)$.

Theorem 4.6.14. *Asymptotically as* $t \to \infty$, *the structure constants of the images of indecomposable objects of the Deligne category* $S_t(\mathcal{C})$ *in the relevant Grothendieck group agree with the structure constants of the* $X_{\vec{\lambda}}$ *in* $\mathcal{G}_{\infty}^{\mathbb{Q}}(\mathcal{C})$.

Proof. The wreath product Deligne categories $S_t(\mathcal{C})$ admit tensor functors to $\mathcal{W}_n(\mathcal{C})$ for $n \in \mathbb{Z}_{\geq 0}$, and their behaviour is discussed in Theorem 5.6 of [19], and restated in our setting in Theorem 3.1 of [14]. The object indexed by a multipartition $\vec{\lambda}$ is mapped to the irreducible object of $\mathcal{W}_n(\mathcal{C})$ indexed by $\vec{\lambda}[n]$, if $\vec{\lambda}[n]$ is a well defined multipartition (i.e. $n - |\vec{\lambda}| \geq \vec{\lambda}(\mathbf{1})_1$), and otherwise it is zero. In [19], Theorem 4.13 demonstrates that for fixed objects of $S_t(\mathcal{C})$ and sufficiently large t, spaces of homomorphisms in $S_t(\mathcal{C})$ can be computed by using the tensor functor to pass to the wreath product categories $\mathcal{W}_n(\mathcal{C})$. The tensor product multiplicities are determined by the homomorphism spaces in the following way. An object M of $S_t(\mathcal{C})$ is determined by the information $\hom_{S_t(\mathcal{C})}(N, M)$ for all objects N (by the Yoneda lemma). This information can be recovered by passing to sufficiently large wreath product categories $\mathcal{W}_n(\mathcal{C})$, and in particular we can choose M to be a tensor product of two objects. In the original setting of $\underline{\mathrm{Rep}}(S_t)$, the result was proved by Deligne in [8]. □

We have a few preliminary facts about the algebra $\mathcal{G}_{\infty}^{\mathbb{Z}}(\mathcal{C})$.

Theorem 4.6.15. *The algebra* $\mathcal{G}_{\infty}^{\mathbb{Z}}(\mathcal{C})$ *is a unital associative algebra satisfying the following:*

1. $\mathcal{G}_{\infty}^{\mathbb{Z}}(\mathcal{C})$ *is commutative if and only if* $\mathcal{G}(\mathcal{C})$ *is commutative.*

2. $\mathcal{G}_{\infty}^{\mathbb{Z}}(\mathcal{C})$ *is generated by the basic hooks* $\overline{e}_n(U)$, *where* $n \geq 1$ *and* $U \in I(\mathcal{C})$.

3. *There is a filtration* $\mathcal{G}_{\infty}^{\mathbb{Z}}(\mathcal{C}) = \cup_{n \in \mathbb{N}} \mathcal{F}_n$, *where* $\mathcal{F}_n$ *is spanned by* $X_{\vec{\lambda}}$ *with* $|\vec{\lambda}| \leq n$.

4. *The associated graded algebra of* $\mathcal{G}_{\infty}^{\mathbb{Z}}(\mathcal{C})$ *with respect to this filtration is isomorphic to* $\bigotimes_{U \in I(\mathcal{C})} \Lambda^{(U)}$, *where* $\Lambda^{(U)}$ *is the ring of symmetric functions. If we write* $f^{(U)}$ *to indicate that the symmetric function* f *is considered as an element of* $\Lambda^{(U)}$, *then the image of* $[R_{\vec{\lambda}}]$ *is* $s_{\vec{\lambda}} = \prod_{U \in I(\mathcal{C})} s^{(U)}_{\vec{\lambda}(U)}$.

All these statements immediately follow for $\mathcal{G}_\infty^{\mathbb{Q}}(\mathcal{C})$ by tensoring up to $\mathbb{Q}$.

Proof. Firstly, the multiplication in $\mathcal{G}_\infty^{\mathbb{Z}}(\mathcal{C})$ is seen to be associative by considering the product $[R_{\vec{\lambda}[n]}][R_{\vec{\mu}[n]}][R_{\vec{\nu}[n]}]$ in the associative algebra $\mathcal{G}(\mathcal{W}_n(\mathcal{C}))$. For n sufficiently large, the coefficient of $[R_{\vec{\rho}[n]}]$ in that element becomes equal to the coefficient of $X_{\vec{\rho}}$ in $X_{\vec{\lambda}}X_{\vec{\mu}}X_{\vec{\nu}}$, regardless of how the latter product is parenthesised. The basis element corresponding to the empty partition is the unit element.

The commutativity or non-commutativity of multiplication can be seen from the proof of Theorem 4.6.10, where (up to conjugation by the double coset representative t) the only change between $[R_{\vec{\mu}}][R_{\vec{\nu}}]$ and $[R_{\vec{\nu}}][R_{\vec{\mu}}]$ is the product $U_i \otimes V_i$ (versus $V_i \otimes U_i$) of objects of $\mathcal{C}$. Proposition 4.3.8 was used to write the result in terms of $[U]$ for $U \in I(\mathcal{C})$, and equal results are obtained for $U_i \otimes V_i$ and $V_i \otimes U_i$ if and only if $[U_i \otimes V_i] = [V_i \otimes U_i]$. This holds for all possible choices of U_i and V_i if and only if $\mathcal{G}(\mathcal{C})$ is commutative.

The filtration is essentially the same as the $|\lambda|$-filtration defined in Definition 2.7 of [14]. In particular, the associated graded algebra (with basis induced from $X_{\vec{\lambda}}$) has structure constants equal to those of the ring of symmetric functions with the Schur function basis. So, the basic hooks $\overline{e}_n(U)$ correspond to elementary symmetric functions $e_n^{(U)}$ in $\Lambda_{\mathbb{Q}}^{(U)}$. This means that the basic hooks generate the associated graded algebra, and hence they generate $\mathcal{G}_\infty^{\mathbb{Z}}(\mathcal{C})$. □

Example 4.6.16. *In the case where $\mathcal{C} = kG-\mathrm{mod}$ for a finite group G, $\mathcal{W}_n(\mathcal{C})$ is equivalent to the category of finite-dimensional representations for the wreath product $G^n \rtimes S_n$. In this case, $\mathcal{G}_\infty^{\mathbb{Z}}(\mathcal{C})$ is commutative, and it follows that $\mathcal{G}_\infty^{\mathbb{Z}}(\mathcal{C})$ is isomorphic to a free polynomial algebra in the basic hooks; see Corollary 2.9 of [14]. In our setting, $\mathcal{G}(\mathcal{C})$ may not be commutative, in which case $\mathcal{G}_\infty^{\mathbb{Z}}(\mathcal{C})$ cannot possibly be a free polynomial algebra. Nevertheless, we will give a description of the algebra structure of the ring in terms of basic hooks, and also give generating functions describing how the $X_{\vec{\lambda}}$ are expressed in terms of basic hooks.*

4.7 The Ring Structure of $\mathcal{G}^{\mathbb{Q}}_{\infty}(\mathcal{C})$

In this section, we determine the algebra structure of $\mathcal{G}^{\mathbb{Q}}_{\infty}(\mathcal{C})$, using a family of elements $T_n(M)$.

4.7.1 The Elements $T_n(M)$

We use the following construction to relate $\mathcal{W}_n(\mathcal{C})$ with $\mathcal{G}^{\mathbb{Q}}_{\infty}(\mathcal{C})$; informally, we take $n \to \infty$.

Definition 4.7.1. *Suppose that H is a subgroup of S_n and M is a module over $\mathcal{R} \wr H$. In $\mathcal{G}(\mathcal{W}_{n+m}(\mathcal{C}))$ We may write*

$$[\mathrm{Ind}_{H\times S_m}^{S_{m+n}} \left(M \boxtimes \left(\mathbf{1}^{\boxtimes m} \otimes \mathbf{1}_{S_m}\right)\right)] = \sum_{\vec{\mu}\in\mathcal{P}^{\mathcal{C}}} c_{\vec{\mu}}(M,m)[R_{\vec{\mu}[n+m]}].$$

Here $\mathbf{1}_{S_m}$ is the trivial representation of S_m and for any fixed n we only sum over $\vec{\mu}$ such that $\vec{\mu}[n+m]$ is defined. By transitivity of induction, we may replace M and H with $\mathrm{Ind}_H^{S_n}(M)$ and S_n respectively. In this case, if $\mathrm{Ind}_H^{S_n}(M)$ is a simple object, it is induced from an object of the form (where $\vec{\rho} \in \mathcal{P}^{\mathcal{C}}$)

$$M = \left(\mathbf{1}^{\boxtimes|\vec{\rho}[n](\mathbf{1})|} \otimes \mathcal{S}^{\vec{\rho}[n](\mathbf{1})}\right) \boxtimes \left(\boxtimes_{U\neq\mathbf{1}} \left(U^{\boxtimes|\vec{\rho}[n](U)|} \otimes \mathcal{S}^{\vec{\rho}[n](U)}\right)\right).$$

Substituting this into the equation defining $c_{\vec{\mu}}(M,m)$, we see that the stability of Littlewood-Richardson coefficients (Proposition 2.1.4) implies that there is a nonzero contribution to only finitely many $c_{\mu}(M,m)$, and the contribution becomes constant for m sufficiently large. We define

$$\lim_{m\to\infty} \mathrm{Ind}_{H\times S_m}^{S_{m+n}} \left(M \boxtimes \left(\mathbf{1}^{\boxtimes m} \otimes \mathbf{1}_{S_m}\right)\right) = \sum_{\mu\in\mathcal{P}_{\mathcal{C}}} \left(\lim_{m\to\infty} c_{\mu}(M,m)\right) X_{\mu}.$$

By the linearity of induction, we may extend this definition to allow M to be not necessarily simple, or indeed a formal difference of objects (when working with Grothendieck groups). Note that this construction only depends on the class of M in the Grothendieck group because *induction is an exact functor. So, we may write* $\lim_{m\to\infty}[\mathrm{Ind}_{H\times S_m}^{S_{m+n}}](-)$ *when the argument is an element of a Grothendieck group (rather than an object of a category), and the operation is still well defined.*

We now define a generating set of $\mathcal{G}_\infty^{\mathbb{Q}}(\mathcal{C})$ with favourable multiplicative properties.

Definition 4.7.2. *For an object* M *of* $\mathcal{C}$ *and* $n \in \mathbb{Z}_{>0}$*, we define the following elements of* $\mathbb{Q} \otimes_{\mathbb{Z}} \mathcal{G}(\mathcal{W}_n(\mathcal{C}))$*:*

$$T_n^f(M) = \frac{1}{n}\sum_{\lambda\vdash n}\chi_{(n)}^{\lambda}[M^{\boxtimes|\lambda|}\otimes\mathcal{S}^{\lambda}].$$

We construct an analogous element of $\mathcal{G}_\infty^{\mathbb{Q}}(\mathcal{C})$ *as follows:*

$$T_n(M) = \frac{1}{n}\sum_{\lambda\vdash n}\chi_{(n)}^{\lambda}\lim_{m\to\infty}\mathrm{Ind}_{S_n\times S_m}^{S_{n+m}}\left(\left(M^{\boxtimes|\lambda|}\otimes\mathcal{S}^{\lambda}\right)\boxtimes\left(\mathbf{1}^{\boxtimes m}\otimes\mathbf{1}_{S_m}\right)\right).$$

We write $[\mathrm{Ind}_{S_n^k\times S_m}^{S_{nk+m}}] : \mathcal{G}(\mathcal{W}_n(\mathcal{C}))^{\otimes k}\otimes\mathcal{G}(\mathcal{W}_m(\mathcal{C})) \to \mathcal{G}(\mathcal{W}_{nk+m}(\mathcal{C}))$ *for the function on Grothendieck groups induced by the induction functor. Now we let*

$$\begin{aligned}&T_n(M_1, M_2, \ldots, M_k)\\ =\ &\lim_{m\to\infty}[\mathrm{Ind}_{S_n^k\times S_m}^{S_{nk+m}}]\left(T_n^f(M_1)\boxtimes T_n^f(M_2)\boxtimes\cdots\boxtimes T_n^f(M_k)\boxtimes\left[\left(\mathbf{1}^{\boxtimes m}\otimes\mathbf{1}_{S_m}\right)\right]\right).\end{aligned}$$

As before, $T_n(M_1, M_2, \ldots, M_k)$ *only depends on the class of the* M_i *in the corresponding Grothendieck groups.*

Remark 4.7.3. *The character orthogonality relation*

$$\frac{1}{n}\sum_{\lambda\vdash n}\chi_{(n)}^{\lambda}\chi_{\nu}^{\lambda} = \delta_{\nu,(n)}$$

suggests that one can think of $T_n(U)$ *as a generalisation of the indicator function of cycle type* (n) *in a copy of the class functions on* S_n *associated to* $U \in I(\mathcal{C})$*. This is based on the fact that the virtual character associated to* $\frac{1}{n}\sum_{\lambda\vdash n}\chi_{(n)}^{\lambda}\mathcal{S}^{\lambda}$ *is the indicator function of* n*-cycles on* S_n*.*

Proposition 4.7.4. *We have the following properties of the $T_n(U)$, for $U \in I(\mathcal{C})$:*

1. *The image of $T_n(U)$ in the associated graded algebra of $\mathcal{G}^{\mathbb{Q}}_\infty(\mathcal{C})$, which we identify with $\bigotimes_{U \in I(\mathcal{C})} \Lambda^{(U)}_{\mathbb{Q}}$, is $p^{(U)}_n/n$. That is, the n-th power sum symmetric function in $\Lambda^{(U)}_{\mathbb{Q}}$, divided by n.*

2. *Fix a total order on $\mathbb{Z}_{>0} \times I(\mathcal{C})$. Consider the monomials in $T_n(U)$ for $(n, U) \in \mathbb{Z}_{>0} \times I(\mathcal{C})$ where the factors occur in order consistent with the total order ("PBW monomials"). These monomials are linearly independent.*

3. *The $T_n(U)$ generate $\mathcal{G}^{\mathbb{Q}}_\infty(\mathcal{C})$.*

4. *$T_n(U)$ lies in the n-th filtered component of $\mathcal{G}^{\mathbb{Q}}_\infty(\mathcal{C})$.*

Proof. The first claim follows from the fact that the virtual character of $\frac{1}{n}\sum_{\lambda \vdash n} \chi^\lambda_{(n)} \mathcal{S}^\lambda$ is the indicator function of n-cycles on S_n. The second follows from the fact that the $p^{(U)}_n$ are algebraically independent in the associated graded algebra. Since the $p^{(U)}_n$ generate the associated graded algebra, the third claim follows. The final claim is immediate if U is different from $\mathbf{1}$, for then $T_n(U)$ becomes a linear combination of $X_{\vec{\lambda}}$ with $|\vec{\lambda}| = n$. The $U = \mathbf{1}$ case follows from the Pieri rule (see also Remark 4.8.6) which describes certain Littlewood-Richardson coefficients; we wish to decompose $\mathrm{Ind}^{S_{n+m}}_{S_n \times S_m}(\mathcal{S}^\lambda \boxtimes \mathbf{1}_{S_m})$ into $\mathcal{S}^{\mu[n+m]}$, with $|\mu| \leq n$. The μ that appear are obtained from λ by adding m boxes, no two in the same column, and then removing the top row. Removing the top row removes one box from each column, so any μ obtained this way satisfies $|\mu| \leq |\lambda| = n$. □

The following lemma will underpin much of what follows. Note that restriction is an exact functor, so it descends to a function between Grothendieck groups.

Lemma 4.7.5. *Any restriction of $T^f_n(U)$ to (the Grothendieck group of) a proper Young subgroup S_λ of S_n is zero.*

Proof. We use Proposition 4.3.6, part 2. It now suffices to understand how the indicator function of n-cycles restricts to S_λ. The result now follows from the fact that the only Young subgroup of S_n containing an n-cycle is all of S_n. □

In order to understand the algebra structure of $\mathcal{G}_\infty^{\mathbb{Q}}(\mathcal{C})$, we determine the commutator of two elements of the form $T_n(U)$ (recall that such elements generate the algebra).

Theorem 4.7.6. *Let $U_1, U_2 \in I(\mathcal{C})$. If $n \neq m$, the commutator of $T_n(U_1)$ and $T_m(U_2)$ vanishes: we have $[T_n(U_1), T_m(U_2)] = 0$. If $N_{U_1,U_2}^{U_3}$ is the structure tensor of the Grothendieck ring of $\mathcal{C}$ (so that $[U_1][U_2] = \sum_{U_3} N_{U_1,U_2}^{U_3}[U_3]$), then we have:*

$$[T_n(U_1), T_n(U_2)] = \sum_{U_3} (N_{U_1,U_2}^{U_3} - N_{U_2,U_1}^{U_3}) T_n(U_3).$$

Proof. To calculate the commutator $[T_n(U_1), T_m(U_2)]$, we calculate the analogous quantity in the ring $\mathbb{Q} \otimes_{\mathbb{Z}} \mathcal{G}(\mathcal{W}_{m+n+k}(\mathcal{C}))$ for k sufficiently large. As $T_n(U_1), T_m(U_2)$ are defined as the image of a linear combination of induced objects in the Grothendieck group, we may apply the Mackey theory formalism to calculate $T_n(U_1)T_m(U_2)$.

Firstly, the number of $(S_n \times S_{m+k}, S_m \times S_{n+k})$-double cosets in S_{m+n+k} is $\min(m,n)+1$. This can be seen from calculating the possible $C_{i,j}(\sigma)$ that can arise. Both i and j may take two different values. Therefore $C_{1,1}(\sigma)$ determines all other $C_{i,j}(\sigma)$ via identities such as $C_{i,1}(\sigma) + C_{i,2}(\sigma) = |B_1| = m$ (similarly $C_{1,j}(\sigma)$ and $C_{2,j}(\sigma)$ determine each other). So, double cosets are determined by a single invariant $C_{1,1}(\sigma)$, namely, the number of elements of $\{1, 2, \ldots, m\}$ that are mapped to the set $\{1, 2, \ldots, n\}$. Clearly $C_{1,1}(\sigma)$ can take any of the values $0, 1, \ldots, \min(m,n)$.

Consider a double coset representative σ (interpreted as a bijection from the set $\{1, 2, \ldots, n+m+k\}$ to itself) such that $\sigma(\{1, 2, \ldots, n\}) \neq \{1, 2, \ldots, m\}$ and $\sigma(\{1, 2, \ldots, n\}) \cap \{1, 2, \ldots, m\} \neq \varnothing$. In the Mackey theoretic computation the summand coming from a twist by σ will involve restricting to $(S_n \times S_{m+k}) \cap \sigma(S_m \times S_{n+k})\sigma^{-1}$, which will not contain the entirety of S_n (considered as a subgroup of $S_n \times S_{m+k}$):

$$\begin{aligned}\mathrm{Ind}_{(S_n \times S_{m+k}) \cap \sigma(S_m \times S_{n+k})\sigma^{-1}}^{S_{n+m+k}} \Big(&\mathrm{Res}_{(S_n \times S_{m+k}) \cap \sigma(S_m \times S_{n+k})\sigma^{-1}}^{S_n \times S_{m+k}} \left(T_n^f(U_1) \boxtimes \left(\mathbf{1}^{\boxtimes(m+k)} \otimes \mathbf{1}_{S_{m+k}} \right)\right) \\ &\otimes \mathrm{Res}_{(S_n \times S_{m+k}) \cap \sigma(S_m \times S_{n+k})\sigma^{-1}}^{\sigma(S_m \times S_{n+k})\sigma^{-1}} \left(\sigma \left(T_m^f(U_2) \boxtimes \left(\mathbf{1}^{\boxtimes(n+k)} \otimes \mathbf{1}_{S_{n+k}} \right)\right)\right)\Big).\end{aligned}$$

In particular, the calculation involves restricting $T_n(U)$ to a proper Young subgroup of S_n, giving zero by Lemma 4.7.5. There are two cases that need to be considered: $C_{1,1}(\sigma) = 0$ and $C_{1,1}(\sigma) = n = m$. The first case gives rise to the following term:

$$\mathrm{Ind}_{S_n \times S_m \times S_k}^{S_{n+m+k}}(T_n^f(U_1) \boxtimes T_m^f(U_2) \boxtimes [\mathbf{1}^{\boxtimes k} \otimes \mathbf{1}_{S_k}]).$$

Since $S_n \times S_m \times S_k$ and $S_m \times S_n \times S_k$ are conjugate subgroups of S_{m+n+k}, it follows that if $T_n^f(U_1)$ and $T_m^f(U_2)$ were interchanged, the contribution would be the same, in particular, the contribution of the term associated to this double coset is cancelled out in the commutator by the corresponding term in $T_m(U_2)T_n(U_1)$. In particular, if $n \neq m$, $[T_n(U_1), T_m(U_2)] = 0$.

If $n = m$, then we consider the contribution from the double coset representative which identifies the symmetric group factors associated to S_n and S_m in the respective Young subgroups. Working in $\mathcal{G}(\mathcal{W}_n(\mathcal{C}))$ and applying Proposition 4.3.5 we find:

$$T_n^f(U_1)T_n^f(U_2) = T_n^f(U_1 \otimes U_2).$$

Here we used the fact that the indicator function of n-cycles (considered as a class function on kS_n) is an idempotent for the tensor product. We may use Proposition 4.3.8 to express $T_n(U_1 \otimes U_2)$ in terms of $T_n(U)$ for $U \in I(\mathcal{C})$. We use the equation in Proposition 4.3.8 multiplied by $T_n^f(\mathbf{1})$ (take N to be a subobject of M in $\mathcal{C}$):

$$\begin{aligned}
T_n^f(M) &= \sum_{\lambda \vdash n} \frac{\chi_{(n)}^{\lambda}}{n}[M^{\boxtimes n} \otimes \mathcal{S}^{\lambda}] \\
&= [M^{\boxtimes n} \otimes \mathbf{1}_{S_n}] \sum_{\lambda \vdash n} \frac{\chi_{(n)}^{\lambda}}{n}[\mathbf{1}^{\boxtimes n} \otimes \mathcal{S}^{\lambda}] \\
&= [M^{\boxtimes n} \otimes \mathbf{1}_{S_n}] T_n^f(\mathbf{1}) \\
&= \sum_{r=0}^{n} [\mathrm{Ind}_{S_r \times S_{n-r}}^{S_n} \left((N^{\boxtimes r} \otimes \mathbf{1}_{S_r}) \boxtimes ((M/N)^{\boxtimes (n-r)} \otimes \mathbf{1}_{S_{n-r}}) \right)] T_n^f(\mathbf{1}).
\end{aligned}$$

Now we use Lemma 4.4.3, giving:

$$\sum_{r=0}^{n}[\mathrm{Ind}_{S_r\times S_{n-r}}^{S_n}\left(\left((N^{\boxtimes r}\otimes \mathbf{1}_{S_r})\boxtimes((M/N)^{\boxtimes(n-r)}\otimes \mathbf{1}_{S_{n-r}})\right)\otimes \mathrm{Res}_{S_r\times S_{n-r}}^{S_n}\left(\sum_{\lambda\vdash n}\frac{\chi_{(n)}^{\lambda}}{n}(\mathbf{1}^{\boxtimes n}\otimes \mathcal{S}^{\lambda})\right)\right)]$$

Note that the argument in Lemma 4.7.5 implies that all terms except for $r = 0, n$ vanish (they involve the restriction of the indicator function of n-cycles to a proper Young subgroup of S_n). We get $T_n^f(M) = T_n^f(N) + T_n^f(M/N)$, and this immediately implies $T_n(M) = T_n(N) + T_n(M/N)$. Iterating this, we get one term for each composition factor of $U_1 \otimes U_2$. If $N_{U_1,U_2}^{U_3}$ is the structure tensor of the Grothendieck ring, then we have

$$[T_n(U_1), T_n(U_2)] = \sum_{U_3}(N_{U_1,U_2}^{U_3} - N_{U_2,U_1}^{U_3})T_n(U_3).$$

□

Remark 4.7.7. *We may summarise these results by saying that the map from the Grothendieck ring of $\mathcal{C}$ (with coefficients in $\mathbb{Q}$) to the $\mathbb{Q}$-span of the $T_n(U)$ defined by $[U] \mapsto T_n(U)$ is a homomorphism of Lie algebras. The fact that $T_n(M) = T_n(N) + T_n(M/N)$ shows linearity, and we have just shown that it preserves the Lie bracket.*

4.7.2 Structure of the Limiting Grothendieck Ring

We are now able to give a presentation of $\mathcal{G}_\infty^{\mathbb{Q}}(\mathcal{C})$. Recall that if A_i is an infinite family of unital algebras over k, then the the tensor product $\bigotimes_i A_i$ is spanned by pure tensors $a_1 \otimes a_2 \otimes \cdots$ $(a_i \in A_i)$ whose factors are the unit elements in their respective algebras for all but finitely many i.

Theorem 4.7.8. *The $\mathbb{Q}$-algebra structure on $\mathcal{G}_\infty^{\mathbb{Q}}(\mathcal{C})$ is as follows:*

$$\mathcal{G}_\infty^{\mathbb{Q}}(\mathcal{C}) = \bigotimes_{i=1}^{\infty} U(\mathcal{G}(\mathcal{C})_i).$$

Here, $U(\mathcal{G}(\mathcal{C})_i)$ is the universal enveloping algebra of the span of $T_i(U)$ for $U \in I(\mathcal{C})$, which *is isomorphic to $\mathcal{G}(\mathcal{C})$ as a Lie algebra ($\mathcal{G}(\mathcal{C})_i$ is contained within the i-th filtered component of $\mathcal{G}^{\mathbb{Q}}_{\infty}(\mathcal{C})$).*

Proof. We have a map that takes $[U] \in U(\mathcal{G}(\mathcal{C})_n)$ to $T_n(U)$. It is a homomorphism by Remark 4.7.7. It is a bijection by Proposition 4.7.4; it is surjective because the $T_n(U)$ generate $\mathcal{G}^{\mathbb{Q}}_{\infty}(\mathcal{C})$ and it is injective because the map is an isomorphism upon taking associated graded algebras. □

Remark 4.7.9. *The previous theorem generalises the result that $\underline{Rep}(\mathcal{R} \wr S_t)$ is the free polynomial algebra generated by basic hooks when $\mathcal{R}$ is cocommutative, as the universal enveloping algebra of an abelian Lie algebra is a free polynomial algebra.*

4.8 Partition Combinatorics

We now focus on finding an expression for $X_{\vec{\lambda}}$ in terms of the $T_n(U)$.

4.8.1 Irreducibles in Terms of $T_n(U)$

Lemma 4.8.1. *If $D_{(n)}$ denotes the class function on S_n which is the indicator function of n-cycles, then the class function $\mathrm{Ind}_{S_n^m}^{S_{mn}}(D_{(n)}^{\otimes m})$ is $m!$ times the indicator function of elements of cycle type (n^m).*

Proof. This must be some multiple of the indicator function of elements of cycle type (n^m). The multiplicity can be found using the Frobenius character formula for induced representations. Take H a subgroup of G, G/H a collection of left coset representatives, then the induction of a character χ from H to G is given by:

$$\mathrm{Ind}_H^G(\chi)(x) = \sum_{g \in G/H, g^{-1}xg \in H} \chi(g^{-1}xg),$$

which demonstrates that the multiplicity is in fact the index of S_n^m in its normaliser in S_{nm}. The normaliser is the wreath product $S_m \ltimes S_n^m$, hence the index is $m!$. □

For now we will fix m, and consider relations between the $T_m(U)$.

Definition 4.8.2. *If λ is a partition of n, let*

$$\begin{aligned} & T_{m,\lambda}(a_1, a_2, \ldots, a_n) \\ = \; & T_m(a_1 a_2 \cdots a_{\lambda_1}) T_m(a_{\lambda_1+1} a_{\lambda_1+2} \cdots a_{\lambda_1+\lambda_2}) \cdots T_m(a_{n-\lambda_{l(\lambda)}+1} a_{n-\lambda_{l(\lambda)}+1} \cdots a_n). \end{aligned}$$

Proposition 4.8.3. *We have the following identity in $\mathcal{G}_\infty^{\mathbb{Q}}(\mathcal{C})$:*

$$T_m(a_1, a_2, \ldots, a_n) = \sum_{\sigma \in S_n} \sum_{\lambda \vdash n} \frac{\varepsilon_\lambda}{z_\lambda} T_{m,\lambda}(a_{\sigma(1)}, a_{\sigma(2)}, \ldots, a_{\sigma(n)}).$$

Before the proof, we note a useful corollary.

Corollary 4.8.4. *In the case where $a_i = a$ for all $1 \leq i \leq n$, writing $\lambda = (1^{m_1} 2^{m_2} \cdots n^{m_n})$ we obtain:*

$$T_m(a, a, \ldots, a) = \sum_{\lambda \vdash n} \frac{\varepsilon_\lambda n!}{z_\lambda} T_m(a)^{m_1} T_m(a^2)^{m_2} \cdots T_m(a^n)^{m_n}.$$

Proof. We use Mackey theory to calculate $T_m(b) T_m(a_1, a_2, \ldots, a_n)$. This can be understood by taking the tensor product of

$$\mathrm{Ind}_{S_m \times S_{(n-1)m+k}}^{S_{nm+k}} (T_m^f(b) \boxtimes (\mathbf{1}^{\boxtimes((n-1)m+k)} \otimes \mathbf{1}_{S_{(n-1)m+k}}))$$

and

$$\mathrm{Ind}_{S_m^n \times S_k}^{S_{mn+k}} (T_m^f(a_1) \boxtimes T_n^f(a_2) \boxtimes \cdots \boxtimes T_n^f(a_n) \boxtimes (\mathbf{1}^{\boxtimes k} \otimes \mathbf{1}_{S_k}).$$

The first step is to understand double-coset representatives. The minimal length $(S_m \times S_{(n-1)m+k}, S_m^n \times S_k)$-double coset representatives either map the elements of $\{1, 2, \ldots, m\}$ to a contiguous block of m elements permuted by a single component group in $S_m^n \times S_k$, or the elements are split between such component groups. In the latter case, the corresponding terms (in the Mackey theory computation) will involve a nontrivial restriction of a T_m^f to a Young subgroup; as in the proof of Theorem 4.7.6, a nontrivial restriction is zero. Thus, we consider the ways to pick a copy of S_m as one of the n given ones, or one contained in S_k.

Analogously to Theorem 4.7.6, each gives rise to a term where the arguments multiply:

$$T_m(b)T_m(a_1, a_2, \ldots, a_n) = T_m(b, a_1, a_2, \ldots, a_n) + \sum_{i=1}^{n} T_m(a_1, \ldots a_{i-1}, ba_i, a_{i+1}, \ldots, a_n).$$

Using this equation, we may decompose the claimed expression for $T_m(a_1, a_2, \ldots, a_n)$ into a linear combination of $T_m(b_1, b_2, \ldots, b_m)$, where each b_j is a product of a_is. We count the coefficient of a term of the following form in the expression on the right hand side of the claimed equality:

$$T_m(a_{r_{1,1}} a_{r_{1,2}} \cdots a_{r_{1,q_1}}, \ldots, a_{r_{k,1}} a_{r_{k,2}} \cdots a_{r_{k,q_k}}).$$

Here, the $r_{i,j}$ for $1 \leq i \leq k$, $1 \leq j \leq q_i$ are exactly the numbers 1 through n in some order.

The argument $a_{r_{1,1}} a_{r_{1,2}} \cdots a_{r_{1,q_1}}$ must arise from a product of terms such as the following:

$$T_m(a_{r_{1,1}} a_{r_{1,2}} \cdots a_{r_{1,\mu_1}}) T_m(a_{r_{1,\mu_1+1}} a_{r_{1,\mu_1+2}} \cdots a_{r_{1,\mu_1+\mu_2}}) \cdots T_m(a_{r_{1,q_1-\mu_l+1}} a_{r_{1,2}} \cdots a_{r_{1,q_k}}).$$

Since all terms $T_m(x)$ in the definition of $T_{m,\lambda}$ occur in non-increasing order of the number of factors in the argument x, we obtain a partition of q_1, $\mu^{(1)} = (\mu_1, \mu_2, \ldots, \mu_l)$, associated to the sequence $r_{1,j}$ which describes the factors $T_m(x)$ contributing to that term. A similar description holds for the other $r_{i,j}$ for other values of i. We obtain a description of all contributions; note that the λ appearing in the sum will be the union of all $\mu^{(i)}$, and that if multiple $\mu^{(i)}$ have parts of some size s, then there is no restriction on the ordering of the corresponding $T_m(b_1 b_2 \cdots b_s)$ terms within $T_{m,\lambda}$ (each possible ordering has an equal contribution). The coefficient of $T_m(a_{r_{1,1}} a_{r_{1,2}} \cdots a_{r_{1,q_1}}, \ldots, a_{r_{k,1}} a_{r_{k,2}} \cdots a_{r_{k,q_k}})$ is:

$$\sum_{\mu^{(1)} \vdash q_1} \sum_{\mu^{(2)} \vdash q_2} \cdots \sum_{\mu^{(k)} \vdash q_k} \frac{\varepsilon_{\cup_i \mu^{(i)}}}{z_{\cup_i \mu^{(i)}}} \prod_{j=1}^{n} \frac{\left(\sum_{i=1}^{k} m_j(\mu^{(i)})\right)!}{\prod_{i=1}^{k} m_j(\mu^{(i)})!}.$$

Here the multinomial coefficient arose because different orderings of factors can give rise to the same term. Using the fact that $\varepsilon_{\mu \cup \nu} = \varepsilon_\mu \varepsilon_\nu$ and the definition of z_μ, our equation becomes:

$$\sum_{\mu^{(1)}\vdash q_1}\sum_{\mu^{(2)}\vdash q_2}\cdots\sum_{\mu^{(k)}\vdash q_k}\prod_{i=1}^{k}\frac{\varepsilon_{\mu^{(i)}}}{z_{\mu^{(i)}}}=\prod_{i=1}^{k}\left(\sum_{\mu^{(i)}\vdash q_i}\frac{\varepsilon_{\mu^{(i)}}}{z_{\mu^{(i)}}}\right).$$

The expansions for elementary and complete symmetric functions in terms of power sum symmetric functions show that we have:

$$\delta_{n,1}=\langle s_{(n)},s_{(1^n)}\rangle=\langle h_n,e_n\rangle=\sum_{\lambda\vdash n}\langle\frac{p_\lambda}{z_\lambda},\frac{\varepsilon_\lambda p_\lambda}{z_\lambda}\rangle=\sum_{\lambda\vdash n}\frac{\varepsilon_\lambda}{z_\lambda}.$$

This means that the expression of interest vanishes unless each $q_i=1$. In that case the constant is 1, and we simply obtain $T(a_1,a_2,\dots,a_n)$ as claimed. □

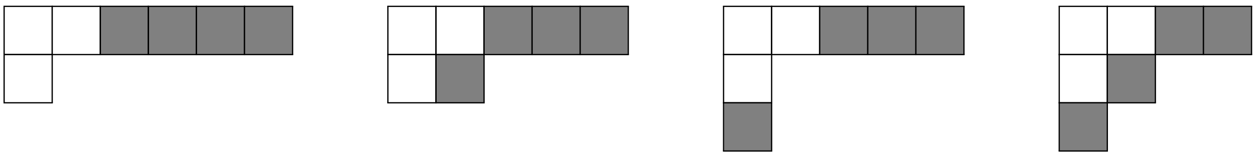
Proposition 4.8.5. *We may relate $U^{\boxtimes|\lambda|}\otimes\mathcal{S}^\lambda$ to the $T_i(U)$ as follows:*

$$\lim_{m\to\infty}\operatorname{Ind}_{S_{|\lambda|}\times S_m}^{S_{|\lambda|+m}}\left(\left(U^{\boxtimes|\lambda|}\otimes\mathcal{S}^\lambda\right)\boxtimes\left(\mathbf{1}^{\boxtimes m}\otimes\mathbf{1}_{S_m}\right)\right)$$

$$=\sum_{\mu\vdash|\lambda|}\chi^\lambda_\mu\frac{T_1(\overbrace{U,U,\dots,U}^{m_1(\mu)})}{m_1(\mu)!}\frac{T_2(\overbrace{U,U,\dots,U}^{m_2(\mu)})}{m_2(\mu)!}\cdots\frac{T_{|\lambda|}(\overbrace{U,U,\dots,U}^{m_{|\lambda|}(\mu)})}{m_{|\lambda|}(\mu)!}.$$

Proof. We decompose $\mathcal{S}^\lambda$ into a linear combination of virtual representations, each having character equal to an indicator function of some cycle type μ. The coefficients are χ^λ_μ. By Lemma 4.8.1, $\frac{T_i(U,U,\dots,U)}{m_i(\mu)!}$ corresponds to the indicator function of cycle type $(i^{m_i(\mu)})$. Multiplying these together corresponds to taking the tensor product of class functions on $S_{im_i(\mu)}$ and inducing up to S_n, which precisely gives the indicator function of cycle type μ. □

Remark 4.8.6. *Suppose that we wish to decompose the expression in Proposition 4.8.5 into X_μ. It is clear that if $U\neq\mathbf{1}$ then we get the definition of $X_{\vec{\mu}}$ where $\vec{\mu}(U)=\lambda$ and $\vec{\mu}(V)$ is the trivial partition for $V\neq U$. If $U=\mathbf{1}$, we use Proposition 4.3.6 to see that we must describe $\operatorname{Ind}_{S_{|\lambda|}\times S_m}^{S_{|\lambda|+m}}(\mathcal{S}^\lambda\boxtimes\mathbf{1}_{S_m})$ for m sufficiently large. Under (the inverse of) the characteristic map between symmetric functions and representations of symmetric groups, calculating the induced representation amounts to calculating the product of symmetric functions $s_\lambda h_m$,* which is described combinatorially by the Pieri rule. The Pieri rule states that we get $\sum_\mu s_\mu$ where the sum across all partitions μ obtained from λ by adding m boxes, no two in the same column. For example, suppose $\lambda = (2,1)$ and $m = 4$. The valid μ are shown below, where the added boxes are highlighted.



When m is larger than the number of columns in the diagram of λ (i.e. the longest part of λ), there is no restriction on the collection of columns that a box may be added to, save that the final result must be a partition. We are interested in the set of partitions obtained by removing the first row of each of the diagrams after performing the above operation. The operation of removing the top row is the same as removing a box from each column. Thus, we are interested in all partitions obtained by adding at most one box to some column, and then removing one box from each column. This is equivalent to removing one box from each of the columns in the diagram of λ that were not chosen. In other words, the set we are interested in consists of all partitions obtained from λ by removing some number of boxes, no two in the same column. If we write $h_r^\perp$ for the operator adjoint to multiplication by h_r with respect to the usual bilinear form form on Λ, then by the Pieri rule, $h_r^\perp s_\lambda$ is $\sum_\mu s_\mu$ across all partitions μ obtained from λ by removing r boxes in the diagram of λ, no two in the same column. Continuing to encode partitions as their associated Schur functions, we find that the desired decomposition is

$$\left(\sum_{r=0}^{\infty} h_r^\perp\right) s_\lambda.$$

This is because, for m sufficiently large, there is no restriction on the number of boxes that could be removed (i.e. $r = 0, 1, 2, \ldots$).

Example 4.8.7. *Suppose that in Proposition 4.8.5,* $\lambda = (1^r)$ *and* $U = \mathbf{1}$*. Since* $h_i^\perp s_{(1^r)} = 0$

for $i \geq 2$, *and* $h_1^\perp s_{(1^r)} = s_{(1^{r-1})}$ *(and* $h_0^\perp s_{(1^r)} = s_{(1^r)}$*), we obtain*

$$\lim_{m\to\infty} \mathrm{Ind}_{S_r \times S_m}^{S_{r+m}} \left(\left(\mathbf{1}^{\boxtimes r} \otimes \mathcal{S}^{(1^r)}\right) \boxtimes \left(\mathbf{1}^{\boxtimes m} \otimes \mathbf{1}_{S_m}\right)\right) = X_{\vec{\mu}^{(1)}} + X_{\vec{\mu}^{(2)}},$$

where $\vec{\mu}^{(1)}(\mathbf{1}) = (1^r)$ *and* $\vec{\mu}^{(2)}(\mathbf{1}) = (1^{r-1})$ *(and all parts of these multipartitions associated to* $U \neq \mathbf{1}$ *are the empty partition).*

Remark 4.8.8. *To reconstruct* $X_{\vec{\lambda}}$ *from the objects in Proposition 4.8.5, we need to invert the operator* $\sum_{r=0}^{\infty} h_r^\perp$*. Recognising it as the adjoint of* $H(1)$ *(where* $H(t) = h_0 + h_1 t + h_2 t^2 + \cdots$ *is the generating function of complete symmetric functions), we may write the inverse as the adjoint of* $E(-1)$ *(recall that* $H(t)E(-t) = 1$*). So, the relevant operator (when we are encoding partitions as Schur functions) is* $\sum_{r=0}^{\infty} (-1)^r e_r^\perp$ *(where* $e_r^\perp$ *is adjoint to multiplication by* e_r*).*

Proposition 4.8.9. *Let* U *be an object of* $\mathcal{C}$*, and* $\varphi_U : \Lambda_{\mathbb{Q}} \to \mathcal{G}_\infty^{\mathbb{Q}}(\mathcal{C})$ *be defined by*

$$\varphi_U(e_i) = \lim_{m\to\infty} \mathrm{Ind}_{S_i \times S_m}^{S_{i+m}} \left(\left(U^{\boxtimes i} \otimes \mathcal{S}^{(1^i)}\right) \boxtimes \left(\mathbf{1}^{\boxtimes m} \otimes \mathbf{1}_{S_m}\right)\right),$$

so that if $U \neq \mathbf{1}$ *is a simple object, then* $\varphi_U(e_i)$ *is the basic hook* $\overline{e}_i(U)$*, whilst Example 4.8.7 shows that if* $U = \mathbf{1}$*, then* $\varphi(e_i) = \overline{e}_i(\mathbf{1}) + \overline{e}_{i-1}(\mathbf{1})$*. Then:*

$$\varphi_U(p_n) = \sum_{d|n} dT_d(U^{\frac{n}{d}}).$$

Proof. We use the generating functions $E(t) = \sum_{n=0}^{\infty} e_n t^n$, and $P(t) = \sum_{n=0}^{\infty} p_{n+1} t^n$. Recall that

$$\frac{d}{dt}\log(E(t)) = P(-t).$$

Additionally, we have the following expression using Proposition 4.8.5, following directly from the character formula for the sign representation ($\chi_\mu^{(1^{|\mu|})} = \varepsilon_\mu$):

$$\begin{aligned}
\varphi_U(e_n) &= \sum_{\lambda \vdash n} \varepsilon_\lambda \frac{T_1(\overbrace{U,U,\ldots,U}^{m_1(\lambda)})}{m_1(\lambda)!} \frac{T_2(\overbrace{U,U,\ldots,U}^{m_2(\lambda)})}{m_2(\lambda)!} \cdots \frac{T_n(\overbrace{U,U,\ldots,U}^{m_n(\lambda)})}{m_n(\lambda)!} \\
&= \sum_{\lambda \vdash n} \prod_{i=1}^{n} \frac{(-1)^{m_i(\lambda)(i-1)}}{m_i(\lambda)!} T_i(\overbrace{U,U,\ldots,U}^{m_i(\lambda)}) \\
&= \sum_{\lambda \vdash n} \prod_{i=1}^{n} (-1)^{m_i(\lambda)(i-1)} \sum_{\mu^{(i)} \vdash m_i(\lambda)} \frac{\varepsilon_{\mu^{(i)}}}{z_{\mu^{(i)}}} \prod_j T_i(U^j)^{m_j(\mu^{(i)})}.
\end{aligned}$$

In the last step, we used Corollary 4.8.4. We now calculate the generating function $E(t)$. Below we abbreviate $m_i(\lambda)$ to m_i, and use the fact that λ is parametrised by the numbers m_i:

$$
\begin{aligned}
\varphi_U(E(t)) &= \sum_{n=0}^{\infty}\sum_{\lambda\vdash n} t^{|\lambda|}\prod_{i=1}^{n}(-1)^{m_i(\lambda)(i-1)}\sum_{\mu^{(i)}\vdash m_i(\lambda)}\frac{\varepsilon_{\mu^{(i)}}}{z_{\mu^{(i)}}}\prod_j T_i(U^j)^{m_j(\mu^{(i)})} \\
&= \sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}\cdots\left(\prod_{i=1}^{\infty} t^{im_i}(-1)^{(i-1)m_i}\sum_{\mu^{(i)}\vdash m_i}\frac{\varepsilon_{\mu^{(i)}}}{z_{\mu^{(i)}}}\prod_j T_i(U^j)^{m_j(\mu^{(i)})}\right) \\
&= \prod_{i=1}^{\infty}\sum_{m_i=0}^{\infty}\left(t^{im_i}(-1)^{(i-1)m_i}\sum_{\mu^{(i)}\vdash m_i}\frac{\varepsilon_{\mu^{(i)}}}{z_{\mu^{(i)}}}\prod_j T_i(U^j)^{m_j(\mu^{(i)})}\right) \\
&= \prod_{i=1}^{\infty}\sum_{m_i=0}^{\infty}\left(t^{im_i}(-1)^{(i-1)m_i}\sum_{\mu^{(i)}\vdash m_i}\prod_j \frac{T_i(U^j)^{m_j(\mu^{(i)})}(-1)^{m_j(\mu^{(i)})(j-1)}}{m_j(\mu^{(i)})!\,j^{m_j(\mu^{(i)})}}\right) \\
&= \prod_{i=1}^{\infty}\left(\sum_{m_i=0}^{\infty}\sum_{\mu^{(i)}\vdash m_i}\prod_{j=1}^{\infty} t^{ijm_j(\mu^{(i)})}(-1)^{(i-1)jm_j(\mu^{(i)})}\frac{T_i(U^j)^{m_j(\mu^{(i)})}(-1)^{m_j(\mu^{(i)})(j-1)}}{m_j(\mu^{(i)})!\,j^{m_j(\mu^{(i)})}}\right) \\
&= \prod_{i=1}^{\infty}\left(\sum_{m_i=0}^{\infty}\sum_{\mu^{(i)}\vdash m_i}\prod_{j=1}^{\infty}\frac{1}{m_j(\mu^{(i)})!}\left(t^{ij}(-1)^{(ij-1)}\frac{T_i(U^j)}{j}\right)^{m_j(\mu^{(i)})}\right) \\
&= \prod_{i=1}^{\infty}\left(\sum_{\mu^{(i)}\in\mathcal{P}}\prod_{j=1}^{\infty}\frac{1}{m_j(\mu^{(i)})!}\left(t^{ij}(-1)^{(ij-1)}\frac{T_i(U^j)}{j}\right)^{m_j(\mu^{(i)})}\right) \\
&= \prod_{i=1}^{\infty}\left(\exp\left(\sum_{j=1}^{\infty} t^{ij}(-1)^{(ij-1)}\frac{T_i(U^j)}{j}\right)\right).
\end{aligned}
$$

In the second last step, we used the fact that the set of integer partitions μ is parametrised by the numbers $m_j(\mu)\in\mathbb{Z}_{\geq 0}$ with all but finitely many being zero. This allows us to sum over the numbers $m_j(\mu^{(i)})$ independently of each other. Now we may take the derivative of the logarithm with respect to t:

$$\begin{aligned}
\varphi_U\left(\frac{E'(t)}{E(t)}\right) &= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} ijt^{ij-1}(-1)^{(ij-1)}\frac{T_i(U^j)}{j} \\
&= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} i(-t)^{ij-1}T_i(U^j) \\
&= \sum_{n=1}^{\infty}\sum_{d|n} d(-t)^{n-1}T_d(U^{n/d}).
\end{aligned}$$

In the last step, the change of variables $i = d$, $ij = n$ was used. Equating the terms of the power series with those of $\varphi_U(P(-t))$ gives the result. □

4.8.2 Generating Function for Irreducibles

We now prove the main theorem of this paper. It provides an generating function for the basis $X_{\vec{\lambda}}$ in terms of the $T_n(U)$ generators. In principle, this gives a way to decompose products $X_{\vec{\mu}}X_{\vec{\nu}}$, and therefore a way to calculate multiplicities of tensor products in wreath product Deligne categories (this calculation is carried out in Section 4.9, and Theorem 4.9.2 in particular).

Theorem 4.8.10. *Write $\Lambda_{\mathbb{Q}}^{(U)}$ for a copy of the ring of symmetric functions with rational coefficients, whose variables we associate with $U \in I(\mathcal{C})$. If f is a symmetric function, we write $f^{(U)}$ to denote f considered as an element of $\Lambda_{\mathbb{Q}}^{(U)}$. We work in $\left(\bigotimes_{U\in I(\mathcal{C})}\Lambda_{\mathbb{Q}}^{U}\right)\hat{\otimes}\mathcal{G}(\mathcal{C})$, the completed tensor product of $\left(\bigotimes_{U\in I(\mathcal{C})}\Lambda_{\mathbb{Q}}^{U}\right)$ with $\mathcal{G}(\mathcal{C})$. If $c_\mu^{(U)}$ ($\mu \in \mathcal{P}$, $U \in I(\mathcal{C})$) are constants, let*

$$T_l\left(\sum_{\mu\in\mathcal{P},U\in I(\mathcal{C})} c_\mu^{(U)}p_\mu^{(U)}[U]\right) = \sum_{\mu\in\mathcal{P},U\in I(\mathcal{C})} c_\mu^{(U)}p_\mu^{(U)}\otimes T_l(U),$$

which is an element of $\left(\bigotimes_{U\in I(\mathcal{C})}\Lambda^U_{\mathbb{Q}}\right)\hat{\otimes}\mathcal{G}^{\mathbb{Q}}_{\infty}(\mathcal{C})$. *We have the following equality:*

$$\sum_{\vec{\lambda}\in\mathcal{P}^{\mathcal{C}}}\left(\prod_{U\in I(\mathcal{C})}s^{(U)}_{\vec{\lambda}(U)}\right)\otimes X_{\vec{\lambda}}=\left(\sum_{r\geq 0}(-1)^r e^{(\mathbf{1})}_r\right)\prod_{l=1}^{\infty}\exp\left(T_l\left(\log\left(1+\sum_{U\in I(\mathcal{C})}p^{(U)}_l[U]\right)\right)\right).$$

Proof. We firstly note that the first factor on the right hand side can be inverted (using the generating function relation $H(t)E(-t)=1$). Moving it to the left had side it acts as an operator on the symmetric functions, but by taking the adjoint, we may make it act of the $X_{\vec{\lambda}}$. Taking into consideration Remark 4.8.8, we see that the effect of this manipulation is to replace $X_{\vec{\lambda}}$ with the expression in Proposition 4.8.5:

$$\lim_{m\to\infty}\mathrm{Ind}^{S_{|\vec{\lambda}|+m}}_{S_{\vec{\lambda}}\times S_m}\left(\boxtimes_{U\in I(\mathcal{C})}\left(U^{\boxtimes|\vec{\lambda}|}\otimes\mathcal{S}^{\vec{\lambda}(U)}\right)\boxtimes\left(\mathbf{1}^{\boxtimes m}\otimes\mathbf{1}_{S_m}\right)\right).$$

The left hand side of the equation becomes

$$\sum_{\vec{\lambda}\in\mathcal{P}^{\mathcal{C}}}\left(\prod_{U\in I(\mathcal{C})}s^{(U)}_{\vec{\lambda}(U)}\right)\otimes\left(\lim_{m\to\infty}\mathrm{Ind}^{S_{|\vec{\lambda}|+m}}_{S_{\vec{\lambda}}\times S_m}\left(\boxtimes_{U\in I(\mathcal{C})}\left(U^{\boxtimes|\vec{\lambda}(U)|}\otimes\mathcal{S}^{\vec{\lambda}(U)}\right)\boxtimes\left(\mathbf{1}^{\boxtimes m}\otimes\mathbf{1}_{S_m}\right)\right)\right),$$

and we are required to prove that it is equal to

$$\prod_{l=1}^{\infty}\exp\left(T_l\left(\log\left(1+\sum_{U}p^{(U)}_l[U]\right)\right)\right).$$

Using the same method as in the proof of Proposition 4.8.5, we seek to write the expression in terms of the elements $T_i(U)$. We have

$$\begin{aligned}&\lim_{m\to\infty}\mathrm{Ind}^{S_{|\vec{\lambda}|+m}}_{S_{\vec{\lambda}}\times S_m}\left(\boxtimes_{U\in I(\mathcal{C})}\left(U^{\boxtimes|\vec{\lambda}(U)|}\otimes\mathcal{S}^{\vec{\lambda}(U)}\right)\boxtimes\left(\mathbf{1}^{\boxtimes m}\otimes\mathbf{1}_{S_m}\right)\right)\\ &=\sum_{\mu^{(1)}\in\mathcal{P}}\sum_{\mu^{(2)}\in\mathcal{P}}\cdots\left(\prod_{U_i\in I(\mathcal{C})}\chi^{\vec{\lambda}(U_i)}_{\mu^{(i)}}\right)\prod_{i=1}^{\infty}\frac{T_i(\overbrace{U_1,U_1,\ldots}^{m_i(\mu^{(1)})},\overbrace{U_2,U_2,\ldots}^{m_i(\mu^{(2)})},\ldots)}{m_1(\mu^{(1)})!m_1(\mu^{(2)})!\cdots}.\end{aligned}$$

Here $\mu^{(i)}\in\mathcal{P}$ describes a cycle type in a symmetric group associated to $U_i\in I(\mathcal{C})$. We now let $\nu = \cup_i \mu^{(i)}$ and use Proposition 4.8.3 to express our equation in terms of $T_i(U)$ (i.e. without any inductions). We obtain

$$\sum_{\mu^{(1)}} \sum_{\mu^{(2)}} \cdots \left(\prod_{U_i \in I(\mathcal{C})} \chi_{\mu^{(i)}}^{\vec{\lambda}(U_i)} \right) \left(\prod_{j=1}^{\infty} \frac{1}{m_j(\mu^{(1)})! m_j(\mu^{(2)})! \cdots} \right)$$
$$\times \sum_{\alpha^{(1)} \vdash m_1(\nu)} \sum_{\alpha^{(2)} \vdash m_2(\nu)} \cdots \left(\frac{\varepsilon_{\alpha^{(1)}}}{z_{\alpha^{(1)}}} \frac{\varepsilon_{\alpha^{(2)}}}{z_{\alpha^{(2)}}} \cdots \right) \prod_{l=1}^{\infty} \sum_{\sigma \in S_{m_l(\nu)}} T_{l,\alpha^{(l)}}(\sigma(\overbrace{U_1, U_1, \ldots,}^{m_l(\mu^{(1)})} \overbrace{U_2, U_2, \ldots,}^{m_l(\mu^{(2)})} \ldots)).$$

Ultimately we are calculating a generating function whose inner product with the symmetric function $\prod_{U \in I(\mathcal{C})} s_{\vec{\lambda}(U)}^{(U)}$ is the above quantity. If σ and ρ are partitions, then the fact that the inner product of s_σ and p_ρ is χ_ρ^σ allows us to replace the character values with power-sum symmetric functions and sum over all possible power-sum symmetric functions. We also note that the subgroup $H_l = S_{m_l(\mu^{(1)})} \times S_{m_l(\mu^{(2)})} \times \cdots$ of $S_{m_l(\nu)}$ fixes the vector $(\overbrace{U_1, U_1, \ldots,}^{m_l(\mu^{(1)})} \overbrace{U_2, U_2, \ldots,}^{m_l(\mu^{(2)})} \ldots)$ (with the usual permutation action). This means that we may restrict the sum to coset representatives of this subgroup, at the cost of multiplying by $m_l(\mu^{(1)})! m_l(\mu^{(2)})! \cdots$ (which cancels out the denominators following the symmetric group characters in the above expression). Our new expression is

$$\sum_{\mu^{(1)}} \sum_{\mu^{(2)}} \cdots \left(\prod_{U_i \in I(\mathcal{C})} p_{\mu^{(i)}}^{(U_i)} \right) \otimes \sum_{\alpha^{(1)} \vdash m_1(\nu)} \sum_{\alpha^{(2)} \vdash m_2(\nu)} \cdots \left(\frac{\varepsilon_{\alpha^{(1)}}}{z_{\alpha^{(1)}}} \frac{\varepsilon_{\alpha^{(2)}}}{z_{\alpha^{(2)}}} \cdots \right)$$
$$\times \prod_{l=1}^{\infty} \sum_{\sigma \in S_{m_l(\nu)}/H_l} T_{l,\alpha^{(l)}}(\sigma(\overbrace{U_1, U_1, \cdots,}^{m_l(\mu^{(1)})} \overbrace{U_2, U_2, \cdots,}^{m_l(\mu^{(2)})} \cdots)).$$

Now, note that the $T_{l,\alpha^{(l)}}(\cdots)$ are summed over all distinct reorderings of their arguments. We now inspect the sum over $\alpha^{(l)}$ and σ more closely (which we take to include the terms

$(p_l^{(U_1)})^{m_l(\mu^{(1)})}(p_l^{(U_2)})^{m_l(\mu^{(2)})}\cdots$ coming from $\prod_{U_i\in I(\mathcal{C})} p^{(U_i)}_{\mu^{(i)}}$):

$$(p_l^{(U_1)})^{m_l(\mu^{(1)})}(p_l^{(U_2)})^{m_l(\mu^{(2)})}\cdots \otimes \sum_{\alpha^{(l)}\vdash m_l(\nu)} \frac{\varepsilon_{\alpha^{(l)}}}{z_{\alpha^{(l)}}}$$

$$\times \sum_{\sigma\in S_{m_l(\nu)}/H_l} T_{l,\alpha^{(l)}}(\sigma(\overbrace{U_1,U_1,\ldots,}^{m_l(\mu^{(1)})}\overbrace{U_2,U_2,\ldots,}^{m_l(\mu^{(2)})}\ldots)).$$

Recalling that $\frac{\varepsilon_{\alpha^{(l)}}}{z_{\alpha^{(l)}}} = \prod_{j=1}^{\infty} \frac{(-1)^{m_j(\alpha^{(l)})(j-1)}}{m_j(\alpha^{(l)})! j^{m_j(\alpha^{(l)})}}$, we have:

$$\left(\prod_{U_i\in I(\mathcal{C})} (p_l^{(U_i)})^{m_l(\mu^{(i)})}\right) \otimes \sum_{\alpha^{(l)}\vdash m_l(\nu)} \prod_{j=1}^{\infty} \frac{(-1)^{m_j(\alpha^{(l)})(j-1)}}{m_j(\alpha^{(l)})! j^{m_j(\alpha^{(l)})}}$$

$$\times \sum_{\sigma\in S_{m_l(\nu)}/H_l} T_{l,\alpha^{(l)}}(\sigma(\overbrace{U_1,U_1,\ldots,}^{m_l(\mu^{(1)})}\overbrace{U_2,U_2,\ldots,}^{m_l(\mu^{(2)})}\ldots)).$$

We sum over all possible values of $m_l(\mu^{(1)}), m_l(\mu^{(2)}), \ldots$, which means that the expression $\sigma(\overbrace{U_1,U_1,\ldots,}^{m_l(\mu^{(1)})}\overbrace{U_2,U_2,\ldots,}^{m_l(\mu^{(2)})}\ldots)$ varies across all finite words W in the U_i without repetition. To calculate $T_{l,\alpha^{(l)}}(W)$ we write $W_{\alpha^{(l)},r}$ for the product of the letters of the subword of W starting at the $(\alpha_1^{(l)}+\alpha_2^{(l)}+\cdots+\alpha_{r-1}^{(l)}+1)$-th place and finishing at the $(\alpha_1^{(l)}+\alpha_2^{(l)}+\cdots+\alpha_r^{(l)})$-th place. This lets us write (by definition of $T_{m,\lambda}$)

$$T_{l,\alpha^{(l)}}(W) = T_l(W_{\alpha^{(l)},1})T_l(W_{\alpha^{(l)},2})\cdots T_l(W_{\alpha^{(l)},l(\alpha^{(l)})}).$$

Now, in $\mathcal{G}(\mathcal{C})$ we may write $[W_{\alpha^{(l)},r}] = \sum_{U\in I(\mathcal{C})} M^U_{W,\alpha^{(l)},r}[U]$, and because $T_l(-)$ is linear,

$$T_l(W_{\alpha^{(l)},r}) = \sum_{U\in I(\mathcal{C})} M^{(U)}_{W,\alpha^{(l)},r} T_l(U).$$

If we write $|W|$ for the length of the word W, and $n_U(W)$ for the number of occurrences of

U in W, then we may rewrite our earlier expression as

$$\sum_W \left(\prod_{U_i \in I(\mathcal{C})} (p_l^{(U_i)})^{n_U(W)} \right) \otimes \left(\sum_{\alpha^{(l)} \vdash |W|} \prod_{j=1}^{\infty} \frac{(-1)^{m_j(\alpha^{(l)})(j-1)}}{m_j(\alpha^{(l)})! j^{m_j(\alpha^{(l)})}} \right) \left(\prod_r T_l(W_{\alpha^{(l)},r}) \right)$$
$$= \sum_W \left(\sum_{\alpha^{(l)} \vdash |W|} \prod_{j=1}^{\infty} \frac{(-1)^{m_j(\alpha^{(l)})(j-1)}}{m_j(\alpha^{(l)})! j^{m_j(\alpha^{(l)})}} \times \prod_r \left(\prod_{U \in I(\mathcal{C})} (p_l^{(U)})^{n_U(W_{\alpha^{(l)},r})} \right) \otimes \prod_r T_l(W_{\alpha^{(l)},r}) \right).$$

We now note that each $W_{\alpha^{(l)},r}$ varies independently over all words in the U_i of length $\alpha_r^{(l)}$. We may therefore remove the sum over W at the cost of replacing

$$\prod_r \left(\prod_{U \in I(\mathcal{C})} (p_l^{(U)})^{n_U(W_{\alpha^{(l)},r})} \right) \otimes \prod_r T_l(W_{\alpha^{(l)},r})$$

with

$$T_l \left(\left(\sum_{U \in I(\mathcal{C})} p_l^{(U)}[U] \right)^{\alpha_r^{(l)}} \right).$$

This leaves us with

$$\sum_{\alpha^{(l)} \in \mathcal{P}} \left(\prod_{j=1}^{\infty} \frac{(-1)^{m_j(\alpha^{(l)})(j-1)}}{m_j(\alpha^{(l)})! j^{m_j(\alpha^{(l)})}} \right) \left(\prod_r T_l \left(\sum_{U \in I(\mathcal{C})} p_l^{(U)}[U] \right)^{\alpha_r^{(l)}} \right)$$
$$= \sum_{\alpha^{(l)} \in \mathcal{P}} \prod_{j=1}^{\infty} \frac{1}{m_j(\alpha^{(l)})!} \left(\frac{(-1)^{(j-1)}}{j} \right)^{m_j(\alpha^{(l)})} \left(T_l \left(\sum_{U \in I(\mathcal{C})} p_l^{(U)}[U] \right)^j \right)^{m_j(\alpha^{(l)})}$$
$$= \prod_{j=1}^{\infty} \sum_{m_j(\alpha^{(l)})=0}^{\infty} \frac{1}{m_j(\alpha^{(l)})!} \left(\frac{(-1)^{(j-1)}}{j} \right)^{m_j(\alpha^{(l)})} \left(T_l \left(\sum_{U \in I(\mathcal{C})} p_l^{(U)}[U] \right)^j \right)^{m_j(\alpha^{(l)})}.$$

Here we used the fact that summing over all partitions $\alpha^{(l)}$ is equivalent to summing over all possible values of $m_r(\alpha^{(l)})$ for all r. Now we recognise the power series for the exponential and then for the logarithm:

$$\begin{aligned}
& \prod_{j=1}^{\infty} \exp\left(\frac{(-1)^{(j-1)}}{j} T_l \left(\sum_{U \in I(\mathcal{C})} p_l^{(U)}[U]\right)^j\right) \\
= \quad & \exp\left(\sum_{j=1}^{\infty} \frac{(-1)^{(j-1)}}{j} T_l \left(\sum_{U \in I(\mathcal{C})} p_l^{(U)}[U]\right)^j\right) \\
= \quad & \exp\left(T_l\left(\log\left(1 + \sum_{U \in I(\mathcal{C})} p_l^{(U)}[U]\right)\right)\right).
\end{aligned}$$

Now we simply multiply this expression for $l \in \mathbb{Z}_{>0}$ to obtain the desired result (since $T_{l_1}(U)$ commutes with $T_{l_2}(V)$ whenever $l_1 \neq l_2$, we do not need to be careful about commuting exponentials). □

In order to obtain expressions for $X_{\vec{\lambda}}$ in terms of basic hooks, we must write $T_n(U)$ in terms of basic hooks.

Proposition 4.8.11. *Recall the setting of Proposition 4.8.9, where for any object U of $\mathcal{C}$, we had*

$$\varphi_U(p_n) = \sum_{d|n} d T_d(U^{\frac{n}{d}}).$$

Let $\mu(n)$ be the Möbius function (defined on positive integers by $\sum_{d|n} \mu(d) = \delta_{n,1}$). We have:

$$T_r(U) = \frac{1}{r} \sum_{d|r} \varphi_{(U^{\frac{r}{d}})}(p_d) \mu(r/d).$$

Proof. We directly calculate:

$$\begin{aligned}
\frac{1}{r}\sum_{d|r}\varphi_{(U^{\frac{r}{d}})}(p_d)\mu(r/d) &= \frac{1}{r}\sum_{d|r}\mu(r/d)\left(\sum_{d'|d} d'T_{d'}((U^{\frac{r}{d}})^{\frac{d}{d'}})\right) \\
&= \frac{1}{r}\sum_{d|r}\mu(r/d)\left(\sum_{d'|d} d'T_{d'}(U^{\frac{r}{d'}})\right) \\
&= \frac{1}{r}\sum_{d'|r}\left(\sum_{d'|d|r}\mu(r/d)\right) d'T_{d'}(U^{\frac{r}{d'}}) \\
&= \frac{1}{r}\sum_{d'|r}\delta_{d',r} d'T_{d'}(U^{\frac{r}{d'}}) \\
&= T_r(U).
\end{aligned}$$

□

This means that to express the $T_r(U)$ in terms of basic hooks, it is enough to decompose $\varphi_{(U^{\frac{r}{d}})}(p_d)$ into basic hooks. This task is complicated by the fact that $\varphi_{(V)}(p_d)$ is not linear in V for $d > 1$. However, this difficulty is mitigated if $U^{\frac{r}{d}}$ is itself a simple object of $\mathcal{C}$, for example when $\mathcal{C} = kG - \text{mod}$ where G is an abelian group (simple objects are precisely one dimensional representations of G, the set of which is closed under taking tensor products). When $\mathcal{C} = kG - \text{mod}$ (for abelian G) the problem amounts to expressing power sum symmetric functions in terms of elementary symmetric functions. The elementary symmetric functions give rise to basic hooks for simple $U \neq \mathbf{1}$, and to a sum of two basic hooks for $U = \mathbf{1}$, as per Proposition 4.8.9.

4.9 Applications to Symmetric Groups and Wreath Products

We discuss a selection of results about the asymptotic representation theory of symmetric groups and wreath products that follow from our results. Recall that the Deligne category

$\underline{\text{Rep}}(S_t)$ is a tensor category that can be thought of as an "interpolation" of the representation categories of finite symmetric groups.

Theorem 4.9.1. *The ring $\mathcal{G}_\infty^{\mathbb{Q}}(\mathcal{C})$ is isomorphic to the Grothendieck ring (with rational coefficients) of the wreath product version of the Deligne category, $S_t(\mathcal{C})$, when $t \notin \mathbb{Z}_{\geq 0}$. The Grothendieck ring with integral coefficients is isomorphic to the integral version, $\mathcal{G}_\infty^{\mathbb{Z}}(\mathcal{C})$.*

Proof. When $t \notin \mathbb{Z}_{\geq 0}$, the simple objects of the category $S_t(\mathcal{C})$ are parametrised by $\vec{\lambda} \in \mathcal{P}^{\mathcal{C}}$. The methods of Theorem 4.6.14 allow one to deduce that the structure constants for non-integral t agree with the corresponding stable limits as $t \in \mathbb{Z}_{\geq 0}$ tends to infinity. □

The wreath product categories are discussed in [19], and various aspects of the theory of Deligne categories are discussed in [11] and [12].

We now give a way for computing a formula for structure constants of $\mathcal{G}_\infty^{\mathbb{Q}}(\mathcal{C})$ with respect to the $X_{\vec{\lambda}}$ basis. Of course, these are also the structure constants in the Grothendieck ring of a Deligne category. We use Theorem 4.8.10 with multiple different sets of symmetric function variables. It will be convenient to write $p_l(\mathbf{x}^{(U)})$ instead of $p_l^{(U)}(\mathbf{x})$.

Theorem 4.9.2. *Write $N_{U,V}^W$ for the structure tensor of $\mathcal{G}(\mathcal{C})$ (so that $[U][V] = \sum_W N_{U,V}^W$). Write $\mathbf{z}^{(U)}$ to denote the family of symmetric function variables $\bigoplus_{V_1,V_2 \in I(\mathcal{C})} \left(\mathbf{x}^{(V_1)}\mathbf{y}^{(V_2)}\right)^{\oplus N_{V_1,V_2}^U}$, where direct sum notation denotes a disjoint union of symmetric function variables, and the direct sum in the exponents denotes the multiplicity of each of the sets of variables. Then the the multiplicity of $X_{\vec{\lambda}}$ in $X_{\vec{\mu}}X_{\vec{\nu}}$ is given by the coefficient of*

$$\left(\prod_{U \in I(\mathcal{C})} s_{\vec{\mu}(U)}(\mathbf{x}^{(U)}) \prod_{V \in I(\mathcal{C})} s_{\vec{\nu}(V)}(\mathbf{y}^{(V)})\right)$$

in

$$\prod_{U,V \in I(\mathcal{C})} \left(\sum_{\rho \in \mathcal{P}} s_\rho(\mathbf{x}^{(U)}) s_\rho(\mathbf{y}^{(V)})\right)^{N_{U,V}^{(\mathbf{1})}} \left(\prod_{U \in I(\mathcal{C})} s_{\vec{\lambda}(U)}\left(\mathbf{x}^{(U)}, \mathbf{y}^{(U)}, \mathbf{z}^{(U)}\right)\right).$$

Proof. We manipulate generating functions, starting with one where the coefficient of

$$\left(\prod_{U\in I(\mathcal{C})} s_{\vec{\mu}(U)}(\mathbf{x}^{(U)}) \prod_{V\in I(\mathcal{C})} s_{\vec{\nu}(V)}(\mathbf{y}^{(V)})\right)$$

is $X_{\vec{\mu}}X_{\vec{\nu}}$. Thus, the problem reduces to understanding the coefficient of $X_{\vec{\lambda}}$ in the resulting generating function:

$$\begin{aligned}
&\sum_{\vec{\mu}\in\mathcal{P}^{\mathcal{C}}}\sum_{\vec{\nu}\in\mathcal{P}^{\mathcal{C}}}\left(\prod_{U\in I(\mathcal{C})} s_{\vec{\mu}(U)}(\mathbf{x}^{(U)}) \prod_{V\in I(\mathcal{C})} s_{\vec{\nu}(V)}(\mathbf{y}^{(V)})\right)\otimes (X_{\vec{\mu}}X_{\vec{\nu}})\\
= &\left(\sum_{\vec{\mu}\in\mathcal{P}_{\mathcal{C}}}\left(\prod_{U\in I(\mathcal{C})} s_{\vec{\mu}(U)}(\mathbf{x}^{(U)})\right)\otimes X_{\vec{\mu}}\right)\left(\sum_{\vec{\nu}\in\mathcal{P}_{\mathcal{C}}}\left(\prod_{V\in I(\mathcal{C})} s_{\vec{\nu}(V)}(\mathbf{y}^{(V)})\right)\otimes X_{\vec{\nu}}\right)\\
= &\left(\sum_{r\geq 0}(-1)^r e_r(\mathbf{x}^{(\mathbf{1})})\right)\prod_{l=1}^{\infty}\exp\left(T_l\left(\log\left(1+\sum_{U\in I(\mathcal{C})} p_l(\mathbf{x}^{(U)})[U]\right)\right)\right)\\
\times &\left(\sum_{r\geq 0}(-1)^r e_r(\mathbf{y}^{(\mathbf{1})})\right)\prod_{l=1}^{\infty}\exp\left(T_l\left(\log\left(1+\sum_{V\in I(\mathcal{C})} p_l(\mathbf{y}^{(V)})[V]\right)\right)\right).
\end{aligned}$$

We now use the Baker-Campbell-Hausdorff formula; it provides an expansion for the quantity $\log(\exp(A)\exp(B))$ as $\mathrm{BCH}(A,B) = A+B+\frac{1}{2}[A,B]+\cdots$, for possibly non-commuting A, B as a linear combination of iterated commutators of A and B (we view the monomials A and B as degenerate commutators). Because $T_l(-)$ respects commutators in the sense of a Lie algebra homomorphism (see Remark 4.7.7), we may write:

$$\begin{aligned}
&\exp\left(T_l\left(\log\left(1+\sum_{U\in I(\mathcal{C})} p_l(\mathbf{x}^{(U)})[U]\right)\right)\right)\exp\left(T_l\left(\log\left(1+\sum_{V\in I(\mathcal{C})} p_l(\mathbf{y}^{(V)})[V]\right)\right)\right)\\
=\;&\exp\left(\mathrm{BCH}\left(T_l\left(\log\left(1+\sum_{U\in I(\mathcal{C})} p_l(\mathbf{x}^{(U)})[U]\right)\right), T_l\left(\log\left(1+\sum_{V\in I(\mathcal{C})} p_l(\mathbf{y}^{(V)})[V]\right)\right)\right)\right)\\
=\;&\exp\left(T_l\left(\mathrm{BCH}\left(\log\left(1+\sum_{U\in I(\mathcal{C})} p_l(\mathbf{x}^{(U)})[U]\right), \log\left(1+\sum_{V\in I(\mathcal{C})} p_l(\mathbf{y}^{(V)})[V]\right)\right)\right)\right)\\
=\;&\exp\left(T_l\left(\log\left(\left(1+\sum_{U\in I(\mathcal{C})} p_l(\mathbf{x}^{(U)})[U]\right)\left(1+\sum_{V\in I(\mathcal{C})} p_l(\mathbf{y}^{(V)})[V]\right)\right)\right)\right).
\end{aligned}$$

In the last step we used the fact that $\mathrm{BCH}(\log(A),\log(B)) = \log(AB)$ (equivalent to $\mathrm{BCH}(A,B) = \log(\exp(A)\exp(B))$). We rewrite our expression to feature only one power-sum symmetric function (albeit with a complicated set of variables). We use the facts that $p_l(\mathbf{x},\mathbf{y}) = p_l(\mathbf{x}) + p_l(\mathbf{y})$ and $p_l(\mathbf{x}\mathbf{y}) = p_l(\mathbf{x})p_l(\mathbf{y})$:

$$\begin{aligned}
&\exp\left(T_l\left(\log\left(\left(1+\sum_{U\in I(\mathcal{C})} p_l(\mathbf{x}^{(U)})[U]\right)\left(1+\sum_{V\in I(\mathcal{C})} p_l(\mathbf{y}^{(V)})[V]\right)\right)\right)\right)\\
=\;&\exp\left(T_l\left(\log\left(1+\sum_{U\in I(\mathcal{C})} p_l(\mathbf{x}^{(U)})[U]+\sum_{V\in I(\mathcal{C})} p_l(\mathbf{y}^{(V)})[V]+\sum_{U,V\in I(\mathcal{C})} p_l(\mathbf{x}^{(U)})p_l(\mathbf{y}^{(V)})[U][V]\right)\right)\right)\\
=\;&\exp\left(T_l\left(\log\left(1+\sum_{U\in I(\mathcal{C})} p_l\left(\mathbf{x}^{(U)},\mathbf{y}^{(U)},\bigoplus_{V_1,V_2\in I(\mathcal{C})}\left(\mathbf{x}^{(V_1)}\mathbf{y}^{(V_2)}\right)^{\oplus N^U_{V_1,V_2}}\right)[U]\right)\right)\right).
\end{aligned}$$

Here we have used direct sum notation to indicate that p_l should have a collection of symmetric function variables as arguments, and the direct sum in the exponents denotes the multiplicity of each of the sets of variables. For convenience we write $\mathbf{z}^{(U)}$ to denote the family of symmetric function variables $\bigoplus_{V_1,V_2\in I(\mathcal{C})}\left(\mathbf{x}^{(V_1)}\mathbf{y}^{(V_2)}\right)^{\oplus N^U_{V_1,V_2}}$. Note that if the variables

$\mathbf{x}$ are indexed as x_i, we have

$$E(t) = \sum_{r\geq 0} e_r(\mathbf{x})t^r = \prod_i (1 + x_i t).$$

So $E(t)$ (and in particular $E(-1)$) is multiplicative with respect to variable sets:

$$\left(\sum_{r\geq 0}(-1)^r e_r(\mathbf{x}^{(\mathbf{1})})\right)\left(\sum_{s\geq 0}(-1)^s e_s(\mathbf{y}^{(\mathbf{1})})\right) = \left(\sum_{r\geq 0}(-1)^r e_r(\mathbf{x}^{(\mathbf{1})}, \mathbf{y}^{(\mathbf{1})})\right).$$

Thus our original generating function becomes

$$\left(\sum_{r\geq 0}(-1)^r e_r(\mathbf{x}^{(\mathbf{1})}, \mathbf{y}^{(\mathbf{1})})\right)\prod_{l=1}^{\infty} \exp\left(T_l\left(\log\left(1 + \sum_U p_l\left(\mathbf{x}^{(U)}, \mathbf{y}^{(U)}, \mathbf{z}^{(U)}\right)[U]\right)\right)\right).$$

This is very close to the generating function of Theorem 4.8.10 in variables $\mathbf{x}^{(U)}, \mathbf{y}^{(U)}, \mathbf{z}^{(U)}$ (only the leading factor is different). Because the leading factor is multiplicative with respect to variable sets, we may write it as

$$\frac{1}{\sum_{r\geq 0}(-1)^r e_r(\mathbf{z}^{(\mathbf{1})})} \sum_{\vec{\lambda}\in\mathcal{P}^{\mathcal{C}}} \left(\prod_{U\in I(\mathcal{C})} s_{\vec{\lambda}(U)}\left(\mathbf{x}^{(U)}, \mathbf{y}^{(U)}, \mathbf{z}^{(U)}\right)\right) \otimes X_{\vec{\lambda}}.$$

If the variables $\mathbf{x}^{(U)}$ and $\mathbf{y}^{(V)}$ are indexed as $x_i^{(U)}$ and $y_j^{(V)}$ respectively, the leading term can also be written

$$\prod_{U,V\in I(\mathcal{C})} \left(\prod_{i,j} \frac{1}{1 - x_i^{(U)} y_j^{(V)}}\right)^{N_{U,V}^{(\mathbf{1})}} = \prod_{U,V\in I(\mathcal{C})} \left(\sum_{\rho\in\mathcal{P}} s_\rho(\mathbf{x}^{(U)}) s_\rho(\mathbf{y}^{(V)})\right)^{N_{U,V}^{(\mathbf{1})}}.$$

Upon considering the coefficient of $X_{\vec{\lambda}}$ in

$$\prod_{U,V\in I(\mathcal{C})} \left(\sum_{\rho\in\mathcal{P}} s_\rho(\mathbf{x}^{(U)}) s_\rho(\mathbf{y}^{(V)})\right)^{N_{U,V}^{(\mathbf{1})}} \sum_{\vec{\lambda}\in\mathcal{P}^{\mathcal{C}}} \left(\prod_{U\in I(\mathcal{C})} s_{\vec{\lambda}(U)}\left(\mathbf{x}^{(U)}, \mathbf{y}^{(U)}, \mathbf{z}^{(U)}\right)\right) \otimes X_{\vec{\lambda}},$$

we obtain the statement of the theorem. □

4.9.1 The Case of $\mathcal{C} = \mathbf{Vect}(k)$

Now we specialise to the case where $\mathcal{C}$ is the category of finite-dimensional vector spaces over k. In that case there is only one isomorphism class of simple objects $U \in I(\mathcal{C})$, namely k, which is idempotent with respect to the tensor structure. As it plays no role, we drop $U = k$ from the notation. Also, $\mathcal{P}^{\mathcal{C}}$ is identified with $\mathcal{P}$. To illustrate how to perform the computation in the statement of Theorem 4.9.2, we prove the following theorem of Littlewood [17].

Theorem 4.9.3. *The reduced Kronecker coefficients satisfy the following identity:*

$$\tilde{k}^{\lambda}_{\mu,\nu} = \sum_{\sigma^{(1)},\sigma^{(2)},\sigma^{(3)} \in \mathcal{P}} \sum_{\rho^{(1)},\rho^{(2)},\rho^{(3)} \in \mathcal{P}} k^{\sigma^{(1)}}_{\sigma^{(2)},\sigma^{(3)}} c^{\lambda}_{\sigma^{(1)},\rho^{(2)},\rho^{(3)}} c^{\mu}_{\rho^{(1)},\sigma^{(2)},\rho^{(3)}} c^{\nu}_{\rho^{(1)},\rho^{(2)},\sigma^{(3)}}.$$

Here $k^{\rho^{(3)}}_{\rho^{(1)},\rho^{(2)}}$ is a Kronecker coefficient, and $c^{\delta}_{\alpha,\beta,\gamma}$ is a (generalised) Littlewood-Richardson coefficient (it is the coefficient of s_δ in $s_\alpha s_\beta s_\gamma$).

Proof. We consider the case where $\mathcal{C}$ is the category of finite-dimensional vector spaces over k as stated above. Thus $\mathbf{z}^{(k)}$ (in the notation of Theorem 4.9.2) is just $\mathbf{xy}$. Below, all sums are over the set of partitions. The coefficient of $s_\mu(\mathbf{x})s_\nu(\mathbf{y})$ in the following quantity is the value we wish to calculate:

$$\begin{aligned}
&\left(\sum_{\rho^{(1)}} s_{\rho^{(1)}}(\mathbf{x}) s_{\rho^{(1)}}(\mathbf{y})\right) \sum_{\lambda} s_\lambda(\mathbf{x},\mathbf{y},\mathbf{xy}) \\
= &\sum_{\rho^{(1)}} s_{\rho^{(1)}}(\mathbf{x}) s_{\rho_1}(\mathbf{y}) \sum_{\lambda,\rho^{(2)},\rho^{(3)}} \sum_{\sigma^{(1)}} c^{\lambda}_{\sigma^{(1)}\rho^{(2)},\rho^{(3)}} \left(s_{\rho^{(3)}}(\mathbf{x}) s_{\rho^{(2)}}(\mathbf{y}) s_{\sigma^{(1)}}(\mathbf{xy})\right) \\
= &\sum_{\rho^{(1)},\rho^{(2)},\rho^{(3)}} s_{\rho^{(1)}}(\mathbf{x}) s_{\rho^{(1)}}(\mathbf{y}) \sum_{\lambda} \sum_{\sigma^{(1)},\sigma^{(2)},\sigma^{(3)}} c^{\lambda}_{\sigma^{(1)},\rho^{(2)},\rho^{(3)}} k^{\sigma^{(1)}}_{\sigma^{(2)},\sigma^{(3)}} \left(s_{\rho^{(3)}}(\mathbf{x}) s_{\rho^{(2)}}(\mathbf{y}) s_{\sigma^{(3)}}(\mathbf{x}) s_{\sigma^{(2)}}(\mathbf{y})\right) \\
= &\sum_{\rho^{(1)},\rho^{(2)},\rho^{(3)}} \sum_{\sigma^{(1)},\sigma^{(2)},\sigma^{(3)}} k^{\sigma^{(1)}}_{\sigma^{(2)},\sigma^{(3)}} \sum_{\lambda} c^{\lambda}_{\sigma^{(1)},\rho^{(2)},\rho^{(3)}} \sum_{\mu} c^{\mu}_{\rho^{(1)},\sigma^{(2)},\rho^{(3)}} s_\mu(\mathbf{x}) \sum_{\nu} c^{\nu}_{\rho^{(1)},\rho^{(2)},\sigma^{(3)}} s_\nu(\mathbf{y}).
\end{aligned}$$

This completes the proof. □

We also point out that Theorem 4.8.10 gives a generating function for a known family of symmetric functions, the *irreducible character basis* $\tilde{s}_\lambda$ from [21], which are the same as the *stable Specht polynomials* $s^\dagger_\lambda$ of [2]. As above, we omit $U = k$ entirely from our notation, as well as the tensor product symbols. Theorem 4.8.10 becomes the following.

Theorem 4.9.4. *We have the following equality of generating functions.*

$$\sum_{\lambda \in \mathcal{P}} s_\lambda X_\lambda = \left(\sum_{i \geq 0} (-1)^i e_i \right) \prod_{l \geq 1} (1 + p_l)^{T_l}.$$

Let the variables of the symmetric functions present in the above expression be $\mathbf{x}$. We introduce a new set of symmetric functions in the variables $\mathbf{y}$ such that $\varphi_{\mathbf{1}}(e_i) = \overline{e}_i(\mathbf{1}) + \overline{e}_{i-1}(\mathbf{1})$ is identified with $e_i(\mathbf{y})$. We write $\tilde{s}_\lambda(\mathbf{y})$ for the the symmetric function obtained by writing X_λ in terms of the variables $\mathbf{y}$. In accordance with Proposition 4.8.11 we have the following equality:

$$\begin{aligned} \sum_{\lambda \in \mathcal{P}} s_\lambda(\mathbf{x}) \tilde{s}_\lambda(\mathbf{y}) &= \left(\sum_{i \geq 0} (-1)^i e_i(\mathbf{x}) \right) \prod_{l \geq 1} (1 + p_l(\mathbf{x}))^{\frac{1}{l} \sum_{d|l} \mu(l/d) p_d(\mathbf{y})} \\ &= \left(\sum_{i \geq 0} (-1)^i e_i(\mathbf{x}) \right) \prod_{l \geq 1} \sum_{r \geq 0} p_l(\mathbf{x})^r \binom{\frac{1}{l} \sum_{d|l} \mu(l/d) p_d(\mathbf{y})}{r}. \end{aligned}$$

The $\tilde{s}_\lambda$ are polynomials in the elementary symmetric functions such that if the i-th elementary symmetric function is replaced with the i-th exterior power of the permutation representation of S_n, and the multiplication is taken to be the tensor product of S_n-representations, then for n sufficiently large, the virtual representation we obtain is the Specht module $\mathcal{S}^{\lambda[n]}$ (this also implies that the characters are obtained by evaluating these symmetric functions at suitable roots of unity, as discussed in [21]). Thus the $\tilde{s}_\lambda$ are fundamental objects in the asymptotic representation theory of symmetric groups. A combinatorial description of them is given in [21]. By comparing the above generating function with their Proposition 11, combined with the description of character polynomials in Example 14 of Section 7 of [18] makes it clear that these are indeed the same symmetric functions.

4.10 Generalisation to Tensor Categories

All our results thus far are valid in the setting where $\mathcal{C}$ is a ring category, as per Definition 4.2.3 of [10], and in particular for any tensor category. That is, $\mathcal{C}$ is an essentially small, locally finite k-linear abelian monodial category satisfying two conditions. Firstly, if $\mathbf{1}$ is the unit object, then $\mathrm{End}_{\mathcal{C}}(\mathbf{1}) = k$. Secondly, the product in $\mathcal{C}$ is exact in both arguments and bilinear with respect to direct sums. The essentially small property allows the construction of the Grothendieck group $\mathcal{G}(\mathcal{C})$, whilst the artinian property implies that the $\mathcal{G}(\mathcal{C})$ is the free abelian group generated by isomorphism classes of simple objects. The exactness of the product in the category implies that it respects the relations of the Grothendieck group and therefore descends to a bilinear distributive multiplication on $\mathcal{G}(\mathcal{C})$. Thus, $\mathcal{G}(\mathcal{C})$, inherits the structure of a ring. Due to a theorem of Takeuchi, an essentially small k-linear Artinian abelian category (in particular, our $\mathcal{C}$) is equivalent to C-comod for some coalgebra C over k [32].

The category of finite-dimensional modules for a bialgebra over k is an example of a ring category (as is the category of finite-dimensional comodules). Generalising this example, the category of finite-dimensional modules over a quasibialgebra is also a ring category.

In order to construct wreath product categories, we make use of Deligne's tensor product for categories, which we briefly describe. If $\mathcal{C}_1$ and $\mathcal{C}_2$ are k-linear artinian categories, then their tensor product, $\mathcal{C}_1 \boxtimes \mathcal{C}_2$ is another artinian category. It is equipped with a bifunctor $\boxtimes : \mathcal{C}_1 \times \mathcal{C}_2 \to \mathcal{C}_1 \boxtimes \mathcal{C}_2$ satisfying a certain universal property; details can be found in [10]. For our purposes, it suffices to know several properties. Firstly, simple objects in $\mathcal{C}_1 \boxtimes \mathcal{C}_2$ are precisely those of the form $S_1 \boxtimes S_2$ where S_1 and S_2 are simple objects of $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively. This is a consequence of the fact that if C_1 and C_2 are coalgebras such that $\mathcal{C}_1$ is equivalent to $C_1 - \mathrm{comod}$ and $\mathcal{C}_2$ is equivalent to $C_2 - \mathrm{comod}$, then $\mathcal{C}_1 \boxtimes \mathcal{C}_2$ is equivalent to $(C_1 \otimes C_2) - \mathrm{comod}$. Secondly, if $\mathcal{C}_1$ and $\mathcal{C}_2$ are tensor categories, then so is $\mathcal{C}_1 \boxtimes \mathcal{C}_2$, with tensor structure arising from $(X_1 \boxtimes Y_1) \otimes (X_2 \boxtimes Y_2) = (X_1 \otimes X_2) \boxtimes (Y_1 \otimes Y_2)$.

Example 4.10.1. *If A_1 and A_2 are finite-dimensional k-algebras, then* $(A_1 - \mathrm{mod}) \boxtimes (A_2 - \mathrm{mod}) = (A_1 \otimes_k A_2) - \mathrm{mod}$.

We may form the n-fold Deligne's tensor product of $\mathcal{C}$ which is itself a ring category, which we denote $\mathcal{C}^{\boxtimes n}$.

Definition 4.10.2. *The equivariantisation of $\mathcal{C}^{\boxtimes n}$ under the natural action of S_n is the* wreath product category $\mathcal{W}_n(\mathcal{C}) = (\mathcal{C}^{\boxtimes n})^{S_n}$*. If $\mathcal{C}$ is a ring category,then $\mathcal{W}_n(\mathcal{C})$ obtains the structure of a ring category.*

Example 4.10.3. *If A is a finite-dimensional k-algebra then $\mathcal{W}_n(A - \mathrm{mod})$ is equivalent to $(A \wr S_n) - \mathrm{mod}$, the category of finite-dimensional modules for the wreath product (although A would need some additional structure for $A - \mathrm{mod}$ to be a ring category).*

There is a theory of induction and restriction completely analogous to the theory discussed for finite groups. If a group G acts on objects of $\mathcal{C}$, so does any subgroup, via restriction. Following Section 3.2 of [19], if $\mathcal{D}$ is an additive category, then for any subgroup H of finite index in G, we have a forgetful functor $\mathrm{Res}_H^G : \mathcal{D}^G \to \mathcal{D}^H$. Additionally there is an induction functor $\mathrm{Ind}_H^G : \mathcal{D}^H \to \mathcal{D}^G$ which is both right adjoint and left adjoint to Res_H^G. The induction functor may be written as a sum over coset representatives of H in G as follows:

$$\mathrm{Ind}_H^G(M) = \bigoplus_{g \in G/H} gM.$$

In the above formula, the action of G is analogous to that of an induced representation of a finite group.

An identical classification of simple objects (i.e. specific objects induced from Young subgroups in the sense described above) of $\mathcal{W}_n(\mathcal{C})$ holds in greater generality. In [19], this is shown in the context of indecomposable objects of an additive category, but the proof in our setting is analogous.

Chapter 5

The Integral Form and its Algebraic Structure

5.1 Introduction

Let R be a ring which is free as a $\mathbb{Z}$-module. In this chapter, we define a ring $\mathcal{G}^{\mathbb{Z}}_{\infty}(R)$ (generalising $\mathcal{G}^{\mathbb{Z}}_{\infty}(\mathcal{C})$ from Chapter 4) which is an integral form for an infinite tensor product of $U(\mathbb{Q} \otimes_{\mathbb{Z}} R)$ (the universal enveloping algebra of $\mathbb{Q} \otimes_{\mathbb{Z}} R$, viewed as a Lie algebra). Given a tensor category (or even a ring category in the sense of Definition 4.2.3 of [10]) $\mathcal{C}$ over an algebraically closed field of characteristic zero, we may form $S_t(\mathcal{C})$, the wreath product Deligne category (as in [19]). Our construction recovers the Grothendieck ring $\mathcal{G}(S_t(\mathcal{C}))$ when R is the Grothendieck ring $\mathcal{G}(\mathcal{C})$.

In the context of representation stability (in the sense of [6]), one studies FI-modules; see [5]. There is a version of FI called FI_G that incorporates a group G, discussed in [28]. In the case where $R = \mathcal{G}(\mathbb{C}G\text{-mod})$ for a finite group G, the ring $\mathcal{G}^{\mathbb{Z}}_{\infty}(R)$ is the Grothendieck ring of the Serre quotient of the category of finitely generated FI_G-modules by the subcategory of torsion modules (where the multiplication comes from the pointwise tensor product).

It was shown in [14] that $\mathcal{G}(S_t(\mathcal{C}))$ is a filtered ring with associated graded ring isomorphic to a free polynomial algebra in certain elements called *basic hooks*, indexed by $\mathbb{Z}_{>0} \times I(\mathcal{C})$, where $I(\mathcal{C})$ is the set of isomorphism classes of simple objects in $\mathcal{C}$. We show an analogous result for our case, indexed by $\mathbb{Z}_{>0} \times I$, where I is a $\mathbb{Z}$-basis of R. If $U \in R$ we define the elements $e_n(U)$ by giving an expression for the generating function of the $e_n(U)$ (see Definition 5.4.19):

$$E_U(t) = \sum_{n \geq 0} e_n(U) t^n,$$

where $e_0(U) = 1$. This allows us to state one family of relations between the elements $e_n(U)$; for $U, V \in R$,

$$E_U(u) E_{VU}(-uv)^{-1} E_V(v) = E_V(v) E_{UV}(-uv)^{-1} E_U(u).$$

We also discuss how to express $e_n(V)$ in terms of $e_n(U)$ where $U \in I$ (where I is any fixed basis of R). For this we define an auxiliary generating function (with rational coefficients):

$$F_U(t) = -\sum_{r \geq 1} \frac{\mu(r)}{r} \log(E_{U^r}(-t^r)).$$

We then have the following relations (for any $U, V \in R$):

$$F_{U+V}(t) = F_U(t) + F_V(t).$$

This equation of generating functions may be rewritten to have integral coefficients (although we do not do this explicitly), leading to a description of $\mathcal{G}^{\mathbb{Z}}_{\infty}(R)$.

We show that $\mathcal{G}^{\mathbb{Z}}_{\infty}(R)$ has a λ-ring structure whenever $\mathcal{G}(R)$ does. In the case where R is the Grothendieck ring of a symmetric tensor category, so that R inherits a λ-ring structure, the λ-ring structure on $\mathcal{G}^{\mathbb{Z}}_{\infty}(R) = \mathcal{G}(S_t(\mathcal{C}))$ is the same as the one induced by the symmetric tensor structure on $S_t(\mathcal{C})$.

This algebra $\mathcal{G}^{\mathbb{Z}}_{\infty}(R)$ has a Hopf algebra structure (when $R = \mathcal{G}(\mathcal{C})$, the comultiplication is induced by certain functors between Deligne categories). The comultiplication Δ, counit

ε, and antipode S, are defined as follows:

$$\begin{aligned}
\Delta(E_U(t)) &= E_U(t) \otimes E_U(t), \\
\varepsilon(E_U(t)) &= 1, \\
S(E_U(t)) &= E_U(t)^{-1}.
\end{aligned}$$

In particular, the generating function $E_U(t)$ is grouplike.

Finally, we prove that $\mathcal{G}_\infty^{\mathbb{Z}}(R)$ is isomorphic to the Hopf algebra of distributions on the formal neighbourhood the identity of $(W \otimes_{\mathbb{Z}} R)^\times$, where W is the ring of Big Witt Vectors.

The structure of the chapter is as follows. In Section 2 we recall some facts about symmetric functions and wreath products of tensor categories. Also in Section 2, we discuss the connection to FI_G-modules. The structure of $\mathbb{Q}\otimes_{\mathbb{Z}}\mathcal{G}(S_t(\mathcal{C}))$ was established in Chapter 4 and is summarised in Section 3. We define the main object of the paper, $\mathcal{G}_\infty^{\mathbb{Z}}(R)$ in Section 4, along with the generators $e_n(U)$. In Sections 5 and 6, we prove certain relations satisfied by $e_n(U)$. Section 7 shows how a λ-ring structure on R passes to a λ-ring structure on $\mathcal{G}_\infty^{\mathbb{Z}}(R)$, while Section 8 constructs a Hopf algebra structure on $\mathcal{G}_\infty^{\mathbb{Z}}(R)$ (note that R is not required to be a Hopf algebra for this). Finally in Section 9 we show that, as a Hopf algebra, $\mathcal{G}_\infty^{\mathbb{Z}}(R)$ admits a description as a certain algebra of distributions.

5.2 Background

We recall some relevant material about Hopf algebra structures on the ring of symmetric functions, and some properties of wreath products.

5.2.1 Symmetric Functions as a Hopf algebra

There is a Hopf algebra structure on Λ. The comultiplication, which we denote $\Delta^{(+)} : \Lambda \to \Lambda \otimes \Lambda$, is given by $\Delta^{(+)}(f) = f(\mathbf{x}, \mathbf{y})$ (here we consider $\Lambda \otimes \Lambda$ as symmetric functions in two sets of variables). This makes the power-sum symmetric functions primitive. When we consider the Big Witt Vectors, we will need the Kronecker comultiplication; $\Delta^{(\times)}(f) = f(\mathbf{xy})$ (it is part of a Hopf algebra structure with a different multiplication which we do not consider in this thesis).

We will consider a tensor product of copies of Λ indexed by a set I:

$$\bigotimes_{U \in I} \Lambda^{(U)}$$

Similarly to the case of $\Lambda \otimes \Lambda$, this is the space of symmetric functions in many sets of variables, which we may denote $\mathbf{x}_U$ for $U \in I$. To indicate that a symmetric function f belongs to $\Lambda^{(U)}$ (or its canonical inclusion into $\bigotimes_{U \in I} \Lambda^{(U)}$), we write either $f(\mathbf{x}_U)$, or $f^{(U)}$ when we are not concerned with the variable sets themselves. When we want to consider a compultiplication on $\bigotimes_{U \in I} \Lambda^{(U)}$ itself, we will write $\mathbf{x}$ and $\mathbf{y}$ to indicate the family of variables sets $\{\mathbf{x}_U\}$ and $\{\mathbf{y}_U\}$ respectively.

5.2.2 Tensor Categories and Wreath Products

The main cases of interest and motivating examples for our results will involve tensor categories. Let k be an algebraically closed field of characteristic zero and $\mathcal{C}$ an artinian tensor category over k. All our results will hold when $\mathcal{C}$ is an artinian ring category over k in the sense of Section 4.2 of [10]. However, a reader who is not interested in this degree of generality may take $\mathcal{C}$ to be the category of finite-dimensional modules over a Hopf algebra $\mathcal{R}$ over k. We write $I(\mathcal{C})$ for the set of isomorphism classes of simple objects of $\mathcal{C}$, and $\mathbf{1}$ for the unit object. Throughout, $\mathcal{G}(\mathcal{D})$ will indicate the Grothendieck group or ring of a category $\mathcal{D}$.

Wreath products underlie the main object of this paper. In the case where $\mathcal{C}$ is $\mathcal{R}-\text{mod}$, we may consider the wreath product $\mathcal{R} \wr S_n$; as a vector space, this is $\mathcal{R}^{\otimes n} \otimes kS_n$. The multiplication on $\mathcal{R} \wr S_n$ is determined by requiring the maps $\varphi : \mathcal{R}^{\otimes n} \to \mathcal{R}^{\otimes n} \otimes kS_n : x \mapsto x \otimes 1_{S_n}$ and $\psi : kS_n \to \mathcal{R}^{\otimes n} \otimes kS_n : y \mapsto 1_{\mathcal{R}} \otimes y$ to be algebra homomorphisms, together with the following commutation relation between the Hopf algebra and symmetric group parts.

If $r_1, r_2, \ldots, r_n \in \mathcal{R}$, and $\sigma \in S_n$,

$$\varphi(r_1 \otimes r_2 \otimes \cdots \otimes r_n)\psi(\sigma) = \psi(\sigma)\varphi(r_{\sigma(1)} \otimes r_{\sigma(2)} \otimes \cdots \otimes r_{\sigma(n)}).$$

This is again a Hopf algebra, with the comultiplication and antipode determined by requiring φ and ψ to be homomorphisms of Hopf algebras. In the case of more general $\mathcal{C}$, we have the following definition. Write $\boxtimes$ for the Deligne product of categories (see Section 1.11 of [10]). We may take the n-fold iterated product of $\mathcal{C}$ with itself, $\mathcal{C}^{\boxtimes n}$. This comes with an action of the symmetric group S_n by permutation of the factors. The equivariantisation with respect to this action is the wreath product category we are concerned with; we write $\mathcal{W}_n(\mathcal{C}) = (\mathcal{C}^{\boxtimes n})^{S_n}$.

The simple objects of the wreath product category $\mathcal{W}_n(\mathcal{C})$ are indexed by the following set:

Definition 5.2.1. *Let $\mathcal{P}_n^{\mathcal{C}}$ denote partition-valued functions on $I(\mathcal{C})$ with total size n:*

$$\mathcal{P}_n^{\mathcal{C}} = \{\overrightarrow{\lambda} : I(\mathcal{C}) \to \mathcal{P} \mid \sum_{U \in I(\mathcal{C})} |\overrightarrow{\lambda}(U)| = n\}$$

We use the vector notation $\overrightarrow{\lambda}$ to denote a partition-valued function on $I(\mathcal{C})$, and write $|\overrightarrow{\lambda}| = \sum_{U \in I(\mathcal{C})} |\overrightarrow{\lambda}(U)|$ for the total size of $\overrightarrow{\lambda}$. We will also write $S_{\overrightarrow{\lambda}}$ for the Young subgroup $\prod_{U \in I(\mathcal{C}} S_{|\overrightarrow{\lambda}(U)|}$ of $S_{|\overrightarrow{\lambda}|}$.

The object corresponding to a multipartition $\overrightarrow{\lambda}$ can be explicitly constructed using induction functors as described in [19]; we briefly summarise this theory. Suppose that a group G acts on the category $\mathcal{C}$. Then any subgroup of G also acts on $\mathcal{C}$, via restriction. As in Section 3.2 of [19], if $\mathcal{D}$ is an additive category, then given any finite-index subgroup H of G, we have a forgetful functor $\mathrm{Res}_H^G : \mathcal{D}^G \to \mathcal{D}^H$ (the superscript group indicates the equivariantisation of $\mathcal{D}$ with respect to that group). Further, there is an induction functor $\mathrm{Ind}_H^G : \mathcal{D}^H \to \mathcal{D}^G$ which is two-sided adjoint to Res_H^G. The induction functor may be written as a sum over coset representatives of H in G as follows:

$$\mathrm{Ind}_H^G(M) = \bigoplus_{g \in G/H} gM.$$

The action of G is completely analogous to the case of induction of representations of finite groups, where M would be a representation of H.

Equipped with this notion, we consider $U^{\boxtimes|\vec{\lambda}(U)|} \otimes \mathcal{S}^{\vec{\lambda}(U)}$; this defines an object of $\mathcal{C}^{\boxtimes|\vec{\lambda}(U)|}$. Then, $S_{|\vec{\lambda}(U)|}$ acts by permuting the tensor factors of $U^{\boxtimes|\vec{\lambda}(U)|}$, whilst also acting on $\mathcal{S}^{\vec{\lambda}(U)}$. Hence, we have an object of $\mathcal{W}_{|\vec{\lambda}(U)|}(\mathcal{C})$. We may take the (Deligne) external product of these objects (for $U \in I(\mathcal{C})$) to get:

$$\boxtimes_{U \in I(\mathcal{C})} \left(U^{\boxtimes|\vec{\lambda}(U)|} \otimes \mathcal{S}^{\vec{\lambda}(U)} \right),$$

which is an object of the category

$$\boxtimes_{U \in I(\mathcal{C})} \left(\mathcal{W}_{|\vec{\lambda}(U)|}(\mathcal{C}) \right) \cong (\mathcal{C}^{\boxtimes|\vec{\lambda}|})^{S_{\vec{\lambda}}}.$$

Upon applying the induction functor $\mathrm{Ind}_{S_{\vec{\lambda}}}^{S_{|\vec{\lambda}|}}$, we obtain the simple object corresponding to the multipartition $\vec{\lambda}$. (In Section 5 of [19] an analogous statement is proved for indecomposable objects when $\mathcal{C}$ is an additive category, but our case is not materially different.)

Since the simple objects are induced from Young subgroups of symmetric groups, the tensor structure of the categories $\mathcal{W}_n(\mathcal{C})$ can be studied using Mackey theory. In Chapter 4, this approach was used to demonstrate certain stability properties. To state them, we need the following definition (analogous to Definition 2.1.1).

Definition 5.2.2. *If $\vec{\lambda} \in \mathcal{P}_m^{\mathcal{C}}$, for $n \geq m + \vec{\lambda}(\mathbf{1})_1$, we define $\vec{\lambda}[n] \in \mathcal{P}_n^{\mathcal{C}}$ to be multpartition constructed in the following way. For $U \in I(\mathcal{C})$ different from $\mathbf{1}$, $\vec{\lambda}[n](U) = \vec{\lambda}(U)$ whilst $\vec{\lambda}[n](\mathbf{1}) = \vec{\lambda}(\mathbf{1})[n]$.*

Write $R_{\vec{\lambda}}$ for the simple object of $\mathcal{W}_n(\mathcal{C})$ indexed by the multipartition $\vec{\lambda} \in \mathcal{P}_n^{\mathcal{C}}$. In the Grothendieck ring $\mathcal{G}(\mathcal{W}_n(\mathcal{C}))$, consider the multiplication induced by the tensor product.

Here, $\vec{\lambda}, \vec{\mu}, \vec{\nu}$ are fixed multipartitions of any sizes, square brackets indicate taking the image in the Grothendieck group, and we take n sufficiently large:

$$[R_{\vec{\mu}[n]}][R_{\vec{\nu}[n]}] = [R_{\vec{\mu}[n]} \otimes R_{\vec{\nu}[n]}] = \sum_{\vec{\lambda}} k^{\vec{\lambda}}_{\vec{\mu},\vec{\nu}}(n)[R_{\vec{\lambda}[n]}].$$

The sum is taken only over those $\vec{\lambda}$ such that $\vec{\lambda}[n]$ is defined. This equation serves to define the structure constants $k^{\vec{\lambda}}_{\vec{\mu},\vec{\nu}}(n)$ for n sufficiently large. Theorem 4.6.10 states the following: $\lim_{n\to\infty} k^{\vec{\lambda}}_{\vec{\mu},\vec{\nu}}(n)$ exists, and is finite. Further, for fixed $\vec{\mu}, \vec{\nu}$, it is nonzero for only finitely many $\vec{\lambda}$. With some further work (Theorem 4.6.15), it follows that we may define an associative algebra $\mathcal{G}^{\mathbb{Z}}_{\infty}(\mathcal{C})$ which is in some sense "the $n \to \infty$ limit of the Grothendieck ring of $\mathcal{W}_n(\mathcal{C})$".

Definition 5.2.3. *Let $\mathcal{G}^{\mathbb{Z}}_{\infty}(\mathcal{C})$ have $\mathbb{Z}$-basis $\{X_{\vec{\lambda}}\}$ indexed by all multipartitions of finite size $\{\vec{\lambda} \mid \vec{\lambda} \in \mathcal{P}^{\mathcal{C}}_n, n \in \mathbb{N}\}$, and multiplication given by:*

$$X_{\vec{\mu}} X_{\vec{\nu}} = \sum_{\vec{\lambda}} \left(\lim_{n\to\infty} k^{\vec{\lambda}}_{\vec{\mu},\vec{\nu}}(n) \right) X_{\vec{\lambda}}.$$

We write $\mathcal{G}^{\mathbb{Q}}_{\infty}(\mathcal{C}) = \mathbb{Q} \otimes_{\mathbb{Z}} \mathcal{G}^{\mathbb{Z}}_{\infty}(\mathcal{C})$, when we take rational coefficients.

By Theorem 4.6.14 it follows that $\mathcal{G}^{\mathbb{Z}}_{\infty}(\mathcal{C})$ is the Grothendieck ring of the wreath-product Deligne category $S_t(\mathcal{C})$ as introduced by [19], with $X_{\vec{\lambda}}$ corresponding to the simple objects (for generic t) of $S_t(\mathcal{C})$. It was proved in [14] that the Grothendieck ring of $S_t(\mathcal{C})$ has a filtration ("$|\lambda|$-filtration"), and a generating set was given (called "basic hooks"). Theorem 4.7.8 asserts that $\mathcal{G}^{\mathbb{Q}}_{\infty}(\mathcal{C})$ is isomorphic to an infinite tensor product of universal enveloping algebras of the Lie algebra obtained by taking the Grothendieck ring of $\mathcal{C}$ with rational coefficients. That is:

$$\mathcal{G}^{\mathbb{Q}}_{\infty}(\mathcal{C}) = \bigotimes_{i=1}^{\infty} U(\mathbb{Q} \otimes_{\mathbb{Z}} \mathcal{G}(\mathcal{C})_i).$$

Here, the subscript i in $\mathcal{G}(\mathcal{C})_i$ means that the elements of this copy of $\mathcal{G}(\mathcal{C})$ lie in filtration degree i. The infinite tensor product is spanned by pure tensors $a_1 \otimes a_2 \otimes \cdots$ such that all but finitely many $a_i = 1$.

The purpose of this chapter is to describe the integral version $\mathcal{G}_\infty^{\mathbb{Z}}(\mathcal{C})$ of the algebra $\mathcal{G}_\infty^{\mathbb{Q}}(\mathcal{C})$, and to elucidate some other properties, such as a λ-ring structure (when there is a λ-ring structure on the Grothendieck ring $\mathcal{G}(\mathcal{C})$), a Hopf-algebra structure, and a realisation as a certain algebra of distributions.

5.2.3 Connection to FI_G-modules

Let FI denote the category of finite sets, with injective maps as morphisms. An FI-module is a functor from FI to the category of vector spaces (we only consider the case where $\mathbb{C}$ is the ground field). Thus the image of the finite set $[n] = \{1, 2, \ldots, n\}$ under the functor is a representation of $\mathrm{End}_{FI}([n]) = S_n$ (and these satisfy certain compatibilities for varying n). This notion was introduced in [5] and has been used to study various phenomena that have come to be termed "representation stability". Following Definition 2.3 of [6], a sequence of vector spaces V_n (where V_n is a representation of S_n) is said to exhibit representation stability if (in addition to certain compatibility conditions),

$$V_n = \bigoplus_{\lambda \in \mathcal{P}} (\mathcal{S}^{\lambda[n]})^{\oplus c_\lambda},$$

for some fixed constants c_λ and $n \gg 0$. This says that the multiplicities of the irreducible representations of symmetric groups in V_n stabilise for large n. One of the key examples is the cohomology of the configuration spaces $\mathcal{M}_n$ of n distinct unordered points on a manifold M. See Section 6 of [5] for the precise statement and assumptions on M. For each i, $H^i(\mathcal{M}_n, \mathbb{C})$ defines an FI-module, with maps between configuration spaces of different numbers of points arising from forgetting some of the points. A key part of proving stability properties amounts to showing that the object being studied is a finitely generated FI-module (we omit the definition of finite generation, see Definition 2.3.4 of [5]); indeed representation stability is approximately equivalent to finite generation of an FI-module by Theorem 1.13 of [5]. Additionally, one may take the "pointwise" tensor product of two FI-modules; if F_1, F_2 are functors $FI \to \mathbb{C}$-mod, one obtains $(F_1 \otimes F_2)([n]) = F_1([n]) \otimes F_2([n])$. If F_1, F_2 are finitely generated, so is $F_1 \otimes F_2$.

Now let us introduce the action of a finite group G. Define the category FI_G as follows (see [28]). Objects are finite sets, and morphisms between finite sets R and S are injective G-equivariant maps $R \times G \to S \times G$ (R, S have trivial G-action). As before, an FI_G-module is a functor from FI_G to vector spaces over $\mathbb{C}$. Similarly to the FI-module case, an FI_G-module consists of a sequence of representations V_n of $\mathrm{End}_{FI_G}([n]) = G \wr S_n$ for varying n. There is an analogous notion of finite generation and tensor product. Further, Proposition 3.1.6 of [28] asserts that a tensor product of two finitely generated FI_G-modules is again finitely generated. In this setting, representation stability manifests in the form

$$V_n = \bigoplus_{\vec{\lambda} \in \mathcal{P}^{\mathbb{C}G-\mathrm{mod}}} (R_{\vec{\lambda}[n]})^{\oplus c_{\vec{\lambda}}}$$

for some fixed constants $c_{\vec{\lambda}}$ and $n \gg 0$. In particular, the multiplicity of each $R_{\vec{\lambda}[n]}$ in V_n stabilises. Now, for large n, the tensor product of the $R_{\vec{\lambda}[n]}$ has the same structure constants as the multiplication of $X_{\vec{\lambda}}$. One would like to say that when $R = \mathcal{G}(\mathbb{C}G - \mathrm{mod})$, $\mathcal{G}^{\mathbb{Z}}_{\infty}(R)$ is the Grothendieck ring of the category of finitely-generated FI_G-modules, but this is not quite correct because of this transient behaviour for small n. To resolve this issue, say an FI_G-module F is a torsion module if $F([n]) = 0$ for all but finitely many n; these form a Serre subcategory, FI_G-mod^{tors}, which is also a tensor ideal. Now, an object of the quotient FI_G-mod$/FI_G$-mod^{tors} is precisely determined by the multiplicities $c_{\vec{\lambda}}$. In fact, the proof of Theorem 3.2.2 of [28] shows that the simple objects of this quotient category are given by functors $[n] \mapsto R_{\vec{\lambda}[n]}$. Thus the Grothendieck ring of this category is $\mathcal{G}^{\mathbb{Z}}_{\infty}(R)$. The author is grateful to the referee of [24] for this observation.

5.3 Structure of the Rational Limiting Grothendieck Ring of Wreath Product Categories

We now summarise the structure of the rational limiting Grothendieck ring $\mathcal{G}_\infty^{\mathbb{Q}}(\mathcal{C})$ from Chapter 4. The main tool is the family of elements $T_n(U)$ (constructed in Section 4.7), where $n \in \mathbb{Z}_{\geq 0}$ and U can be any element of the Grothendieck ring $\mathcal{G}(\mathcal{C})$. The elements $T_n(U)$ are first defined for the case where U is the image of an object of $\mathcal{C}$ in the Grothendieck ring. It is then shown that if N is a subobject of M (so that $0 \to N \to M \to M/N \to 0$ is a short exact sequence), then $T_n(M) = T_n(N) + T_n(M/N)$. This means that $T_n(U)$ is linear in U for U that are the images of objects of $\mathcal{C}$ in the Grothendieck ring. It also means that $T_n(U)$ only depends on the composition factors of U (with multiplicity), so we will not worry about distinguishing U from its image in the Grothendieck group. This linearity property justifies defining $T_n(U)$ for arbitrary $U \in \mathcal{G}(\mathcal{C})$ by writing $U = U_1 - U_2$ where U_1 and U_2 are the images of objects of $\mathcal{C}$, and letting $T_n(U) = T_n(U_1) - T_n(U_2)$. It is then shown that

$$T_n(U)T_n(V) - T_n(V)T_n(U) = T_n(UV) - T_n(VU),$$

where the multiplication of U and V takes place in $\mathcal{G}(\mathcal{C})$. Also, $T_m(U)$ and $T_n(V)$ commute when $m \neq n$. Although the $T_n(U)$ are not elements of $\mathcal{G}_\infty^{\mathbb{Z}}(\mathcal{C})$ unless $n = 1$, they do generate $\mathcal{G}_\infty^{\mathbb{Q}}(\mathcal{C})$.

The way to express an element $X_{\vec{\lambda}}$ in terms of the $T_n(U)$ is encoded in a certain identity of generating functions. It is given by Theorem 4.8.10:

Theorem 5.3.1. *Write $\Lambda_{\mathbb{Q}}^{(U)}$ for a copy of the ring of symmetric functions with rational coefficients, whose variables we associate with $U \in I(\mathcal{C})$. We work in the completed tensor product $\left(\bigotimes_{U\in I(\mathcal{C})} \Lambda_{\mathbb{Q}}^{(U)}\right) \hat{\otimes} \mathcal{G}(\mathcal{C})$. If f is a symmetric function, we write $f^{(U)}$ to denote f considered as an element of $\Lambda_{\mathbb{Q}}^{(U)}$. If $c_{\vec{\mu}}^{(U)}$ are constants, we define*

$$T_l\left(\sum_{\vec{\mu}\in\mathcal{P}^{\mathcal{C}},U\in I(\mathcal{C})} c_{\vec{\mu}}^{(U)}\left(\prod_{V\in I(\mathcal{C})} p_{\vec{\mu}(V)}^{(V)}\right) U\right) = \sum_{\vec{\mu}\in\mathcal{P},U\in I(\mathcal{C})} c_{\vec{\mu}}^{(U)}\left(\prod_{V\in I(\mathcal{C})} p_{\vec{\mu}(V)}^{(V)}\right) T_l(U)$$

which is an element of $\left(\bigotimes_{U\in I(\mathcal{C})}\Lambda_{\mathbb{Q}}^{(U)}\right)\hat{\otimes}\mathcal{G}_\infty(\mathcal{C})$. *We have the following equality:*

$$\sum_{\vec{\lambda}\in\mathcal{P}^{\mathcal{C}}}\left(\prod_{U\in I(\mathcal{C})}s^{(U)}_{\vec{\lambda}(U)}\right)\otimes X_{\vec{\lambda}}=\left(\sum_{r\geq 0}(-1)^r e_r^{(\mathbf{1})}\right)\prod_{l=1}^{\infty}\exp\left(T_l\left(\log\left(1+\sum_{U\in I(\mathcal{C})}p_l^{(U)}U\right)\right)\right).$$

This theorem encodes how to express the $X_{\vec{\lambda}}$ (quantities of representation-theoretic interest) in terms of the $T_n(U)$ (amenable to computation). For our purposes we will be interested in a slightly different generating function which encodes a slightly different basis of $\mathcal{G}_\infty^{\mathbb{Z}}(\mathcal{C})$. Consider the elements $Z_{\vec{\lambda}}$ defined by the following equation:

$$\sum_{\vec{\lambda}\in\mathcal{P}^{\mathcal{C}}}\left(\prod_{U\in I(\mathcal{C})}s^{(U)}_{\vec{\lambda}(U)}\right)\otimes Z_{\vec{\lambda}}=\prod_{l=1}^{\infty}\exp\left(T_l\left(\log\left(1+\sum_{U}p_l^{(U)}U\right)\right)\right).$$

Because the omitted factor $\left(\sum_{r\geq 0}(-1)^r e_r^{(\mathbf{1})}\right)$ is equal to 1 plus higher order terms, this means that $Z_{\vec{\lambda}}$ and $X_{\vec{\lambda}}$ agree up to lower order terms in the filtration on $\mathcal{G}_\infty^{\mathbb{Z}}(\mathcal{C})$. In particular the $Z_{\vec{\lambda}}$ form a $\mathbb{Z}$-basis of $\mathcal{G}_\infty^{\mathbb{Z}}(\mathcal{C})$.

Remark 5.3.2. *The relation between the two bases $X_{\vec{\lambda}}$ and $Z_{\vec{\lambda}}$ is calculated in Chapter 4, where elements denoted* $\lim_{m\to\infty}\operatorname{Ind}_{S_{\vec{\lambda}}\times S_m}^{S_{|\vec{\lambda}|+m}}\left(\boxtimes_{U\in I(\mathcal{C})}\left(U^{\boxtimes|\vec{\lambda}|}\otimes\mathcal{S}^{\vec{\lambda}(U)}\right)\boxtimes\left(\mathbf{1}^{\boxtimes m}\otimes\mathbf{1}_{S_m}\right)\right)$ *are shown to have the same generating function as the $Z_{\vec{\lambda}}$ (see the proof of Theorem 4.8.10), so they are equal to our $Z_{\vec{\lambda}}$. Then, in Remark 4.8.6, the decomposition into $X_{\vec{\lambda}}$ is explained using the Pieri rule.*

It was shown in [14] that the associated graded algebra of $\mathcal{G}_\infty^{\mathbb{Z}}(\mathcal{C})$ is a free polynomial algebra in basic hooks, which are defined as $X_{\vec{\lambda}}$, where $\vec{\lambda}(U)$ is nonzero for a unique $U\in I(\mathcal{C})$, and for this U, $\vec{\lambda}(U)=(1^i)$.

5.4 General Setup

The algebra $\mathcal{G}_\infty^{\mathbb{Z}}(\mathcal{C})$ depends on the category $\mathcal{C}$ only via the Grothendieck ring $\mathcal{G}(\mathcal{C})$. Furthermore, the construction is still possible after replacing $\mathcal{G}(\mathcal{C})$ with a suitable ring, even if that ring is not the Grothendieck ring of a ring category. To emphasise this, we work only with rings rather than categories. To take the place of $\mathcal{G}(\mathcal{C})$, $I(\mathcal{C})$, and $\mathcal{P}^{\mathcal{C}}$, we will use:

Definition 5.4.1. *Let R be a ring which is free as a $\mathbb{Z}$-module, and let I be a $\mathbb{Z}$-basis of R. We define $\mathcal{P}^I = \{f : I \to \mathcal{P} \mid \sum_{U \in I} |f(U)| < \infty\}$, the set of I-indexed multipartitions.*

Definition 5.4.2. *Define*

$$\mathcal{G}_\infty^{\mathbb{Q}}(R) = \bigotimes_{i \geq 1} U(\mathbb{Q} \otimes R_i),$$

where $U(\mathbb{Q} \otimes R_i)$ is a copy of the universal enveloping algebra of $\mathbb{Q} \otimes R$ (where the Lie algebra structure is inherited from the associative algebra structure). Given $r \in \mathbb{Q} \otimes R$, we write $T_i(r)$ for the image of r under the canonical map $\mathbb{Q} \otimes R \to U(\mathbb{Q} \otimes R_i)$.

Proposition 5.4.3. *There is a filtration on $\mathcal{G}_\infty^{\mathbb{Q}}(R)$ uniquely determined by making the element $T_{i_1}(r_1)T_{i_2}(r_2) \cdots T_{i_n}(r_n)$ lie in filtration degree $i_1 + i_2 + \cdots + i_n$.*

Proof. Uniqueness follows because the monomials $T_{i_1}(r_1)T_{i_2}(r_2) \cdots T_{i_n}(r_n)$ span the algebra by the PBW theorem. Existence follows from observing that the filtration is constructed in the following way. Modify the usual PBW filtration on $U(\mathbb{Q} \otimes R_i)$ by multiplying all filtration degrees by i, and take the induced filtration on the tensor product

$$\bigotimes_{i \geq 1} U(\mathbb{Q} \otimes R_i).$$

□

Proposition 5.4.4. *The associated graded algebra of $\mathcal{G}_\infty^{\mathbb{Q}}(R)$ is isomorphic to $\bigotimes_{U \in I} \Lambda_{\mathbb{Q}}^{(U)}$ where the image of $T_l(U)$ is $\frac{p_l^{(U)}}{l}$.*

Proof. This follows from the fact that the associated graded algebra of $\mathcal{G}_\infty^{\mathbb{Q}}(R)$ is the free polynomial algebra generated by $T_i(U)$ for $i \in \mathbb{Z}_{>0}$ and $U \in I$, together with the fact that $\Lambda_{\mathbb{Q}}$ is generated by $\frac{p_i}{i}$ for $i \in \mathbb{Z}_{>0}$. □

Definition 5.4.5. *Define $Z_{\vec{\lambda}} \in \mathcal{G}_\infty^{\mathbb{Q}}(R)$ (implicitly depending on I) by the following equality of generating functions:*

$$\sum_{\vec{\lambda}\in\mathcal{P}^{\mathcal{C}}} \left(\prod_{U\in I} s^{(U)}_{\vec{\lambda}(U)}\right) \otimes Z_{\vec{\lambda}} = \prod_{l=1}^{\infty} \exp\left(T_l\left(\log\left(1+\sum_{U\in I} p_l^{(U)} U\right)\right)\right),$$

where, as before,

$$T_l\left(\sum_{\vec{\mu}\in\mathcal{P}^{\mathcal{C}}, U\in I(\mathcal{C})} c^{(U)}_{\vec{\mu}} \left(\prod_{V\in I(\mathcal{C})} p^{(V)}_{\vec{\mu}(V)}\right) U\right) = \sum_{\vec{\mu}\in\mathcal{P}, U\in I(\mathcal{C})} c^{(U)}_{\vec{\mu}} \left(\prod_{V\in I(\mathcal{C})} p^{(V)}_{\vec{\mu}(V)}\right) T_l(U).$$

Also define $\mathcal{G}_\infty^{\mathbb{Z}}(R)$ as the $\mathbb{Z}$-span of the $Z_{\vec{\lambda}}$.

Theorem 5.4.6. *We have that $\mathcal{G}_\infty^{\mathbb{Z}}(R)$ is independent of the choice of basis $I \subset R$.*

Proof. Fix two bases I and I' of R, writing $Z_{\vec{\lambda}}(I)$ and $Z_{\vec{\lambda}}(I')$ for the elements $Z_{\vec{\lambda}}$ defined using the bases I and I' respectively. Consider the symmetric functions $p_l^{(V)}$ defined by

$$\sum_{V\in I'} p_l^{(V)} V = \sum_{U\in I} p_l^{(U)} U.$$

If the transition matrix between I' and I is a_{VU}, $V = \sum_{U\in I} a_{VU} U$. This gives

$$p_l^{(V)} = \sum_{U\in I} a_{VU} p_l^{(U)}.$$

This extends from power-sum symmetric functions to arbitrary symmetric functions via

$$f^{(V)} = f\left(\bigoplus_{U\in I} \mathbf{x}_U^{\oplus a_{VU}}\right)$$

where the meaning of negative multiplicities of variable sets is as in Example 23 in Section 1.3 of [18] (where $f(\mathbf{x}^{\oplus 1}, \mathbf{y}^{\oplus(-1)})$ is considered, but denoted $f(\mathbf{x}/\mathbf{y})$). Because a_{VU} is invertible, we have constructed an isomorphism

$$\theta : \bigotimes_{V\in I'} \Lambda^{(V)} \to \bigotimes_{U\in I} \Lambda^{(U)}$$

which moreover satisfies

$$\begin{aligned}
\sum_{\vec{\lambda}\in\mathcal{P}^{I'}} \theta\left(\prod_{V\in I'} s^{(V)}_{\vec{\lambda}(V)}\right) \otimes Z_{\vec{\lambda}}(I') &= \prod_{l=1}^{\infty} \exp\left(T_l\left(\log\left(1+\sum_{V\in I'}\theta(p_l^{(V)})V\right)\right)\right)\\
&= \prod_{l=1}^{\infty} \exp\left(T_l\left(\log\left(1+\sum_{U\in I} p_l^{(U)}U\right)\right)\right)\\
&= \sum_{\vec{\lambda}\in\mathcal{P}^{I}} \left(\prod_{U\in I} s^{(U)}_{\vec{\lambda}(U)}\right) \otimes Z_{\vec{\lambda}}(I).
\end{aligned}$$

To recover $Z_{\vec{\lambda}}(I)$ in terms of $Z_{\vec{\lambda}}(I')$ it suffices to take the inner product of both sides with $\prod_{U\in I} s^{(U)}_{\vec{\lambda}(U)}$. In particular, this means that the $\mathbb{Z}$-linear span of $Z_{\vec{\lambda}}(I)$ is independent of the choice of basis I. □

Definition 5.4.7. *We let*

$$s_{\vec{\lambda}} = \prod_{U\in I} s^{(U)}_{\vec{\lambda}(U)},$$

which is an element of $\bigotimes_{U\in I} \Lambda^{(U)}$*, implicitly depending on* I*.*

Proposition 5.4.8. *In the associated graded algebra of* $\mathcal{G}^{\mathbb{Q}}_{\infty}(R)$*, isomorphic to* $\bigotimes_{U\in I} \Lambda^{(U)}_{\mathbb{Q}}$ *(as discussed in proposition Proposition 5.4.4), the image of* $Z_{\vec{\lambda}}$ *is* $s_{\vec{\lambda}}$*.*

Proof. Recall that $Z_{\vec{\lambda}}$ was defined by the equation

$$\sum_{\vec{\lambda}\in\mathcal{P}^{\mathcal{C}}} s_{\vec{\lambda}} \otimes Z_{\vec{\lambda}} = \prod_{l=1}^{\infty} \exp\left(T_l\left(\log\left(1+\sum_{U\in I} p_l^{(U)}U\right)\right)\right).$$

To calculate the leading coefficient of $\prod_{U\in I} s^{(U)}_{\vec{\lambda}(U)}$ as a polynomial in the $T_i(U)$, we approximate

$$T_l\left(\log\left(1+\sum_{U\in I} p_l^{(U)}U\right)\right)$$

by

$$\sum_{U\in I} T_l(U)p_l^{(U)}.$$

This turns the generating function

$$\prod_{l=1}^{\infty}\exp\left(T_l\left(\log\left(1+\sum_{U\in I}p_l^{(U)}U\right)\right)\right)$$

into

$$\prod_{l=1}^{\infty}\exp\left(\sum_{U\in I}p_l^{(U)}T_l(U)\right).$$

Let the symmetric function variables be $\mathbf{x}_U$, and think of $T_l(U)$ as $\frac{p_l(\mathbf{y}_U)}{l}$ (as the $T_l(U)$ are algebraically independent in the associated graded algebra). Then, the generating function becomes

$$\prod_{U\in I}\exp\left(\sum_{l\geq 1}\frac{p_l(\mathbf{x}_U)p_l(\mathbf{y}_U)}{l}\right).$$

which we recognise as the product of instances of the Cauchy identity (one for each element of I). The coefficient of $s_{\vec{\lambda}}(\mathbf{x}) = \prod_{U\in I} s_{\vec{\lambda}(U)}(\mathbf{x}_U)$ is therefore $s_{\vec{\lambda}}(\mathbf{y}) = \prod_{U\in I} s_{\vec{\lambda}(U)}(\mathbf{y}_U)$. □

Corollary 5.4.9. *The $Z_{\vec{\lambda}}$ are a basis of $\mathcal{G}_\infty^{\mathbb{Q}}(R)$.*

Proof. This follows from the fact that the images of the $Z_{\vec{\lambda}}$ in the associated graded algebra (see Proposition 5.4.8) form a basis. □

Definition 5.4.10. *Let us write*

$$\Theta_l(x) = \exp(T_l(\log(x))),$$

where x may be any series in the completed tensor product $\left(\bigotimes_{U\in I}\Lambda_{\mathbb{Q}}^{(U)}\right)\hat{\otimes}R$ whose lowest order term is 1.

Remark 5.4.11. *With this notation we may rewrite the equation in Definition 5.4.5 as*

$$\sum_{\vec{\lambda}\in\mathcal{P}^I} s_{\vec{\lambda}}\otimes Z_{\vec{\lambda}} = \prod_{l=1}^{\infty}\Theta_l\left(1+\sum_{U\in I}p_l^{(U)}U\right).$$

Proposition 5.4.12. *We have the following identity:*

$$\Theta_l(x)\Theta_l(y) = \Theta_l(xy).$$

Proof. We make use of the Baker-Campbell-Hausdorff formula. Let

$$\mathrm{BCH}(A, B) = \log(\exp(A)\exp(B)) = A + B + \frac{1}{2}[A, B] + \cdots,$$

this is a sum of iterated commutators of A and B (the monomials A and B are thought of as a commutator iterated zero times). We have the relation $T_l([U, V]) = [T_l(U), T_l(V)]$ in the universal enveloping algebra. Let $\widehat{\otimes}$ denote the completed tensor product of graded algebras (completion of the usual tensor product with respect to the grading). Thus T_l defines a Lie algebra homomorphism $\left(\bigotimes_{U\in I}\Lambda_{\mathbb{Q}}^{(U)}\right)\widehat{\otimes}R \to \left(\bigotimes_{U\in I}\Lambda_{\mathbb{Q}}^{(U)}\right)\widehat{\otimes}\mathcal{G}_\infty^{\mathbb{Q}}(R)$ (where the Lie algebra structure is inherited from the associative algebra structure, and the right hand factors have trivial gradings). We have that $T_l(\mathrm{BCH}(-,-)) = \mathrm{BCH}(T_l(-), T_l(-))$ (as BCH is a sum of iterated commutators and $T_l(-)$ is a Lie algebra homomorphism). Now we calculate:

$$\begin{aligned}
&\Theta_l(x)\Theta_l(y)\\
= \;& \exp(T_l(\log(x)))\exp(T_l(\log(y)))\\
= \;& \exp(\mathrm{BCH}(T_l(\log(x)), T_l(\log(y))))\\
= \;& \exp(T_l(\mathrm{BCH}(\log(x), \log(y))))\\
= \;& \exp(T_l(\log(xy)))\\
= \;& \Theta_l(xy).
\end{aligned}$$

□

Theorem 5.4.13. *Let $N_{V,W}^{U}$ be the structure tensor of R with respect to the basis I (so that*

$VW = \sum_U N_{V,W}^U U$ *for* $U, V, W \in I$*). We have the following equality of generating functions:*

$$\left(\sum_{\vec{\mu}\in\mathcal{P}^I} s_{\vec{\mu}}(\mathbf{x}) \otimes Z_{\vec{\mu}}\right)\left(\sum_{\vec{\nu}\in\mathcal{P}^I} s_{\vec{\nu}}(\mathbf{y}) \otimes Z_{\vec{\nu}}\right)$$
$$= \sum_{\vec{\lambda}\in\mathcal{P}^I}\left(\prod_{U\in I} s_{\vec{\lambda}(U)}(\mathbf{x}_U, \mathbf{y}_U, \bigoplus_{V,W\in I}(\mathbf{x}_V\mathbf{y}_W)^{\oplus N_{V,W}^U})\right) \otimes Z_{\vec{\lambda}}.$$

Proof. We use the multiplicative expression for these generating functions from Definition 5.4.5.

$$\begin{aligned}
&\left(\sum_{\vec{\mu}\in\mathcal{P}^I}\left(\prod_{U\in I} s_{\vec{\mu}(U)}(\mathbf{x}_U)\right)\otimes Z_{\vec{\mu}}\right)\left(\sum_{\vec{\nu}\in\mathcal{P}^I}\left(\prod_{U\in I} s_{\vec{\nu}(U)}(\mathbf{y}_U)\right)\otimes Z_{\vec{\nu}}\right)\\
=\;& \prod_{l=1}^{\infty}\Theta_l\left(1+\sum_{U\in I} p_l(\mathbf{x}_U)U\right)\prod_{l=1}^{\infty}\Theta_l\left(1+\sum_{U\in I} p_l(\mathbf{y}_U)U\right)\\
=\;& \prod_{l=1}^{\infty}\Theta_l\left((1+\sum_{U\in I} p_l(\mathbf{x}_U)U)(1+\sum_{U\in I} p_l(\mathbf{y}_U)U)\right)\\
=\;& \prod_{l=1}^{\infty}\Theta_l\left(1+\sum_{U\in I}\left(p_l(\mathbf{x}_U)+p_l(\mathbf{y}_U)+\sum_{V,W\in I} N_{V,W}^U p_l(\mathbf{x}_V)p_l(\mathbf{y}_W)\right)U\right)\\
=\;& \prod_{l=1}^{\infty}\Theta_l\left(1+\sum_{U} p_l\left(\mathbf{x}_U,\mathbf{y}_U,\bigoplus_{V,W\in I}(\mathbf{x}_V\mathbf{y}_W)^{\oplus N_{V,W}^U}\right)U\right)\\
=\;& \sum_{\vec{\lambda}\in\mathcal{P}^I}\left(\prod_{U\in I} s_{\vec{\lambda}(U)}\left(\mathbf{x}_U,\mathbf{y}_U,\bigoplus_{V,W\in I}(\mathbf{x}_V\mathbf{y}_W)^{\oplus N_{V,W}^U}\right)\right)\otimes Z_{\vec{\lambda}}
\end{aligned}$$

□

By comparing coefficients of Schur functions on each side of the previous theorem, we obtain the following corollary.

Corollary 5.4.14. *We have*

$$Z_{\vec{\mu}}Z_{\vec{\nu}} = \sum_{\vec{\lambda}} a_{\vec{\mu},\vec{\nu}}^{\vec{\lambda}} Z_{\vec{\lambda}},$$

where $a^{\vec{\lambda}}_{\vec{\mu},\vec{\nu}}$ is the coefficient of $s_{\vec{\mu}}(\mathbf{x})s_{\vec{\nu}}(\mathbf{y})$ in

$$\prod_{U\in I} s_{\vec{\lambda}(U)}\left(\mathbf{x}_U, \mathbf{y}_U, \bigoplus_{V,W\in I}(\mathbf{x}_V\mathbf{y}_W)^{\oplus N^U_{V,W}}\right).$$

Theorem 5.4.15. *The multiplication induced from $\mathcal{G}^{\mathbb{Q}}_\infty(R)$ makes $\mathcal{G}^{\mathbb{Z}}_\infty(R)$ into a ring.*

Proof. If $\vec{\lambda}$ is the empty multipartition, then $Z_{\vec{\lambda}}$ is the identity. Thus, it suffices to show that $\mathcal{G}^{\mathbb{Z}}_\infty(R)$ is closed under multiplication, which follows from Corollary 5.4.14 □

Remark 5.4.16. *We have shown that the ring $\mathcal{G}^{\mathbb{Z}}_\infty(R)$ is an integral form of $\mathcal{G}^{\mathbb{Q}}_\infty(R)$.*

Definition 5.4.17. *Let $e_r(U) = Z_{\vec{\lambda}}$, where $\vec{\lambda}(U) = (1^r)$ for a single $U \in I$ and $\vec{\lambda}(U)$ is the empty partition for all other U. For fixed U, we write $E_U(t) = \sum_{r\geq 0} e_r(U)t^r$.*

The following proposition shows that $e_r(U)$ does not depend on the choice of basis I.

Proposition 5.4.18. *Fix $U \in I$. We have the following equality:*

$$E_U(t) = \prod_{i\geq 1}\Theta_i(1-(-t)^i U).$$

Proof. Consider the generating function defining the $Z_{\vec{\lambda}}$:

$$\sum_{\vec{\lambda}\in\mathcal{P}^I} s_{\vec{\lambda}}\otimes Z_{\vec{\lambda}} = \prod_{l=1}^{\infty}\Theta_l\left(1+\sum_{U\in I} p_l^{(U)}U\right).$$

For all $V \in I$ different from U, we set the variable set $\mathbf{x}_V$ to zero. This has the effect of sending any term with $|\vec{\lambda}(V)| > 0$ to zero. If we write $Z_{\lambda,U}$ for $Z_{\vec{\mu}}$ where $\vec{\mu}(U) = \lambda$ and $\vec{\mu}(V)$ is the empty partition for $V \neq U$, this gives

$$\sum_{\lambda\in\mathcal{P}} s_\lambda^{(U)}\otimes Z_{\lambda,U} = \prod_{l=1}^{\infty}\Theta_l\left(1+p_l^{(U)}U\right).$$

We apply the involution ω in the symmetric function variables $\mathbf{x}_U$. Recall that $\omega(s_\lambda) = s_{\lambda'}$

and $\omega(p_l) = (-1)^{l-1}p_l$. Hence,

$$\sum_{\lambda\in\mathcal{P}} s_{\lambda'}^{(U)} \otimes Z_{\lambda,U} = \prod_{l=1}^{\infty} \Theta_l \left(1 + (-1)^{l-1} p_l^{(U)} U\right).$$

We evaluate the set of variables $\mathbf{x}_U = (t, 0, 0, \ldots)$. Note that $s_{\lambda'}(t, 0, 0, \ldots)$ is equal to t^n if $\lambda' = (n)$, and zero if λ' has more than one part, while $p_l(t, 0, 0, \ldots) = t^l$. Now,

$$\sum_{r\geq 0} e_r(U)t^r = \prod_{l=1}^{\infty} \Theta_l \left(1 - (-t)^l U\right).$$

□

Definition 5.4.19. *For arbitrary $U \in R$ (not necessarily an element of I), we define $e_r(U)$ and $E_U(t)$ via the preceding series:*

$$E_U(t) = \sum_{r\geq 0} e_r(U)t^r = \prod_{l\geq 1} \Theta_l(1 - (-t)^l U).$$

Proposition 5.4.20. *For $r \in \mathbb{Z}_{>0}$, and $U \in I$, the $e_r(U)$ generate $\mathcal{G}_\infty^{\mathbb{Z}}(R)$.*

Proof. Passing to the associated graded algebra, the image of $e_r(U)$ is $e_r^{(U)} \in \bigotimes_{U\in I} \Lambda^{(U)}$. Because the $e_r^{(U)}$ generate the associated graded algebra, it follows the $e_r(U)$ generate $\mathcal{G}_\infty^{\mathbb{Z}}(R)$.

□

5.5 The Commutation Relation for $E_U(t)$

We now concern ourselves with understanding the relations between the $e_r(U)$.

Lemma 5.5.1. *If U and V commute in R, then $e_i(U)$ and $e_j(V)$ commute in $\mathcal{G}_\infty^{\mathbb{Z}}(R)$ for any i and j.*

Proof. It suffices to check that the generating functions $E_U(t)$ and $E_V(s)$ commute:

$$\begin{aligned} E_U(t)E_V(s) &= \prod_l \Theta_l(1-(-t)^l U)\Theta_l(1-(-s)^l V) \\ &= \prod_l \Theta_l\left((1-(-t)^l U)(1-(-s)^l V)\right) \\ &= \prod_l \Theta_l\left((1-(-s)^l V)(1-(-t)^l U)\right) \\ &= \prod_l \Theta_l(1-(-s)^l V)\Theta_l(1-(-t)^l U) \\ &= E_V(s)E_U(t). \end{aligned}$$

□

The following lemma follows immediately from the multiplicativity of Θ_l.

Lemma 5.5.2. *The multiplicative inverse of $\Theta_l(1+tU)$ is equal to $\Theta_l((1+tU)^{-1})$.*

Proposition 5.5.3. *Let X and Y be non-commuting variables. Then, the sum of all monomials that alternate in X and Y can be written in the following two ways:*

$$(1+X)(1-YX)^{-1}(1+Y) = (1+Y)(1-XY)^{-1}(1+X).$$

Proof. Let $A = (1+X)$ and $B = (1+Y)$. Then we must show that

$$A(A+B-BA)^{-1}B = B(A+B-AB)^{-1}A.$$

But the inverse of this equation is

$$B^{-1}(A+B-BA)A^{-1} = A^{-1}(A+B-AB)B^{-1},$$

and both sides equal $A^{-1}+B^{-1}+1$. □

Theorem 5.5.4. *We have the following relation between generating functions (in $\mathcal{G}_\infty^{\mathbb{Z}}(R)[[u,v]]$):*

$$E_U(u)E_{VU}(-uv)^{-1}E_V(v) = E_V(v)E_{UV}(-uv)^{-1}E_U(u).$$

Proof. Recall that $E_U(u) = \prod_{l\geq 1}\Theta_l(1-(-u)^iU)$. We will work with terms corresponding to a fixed l, and then multiply these together to recover the result for the full generating functions. With Proposition 5.5.3 we obtain:

$$\begin{aligned}
& \Theta_l(1-(-u)^lU)\Theta_l((1-(-v)^l(-u)^lVU)^{-1})\Theta_l(1-(-v)^lV) \\
= \;& \Theta_l((1-(-u)^lU)(1-(-v)^l(-u)^lVU)^{-1}(1-(-v)^lV)))) \\
= \;& \Theta_l((1-(-v)^lV)(1-(-u)^l(-v)^lUV)^{-1}(1-(-u)^lU)) \\
= \;& \Theta_l(1-(-v)^lV)\Theta_l((1-(-u)^l(-v)^lUV)^{-1})\Theta_l(1-(-u)^lU))).
\end{aligned}$$

We may now take the product of these across $i \geq 1$. Because $\Theta_{l_1}(-)$ and $\Theta_{l_2}(-)$ commute whenever $l_1 \neq l_2$, it does not matter in what order we take the product. By taking the product over l, we may use Definition 5.4.19 to write:

$$E_U(u)E_{VU}(-vu)^{-1}E_V(v) = E_V(v)E_{UV}(-uv)^{-1}E_U(u).$$

□

Definition 5.5.5. *For $W \in R$ and $n \geq 0$, let $h_n(W) \in \mathcal{G}_\infty^{\mathbb{Z}}(R)$ be given by determinant of the matrix $(e_{1+j-i}(W))_{i,j=1}^n$ (where $e_r(W) = 0$ if $r < 0$).*

Note that $h_n(W)$ is a polynomial in $e_1(W), e_2(W), \ldots, e_n(W)$ that lies in filtration degree n of $\mathcal{G}_\infty^{\mathbb{Z}}(R)$.

Lemma 5.5.6. *The series $\sum_{n\geq 0} h_n(W)t^n$ is the multiplicative inverse of $E_W(-t)$.*

Proof. Recall that in the theory of symmetric functions, we have the relation $H(t)E(-t) = 1$, where $H(t)$ is the generating function of complete symmetric functions, and $E(t)$ is the generating function of elementary symmetric functions. Since the elementary symmetric functions are algebraically independent, we may specialise the r-th elementary symmetric function to $e_r(W)$ (which commute with each other by Lemma 5.5.1), which specialises $E(-t)$ to $E_W(-t)$. The coefficients of $H(t)$ are given by the Jacobi-Trudi formula: $h_n = \det(e_{1+j-i})_{i,j=1}^n$, which specialise to $h_n(W)$ as defined above. □

Corollary 5.5.7. *The commutation relation between $e_i(U)$ and $e_j(V)$ is given by the following:*

$$\sum_{k=0}^{\min(i,j)} e_{i-k}(U)h_k(VU)e_{j-k}(V) = \sum_{k=0}^{\min(i,j)} e_{j-k}(V)h_k(UV)e_{i-k}(U).$$

More explicitly:

$$[e_i(U), e_j(V)] = \sum_{k=1}^{\min(i,j)} e_{j-k}(V)h_k(UV)e_{i-k}(U) - e_{i-k}(U)h_k(VU)e_{j-k}(V).$$

Note in particular that because $h_k(W)$ is in filtration degree k, the right hand side is contained in filtration degree $i + j - 1$.

Proof. This equation is simply the coefficient of $u^i v^j$ in the equation in Theorem 5.5.4. □

Example 5.5.8. *The case of $i = j = 1$ in Corollary 5.5.7 is $e_1(U)e_1(V) + e_1(VU) = e_1(V)e_1(U) + e_1(UV)$.*

Corollary 5.5.9. *Suppose that R has a $\mathbb{Z}$-basis I such that the product of any two basis elements is either zero or another basis element (a "monomial algebra"). Then $\mathcal{G}_\infty^{\mathbb{Z}}(R)$ is generated by $e_i(U)$ for $U \in I$ and $i \geq 1$. If we let $U, V \in I$, the relations are simply:*

$$E_U(u)E_{VU}(-vu)^{-1}E_V(v) = E_V(v)E_{UV}(-uv)^{-1}E_U(u)$$

where $E_0(t) = 1$.

Proof. Recall that the associated graded algebra of $\mathcal{G}_\infty^{\mathbb{Z}}(R)$ is generated as a free polynomial algebra by the $e_i(U)$. Hence, the $e_i(U)$ generate $\mathcal{G}_\infty^{\mathbb{Z}}(R)$, and to describe the relations, it suffices to give expressions for the commutators $[e_i(U), e_j(V)]$ that lie in filtration degree

$i+j-1$. This is done by Corollary 5.5.7, where the essential point is that UV and VU are either zero or in I, so that $h_k(UV)$ and $h_k(VU)$ are polynomials in elements of our generating set. □

Example 5.5.10. *Suppose that $R = \mathbb{Z}G$, the integral group algebra of G. Taking $I = G$, we are in the case of Corollary 5.5.9. We conclude that $\mathcal{G}_\infty^{\mathbb{Z}}(kG)$ is generated by $e_i(g)$ for $i \in \mathbb{Z}_{>0}$ and $g \in G$, subject to the relations arising from the following equations of generating functions for $g, h \in G$ (where, as always, $E_k(t) = \sum_{i\geq 0} e_i(k)t^i$):*

$$E_g(u)E_{hg}(-vu)^{-1}E_h(v) = E_h(v)E_{gh}(-uv)^{-1}E_g(u).$$

Example 5.5.11. *Suppose that $R = \mathrm{Mat}_n(\mathbb{Z})$, and I is the set of elementary matrices $\{\mathbf{E}_{i,j}\}_{i,j=1}^n$. Because $\mathbf{E}_{i,k}\mathbf{E}_{k,l} = \delta_{j,k}\mathbf{E}_{i,l}$, this is a monomial algebra. It follows $\mathcal{G}_\infty^{\mathbb{Z}}(\mathrm{Mat}_n(\mathbb{Z}))$ has generators $e_m(\mathbf{E}_{ij})$ and relations*

$$E_{i,j}(u)E_{k,j}(-vu)^{\delta_{li}}E_{k,l}(v) = E_{k,l}(v)E_{i,l}(-uv)^{\delta_{jk}}E_{i,j}(u)$$

where $E_{i,j}(t) = \sum_{m\geq 0} e_m(\mathbf{E}_{ij})t^m$.

5.6 Decomposing $E_U(t)$ when $U \notin I$

In the previous section, we computed the commutation relations of the generators $e_i(U)$ at the cost of introducing terms of the form $e_i(W)$, where W might not be an element of I. In this section, we express $e_i(W)$ in terms of $e_i(U)$ for $U \in I$.

Definition 5.6.1. *Let $\mu(r)$ denote the usual Möbius function (so $\sum_{d|n}\mu(d) = \delta_{1,n}$, the Kronecker delta). For $U \in R$, let $F_U(t)$ be defined by:*

$$F_U(t) = -\sum_{r\geq 1} \frac{\mu(r)}{r} \log(E_{U^r}(-t^r)).$$

Note that this series does not have integral coefficients; it lies in $\mathcal{G}_\infty^{\mathbb{Q}}(R)[[t]]$. We now express $F_U(t)$ in terms of $T_i(U^r)$.

Proposition 5.6.2. *We have the following equality of elements of $\mathcal{G}_\infty^{\mathbb{Q}}(R)[[t]]$:*

$$F_U(t) = \sum_{i\geq 1} T_i(U)t^i.$$

Proof. We substitute Definition 5.4.19 into the definition of $F_U(t)$:

$$\begin{aligned}
F_U(t) &= -\sum_{r\geq 1} \frac{\mu(r)}{r} \log(E_{U^r}(-t^r)) \\
&= -\sum_{r\geq 1} \frac{\mu(r)}{r} \sum_{i\geq 1} T_i(\log(1 - t^{ir}U^r)) \\
&= -\sum_{r\geq 1} \frac{\mu(r)}{r} \sum_{i\geq 1} T_i(-\sum_{d\geq 1} \frac{t^{ird}U^{rd}}{d}) \\
&= \sum_{r\geq 1} \sum_{i\geq 1} \sum_{d\geq 1} \frac{\mu(r)t^{ird}}{rd} T_i(U^{rd}) \\
&= \sum_{i\geq 1} \sum_{n\geq 1} \sum_{r|n} \frac{\mu(r)t^{in}}{n} T_i(U^n) \qquad \text{(change variables to } n = rd) \\
&= \sum_{i\geq 1} \sum_{n\geq 1} \delta_{1,n} \frac{t^{in}}{n} T_i(U^n) \\
&= \sum_{i\geq 1} t^i T_i(U).
\end{aligned}$$

In the second-last step, we used that $\sum_{r|n} \mu(r) = \delta_{1,n}$ (Kronecker delta). □

Theorem 5.6.3. *Suppose that $W = \sum_i a_i U_i$, with $a_i \in \mathbb{Z}$ and $U_i \in R$. Then we have:*

$$F_W(t) = \sum_i a_i F_{U_i}(t).$$

Proof. This immediately follows from the fact that $T_r(\sum_i a_i U_i) = \sum_i a_i T_r(U_i)$ and Proposition 5.6.2, expressing $F_U(t)$ in terms of $T_r(U)$. □

If we write $W = \sum_{U\in I} a_U U$, we may extract the coefficient of t^n in the equation in Theorem 5.6.3 to express $e_n(W)$ in terms of $e_n(U)$ (our generators of interest) plus terms with smaller values of n. By induction, this allows us to express $e_n(W)$ for arbitrary W in terms of $e_r(U)$ for $U \in I$.

Example 5.6.4. *We calculate $F_U(t)$ to order t^2:*

$$F_U(t) = e_1(U)t + \frac{1}{2}(e_1(U)^2 - e_1(U^2) - 2e_2(U))t^2 + \cdots$$

Let us take $W = U_1 + U_2$. Then the degree 1 and 2 terms of the equation in Theorem 5.6.3 become:

$$\begin{aligned} e_1(U_1+U_2) &= e_1(U_1)+e_1(U_2) \\ \frac{1}{2}(e_1(U_1+U_2)^2 - e_1((U_1+U_2)^2) - 2e_2(U_1+U_2)) &= \frac{1}{2}(e_1(U_1)^2 - e_1(U_1^2) - 2e_2(U_1)) \\ &\quad + \frac{1}{2}(e_1(U_2)^2 - e_1(U_2^2) - 2e_2(U_2)). \end{aligned}$$

These rearrange to the following:

$$\begin{aligned} e_1(U_1+U_2) &= e_1(U_1)+e_1(U_2) \\ e_2(U_1+U_2) &= \frac{1}{2}(e_1(U_1)e_1(U_2) + e_1(U_2)e_1(U_1) - e_1(U_1U_2) - e_1(U_2U_1)) + e_2(U_1) + e_2(U_2) \end{aligned}$$

Note that the previous equation for $e_2(U_1+U_2)$ is not manifestly integral. However, the relation $e_1(U_1)e_1(U_2) - e_1(U_1U_2) = e_1(U_2)e_1(U_1) - e_1(U_2U_1)$ from Example 5.5.8 gives the following:

$$\begin{aligned} e_2(U_1+U_2) &= e_1(U_1)e_1(U_2) - e_1(U_1U_2) + e_2(U_1) + e_2(U_2) \\ &= e_1(U_2)e_1(U_1) - e_1(U_2U_1) + e_2(U_1) + e_2(U_2). \end{aligned}$$

Thus $e_2(U_1+U_2)$ can be written in terms of $e_i(V)$ for $V \in I$ (after possibly decomposing $e_1(U_1U_2)$ or $e_1(U_2U_1)$), although not in a canonical way.

We have the following theorem.

Theorem 5.6.5. *The $\mathbb{Q}$-algebra $\mathcal{G}_\infty^{\mathbb{Q}}(R)$ admits the following presentation. The generators $e_r(U)$ are $U \in I$ and $r \in \mathbb{Z}_{>0}$. We write $E_U(t) = \sum_{i\geq 0} e_i(U)t^i$, where $e_0(U)$ is taken to be the multiplicative identity. The relations between the generators are given by the following equalities of generating functions.*

$$E_U(u)E_{VU}(-uv)^{-1}E_V(v) = E_V(v)E_{UV}(-uv)^{-1}E_U(u)$$

If $W = \sum_{U\in I} a_U U$ in R (where $a_U \in \mathbb{Z}$), we also have:

$$\sum_{r\geq 1} \frac{\mu(r)}{r} \log(E_{W^r}(-t^r)) = \sum_{U\in I} a_U \sum_{r\geq 1} \frac{\mu(r)}{r} \log(E_{U^r}(-t^r))$$

Further $\mathcal{G}_\infty^{\mathbb{Z}}(R)$ is the $\mathbb{Z}$-subalgebra of $\mathcal{G}_\infty^{\mathbb{Q}}(R)$ generated by the $e_i(U)$.

Proof. As in Corollary 5.5.9, we use the fact that the $e_i(U)$ (for $U \in I$) generate the associated graded algebra, and the fact that Theorem 5.5.4 expresses the commutator $[e_i(U), e_j(V)]$ as an element in filtration degree $i+j-1$. However, unlike Corollary 5.5.9, this now involves the generators $e_k(UV)$, where UV may not be an element of I. To provide a presentation, it suffices to express $e_k(UV)$ in terms of $e_r(W)$ for $W \in I$, which is precisely what is achieved by Theorem 5.6.3. □

Remark 5.6.6. *This falls short of giving a presentation of $\mathcal{G}_\infty^{\mathbb{Z}}(R)$ because the second family of relations are not manifestly integral. One may extract the coefficient of t^r and rearrange for $e_r(U)$, as was done for $r = 2$ in Example 5.6.4. Because the resulting expression is in $\mathcal{G}_\infty^{\mathbb{Z}}(R)$, it must be possible to rewrite $e_r(U)$ as a linear combination of monomials in $e_s(V)$ for $s \leq r$ and $V \in I$. Knowing these relations would be sufficient to give a presentation. It is reasonable to expect that it is possible to give an integral expression in terms of monomials in $e_s(U)$ (where the U are not necessarily in I) as in Example 5.6.4 for $r = 2$. However, as was already noted, there is no canonical choice of such a decomposition.*

5.7 λ-Ring Structure

Recall that a λ-ring structure on a commutative ring R is given by certain operations $\lambda^i : R \to R$ satisfying certain relations that make the λ^i analogous to exterior powers (a precise definition is given later in this section). In particular, a typical example of a λ-ring is the Grothendieck ring of representations of a group, with $\lambda^i([V]) = [\bigwedge^r(V)]$. In this section, we show that if R is a λ-ring, then so is $\mathcal{G}_\infty^{\mathbb{Z}}(R)$. Moreover, if R is the Grothendieck ring of a symmetric tensor category (with induced λ-operations), we show that our λ-ring structure on $\mathcal{G}_\infty^{\mathbb{Z}}(R)$ agrees with the λ-ring structure induced from the realisation of $\mathcal{G}_\infty^{\mathbb{Z}}(R)$ as the Grothendieck ring of a wreath product Deligne category.

First, we prove a statement about exterior powers for wreath products.

Lemma 5.7.1. *Let U be an object of a symmetric tensor category $\mathcal{C}$ (over an algebraically closed field of characteristic zero). Consider the following object of the wreath product category $\mathcal{W}_n(\mathcal{C}) = (\mathcal{C}^{\boxtimes n})^{S_n}$:*

$$V = \mathrm{Ind}_{S_1 \times S_{n-1}}^{S_n} (U \boxtimes \mathbf{1}^{\boxtimes (n-1)}).$$

Then, the r-th exterior power of V is given by the following formula:

$$\bigwedge^r(V) = \bigoplus_{\substack{\lambda \vdash r \\ l(\lambda) \le n}} \mathrm{Ind}_{S_{n-l(\lambda)} \times \prod_i S_{m_i}}^{S_n} (\mathbf{1}^{n-l(\lambda)} \boxtimes (\boxtimes_i (\bigwedge^i(U)^{\otimes m_i} \otimes \sigma_{m_i}^{(i)}))).$$

Here $\sigma_{m_i}^{(i)}$ is the trivial representation of the symmetric group S_{m_i} if i is even, and the sign representation of i is odd.

Proof. We sketch the proof. We perform the calculation by describing an exterior power of V as the image of an antisymmetrising morphism of a tensor power of V. More precisely, $V^{\otimes r}$ carries an action of the symmetric group S_r by permuting the tensor factors, and $\bigwedge^r(V)$ is the image of the endomorphism defined by $\sum_{g \in S_r} \varepsilon(g) g$, where $\varepsilon(g)$ is the sign of g.

Consider $V^{\otimes r}$, which we simplify using the Mackey formula. Recall that a set partition of a set X is a set of subsets of X whose disjoint union is equal to X. Then, we claim:

$$V^{\otimes r} = \bigoplus_{\substack{\text{set partitions } D \text{ of } \{1,2,\ldots,r\} \\ \text{having at most } n \text{ parts}}} \mathrm{Ind}_{S_{n-|D|}\times\prod_{d\in D} S_1}^{S_n}(\mathbf{1}^{\boxtimes(n-|D|)} \boxtimes (\boxtimes_{d\in D} U^{\otimes|d|})) \tag{5.1}$$

We demonstrate this by induction on r. When $r = 1$, there is only one set partition of $\{1\}$, namely $\{\{1\}\}$, and this gives rise to a single summand which is isomorphic to V. For the inductive step, we must tensor with V. To use the Mackey formula

$$\mathrm{Ind}_H^G(M) \otimes \mathrm{Ind}_K^G(N) = \bigoplus_{s\in H\backslash G/K} \mathrm{Ind}_{H\cap sKs^{-1}}^G(M \otimes sN),$$

we must know the double cosets $(S_{n-|D|} \times \prod_{d\in D} S_1)\backslash S_n/(S_1 \times S_{n-1})$. If we view $S_1 \times S_{n-1}$ as the subgroup of S_n fixing $1 \in \{1, 2, \ldots, n\}$, then the double cosets are in bijection with the nontrivial factors in the product $(S_{n-|D|} \times \prod_{d\in D} S_1)$ (so there are $|D| + 1$ if $n > |D|$ and $|D|$ otherwise) where the double coset associated to a factor group is the set of all elements in S_n mapping $1 \in \{1, 2, \ldots, n\}$ to an element permuted by that factor group (when considered as a subgroup of S_n). Thus by the Mackey formula, tensoring an object associated to a set partition D with V gives a sum of induced objects associated to all set partitions obtained from D by adding $r + 1$ to any part $d \in D$, or in a part of its own if $|D| < n$. When we consider all set partitions D of $\{1, 2, \ldots, r\}$ of length at most n, this process of appending $r + 1$ gives all set partitions of $\{1, 2, \ldots, r + 1\}$ of length at most n.

The action of S_r on our decomposition in Equation 5.1 is given by the obvious action of S_r on set partitions of $\{1, 2, \cdots, r\}$, provided that in the product $\boxtimes_{d\in D} U^{\otimes|d|}$, we identify each tensor factor of $U^{\otimes|d|}$ with an element of d, and that these tensor factors are also permuted according to the symmetric group action. Two summands lie in the same orbit of S_r if and only if they have the same multisets $\{|d| \mid d \in D\}$ for their respective set partitions D. Such a multiset defines a partition λ by reordering the elements of the multiset in nonincreasing order. It remains to show that the image of the antisymmetrising morphism on the summands in the orbit corresponding to λ gives the term indexed by λ in the statement of the Lemma.

Note that because the sign of a permutation g, $\varepsilon(g)$, is multiplicative, we have the following factorisation property. Let H be a subgroup of G (itself a subgroup of S_r), and G/H a collection of left coset representatives. Then:

$$\sum_{g\in G}\varepsilon(g)g = \left(\sum_{c\in G/H}\varepsilon(c)c\right)\left(\sum_{h\in H}\varepsilon(h)h\right).$$

We consider a set partition D with associated partition λ, and take $H = \prod_i S_i \wr S_{m_i}$ (the wreath product $S_i \wr S_{m_i}$ is considered as a subgroup of S_{im_i} in the usual way). To find the image of $U^{\boxtimes|D|}$ under this partial antisymmetrisation consider the case of a single factor $S_i \wr S_{m_i}$ acting on $(U^{\otimes i})^{\boxtimes m_i}$. We further factorise the antisymmetrising morphism taking H to be the subgroup of $S_i^{m_i}$ in the wreath product $G = S_i \wr S_{m_i}$. This acts by antisymmetrising each of the $\boxtimes$-tensor factors; the image is $\bigwedge^i(U)^{\boxtimes m_i}$. The complementary antisymmetriser $\sum_{c\in G/H}\varepsilon(c)c$ coming from coset representatives $(S_i \wr S_{m_i})/S_i^{m_i}$ is indexed by elements of S_{m_i}, and the sign of an element g is found as follows. Note that a transposition in S_{m_i} permutes each the factor groups of $S_i^{m_i}$. As an element of S_{im_i}, this has cycle type $1^{i(m_i-2)}2^i$, which is odd if i is odd and is even if i is even. Because transpositions generate S_{m_i} and signs are multiplicative, it follows that the complementary antisymmetriser, viewed as an element of the group algebra of S_{m_i} is (up to a scalar) the antisymmetriser if i is odd, and the symmetriser if i is even. Thus, after antisymmetrising over $S_i \wr S_{m_i}$, we obtain $\bigwedge^i(U)^{\otimes m_i} \otimes \sigma^{(i)}_{m_i}$. Performing this calculation for all i simultaneously gives a term isomorphic to the summand indexed by λ in the statement of the lemma. It remains to explain how the remaining antisymmetrisation gives rise to exactly one summand for each λ.

For each λ, the number of summands we have is $N = |S_r/\prod_i S_i \wr S_{m_i}|$. The antisymmetriser

$$\sum_{c\in S_n/\prod_i S_i \wr S_{m_i}}\varepsilon(c)c$$

defines a map from one summand to the direct sum of all the N summands associated to λ. Taking into consideration that the choice of the subgroup $\prod_i S_i \wr S_{m_i}$ depended on the choice of the original set partition D, one can check that all N maps obtained in this way have the same image (which can be identified with the summand corresponding to λ in the statement of the lemma). □

Recall that a λ-ring structure on a commutative ring R is a collection of operations $\lambda^i : R \to R$ (indexed by $i \in \mathbb{Z}_{\geq 0}$). These operations are required to satisfy $\lambda^0(x) = 1$, $\lambda^1(x) = x$, $\lambda^i(1) = 0$ for $i \geq 2$, as well as the following three compatibility conditions [16]:

$$\begin{aligned}
\lambda^n(x+y) &= \sum_{i+j=n} \lambda^i(x)\lambda^j(y) \\
\lambda^n(xy) &= P_n(\lambda^1(x), \lambda^2(x), \ldots, \lambda^n(x), \lambda^1(y), \lambda^2(y), \ldots, \lambda^n(y)) \\
\lambda^m(\lambda^n(x)) &= Q_{m,n}(\lambda^1(x), \lambda^2(x), \ldots, \lambda^{mn}(x)).
\end{aligned}$$

Here P_n and $Q_{m,n}$ are the integer polynomials defined by the following equations inside the ring of symmetric functions.

$$\begin{aligned}
P_n(e_1(\mathbf{x}), e_2(\mathbf{x}), \ldots, e_n(\mathbf{x}), e_1(\mathbf{y}), e_2(\mathbf{y}), \ldots, e_n(\mathbf{y}) &= e_n(\mathbf{xy}) \\
Q_{m,n}(e_1, e_2, \ldots, e_{mn}) &= e_m[e_n],
\end{aligned}$$

where $e_m[e_n]$ denotes the plethysm of the symmetric functions e_m and e_n. The first equation relates the exterior power of the tensor product of two modules to the exterior powers of the two tensor factors. The second equation expresses the composition of two exterior powers in terms of exterior powers.

Let a be an element of a λ-ring, and $\lambda_t(a) = \sum_{n\geq 0} \lambda^n(a)t^n$, the generating function of the λ operations applied to a. Also let $\psi_t(a) = \sum_{n\geq 1} \psi^n(a)t^{n-1}$ be the generating function defined by the following equation:

$$\frac{d}{dt}\log(\lambda_t(a)) = \psi_{-t}(a).$$

The ψ^n are called Adams operations; they are are commuting endomorphisms of R satisfying

$\psi^n \circ \psi^m = \psi^{nm}$ [16]. In fact, we have the following fact. Suppose that R is a commutative $\mathbb{Q}$-algebra, equipped with operations $\psi^i : R \to R$ (for $i \in \mathbb{Z}_{>0}$) satisfying the following conditions:

$$\begin{aligned} \psi^1(x) &= x \\ \psi^i(1) &= 1 \\ \psi^i(x+y) &= \psi^i(x) + \psi^i(y) \\ \psi^i(xy) &= \psi^i(x)\psi^i(y) \\ \psi^i(\psi^j(x)) &= \psi^{ij}(x). \end{aligned}$$

Then R is a λ-ring, where the λ-operations are obtained algebraically from the ψ^i in the same way that elementary symmetric functions are obtained from power-sum symmetric functions. We use this to describe a λ-ring structure on $\mathcal{G}^{\mathbb{Z}}_{\infty}(R)$.

Theorem 5.7.2. *We have the following:*

1. *Suppose that there is a λ-ring structure on R. Then, there is a λ-ring structure on $\mathcal{G}^{\mathbb{Z}}_{\infty}(R)$ defined on $e_1(U)$ by:*

$$\sum_{n\geq 0} \lambda^n(e_1(U))t^n = \prod_{l\geq 1} \Theta_l \left(\sum_{r\geq 0} (-1)^{r(l-1)} t^{rl} \lambda^r(U) \right).$$

   Together with the λ-ring axioms, this determines the λ-ring structure on all of $\mathcal{G}^{\mathbb{Z}}_{\infty}(R)$. Over $\mathcal{G}^{\mathbb{Q}}_{\infty}(R)$, the λ-ring structure is given by the following Adams operations (we write ψ^d for the Adams operations on R, and Ψ^d for those on $\mathcal{G}^{\mathbb{Q}}_{\infty}(R)$):

$$\Psi^m(T_l(U)) = \sum_{\substack{d|m \\ \gcd(d,l)=1}} \frac{m}{d} T_{l\frac{m}{d}}(\psi^d(U)).$$

2. *Consider the case where $\mathcal{C}$ is a symmetric tensor category, so that the wreath product Deligne category $S_t(\mathcal{C})$ is also a symmetric tensor category, inducing a λ-ring structure*

on $\mathcal{G}(S_t(\mathcal{C})) = \mathcal{G}^{\mathbb{Z}}_{\infty}(\mathcal{G}(\mathcal{C}))$. This λ-ring structure is obtained in the above way from the λ-ring structure induced on $\mathcal{G}(\mathcal{C})$ by the symmetric tensor structure.

Proof. We initially work over $\mathcal{G}^{\mathbb{Q}}_{\infty}(R)$, and deduce integrality at the end.

When R is a λ-ring, it is in particular commutative. This means that $\mathcal{G}^{\mathbb{Q}}_{\infty}(R)$ is the free polynomial algebra generated by $T_l(U)$ for $l \in \mathbb{Z}_{>0}$ and $U \in I$ (the universal enveloping algebra of $\mathbb{Q} \otimes_{\mathbb{Z}} \mathcal{G}(R)$ is itself a free polynomial algebra, and $\mathcal{G}^{\mathbb{Q}}_{\infty}(R)$ is a tensor product of these). Thus, to define an endomorphism of $\mathcal{G}^{\mathbb{Q}}_{\infty}(R)$, it is enough to define the image of each $T_l(U)$. We consider Ψ^m as in the statement of the theorem. Note that Ψ^1 is the identity map. We also have:

$$\begin{aligned}
\Psi^m(\Psi^n(T_l(U))) &= \sum_{\substack{d|m \\ \gcd(d,l)=1}} \frac{m}{d}\Psi^n(T_{l\frac{m}{d}}(\psi^d(U))) \\
&= \sum_{\substack{d|m \\ \gcd(d,l)=1}} \frac{m}{d} \sum_{\substack{e|n \\ \gcd(e,lm/d)=1}} \frac{n}{e} T_{l\frac{m}{d}\frac{n}{e}}(\psi^e(\psi^d(U))) \\
&= \sum_{\substack{d|m \\ \gcd(d,l)=1}} \sum_{\substack{e|n \\ \gcd(e,lm/d)=1}} \frac{mn}{de} T_{l\frac{mn}{de}}(\psi^{de}(U)) \\
&= \sum_{\substack{f|mn \\ \gcd(f,l)=1}} \frac{mn}{f} T_{l\frac{mn}{f}}(\psi^f(U)) \qquad \text{(change variables to } f = de) \\
&= \Psi^{mn}(T_l(U)).
\end{aligned}$$

Note that Ψ^n is multiplicative and additive by construction; it is also immediate that $\Psi^n(1) = 1$. This means that we have a valid collection of Adams operations, and thus a unique λ-ring structure.

We set $a = e_1(U) = T_1(U)$. Then $\Psi^n(a) = \sum_{d|n} \frac{n}{d} T_{\frac{n}{d}}(\psi^d(U))$, and we may use this to calculate the generating function of the Adams operations:

$$\begin{aligned}
\Psi_t(T_1(U)) &= \sum_{n\geq 1}\sum_{d|n}\frac{n}{d}T_{\frac{n}{d}}(\psi^d(U))t^{n-1} \\
&= \sum_{d\geq 1}\sum_{r\geq 1} rT_r(\psi^d(U))t^{rd-1} && \text{(change variables to } rd=n) \\
&= t^{r-1}\sum_{r\geq 1} rT_r(\psi_{t^r}(U)) \\
&= t^{r-1}\sum_{r\geq 1} rT_r(\frac{d}{d(-t^r)}\log(\lambda_{-t^r}(U))) \\
&= -\frac{d}{dt}\sum_{r\geq 1} T_r(\log(\lambda_{-t^r}(U))). && \text{(chain rule)}
\end{aligned}$$

We may now integrate and exponentiate to obtain a generating function for $\lambda^r(T_1(U))$.

$$\begin{aligned}
\sum_{n\geq 0}\lambda^n(T_1(U))t^n &= \exp(\int \Psi_{-t}(T_1(U))dt) \\
&= \exp(\int \frac{d}{dt}\sum_{r\geq 1} T_r(\log(\lambda_{-(-t)^r}(U)))dt) \\
&= \exp(\sum_{r\geq 1} T_r(\log(\lambda_{-(-t)^r}(U)))) \\
&= \prod_{l\geq 1}\Theta_l\left(\sum_{r\geq 0}\lambda^r(U)(-(-t)^l)^r\right)
\end{aligned}$$

This is the claimed formula for the λ-ring structure. To see that $\lambda^r(e_1(U))$ is integral (i.e. an element of $\mathcal{G}^{\mathbb{Z}}_\infty(R)$), we proceed as follows. Let $c(r,V)$ ($r\in\mathbb{Z}_{\geq 0}$, $V\in I$) be defined by

$$\lambda^r(U) = \sum_{V\in I} c(r,V)V.$$

Consider an infinite collection of symmetric function variables sets, denoted $\mathbf{x}_{r,V}$ for $V\in I$

and $r \in \mathbb{Z}_{>0}$. Take the definition of the $Z_{\vec{\lambda}}$,

$$\sum_{\vec{\lambda}\in\mathcal{P}^I} \left(\prod_{V\in I} s^{(V)}_{\vec{\lambda}(V)}\right) \otimes Z_{\vec{\lambda}} = \prod_{l\geq 1} \Theta_l(1+\sum_{V\in I} p_l^{(V)} V),$$

and make the substitution $\mathbf{x}_V = \bigoplus_{r\in\mathbb{Z}_{>0}} \mathbf{x}_{r,V}^{\oplus c(r,V)}$. The right hand side becomes

$$\prod_{l\geq 1} \Theta_l(1+\sum_{V\in I} p_l(\bigoplus_{r\in\mathbb{Z}_{>0}} \mathbf{x}_{r,V}^{\oplus c(r,V)})V) = \prod_{l\geq 1}\Theta_l(1+\sum_{V\in I}\sum_{r\in\mathbb{Z}_{>0}} c(r,V)p_l(\mathbf{x}_{r,V})V).$$

For each variable set $\mathbf{x}_{r,V}$, we apply the involution ω r times. This has the effect of multiplying $p_l(\mathbf{x}_{r,V})$ by $(-1)^{r(l-1)}$. We now evaluate each variable set $\mathbf{x}_{r,V}$ at the values $t^r, 0, 0, \ldots$ (this maps $p_l(x_{r,V})$ to t^{rl}). Our expression becomes

$$\begin{aligned}\prod_{l\geq 1}\Theta_l(1+\sum_{V\in I}\sum_{r\in\mathbb{Z}_{>0}} c(r,V)(-1)^{r(l-1)}t^{rl}V) &= \prod_{l\geq 1}\Theta_l(1+\sum_{r\in\mathbb{Z}_{>0}}(-1)^{r(l-1)}t^{rl}\sum_{V\in I}c(r,V)V) \\ &= \prod_{l\geq 1}\Theta_l(1+\sum_{r\in\mathbb{Z}_{>0}}(-(-t)^l)^r\lambda^r(U)).\end{aligned}$$

We recognise this as the generating function defining the λ-operations on $e_1(U)$. Because we began with a generating function describing integral elements and performed operations that preserve integrality, we conclude that $\lambda^i(e_1(U))$ is integral.

To see that the $\lambda^i(e_1(U))$ uniquely determine the λ-ring structure, we prove by induction on r that $e_r(U)$ is a polynomial in the variables $\lambda^i(e_1(V))$ (where $i \leq r$ and any V is permitted), then the λ-ring axioms guarantee that $\lambda^j(e_r(U))$ is uniquely determined and integral. The base case, $r = 1$, is immediate. By inspecting the coefficient of t^n in the generating function in Definition 5.4.19, we see that $e_n(U)$ is equal to $T_n(U)$ plus a polynomial in $T_s(V)$ for $s < n$ and V arbitrary. A similar inspection for the generating function in the statement of the theorem shows that $\lambda^r(e_1(U))$ equals $T_r(U)$ plus lower order terms. Hence, $\lambda^r(e_1(U)) - e_r(U)$ is a (necessarily integral) polynomial in $T_s(V)$ for $s < r$ and arbitrary V. By an upper triangularity argument, $\lambda^r(e_1(U)) - e_r(U)$ is a polynomial in $e_s(V)$ for $s < r$ and V arbitrary. By induction $e_r(U)$ is a polynomial in $\lambda^i(e_1(V))$. We conclude that the

λ-ring axioms (which define how λ-operations behave on sums and products) determine the λ-operations on $e_r(U)$ and hence all of $\mathcal{G}_\infty^{\mathbb{Q}}(\mathcal{C})$.

Now we prove part (2). Consider the following generating function in the variable sets $\mathbf{x}_r$ for $r \in \mathbb{Z}_{>0}$:

$$\prod_{l\geq 1} \Theta_l(1 + \sum_{r\geq 1} \lambda^r(U) p_l(\mathbf{x}_r)) = \sum_{\vec{\lambda}\in\mathcal{P}^{\mathbb{Z}}} \prod_{n\in\mathbb{Z}_{>0}} s^{(n)}_{\vec{\lambda}(n)} \otimes Z_{\vec{\lambda}}.$$

The $Z_{\vec{\lambda}}$ defined by the above are not necessarily a basis of $\mathcal{G}_\infty^{\mathbb{Z}}(R)$ because $\lambda^r(U)$ may not be a basis of R. They are nevertheless elements of $\mathcal{G}_\infty^{\mathbb{Z}}(R)$ (i.e. integral). By applying the involution ω to $\mathbf{x}_r$ r times, and then evaluating at $\mathbf{x}_r = (t^r, 0, 0, \ldots)$, our generating function becomes

$$\prod_{l\geq 1} \Theta_l(1 + \sum_{r\geq 1} \lambda^r(U)(-(-t^l)^r)) = \sum_{\vec{\lambda}} t^{\sum_{r\geq 1} r|\vec{\lambda}(r)|} Z_{\vec{\lambda}}.$$

Here, the sum over $\vec{\lambda}$ only includes terms where $\vec{\lambda}(r)$ consists of a single row when r is even, and consists of a single column when r is odd. The left hand side is the generating function of $\lambda^i(e_1(U))$. The terms in the right hand side are identified with the terms in Lemma 5.7.1 (where n is taken sufficiently large). This shows that this λ-ring structure is inherited from the implied symmetric tensor category structure on $S_t(\mathcal{C})$, if the λ-ring structure on $\mathcal{G}(\mathcal{C})$ comes from a symmetric tensor category structure on $\mathcal{C}$. This is because $S_t(\mathcal{C})$ (as defined in [19]) is defined as an interpolation of finite wreath products. □

To summarise, suppose that we are working in the setting where $R = \mathcal{G}(\mathcal{C})$ is the Grothendieck ring of a symmetric tensor category $\mathcal{C}$. Then $\mathcal{G}_\infty^{\mathbb{Z}}(R) = \mathcal{G}(S_t(\mathcal{C}))$ inherits a λ-ring structure because $S_t(\mathcal{C})$ inherits a symmetric tensor structure from $\mathcal{C}$. The λ-ring structure on $\mathcal{G}(S_t(\mathcal{C}))$ is obtained in a formulaic way from the one on $\mathcal{G}(\mathcal{C})$ (in a way that is made precise in the previous theorem).

5.8 Hopf Algebra Structure

It turns out that $\mathcal{G}_\infty^{\mathbb{Z}}(R)$ is a Hopf algebra; this construction generalises the Hopf algebra structure on the ring of symmetric functions.

Theorem 5.8.1. *There is a Hopf algebra structure* (Δ, ϵ, S) *on* $\mathcal{G}^{\mathbb{Z}}_{\infty}(\mathcal{C})$ *defined on the generating functions* $E_U(t)$ *by:*

$$\Delta(E_U(t)) = E_U(t) \otimes E_U(t), \tag{5.2}$$

$$\epsilon(E_U(t)) = 1, \tag{5.3}$$

$$S(E_U(t)) = E_U(t)^{-1}. \tag{5.4}$$

When working over $\mathcal{G}^{\mathbb{Q}}_{\infty}(\mathcal{C})$*, we have that the* $T_i(U)$ *are primitive.*

Proof. The fact that ϵ and S are well defined (respect the algebra structure) and satisfy the Hopf algebra axioms easily follows from the fact that $E_U(t)$ is grouplike and $\log(E_U(t))$ is primitive with respect to Δ when using the presentation in Theorem 5.6.5. Note that $F_U(t)$ is a linear combination of $\log(E_{U^r}(-t^r))$, and therefore primitive. By Proposition 5.6.2, we may take the t^i coefficient of $F_U(t)$ to deduce that $T_i(U)$ is primitive. $\square$

Remark 5.8.2. *The above Hopf algebra structure is similar to that of the ring of symmetric functions, and coincides with it when* $\mathcal{C} =$ *Vect. A succinct way of characterising the Hopf algebra structure is to say that the generating functions* $E_U(t)$ *are grouplike.*

Remark 5.8.3. *The comultiplication is categorified by a "restriction" functor*

$$S_{t_1+t_2}(\mathcal{C}) \to S_{t_1}(\mathcal{C}) \boxtimes S_{t_2}(\mathcal{C}).$$

As the Grothendieck ring $G^{\mathbb{Z}}_{\infty}(\mathcal{C}) = \mathcal{G}(S_t(\mathcal{C}))$ *does not depend on* t*, this gives rise to a comultiplication on* $\mathcal{G}^{\mathbb{Z}}_{\infty}(\mathcal{C})$*. Consider the case* $\mathcal{C} =$ *Vect, so that* $S_t(\mathcal{C})$ *is the usual Deligne category* $\underline{\mathrm{Rep}}(S_t)$*. Recall that this category is the Karoubian envelope of a category* $\underline{\mathrm{Rep}}_0(S_t)$ *whose objects are* $V^{\otimes n}$ *as discussed in [7]. The functor we are interested in,* $\mathcal{F}$*, satisfies* $\mathcal{F}(V) = V \boxtimes 1 \oplus 1 \boxtimes V$*. The wreath product case generalises this. See definition 4.21 in [19].*

One easily checks that the functor takes the antisymmetrising endomorphism of $U^{\boxtimes(m+n)}$ to the tensor product of the antisymmetrising endomorphisms of $U^{\boxtimes m}$ and $U^{\boxtimes n}$ (for all possible values of m and n). Passing to the Grothendieck ring, we obtain a comultiplication Δ, which satisfies the property that $\Delta(e_n(U)) = \sum_{i=0}^{n} e_i(U) \otimes e_{n-i}(U)$, namely our comultiplication.

We now express the comultiplication in terms of generating functions analogously to the how the multiplication was expressed in Theorem 5.4.13 and Corollary 5.4.14.

Theorem 5.8.4. *We have the following equality of generating functions:*

$$\begin{aligned}
&\sum_{\vec{\lambda}\in\mathcal{P}^I} s_{\vec{\lambda}} \otimes \Delta(Z_{\vec{\lambda}}) \\
= &\left(\sum_{\vec{\mu}\in\mathcal{P}^I} s_{\vec{\mu}} \otimes (Z_{\vec{\mu}} \otimes 1)\right)\left(\sum_{\vec{\nu}\in\mathcal{P}^I} s_{\vec{\nu}} \otimes (1 \otimes Z_{\vec{\nu}})\right)
\end{aligned}$$

Proof. Recall that $\Delta(T_l(U)) = T_l(U) \otimes 1 + 1 \otimes T_l(U)$ for arbitrary U. Therefore:

$$\begin{aligned}
&\sum_{\vec{\lambda}\in\mathcal{P}^I} s_{\vec{\lambda}} \otimes \Delta(Z_{\vec{\lambda}}) \\
= &\prod_{l=1}^{\infty} \exp\left(\Delta\left(T_l\left(\log\left(1+\sum_{U\in I} p_l(\mathbf{x}_U)U\right)\right)\right)\right) \\
= &\prod_{l=1}^{\infty} \exp\left(T_l\left(\log\left(1+\sum_{U\in I} p_l(\mathbf{x}_U)U\right)\right) \otimes 1 + 1 \otimes T_l\left(\log\left(1+\sum_{U\in I} p_l(\mathbf{x}_U)U\right)\right)\right) \\
= &\prod_{l=1}^{\infty} \exp\left(T_l\left(\log\left(1+\sum_{U\in I} p_l(\mathbf{x}_U)U\right)\right) \otimes 1\right) \prod_{l=1}^{\infty} \exp\left(1 \otimes T_l\left(\log\left(1+\sum_{U\in I} p_l(\mathbf{x}_U)U\right)\right)\right) \\
= &\left(\sum_{\vec{\mu}\in\mathcal{P}^I} s_{\vec{\mu}} \otimes (Z_{\vec{\mu}} \otimes 1)\right)\left(\sum_{\vec{\nu}\in\mathcal{P}^I} s_{\vec{\nu}} \otimes (1 \otimes Z_{\vec{\nu}})\right).
\end{aligned}$$

□

Corollary 5.8.5. *The coefficient of $Z_{\vec{\mu}} \otimes Z_{\vec{\nu}}$ in $\Delta(Z_{\vec{\lambda}})$ is equal to the coefficient of $s_{\vec{\lambda}}$ in $s_{\vec{\mu}} s_{\vec{\nu}}$.*

Proof. This immediately follows from Theorem 5.8.4 by extracting the coefficients of the relevant Schur functions. □

Remark 5.8.6. *Recall that the basis $X_{\vec{\lambda}}$ from Remark 5.3.2 corresponds to irreducible objects of the wreath product Deligne category. Because the comultiplication Δ corresponds to restriction (by Remark 5.8.3), the structure constants of Δ with respect to the basis $X_{\vec{\lambda}}$ may be thought of as stable Littlewood-Richardson coefficients (in the wreath product setting). The author is grateful to the referee of [24] for this observation.*

We conclude this section by defining some subalgebras that define integral forms of tensor powers of the algebra $U(\mathbb{Q} \otimes_{\mathbb{Z}} R)$.

Definition 5.8.7. *Let $\mathcal{G}_k^{\mathbb{Z}}(R)$ be the subalgebra of $\mathcal{G}_\infty^{\mathbb{Z}}(R)$ generated by $e_i(U)$ with $i \leq k$ and arbitrary U.*

It easily follows that $\mathcal{G}_k^{\mathbb{Z}}(R)$ is a sub-Hopf algebra of $\mathcal{G}_\infty^{\mathbb{Z}}(R)$. The following proposition describes these subalgebras.

Proposition 5.8.8. *The algebra $\mathcal{G}_1^{\mathbb{Z}}(R)$ is an integral form of the universal enveloping algebra of $\mathbb{Q} \otimes_{\mathbb{Z}} R$ (considered as a Lie algebra). This integral form has a $\mathbb{Z}$-basis consisting of PBW monomials in a basis of R. Similarly, $\mathcal{G}_k^{\mathbb{Z}}(R)$ is an integral form of the k-fold tensor product of the universal enveloping algebra $U(\mathbb{Q} \otimes_{\mathbb{Z}} R)$. The Hopf algebra structures coming from $\mathcal{G}_\infty^{\mathbb{Z}}(R)$ and from the universal enveloping algebra coincide. Finally, if R is of finite rank over $\mathbb{Z}$, each $\mathcal{G}_k^{\mathbb{Z}}(R)$ is noetherian.*

Proof. Note that $\mathbb{Q} \otimes_{\mathbb{Z}} \mathcal{G}_k^{\mathbb{Z}}(R)$ is the subalgebra of $\mathcal{G}_\infty^{\mathbb{Q}}(R)$ generated by the $e_i(U)$ for $i \leq k$. Because $e_n(U)$ is equal to $T_n(U)$ plus a polynomial in $T_m(V)$ with $m < n$ (this follows from extracting the coefficient of t^n from the generating function defining $e_n(U)$), this is the same as the subalgebra generated by the $T_i(U)$ for $i \leq k$. By Theorem 8.8 of [23], this is the k-fold tensor product of the universal enveloping algebra of $\mathbb{Q} \otimes_{\mathbb{Z}} \mathcal{G}(R)$, and as $T_i(U)$ is primitive, the Hopf algebra structures agree. This proves the statement about $\mathcal{G}_k^{\mathbb{Z}}(R)$ being an integral form. To show that $\mathcal{G}_k^{\mathbb{Z}}(R)$ is noetherian, it suffices to consider the associated graded algebra. Note that the $e_i(U)$ for $U \in I$ (I a $\mathbb{Z}$-basis of R) are free polynomial generators of the associated graded algebra of $\mathcal{G}_\infty^{\mathbb{Z}}(R)$ (as per [14]). Because Corollary 5.5.7 expresses $[e_i(U), e_j(V)]$ in terms of $e_k(W)$ with $k < \min(i,j)$, it follows that the associated graded algebra of $\mathcal{G}_k^{\mathbb{Z}}(R)$ is a free polynomial algebra generated by $e_i(U)$ for $i \leq k$ and $U \in I$ (which is noetherian, as I was assumed to be finite). It follows that $\mathcal{G}_k^{\mathbb{Z}}(R)$ is noetherian. $\square$

5.9 Dual of the Hopf algebra Structure

In this section we discuss the dual of the Hopf algebra defined in Section 8. Our main result is that $\mathcal{G}_\infty^{\mathbb{Z}}(R)$ is isomorphic to the Hopf algebra of distributions on the formal neighbourhood of the identity in $(W \otimes_{\mathbb{Z}} R)^\times$ (where W is the ring of Big Witt Vectors) that are supported at the identity. In this section, we require that the unit of R, denoted $\mathbf{1}$, is an element of I.

Definition 5.9.1. *Let $\mathcal{G}_\infty^{\mathbb{Z}}(R)^*$ be the (full) dual space of $\mathcal{G}_\infty^{\mathbb{Z}}(R)$. We write $Y_{\vec{\lambda}}$ for the elements of $\mathcal{G}_\infty^{\mathbb{Z}}(R)^*$ that are dual to $Z_{\vec{\lambda}}$ (i.e. $Y_{\vec{\lambda}}(Z_{\vec{\mu}}) = \delta_{\vec{\lambda},\vec{\mu}}$, where we have used the Kronecker delta).*

Proposition 5.9.2. *There is a Hopf algebra structure on $\mathcal{G}_\infty^{\mathbb{Z}}(R)^*$ coming from the dual of the multiplication, unit, comultplication and counit on $\mathcal{G}_\infty^{\mathbb{Z}}(R)$ in Theorem 5.8.1. Additionally, $\mathcal{G}_\infty^{\mathbb{Z}}(R)^*$ is isomorphic to the ring Q of formal power series in $e_r(U)$ ($r \geq 1$, $U \in I$):*

$$Q = \mathbb{Z}[[e_1(\mathbf{1}), e_2(\mathbf{1}), \ldots, e_1(U), e_2(U), \ldots, \ldots]],$$

where $Y_{\vec{\lambda}} \mapsto s_{\vec{\lambda}}$.

Proof. Consider the equations in Corollary 5.8.5. Dualising the comultiplication on $\mathcal{G}_\infty^{\mathbb{Z}}(R)$ gives a multiplication on $\mathcal{G}_\infty^{\mathbb{Z}}(R)^*$, where the coefficient of $Y_{\vec{\lambda}}$ in $Y_{\vec{\mu}}Y_{\vec{\nu}}$ coincides with the coefficient of $s_{\vec{\lambda}}$ in $s_{\vec{\mu}}s_{\vec{\nu}}$. Note that this gives a well defined multiplication because for any fixed $\vec{\lambda}$, there are only finitely many pairs $(\vec{\mu}, \vec{\nu})$ for which the coefficient above is nonzero. It immediately follows that $\mathcal{G}_\infty^{\mathbb{Z}}(R)^*$ is isomorphic to Q as an algebra, with the claimed isomorphism. From now on, we prefer to work with the symmetric function realisation. To determine the comultiplication, we write $Q \otimes Q$ with two sets of symmetric function variables: $\{\mathbf{x}_U\}_{U\in I}$ and $\{\mathbf{y}_U\}_{U\in I}$. Then Corollary 5.4.14 gives that the comultiplication in Q applied to $s_{\vec{\lambda}}$ is equal to

$$\prod_{U\in I} s_{\vec{\lambda}(U)}(\mathbf{x}_U, \mathbf{y}_U, \bigoplus_{V,W\in I} (\mathbf{x}_V \mathbf{y}_W)^{\oplus N^U_{V,W}}).$$

Note that this is well defined because any $s_{\vec{\mu}}(\mathbf{x}) s_{\vec{\nu}}(\mathbf{y})$ can occur in the image of only finitely many $s_{\vec{\lambda}}$. □

In view of the convenient description using symmetric functions, we prefer to work with $\mathcal{G}^{\mathbb{Z}}_{\infty}(R)^*$ in terms of symmetric functions (specifically, we view Q as a completion of a tensor product of copies of the ring of symmetric functions) rather than the dual of $\mathcal{G}^{\mathbb{Z}}_{\infty}(R)$.

Proposition 5.9.3. *The antipode S on Q is implicitly defined by the following equation:*

$$\sum_{U\in I} S(p_l(\mathbf{x}_U))U = \sum_{r\geq 1}(-1)^r (\sum_{U\in I} p_l(\mathbf{x}_U)U)^r$$

Proof. Because $T_l(U)$ is primitive, if S is the antipode on $\mathcal{G}^{\mathbb{Z}}_{\infty}(R)$, $S(T_l(U)) = -T_l(U)$. This implies:

$$\begin{aligned}
\sum_{\vec{\lambda}\in\mathcal{P}^I} s_{\vec{\lambda}} \otimes S(Z_{\vec{\lambda}}) &= \prod_{l=1}^{\infty} \exp\left(S\left(T_l\left(\log\left(1+\sum_{U\in I} p_l(\mathbf{x}_U)U\right)\right)\right)\right) \\
&= \prod_{l=1}^{\infty} \exp\left(-T_l\left(\log\left(1+\sum_{U\in I} p_l(\mathbf{x}_U)U\right)\right)\right) \\
&= \prod_{l=1}^{\infty} \exp\left(T_l\left(\log\left((1+\sum_{U\in I} p_l(\mathbf{x}_U)U)^{-1}\right)\right)\right) \\
&= \prod_{l=1}^{\infty} \exp\left(T_l\left(\log\left(1+\sum_{r\geq 1}(-1)^r(\sum_{U\in I} p_l(\mathbf{x}_U)U)^r\right)\right)\right).
\end{aligned}$$

From this equation, it follows that the dualised antipode (i.e. the antipode on the ring Q) on power sum symmetric functions is defined by the following equation (similarly to the proof of Proposition 5.9.2):

$$\sum_{U\in I} S(p_l(\mathbf{x}_U))U = \sum_{r\geq 1}(-1)^r(\sum_{U\in I} p_l(\mathbf{x}_U)U)^r.$$

□

Example 5.9.4. *This sum defining the antipode is not finite. For example, when $R=\mathbb{Z}$, we take $I=\{\mathbf{1}\}$; this gives*

$$S(p_l^{(\mathbf{1})}) = \sum_{r\geq 1}(-1)^r(p_l^{(\mathbf{1})})^r.$$

Proposition 5.9.5. *The comultiplication $\Delta : Q \to Q\otimes Q$ defines a formal group law F in the variables $e_i^{(U)}$ $(U\in I)$ (essentially by dualisation). The definition is*

$$F(\{e_i(\mathbf{x}_U)\}_{i\geq 1, U\in I}, \{e_i(\mathbf{y}_U)\}_{i\geq 1, U\in I}) = \{e_i(\mathbf{x}_U, \mathbf{y}_U, \bigoplus_{V,W\in I(\mathcal{C})}(\mathbf{x}_V\mathbf{y}_W)^{\oplus N_{V,W}^U})\}_{i\geq 1, U\in I}.$$

Proof. Coassociativity of the comultiplication implies associativity of the formal group law. The fact that F is addition to first order can be seen by using the fact that $e_i(\mathbf{x},\mathbf{y}) = \sum_{j=0}^i e_j(\mathbf{x})e_{i-j}(\mathbf{y})$:

$$e_i(\mathbf{x}_U, \mathbf{y}_U, \bigoplus_{V,W\in I}(\mathbf{x}_V\mathbf{y}_W)^{\oplus N_{V,W}^U}) = \sum_{a+b+c=i} e_a(\mathbf{x}_U)e_b(\mathbf{y}_U)e_c(\bigoplus_{V,W\in I}(\mathbf{x}_V\mathbf{y}_W)^{\oplus N_{V,W}^U}).$$

The only summands that give rise to (a scalar multiple of) an elementary symmetric function are those for which $c=0$ and either $a=0$ or $b=0$. So we get $e_i(\mathbf{x}_U)+e_i(\mathbf{y}_U)$ plus higher order terms, as required. □

Definition 5.9.6. *Let the affine group scheme represented by the commutative ring Λ be called the Big Witt Vectors, and denoted W. Thus for a commutative ring A, the Big Witt Vectors associated to A, denoted $W(A)$, are defined as follows. As a set, $W(A) = \hom_{alg}(\Lambda, A)$ is the set of algebra homomorphisms from the ring of symmetric functions to A. The addition is induced by the usual comultiplication on Λ, $\Delta^{(+)}(e_n) = \sum_{i=0}^n e_i\otimes e_{n-i}$. The multiplication is induced by the Kronecker comultiplication $\Delta^{(\times)}(e_n) = \sum_{\lambda\vdash n} s_\lambda\otimes s_{\lambda'}$.*

Note in particular that the underlying additive group of the Big Witt Vectors is the affine group scheme over $\mathbb{Z}$ represented by the ring of symmetric functions Λ (with the usual Hopf algebra structure). The addition and multiplication on W satisfy distributivity as discussed in Section 10 of [15].

Since Λ is the free polynomial algebra in the elementary symmetric functions e_i ($i \in \mathbb{Z}_{>0}$), an element of $W(A)$ is the same thing as a choice of an element of A for each e_i. This may be represented as an infinite sequence $(a_1, a_2, \ldots)$, where $a_i \in A$ is the image of e_i. Then, the additive identity in $W(A)$ is $(0, 0, \ldots)$, and the multiplicative identity is $(1, 0, 0, \ldots)$ (this is equivalent to Equation 10.24 of [15], although complete symmetric functions are used rather than elementary symmetric functions).

Theorem 5.9.7. *The algebra $Q = \mathcal{G}_\infty^{\mathbb{Z}}(R)^*$ is isomophic to the Hopf algebra of functions on the formal neighbourhood of the multiplicative identity of $(W \otimes_{\mathbb{Z}} R)^\times$.*

Proof. Note that the underlying additive group of $W \otimes_{\mathbb{Z}} R$ is represented (as an affine group scheme) by the ring $Q = \bigotimes_{U \in I} \Lambda^{(U)}$ (with comultiplication coming from the usual comultiplication on Λ). The multiplication comes from the following comultiplication (obtained by combining $\Delta^\times$ with the dual of the multiplication on R):

$$f^{(U)} \mapsto f\Big(\bigoplus_{V,W \in I} (\mathbf{x}_V \mathbf{y}_W)^{\oplus N_{V,W}^U}\Big).$$

The multiplicative identity in W is given by the sequence $(1, 0, 0, \ldots)$. This maps e_1 to 1 and all other elementary symmetric functions to zero, giving $(1, 0, 0, \ldots) \in W$; this is the same as evaluating a symmetric function on the variable set $\{1, 0, 0, \ldots\}$. It then follows that the maximal ideal of Λ corresponding to this point is defined by the homomorphism $\varphi : \Lambda \to \mathbb{Z}$ given by $\varphi(s_{(n)}) = 1$ and $\varphi(s_\lambda) = 0$ for λ with at least two parts. It also follows that the maximal ideal of Q corresponding to the identity in $W \otimes_{\mathbb{Z}} R$ is the ideal which is the kernel of the homomorphism $\psi : Q \to \mathbb{Z}$ given by $\psi(s_{\vec{\lambda}}) = 1$ when $\vec{\lambda}(\mathbf{1}) = (n)$ for some n, and $\vec{\lambda}(U)$ is the empty partition for $U \neq \mathbf{1}$, and $\psi(s_{\vec{\lambda}}) = 0$ otherwise. Let this maximal ideal be called J. Then, we must complete Q with respect to the ideal J. To do this, consider the automorphism θ of $\Lambda^{(\mathbf{1})}$ defined by $\theta(e_i^{(\mathbf{1})}) = e_i^{(\mathbf{1})} - e_{i-1}^{(\mathbf{1})} + e_{i-2}^{(\mathbf{1})} - \cdots$. Note that $\theta(J)$ is the ideal of positive degree elements of Q under the usual grading (it suffices to notice that evaluating $\theta(e_i^{(\mathbf{1})})$ at the variable set $\{1, 0, 0, \ldots\}$ gives zero for all i). Completing with respect to this ideal, we obtain:

$$\widehat{\bigotimes}_{U\in I}\Lambda^{(U)} = \mathbb{Z}[[e_1^{(\mathbf{1})}, e_2^{(\mathbf{1})}, \ldots, e_1^{(U)}, e_2^{(U)}, \ldots, \ldots]] = Q.$$

This algebra is isomorphic to $\mathcal{G}_\infty^{\mathbb{Z}}(R)^*$ by letting $Y_{\vec{\lambda}} \mapsto s_{\vec{\lambda}}$. To check that this is an isomorphism of Hopf algebras, it remains to check that the comultiplication and antipode agree (here we must take into account the twist by θ). The comultiplication is determined by taking the comultiplication on Q and twisting it by θ. The comultiplication on Q is defined by the following formula (where we interpret $Q \otimes Q$ as being complete symmetric functions in the variables $\mathbf{y}_U$ and $\mathbf{z}_U$ for $U \in I$):

$$f(\mathbf{x}_U) \mapsto f(\bigoplus_{V,W\in I(\mathcal{C})} (\mathbf{y}_V \mathbf{z}_W)^{\oplus N_{V,W}^U}).$$

Thus, θ has the effect of changing the variable sets where either V or W is $\mathbf{1}$. For examples $\mathbf{y_1}$ becomes $\{1\} \cup \mathbf{y_1}$ (we have appended 1 to the variable set), and so $\mathbf{y_1 z}_W$ becomes $\mathbf{y_1 z}_W \cup \mathbf{z}_W$. Hence, we obtain a formula for the comultiplication twisted by θ:

$$f(\mathbf{x}_U) \mapsto f(\mathbf{y}_U, \mathbf{z}_U, \bigoplus_{V,W\in I} (\mathbf{y}_V \mathbf{z}_W)^{\oplus N_{V,W}^U}).$$

This is in agreement with the formula in Corollary 5.8.5. Because a bialgebra admits at most one antipode, it automatically follows that the antipodes agree. This completes the proof. □

Remark 5.9.8. *The isomorphism constructed in Theorem 5.9.7 depends on I.*

Theorem 5.9.9. *The algebra $\mathcal{G}_\infty^{\mathbb{Z}}(R)$ is isomorphic to the Hopf algebra of distributions on the formal neighbourhood of the identity in $(W \otimes_{\mathbb{Z}} R)^\times$ that are supported at the identity.*

Proof. It is well known that distributions supported at a point are given by differential operators. Moreover, the Hopf algebra structure comes from the group structure on the formal neighbourhood, and is dual to that of functions on the formal neighbourhood. Thus it suffices to show that differential operators on $Q = \mathcal{G}_\infty^{\mathbb{Z}}(R)^*$ give rise to $\mathcal{G}_\infty^{\mathbb{Z}}(R)$. We think of $\mathcal{G}_\infty^{\mathbb{Z}}(R)^*$ as $\mathbb{Z}[[e_1^{(\mathbf{1})}, e_2^{(\mathbf{1})}, \dots, e_1^{(U)}, e_2^{(U)}, \dots, \dots]]$, meaning that differential operators are given by linear combinations of the functionals $f_{\vec{\lambda}}$ which extract the coefficient of $s_{\vec{\lambda}}$ in an element of $\mathbb{Z}[[e_1^{(\mathbf{1})}, e_2^{(\mathbf{1})}, \dots, e_1^{(U)}, e_2^{(U)}, \dots, \dots]]$. The comultiplication on the $f_{\vec{\lambda}}$ has structure constants equal to those defined by multiplication of the $s_{\vec{\lambda}}$:

$$\Delta(f_{\vec{\lambda}}) = \sum_{\vec{\mu},\vec{\nu} \in \mathcal{P}^I} \left(\prod_{U \in I} c^{\vec{\lambda}(U)}_{\vec{\mu}(U),\vec{\nu}(U)}\right) f_{\vec{\mu}} \otimes f_{\vec{\nu}}.$$

In particular, when $\vec{\lambda}$ is a basic hook of size i associated to $U \in I$, we obtain the same comultiplication as for $e_i(U)$ in $\mathcal{G}_\infty^{\mathbb{Z}}(R)$. The multiplication is obtained by dualising the comultiplication on $\mathcal{G}_\infty^{\mathbb{Z}}(R)^*$. This is the same as the formal group law in Proposition 5.9.5. It then follows that the multiplications agree. Finally, because bialgebras admit at most one antipode, the equality of antipodes is automatic. □

Remark 5.9.10. *Although we only considered rings R that are free as $\mathbb{Z}$-modules, in view of Theorem 5.9.9, we may extend the definition to arbitrary R in the following way. Define $\mathcal{G}_\infty^{\mathbb{Z}}(R)$ as the Hopf algebra of distributions on the formal neighbourhood of the identity in $(W \otimes_{\mathbb{Z}} R)^\times$ that are supported at the identity. This definition is functorial in R.*

Chapter 6

Littlewood Complexes for Symmetric Groups

6.1 Introduction

In this chapter, we work over an arbitrary field of characteristic zero, k. Let V be a vector space, and write $T(V)$ for the tensor algebra of V, and $\mathrm{Sym}(V)$ for the symmetric algebra of V. Because $\mathrm{Sym}(V)$ is a quotient of $T(V)$ as an algebra, $\mathrm{Sym}(V)$ is in particular a module over $T(V)$. In fact, $\mathrm{Sym}(V)$ is a module for $T(V)^{\otimes n}$, for any n, where each tensor factor acts in the usual way. The main technical result of this paper is the computation of

$$\mathrm{Tor}_i^{T(V)^{\otimes n}}(k, \mathrm{Sym}(V))$$

as a $GL(V) \times S_n$ module (where the symmetric group S_n acts by permutation on $T(V)^{\otimes n}$, and trivially on $\mathrm{Sym}(V)$). These Tor groups are the homology of a certain Chevalley-Eilenberg complex, whose $GL(V)$-isotypic components we call Littlewood complexes and denote $\mathcal{L}^\lambda_\bullet$. The terms of $\mathcal{L}^\lambda_\bullet$ are representations of S_n restricted from $GL_n(k)$ (where S_n embeds as permutation matrices), which may be thought of as an extension to S_n of the work of Sam, Snowden, and Weyman on Littlewood Complexes for classical groups [29]. Although there are substantial similarities to their work (e.g. computing Tor groups with Koszul complexes), there are also significant differences (e.g. working with noncommutative algebras).

For large n, the (co)homology of $\mathcal{L}^\lambda_\bullet$ is the Specht module $\mathcal{S}^{\lambda[n]}$ concentrated in degree zero. The complex $\mathcal{L}^\lambda_\bullet$ is defined for all $n \in \mathbb{Z}_{\geq 0}$ (in fact also $n = \infty$), and the (co)homology is either zero, or given by an irreducible representation of S_n concentrated in a single degree, which can be computed using an algorithm similar to the Borel-Weil-Bott theorem.

The complexes $\mathcal{L}^\lambda_\bullet$ yield injective resolutions of simple objects in the category $\mathrm{Rep}(\mathfrak{S})$ of "algebraic" representations of S_∞ studied by Sam and Snowden in [27] and also [26]. Additionally, the Euler characteristic of the complex recovers an identity of Assaf and Speyer [2] about stable Specht polynomials (these are the same as the irreducible character basis of Orellana and Zabrocki [21]), and allows us to determine how these polynomials behave for all n, in particular, when n is small.

The structure of the paper is as follows. In Section 2, we express some Tor groups computed in [29] combinatorially. We then compute the Tor groups mentioned in the introduction in Section 3. In Section 4 we define the Littlewood complex $\mathcal{L}^\lambda_\bullet$, compare it to the version for classical groups in [29], and explain its connection to $\mathrm{Rep}(S_\infty)$. In Section 5, we explain how $\mathcal{L}^\lambda_\bullet$ categorifies stable Specht polynomials, and hence determine evaluations of stable Specht polynomials for small n.

6.2 Some Tor Groups

Let U, V, W be vector spaces, and let $d = \dim(U)$. Say that a pair of partitions (λ, μ) is admissible if $l(\lambda) + l(\mu) \leq d$. In that case $(\lambda_1, \ldots, \lambda_r, 0, \ldots, 0, -\mu'_s, \ldots, \mu'_1)$ (where $r = l(\lambda)$ and $s = l(\mu)$) is a well-defined weight of $GL(U)$, and we denote the associated highest-weight irreducible representation $\mathbb{S}^{[\lambda,\mu]}(U)$. For example, when μ is the empty partition, we have $\mathbb{S}^{[\lambda,\mu]} = \mathbb{S}^\lambda(U)$.

Consider the algebras $A = \mathrm{Sym}(V \otimes W)$ and $B = \mathrm{Sym}(V \otimes U \oplus W \otimes U^*)$. Note that B

is an A-algebra via the canonical map

$$V \otimes W \to V \otimes U \otimes W \otimes U^* \subseteq \mathrm{Sym}^2(V \otimes U \oplus W \otimes U^*),$$

hence B is in particular an A-module. Corollary 5.15 of [29] states that as a representation of $GL(V) \times GL(W) \times GL(U)$,

$$\mathrm{Tor}_i^A(B, k) = \bigoplus_{i_d(\lambda,\mu)=i} \mathbb{S}^\lambda(V) \otimes \mathbb{S}^\mu(W) \otimes \mathbb{S}^{[\tau_d(\lambda,\mu)]}(U),$$

where the quantities $i_d(\lambda, \mu)$ and $\tau_d(\lambda, \mu)$ are defined via the following recursion (see Subsection 5.4 of [29], but note that the notation is slightly different).

Definition 6.2.1. *If (λ, μ) is admissible, set $i_d(\lambda, \mu) = 0$ and $\tau_d(\lambda, \mu) = (\lambda, \mu)$. Otherwise, consider the Young diagrams of λ and μ. Let R_λ and R_μ be the border strips of length $l(\lambda) + l(\mu) - d - 1$ starting in the intersection of the first column and final row of λ and μ respectively, if they exist. If both exist, are nonempty, and $\lambda\backslash R_\lambda$ and $\mu\backslash R_\mu$ are partitions, put $\tau_d(\lambda, \mu) = \tau_d(\lambda\backslash R_\lambda, \mu\backslash R_\mu)$, and*

$$i_d(\lambda, \mu) = c(R_\lambda) + c(R_\mu) - 1 + i_d(\lambda\backslash R_\lambda, \mu\backslash R_\mu),$$

where $c(R)$ indicates the number of columns that the border strip R intersects. If either border strip fails to exist, or is empty, or either $\lambda\backslash R_\lambda$ or $\mu\backslash R_\mu$ is not a partition, set $i_d(\lambda, \mu) = \infty$, and leave $\tau_d(\lambda, \mu)$ undefined.

The case $d = 1$ will be important to us; we will characterise the functions i_1 and τ_1. For this, we need a special case of the Bott algorithm.

Definition 6.2.2. *Given a partition λ, and $n \in \mathbb{Z}$, we define $\delta_n(\lambda)$ and $\lambda[n]$ as follows. Let $r \geq l(\lambda) + 1$, and consider the vector $v = (n - |\lambda|, \lambda_1, \ldots, \lambda_{r-1})$ (where $\lambda_i = 0$ if $i > l(\lambda)$). Let $\rho = (r - 1, r - 2, \ldots, 0)$ be the Weyl vector. If $v + \rho$ has no repeated entries, there is a unique permutation $w \in S_r$ such that $w(v + \rho)$ has decreasing entries. If $w(v + \rho) - \rho$ has* *non-negative entries, it defines a partition, and we set* $\lambda[n] = w(v+\rho) - \rho$ *and* $\delta_n(\lambda) = l(w)$ *(using the usual length function on* S_r*). If* $w(v+\rho)-\rho$ *has a negative entry, or if* $v+\rho$ *has a repeated entry, take* $\delta_n(\lambda) = \infty$ *and leave* $\lambda[n]$ *undefined.*

Remark 6.2.3. *We make a few observations.*

1. *This definition is independent of the value of* r*.*

2. *When* n *is sufficiently large,* $(n-|\lambda|, \lambda_1, \ldots, \lambda_{r-1}) + \rho$ *is decreasing, so* $\delta_n(\lambda) = 0$*, and* $\lambda[n]$ *is the partition obtained by appending a long top row to* λ *so that the total size of the resulting partition is* n*.*

3. *If we disregard the first entry of* $v+\rho$*, the entries are already ordered, so* $\delta_n(\lambda)$ *counts the number of indices* i *such that* $\lambda_i + r - (i+1) > n - |\lambda| + r - 1$*, which reduces to* $\lambda_i - i > n - |\lambda|$*.*

When $d = 1$, a weight is an integer n. We restrict ourselves to the case $n \geq 0$, so in an admissible pair of partitions (λ, μ), we take μ to be the trivial partition.

Theorem 6.2.4. *We have that* $\tau_1(\lambda, \mu)$ *is defined and equal to* n *if and only if there is a partition* ν *such that* $\lambda = \nu[n+|\nu|]$ *and* $\mu = \nu'$*. In this case,* $i_1(\lambda, \mu) = |\nu| - \delta_{n+|\nu|}(\lambda)$*.*

Proof. We recall some facts about a partition ν. Let $r, s \in \mathbb{Z}_{\geq 0}$ such that $r \geq l(\nu)$ and $s \geq l(\nu')$. We take them to be sufficiently large so that subsequent calculations are well defined. We write $\rho_m = (m-1, m-2, \ldots, 0)$ for the Weyl vector of size m.

1. The disjoint union

$$\{(\nu+\rho_r)_i\}_{i=1}^{r} \coprod \{r+s-1-(\nu'+\rho_s)_j\}_{j=1}^{s}$$

   is equal to $\{0, 1, \ldots, r+s-1\}$ (see 1.7 in Chapter 1 Section 1 of [18]).

2. Let σ be obtained from ν by adding a border strip of size p. Then $\sigma + \rho_r$ is obtained from $\nu + \rho_r$ by adding p to an entry, and rearranging the entries in decreasing order.

(See Example 8 (a) in Chapter 1 Section 1 of [18].) Moreover, suppose the border strip starts in the i-th row and ends in the j-th row, with $i > j$. Then, the i-th entry of $\nu + \rho_r$ was the one that p was added to, and after reordering, it became the j-th entry of $\sigma + \rho_r$.

3. If R is a border strip and $r(R)$ is the number of rows that R intersects, we have the equation $|R| = r(R) + c(R) - 1$. This is clear when $|R| = 1$ and incrementing the size of $|R|$ extends R into a new row or column.

4. If $\nu[n + |\nu|]$ is defined, then $l(\nu[n + |\nu|]) = l(\nu) + 1$.

The proof is by induction on the number of steps of the recursion defining $\tau_n(\lambda, \mu)$ required to reach the base case of an admissible pair. The base case of our induction is zero steps, in which case the statement holds with ν being the trivial partition.

Now let us consider how border strips may be removed in accordance with the recursion. To prove the "if" direction, suppose we are given $(\nu[n+|\nu|], \nu')$. According to the algorithm, we must remove border strips of size

$$|R| = l(\nu[n + |\nu|]) + l(\nu') - 1 - 1 = l(\nu) + \nu_1 - 1.$$

In the case of ν', this is the length of the unique largest border strip since it intersects every row and column of ν' by fact (3). By fact (2), removing this border strip amounts to subtracting $|R|$ from the first entry of $\nu' + \rho_s$, turning it from $\nu'_1 + s - 1 = l(\nu) + s - 1$ into $s - \nu_1$. We use fact (1) to turn this into a statement about ν rather than ν'.

The second part of the set partition of $\{0, 1, \ldots, r + s - 1\}$ contained

$$r + s - 1 - (l(\nu) + s - 1) = r - l(\nu)$$

before removing the border strip, while afterwards, that element was replaced by

$$r + s - 1 - (s - \nu_1) = \nu_1 + r - 1.$$

Consequently, the first part of the set partition must have contained ν_1+r-1 before removing the border strip, and $r - l(\nu)$ afterwards. Indeed, this can be achieved by subtracting $|R|$ from the first entry of $\nu + \rho_r$. Let us write σ for the partition obtained by removing this border strip from ν. We must check that removing a particular border strip of length $|R|$ turns $\nu[n + |\nu|]$ into $\sigma[n + |\sigma|]$. We do this by verifying that adding the border strip to $\sigma[n + |\sigma|]$ yields $\nu[n + |\nu|]$.

The entries of $\sigma[n + |\sigma|] + \rho_{r+1}$ are (in some order)

$$n + r, r - l(\nu), \nu_2 + r - 2, \nu_3 + r - 3, \ldots, \nu_r.$$

We check that adding the border strip by adding $|R|$ to the entry $r - l(\nu)$ is permitted by the algorithm. For this to be the case, the border strip must start in the intersection of the first column and bottom row of the resulting partition diagram. This means that it must be constructed by adding $|R|$ to an entry corresponding to a part of $\sigma[n + |\sigma|]$ of size zero. We check that the entry $r - l(\nu)$ satisfies this requirement. Note that

$$\nu_{l(\nu)} + r - l(\nu) > r - l(\nu) > \nu_{l(\nu)+1} + r - (l(\nu) + 1) = r - l(\nu) - 1$$

as $\nu_{l(\nu)} > 0$ and $\nu_{l(\nu)+1} = 0$ by definition. Because $n \geq 0$, $n + r > r - l(\nu)$, so it follows that $r - l(\nu)$ is smaller than $n + r$ and exactly $l(\nu) - 1$ other entries. Hence after sorting, the $l(\nu)$-th entry is $r - l(\nu)$. When we subtract $(\rho_{r+1})_{l(\nu)} = r - l(\nu)$, we obtain zero, as required.

To complete the proof of the "if" direction, we must check that $i_n(\lambda, \mu)$ behaves as claimed. The amount by which $i_n(\nu[n + |\nu|], \nu')$ changes in this recursion step is

$$c(R_{\nu[n+|\nu|]}) + c(R_{\nu'}) - 1 = |R| - r(R_{\nu[n+|\nu|]}) + c(R_{\nu'}) = |R| - (r(R_{\nu[n+|\nu|]}) - l(\nu)),$$

where we use fact (3) and the fact that $R_{\nu'}$ intersects all $l(\nu)$ columns of ν'. Note that $|R|$ is precisely $|\nu| - |\sigma|$, so it suffices to check that $\delta_{n+|\nu|}(\nu) - \delta_{n+|\sigma|}(\sigma) = r(R_{\nu[n+|\nu|]}) - l(\nu)$. Recall that $R_{\nu[n+|\nu|]}$ was added to $\sigma[n + |\sigma|]$ by adding $|R|$ to the entry of $\sigma[n + |\sigma|] + \rho_{r+1}$ at index $l(\nu) + 1$ (which was equal to $r - l(\nu)$). This entry became $\nu_1 + r - 1$ which is either the second largest entry, if $n + r > \nu_1 + r - 1$, or the largest entry, if $n + r < \nu_1 + r - 1$. Fact (2) tells us that $r(R_{\nu[n+|\nu|]}) - l(\nu)$ is equal to 0 if $n + r > \nu_1 + r - 1$ (i.e. $n > \nu_1 - 1$), and equal to 1 if $n + r < \nu_2 + r - 1$ (i.e. $n < \nu_1 - 1$). However, $\delta_{n+|\nu|}(\nu)$ counts the number of i such that $\lambda_i - i > n$, while $\delta_{n+|\sigma|}(\sigma)$ counts only the number of such i with $i > 1$. The difference is therefore 0 if $n > \lambda_1 - 1$ and 1 if $n < \lambda_1 - 1$, as required.

The proof of the "only if" part is essentially the same, and so we omit it. □

6.3 Calculating the Homology

In this section, we prove our main theorem. First we begin with a somewhat simpler version that has independent utility.

Proposition 6.3.1. *For $0 \leq i \leq n$:*

$$\mathrm{Ext}^i_A(k,k) = \mathrm{Ind}^{S_n}_{S_i \times S_{n-i}}((V^*)^{\otimes i} \otimes \varepsilon_i \boxtimes k),$$

where ε_i is the sign representation of S_i, and k is the trivial representation of S_{n-i}. Further, for $i > n$, $\mathrm{Ext}^i_A(k,k)$ vanishes.

Proof. From the discussion of Koszul duality in Chapter 3, we have that $\mathrm{Ext}^{\bullet}_{T(V)}(k,k) = T(V)^! = k \oplus V^*$. Noting that A is the n-fold tensor product of $T(V)$, we apply the Künneth theorem in an S_n-equivariant way. The sign representation ε_i arises because of the Koszul sign rule (cohomology is only graded commutative). □

We will have to work considerably harder for the following theorem.

Theorem 6.3.2. *Let $A = T(V)^{\otimes n}$ as before. Then,*

$$\mathrm{Tor}^A_i(k, \mathrm{Sym}(V)) = \bigoplus_{|\lambda| - \delta_n(\lambda) = i} \mathbb{S}^{\lambda'}(V) \otimes \mathcal{S}^{\lambda[n]}$$

as representations of $GL(V) \times S_n$.

Proof. Because $T(V)$ is Koszul with $T(V)_1 = V$, it follows that A is also Koszul, with generating set $A_1 = V^{\oplus n} = V \otimes k^n$. We will write $v^{(r)}$ to indicate the vector $v \in V$ inside the r-th summand. The Koszul complex for A is the n-th tensor power of the Koszul complex for $T(V)$:

$$0 \to T(V) \otimes V \to T(V) \to 0.$$

Writing the total space of this complex as $T(V) \otimes (k \oplus V)$, where $T(V)$ and k are in degree zero and v is in degree one, the n-th tensor power is

$$A \otimes (k \oplus V)^{\otimes n}.$$

The differential is defined as follows. Let $\{v_i\}$ be a basis of V, and $\{v_i^*\}$ be the dual basis of V^*. Then,

$$d = \sum_{r,i} v_i^{(r)} \otimes v_i^{*(r)},$$

where $(v_i^*)^{(r)}$ acts on the r-th tensor factor of $(k \oplus V)^{\otimes n}$ by annihilating k, and mapping V to k in the natural way.

To calculate $\mathrm{Tor}_i^A(k, \mathrm{Sym}(V))$, we apply the functor $- \otimes_A \mathrm{Sym}(V)$ to the projective resolution. We obtain the chain groups

$$\mathrm{Sym}(V) \otimes (k \oplus V)^{\otimes n}$$

and differential

$$d = \sum_i v_i \otimes \left(\sum_r v_i^{*(r)} \right),$$

because $v^{(r)} \in A$ acts on $\mathrm{Sym}(V)$ by multiplying by v.

Fix an auxiliary vector space W of dimension at least n, and let us apply the Schur-Weyl functor $(W^{\otimes n} \otimes -)^{S_n}$ to the complex. This turns it from a complex of $GL(V) \times S_n$-modules into a complex of $GL(V) \times GL(W)$-modules. We obtain

$$\begin{aligned} & \mathrm{Sym}(V) \otimes \left(W^{\otimes n} \otimes (k \oplus V)^{\otimes n}\right)^{S_n} \\ = \; & \mathrm{Sym}(V) \otimes \left((W \oplus W \otimes V)^{\otimes n}\right)^{S_n} \\ = \; & \mathrm{Sym}(V) \otimes \mathrm{Sym}^n(W \oplus W \otimes V) \\ = \; & \mathrm{Sym}(V) \otimes \bigoplus_i \mathrm{Sym}^{n-i}(W) \otimes \mathrm{Sym}^i(W \otimes V), \end{aligned}$$

where $W \otimes V$ is an odd superspace, so its i-th symmetric power is equal to the i-th exterior power of the underlying vector space. We may therefore write the i-th chain group as

$$\mathrm{Sym}(V) \otimes \bigoplus_i \mathrm{Sym}^{n-i}(W) \otimes \bigwedge^i (W \otimes V).$$

We observe that the action of $\sum_r v_i^{*(r)}$ becomes "contracting with v_i^*", meaning

$$\sum_j w_j \otimes (w_j^* \otimes v_i^*),$$

where w_j is a basis of W and w_j^* is the dual basis of W^*. Thus the differential becomes

$$\sum_{i,j} v_i \otimes w_j \otimes (w_j^* \otimes v_i^*).$$

At this point it is convenient to consider all n simultaneously. Let us take the direct sum of these complexes over all n, which has the effect of replacing $\mathrm{Sym}^{n-i}(W)$ by $\mathrm{Sym}(W)$. After identifying $\mathrm{Sym}(W) \otimes \mathrm{Sym}(V) = \mathrm{Sym}(W \oplus V)$, we recognise this as the Koszul complex computing

$$\mathrm{Tor}_i^{\mathrm{Sym}(W \otimes V)}(k, \mathrm{Sym}(W \oplus V)),$$

where the action of $w \otimes v \in \mathrm{Sym}(W \otimes V)$ on $\mathrm{Sym}(W \oplus V)$ is my multiplying by wv. This is the $d = 1$ case of what is computed by [29] as discussed in Section 2. Applying Theorem

6.2.4 gives

$$\mathrm{Tor}_i^{\mathrm{Sym}(W\otimes V)}(k, \mathrm{Sym}(W \oplus V)) = \bigoplus_{m\in\mathbb{Z}_{\geq 0}} \bigoplus_{|\nu|-\delta_{m+|\nu|}(\nu)=i} \mathbb{S}^{\nu'}(V) \otimes \mathbb{S}^{\nu[m+|\nu|]}(W).$$

We now undo Schur-Weyl duality, which has the effect of picking out only the Schur functors of degree n in W, and converting them back in to representations of S_n. This immediately yields the result. □

6.4 A Littlewood Complex for Symmetric Groups

Recall that the tensor algebra $T(V)$ is the universal enveloping algebra of the free Lie algebra $L(V)$. Therefore $A = T(V)^{\otimes n}$ is the universal enveloping algebra of $L(V)^{\oplus n} = L(V) \otimes k^n$. So, we have computed the Lie algebra homology

$$H_i(L(V) \otimes k^n; \mathrm{Sym}(V)) = \mathrm{Tor}_i^A(k, \mathrm{Sym}(V)).$$

Because the action of S_n on $L(V)^{\oplus n} = L(V) \otimes k^n$ by permutation of summands (equivalently, permutation on k^n) preserves the Lie algebra structure, this passes to an action of S_n on the Lie algebra homology (as before, S_n acts trivially on $\mathrm{Sym}(V)$). Our proof relied on computing the relevant Koszul complex, but we could have also considered the Chevalley-Eilenberg complex. Here, the chain groups are

$$\begin{aligned} &\bigwedge^i (L(V) \otimes k^n) \otimes \mathrm{Sym}(V) \\ = &\bigoplus_{\mu\vdash i} \mathbb{S}^{\mu'}(L(V)) \otimes \mathbb{S}^{\mu}(k^n) \otimes \mathrm{Sym}(V), \end{aligned}$$

where the action of S_n on $\mathbb{S}^\lambda(k^n)$ is implicitly restricted from $GL_n(k)$; S_n is embedded in $GL_n(k)$ as permutation matrices.

By applying $\mathrm{Hom}_{GL(V)}(\mathbb{S}^{\lambda'}(V), -)$, this provides a resolution of $\mathcal{S}^{\lambda[n]}$ by modules restricted from $GL_n(k)$, as we now explain. If we define the multiplicity space

$$M(\lambda,\mu) = \mathrm{Hom}_{GL(V)}(\mathbb{S}^{\lambda'}(V), \mathbb{S}^{\mu'}(L(V)) \otimes \mathrm{Sym}(V)),$$

we may write the i-th chain module as

$$\bigoplus_{\mu \vdash i} M(\lambda,\mu) \otimes \mathbb{S}^{\mu}(k^n).$$

Because $L(V)$ is equal to V plus terms of higher degree as representations of $GL(V)$, $\mathbb{S}^{\mu'}(L(V))$ is isomorphic to $\mathbb{S}^{\mu'}(V)$ plus terms of higher degree. Hence $M(\lambda,\mu) = 0$ unless $|\mu| \leq |\lambda|$, and when $|\mu| = |\lambda|$, $M(\lambda,\mu)$ is one-dimensional if $\mu = \lambda$ and zero otherwise. Thus our complex becomes

$$0 \to \mathbb{S}^{\lambda}(k^n) \to \bigoplus_{\mu \vdash |\lambda|-1} M(\lambda,\mu) \otimes \mathbb{S}^{\mu}(k^n) \to \cdots \to \bigoplus_{\mu \vdash 0} M(\lambda,\mu) \otimes \mathbb{S}^{\mu}(k^n) \to 0.$$

The homology is $\mathcal{S}^{\lambda[n]}$, concentrated in degree $|\lambda| - \delta_n(\lambda)$ or zero, if $\delta_n(\lambda) = \infty$.

Definition 6.4.1. *Let us instead view the above chain complex as a cochain complex (i.e. change the degree indexing from descending to ascending). We call this cochain complex the Littlewood complex associated to the partition λ, and denote it $\mathcal{L}^{\lambda}_{\bullet}$.*

The cohomology of $\mathcal{L}^{\lambda}_{\bullet}$ is $\mathcal{S}^{\lambda[n]}$, concentrated in degree $\delta_n(\lambda)$ or zero, if $\delta_n(\lambda) = \infty$.

The name "Littlewood complex" is taken from [29], where it has the following meaning. Let V be a vector space, and let $G(V) \subseteq GL(V)$ be either the symplectic group on V (for a choice of symplectic form), or the orthogonal group on V (for a choice of orthogonal form). An irreducible representation of $G(V)$ is not typically a representation of $GL(V)$, however it turns out that that any irreducible representation of $G(V)$ may be resolved by representations that extend to $GL(V)$. The Littlewood complex $L^{\lambda}_{\bullet}$ (associated to a partition λ defining an irreducible representation) is the minimal such resolution.

Key features of the Littlewood complexes $L^{\lambda}_{\bullet}$ discussed in Section 2 of [29] include:

1. One can define $L^\lambda_\bullet$ even if $l(\lambda) > \dim(V)$, in which case the homology is either zero, or is equal to an irreducible representation of $G(V)$ concentrated in a single degree. There is a "modification rule" for determining the irreducible and the degree.

2. The modification rule is similar to the Borel-Weil-Bott theorem; it involves a dotted Weyl group action.

3. The Littlewood complex $L^\lambda_\bullet$ may be viewed as an isotypic component of a $GL(E)$-equivariant (for an auxiliary vector space E) Koszul complex.

4. The homology may be computed using algebraic geometry and the cohomology of certain vector bundles on Grassmannians.

5. There is a category of "algebraic" representations of $G(\infty)$ (i.e. the infinite symplectic group, or infinite orthogonal group), denoted $\mathrm{Rep}(G(\infty))$. There is a specialisation functor

$$\Gamma_V : \mathrm{Rep}(G(\infty)) \to \mathrm{Rep}(G(V)).$$

   The Littlewood complex $L^\lambda_\bullet$ computes the derived functors on the simple objects of $\mathrm{Rep}(G(\infty))$ which are indexed by partitions λ of any size. See Subsection 1.3 of [29] for more details.

In our case, the first two points apply verbatim. The third point is slightly different. Our Littlewood complex is an isotypic component of a Chevalley-Eilenberg complex. In the case of an abelian Lie algebra, the Chevalley-Eilenberg complex reduces to a Koszul complex for a polynomial algebra. Hence our situation may be viewed as analogous in the noncommutative world. The fourth point has no clear comparison. It would be interesting to have a "geometric" realisation of our Littlewood complex. There is a version of the fifth point, which we now briefly explain.

There is a category $\mathrm{Rep}(\mathfrak{S})$ of "algebraic" representations of the infinite symmetric group which consists of permutations of $\mathbb{Z}_{>0}$ fixing all but finitely many elements. We direct the interested reader to Section 6 of [27] for more information about this category. Objects of

Rep($\mathfrak{S}$) are subquotients of direct sums of tensor powers of k^∞ (the permutation representation of S_∞). Unlike its finite counterparts, this category is not semisimple.

The simple objects are indexed by partitions λ of any size, and are directed limits of $\mathcal{S}^{\lambda[n]}$ as $n \to \infty$. The objects $\mathbb{S}^\mu(k^\infty)$ are injective, although they are not indecomposable. We may set $n = \infty$ in the Littlewood complex, which becomes the directed limit of $\mathcal{L}^\lambda_\bullet$ for finite n under the inclusions $k^n \to k^{n+1}$ defined by appending a zero entry to a vector. The cohomology is the directed limit of $\mathcal{S}^{\lambda[n]}$ in degree zero. So the $n = \infty$ Littlewood complex is an injective resolution of the simple object of Rep($\mathfrak{S}$) indexed by λ.

There is a specialisation functor

$$\Gamma_n : \mathrm{Rep}(\mathfrak{S}) \to \mathrm{Rep}(S_n),$$

such that $\Gamma_n(k^\infty) = k^n$ and Γ_n is a left-exact tensor functor. Applying this functor to the $n = \infty$ Littlewood complex yields $\mathcal{L}^\lambda_\bullet$ for finite n. Therefore $\mathcal{L}^\lambda_\bullet$ provides a computation of the derived specialisation functors on simple objects. This was originally done in Proposition 7.4.3 of [26] using the indecomposable injective objects instead of $\mathbb{S}^\mu(k^\infty)$.

6.5 Application to $\mathcal{F}$-modules

We explain how constructing the analogous version of the Littlewood complex of Section 6.4 that computes the Ext groups in Proposition 6.3.1 yields projective resolutions in the category of $\mathcal{F}$-modules (a variant of the category of FI-modules).

Writing down the Chevalley-Eilenberg complex computing Lie algebra cohomology, the chain groups are

$$\hom_k\left(\bigwedge^i (L(V)^{\oplus n}, k\right).$$

As noted before, as a representation of $GL(V)$, the chain groups are concentrated in degrees $\leq -i$. Recall that the cohomology is given by Proposition 6.3.1:

$$H^i(L(V) \otimes k^n; k) = \mathrm{Ind}^{S_n}_{S_i \times S_{n-i}}((V^*)^{\otimes i} \otimes \varepsilon_i \boxtimes k),$$

which is precisely of degree $-i$ as a representation of $GL(V)$. Hence, to calculate the i-th cohomology, we may truncate the complex at the i-th term, and take only the degree $-i$ component as a representation of $GL(V)$, which we denote with a subscript "$-i$". This gives us the following result.

Proposition 6.5.1. *The complex (with differential inherited from the Chevalley-Eilenberg complex)*

$$\begin{array}{cccccc} 0 \leftarrow & \hom_k\left(\bigwedge^i(L(V)\otimes k^n),k\right)_{-i} \leftarrow & \hom_k\left(\bigwedge^{i-1}(L(V)\otimes k^n),k\right)_{-i} \leftarrow & \cdots \\ \leftarrow & \hom_k\left(\bigwedge^1(L(V)\otimes k^n),k\right)_{-i} \leftarrow & \hom_k\left(\bigwedge^0(L(V)\otimes k^n),k\right)_{-i} \leftarrow & 0 \end{array}$$

has cohomology $\mathrm{Ind}_{S_i\times S_{n-i}}^{S_n}((V^*)^{\otimes i}\otimes\varepsilon_i\boxtimes k)$ *on the far left, and zero elsewhere.*

Let us take the multiplicity space of the $GL(V)$-irreducible $\mathbb{S}^{\mu'}(V^*)$.

Proposition 6.5.2. *The* $\mathbb{S}^{\mu'}(V^*)$ *multiplicity space in* $\mathrm{Ext}^i_A(k,k)$ *is*

$$M_n^\mu = \mathrm{Ind}_{S_i\times S_{n-i}}^{S_n}(\mathcal{S}^\mu\boxtimes k).$$

Proof. We apply Schur-Weyl duality to Proposition 6.3.1, noting that $S^\lambda\otimes\varepsilon_i = S^{\lambda'}$:

$$\mathrm{Ext}^i_A(k,k) = \mathrm{Ind}_{S_i\times S_{n-i}}^{S_n}((V^*)^{\otimes i}\otimes\varepsilon_i\boxtimes k) = \mathrm{Ind}_{S_i\times S_{n-i}}^{S_n}(\bigoplus_{\lambda\vdash i}\mathbb{S}^\lambda(V^*)\otimes\mathcal{S}^{\lambda'}\boxtimes k).$$

Hence, the $\mathbb{S}^{\mu'}(V^*)$ multiplicity space is $\mathrm{Ind}_{S_i\times S_{n-i}}^{S_n}(\mathcal{S}^\mu\boxtimes k)$. □

Because the complex constructed in Proposition 6.5.1 is $GL(V)$ equivariant, taking cohomology commutes with taking the $\mathbb{S}^{\mu'}(V^*)$ multiplicity space. We immediately obtain the following.

Theorem 6.5.3. *Consider the complex of* S_n*-representations*

$$\hom_{GL(V)}\left(\mathbb{S}^{\mu'}(V^*),\hom_k\left(\bigwedge^i(L(V)\otimes k^n),k\right)\right)$$

for $|\mu| \geq i \geq 0$ with maps induced by the differential of the Chevalley-Eilenberg complex. This is a resolution of M_n^μ by representations restricted from $GL_n(k)$.

Proof. This is immediate from Proposition 6.5.2 and Proposition 6.5.1. □

Let $\mathcal{F}$ denote the category of finite sets. An $\mathcal{F}$-module is a functor from $\mathcal{F}$ to vector spaces over a fixed field. These were introduced in [34], and their homological algebra was studied over $\mathbb{Q}$.

An $\mathcal{F}$-module consists of a S_n-module for each n together with suitably compatible maps between them. This is because the image of an n-element set carries an action of $\mathrm{Aut}(\{1, 2, \ldots, n\}) = S_n$. When μ is a partition different from (1^k), i.e. not a single column, M_n^μ (considered for fixed μ but varying n) defines an irreducible $\mathcal{F}$-module, by demanding that an n-element set in $\mathcal{F}$ map to M_n^μ (see Theorem 5.5 of [34]). Furthermore, in this category, objects obtained by restricting $\mathbb{S}^\lambda(\mathbb{Q}^n)$ to S_n are projective (see Definition 4.8 and Proposition 4.12 of [34]). When $k = \mathbb{Q}$, our resolution therefore gives a projective resolution of these simple $\mathcal{F}$-modules M_n^μ.

This resolution is in fact minimal as a resolution of M_n^μ by Schur modules (in particular, there are no maps $\mathbb{S}^\lambda(\mathbb{Q}^{\oplus n}) \to \mathbb{S}^\lambda(\mathbb{Q}^{\oplus n})$). This follows from the following two facts. Firstly, the r-th term in the resolution of M_n^μ is a sum of $\mathrm{Res}_{S_n}^{GL_n(\mathbb{Q})}(\mathbb{S}^\lambda(\mathbb{Q}^n))$ with $|\lambda| = |\mu| - r$. In particular, such a module with fixed λ can only appear in one step of the resolution. Secondly, a theorem of Littlewood (Theorem XI of [17]), states that the restriction multiplicity a_μ^λ is equal to $\delta_{\mu,\lambda}$ if $|\mu| \geq |\lambda|$. Thus, $[\mathrm{Res}_{S_n}^{GL_n(\mathbb{Q})}(\mathbb{S}^\lambda(\mathbb{Q}^n)]$ are linearly independent elements of the Grothendieck ring of S_n-modules, provided n is sufficiently large. Furthermore, the $[\mathrm{Res}_{S_n}^{GL_n(\mathbb{Q})}(\mathbb{S}^\lambda(\mathbb{Q}^n)]$ should only occur in the resolution in order of decreasing $|\lambda|$, as in our resolution. Together with Observation 4.25 of [34], which provides a projective resolution of certain $\mathcal{F}$-modules D_k (which can be thought of as substitutes for M_n^μ when $\mu = (1^k)$), we obtain projective resolutions of all finitely-generated $\mathcal{F}$-modules over $\mathbb{Q}$.

Remark 6.5.4. *The $\mathcal{F}$-modules corresponding to $\mathbb{S}^\lambda(\mathbb{Q}^{\oplus n})$ are not in general indecomposable. As a result, the complex itself may be decomposable and hence not minimal as a projective* *resolution in the category of $\mathcal{F}$-modules. For example, consider the case of $\lambda = (2)$:*

$$0 \to \mathbb{Q}^{\oplus n} \xrightarrow{\pi} \mathrm{Sym}^2(\mathbb{Q}^{\oplus n}) \to M_n^{(2)} \to 0.$$

If $\{e_i\}_{i=1}^n$ denotes the standard basis of $\mathbb{Q}^{\oplus n}$, then $\pi(e_i) = e_i^2$. However, there is a splitting defined by $\psi(e_ie_j) = \frac{1}{2}(e_i + e_j)$. It follows that $Sym^2(\mathbb{Q}^{\oplus n}) = \mathbb{Q}^{\oplus n} \oplus M_n^{(2)}$, and hence that $M_n^{(2)}$ is already projective. The author is grateful to the referee of [25] for this remark.

6.6 Categorification of Stable Specht Polynomials

Suppose now that n is sufficiently large, so that $\delta_n(\lambda) = 0$, and the cohomology of $\mathcal{L}_\bullet^\lambda$ is $\mathcal{S}^{\lambda[n]}$ in degree zero. We may compute the character of $\mathcal{S}^{\lambda[n]}$ as a representation of S_n by taking the trace of a permutation matrix on the Euler characteristic of $\mathcal{L}_\bullet^\lambda$.

Recall that the trace of $g \in GL_n(k)$ acting on $\mathbb{S}^\mu(k^n)$ is equal to the Schur function s_μ evaluated at the eigenvalues of g (viewed as an $n \times n$ matrix). Taking the Euler characteristic of $\mathcal{L}_\bullet^\lambda$, we obtain the symmetric function

$$s_\lambda^\dagger = \sum_{\mu \leq |\lambda|} (-1)^{|\lambda|-|\mu|} \dim(M(\lambda, \mu)) s_\mu.$$

This symmetric function satisfies the property that when it is evaluated at the eigenvalues of a permutation matrix, the value is the character of $\mathcal{S}^{\lambda[n]}$ evaluated at that permutation. This is the defining property of the irreducible character basis, $\tilde{s}_\lambda$, of the ring of symmetric functions as found in [21], introduced independently in [2] under the name stable Specht polynomials. Thus the Littlewood complexes categorify the symmetric functions $s_\lambda^\dagger$.

We may express $\dim(M(\lambda, \mu))$ in terms of symmetric functions. Let $\langle -, - \rangle$ be the usual inner product on symmetric functions. Also define the two symmetric functions

$$H = \sum_{m \geq 0} h_m, \qquad L = \sum_{m \geq 1} \frac{1}{m} \sum_{d|m} \mu(d) p_d^{m/d}$$

to be the sum of all complete symmetric functions, and Lyndon symmetric functions, respectively. Here, μ is the usual Möbius function, and p_d are the power-sum symmetric functions. Finally, let us write $s_{\mu'}[L]$ to indicate the plethysm of $s_{\mu'}$ with L. Then we have:

$$\dim(M(\lambda, \mu)) = \langle s_{\lambda'}, s_{\mu'}[L]H \rangle.$$

After applying the Pieri rule, this is precisely Theorem 2 of [2].

In both [21] and [2], these symmetric functions were defined by considering representations of S_n when n is sufficiently large with respect to λ. Our description allows us to understand how they behave for all n.

Theorem 6.6.1. *The value of $s_\lambda^\dagger$ evaluated on the eigenvalues of a permutation matrix of size n is equal to $(-1)^{\delta_n(\lambda)}$ times the character of $\mathcal{S}^{\lambda[n]}$ if $\delta_n(\lambda)$ is finite, and zero otherwise.*

Proof. This is immediate from the equality of $s_\lambda^\dagger$ with the Euler characteristic of the Littlewood complex, together with the characterisation of the cohomology of the Littlewood complex. □

Bibliography

[1] Inna Entova Aizenbud. Deligne categories and reduced Kronecker coefficients. *Journal of Algebraic Combinatorics*, 44(2):345–362, 2016.

[2] Sami Assaf and David Speyer. Specht modules decompose as alternating sums of restrictions of Schur modules. *Proceedings of the American Mathematical Society*, 148(3):1015–1029, 2020.

[3] Georgia Benkart, Tom Halverson, and Nate Harman. Dimensions of irreducible modules for partition algebras and tensor power multiplicities for symmetric and alternating groups. *Journal of Algebraic Combinatorics*, 46(1):77–108, 2017.

[4] Christopher Bowman, Maud De Visscher, and Rosa Orellana. The partition algebra and the Kronecker coefficients. *Transactions of the American Mathematical Society*, 367(5):3647–3667, 2015.

[5] Thomas Church, Jordan S Ellenberg, and Benson Farb. FI-modules and stability for representations of symmetric groups. *Duke Mathematical Journal*, 164(9):1833–1910, 2015.

[6] Thomas Church and Benson Farb. Representation theory and homological stability. *Advances in Mathematics*, 245:250–314, 2013.

[7] Jonathan Comes and Victor Ostrik. On blocks of Deligne's category $\underline{Rep}(S_t)$. *Advances in Mathematics*, 226(2):1331 – 1377, 2011.

[8] Pierre Deligne. La catégorie des représentations du groupe symétrique S_t, lorsque t n'est pas un entier naturel. *Algebraic groups and homogeneous spaces*, 19:209–273, 2007.

[9] Yoav Dvir. On the Kronecker Product of S_n Characters. *Journal of Algebra*, 154(1):125 – 140, 1993.

[10] P. Etingof, S. Gelaki, D. Nikshych, and V. Ostrik. *Tensor Categories*, volume 205 of *Mathematical Surveys and Monographs*. American Mathematical Society, 2015.

[11] Pavel Etingof. Representation theory in complex rank, I. *Transformation Groups*, 19(2):359–381, 2014.

[12] Pavel Etingof. Representation theory in complex rank, II. *Advances in Mathematics*, 300:473–504, 2016.

[13] David A. Gay. Characters of Weyl group of $SU(n)$ on zero weight spaces and centralizers of permutation representations. *The Rocky Mountain Journal of Mathematics*, 6(3):449–455, 09 1976.

[14] Nate Harman. Generators for the representation rings of certain wreath products. *Journal of Algebra*, 445:125 – 135, 2016.

[15] Michiel Hazewinkel. Witt vectors. Part 1. In *Handbook of algebra*, volume 6, pages 319–472. Elsevier, 2009.

[16] Donald Knutson. *Lambda-rings and the Representation Theory of the Symmetric Group*, volume 308. Springer, 1973.

[17] D. E. Littlewood. Products and Plethysms of Characters with Orthogonal, Symplectic and Symmetric Groups. *Canadian Journal of Mathematics*, 10:17–32, 1958.

[18] I . G. Macdonald. *Symmetric functions and Hall polynomials.* Oxford Mathematical Monographs. Clarendon Press, second edition, 1998.

[19] Masaki Mori. On representation categories of wreath products in non-integral rank. *Advances in Mathematics*, 231(1):1 – 42, 2012.

[20] Francis D Murnaghan. The Analysis of the Kronecker Product of Irreducible Representations of the Symmetric Group. *American Journal of Mathematics*, 60(3):761–784, 1938.

[21] Rosa Orellana and Mike Zabrocki. Symmetric group characters as symmetric functions. *arXiv preprint arXiv:1605.06672*, 2016.

[22] Christophe Reutenauer. *Free Lie Algebras.* London Mathematical Society Monographs. Clarendon Press, 1993.

[23] Christopher Ryba. Stable Grothendieck rings of wreath product categories. *Journal of Algebraic Combinatorics*, 49(3):267–307, 2019.

[24] Christopher Ryba. The Structure of the Grothendieck Rings of Wreath Product Deligne Categories and their Generalisations. *International Mathematics Research Notices*, 09 2019. rnz144.

[25] Christopher Ryba. Resolving Irreducible $\mathbb{C}S_n$-Modules by Modules Restricted from $GL_n(\mathbb{C})$. *Representation Theory*, to appear.

[26] Steven Sam and Andrew Snowden. GL-equivariant modules over polynomial rings in infinitely many variables. *Transactions of the American Mathematical Society*, 368(2):1097–1158, 2016.

[27] Steven V Sam and Andrew Snowden. Stability patterns in representation theory. In *Forum of Mathematics, Sigma*, volume 3. Cambridge University Press, 2015.

[28] Steven V Sam and Andrew Snowden. Representations of categories of G-maps. *J. Reine Agnew Math.*, 2019(750):197–226, 2019.

[29] Steven V Sam, Andrew Snowden, and Jerzy Weyman. Homology of Littlewood complexes. *Selecta Mathematica*, 19(3):655–698, 2013.

[30] Wilhelm Specht. Die charaktere der symmetrischen gruppe. *Mathematische Zeitschrift*, 73(4):312–329, 1960.

[31] Richard P. Stanley and Sergey Fomin. *Enumerative Combinatorics, Volume 2*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1997.

[32] Mitsuhiro Takeuchi. Morita theorems for categories of comodules. *Journal of the Faculty of Science, the University of Tokyo. Sect. 1 A, Mathematics*, 24(3):629–644, 1977.

[33] C.A. Weibel. *An Introduction to Homological Algebra*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1995.

[34] John D Wiltshire-Gordon. Uniformly presented vector spaces. *arXiv preprint arXiv:1406.0786*, 2014.

[35] Alexander Zimmermann. *Representation Theory: A Homological Algebra Point of View*. Algebra and Applications. Springer, 2014.