THREE ESSAYS IN ECONOMIC THEORY AND PUBLIC FINANCE

by

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ABSTRACT

This thesis contains three parts. Part 1 presents a model of negotiation under bounded creativity. New ideas for proposals occur to the parties as negotiations progress. Under bounded creativity, proposing an early settlement can reveal a high valuation of the actions the other player would perform in the settlement. This can encourage the other player to hold out for a better agreement. Bounded creativity also provides a new explanation of delay. Patient negotiators may delay making serious proposals in the hope that time will produce ideas that are more favorable to them.

Part 2 presents a model of sequential search in which a buyer can call a firm, before visiting, to ask for a price quote. Because of product differentiation, buyers learn their valuations only when they visit. There are outcomes in which all, some, or no buyers call ahead. Outcomes in which more buyers call ahead have lower prices. Because search is sequential, prices do not converge to marginal cost as search costs shrink. Results also suggest that a fall in traveling costs need not improve consumer welfare and may actually reduce it. A fall in marginal calling costs also need not make consumers better off.

Part 3 examines the effect of commitment on the optimal taxation of capital income and wages in a closed, representative agent economy. A government with commitment is able eventually to substitute wage taxes for capital income taxes. A government with commitment may actually subsidize future capital income. Without commitment, a government must tax capital income at 100% as long as revenue is needed. With commitment, for a class of utility functions. the wage tax increases when capital income is taxed and decreases when it is subsidized.

Part I

NEGOTIATION AND BOUNDED CREATIVITY

Chapter 1

INTRODUCTION

Negotiating parties are usually not aware of the full range of actions they or others may take to address the issues under negotiation. New ideas for such actions occur to the parties as negotiations progress, typically during adjournments. Lincoln and O'Donnell [16, pages 36,38] hold that a mediator should meet with negotiating teams during adjournments, in part to come up with ideas for new positions and offers. Karrass [14, page 235] also stresses the importance of adjournments to give negotiators time to think how best to respond to proposals or demands.[1] He counsels patience and states [14, page 143]:

> Before a negotiation begins it is not possible for either to know the best way to resolve problems, issues and risks.

Kennedy *et al* [15, page 91] present an example that illustrates the use of adjournments to generate new ideas and responses. (These are summaries, provided in Kennedy *et al* [15], of the negotiators' actual words.)

[1] Winham [26, page 194] also stresses the importance of creativity in large scale, complex negotiations:

> Mediation in [such negotiations] is a matter of puzzle solving; that is, helping the parties to find a solution that will accommodate their overlapping and conflicting interests.

Mgt: If you agree to drop the claims for meal allowances, shift payment improvements and increased holidays, then we are prepared to make an improved offer on basic rates of pay.

Union: We would be prepared to consider dropping these items but this would be dependent upon the size of your offer on basic rates ... *(adjournment)*

Mgt: If you confirm your willingness to remove these items from the table, then we will improve our offer from 10 to 12 percent.

Union: That proposal is not acceptable. However, if you would be prepared to consider an increase to 15 percent then we might be in a position to reach some accommodation with you ... *(adjournment)*

Mgt: We cannot accept your proposal. However, [...] we would be prepared to improve our last position on condition that your side unanimously recommended acceptance of the total package; that the agreement would have a duration of 12 months and you were able to accept the additional payment 3 months after implementation of the deal.

Beginning with Nash [18, 19], economists have assumed *perfect creativity.* In formulating proposals, agents have access to the full set of Pareto dominant agreements. Every proposal is on the Pareto frontier. This paper presents a model of bargaining under *bounded creativity.* The parties do not have immediate access to the Pareto frontier of possible agreements. As they negotiate, they get new ideas for actions that could be part of a final settlement. To distinguish it from the perfect creativity case, we refer to bargaining under bounded creativity as *negotiation.*[2]

Some actions are costly to one party and benefit the other. We will call these actions *demands* or *offers*, depending on whether or not it is the beneficiary who first mentions the idea. In the above example, the union's *demands* include the "claims for meal allowances, shift payment improvements and increased holidays", and the 15 percent wage increase. Management's demands include the unanimous acceptance recommendation, the deferred onset of the

[2]This differs from the definition some other authors have used (e.g., Myerson [17]).

raise, and the implicit promise of 12 months of labor peace. Management also *offers* a 12% wage increase.

There are also actions that either hurt or benefit both parties. These are not usually the subject of negotiations. However, a party could threaten to perform a mutually harmful action unless the other party makes certain concessions. A party could also threaten not to perform a mutually beneficial action. For simplicity, we assume that parties do not think of mutually beneficial or harmful actions.

In principle, a party can get an idea for either an offer or a demand. There is some evidence that parties originate demands more often than offers. The union-management negotiation above provides one example. Another appears in Kennedy *et al* [15, page 84]. A training course was being held in a hotel. The hotel was fully booked. The hotel manager asked the course trainer to permit some guests to use two rooms that had been reserved for a course segment that was finished. The course trainer agreed on condition that the hotel provide a champagne lunch for the seminar participants. The manager accepted this proposal.

Parties do sometimes originate offers. Sometimes this happens because one party has experience in similar negotiations. Nierenberg [20, page 11] recounts a negotiation in which a "broad-ranging investor and speculator" offers to buy out a businessman with stock rather than cash. A party may originate some offers if the other party has a strong emotional commitment to its initial demands. Kennedy *et al* [15, pages 32–33] cite an example in which a firm originates several offers in a negotiation with irate workers. In our model, we will assume for simplicity that the parties can think only of demands.

We also assume that parties cannot propose monetary transfers. This usually holds when the parties hold joint assets. A married couple is one example. Monetary transfers are usually prohibited between two parties in an organization, neither of whom has salary or budgetary authority over the other. Kennedy *et al* [15, page 49] give an example of such a negotiation. A computer company was moving. The heads of two departments each wanted to maximize their floor space. Kennedy *et al* [15, page 114] note that monetary transfers are often excluded in negotiations "between governments on diplomatic or political matters".[3]

We also assume that parties are negotiating under an exogenous deadline.[4] For example, two parties may negotiate over what to do at or before a certain date. They cannot continue to negotiate after the date passes. Different members of an organization may negotiate under a deadline imposed from above. Deadlines are common in international negotiations [15, page 135]. Nuclear arms treaties are often completed in the waning hours before an impending summit meeting.

We also assume that the cost of an action to one player is independent of its value to the other. It would be straightforward to extend the model to the more realistic case of conditional independence. By "conditional" we mean the following. Consider a negotiator who thinks of an action that her opponent could perform. The negotiator knows her own valuation of the action. She also has certain beliefs about the action's cost to her opponent.

Suppose her opponent can infer these beliefs from a description of the action alone. Suppose also that the first player knows what her opponent will believe about her (the first player's) valuation. These two conditions hold if and only if each action is associated with a distribution of costs and values in which cost is independent of value. Cost and value can be correlated across actions, but not for a given action. It would be routine to extend the analysis to this more realistic case. We think that little would be gained from the exercise, however.

Some of the literature on bargaining with perfect creativity has been concerned with explaining delay tactics. If a player places a low value on an agreement, she may delay in order to signal her valuation and thus obtain better terms [4, 23, 10, 1, 12].

Bounded creativity is another explanation of delay. In our model, a player may refrain from proposing a settlement that would give her a positive payoff. She does so hoping that, in the next period, the other player will propose an agreement that is even better for her.

Under bounded creativity, if one player proposes a settlement early in the game, the other may become more reluctant to accept. By proposing early, the first player reveals that she places a high valuation on the actions the

[3]Kennedy *et al* [15, page 114] note several exceptions: "Common Market member negotiations, trade deals and funding of foreign currency transactions".

[4]Parties may also negotiate under a deadline that one of the parties imposes [14, pages 44–45]. Since such deadlines are endogenous, our model does not apply.

second player would perform in the proposed settlement. This makes the second player more likely to hold out for a more favorable agreement.

An early proposal may signal that the other player's initial demands could be satisfied at low cost. It may signal that the proposal would give the proposer a positive payoff.[5] Or it may signal nothing. However, there are some things an early proposal never signals in our model. It never signals that the proposal would give the proposer a negative payoff. It never signals that the other player's initial demands would be hard to satisfy.

[5]The other player's demands may be high cost in this case, as long as the proposer places an even higher value on her own demands.

Chapter 2

THE MODEL

The model has four periods. Player 1 is active in periods 1 and 3. Player 2 is active in periods 2 and 4. Each period, including the first, begins with an adjournment. During the adjournment, the active player (say, j) thinks of a new action that the inactive player (say, k) might perform as part of a final settlement. Player k does not get any ideas during this adjournment. We refer to the action conceived in adjournment $n \in \{1, 2, 3, 4\}$ as i^n.

Associated with i^n is a pair (v^n, c^n). v^n is the value of the action to player j. c^n is the cost of the action to player k. Player j knows v^n immediately upon thinking of the action. Player k learns c^n if and when player j describes the action to k. Player j never observes c^n, although player k's actions may reveal it. Likewise, player k never observes v^n.

We also assume that $c^n \in \{c_L, c_H\}$ and $v^n \in \{v_L, v_H\}$, where

$$0 < c_L < c_H \tag{2.1}$$

and

$$0 < v_L < v_H \tag{2.2}$$

$c^n = c_L$ with probability p. $v^n = v_L$ with probability q. c^n is independent of v^n, as well as of c^m and v^m for all $m \neq$ n. Likewise, v^n is independent of c^n and of c^m and v^m for all $m \neq n$.

There are several possible orders of c_L, c_H, v_L, and v_H. We impose two desiderata. First, between the two costs there should be a value. Second, between the two values there should be a cost. These conditions are desirable because they approximate continuous distributions in which v and c have overlapping support.

Two orders satisfy these conditions:

$$c_L < v_L < c_H < v_H \tag{2.3}$$

$$v_L < c_L < v_H < c_H \tag{2.4}$$

Under order (2.4), an agreement would have a positive payoff for a player only if the cost of her action were c_L and the value of the other player's action were v_H. All other combinations would be worse than no agreement. With order (2.3), there would be three ways that an agreement could have a positive payoff. This is likely to give richer results. We select order (2.3).

After the adjournment, the players meet. The active player, j, can do one of several things. She can accept the suggestion, if any, that k made in the prior period. A *suggestion* is a proposal, a demand, or an offer. A ***proposal*** is a statement by one party of the form "if you do action A, I will do action B." A *demand* is a statement of the form "I want you to do action A." An *offer* is a statement of the form "I will do action B if you desire."

Note that we do not permit suggestions to include more than one action of a given player. This restriction is appropriate when a given player's actions are mutually exclusive. More generally, the marginal value of a second action by one player must always be less than c_L. Even under these assumptions, our restriction does limit the signals players can send.

If j accepts k's suggestion, the game ends. Suppose the suggestion commits player j to perform an action with cost c and value v. (Set $c = v = 0$ if the agreement contains no action for j to perform.) Suppose also that player k is to perform an action with cost c' and value v'. (Set $c' = v' = 0$ if the agreement contains no action for k to perform.) Then j's payoff is

$$u_j = \delta_j^n(v' - c) \tag{2.5}$$

and k's payoff is

$$u_k = \delta_k^n(v - c') \tag{2.6}$$

where n is the period. We assume that the discount factors δ_j, δ_k fall in the interval [0,1]. If the game ends without any suggestion having been accepted, both players receive zero payoffs.

If j rejects k's suggestion, then j must make a suggestion of her own. (Player j cannot 'pass'.) The suggestion can include any action that j has thought of, as well as any action that k has mentioned in a prior suggestion.

The fourth period is like the first three. However, if player 2 makes a suggestion in period 4, there is no period 5 in which player 1 could respond. For simplicity, we assume that player 2 can accept or reject any period 3 suggestion of player 1, but cannot make a new suggestion.

We also assume that

$$v_L > v_H - c_L \tag{2.7}$$

This guarantees that any offer would be accepted, giving the offerer a negative payoff. Since a player can guarantee herself a payoff of zero, players will never make offers.

Chapter 3

FINDING THE EQUILIBRIA

We look for pure strategy perfect Bayesian equilibria [7, pages 331 ff.]. We also restrict attention to games with generic extensive forms. That is, we ignore equilibria that occur only on parameter sets of measure zero.

The concept of *equilibrium* assumes that players can be certain about their opponents' strategies. While such certainty may be plausible in our game, it seems a poor approximation to the intricate and uncertain activity that we are trying to model. Thus, we also assume that players do not choose weakly dominated strategies. This assumption gives outcomes that rely less heavily on common knowledge.

Everything in the game is discrete. Completeness would require a description of the mixed strategy equilibria as well. Although the mixed equilibria are easy to find with our technique, we do not describe them here. Most of the important features are already present in the pure equilibria. In addition, a grid search shows that some pure equilibrium occurs in (approximately) 97.5% of the parameter space.

Lemma 1 describes common features of the pure equilibria.

Lemma 1 *In any pure strategy perfect Bayesian equilibrium,*

1. *Players never make offers.*
2. *Players never accept demands.*
3. *In period 1, player 1 demands idea i^1.*

4. *In period 2, player 2 either demands i^2 or proposes (i^1, i^2).*

5. *If 2 proposes (i^1, i^2) in period 2, 1 may accept in period 3. If 1 accepts, the game ends. If 1 rejects, or if 2 demands i^2 in period 2, then 1 either demands i^1 or i^3, or proposes (i^1, i^2) or (i^3, i^2).*

6. *In period 4, player 2 accepts the proposal (i^1, i^2) iff $c^1 < v^2$. Player 2 accepts the proposal (i^3, i^2) iff $c^3 < v^2$. Player 2 rejects any demand.*

Proof

1 *Any offer would be accepted by (2.7). The offerer's payoff would be negative. But a player can guarantee a payoff of 0 by always demanding and never accepting.*

2 *To accept a demand would also give a negative payoff.*

3 *Player 1 has no alternative.*

4 *Claims 1 and 2 of this Lemma rule out player 2's other choices.*

5 *Player 1 rejects any demand by Claim 2. The rest follows from Claim 1.*

6 *This follows from Claim 2, (2.5), and (2.6).*

Lemma 1 specifies players' strategies in periods 1 and 4. Strategies can vary in periods 2 and 3 only. We now describe the form these strategies take.

Player 2's period 2 action may signal something about 2's private information, or "type", (c^1, v^2). Let D be the action of demanding i^2 in period 2. Let P be the action of proposing (i^1, i^2). Suppose player 2 demands i^2 in period 2. Player 1 first rejects 2's demand, by Lemma 2. If $v^1 < c^2$ and $v^3 < c^2$, then player 1 demands i^1 or i^3, since any other response is weakly dominated. Otherwise, player 1 proposes either (i^1, i^2) or (i^3, i^2), depending on whether or not

$$\begin{aligned} &\delta_1(v^1 - c^2) \text{ Prob}\left(v^2 > c^1 \mid \text{D}\right) \\ &> \ \delta_1(v^3 - c^2) \text{ Prob}\left(v^2 > c^3 \mid \text{D}\right) \end{aligned} \tag{3.1}$$

where these probabilities are based on 1's updated beliefs about 2's type, (v^2, c^1).

Now suppose 2 proposes (i^1, i^2) in period 2. If $v^1 < c^2$ and $v^3 < c^2$, then player 1 rejects and again demands either i^1 or i^3, by weak dominance. Otherwise, player 1 either accepts or proposes (i^3, i^2), depending on whether or not

$$v^1 - c^2 > \delta_1(v^3 - c^2) \text{ Prob}\left(v^2 > c^3 \mid \text{P}\right)$$

Player 1 will never reject (i^1, i^2) and then *propose* (i^1, i^2). If $v^1 < c^2$, it is at least weakly better to demand something. If $v^1 > c^2$, then

$$v^1 - c^2 > \delta_1(v^1 - c^2) \text{ Prob}\left(v^2 > c^1 \mid \text{P}\right)$$

so it is strictly better to accept 2's initial proposal.

We can also say something about player 2's optimal period 2 action. Given 1's strategy and the distribution over 1's period 3 information, or type, (v^1, c^2, v^3), player 2 can compute the probabilities of 1's various period 3 responses. Let

$$r = \text{ Prob}(\text{ 1 accepts} \mid \text{P}) \tag{3.2}$$

$$s = \text{ Prob}(\text{ 1 rejects and proposes}(i^3, i^2) \mid \text{P}) \tag{3.3}$$

$$t = \text{ Prob}(\text{ 1 proposes}(i^1, i^2) \mid \text{D}) \tag{3.4}$$

$$u = \text{ Prob}(\text{ 1 proposes}(i^3, i^2) \mid \text{D}) \tag{3.5}$$

If 2 proposes, 2 gets

$$\begin{aligned} U_2^P &= (v^2 - c^1)r \\ &+ \delta_2 s \text{E}_{c^3}\left(\max\langle v^2 - c^3, 0\rangle \,\middle|\, \begin{array}{l} \text{1 proposes } (i^3, i^2) \\ \text{in response to} \\ \text{2's proposal} \end{array} \right) \end{aligned} \tag{3.6}$$

where E_{c^3} denotes the expectation over c^3. If 2 demands, 2 gets

$$\begin{aligned} U_2^D &= \\ &\delta_2 t\left[\max\langle v^2 - c^1, 0\rangle\right] \\ &+\delta_2 u \text{E}_{c^3}\left(\max\langle v^2 - c^3, 0\rangle \,\middle|\, \begin{array}{l} \text{1 proposes } (i^3, i^2) \\ \text{in response to} \\ \text{2's demand} \end{array} \right) \end{aligned} \tag{3.7}$$

Player 2 proposes if $U_2^P > U_2^D$ and demands if $U_2^P < U_2^D$.

We have seen how player 1's posteriors determine 1's best response, as a function of (v^1, c^2, v^3). 1's best response function determines player 2's expectations, which are summarized by r, s, t, and u. Player 2's expectations determine 2's best period 2 action, as a function of (v^2, c^1). This function in turn generates posteriors for 1.[1] Every perfect Bayesian equilibrium is a fixed point of this circular process. To find the equilibria, we must explore the details of the best response correspondences. We do so in Appendix A. The equilibrium actions are given in Tables 3.1 and 3.2. Equilibrium codes such as "ZS" will be defined in Theorems 1-7. The full list of equilibria is ZS, SS, TSZ, DSZ, TZZ, DZZ, and B.

We now describe all pure equilibria. We use colorful names to make the equilibria easier to remember. In the first equilibrium, player 1's discount factor is large, so 1 is patient. In Yiddish, a patient person is said to have good *zitzflaysh* (literally, "sitting flesh"). Player 2's discount factor is small, so 2 is anxious for an agreement. In Yiddish, a restless, antsy person is said to be on *shpilkes* (literally, "pins").

Theorem 1 (The Zitzflaysh–Shpilkes (ZS) Equilibrium)

The actions given in Tables 3.1 and 3.2 under "ZS" are a perfect Bayesian equilibrium if

$$\delta_1 > \frac{(v_L - c_L)(1 - q + pq)}{(v_H - c_L)(1 - q + p^2 q)} \tag{3.8}$$

and

$$\delta_2 < \max\left\{ \frac{v_H - c_H}{v_H - (pc_L + (1-p)c_H)}, \frac{(v_H - c_H)(1 - q + pq^2)}{(v_H - c_H)(1 - q + pq^2) + (c_H - c_L)p[(1-q)^2 + pq]} \right\} \tag{3.9}$$

The two ratios on the right hand side of (3.9) are both in the interval (0,1).

There is a nonempty, open region of the parameter space in which both of the exogenous requirements are satisfied.

Proof

Appendix B.

[1] Of course, player 1's posteriors are indeterminate following a probability zero action of player 2.

2's Type: (v^2, c^1)	Equilibria	2's Period 2 Action
(v_H, c_H)	ZS,SS,B	Propose (i^1, i^2)
	TSZ,DSZ,TZZ,DZZ	Demand i^2
(v_H, c_L)	All	Propose (i^1, i^2)
(v_L, c_H)	ZS,SS,TSZ,DSZ,TZZ,DZZ	Demand i^2
	B	Propose (i^1, i^2)
(v_L, c_L)	All	Propose (i^1, i^2)

Table 3.1: PLAYER 2's ACTIONS IN ALL EQUILIBRIA

		1's Period 3 Action if 2:	
1's Type: (v^1, c^2, v^3)	Equilibria	Proposes	Demands
(v_H, c_H, v_H)	All but B	Accept	Propose (i^3, i^2)
	B	Accept	Propose (i^1, i^2)
(v_H, c_H, v_L)	All	Accept	Propose (i^1, i^2)
(v_H, c_L, v_H)	All but B	Accept	Propose (i^3, i^2)
	B	Accept	Propose (i^1, i^2)
(v_H, c_L, v_L)	ZS,SS,TSZ,TZZ	Accept	Propose (i^3, i^2)
	DSZ,DZZ,B	Accept	Propose (i^1, i^2)
(v_L, c_H, v_H)	All	Propose (i^3, i^2)	Propose (i^3, i^2)
(v_L, c_H, v_L)	All	Demand	Demand
(v_L, c_L, v_H)	ZS,TZZ,DZZ	Propose (i^3, i^2)	Propose (i^3, i^2)
	SS,TSZ,DSZ	Accept	Propose (i^3, i^2)
	B	Propose (i^3, i^2)	Propose (i^1, i^2)
(v_L, c_L, v_L)	All but B	Accept	Propose (i^3, i^2)
	B	Accept	Propose (i^1, i^2)

Table 3.2: PLAYER 1's PERIOD 3 ACTION IN ALL EQUILIBRIA

In the next equilibrium, both players' discount factors must be small. Both players are on *shpilkes.*

Theorem 2 (The Shpilkes–Shpilkes (SS) Equilibrium)

The actions given in Tables 3.1 and 3.2 under "SS" are a perfect Bayesian equilibrium if

$$\delta_1 < \frac{(v_L - c_L)(1 - q + pq)}{(v_H - c_L)(1 - q + p^2 q)} \tag{3.10}$$

and

$$\delta_2 < \max\left\{ \frac{v_H - c_H}{v_H - (pc_L + (1-p)c_H)}, \frac{(v_H - c_H)(1 - q + pq)}{(v_H - c_H)(1 - q + pq) + (c_H - c_L)p[(1-q)^2 + pq(2-q))]} \right\} \tag{3.11}$$

The two ratios on the right hand side of (3.11) are both in (0,1).

There is a nonempty, open region of the parameter space in which both of the exogenous requirements are satisfied.

Proof

Appendix B.

A comparison between ZS and SS shows the effect of 1's discount factor. If player 2 proposes, player 1 must choose between (i^1, i^2) right away and (i^3, i^2) (together with the risk that 2 will reject) one period later. If player 1's type is $(v^1, c^2, v^3) = (v_L, c_L, v_H)$, player 1 prefers (i^3, i^2) to (i^1, i^2). However, player 1 prefers either proposal to no agreement. With these preferences, player 1 rejects and proposes (i^3, i^2) in ZS, where 1 has good *zitzflaysh.* But player 1 accepts 2's proposal in SS, because 1 is on *shpilkes.*

In the next four equilibria, 2 proposes if 1's initial demand is low cost $(c^1 = c_L)$ and demands if it is high cost $(c^1 = c_H)$. Player 2's period 2 action reveals c^1 perfectly, but tells 1 nothing about v^2. In period 3, player 1 still believes that $v^2 = v_L$ with probability q. In the next equilibrium, q cannot be too small. Suppose player 1's type is (v_H, c_L, v_L). Player 1 likes (i^1, i^2) more than (i^3, i^2), but prefers either to no agreement. If 2 demands rather than proposes, player 1 does not propose his preferred (i^1, i^2) because $c^1 = c_H$ for sure and v^2 is very likely v_L. Thus, we say that 1 is *timid* about proposing (i^1, i^2).

Player 1's discount factor cannot be too close to 1 in this equilibrium. Player 1 is on *shpilkes*. When player 1's type is (v_L, c_L, v_H) and 2 proposes (i^1, i^2), player 1 impatiently accepts, even though he prefers (i^3, i^2).

Finally, player 2's discount factor cannot be close to 0. Player 2 has good *zitzflaysh*.

Theorem 3 (The Timid Shpilkes–Zitzflaysh (TSZ) Equilibrium)

The actions given in Tables 3.1 and 3.2 under "TSZ" are a perfect Bayesian equilibrium if

$$\delta_1 < \frac{v_L - c_L}{(1 - q + pq)(v_H - c_L)} \tag{3.12}$$

$$q > \frac{v_H - v_L}{v_H - (1-p)v_L - pc_L} \tag{3.13}$$

and

$$\delta_2 > \max\left\{\frac{v_H - c_H}{v_H - (pc_L + (1-p)c_H)}, \frac{(v_H - c_H)(1 - q + pq)}{(v_H - c_H)(1 - q + pq) + (c_H - c_L)p[(1-q)^2 + pq(2-q)]}\right\} \tag{3.14}$$

The ratios on the right hand side of (3.14) both lie in (0,1).

There is a nonempty, open region of the parameter space in which all the exogenous requirements are satisfied.

Proof

Appendix B.

This equilibrium is the first to illustrate the use of delay to obtain a more favorable settlement. When player 2's type is $(v^2, c^1) = (v_H, c_H)$, the agreement (i^1, i^2) would give her a positive payoff. However, if player 2 demands i^2, player 1 will be more likely to propose (i^3, i^2). And while i^1 is a high cost action for player 2, there is some chance that i^3 will not be. Since player 2 is patient in TSZ, she does demand i^2 in the hope that player 1 will propose (i^3, i^2). Player 2 is willing to delay because she has *zitzflaysh.*

In the next equilibrium, as in TSZ, player 1 always thinks that $v^2 = v_L$ with probability q in period 3. But q is small in the next equilibrium, unlike in TSZ. Thus, player 1 thinks it likely that 2 puts a high value on i^2. When 1 is of type (v_H, c_L, v_L) and player 2 demands i^2, this belief leads player 1 to propose his preferred (i^1, i^2). Player 1 does so even though he knows $c^1 = c_H$. Player 1 is very *daring.*

In addition, player 1's discount factor is small in the next equilibrium, so 1 is on *shpilkes.* When player 1's type is (v_L, c_L, v_H) and 2 proposes (i^1, i^2), player 1 impatiently accepts, even though he prefers (i^3, i^2).

Player 2's discount factor cannot be close to 0, so 2 has *zitzflaysh.* Player 2 displays the clever delay behavior we saw in TSZ when $(v^2, c^1) = (v_H, c_H)$.

Theorem 4 (The Daring Shpilkes–Zitzflaysh (DSZ) Equilibrium)

The actions given in Tables 3.1 and 3.2 under "DSZ" are a perfect Bayesian equilibrium if

$$\delta_1 < \frac{v_L - c_L}{(1 - q + pq)(v_H - c_L)} \tag{3.15}$$

$$q < \frac{v_H - v_L}{v_H - (1 - p)v_L - pc_L} \tag{3.16}$$

and

$$\delta_2 > \max\left\{ \frac{v_H - c_H}{v_H - (pc_L + (1 - p)c_H)}, \frac{(v_H - c_H)(1 - q + pq)}{(v_H - c_H)(1 - q + pq) + (c_H - c_L)p[(1 - q)^2 + pq]} \right\} \tag{3.17}$$

The ratios on the right hand side of (3.17) both lie in (0,1).

There is a nonempty, open region of the parameter space in which all the exogenous requirements are satisfied.

Proof

Appendix B.

In the next equilibrium, q is large. As in TSZ, this makes player 1 *timid* about proposing (i^1, i^2) when player 1's type is (v_H, c_L, v_L) and 2 demands i^2.

Player 2 has *zitzflaysh* in TZZ, so she cleverly delays agreement by demanding i^2 when $(v^2, c^1) = (v_H, c_H)$. Player 1 also has *zitzflaysh*, so when his type is (v_L, c_L, v_H) and 2 proposes, 1 does not accept. Instead, he shows his patience by proposing (i^3, i^2), which he prefers.

Theorem 5 (The Timid Zitzflaysh–Zitzflaysh (TZZ) Equilibrium)

The actions given in Tables 3.1 and 3.2 under "TZZ" are a perfect Bayesian equilibrium if

$$q > \frac{v_H - v_L}{v_H - (1-p)v_L - pc_L} \tag{3.18}$$

as well as

$$\delta_1 > \frac{v_L - c_L}{(1 - q + pq)(v_H - c_L)} \tag{3.19}$$

and

$$\begin{aligned} \delta_2 \quad > \quad \max\Bigg\{ & \frac{v_H - c_H}{v_H - (pc_L + (1-p)c_H)}, \\ & \frac{(v_H - c_H)(1 - q + pq^2)}{(v_H - c_H)(1 - q + pq^2) + (c_H - c_L)p[(1-q)^2 + pq]} \Bigg\} \end{aligned} \tag{3.20}$$

Both ratios in (3.20) lie in (0,1).

There is a nonempty, open region of the parameter space in which all the exogenous requirements are satisfied.

Proof

Appendix B.

In the next equilibrium, q is small. This makes player 1 *daring* as we discussed in DSZ. As in TSZ, DSZ, and TZZ, player 2 has *zitzflaysh* in DZZ and shrewdly demands i^2 when $(v^2, c^1) = (v_H, c_H)$. Player 1 also has *zitzflaysh*. When 1's type is (v_L, c_L, v_H) and 2 proposes, 1 rejects and proposes (i^3, i^2), as in ZS and TZZ.

Theorem 6 (The Daring Zitzflaysh–Zitzflaysh (DZZ) Equilibrium)

The actions given in Tables 3.1 and 3.2 under "DZZ" are a perfect Bayesian equilibrium if

$$q < \frac{v_H - v_L}{v_H - (1-p)v_L - pc_L} \tag{3.21}$$

$$\delta_1 > \frac{v_L - c_L}{(1-q+pq)(v_H - c_L)} \tag{3.22}$$

and

$$\delta_2 > \max\left\{ \frac{v_H - c_H}{v_H - (pc_L + (1-p)c_H)}, \frac{(v_H - c_H)(1-q+pq^2)}{(v_H - c_H)(1-q+pq^2) + (c_H - c_L)p[(1-q)^2 + pq^2]} \right\} \tag{3.23}$$

Both ratios in (3.23) lie in (0,1).

There is a nonempty, open region of the parameter space in which all the exogenous requirements are satisfied.

Proof

Appendix B.

The next equilibrium is unlike all the others. It is a bit *bizarre.*

Theorem 7 (The Bizarre (B) Equilibrium)

The strategies given in Tables 3.1 and 3.2 under "B" are a perfect Bayesian equilibrium if

$$\delta_1 > \frac{v_L - c_L}{(1-q+pq)(v_H - c_L)} \tag{3.24}$$

$$p < \frac{v_L - c_L}{v_H - c_L} \tag{3.25}$$

and

$$\delta_2 \in \left(\frac{(c_H - v_L)(1-q+pq^2)}{p^2q(1-q)(v_L - c_L)}, \frac{1-q+pq^2}{1-q+pq^2+pq(1-p)(1-q)} \right) \tag{3.26}$$

This interval may be empty.

There is a nonempty, open region of the parameter space in which all the exogenous requirements are satisfied.

Proof

Appendix B.

One unique feature of equilibrium B is that player 2 proposes even when $(v^2, c^1) = (v_L, c_H)$, so that (i^1, i^2) would give 2 a negative payoff. The reason is that player 2 is actually more likely to obtain the agreement (i^3, i^2) by proposing (i^1, i^2) than by demanding i^2. When $(v^2, c^1) = (v_L, c_H)$, player 2 is willing to accept the risk that 1 will accept (i^1, i^2) in the hope that 1 will propose (i^3, i^2). Of course, $|v_L - c_H|$ must be very small relative to $v_L - c_L$ for this tradeoff to be desirable. This is implicit in (3.26).

Why does player 1 propose (i^1, i^2) so insistently if 2 demands (which 2 never does)? If 2 mistakenly demands, player 1 concludes that $(v^2, c^1) = (v_L, c_L)$ is quite likely. Thus, 1 believes that 2 is likely to accept (i^1, i^2) but, since v^2 is probably v_L, 2 is less likely to accept (i^3, i^2). This leads 1 to propose (i^1, i^2) in all cases but two. The exceptions are when

$$(v^1, c^2, v^3) \in \{(v_L, c_H, v_H), (v_L, c_H, v_L)\}$$

so that (i^1, i^2) would give 1 a negative payoff.

Player 1's strategy is remarkable when $(v^1, c^2, v^3) = (v_L, c_L, v_H)$. In this case, 1 prefers (i^3, i^2) to (i^1, i^2), but he likes either better than no agreement. When 2 proposes (i^1, i^2), player 1 rejects and proposes (i^3, i^2), which 1 prefers. However, when 2 demands i^2, 1 actually proposes (i^1, i^2) rather than (i^3, i^2). Player 1 does this because he is convinced that 2 is especially likely to demand i^2 when $(v^2, c^1) = (v_L, c_L)$, so that 2 is much more likely to accept (i^1, i^2) than (i^3, i^2).

Equilibrium B raises the exotic possibility that a player may be more likely to obtain a given settlement by *not* proposing it. Below, we discuss a pair of multiple equilibria that jointly display a similar feature.

Now we show that there are no other pure perfect Bayesian equilibria.

Theorem 8 ("That's All, Folks")

Theorems 1-7 describe the only pure perfect Bayesian equilibria.

Proof

Appendix B.

Chapter 4

MULTIPLE EQUILIBRIA

The parameter restrictions are too complex to capture in a 2 dimensional chart. We performed a grid search to determine which equilibria can occur for the same parameters. The search turned up open regions in which either equilibrium ZS or TSZ may occur; in which either TZZ or B may occur; and in which DZZ or B may occur. We give examples in the following table. The search turned up no other instances of multiple equilibria for generic parameters. (Without loss of generality, we assumed $c_L = 1$ and $v_H = 2$.)

In equilibrium B, player 2 was more likely to obtain the settlement (i^1, i^2) by demanding i^2 than by proposing (i^1, i^2) itself. In a given equilibrium, making a proposal early on can make the players less likely to settle on the proposal.

Comparing ZS and TSZ reveals a different but analogous property. In either ZS or TSZ taken alone, making a proposal early on does make the players more likely to settle on the proposal. However, player 2 is more likely to propose early in ZS than in TSZ. Since the types of player 2 who

Pair of Equilibria	Parameter Examples							
	p	q	δ_1	δ_2	c_L	v_L	c_H	v_H
ZS,TSZ	0.353	0.941	0.412	0.882	1.000	1.176	1.294	2.000
TZZ,B	0.529	0.588	0.882	0.882	1.000	1.588	1.647	2.000
DZZ,B	0.588	0.353	0.765	0.882	1.000	1.647	1.706	2.000

Table 4.1: Examples of Multiple Equilibria

propose early are different, proposing early has a different *meaning* (signalling content) in the two equilibria. This difference in meanings actually makes player 1 *less* likely to accept in ZS than in TSZ.

In particular, an early proposal is a stronger signal in ZS than in TSZ that player 2 places a high valuation on her *demand* (the action she wants player 1 to perform). If player 1 does not like 2's *offer* (the action 2 offers to perform), player 1 knows 2 is more likely (in ZS than in TSZ) to accept a substitute for this offer. This makes player 1 less likely to accept an early proposal.

Why is player 2 more likely to propose in ZS, even though 1 is less likely to accept? Player 2's action differs only when $(v^2, c^1) = (v_H, v_H)$. For the parameters in which both ZS and TSZ can occur, player 2 has the following preferences over outcomes when $(v^2, c^1) = (v_H, v_H)$:[1]

- Best Outcome: 1 proposes (i^3, i^2).
- Second Best: 1 accepts (i^1, i^2).
- Third Best: 1 proposes (i^1, i^2).

Thus, player 2 actually *wants* 1 to reject her proposal and then propose (i^3, i^2) when $(v^2, c^1) = (v_H, v_H)$. Since 1 is more likely to do so in ZS than TSZ, 2 is willing to propose in ZS but not in TSZ when $(v^2, c^1) = (v_H, v_H)$.

B and TZZ can also occur for the same parameters, as can B and DZZ. These multiplicities underscore the finding that, for given parameters, the meaning of an early proposal may depend on the equilibrium. In TZZ and DZZ, an early proposal signals that player 2 likes 1's initial demand (i.e., that $c^1 = c_L$). This is what we would expect, since player 2 is more likely to obtain the settlement (i^1, i^2) by proposing it and since 2 is somewhat impatient.[2]

[1]Player 1's change in ZS (relative to TSZ) raises s, lowers r, and leaves t and u unchanged. From the formulae in the proof of Lemma 2, this can raise U_2^P (2's payoff to proposing) when $(v^2, c^1) = (v_H, c_H)$ only if

$$v_H - c_H < \delta_2 [v_H - (pc_L + (1 - p)c_H)]$$

(This uses the fact that r + s is constant [Lemma 4].) This can happen in both equilibria. It implies the preferences over outcomes that we claim for 2, by the proof of Lemma 4.

[2]By equation (3.26). We are talking about parameters for which B is also possible.

But in B, player 1 changes his response to a demand because his beliefs change. Given the parameters, 1's strategy in B makes it unwise for player 2 to demand, regardless of her type. This changes the meaning of an early proposal. An early proposal signals *nothing* in B, in the sense that 1's posteriors equal his priors following a proposal.

Chapter 5

DISCUSSION

Bounded creativity offers a new explanation of delay in bargaining. It is common for a player to refrain from making an early proposal because she knows the other player may come up with an even more favorable proposal. If she is patient, she may do so even if the prefers the early proposal to no agreement.

We also find that an excessive eagerness for a particular proposal can make the proposal a less likely outcome of negotiations. In equilibrium B, player 2 is actually more likely to obtain the agreement (i^1, i^2) by demanding i^2 than by proposing (i^1, i^2) directly. (She proposes anyway.) A similar property holds of the multiple equilibria ZS and TSZ. Player 2 is more likely to propose in ZS than in TSZ. Proposing is also a stronger signal in ZS (than in TSZ) that player 2 places a high value on the action she wants player 1 to perform. If player 1 can think of proposal he likes better, he will suggest it in ZS. In TSZ, player 1 is less optimistic that 2 would accept a different proposal, so he accepts 2's initial proposal.

In general, an early proposal can signal one of three things in the pure equilibria of the model. It can signal that the other player's initial demands could be satisfied at low cost (equilibria TSZ, DSZ, TZZ, and DZZ). It can signal that the proposal would give the proposer a positive payoff (equilibria ZS and SS). It can also signal nothing (equilibrium B). In the pure equilibria of this model, an early proposal never signals that the pr^posal would give the proposer a negative payoff, or that the other player's initial demands would be hard to satisfy.

Appendix A

Properties of the Best Response

This appendix derives the properties of the best response correspondence in a sequence of Lemmas.

Player 2's period 2 action may signal something about 2's private information, (c^1, v^2). Let D be the action of demanding i^2 in period 2. Let P be the action of proposing (i^1, i^2). Let $\mu(c, v|a)$ be the posterior probability that 1 places on the event $(c^1, v^2) = (c, v)$ when 2 selects action $a \in \{D, P\}$. For brevity, define

$$\alpha_D = \mu(c_L, v_L|D) \tag{A.1}$$

$$\alpha_P = \mu(c_L, v_L|P) \tag{A.2}$$

$$\gamma_D = \mu(c_H, v_L|D) \tag{A.3}$$

$$\gamma_P = \mu(c_H, v_L|P) \tag{A.4}$$

The following Lemmas give player 1's period 3 action as a function of these beliefs.

Lemma 2 *Suppose player 2 demands i^2 in period 2.*

1. *Player 1 first rejects 2's demand.*

2. *If $v^1 < c^2$ and $v^3 < c^2$ then player 1 demands i^1 or i^3.*

3. If $v^1 > c^2$ *and*

$$(v^1 - c^2)(1-\gamma_D) > (v^3 - c^2)[(\alpha_D + \gamma_D)p + 1 - \alpha_D - \gamma_D]$$

then player 1 proposes (i^1, i^2).

4. If $v^3 > c^2$ *and*

$$(v^1 - c^2)(1-\gamma_D) < (v^3 - c^2)[(\alpha_D + \gamma_D)p + 1 - \alpha_D - \gamma_D]$$

then player 1 proposes (i^3, i^2).

Proof

1 *Follows from Lemma 1.*

2 *If* $(v^1, c^2, v^3) = (v_L, c_H, v_L)$, *player 1 strictly prefers that there be no agreement. By weak dominance, player 1 demands* i^1 *or* i^3 *rather than proposing anything.*

3 *By demanding* i^1 *or* i^3, 1 *receives a payoff of zero. By proposing* (i^1, i^2), *1 receives the payoff*

$$\delta_1(v^1 - c^2) \ \mathit{Prob}\ (v^2 > c^1 \mid D)$$

where this probability is based on 1's posteriors. By proposing (i^3, i^2), *1 receives the payoff*

$$\delta_1(v^3 - c^2) \ \mathit{Prob}\ (v^2 > c^3 \mid D)$$

By definition, $1 - \gamma_D$ *is the probability 1 places on* $v^2 > c^1$ *when 2 selects action D.* $(\alpha_D + \gamma_D)p + 1 - \alpha_D - \gamma_D$ *is the probability 1 places on* $v^2 > c^3$ *when 2 selects D. Lemma 3 follows by substituting these probabilities into 1's payoffs.*

4 *See proof of 3.*

Lemma 3 *Suppose player 2 proposes* (i^1, i^2) *in period 2.*

1. *If $v^1 < c^2$ and $v^3 < c^2$ then player 1 rejects and demands either i^1 or i^3.*

2. *If $v^1 > c^2$ and*

$$v^1 - c^2 > \delta_1(v^3 - c^2)\left[(\alpha_P + \gamma_P)p + 1 - \alpha_P - \gamma_P\right]$$

then player 1 accepts 2's proposal.

3. *If $v^3 > c^2$ and*

$$v^1 - c^2 < \delta_1(v^3 - c^2)\left[(\alpha_P + \gamma_P)p + 1 - \alpha_P - \gamma_P\right]$$

then player 1 rejects 2's proposal and proposes (i^3, i^2).

Proof

1 *Follows by weak dominance.*

2 *By accepting, 1 gets $v^1 - c^2$. By rejecting and proposing $(i^1, i^2), 1$ gets*

$$\delta_1(v^1 - c^2)\ Prob\ (v^2 > c^1 \mid P)$$

By rejecting and proposing (i^3, i^2), 1 gets

$$\delta_1(v^3 - c^2)\ Prob\ (v^2 > c^3 \mid P)$$

Player 1 will never reject and propose (i^1, i^2). If $v^1 < c^2$, it is at least weakly better to demand something. If $v^1 > c^2$, it is strictly better to accept 2's initial proposal. Player 1 either accepts or rejects and proposes (i^3, i^2).

The probability that $v^2 > c^3$ given that 2 proposes is $(\alpha_P + \gamma_P)p + 1 - \alpha_P - \gamma_P$. 2 follows by substitution.

3 *See proof of 2.*

We can also say something about player 2's optimal period 2 action. Let r, s, t, and u be as defined in (3.2)-(3.5). Also define the function $\phi(v, c)$ by

$$\begin{aligned}\phi(v,c) \;&=\; (v-c)\left[r - \delta_2 t \mathbb{1}(v > c)\right] \\ &\quad + \delta_2(t-r)\left[p(v - c_L) + (1-p)(v - c_H)\mathbb{1}(v = v_H)\right] \qquad \text{(A.5)}\end{aligned}$$

where $\mathbb{1}$ is the indicator function, which equals 1 if its argument is true and 0 otherwise.

Lemma 4 *1. $r + s = t + u$.*

2. Player 2 proposes (i^1, i^2) if $\phi(v^2, c^1) > 0$ and demands i^2 if $\phi(v^2, c^1) < 0$.

Proof

1 *$1 - r - s$ is the probability that player 1 demands something in response to a proposal of (i^1, i^2). By Lemma 3, this happens if and only if $(v^1, c^2, v^3) = (v_L, c_H, v_L)$. Similarly, $1 - t - u$ is the probability that player 1 demands something in response to a demand of i^2. By Lemma 2, this happens if and only if $(v^1, c^2, v^3) = (v_L, c_H, v_L)$. Thus, $1-r-s = 1 - t - u$.*

2 *If 2 proposes, 2 gets U_2^P, defined in (3.6). Since c^3 and v^3 are independent,*

$$U_2^P = (v^2 - c^1)r + \delta_2 s\left[p(v^2 - c_L) + (1-p)(v^2 - c_H)\mathbb{1}(v^2 = v_H)\right] \tag{A.6}$$

If 2 demands, 2 gets U_2^D, defined in (3.7). Again, c^3 and v^3 are independent. Using this and the definition of $\mathbb{1}$, 2's payoff becomes

$$\begin{aligned} U_2^D &= \delta_2 \Big[t[v^2 - c^1]\mathbb{1}(v^2 > c^1) \\ &\quad + u\Big(p(v^2 - c_L) + (1-p)(v^2 - c_H)\mathbb{1}(v^2 = v_H)\Big)\Big] \end{aligned} \tag{A.7}$$

Now subtract A.7 from A.6. By Claim 1 of this Lemma, we can substitute $t - r$ for $s - u$. This establishes that $U_2^P - U_2^D = \phi(v^2, c^1)$.

Lemmas 3 and 2 show how player 1's beliefs α_P, γ_P, α_D, and γ_D determine 1's best response, as a function of (v^1, c^2, v^3). This function determines player 2's expectations, which are summarized by t and r. Lemma 4 shows how t and r determine 2's best period 2 action, as a function of (v^2, c^1). This function generates values of α_P, γ_P, α_D, and γ_D. Every perfect Bayesian equilibrium is a fixed point of this circular process. To find the equilibria, we must explore the details of the best response correspondences.

Lemma 5 explores player 1's best response when 2 demands i^2. First define the functions

$$w(\gamma_D) = \frac{-(v_H - v_L) + \gamma_D[v_H - (1-p)v_L - pc_L]}{(1-p)(v_L - c_L)} \tag{A.8}$$

$$x(\gamma_D) = \frac{v_H - v_L - \gamma_D[(1-p)v_H - v_L + pc_L]}{(1-p)(v_H - c_L)} \tag{A.9}$$

$$y(\gamma_D) = \left[\frac{p}{1-p}\right]\gamma_D \tag{A.10}$$

Lemma 5 *Suppose player 2 demands i^2 and player 1's beliefs are α_D and γ_D.*

1. *Suppose*

   $$(v^1, c^2, v^3) \in \{(v_L, c_L, v_L), (v_H, c_L, v_H), (v_H, c_H, v_H)\}$$

   From 2's perspective, this has probability $pq^2 + (1-q)^2$. 1 will propose (i^1, i^2) if $\alpha_D > y(\gamma_D)$. 1 will propose (i^3, i^2) if $\alpha_D < y(\gamma_D)$. 1 will mix between the two if the two quantities are equal.

2. *Suppose $(v^1, c^2, v^3) = (v_L, c_H, v_L)$ (probability $(1-p)q^2$). 1 will demand i^1 or i^3.*

3. *Suppose $(v^1, c^2, v^3) = (v_L, c_L, v_H)$ (probability $pq(1-q)$). Player 1 will propose (i^1, i^2) if $\alpha_D > x(\gamma_D)$. Player 1 will propose (i^3, i^2) if $\alpha_D < x(\gamma_D)$. 1 will mix between the two if $\alpha_D = x(\gamma_D)$.*

4. *Suppose $(v^1, c^2, v^3) = (v_H, c_L, v_L)$ (probability $pq(1-q)$). Player 1 will propose (i^1, i^2) if $\alpha_D > w(\gamma_D)$. Player 1 will propose (i^3, i^2) if $\alpha_D < w(\gamma_D)$. 1 will mix between the two if $\alpha_D = w(\gamma_D)$.*

5. *Suppose $(v^1, c^2, v^3) = (v_L, c_H, v_H)$ (probability $q(1-p)(1-q)$). 1 will propose (i^3, i^2).*

6. *Suppose* $(v^1, c^2, v^3) = (v_H, c_H, v_L)$ *(probability* $q(1-p)(1-q)$*). 1 will propose* (i^1, i^2)*.*

Proof

The probabilities are all derived from the independence of v^1, c^2, *and* v^3 *from each other and from any of 2's information.*

1 *Since* $v^1 - c^2 = v^3 - c^2 > 0$ *in each case, 1 will propose* (i^1, i^2) *if* $1 - \gamma_D > 1 - (1-p)(\alpha_D + \gamma_D)$*. (See Lemma 2). 1 will propose* (i^3, i^2) *if* $1 - \gamma_D < 1 - (1-p)(\alpha_D + \gamma_D)$*. 1 will mix between the two if the two sides are equal.*

2 *Lemma 1.*

3 $v^1 > c^2$ *and* $v^3 > c^2$*, so 1 will propose* (i^1, i^2) *if* $(v_L - c_L)(1-\gamma_D) > (v_H - c_L)(1-(1-p)(a_D+\gamma_D))$ *and* (i^3, i^2) *if the opposite holds (Lemma 2). 1 will mix between the two if the two sides are equal.*

4 *Analogous to 3.*

5,6 *In each case, one proposal gives either zero or less while the other gives either zero or more. Weak dominance implies that player 1 will always pick the latter.*

Lemma 6 explores player 1's best response when 2 proposes (i^1, i^2). For future convenience, define

$$z = \frac{\delta_1 v_H - v_L + (1-\delta_1)c_L}{(1-p)\delta_1(v_H - c_L)} \tag{A.11}$$

Lemma 6 *Suppose 2 proposes* (i^1, i^2) *in period 2.*

1. *Suppose*

$$\begin{aligned}(v^1, c^2, v^3) \;\in\; & \{(v_L, c_L, v_L), (v_H, c_L, v_H), (v_H, c_H, v_H),\\ & (v_H, c_L, v_L), (v_H, c_H, v_L)\}\end{aligned}$$

The probability of this is $1 - q + pq^2$*. Player 1 will accept 2's proposal.*

2. Suppose $(v^1, c^2, v^3) = (v_L, c_H, v_L)$ (probability $(1-p)q^2$). 1 will demand something.

3. Suppose $(v^1, c^2, v^3) = (v_L, c_L, v_H)$ (probability $pq(1-q)$). Player 1 will accept if $\alpha_P + \gamma_P > z$, where z is defined above. 1 will propose (i^3, i^2) if $\alpha_P + \gamma_P < z$. 1 will mix between the two if $\alpha_P + \gamma_P = z$.

4. Suppose $(v^1, c^2, v^3) = (v_L, c_H, v_H)$ (probability $(1-p)q(1-q)$). 1 will always propose (i^3, i^2).

Proof

1 *In each case, 1 will accept by Lemma 2.*

2 *Lemma 1.*

3 *By Lemma 3, 1 will accept if*

$$v_L - c_L > \delta_1(v_H - c_L)(1 - (1-p)(\alpha_P + \gamma_P)$$

1 will propose (i^3, i^2) if the opposite holds and mix if an equality holds.

4 *This holds by Lemma 3, using $v^1 < c^2 < v^3$.*

We can use Lemmas 5 and 6 to express player 1's best response as a function from $(\alpha_P, \gamma_P, \alpha_D, \gamma_D)$ to (r, s, t, u). Lemmas 7, 8, and 9 do this.

Lemma 7

$$(r,s) = \begin{cases} (1-q+pq^2, q(1-q)) & \text{if } \alpha_P + \gamma_P < z \\ (1-q+pq, q(1-p)(1-q)) & \text{if } \alpha_P + \gamma_P > z \\ (1-q+pq^2+\nu pq(1-q), q(1-q) - \nu pq(1-q)) & \text{for any } \nu \in [0,1] \\ & \text{if } \alpha_P + \gamma_P = z \end{cases}$$

Proof

This follows directly from Lemma 6.

To characterize t and u as functions of α_D and γ_D, we first need to establish some properties of the conditions on α_D and γ_D given in Lemma 5.

Consider the functions $w(\cdot)$, $x(\cdot)$, and $y(\cdot)$ in (α_D, γ_D) space. Let α_D take the vertical axis. See Chart A.

Lemma 8 *1. $\alpha_D = w(\gamma_D)$ is a line extending from the α_D axis, below the origin, to the point $(\frac{p}{1-p}, \; 1)$.*

2. $\alpha_D = x(\gamma_D)$ is a line extending from the α_D axis, above the origin, to the point $(\frac{p}{1-p}, \; 1)$.

3. $\alpha_D = y(\gamma_D)$ is a line extending from the origin to the point $(\frac{p}{1-p}, \; 1)$.

Proof

This can be verified with simple algebra.

We can now specify t and u as functions of α_D and γ_D. Let the open regions **W**, **X**, **Y**, and **Z** be as given in Chart A. The slope of $x(\gamma_D)$ may be either positive or negative. Define **T** as the triangular region $\{(\alpha_D, \gamma_D) \in [0,1]x[0,1]; \alpha_D + \gamma_D \leq 1\}$. The regions are defined by:

$$\mathbf{W} = \{(\alpha_D, \gamma_D) \in \mathbf{T} : \alpha_D \in [0, w(\gamma_D))\} \tag{A.12}$$

$$\mathbf{X} = \{(\alpha_D, \gamma_D) \in \mathbf{T} : \alpha_D \in (w(\gamma_D), y(\gamma_D))\} \tag{A.13}$$

$$\mathbf{Y} = \{(\alpha_D, \gamma_D) \in \mathbf{T} : \alpha_D \in (y(\gamma_D), x(\gamma_D))\} \tag{A.14}$$

$$\mathbf{Z} = \{(\alpha_D, \gamma_D) \in \mathbf{T} : \alpha_D \in (x(\gamma_D), 1]\} \tag{A.15}$$

From Chart A, Region **Z** is nonempty if and only if $x(0) < 1$.

In all regions, 1 demands i^1 or i^3 if and only if $(v^1, c^2, v^3) = (v_L, c_H, v_L)$. In Lemma 9, we will specify when 1 proposes (i^1, i^2) and when 1 proposes (i^3, i^2). Understand "all other cases" to exclude the case $(v^1, c^2, v^3) = (v_L, c_H, v_L)$.

Lemma 9 *If (α_D, γ_D) is contained in:*

1. *Region* **W**, *then $(t, u) = (q(1-p)(1-q), 1-q+pq)$. 1 proposes (i^1, i^2) iff $(v^1, c^2, v^3) = (v_H, c_H, v_L)$. 1 proposes (i^3, i^2) in all other cases.*

2. *The* **W/X** *Border, then 1 proposes* (i^1, i^2) *iff* $(v^1, c^2, v^3) = (v_H, c_H, v_L)$ *and mixes between the two proposals iff* $(v^1, c^2, v^3) = (v_H, c_L, v_L)$. *1 proposes* (i^3, i^2) *in all other cases.*

3. *Region* **X**, *then* $(t, u) = (q(1-q), 1-q+pq^2)$. *1 proposes* (i^1, i^2) *iff*

$$(v^1, c^2, v^3) \in \{(v_H, c_H, v_L), (v_H, c_L, v_L)\}$$

   1 proposes (i^3, i^2) *in all other cases.*

4. *The* **X/Y** *Border, then 1 proposes* (i^1, i^2) *iff*

$$(v^1, c^2, v^3) \in \{(v_H, c_H, v_L), (v_H, c_L, v_L)\}$$

   and mixes between the two proposals iff

$$(v^1, c^2, v^3) \in \{(v_L, c_L, v_L), (v_H, c_L, v_H), (v_H, c_H, v_H)\}$$

   1 proposes (i^3, i^2) *in all other cases.*

5. *Region* **Y**, *then* $(t, u) = (1-q+pq^2, q(1-q))$. *1 proposes* (i^3, i^2) *iff*

$$(v^1, c^2, v^3) \in \{(v_L, c_L, v_H), (v_L, c_H, v_H)\}$$

   1 proposes (i^1, i^2) *in all other cases.*

6. *The* **Y/Z** *Border, then 1 proposes* (i^3, i^2) *iff* $(v^1, c^2, v^3) = (v_L, c_H, v_H)$ *and mixes between the two proposals when* $(v^1, c^2, v^3) = (v_L, c_L, v_H)$. *1 proposes* (i^1, i^2) *in all other cases.*

7. *Region* **Z**, *then* $(t, u) = (1-q+pq, q(1-q)(1-p))$. *1 proposes* (i^3, i^2) *iff* $(v^1, c^2, v^3) = (v_L, c_H, v_H)$. *1 proposes* (i^1, i^2) *in all other cases.*

Proof

This follows directly from Lemmas 5 and 8.

Lemmas 7 and 9 completely specify (r, s, t, u) as functions of $(\alpha_P, \gamma_P, \alpha_D, \gamma_D)$. It now remains to specify $(\alpha_P, \gamma_P, \alpha_D, \gamma_D)$ as functions of (r, s, t, u). For brevity, define

$$\eta = \frac{p(c_H - c_L)}{c_H - (1-\delta_2)v_H - \delta_2(pc_L + (1-p)c_H)} \tag{A.16}$$

$$\iota = \frac{(1-p)(c_H - c_L)}{(1-\delta_2)v_H - c_L + \delta_2(pc_L + (1-p)c_H)} \tag{A.17}$$

$$\Psi = \frac{p(v_L - c_L)}{c_H - v_L + \delta_2 p(v_L - c_L)} \tag{A.18}$$

$$\kappa = \frac{1-p}{1-\delta_2 p} \tag{A.19}$$

Lemma 10 *1. η may be positive or negative. If positive, it exceeds $1/\delta_2$. $\phi(v_H, c_H) > 0$ if either $\eta < 0$ or $\eta > \frac{r}{\delta_2 t}$. $\phi(v_H, c_H) < 0$ if $\frac{r}{\delta_2 t} > \eta > 0$.*

2. $\iota \in (0,1)$. $\phi(v_H, c_L) > (<)\, 0$ if $\iota < (>)\, \frac{r}{\delta_2 t}$.

3. $\psi \in (0, 1/\delta_2)$. $\phi(v_L, c_H) > (<)\, 0$ if $\psi > (<)\, \frac{r}{\delta_2 t}$.

4. $\kappa \in (0,1)$. $\phi(v_L, c_L) > (<)\, 0$ if $\kappa < (>)\, \frac{r}{\delta_2 t}$. Also, $\kappa > \iota$.

Proof

These are simple applications of algebra. One can derive the properties of η, ι, ψ, and κ using (2.3) and the facts that p, q, δ_1, $\delta_2 \in (0,1)$. One can express the condition $\phi > (<)\, 0$ in terms of $\frac{r}{\delta_2 t}$ and η, ι, ψ, and κ by manipulating the definition of ϕ in (A.5).

Appendix B

Proofs of Theorems

To prove that the list of equilibria is complete, we classify equilibria according to the relations between r, t, δ_2, η, ι, ψ, and κ. Equations (A.16), (A.17), (A.18), and (A.19) define η, ι, ψ, and κ, respectively. Theorems 1-6 discuss the cases in which $\frac{r}{\delta_2 t} > \iota, \psi$, and κ. Theorem 7 discusses one equilibrium in which this does not hold. Theorem 8 shows that there is no other such equilibrium.

Proof of Theorem 1 *We will show that ZS is the only pure equilibrium that satisfies the three properties*

$$\frac{r}{\delta_2 t} > \iota, \psi, \text{ and } \kappa \tag{B.1}$$

$$\text{Either } \eta < 0 \text{ or } \eta > \frac{r}{\delta_2 t} > 0 \tag{B.2}$$

and (3.8). Then we will show that (3.9) is equivalent to (B.2).

(B.1) and (B.2) imply the description of 2's actions in Table 3.1, by Lemma 9.

Given 2's actions, we compute:

$$\begin{aligned}
\alpha_P &= \frac{pq}{1-q+pq} \\
\gamma_P &= 0 \\
\alpha_D &= 0 \\
\gamma_D &= 1
\end{aligned}$$

These use Bayes's rule and the definitions in (A.1),(A.2),(A.3), and (A.4). *(3.8) is equivalent to*

$$z > \frac{pq}{1-q+pq}$$

Thus, $\alpha_P + \gamma_P < z$. *Using this, Lemma 6 provides 1's responses when 2 proposes. From Lemma 7 we find*

$$r = 1 - q + pq^2$$

$(\alpha_D, \gamma_D) = (0,1)$ *must fall in region* $\mathbf{W}$ *in Chart A. Lemma 9 then provides 1's responses when 2 demands, as well as:*

$$t = q(1-p)(1-q)$$

Finally, (3.9) is an algebraic transformation of (B.2) that uses (A.16) and the formulae for r *and* t. *By (2.3), the two ratios in (3.9) fall in (0,1).*

We know the exogenous requirements hold in a nonempty, open region of the parameter space because they hold strictly at, for example,

$$(p, q, \delta_1, \delta_2, c_L, v_L, c_H, v_H) = (0.2, 0.2, 0.4, 0.2, 1.0, 1.2, 1.4, 2.0)$$

Proof of Theorem 2

We will show that SS is the only pure equilibrium that satisfies the three properties

$$\frac{r}{\delta_2 t} > \iota, \psi, \text{ and } \kappa \tag{B.3}$$

$$\text{Either } \eta < 0 \text{ or } \eta > \frac{r}{\delta_2 t} > 0. \tag{B.4}$$

and (3.10). The proof is analogous to that of Theorem 1. The only difference is (3.10), which is equivalent to

$$z < \frac{pq}{1-q+pq}$$

This condition confirms 1's acceptance of 2's proposal when $(v^1, c^2, v^3) = (v_L, c_L, v_H)$ *by Lemma 3. By Lemmas 7 and 9, respectively, we find*

$$r = 1 - q + pq$$

and

$$t = q(1-p)(1-q)$$

(3.11) is an algebraic transformation of (B.4) that uses (A.16) and the formulae for r *and* t*. The two ratios in (3.11) are in (0,1) by (2.3).*

We know the exogenous requirements hold in a nonempty, open region of the parameter space because they hold strictly at, for example,

$$(p, q, \delta_1, \delta_2, c_L, v_L, c_H, v_H) = (0.2, 0.2, 0.2, 0.2, 1.0, 1.2, 1.4, 2.0)$$

Proof of Theorem 3

We will show that TSZ is the only pure equilibrium that satisfies

$$\eta > 0 \tag{B.5}$$

$$\frac{r}{\delta_2 t} > \eta \tag{B.6}$$

and (3.12). Then we will show that (B.5) and (B.6) are equivalent to (3.14).

By Lemma 10, when $\eta > 0$, η *also exceeds* ι, ψ, *and* κ*. By (B.6), then,* $\frac{r}{\delta_2 t}$ *exceeds* η, ι, ψ, *and* κ*. Given this, Lemma 10 verifies that 2's strategy is as claimed.*

Given 2's strategy, 1's beliefs must be $(\alpha_D, \gamma_D) = (0, q)$ *and* $(\alpha_P, \gamma_P) = (q, 0)$ *by Bayes's rule. One can show that (3.13) is equivalent to* $w(q) > 0$*. Thus,* (α_D, γ_D) *must be in region* $\mathbf{W}$*(see Chart A). This establishes 1's strategy when 2 demands* i^2*. In addition, (3.12) is equivalent to* $q > z$*. But* $\alpha_P + \gamma_P = q > z$*. This establishes 1's strategy when 2 proposes* (i^1, i^2)*. Inspection shows that 1's strategy is the same as in Theorem 2. Since* r *and* t *depend on 1's strategy only, they must be the same as well:*

$$r = 1 - q + pq$$

$$t = q(1-p)(1-q)$$

(3.14) is equivalent to (B.5) and (B.6) by an algebraic manipulation.

We know the exogenous requirements hold in a nonempty, open region of the parameter space because they hold strictly at, for example,

$$(p, q, \delta_1, \delta_2, c_L, v_L, c_H, v_H) = (0.6, 0.8, 0.2, 0.6, 1.0, 1.4, 1.6, 2.0)$$

Proof of Theorem 4

We will show that DSZ is the only pure equilibrium that satisfies

$$\eta > 0 \tag{B.7}$$

$$\frac{r}{\delta_2 t} > \eta \tag{B.8}$$

and (3.15). The proof is directly analogous to that of Theorem 3. Player 2's strategy is the same, so $(\alpha_P, \gamma_P, \alpha_D, \gamma_D)$ *is also the same. However, (3.16) is equivalent to* $w(q) < 0$. *Thus,* (α_D, γ_D) *must now be in region* $\mathbf{X}$*(see Chart A). Lemma 9 then gives 1's strategy when 2 demands* i^2. *Lemma 9 also gives*

$$t = q(1-q)$$

(3.15) is equivalent to $q > z$. *Thus, we still have* $\alpha_P + \gamma_P = q > z$ *as in Theorem 3, so 1's strategy when 2 proposes* (i^1, i^2) *is as in Theorem 3. Thus,*

$$r = 1 - q + pq$$

(3.17) is equivalent to (B.7)-(B.8) by the formulae for r and t and the usual algebraic manipulation. The two ratios in (3.17) fall in (0,1) by (2.3).

We know the exogenous requirements hold in a nonempty, open region of the parameter space because they hold strictly at, for example,

$$(p, q, \delta_1, \delta_2, c_L, v_L, c_H, v_H) = (0.6, 0.2, 0.2, 0.8, 1.0, 1.2, 1.4, 2.0)$$

Proof of Theorem 5

We will show that TZZ is the only pure equilibrium that satisfies

$$\eta > 0 \tag{B.9}$$

$$\frac{r}{\delta_2 t} > \eta \tag{B.10}$$

as well as (3.18) and (3.19). The proof is analogous to that of Theorem 3. However, (3.19) is equivalent to $q < z$. *This implies that, if 2 proposes* (i^1, i^2) *and* $(v^1, c^2, v^3) = (v_L, c_L, v_H)$, *player 1 will propose* (i^3, i^2). *This is the only difference in strategies from Theorem 3. Coincidentally, 1's strategy is now equal to 1's strategy in Theorem 1. This implies that r and t will be the same as in Theorem 1:*

$$r = 1 - q + pq^2$$

$$t = q(1-p)(1-q)$$

(3.20) is equivalent to (B.9) and (B.10), using the formulae for r and t and the usual algebraic manipulation.

We know the exogenous requirements hold in a nonempty, open region of the parameter space because they hold strictly at, for example,

$$(p, q, \delta_1, \delta_2, c_L, v_L, c_H, v_H) = (0.6, 0.8, 0.6, 0.6, 1.0, 1.4, 1.6, 2.0)$$

Proof of Theorem 6

We will show that DZZ is the only pure equilibrium that satisfies

$$\eta > 0 \tag{B.11}$$

$$\frac{r}{\delta_2 t} > \eta \tag{B.12}$$

as well as (3.21) and (3.22). This proof is also analogous to that of Theorem 3, except that (3.21) and (3.22) are the opposites of (3.12) and (3.13), respectively. Since $(\alpha_P, \gamma_P) = (q, 0)$, *(3.22) implies that 1 will propose* (i^3, i^2) *if 2 proposes* (i^1, i^2). *Thus, 1's response to a proposal is the same as in Theorem 1, and*

$$r = 1 - q + pq^2$$

(3.21) implies that $(\alpha_D, \gamma_D) = (0, q)$ *is in Region* **X** *on Chart A. This establishes that 1's response to a demand will be as in Theorem 1, with the exception that* (v_H, c_L, v_L) *proposes* (i^1, i^2) *rather than* (i^3, i^2). *Lemma 9 also implies:*

$$t = q(1-q)$$

(3.23) is equivalent to (B.11) and (B.12) by the formulae for r and t, using the usual algebraic manipulation.

We know the exogenous requirements hold in a nonempty, open region of the parameter space because they hold strictly at, for example,

$$(p, q, \delta_1, \delta_2, c_L, v_L, c_H, v_H) = (0.6, 0.2, 0.4, 0.8, 1.0, 1.2, 1.4, 2.0)$$

Proof of Theorem 7

We will show that B is the only pure equilibrium in which

$$0 < \iota, \kappa < \frac{r}{\delta_2 t} < \psi \tag{B.13}$$

and that (3.24)-(3.26) are both necessary and sufficient for B to be an equilibrium.

(B.13) implies that 2 always proposes, by Lemma 10.

(B.13) holds only if $r < t$, since $\psi < 1/\delta_2$ by Lemma (10. But the only way to obtain $r < t$ in a pure equilibrium is

$$r = 1 - q + pq^2$$

and

$$t = 1 - q + pq$$

by Lemmas 7 and 9. These values of r and t imply a particular strategy for player 1. First, $\alpha_P + \gamma_P < z$, by Lemma 7. By Bayes's rule, $(\alpha_P, \gamma_P) = (pq, (1-p)q)$. This confirms that $q < z$ is necessary. And $q < z$ is equivalent to (3.24).

Given that $\alpha_P + \gamma_P < z$, Lemma 6 shows that 1's strategy is as claimed when 2 proposes. The value for t implies

$$(\alpha_D, \gamma_D) \in \mathbf{Z}$$

as well as 1's response when 2's demands, by Lemma 7.

Player 1's posteriors (α_D, γ_D) are arbitrary following the probability zero event of 2 demanding i^2. Thus, $(\alpha_D, \gamma_D) \in \mathbf{Z}$ implies only that $\mathbf{Z}$ be nonempty. But by (A.15) or Chart A, $\mathbf{Z}$ is nonempty if and only if

$$\frac{v_H - v_L}{(1-p)(v_H - c_L)} < 1$$

which is equivalent to (3.25).

(3.26) follows from (B.13), the necessary values of r and t, and the fact that $\kappa > \iota$ (Lemma 4).

We know the exogenous requirements hold in a nonempty, open region of the parameter space because they hold strictly at, for example,

$$\begin{aligned}&(p, q, \delta_1, \delta_2, c_L, v_L, c_H, v_H) = \\ &\quad (0.529, 0.588, 0.882, 0.882, 1.000, 1.588, 1.647, 2.000)\end{aligned}$$

Proof of Theorem 8

In Lemma 10, we show that $\eta > \iota$, ψ, and κ. Thus, Theorems 1 through 6 cover all equilibria in which $\frac{r}{\delta_2 t} > \iota, \psi$, and κ. (See the proofs of these Theorems.) We claim that Theorem 7 covers the only pure equilibrium in which $\frac{r}{\delta_2 t}$ is less than either ι, ψ, or κ.

Suppose $\frac{r}{\delta_2 t}$ is less than at least one of ι, ψ, and κ. Since ι, ψ, and κ are all less than $\frac{1}{\delta_2}$, we must have $r < t$. This can happen only if

$$r = 1 - q + pq^2 \tag{B.14}$$

$$t = 1 - q + pq \tag{B.15}$$

in a pure strategy equilibrium (Lemmas 7 and 9). By Lemma 7, we must have

$$\alpha_P + \gamma_P < z \tag{B.16}$$

*By Lemma 9, we must have (α_D, γ_D) contained in Region **Z**. This implies*

$$\alpha_D \in (x(\gamma_D), 1] \tag{B.17}$$

where $x(\cdot)$ is defined in (A.9).

Since $\iota < \kappa$ by Lemma 4, there are only 5 possibilities:

1. $0 < \iota,\ \psi < \frac{r}{\delta_2 t} < \kappa$
2. $0 < \iota,\ \kappa < \frac{r}{\delta_2 t} < \psi$
3. $0 < \iota < \frac{r}{\delta_2 t} < \psi,\ \kappa$
4. $0 < \psi < \frac{r}{\delta_2 t} < \iota,\ \kappa$
5. $0 < \frac{r}{\delta_2 t} < \iota,\ \psi,\ \kappa$

(Commas indicate that an inequality applies to several variables. For example, (1) means that $0 < \iota < \frac{r}{\delta_2 t} < \kappa$ and $0 < \psi < \frac{r}{\delta_2 t} < \kappa$.)

Order (2) can occur in equilibrium (Theorem 7). The others cannot:

1. *By (B.14), (B.15), (A.17), and (A.19), $\iota < \frac{r}{\delta_2 t} < \kappa$ can occur only if*

$$\delta_2 \in \left(\frac{1 - q + pq^2}{1 - q + pq^2 + pq(1-p)(1-q)}, \frac{(1 - q + pq^2)(v_L - c_L)}{(1 - q + pq^2)(v_L - c_L) + pq(1-q)(1-p)(c_H - c_L)} \right)$$

A bit of manipulation shows that this interval is nonempty only if $v_L > c_H$*, contrary to assumption.*

2. *This can occur. See Theorem 7.*

3. *See (1).*

4. *By Lemma 10, 2 proposes only if* $(v^2, c^1) = (v_H, c_H)$*. 2 demands in the other cases. By Bayes's rule,*

$$(\alpha_D, \gamma_D) = \left(\frac{pq}{p+q-pq}, \frac{q-pq}{p+q-pq}\right)$$

   By (B.17), we must have $\alpha_D > x(\gamma_D)$*. Using the definition of* $x(\cdot)$ *in (A.9), this can be shown to require* $p < 0$*.*

5. *By Lemma 10, player 2 proposes when* $(v^2, c^1) \in \{(v_H, c_H), (v_L, c_H)\}$ *and demands in the other cases. Bayes's rule implies that* $(\alpha_P, \gamma_P) = (0, q)$*. By (B.16), we must have* $q < z$*. Bayes's rule also implies that* $(\alpha_D, \gamma_D) = (q, 0)$*. By (B.17) and (A.9), this implies that*

$$q \in \left(\frac{v_H - v_L}{(1-p)(v_H - c_L)}, 1\right]$$

   But z *falls strictly below this interval since we can rewrite (A.11) as*

$$z = \frac{\delta_1(v_H - v_L) + (1-\delta_1)(c_L - v_L)}{\delta_1(1-p)(v_H - c_L)}$$

   and since $(1-\delta_1)(c_L - v_L) < 0$*. This contradicts* $q < z$*.*

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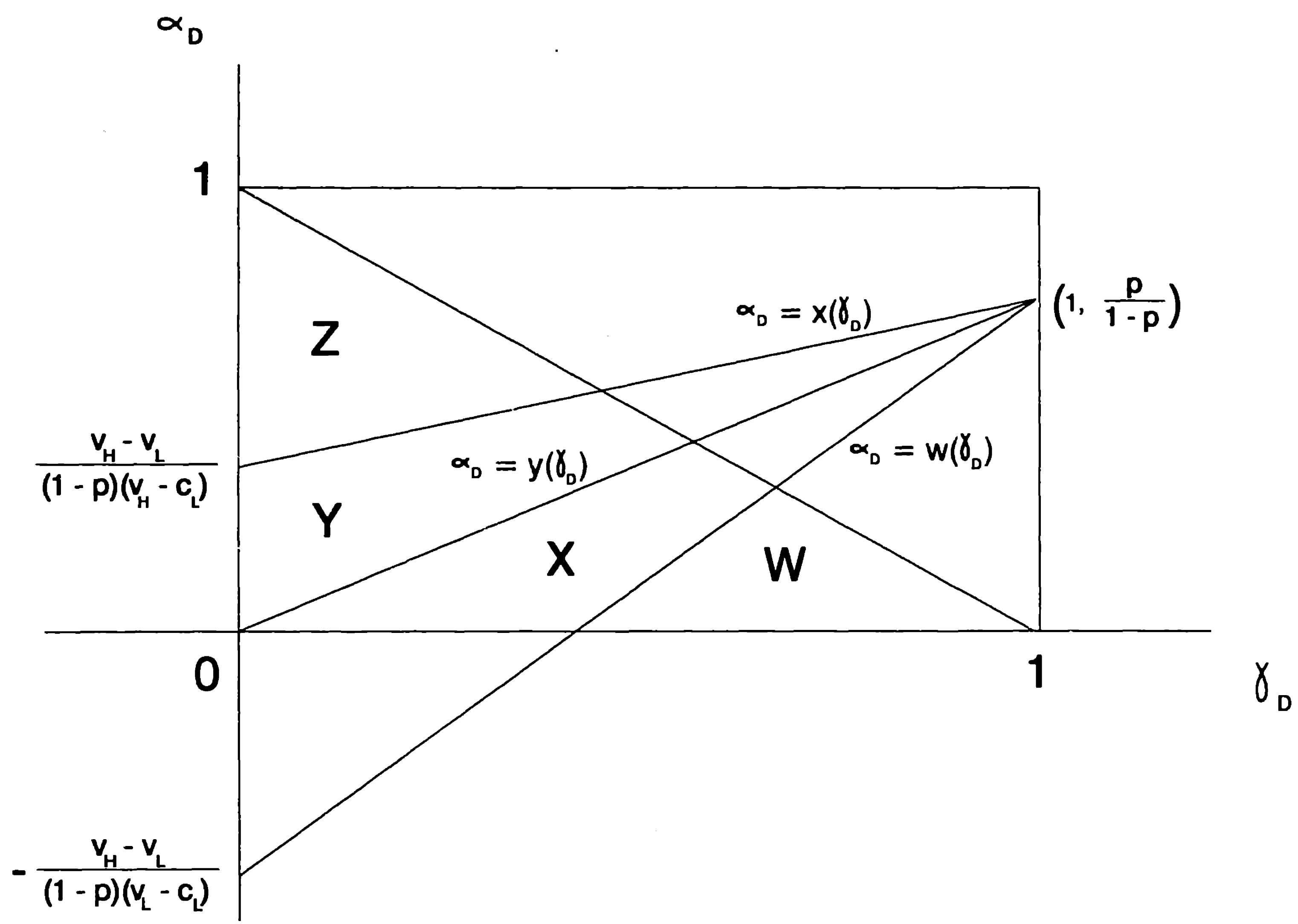

α_D
1
$\frac{v_H - v_L}{(1-p)(v_H - c_L)}$
0
$-\frac{v_H - v_L}{(1-p)(v_L - c_L)}$
1
γ_D
$(1, \frac{p}{1-p})$
$\alpha_D = x(\gamma_D)$
$\alpha_D = y(\gamma_D)$
$\alpha_D = w(\gamma_D)$
Z
Y
X
W

Chart A

Part II

SEARCH WITH TELEPHONES AND DIFFERENTIATED PRODUCTS

1. INTRODUCTION

Before Stigler (1961) and Diamond (1971), the auction was the dominant paradigm for market activity. Economists assumed that markets behaved as if there was an auctioneer who, with full knowledge of all demand and supply functions, selected a market clearing price vector.

In fact, most consumer markets are decentralized. Buyers must collect price information at a cost. They may do this by visiting firms, calling, reading advertisements, or talking to friends. Conversely, firms cannot spread information about their prices costlessly. They must pay for advertising or rely on word of mouth.

Search theory tries to create a new foundation for economics while explicitly modelling the costs of gathering price information. Most search theorists have assumed, for simplicity, that buyers have only one method of search. A few authors have relaxed this assumption to include advertising (Butters 1977; Salop and Stiglitz 1977; Varian 1980). This paper presents a model in which buyers can also call firms on the telephone prior to visiting. This permits a buyer to visit only firms that have sufficiently low prices.

Calling ahead is most common in markets for consumer durables and services. There are four assumptions that are appropriate to most such markets. The first is that buyers have unit demands. This means that each buyer wants to buy only one good or service. The second assumption is that a firm makes a take it or leave it offer to a buyer. The buyer cannot "haggle". Third, we assume that each search has a discrete, positive cost. This is appropriate when search has costs other than delayed consumption.

Finally, we assume sequential search. Buyers collect prices from one firm after another, rather than several firms at once. Search may be nonsequential if buyers can obtain a concise list of prices (Salop and Stiglitz 1977; Varian 1980). Without such a list, a buyer cannot usually commit to continue searching if her first search turns up a particularly good price. In a market with many anonymous buyers, such a commitment would also be inferior to a sequential strategy. Thus, sequential search is appropriate when a concise price list is not available.

These four assumptions fit many markets for consumer durables and services. But as Salop and Stiglitz (1976) first noted, together the assumptions lead to complete market failure. Why? Let θ be the lowest valuation among buyers who search. No firm will charge less than θ. Else the lowest price firm could raise its price just a bit. Since prices elsewhere would be at best only slightly lower, all the firm's visitors would still purchase. So the lowest price must be at least θ. But then, since search is costly, a buyer with valuation θ would not even enter the market. Therefore, there cannot *be* a lowest valuation buyer who searches. All buyers stay out. There is no trade (Salop and Stiglitz 1976).

This irritating but profound result spurred research in search theory. We now know that trade can occur if *any* of the four assumptions is relaxed. Trade is sustainable if buyers have demand curves rather than unit demands (Diamond 1971; Reinganum 1979; Salop and Stiglitz 1982; Benabou 1990), can bargain with firms (Rubinstein and Wolinsky 1985; Wolinsy 1987; Bester 1988), or can search nonsequentially (Butters 1977; Salop and Stiglitz 1977; Varian 1980; Burdett and Judd 1980; MacMinn 1980). Trade can also occur if the only cost of search is delayed consumption (e.g., Diamond [1982a]).

Although they give better outcomes, these approaches do not apply to all retail markets for consumer durables and services. Such markets often feature unit demands, posted prices, sequential search, and search costs other than delay. Our explanation is that buyers do not know their valuations *ex ante*. A buyer does not know precisely how much she is willing to pay for an item until she sees it.[1] If her valuation at a given store is low enough, her realized payoff will be negative. She accepts this risk if her expected payoff is positive. Other models with unknown valuations appear in Wolinsky (1987), Bester (1988), Diamond (1990), and Daughety and Reinganum (1991).

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1 This reduces the relative value of calling. A buyer can learn more about a product by visiting than by calling.

Prior work on alternate modes of search has concentrated on advertising. Butters (1977) presents a model in which firms first send price offers to buyers by mail. Buyers then purchase at the lowest price they receive. Trade is sustained because search is nonsequential and costless. There can be price dispersion.

Salop and Stiglitz (1977) and Varian (1980) assume that buyers can purchase a newspaper or magazine that contains every firm's price. With this information, a buyer can visit the lowest price firm. Without the information, a buyer can visit only a random firm. Trade occurs in these models because some buyers search nonsequentially. There can be price dispersion as well.

In our model, each buyer is matched to a single firm in each period. The buyer has a valuation, θ, for the firm's product. θ takes one of two possible values. The firm never observes θ directly. The buyer does not know θ until she visits the firm.

At a cost, the buyer may call the firm to ask for a price quote. The firm may refuse to quote. If the firm quotes, it incurs a small cost of price retrieval. Whether or not she called ahead, the buyer may then visit the firm. Visiting is also costly. If she visits, the firm reveals its price. The buyer then discovers θ. She decides whether or not to buy. If she buys, she leaves the market. If not, she can search again in the next period.

The model has four equilibrium outcomes. Purchases may take place at one, two, or three different prices. There is also a no trade outcome.

In the three price outcome, some firms select the reservation price. (A caller who is quoted the reservation price is indifferent about visiting. We assume all callers will visit if quoted this price.) Another group of firms picks a lower price. Both groups of firms are willing to give telephone quotes. A third group of firms selects a price above the reservation price. These firms do not quote. Some buyers call ahead and visit only firms that quote. Other buyers visit firms without calling ahead.

In the two price outcome, some firms select a price below the reservation price, and some pick a price above it. No buyers call ahead in this outcome. Firms believe that callers are likely to visit even if a firm refuses to quote. This belief leads all firms to refuse to quote. Thus, calling is not worthwhile. Since refusing is not a signal of price, a buyer who did call ahead would indeed be willing to visit a refusing firm.

There is a range of parameters for which either outcome can occur. Holding parameters constant, prices are higher in the two price outcome than in the three price outcome. Buyer payoffs are lower and profits are higher in the two price outcome. As Salop and Stiglitz (1977) note, informed buyers create a positive externality. The more buyers who call ahead, the more sensitive a firm's volume is to its price. This leads firms to select lower prices, which helps all buyers. Conversely, buyers who do not call ahead hurt other buyers by encouraging firms to charge higher prices.

There is also an outcome in which all buyers call ahead. Some firms quote and some refuse. Firms that quote set price equal to marginal cost, including the cost of quoting. Firms that refuse select from any price distribution that is high enough to make buyers prefer to call ahead. Buyers visit only firms that quote. Thus, all purchases take place at a marginal cost. This is lower than any price in the two and three price outcomes. For fixed parameters, buyer payoffs are also higher in this outcome and firm profits are lower. Since all buyers call ahead, the positive externality is stronger here than in the other outcomes.

Finally, there is a no trade outcome. This outcome can occur for any parameters. There is a range of parameters for which no other outcome is possible. The explanation for this may lie in our restrictive assumptions about what firms can do. If there were a potentially profitable market that would not exist with calling and visiting alone, firms could advertise.

We have sketched how prices and payoffs vary across outcomes, for fixed parameters. Prices and payoffs also vary with parameters in a given outcome. These results are sensitive to the assumption that θ has only two possible values. However, they do illustrate some unusual possibilities.

Calling costs have no effect on payoffs in any of the outcomes. Buyers are indifferent to a decrease in calling costs because more firms select high prices in the distribution. However, the support of the price distribution is unchanged. This leaves firms indifferent. This suggests that a fall in marginal calling costs due to the spread of telephones or "Yellow Pages" need not make consumers better off or firms worse off.

Lower visiting costs have the same effects, with one exception. In the three price outcome, lower visiting costs raise the support of the price distribution. Firms benefit, while buyers are actually worse off. In the real world, this result suggests that retail expansion may sometimes be funded in part by a reduction in consumer surplus due to higher prices.

Our model differs from models of search with advertising in a critical respect. In the two and three price outcomes, the lowest price is bounded above marginal cost as search and quoting costs shrink to zero. In models of search with advertising, the lowest price always equals marginal cost (Butters 1977; Salop and Stiglitz 1977).

Since we model both calling and visiting as sequential processes, a firm that undercut the lowest price would not increase its sales volume. Butters (1977) and Salop and Stiglitz (1977) assume that buyers read advertisements nonsequentially. Under this assumption, the same firm would increase its volume. This creates a competitive pressure that drives the lowest price to marginal cost in the advertising models.

2. THE MODEL

There is a countably infinite number of firms. The firms exist forever. There is also a countably infinite number of buyers. Buyers enter the market and remain until they buy or voluntarily exit. A buyer who exits in one period, for either reason, is replaced in the next.

Corresponding to each buyer-firm pair, there is a valuation θ of the buyer for the firm's product. We assume that $\theta \in \{\theta_1, \theta_2\}$, where $\theta_1 < \theta_2$. For each firm, the probability that a buyer will have valuation θ_1 is $\rho \in (0,1)$. For each buyer, the probability that $\theta = \theta_1$ at a firm is also ρ. A buyer can discover her valuation at a firm only by visiting that firm.

In each period, firms first select prices.[2] Then buyers are matched to stores. Each buyer has a single store, and vice versa. The matching scheme is described in Appendix A. The buyer may first call the store to request a price quote. Calling costs the buyer s^c. The firm may quote its price. Quoting costs the firm q. The firm may

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[2] We assume that firms set their prices before calls or visits are received. This forces the firm to quote the same price that it offers. This consequence is realistic because the cost of tracking different quotes is likely to be high in a large store.

also refuse to quote, at zero cost.[3]

Whether or not she called ahead, the buyer may then visit the store. Visiting costs the buyer s^v. If a buyer visits, the firm costlessly[4] reveals its price. We call this action an "offer", as opposed to a "quote", which is given over the telephone. The buyer then inspects the good and discovers her valuation, θ. If the buyer purchases the good, the firm incurs a constant production cost of c.

Buyers cannot call or visit firms from prior periods. This assumption is not essential. Since the distribution of firm strategies is stationary, callers would have fixed reservation prices with recall as well.

If a buyer visits n^v times and calls n^c times before paying p for an item for which she has valuation θ, her payoff is

$$\theta - p - s^v n^v - s^c n^c$$

Buyers maximize expected payoffs.

For the firm, let δ_q equal 1 if a firm quotes to a given buyer, and 0 otherwise. Let δ_b equal 1 if the buyer

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[3] q includes time costs of retrieving and dispensing information about the price and what exactly it buys. These costs can be considerable for firms with many products. We normalize the cost of refusing to quote at zero. (Firms must answer the telephone.)

[4] In practice, firms often post prices. Each offer thus incurs no additional cost.

visits and buys, and 0 otherwise. Let the firm's price be p. Then the firm's payoff in a period is

$$\delta_b(p - c) - \delta_q q$$

In each period, a firm maximizes the expectation of its payoff in that period.

We assume that a given player uses a pure strategy. Different players may have different pure strategies. We also impose the requirements of Markov perfection (Fudenberg and Tirole 1991, pp. 501 ff.; Maskin and Tirole 1993) and perfect Bayesian equilibrium, or "PBE" (Fudenberg and Tirole 1991, pp. 331 ff.). In our model, equilibria always exist even with these restrictions.

Markov perfection requires that strategies can depend only on payoff-relevant aspects of histories. Since buyers cannot revisit or recall firms, players' experiences in prior periods are not relevant to their payoffs. In addition, there are no enduring, payoff-relevant quantities in this model. Thus, Markov perfection implies that the equilibria will all be steady states.

We also assume that a buyer always buys or visits if offered or quoted, respectively, a price that makes her indifferent. Any lower price would always secure a sale or a visit, so this assumption is necessary for firms to have optimal prices.

3. CHARACTERIZATION OF EQUILIBRIA

The model has parameters $(s^c, s^v, \theta_1, \theta_2, \rho, q, c) \in \mathbb{R}^7$. We ignore equilibria that can occur only on parameter sets of measure zero in $\mathbb{R}^7$.

We assume:

(3.1) $\quad s^v > 0$

(3.2) $\quad s^c > 0$

(3.3) $\quad q > 0$

(3.4) $\quad 0 < \rho < 1$

Since quoting costs are likely to be small in most markets, we assume

(3.5) $$q < \frac{1 - \rho}{\rho}\left[\theta_2 - \theta_1\right]$$

Finally, we assume that

(3.6) $\quad \theta_1 > c + q$

Let V be a buyer's expected payoff. Let π be a firm's expected per-period payoff. In equilibrium, all buyers must receive V and all firms must receive π. Also define

(3.7) $\quad g = (1 - \rho)(\theta_2 - \theta_1)$

This is the gap between a buyer's expected valuation, $\rho\theta_1 + (1 - \rho)\theta_2$, and a buyer's lowest possible valuation, θ_1.

The first lemma derives formulae for the reservation price, p*. It also shows that p* is unique.

LEMMA 3.1

Let p* be a price such that, if quoted p*, a buyer is indifferent about visiting. Then

a) if $0 < s^V < g$, then $p^* = \theta_2 - V - \dfrac{s^V}{1 - \rho}$ and $\theta_1 - V < p^* < \theta_2 - V$

b) if $0 < g < s^V$, then $p^* = g + \theta_1 - V - s^V$ and $p^* < \theta_1 - V$

PROOF

Appendix B.■

The next lemma establishes an upper bound on buyers' payoffs. We use this result in the proof of Lemma 3.3.

LEMMA 3.2

In any equilibrium, $V < \theta_2 - c$.

PROOF

Appendix B.■

If a firm has the price $p = \theta - V$, then a visitor with valuation θ would have continuation payoff $\theta - p = V$ from purchasing. But V is also the continuation payoff from not purchasing. Thus, such a visitor would be just willing to purchase.

This shows that there are only three prices at which transactions can take place in any equilibrium. One is $\theta_1 - V$, the price at which low valuation visitors are just willing to buy. Another is $\theta_2 - V$, the price at which high valuation visitors are just willing to buy. The third is p^*, the price at which callers are just willing to visit.

Any other price could be increased without affecting the visiting or purchasing decisions of any buyer. (Note that there may be other prices at which no units are sold, since firms are indifferent among all such prices.)

Firms that select among these three prices may or may not be willing to quote. Lemma 3.3 gives the relation between possible prices and willingness to quote.

LEMMA 3.3

(a) If there are visitors who do not call ahead, then any firm that would refuse to quote, should a buyer call, selects a price in the set $\{\theta_1 - V, \theta_2 - V\}$.

(b) If there are buyers who call ahead, then a firm that quotes selects either $\theta_1 - V$ or p^*.

(c) If there are <u>only</u> visitors who do not call ahead, then any firm that would be willing to quote if a buyer <u>were</u> to call selects $\theta_1 - V$.

PROOF

(a) Consider a firm that receives visitors, but that would refuse to quote if a buyer called. Lemma 3.2 shows that selecting $\theta_2 - V$ would give positive profits. Thus, the firm will select a price at which some visitors purchase. By the above argument, the price must be either $\theta_1 - V$, p^*, or $\theta_2 - V$. But the firm will not select p^* because it could raise its price without changing the purchase decision of any visitor.

(b) Quoting has a positive cost. Thus, a firm will quote only if it has a positive chance of a transaction. This shows that the firm must select $\theta_1 - V$, p^*, or $\theta_2 - V$. But if the firm selects $\theta_2 - V$, it will subsequently refuse to quote, since $p^* < \theta_2 - V$ (Lemma 3.1).

(c) Now suppose a firm receives visitors and no callers, but would be willing to quote. As in (a), such a firm must select either $\theta_2 - V$, p^*, or $\theta_1 - V$. It cannot select p^* because a higher price would not change the behavior of any buyer in equilibrium. And if the firm selected $\theta_2 - V$, it would not be willing to quote should a buyer call.■

Let V^c be the expected payoff to a buyer who first calls a store and, after the call, follows an optimal strategy. Let V^v be the expected payoff to a buyer who first visits a store without calling ahead and, once in the store, follows an optimal strategy. Since buyers can also exit,

(3.8) $$V = \max \langle V^c, V^v, 0\rangle$$

Let π^q be the optimal per-period payoff to a firm that is forced to quote. Let π^{nq} be the optimal per-period payoff to a firm that is prevented from quoting. Due to the pure strategy and Markov assumptions, firms will either always quote or never quote. Thus, (3.9) $\pi = \max \langle \pi^{q}, \pi^{nq} \rangle$

We present the four equilibrium outcomes in Theorems 4.1-7.1. Proofs are in Appendix B. The outcomes are classified according to values of V^{v}, V^{c}, π^{q}, and π^{nq}. In Theorem 8.1, this classification leads to a very brief proof that there are no other outcomes.

4. A THREE PRICE EQUILIBRIUM

For some parameters, there is an intuitive three price equilibrium. Some buyers visit and some call. Some firms quote. These firms mix between the prices $\theta_1 - V$ and p^*. Some firms select $\theta_2 - V$ and refuse to quote. Callers incur calling costs but avoid wandering into high-price stores.

Theorem 1 describes the three price equilibrium. It also gives restrictions that the parameters must satisfy. We use the following notation:

(4.1) $$\beta = \frac{s^c}{g - s^v}$$

(4.2) $$\nu = \frac{\rho s^v}{g - (1 - \rho)s^v - \rho q}$$

THEOREM 4.1

There is a unique equilibrium that satisfies

(4.3) $$V^v = V^c \geq 0$$

and

(4.4) $$\pi^q = \pi^{nq} > 0$$

In this equilibrium,

(4.5) β of firms select $\theta_1 - V$

(4.6) $1 - \beta - \dfrac{s^c}{s^v}$ of firms select $p^* = \theta_2 - V - \dfrac{s^v}{1 - \rho}$

Both groups of firms quote. There is a third group of firms:

(4.7) $\dfrac{s^c}{s^v}$ of firms select $\theta_2 - V$

These firms do not quote.

A proportion ν of buyers always call ahead and $1 - \nu$ never call ahead. Callers have reservation price $p*$ for visiting. Callers never visit firms that refuse to quote. Visitors with valuation θ, facing price p, buy if and only if $\theta - p \geq V$.

Payoffs in this equilibrium are:

$$(4.8) \qquad V = \theta_1 - c - \frac{g - s^V}{\rho}$$

$$(4.9) \qquad \pi = \frac{g - s^V}{\rho} - \nu q$$

This strategy profile is an equilibrium if and only if the following conditions hold:

$$(4.10) \qquad s^C + s^V < g$$

$$(4.11) \qquad s^C < s^V\left[1 - \frac{s^V}{g}\right]$$

$$(4.12) \qquad s^V > g + \rho(c - \theta_1)$$

$$(4.13) \qquad s^V < g - \rho q$$

■

The lowest price, $\theta_1 - V$, equals

$$c + \frac{g - s^V}{\rho}$$

It rises as s^V falls and is insensitive to s^C. Prices do not converge to marginal cost as frictions dwindle.

Calling costs have no effects on payoffs. However, higher visiting costs leave buyers better off and firms worse off. This occurs because the support of the price distribution falls.

5. A TWO PRICE OUTCOME

The model also has a two price outcome. No buyers call ahead. Two equilibria support this outcome. In one, low price firms are willing to quote, but buyers do not call because s^c is too high. In the other equilibrium, no firms are willing to quote. Firms believe that, even if they refuse to quote, a caller will visit anyway with a "sufficiently high" probability. Since refusing signals nothing, visiting a refuser is indeed a weak best response, conditional on calling.

For the second equilibrium, parameters need satisfy only two requirements. First, search costs cannot be too high:

$$s^v < g$$

Second, the probability of a low valuation must be large relative to the dispersion of valuations:

$$\frac{1 - \rho}{\rho} < \frac{\theta_1 - c}{\theta_2 - \theta_1}$$

Otherwise, prices would be so high that buyers would prefer not to search.

The first equilibrium has the additional condition:

$$s^c > s^v\left[1 - \frac{s^v}{g}\right]$$

Theorem 5.1 omits this requirement because the outcome can occur without it.

THEOREM 5.1

There is a unique outcome satisfying

(5.1) $$V^v > 0;\ V^v > V^c$$

In this outcome,

(5.2) $$\frac{s^v}{g} \text{ of firms offer } \theta_1 - V$$

(5.3) $$1 - \frac{s^v}{g} \text{ of firms offer } \theta_2 - V$$

Buyers visit without calling ahead. A buyer who is offered price p and has valuation θ buys if and only if $p \le \theta - V$.

Payoffs are

(5.4) $$V = \theta_1 - c - \frac{g}{\rho}$$

(5.5) $$\pi = \frac{g}{\rho}$$

This outcome is supported by an equilibrium if and only if

(5.6) $$\frac{1 - \rho}{\rho} < \frac{\theta_1 - c}{\theta_2 - \theta_1}$$

and

(5.7) $$s^v < g$$

■

The lowest price, $c + g/\rho$, exceeds c and does not approach it as frictions decrease. In addition, the price distribution has a higher support than in the three price outcome. (That is, the lowest price and the highest price are both higher in the two price outcome than in the three price outcome.) This is because no buyers call ahead, so firms face a less elastic demand than in the three price outcome.

A change in visiting costs does not affect payoffs, profits, or the support of the price distribution. This contrasts with the three price outcome. An increase in s^v does lead more firms to charge the low price. Since no buyers call, a small change in s^c has no effect on the outcome.

6. A ONE PRICE OUTCOME

The model also has an outcome in which all transactions take place at price c + q. Some firms do select other prices. These firms do not quote. Since all buyers call ahead and no buyers visit refusers, these firms have no sales. The prices of refusers must be high enough to make buyers call ahead. These prices are not uniquely determined.

THEOREM 6.1

There is an outcome satisfying

(6.1) $V^c \geq V^v$; $V^c > 0$

(6.2) $\pi^q = \pi^{nq} = 0$

All buyers call ahead. No buyers visit refusers. β of firms quote. These firms select the price $\theta_1 - V = c + q$. $1 - \beta$ of firms do not quote. These firms select any distribution of prices that guarantees that $V^v \leq V^c$.

Payoffs are

(6.3) $V = \theta_1 - c - q$

(6.4) $\pi = 0$

This outcome is supported by equilibria if and only if

(6.5) $s^c + s^v < g$

(6.6) $s^v > g - \rho q$

(6.7) $s^c < s^v\left[1 - \frac{s^v}{g}\right]$

■

Since all buyers call ahead, demand is more elastic than in the prior two outcomes. This leads to the lowest price distribution, the highest buyer payoffs, and the lowest profits. In particular, the price, $c + q$, is lower than the lowest price in the three price outcome, $c + (g-s^V)/\rho$, when that outcome can occur, by (4.13). And $c + q$ is always lower than the lowest price in the two-price outcome, $c + g/\rho$, by (3.5).

Also by (3.5), the one price buyer payoff, $\theta_1 - c - q$, is greater than the two price buyer payoff, given in (5.4). By (4.13), the one price buyer payoff is also greater than the three price buyer payoff, given in (4.8), when the three price outcome can occur. Firm profits are also clearly the lowest in the one price outcome.

7. A NO TRADE OUTCOME

The model also has a no trade outcome.

THEOREM 7.1

There is a unique outcome that satisfies either

(7.1) $V^c < 0,\ V^v < 0$

or the pair

(7.2) $V^c = V^v = 0$

(7.3) $\pi^q = \pi^{nq} = 0$

All buyers exit immediately. There is no trade. $V = \pi = 0$.

There is an equilibrium satisfying (7.1) and supporting this outcome for any parameters in $\mathbb{R}^7$.■

The no trade outcome occurs if firms select a distribution of prices that is high enough that V^v and V^c are both strictly negative. Firms are willing to select such prices because they do not expect to receive any callers or visitors. Obviously, this outcome can occur for any parameters.

8. NONEXISTENCE OF OTHER OUTCOMES

Theorem 8.1 shows that there are no outcomes other than those we have already described.

THEOREM 8.1

Theorems 4.1–7.1 give a complete list of possible outcomes, except on a parameter set of measure zero.■

9. A MAP OF THE OUTCOMES IN PARAMETER SPACE

Charts 1 and 2 map which outcomes occur in the various regions of parameter space. The curve in each chart, which is given by

(9.1) $$s^c < s^v\left[1 - \frac{s^v}{g}\right],$$

is strictly concave and has a slope of 1 at the origin. This shows that outcomes with calling occur only if $s^c < s^v$. Since calling gives a buyer less information than visiting, visiting dominates calling if $s^c \geq s^v$.

The widest range of outcomes occurs under the curve, when s^c is small and s^v is of moderate size. These are plausible conditions for most retail markets. Trade is not sustainable in a neighborhood of the origin in Chart 2. This may be due to our assumption that there are no other ways of conveying and gathering information, such as advertising.

10. DISCUSSION

We have presented a model of sequential search with telephones and differentiated products. The model has some unusual features. First, a decline in calling costs never affects buyer or firm payoffs. In the two price outcome, this is because no one calls ahead. In the three price outcome, lower calling costs lead more firms to charge the middle price. Since the support of the price distribution does not change, firm profits are unchanged. Curiously, the shift in price weights leaves buyer payoffs unchanged as well. In the one price outcome, a fall in calling costs leads precisely enough firms stop quoting to leave buyer payoffs unchanged.

Changes in visiting costs also have unusual effects. In the two price outcome, lower visiting costs lead exactly enough firms to switch to the high price to leave buyer payoffs unaffected. In the one price outcome, lower visiting costs lead precisely enough firms to switch to marginal cost and to quote that buyers are neither harmed nor helped. In the three price outcome, higher visiting costs lower the support of the price distribution. Firm profits are lower. Buyers' payoffs actually increase. Salop and Stiglitz (1982, p. 126) report the same finding in a different model.

We can provide no obvious dynamic story for these results. Since we assume perfect foresight and have no state variables, the players simply jump to a new steady state in response to a change in the environment. The results are novel enough that work on the dynamics might be worthwhile.

We have found price distributions that differ markedly from the competitive model and even from other search models. In the two and three price outcomes, the lowest price is bounded above marginal cost as search costs shrink to zero. Models of search with advertising have quite different features. In Butters (1977), the lowest price always equals marginal cost even with positive advertising costs (p. 470). All purchases take place at marginal cost in the limit as advertising costs go to zero (Butters 1977, p. 471). The model of Salop and Stiglitz (1977, pp. 494, 503) has the same properties.

The difference comes from these authors' assumption that buyers can search nonsequentially. In Butters (1977), buyers may receive several advertisements simultaneously. In Salop and Stiglitz (1977), buyers can purchase a complete price list. With nonsequential search, a firm can always increase its expected sales volume by lowering its price. This drives the lowest price to marginal cost.

In our model, calling and visiting are both sequential. Thus, a firm that charged the lowest price would not receive any new callers or visitors if it lowered its price. (It receives no new visitors in the two and three price outcomes because the lowest price is already below the reservation price.) This makes demand perfectly inelastic below the lowest price. The lowest price can easily exceed marginal cost.

When search costs shrink under nonsequential search, the price distribution converges to a competitive uniformity. As the cost of the price list goes to zero in Salop and Stiglitz (1977), all buyers purchase the list. The resulting competition leads all firms to charge marginal cost. In Butters (1977), lower advertising costs lead firms to send out more price offers. In the limit, each buyer receives at least one offer of marginal cost. Firms do advertise higher prices in the limit, but all purchases are at marginal cost (Butters 1977, pp. 470-471).

In our model, demand remains perfectly inelastic below the lowest price even as search costs go to zero. The lowest price does not converge to marginal cost. Sequential search blocks convergence to the competitive outcome as search costs shrink.

APPENDIX A

THE MATCHING SCHEME

Assume ρ is rational. We suppose that there is one firm for each element of $\mathbb{N} = \{1,2,...\}$. Similarly, there is one buyer for each element of $\mathbb{N}$. When buyers leave the market, they are replaced by new buyers.

Let $\rho = m/n$ for some $m \in \{0,1,...\}$ and $n \in \{1,2,...\}$. Each firm offers a product with a bundle of features in the set $\{p_1,...,p_n\} = P$. The differences between the feature-bundles can only be distinguished in person and cannot be described over the telephone. Each bundle p_i is offered by an infinite number of firms. Let $\alpha^f:\mathbb{N} \to P$ be a function such that $\alpha^f(i)$ is the feature-bundle carried by firm i.

Buyers have taste parameters that take on one of the n possible values in the set $\{t_1,...,t_n\} = T$. For any t_i, there is an infinite number of buyers with tastes t_i. The function $\alpha^b:\mathbb{N} \to T$ gives the taste parameter of each buyer. For $i \in \{1,...,n\}$, a buyer with tastes t_i has valuation θ_1 for the m consecutive features that start with p_i and wrap around after p_n is reached. The buyer has valuation θ_2 for the other features. There are many other assignments that would work.

The random matching procedure is as follows. First, each buyer rolls a fair n-sided die, as does each firm. Let the outcome of this process be given by $\{\sigma^f(\cdot),\sigma^b(\cdot)\}$, where σ^f,σ^b are functions from $\mathbb{N}$ to $\{1,\ldots,n\}$. $\sigma^f(i) = j$ if firm i rolls j and $\sigma^b(i) = j$ if buyer i rolls j.

We seek to match each buyer $i \in \mathbb{N}$ with a firm of type $\sigma^b(i)$, and each firm $i \in \mathbb{N}$ with a buyer of type $\sigma^f(i)$, such that no buyer is matched with a firm that she visited in a prior period. For $i = 1,2,\ldots$, match buyer i with the lowest-indexed firm j such that $\sigma^b(i) = \alpha^f(j)$, $\sigma^f(j) = \alpha^b(i)$, and buyer i has not yet visited firm j. Then each buyer has probability ρ of having valuation θ_1, and each firm has probability ρ that a visitor or caller will have valuation θ_1. This matching procedure is a variation of one given by Boylan (1992, p. 482).

APPENDIX B

PROOFS OF LEMMAS AND THEOREMS

PROOF OF LEMMA 3.1

When quoted p^*, a buyer is indifferent between visiting and not visiting. Visiting must give the buyer expected payoff V. If the buyer visits and buys, she has payoff $-s^V + \theta - p^*$. If she visits and does not buy, she has payoff $-s^V + V$. Once in the store, the buyer will choose the higher of these two payoffs. Thus,

$$V = -s^V + \rho \max \langle \theta_1 - p^*, V \rangle + (1 - \rho) \max \langle \theta_2 - p^*, V \rangle$$

so

$$0 = -s^V + \rho \max \langle \theta_1 - p^* - V, 0 \rangle + (1 - \rho) \max \langle \theta_2 - p^* - V, 0 \rangle$$

Since $\theta_2 > \theta_1$ and $s^V > 0$, the second maximum is nonzero. There are two cases:

a) If $\theta_1 - p^* - V < 0$, then $p^* = \theta_2 - V - \dfrac{s^V}{1 - \rho}$.
Substituting the second equation into the first, we obtain $s^V < g$.

b) If $\theta_1 - p^* - V > 0$, then $p^* = g + \theta_1 - p^* - V$.
Substituting, we obtain $s^V > g$.■

PROOF OF LEMMA 3.2

Suppose $V \geq \theta_2 - c$. By (3.6), $V > 0$. Buyers do not exit until they purchase. Firms that quote or receive visitors in equilibrium will pick only prices $p \geq \theta_2 - V$.

Hence a visitor will never buy if her valuation is θ_1. A visitor with valuation θ_2 may buy, but only if

$$\theta_2 - p = V$$

Both types have continuation payoff V after sinking visiting costs. Calling ahead cannot improve these payoffs, since no quoted price is below $\theta_2 - V$. But a buyer's payoff when entering the game is V. This implies

$$V \leq -s^V + V$$

which is impossible.■

PROOF OF THEOREM 4.1

Claims 4.1 through 4.10 show that (4.3-4.4) imply the given strategies, and that (4.10-4.13) are necessary conditions for the strategies to form an equilibrium. Claim 4.11 shows that (4.10-4.13) are sufficient as well.

<u>Claim 4.1</u>

If (4.3) holds, then some firms must quote and some must refuse.

<u>Proof</u>

By perfection, a firm can quote only if its price would attract the caller. Thus, all firms quote only if all firms have prices less than or equal to p*. But then every call will lead to a visit, so $V^c < V^v$. Conversely, if no firms quote, then $V^c < V^v$ because calling does not reveal any information about the firm. These contradict (4.3).■

Claim 4.2

If (4.3) holds, buyers strictly prefer not to visit firms that refuse to quote.

Proof

Let V^q be the expected payoff from visiting a quoter and following an optimal strategy once in the store. V^q is the weighted average expected payoff over all prices that are quoted. Let V^r be the expected payoff from visiting a refuser and following an optimal strategy once in the store. V^q and V^r do not include the initial calling cost $-s^c$.

Since quoters quote only prices $p \le p^*$, callers always visit quoters. A caller who is refused can either visit anyway and obtain V^r or follow an optimal strategy and obtain V. Thus,

$$V^c = -s^c + \delta V^q + (1-\delta)\max \langle V^r, V\rangle$$

Since a buyer who does not call ahead will visit a quoter with probability δ and a refuser with probability $1 - \delta$,

$$V^v = \delta V^q + (1-\delta)V^r$$

If $V^r \ge V$, then $V^c < V^v$, contradicting (4.3). Thus, $V^r < V$. The claim follows.■

Claim 4.3

Assuming (4.4),

(a) Firms that would not quote on receiving a caller select either $\theta_1 - p$ or $\theta_2 - p$.

(b) Firms that would quote select either $\theta_1 - p$ or p^*.

Proof

Since $\pi^{nq} > 0$ and by Claim 4.2, there must be visitors who do not call ahead in equilibrium. Both claims follow by Lemma 3.3.■

The intuition for the next claim is that a buyer who is willing to visit a refuser does not obtain any advantage from calling ahead. Let $\delta \in (0,1)$ be the proportion of firms that quote.

<u>Claim</u> <u>4.4</u>

Assuming (4.3-4.4),

(a) Some firms that are willing to quote select the price $\theta_1 - V$.

(b) $p^* > \theta_1 - V$.

(c) $s^V < g$.

<u>Proof</u>

Some firms that are willing to quote must select a price strictly less than p*. If not, all such firms select p*. (Firms will only quote a price that draws buyers in.) The payoff to visiting a firm that quotes p* is just V (see Lemma 3.1). Since $V > V^r$,

$$V^c = -s^c + \delta V + (1-\delta)V = -s^c + V < V$$

which contradicts (4.3) and (3.8). So some that are willing to quote must select a price below p*. With this, (a) and (b) follow from Claim 4.3. Claim (c) then follows from Lemma 3.1.■

Let $\pi(p)$ denote the expected profit in one period for a firm that selects price p and then behaves optimally. Also let $\eta \in [0,1]$ be the proportion of buyers who always call ahead. $1 - \eta$ of buyers never call before visiting. Because strategies are Markov, a given buyer does not call ahead in some periods but not in others.

<u>Claim 4.5</u>

Assuming (4.3-4.4), all firms that select $\theta_1 - V$ strictly prefer to quote if called.

<u>Proof</u>

Suppose not. By Claim 4.4(a), firms that select $\theta_1 - V$ must be indifferent about quoting. By Claim 4.2, buyers will not visit if a firm does not quote. If a firm quotes $\theta_1 - V$, all the firm's callers will ultimately purchase. Thus, if a buyer calls, profits from quoting are

$$\theta_1 - V - c - q$$

Indifference about quoting implies that

$$V = \theta_1 - c - q$$

Firms that charge $\theta_1 - V$ must earn their profits from visitors who do not call ahead:

$$\pi(\theta_1 - V) = (1 - \eta)(\theta_1 - V - c) = (1 - \eta)q$$

Some firms must select $\theta_2 - V$. Else all prices are less than or equal to p^*. Thus any caller will visit, so $V^c < V^v$. This contradicts (4.3). A firm that selects $\theta_2 - V$ obtains profits

$$\pi(\theta_2 - V) = (1 - \eta)(1 - \rho)(\theta_2 - V - c)$$

Substituting for V,

$$\pi(\theta_2 - V) = (1 - \eta)(g + (1 - \rho)q)$$

Since there are such firms,

$$\pi(\theta_2 - V) = \pi(\theta_1 - V)$$

If $\eta < 1$, this implies a strict equality on the parameters:

$$g = \rho q$$

We are not considering such parameter sets. If $\eta = 1$, $\pi^{nq} = 0$, which contradicts (4.4). This proves the claim.■

Suppose that a proportion $\gamma > 0$ of firms offer $\theta_1 - V$ and are willing to quote. Suppose that μ offer $\theta_2 - V$ and are not willing to quote, and $1 - \gamma - \mu$ offer p* and are willing. By Claims 4.3 and 4.5, these exhaust the possible firm strategies. We now compute γ and μ, verifying equations (4.5-4.7). Claim 4.6(b) also verifies equation (4.10).

<u>Claim</u> <u>4.6</u>

(4.3-4.4) imply that

(a) $\gamma = \beta$ (β is defined in [4.4])

(b) $s^c + s^v < g$

<u>Proof</u>

Let V^ℓ be the continuation payoff of a caller after being quoted the price $\theta_1 - V$. Since the buyer will always visit and purchase,

$$V^\ell = -s^v + \rho\theta_1 + (1 - \rho)\theta_2 - (\theta_1 - V)$$
$$= -s^v + g + V$$

A caller has continuation payoff V after being quoted p* (see Lemma 3.1). A buyer also has continuation payoff V after calling a firm that refuses to quote (Claim 4.2). Using these facts,

$$V^C = -s^C + \gamma(-s^V + g + V) + (1 - \gamma)V$$

Since $V^C = V$, we obtain (a). $\gamma > 0$ by equation (3.2) and Claim 4.4(c). Claim (b) follows from the requirement that $\gamma \leq 1$ and the rejection of the measure zero parameter set given by $\gamma = 1$.■

<u>Claim 4.7</u>

If (4.3-4.4),

(a) $\mu = \dfrac{s^C}{s^V}$

(b) $0 < s^C < s^V$

<u>Proof</u>

Since $V^\ell = -s^V + g + v$, a visitor who is offered $\theta_1 - V$ has continuation payoff g + V. A caller who is quoted p* has continuation payoff V and will visit. Thus, a <u>visitor</u> who discovers that the price is p* must have continuation payoff $V + s^V$. A visitor of either valuation has continuation payoff V if the price is $\theta_2 - V$. Therefore,

$$V^V = -s^V + \gamma(g + V) + \mu V + (1 - \gamma - \mu)(V + s^V)$$

Because $V^V = V$, this implies

$$(\gamma + \mu)s^V = \gamma g$$

so

$$\mu = \beta(g/s^V - 1) = \frac{s^C}{s^V}$$

(b) holds because $\mu \in [0,1]$ and because of equations (3.1) and (3.2).■

We now verify equations (4.8) and (4.9). Claim 4.8(c) also verifies equation (4.12). Recall that η is the proportion of buyers who call ahead.

<u>Claim 4.8</u>

Assuming (4.3-4.4),

(a) $V = \theta_1 - c - \dfrac{g - s^V}{\rho}$

(b) $\pi = \dfrac{g - s^V}{\rho} - \eta q$

(c) $s^V > g + \rho(c - \theta_1)$

<u>Proof</u>

Profits from selecting $\theta_1 - V$ are

$$\pi(\theta_1 - V) = \theta_1 - V - c - \eta q$$

Since $p^* > \theta_1 - V$ by Claim 4.4(b), profits from selecting p^* are

$$\pi(p^*) = (1 - \rho)(p^* - c) - \eta q$$

By Claim 4.4(c) and Lemma 3.1,

$$\pi(p^*) = (1 - \rho)(\theta_2 - V - c) - s^V - \eta q$$

Claims 4.6 and 4.7 together imply that $\gamma \neq 0$ and $\gamma + \mu \neq 1$ except on sets of measure zero, which we ignore. Thus, some firms select p^* and some select $\theta_1 - V$. This implies that $\pi(\theta_1 - V) = \pi(p^*)$. Claim (a) follows. Substitute V from (a) into the formula for $\pi(\theta_1 - V)$ to obtain (b). Claim (c) holds because $V > 0$.■

We now verify (4.13) and compute η.

Claim 4.9

(4.3-4.4) hold only if

(a) $\eta = \nu$ (ν is defined in [4.7])

(b) $s^V < g - \rho q$

Proof

Claim (b) follows from (a), $\eta \leq 1$, and the rejection of measure zero parameter sets. A firm that selects $\theta_2 - V$ sells only to high valuation buyers who do not call ahead. Profits are

$$\pi(\theta_2 - V) = (1 - \eta)(1 - \rho)(\theta_2 - V - c)$$

Profits from selecting $\theta_1 - V$ are

$$\pi(\theta_1 - V) = \theta_1 - V - c - \eta q$$

Since $\gamma > 0$ and $\mu > 0$, these must be equal. Using Claim 4.8(a) to eliminate V, we obtain (a).■

We must still verify (4.11).

Claim 4.10

(4.3-4.4) require that

$$s^c < s^v\left[1 - \frac{s^v}{g}\right]$$

Proof

For $1 - \gamma - \mu \geq 0$, we require

$$s^c\left[\frac{1}{s^v} + \frac{1}{g - s^v}\right] \leq 1$$

We can replace this with a strict inequality by the rejection of sets of measure zero. The claim follows.■

Claims 4.1 through 4.10 show that the three price equilibrium is the only one that satisfies (4.3) and (4.4). They show also that (4.10) through (4.13) are necessary conditions for the equilibrium. We must also verify that (4.10) through (4.13) are sufficient for the strategy profile to be an equilibrium.

<u>Claim 4.11</u>

Equations (4.10) through (4.13) are sufficient for the strategy profile described in Theorem 1 to be an equilibrium satisfying (4.3) and (4.4).

<u>Proof</u>

For the strategy profile to be an equilibrium satisfying (4.3) and (4.4), we require:

(a) γ, μ, $1 - \gamma - \mu$, and η must all lie between 0 and 1, inclusive.

(b) Buyer indifference: $V^C = V^V = V > 0$

(c) Firm indifference: $\pi(\theta_1 - V) = \pi(p^*) = \pi(\theta_2 - V) > 0$

(d) Firms that select $\theta_1 - V$ and p^* must strictly prefer to quote.

(e) Firms that select $\theta_2 - V$ must strictly prefer not to quote.

For (a), $\gamma \in [0,1]$ is guaranteed by (3.2) and (4.10), as noted in Claim 4.6. $\mu \in [0,1]$ is guaranteed by (3.1) and (3.2), as well as (4.11) which implies $s^c < s^v$. This uses the fact, from (3.2) and (4.10), that $s^v < g$. Claim 4.10 shows that $1 - \gamma - \mu \in [0,1]$ is guaranteed by (4.11). By Claim 4.9, $\eta \leq 1$ if (4.13) holds. One can verify that (4.13) implies $\eta \geq 0$ as well.

For (b), the proofs of Claims 4.6 and 4.7 can be reversed to show that, if γ and μ are as in Claims 4.6(a) and 4.7(a), then

$$V^c = V^v = V$$

The proof of Claim 4.8 shows that equation (4.12) implies V > 0.

For (c), the proof of Claim 4.8 shows that $\pi(\theta_1 - V) = \pi(p^*)$ if (4.8) holds. The proof of Claim 4.9 shows that $\pi(\theta_1 - V) = \pi(\theta_2 - V)$ if (4.2) holds. By Claim 4.8(b), $\pi > 0$ if

$$\frac{g - s^v}{\rho} - \nu q > 0$$

Using (4.2) to substitute for ν, this is equivalent to

$$\frac{(g - s^v)(g - (1 - \rho)s^v - \rho q) - q\rho^2 s^v}{\rho(g - (1 - \rho)s^v - \rho q)} > 0 \tag{4.11.1}$$

The denominator is positive by (4.13). (4.13) and (4.10) also imply that

$$(g - s^v)(g - s^v - \rho q) > 0$$

and so the numerator of (4.11.1) is positive if

$$(g - s^v)\rho s^v - q\rho^2 s^v > 0$$

which also follows from (4.13), using (3.1) to cancel s^v.

For (d), the claim respecting $\theta_1 - V$ was verified in

Claim 4.5. For p^*, the expected profit from quoting p^*, conditional on a buyer calling, is

$$(1 - \rho)(\theta_2 - V - c) - s^V - q$$

Using (4.8), this is positive if and only if (4.13) holds.

(e) holds because no buyer visits if $\theta_2 - V$ is quoted.■

END OF PROOF

PROOF OF THEOREM 5.1

Note that (5.1) implies $V^V = V$. By (5.1), no buyers call ahead. Firms select either $\theta_1 - V$ or $\theta_2 - V$ (Lemma 3.1).

Claim 5.1 verifies (5.2-5.3) and that (5.7) is necessary.

<u>Claim 5.1</u>

If (5.1) holds, then

(a) $\frac{s^V}{g}$ of firms offer $\theta_1 - V$

(b) $1 - \frac{s^V}{g}$ of firms offer $\theta_2 - V$

(c) $s^V < g$

<u>Proof</u>

Let α be the proportion of firms that offer $\theta_1 - V$. Then

$$V^V = -s^V + \alpha(g + V) + (1 - \alpha)V$$

This proves (a). (b) and (c) follow from (a).■

The next claim verifies that payoffs are given by (5.4-5.5) and that (5.6) is necessary.

<u>Claim</u> <u>5.2</u>

If (5.1) holds,

(a) $V = -c + \frac{1}{\rho}\left[\theta_1 - (1 - \rho)\theta_2\right]$

(b) $\pi = \frac{g}{\rho}$

(c) $\frac{\rho}{1 - \rho} > \frac{\theta_2 - \theta_1}{\theta_1 - c}$

<u>Proof</u>

Profits from selecting $\theta_1 - V$ are

$$\pi(\theta_1 - V) = \theta_1 - V - c$$

while profits from $\theta_2 - V$ are

$$\pi(\theta_2 - V) = (1 - \rho)(\theta_2 - V - c)$$

By Claim 5.1, these must be equal. Equating them gives (a). (b) follows by substituting V from (a) into the expression for $\pi(\theta_1 - V)$. (c) comes from rearranging the equation V > 0.■

Claims 5.1-5.2 show that, if (5.1) holds, then any equilibrium outcome must be of the given form, and (5.6-5.7) are necessary conditions for the outcome to be supported by an equilibrium. We must also show that (5.6-5.7) are sufficient for the outcome to be supported and to satisfy (5.1).

There is one equilibrium in which low price firms quote. No buyers call ahead because calling costs are too high. This equilibrium imposes additional requirements on the parameters. We present instead a set of equilibria in which no firms quote because firms believe callers are likely to visit anyway. No buyers call ahead because no firms quote.

Claim 5.3

If (5.6-5.7) hold, there is a two dimensional continuum of equilibria that support the two price outcome.

Proof

If no firm is willing to quote, refusing would signal nothing. If a buyer called ahead, she would thus be indifferent about visiting the firm if it refused. Suppose that some proportion $\alpha \in (0,1)$ of buyers would visit a firm if they called and the firm refused.

If no firm quotes, noone calls in equilibrium. Since $\alpha \in (0,1)$, if a buyer did call, a firm could put any probability on the buyer's willingness to visit the firm if it refused. Suppose all firms put probability ζ on this event.

Firms with price $\theta_2 - V$ will never quote. If a buyer calls and a firm with price $\theta_1 - V$ quotes, the buyer will visit. The firm obtains

$$\theta_1 - V - c - q$$

If the firm refuses, its expected payoff is

$$\zeta(\theta_1 - V - c)$$

Use (5.4) to eliminate V. The firm prefers to refuse for any

$$\zeta \in \left[\frac{g - \rho q}{g}, 1\right]$$

Thus, every such ζ, together with an $\alpha \in (0,1)$, gives an equilibrium that supports the two price outcome.■

END OF PROOF

PROOF OF THEOREM 6.1

(6.4) follows from (6.2). The first claim proves some basic properties that (6.1-6.2) imply.

Claim 6.1

Assuming (6.1-6.2), all buyers call ahead and no buyers visit refusers. Some firms are willing to quote.

Proof

No buyers exit before purchasing because $V = V^c > 0$. Let $\zeta \geq 0$ be the proportion of buyers who call and who visit refusers. Let $\xi \geq 0$ be the proportion of buyers who visit without calling ahead. If a firm selected the price $\theta_2 - V$ and refused to quote, its profits would be no less than

$$(\zeta + \xi)(1 - \rho)(\theta_2 - V - c)$$

Since $\pi^{nq} = 0$, this cannot be positive. By Lemma 3.2, $\zeta = \xi = 0$.

Some firms must quote since $V^c > 0$ and no callers visit refusers.■

Let δ be the proportion of firms that quote. Claim 6.2 shows what price these firms select and verifies (6.3). Claim 6.3 verifies that $\delta = \beta$.

Claim 6.2

Assuming (6.1-6.2),

(a) $p^* > \theta_1 - V$;

(b) All quoting firms select $\theta_1 - V$.

(c) $V = \theta_1 - V - c$

Proof

Some firms that are willing to quote must select a price below p^*. If not,

$$V^c = -s^c + \delta V + (1-\delta)V \quad = \quad -s^c + V < V$$

since no buyers visit refusers. But $V^c = V$ by (6.1). By Lemma 3.3, this establishes (a) and shows that some quoting firms select $\theta_1 - V$.

Suppose some quoting firms select p^*. By (a) and Lemma 3.1, $p^* = \theta_1 - V - s^v/(1 - \rho)$. Firms that select $\theta_1 - V$ and p^* earn equal profits:

$$\theta_1 - V - c - q \quad = \quad (1 - \rho)(\theta_2 - V - c) - s^v - q$$

But $\pi^q = 0$, so both sides equal zero. Since only V is endogenous, this can happen only on a parameter set of measure zero. Thus, quoting firms do not select p^* except perhaps on such a parameter set. This shows (b). Setting the left hand side to zero, we obtain (c).■

Claim 6.3

$\delta = \beta$.

Proof

By Claim 6.1 and 6.2(b),

$$V^c = -s^c + \delta(-s^v + g + V) + (1 - \delta)V$$

The claim follows by setting $V^c = V$.■

We now prove that (6.5) and (6.6) are necessary for the one price outcome.

Claim 6.4

Assuming (6.1-6.2),

(a) $s^c + s^v < g$

(b) $s^v > g - \rho q$

Proof

Since $\delta = \beta \leq 1$ and we ignore measure zero parameter sets, (a) holds. For quoters to prefer $\theta_1 - V$ over p^*, we must have

$$\theta_1 - V - c - q \quad > \quad (1 - \rho)(\theta_2 - V - c) - s^v - q$$

Substituting V from (6.3), we obtain (b).■

We now verify that (6.7) is necessary as well.

Claim 6.5

If (6.5-6.6), then

$$s^c < s^v\left[1 - \frac{s^v}{g}\right]$$

Proof

Since $\pi = 0$, firms are willing to pick any price and then to refuse to quote it. Any visitor to one of the $1 - \beta$ of firms that refuse can leave without purchasing. Thus, such a visitor receives a continuation payoff of no less than V after sinking visiting costs. V^V must satisfy

$$V^V \geq -s^V + \beta(g + V) + (1 - \beta)V$$

Because $V^V \leq V$, we must have $s^V \geq \beta g$. Multiply both sides by $g - s^V$, which is positive by (6.5). (6.7) follows.■

We have shown that (6.1-6.2) imply that any equilibrium outcome must be the one price outcome, and that (6.5-6.7) are necessary conditions for this outcome to be supported by an equilibrium. We now show that, if (6.5-6.7) hold, then there is an equilibrium supporting the one price outcome and satisfying (6.1-6.2).

Consider the strategy profile in which β of firms quote $c + q$ and $1 - \beta$ select $\theta_2 + \varepsilon$ and refuse to quote, where $\varepsilon > 0$. Suppose all buyers call ahead. No sale can occur at the price $\theta_2 + \varepsilon$, so no buyers visit refusers.

Claim 6.6

If (6.5-6.7) hold, then under the given strategy profile,

(a) $V^V \leq V^C$

(b) $V = V^C$

(c) $V^c = \theta_1 - c - q$

Proof

First, we show (a). Let V^q be the expected payoff to visiting a quoter, including visiting costs:

$$V^q = -s^v + \rho \max \langle\theta_1 - c - q, V\rangle$$
$$+ (1 - \rho) \max \langle\theta_2 - c - q, V\rangle$$

Note that

$$V^q \geq -s^v + \rho\theta_1 + (1 - \rho)\theta_2 - c - q$$

which is strictly positive by (6.5) and (3.6). Thus, visiting a quoter must be strictly preferable to exiting. Since no sale at $\theta_2 + \varepsilon$ is possible, $V^q > V^v$ as well. Thus, callers always visit quoters. This implies

$$V^c = -s^c + \beta V^q + (1 - \beta)V$$

Also,

$$V^v = \beta V^q + (1 - \beta)(V - s^v)$$

After some manipulation of the equation $V^c \geq V^v$, we find that (6.7) implies it.

Now, (b). By (a), $V = \max \langle 0, V^c\rangle$. If $V = 0$ then, using (3.6) to simplify the formula for V^q,

$$V^c = -s^c + \beta(-s^v + g + \theta_1 - c - q)$$
$$= \beta(\theta_1 - c - q)$$

which is positive by (3.6) and since $s^v < g$ by (6.5). So $V \neq 0$. This shows $V = V^c$.

We now show (c). A buyer with valuation θ_2, facing price $c + q$, will buy, since the buyer's surplus is positive and cannot be higher. Thus,

$$V^c = -s^c + \beta\Big[-s^v + \rho \max \langle\theta_1 - c - q, V^c\rangle$$

$$+ (1 - \rho)(\theta_2 - c - q)\Big] + (1 - \beta)V^c$$

Regardless of which value the max takes, (c) must hold.■

Claim 6.6 shows that, under the given strategy profile, it is indeed optimal to call ahead. By Lemma 3.3, quoters quote either $\theta_1 - V$ or p^*. Claim 6.7 shows that $\theta_1 - V$ is strictly better than p^*.

Claim 6.7

Assuming (6.5-6.7), firms earn negative profits from quoting p^* under the given strategy profile.

Proof

By (6.5) and Lemma 3.1, $p^* = \theta_2 - V - s^V/(1 - \rho) > \theta_1 - V$. Profits from quoting p^* are thus

$$(1 - \rho)(\theta_2 - \theta_1 + q) - s^V - q$$

$$= g - \rho q - s^V$$

which is negative by (6.6).■

Since quoters strictly prefer $\theta_1 - V = c + q$ to any other price, $\pi^q = 0$. Thus, firms are willing to select $\theta_2 + \varepsilon$ and not quote. Together with Claim 6.6, this verifies that the strategy profile is an equilibrium satisfying (6.1-6.2). We have shown that (6.5-6.7) are sufficient as well as necessary.

END OF PROOF

PROOF OF THEOREM 7.1

First, assume (7.1). All buyers must exit immediately. This shows that only the no trade outcome is possible.

There are no visitors or callers, so firms are willing select any prices. Suppose all firms select θ_2. This supports the outcome and satisfies (7.1).

Now assume (7.2-7.3). No buyers can call or visit. Why? By (7.2), $V = 0$. If a visitor encounters the price θ_1, she will buy. But $\theta_1 > c$. The firm would earn positive profits from the visitor. Now suppose there are callers. The argument in Claim 4.4 applies here also. Thus, some quoters select $\theta_1 < p^*$. Since callers always visit and purchase at θ_1, these quoters earn $\theta_1 - c - q > 0$. This again contradicts (7.3).

END OF PROOF

PROOF OF THEOREM 8.1

We divide all possible pairs (V^V, V^C) into five sets:

(a) $V^V < 0$, $V^C < 0$;

(b) $V^V > 0$, $V^V > V^C$;

(c) $V^C > 0$, $V^C > V^V$;

(d) $V^V = V^C > 0$;

(e) $V^V = V^C = 0$.

We also partition pairs (π^q, π^{nq}) into five sets. We use the fact that $\pi^{nq} \geq 0$. (Any price over θ_2 guarantees zero profits.) There are three sets in which $\pi^{nq} > 0$:

(1) $\pi^{nq} > 0$, $\pi^{nq} > \pi^q$;

(2) $\pi^{nq} > 0$, $\pi^{nq} = \pi^q$;

(3) $\pi^{nq} > 0$, $\pi^{nq} < \pi^{q}$.

There are also two sets in which $\pi^{nq} = 0$:

(4) $\pi^{nq} = 0$, $\pi^{nq} \neq \pi^{q}$;

(5) $\pi^{nq} = 0$, $\pi^{nq} = \pi^{q}$.

Theorem 1 covers (d,2) and (e,2). Theorem 2 covers (b,1-5). Theorem 6.1 covers (c,5) and (d,5). Theorem 7.1 covers (a,1-5) and (e,5).

No other pairs are possible. Firms will only quote prices that draw callers in. But if $\pi^{q} \neq \pi^{nq}$, then either all firms are willing to quote or none are. This implies $V^{v} > V^{c}$ (Claim 4.1). This rules out any pairing of 1, 3, or 4 with a, c, d, or e. We are left only with (c,2).

(c) implies that all buyers call ahead. If buyers are willing to visit refusers, then we cannot have $V^{c} > V^{v}$ (Claim 4.2). If buyers are not willing to visit refusers, then $\pi^{nq} = 0$. This contradicts (2). So (c,2) is impossible.

END OF PROOF

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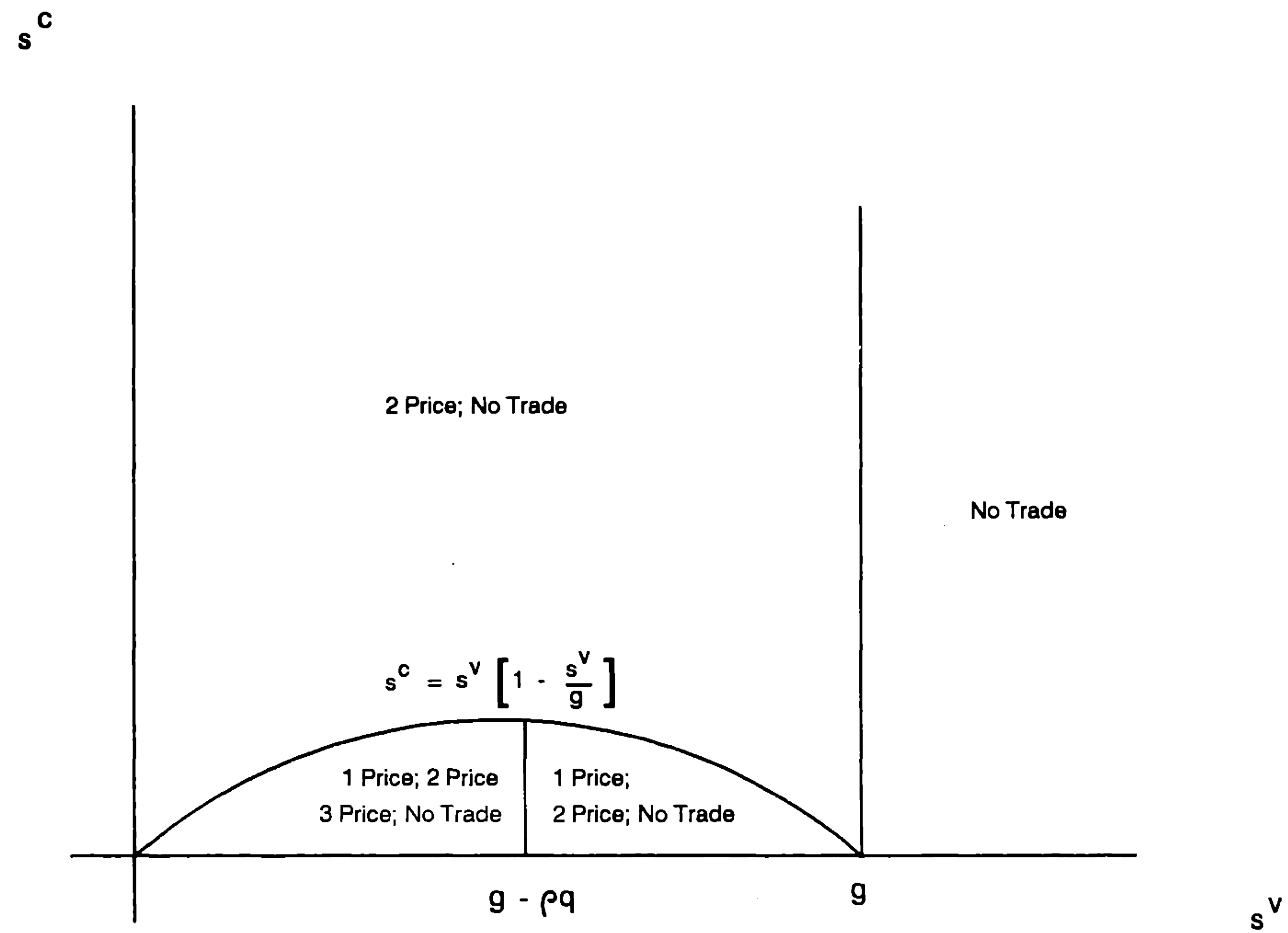

Chart 1. $\frac{1-\rho}{\rho} < \frac{\theta_1 - c}{\theta_2 - \theta_1}$

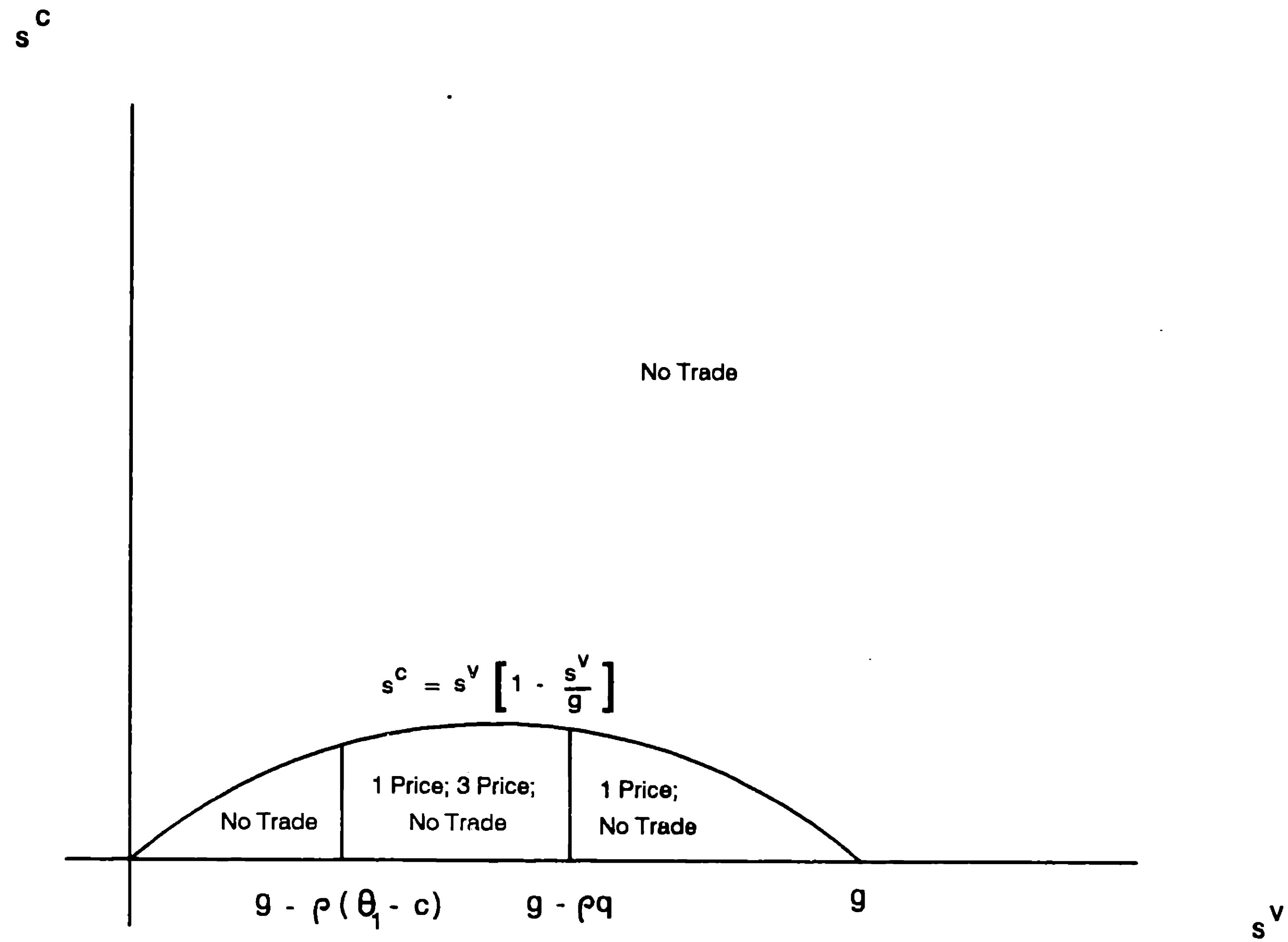

Chart 2. $\frac{1-\rho}{\rho} > \frac{\theta_1 - c}{\theta_2 - \theta_1}$

Part III

COMMITMENT AND OPTIMAL CAPITAL TAXATION

1. INTRODUCTION

Commitment often permits government policies that are better for the private sector. However, the optimal policy with commitment is not time consistent, unless it coincides with the optimal policy without commitment. (Kydland and Prescott 1977; Calvo 1978). Most prior work on optimal capital taxation with a representative agent has assumed commitment (Brock and Turnovsky 1981; Judd 1985a; Chamley 1986; Lucas 1990). This paper contrasts the commitment case with the no-commitment case. We find that optimal tax policies look very different in the two cases.

Governments do have a limited ability to commit to policies for some period of time. A president or governor can repeat a campaign promise over and over, until a reversal would amount to political suicide. A legislature can commit itself not to change a law for a few months by going out of session.[1] For longer periods, a legislature can pass a constitutional amendment. Amendments have commitment value because they require overwhelming opposition to reverse. They also require overwhelming support to pass. The doctrine of precedent gives the courts commitment power. However, judges have neither the mandate nor the resources needed to formulate fiscal policy.

We contrast optimal policies in the polar cases of

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1 If laws can be retroactive, this will not work.

perfect commitment and no commitment. The results are not directly applicable to the real world, which lies somewhere between the two cases. The value of the contrast is to illuminate the effects of greater commitment power on optimal policy and social welfare.

Our capital income tax is equivalent to a uniform personal tax on all investment income, including accrued capital gains. It would be desirable to distinguish between old and new capital by giving the government an investment tax credit as well. We do not do so here.

In optimal capital taxation, one must take account of firms' and individuals' ability to avoid and evade capital income taxes. Otherwise, there is no finite optimal capital income tax (Chamley 1986). For tractability, we permit tax avoidance rather than evasion. Firms can reduce or suspend operations if they desire. They must pay taxes on all capital income actually earned. If the marginal product of capital is positive, then profit maximizing firms will employ all of their capital so long as the capital income tax rate does not exceed 100%.[2] If it exceeds 100%, firms will suspend operations. An optimal capital income tax rate will never exceed 100%.

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[2] The marginal product of capital is measured net of depreciation throughout this paper. We assume that economic depreciation is fully deductible from capital income for tax purposes.

We assume that the government does not initially possess a stock of capital sufficient to fund all of future expenditures. Chamley (1986) shows that the capital income tax rate is 100% initially with commitment. He shows also that there is no capital income tax in the steady state if the optimal path is convergent. This is true for general utility functions. When the utility function is additively separable in consumption and labor and CRRA in consumption, Chamley shows that the tax rate is 100% for an initial period. At some point, the tax jumps to zero, and remains there ever after.[3]

We add some new results in the commitment case. First, the optimal capital income tax does not usually jump from 100% to zero. Assume the utility function is separable in consumption and leisure. Suppose utility is IRRA in consumption. Let θ be the rate of time preference. Let F_k be the marginal product of capital. Let τ_r be the capital tax per unit of capital income. A 100% capital income tax corresponds to $\tau_r = F_k$. Suppose that $\tau_r < F_k$. Then the signs of $F_k - \theta$ and τ_r are the same. One is zero only if the other is.

The intuition? If utility is IRRA, then consumption is less sensitive to its price when consumption is high than when it is low. Goods that are less price elastic should be

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[3] Chamley (1986, p. 608) notes that this does not hold for general utility functions.

taxed more heavily than others. Let $r = F_k - \tau_r$, the after tax interest rate. If $r > \theta$, then consumption grows. Later consumption should be taxed more heavily than earlier consumption. However, a capital income tax at time t is equivalent to a tax on all consumption that occurs after t. Thus, there should be a positive capital income tax whenever $r > \theta$. In the IRRA case, $r - \theta$ and $F_k - \theta$ are of the same sign, so the result follows.

On the other hand, suppose the utility function is separable and DRRA in consumption. Whenever $\tau_r < F_k$, τ_r is zero if and only if $F_k = \theta$. As before, consumption is taxed the same in different periods if and only if its price elasticity is constant.

One might expect the relation between the signs of τ_r and $F_k - \theta$ to be the opposite of the IRRA case. Unfortunately, we could not verify this. The difficulty is that $r - \theta$ and $F_k - \theta$ need not be of the same sign when utility is DRRA in consumption.

This paper also contains some new results about the wage tax in the commitment case. Let F_ℓ be the marginal product of labor. Let τ_w be the wage tax per unit of labor, so that a 100% wage tax would correspond to $\tau_w = F_\ell$. τ_w is strictly positive whenever $\tau_r < F_k$ (whenever $r > 0$). This holds for any utility function. If $\theta > 0$, this result implies that there is a positive wage tax in steady state.[4]

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[4] This is because $r = \theta$ is necessary for consumption to be

The intuition is that there is no first order deadweight loss from an infinitesimal tax on labor, no matter how far in the future. Since revenue is needed, marginal deadweight losses must be positive on all commodities. This principle is consistent with the lack of a capital income tax in steady state. A capital income tax at date t is a tax on all consumption that occurs after t. Steady state consumption bears the burden of all *prior* taxes on capital income.

We present more results in the case of a utility function that is additively separable and CRRA in labor supply. With such a function, τ_w rises when $\tau_r > 0$ and falls whenever $\tau_r < 0$. The condition that the utility function be CRRA in labor supply means that

$$\frac{\ell u_{\ell\ell}}{u_\ell}$$

is constant, where u is the utility function, ℓ is labor supply, and subscripts indicate a partial derivative. An example of such a function is

$$u(c,\ell) \quad = \quad v(c) - \alpha\ell^{\beta}$$

where α and β are constants.

Why does this result hold? A wage tax cut stimulates labor supply. This leads to increased production and thus adds to national capital (private wealth less government debt). Higher labor supply also lowers *instantaneous*

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constant.

utility, since people prefer not to work (we assume!). Finally, a wage tax cut increases private wealth through both the labor supply response and a transfer on existing labor supply. (Controlling for national capital, an increase in private wealth is bad because it amounts to a loss to the treasury and leads to higher taxes later.)

While $\tau_r > 0$, the social rate of return exceeds the private rate. Optimizing behavior implies that the shadow values of social quantities fall over time, relative to the shadow values of private quantities. In particular, the shadow value of national capital falls relative to the shadow value of marginal utility *and* the (negative) shadow value of private wealth. To equate the value of a small change in τ_w over time, the effect of such a change on national capital must increase, relative to its effects on instantaneous utility and on private wealth. When utility is separable in consumption and labor supply, and CRRA in labor supply, the *only way* this can happen is for τ_w itself to increase over time. A higher wage tax increases the effect of labor supply on social quantities relative to private quantities because of a larger tax revenue externality. The same intuition works in reverse when $\tau_r < 0$.

We also examine the no-commitment case. Capital income is taxed at 100% until the government has collected enough revenue to fund all future spending. There is no taxation at all thereafter. This contrasts with the commitment case, in which wages are taxed, and capital income may be taxed or subsidized, when the capital income tax rate is less than 100%.

An immediate capital income tax is as efficient as a lump-sum tax. If the government requires revenue, it must fully exploit this efficient revenue source. Commitment raises welfare by permitting the government to lower capital income taxes, and to subsidize capital income if desirable, in the future. This difference suggests that the social value of commitment may be high. There may be substantial social value in devising new ways for governments to commit to future tax rates.

The representative agent approach of this paper captures the dual role of capital as a factor of production and a store of wealth. This makes it suitable for examining the efficient taxation of capital income. However, the representative agent approach neglects equity issues. One such issue is the distribution of wealth between capitalists and workers, broadly defined. A model like Judd's (1985a) two-class, capitalist-worker model would allow one to examine optimal taxation without commitment while taking this issue into account.

The representative agent model also presumes that families will offset the 100% capital income tax through smaller bequests or gifts from children to parents. In reality, any such adjustments would almost surely be inadequate. To address this objection, it may be useful to revisit the question of optimal taxation in overlapping generations models (Diamond 1973; Atkinson and Sandmo 1980).

A 100% capital income tax rate would also give rise to widespread tax evasion and costly allocative distortions. A more realistic model, with costly evasion, would undoubtably feature a lower capital income tax.

The rest of the paper is organized as follows. We describe the economy and the models in Sections 2 and 3. We prove what we can about the optimal policies in Sections 4 and 5. We end with a brief discussion in Section 6.

2. COMMON FEATURES OF THE MODELS

The private sector is represented by a single, infinitely-lived agent. At time t, the agent supplies labor ℓ_t ($0 \le \ell_t \le 1$) and consumes c_t units of a homogeneous good. The economy has a twice continuously differentiable, CRTS, per capita, net production function $F(k,\ell)$. k is per-capita national capital. To ensure an interior solution for labor supply, we also assume that, for $k > 0$,

$$\lim_{\ell \to 0} F_\ell(k,\ell) = \infty \tag{2.1}$$

The government can borrow or lend. Government capital is a perfect substitute for private capital in the production function. Thus, national capital k equals private wealth, k^P, plus government capital k^G (both per capita).[5]

The government can also levy linear taxes on wages and on capital income. We assume perfect competition, so

$$r = F_k - \tau_r \tag{2.2}$$

$$w = F_\ell - \tau_w \tag{2.3}$$

where r and w are the after tax interest rate and wage, respectively, and τ_r and τ_w are the taxes.

The government must finance time t per capita public

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[5] This is equivalent to the conventional formulation in which national capital equals private capital less government debt. Government capital is just the negative of government debt.

consumption at an exogenous rate G_t. The government purchases the consumption good only. Government spending affects neither marginal utility nor marginal factor productivity.

The agent and the government share the objective function

$$(2.4) \qquad \int_{t=0}^{\infty} e^{-\theta t} u(c_t, \ell_t) dt$$

The felicity function, $u(c,\ell)$, is twice differentiable with respect to both c and ℓ. As usual, $u_c > 0$, $u_{cc} < 0$, $u_\ell < 0$, and $u_{\ell\ell} < 0$.

To ensure an interior solution, we also assume that

$$(2.5) \qquad \lim_{c \to 0} u_c(c,\ell) = \infty$$

for all $\ell \in [0,1]$, and

$$(2.6) \qquad \lim_{\ell \to 1} u_\ell(c,\ell) = -\infty$$

for all $c \geq 0$.

The felicity function is strictly concave, so the determinant of its Hessian is positive:

$$(2.7) \qquad u_{cc}u_{\ell\ell} - u_{c\ell}u_{\ell c} > 0$$

Both leisure and consumption are normal goods. For consumption, this holds if

$$(2.8) \qquad u_{c\ell}u_\ell - u_{\ell\ell}u_c > 0$$

For leisure, normality holds if

$$(2.9) \qquad u_{cc}u_\ell - u_{\ell c}u_c > 0$$

(Killingsworth 1988, p. 223).

Firms maximize after-tax profits. Firms can limit or suspend operations. Since the capital income tax is levied on income net of depreciation, firms will suspend operations if the tax rate exceeds 100%. We assume that firms do not limit operations if the capital income tax rate is exactly 100%. This is necessary for there to be an optimal policy.

We use a short cut to model tax avoidance. Capital income tax revenue is zero if the tax exceeds 100%. Thus, permitting tax avoidance is equivalent to requiring that $r \geq 0$. This restriction is easier to work with.

In this paper, time subscripts (if any) appear after any partial derivative or descriptor subscripts. For example, the marginal product of capital at time t is written

$$F_{kt}$$

and the capital income tax at time v is written

$$\tau_{rv}$$

Time subscripts will often be omitted.

3. THE GOVERNMENT'S PROBLEM

If the government has a commitment technology, then at time zero it chooses taxes $\{\tau_{rt}, \tau_{wt}: t \geq 0\}$ to maximize private utility

$$(3.1) \qquad \int_{t=0}^{\infty} e^{-\theta t} u(c_t, \ell_t) dt$$

subject to initial conditions:

$$(3.2) \qquad k_0, \; k_0^P \text{ given}$$

the transversality condition for national capital:

$$(3.3) \qquad \lim_{t \to \infty} e^{-\int_0^t F_{kv} dv} k_t = 0$$

the solution to the private problem[6]:

$$(3.4) \qquad \lim_{t \to \infty} e^{-\int_0^t r_v dv} k_t^P = 0$$

$$(3.5) \qquad \dot{u}_c = (\theta - r) u_c$$

$$(3.6) \qquad u_\ell = -w u_c$$

the equations of motion for national and private capital:

$$(3.7) \qquad \dot{k} = F(k, \ell) - c - G$$

$$(3.8) \qquad \dot{k}^P = r k^P + w\ell - c$$

the conditions for market clearance:

$$(3.9) \qquad r = F_w - \tau_r$$

$$(3.10) \qquad w = F_\ell - \tau_w$$

and the tax avoidance constraint:

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[6] Assuming there is an interior solution that satisfies the usual first order conditions (see Appendix A).

(3.11) $r \geq 0$

If the government does not have a commitment technology, Cohen and Michel (1988) show that one can obtain the optimal policy by requiring the government to play a stationary strategy.[7] The state variables are k and k^P. Any equilibrium strategy in such a game is time consistent. Since it is optimal starting from any k and k^P, it also remains optimal as k and k^P change over time. Since the government plays a stationary strategy, the agent has an optimal strategy that is stationary as well.

We assume that the government moves first at each point in time. That is, the agent observes τ_{rt} and τ_{wt} before deciding on c_t and ℓ_t. This is a conventional assumption for fiscal policy, though not for monetary policy (Cohen and Michel 1988). The government's strategy is a pair

$$\left(\tau_r(k,k^P),\ \tau_w(k,k^P)\right)$$

of functions that depend only on the state variables k and k^P. The agent's strategy is a pair

$$\left(c(k,k^P,\tau_r,\tau_w),\ \ell(k,k^P,\tau_r,\tau_w)\right)$$

of functions that depend on the state variables k and k^P, as well as on the current observed taxes.

The government without commitment solves the problem

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7 This is also known as a Markov strategy. The equilibrium concept we are using is also known as Markov perfection (Fudenberg and Tirole 1991, pp. 513-515; Maskin and Tirole 1993).

$$(3.12) \qquad \max_{\tau_w(\cdot),\tau_r(\cdot)} \int_{t=0}^{\infty} e^{-\theta t} u(c_t, \ell_t) dt$$

subject to initial conditions:

$$(3.13) \qquad k_0,\ k_0^P \text{ given}$$

the transversality condition for national capital:

$$(3.14) \qquad \lim_{t\to\infty} e^{-\int_0^t F_{kv} dv} k_t = 0$$

the equations of motion for national and private capital:

$$(3.15) \qquad \dot{k} = F(k,\ell) - c - G$$

$$(3.16) \qquad \dot{k}^P = rk^P + w\ell - c$$

the conditions for market clearance:

$$(3.17) \qquad r = F_w - \tau_r$$

$$(3.18) \qquad w = F_\ell - \tau_w$$

the representative agent's equilibrium strategy:

$$(3.19) \qquad c = c(k,k^P,\tau_r,\tau_w)$$

$$(3.20) \qquad \ell = \ell(k,k^P,\tau_r,\tau_w)$$

and the tax avoidance constraint:

$$(3.21) \qquad r \geq 0$$

Note that the solution equations for the private problem, (3.4-3.6), are replaced by (3.19) and (3.20).

The agent takes aggregate variables as given. To formalize this, suppose there are many agents, all playing $\left(c(\cdot),\ \ell(\cdot)\right)$ with wealth k^P, and consider a particular agent who plays $\left(\underset{\sim}{c}(\cdot),\ \underset{\sim}{\ell}(\cdot)\right)$ and has wealth $\underset{\sim}{k}^P$. At time t, the agent observes $(\tau_{rt},\tau_{wt},k_t,k_t^P)$ and uses knowledge of the equilibrium strategies, $\left(c(\cdot),\ \ell(\cdot)\right)$ and $\left(\tau_r(\cdot),\ \tau_w(\cdot)\right)$, to predict the current and future interest rates and wages she will face, $\{r_s, w_s\colon s \geq t\}$. The agent maximizes

$$(3.22) \qquad \int_{t=0}^{\infty} e^{-\theta t} u(\underset{\sim}{c}_t, \underset{\sim}{\ell}_t)\,dt$$

given these current and predicted interest rates and wages, as well as the law of motion for the agent's wealth:

$$(3.23) \qquad \dot{\underset{\sim}{k}}^P = r\underset{\sim}{k}^P + w\underset{\sim}{\ell} - \underset{\sim}{c}$$

and the usual transversality condition for $\underset{\sim}{k}^P$. Finally, all agents are identical, so we set $c(\cdot) = \underset{\sim}{c}(\cdot)$, $\ell(\cdot) = \underset{\sim}{\ell}(\cdot)$, and $k^P = \underset{\sim}{k}^P$.

4. OPTIMAL POLICY: COMMITMENT CASE

The following theorems address the commitment case. The proof of these theorems apply the first order conditions of Pontryagin's Maximum Principle (Pontryagin et al 1962). As Seierstad and Sydsæter (1987, pp. 85-87) note, there are degenerate systems in which these first order conditions are not necessary for an optimum. Degeneracy is analogous to the failure of constraint qualification in the Kuhn-Tucker theorem (Seierstad and Sydsæter 1987, p. 87). The problem could be degenerate if, for example, there were only one path of taxes that raised enough revenue to cover planned spending.[8]

We assume the system given in (3.1-3.11) is not degenerate. We also assume nondegeneracy in the solution to the private problem (Appendix A) and in no-commitment problem (Appendix C).

Finally, we assume that $k_t^P > 0$ for all $t \geq 0$ in the optimal path. If k^P were negative, r would become the interest rate the agent payed on her debt to the government. The goverment could raise arbitrary revenue by letting r approach infinity if k^P ever dropped below 0.[9] This would

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[8] I am grateful to Peter Diamond for bringing this example, as well as the general issue, to my attention.

[9] The government is increasing the agent's debt essentially by fiat.

clearly be efficient. Rather than adding another constraint on r, we restrict attention to economies in which $k_t^P > 0$ for all t on the optimal path.

Theorem 4.1, parts (a) and (b), restate Chamley's (1986) results. Part (c) is new.

Theorem 4.1

Suppose the government's problem is given by (3.1-3.11). The following hold under the assumptions of Section 2.

(a) $r_0 = 0$ *if the government initially requires revenue (Chamley 1986).*

(b) *If the optimal policy gives rise to a convergent path, then* $\tau_r = 0$ *and* $F_k = \theta$ *in steady state (Chamley 1986).*

(c) *If the government initially requires revenue, then the wage tax is strictly positive whenever* $\tau_r < F_k$.■

Proof

Appendix B.

As Chamley (1986) remarks, an immediate capital income tax is equivalent to a capital levy. This explains (a). In steady state, consumption at different dates has the same price elasticity. It should be taxed equally. A brief capital income tax at time t is equivalent to a tax on all consumption after t. The only way to tax all consumption equally is to set $\tau_r = 0$. This explains (b).[10]

Small taxes have no first order deadweight loss. If future wage taxes were nonpositive, one could therefore reduce the excess burden of taxation by raising future wage taxes and lowering other taxes. This explains (c).

Theorem 4.2 derives the wage tax path in a special case.

Theorem 4.2

Suppose the utility function is additively separable:

$$(4.1) \qquad u(c,\ell) = \alpha(c) + \beta(\ell)$$

and CRRA in labor supply. *The second condition means that*[11]

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[10] As noted earlier, the total of earlier capital income taxes constitute a tax on steady state consumption.

[11] There is a class of utility functions that satisfy both (4.2) and (2.6). First it is necessary to redefine labor supply so that it ranges from 0 to ∞. The transformed variable $\hat{\ell}$ satisfies this:

$$\hat{\ell} = \frac{\ell}{1-\ell}$$

We then define a new production function $\hat{F}(k,\hat{\ell})$:

$$\hat{F}(k,\hat{\ell}) = F\left(k, \frac{\hat{\ell}}{1+\hat{\ell}}\right)$$

It can be verified that $\hat{F}$ satisfies (2.1) and the usual concavity conditions, if F does. Then any utility function of the following form is CRRA in transformed labor, $\hat{\ell}$:

(4.2) $$\frac{d}{d\ell}\left[\frac{\ell\beta_{\ell\ell}}{\beta_\ell}\right] = 0$$

Then the wage tax for any time t ≥ 0 *is given by:*

(4.3) $$\frac{\tau_{wt}}{w_t} = A*\exp\left(\int_{v=0}^{t}\tau_{rv}dv\right) - 1$$

where A *is a positive constant and* τ_{rv} *is the capital income tax at time* v ∈ [0,t]. *This holds regardless of whether* $\tau_{rt} = F_{kt}$ *or not.*

Proof

Appendix B.

In this special case, the wage tax rises while capital income is taxed and falls while it is subsidized. An increase in w affects social welfare through three effects:

(a) Current labor supply rises, lowering instantaneous utility.

(b) Private wealth, k^P, increases directly through a transfer on existing labor supply, as well as indirectly through a labor supply increase. Holding national capital constant, any such increase represents lost potential revenue and is undesirable.

(c) National capital increases through a labor supply response. When the wage tax is high, more of the

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$$u(c,\hat{\ell}) = \alpha(c) - \beta\hat{\ell}^{\gamma}$$

where $\gamma > 1$ and $\beta > 0$.

increase in production accrues to the treasury than to the private sector, so the effect on national capital is large relative to the effect on private wealth in (b).

Each of these effects may be countered by an increase in consumption if consumption and leisure are substitutes. Since we assume utility is separable in consumption and labor supply, we ignore such effects.

While capital income is taxed, the social rate of return exceeds the private rate. Therefore, when both the government and the agent are maximizing, the social value of an increase in national capital in (c), resulting from an increase in w, falls over time, *relative* to the instantaneous disutility from greater labor supply in (a) and the welfare loss from the transfer to the private sector in (b) that also result from the increase in w.

On the other hand, the maximum principle implies that a small change in w (or, equivalently, in τ_w) should have no effect on social welfare at any time. This implies that τ_w must rise over time if capital income is taxed. Why? A higher wage tax strengthens the external effect of a *change* in the wage tax on national capital, relative to its private effects on either private wealth or instantaneous utility. This greater externality balances the lower shadow *value* of national capital relative to the cost of a transfer to the private sector or a decrease in instantaneous utility. For analogous reasons, τ_w must fall if capital income is subsidized.

The strong assumptions on the utility function in (4.1) and (4.2) ensure that the relative effects of a change in τ_w on national capital, private wealth, and instantaneous utility remain constant over time, for given τ_w. That is, the changes over time in variables *other than* τ_w (such as c and ℓ) must not alter the relation between these effects.

We now consider what happens to the capital income tax rate if and when it falls below 100%. Suppose the utility function is separable (4.1) and CRRA in consumption:

$$(4.4) \qquad \frac{d}{dc}\left[-\frac{c\alpha_{cc}}{\alpha_c}\right] = 0$$

Chamley (1986) shows that, if the capital income tax rate ever falls below 100%, it must jump directly to zero and remain there forever. Theorem 4.3 examines what happens if utility is IRRA or DRRA in consumption. Our results for DRRA utility are rather limited.

Theorem 4.3

Suppose the utility function is separable (4.1) and IRRA in consumption:

$$(4.5) \qquad \frac{d}{dc}\left[-\frac{c\alpha_{cc}}{\alpha_c}\right] > 0$$

Then τ_r *and* $F_k - \theta$ *are of the same sign whenever* $\tau_r < F_k$, *and one is zero only if the other is.*

On the other hand, if the utility function is separable (4.1) and <u>DRRA</u> *in consumption:*

(4.6) $$\frac{d}{dc}\left[-\frac{c\alpha_{cc}}{\alpha_c}\right] < 0$$

then whenever $\tau_r < F_k$, $\tau_r = 0$ *only if* $F_k = \theta$.

Proof

Appendix B.

A brief capital income tax at time t is equivalent to a tax on all consumption that occurs after t. If utility is IRRA, then consumption is less sensitive to its price when consumption is high than when it is low. Ceteris paribus, consumption should be taxed more heavily when it is less price elastic. Consider what would happen if τ_r were to drop directly to 0 when $F_k > \theta$. Since $r > \theta$, consumption would then grow. The sensitivity of consumption to its price would decline. Future consumption should be taxed more heavily than current consumption. τ_r should be positive. The reverse ($\tau_r < 0$) holds if $F_k < \theta$, since consumption would decline if τ_r were zero.

In the DRRA case, the preceding argument is not conclusive. Consumption is more sensitive to its price when consumption is high than when it is low. Suppose $F_k > \theta$. One possibility is that $r < \theta$. Consumption decreases. Since $\tau_r > 0$, later consumption is taxed more heavily than early consumption. Fine so far. But suppose $r > F_k$. Since $r > \theta$, consumption rises. Because $\tau_r < 0$, later consumption is taxed less heavily than early consumption. These are also consistent. Thus, this line of reasoning does not establish the sign of τ_r.

5. OPTIMAL POLICY: NO-COMMITMENT CASE

We now turn to the optimal policy without commitment. The problem is not very tractable and our results are modest. We continue to assume that the first order conditions of Pontryagin's Maximum Principle (Pontryagin _et al_ 1962) give necessary conditions for an optimum.[12] We also assume that an optimum exists. We assume that a solution exists to the agent's problem in (3.22-3.23) and in the accompanying discussion. We also continue to assume that $k_t^P > 0$ for all $t \geq 0$ in the optimal path.

Theorem 5.1

Under the above assumptions, the capital income tax rate is 100% until the government has raised enough revenue to finance all subsequent expenditures. There is no taxation after this point.

Proof

Appendix C.

The chief contrast with the commitment case is that an immediate capital income tax must be used as long as revenue is needed, due to the short run efficiency of such a tax.

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[12] See the discussion preceding Theorem 4.1.

6. DISCUSSION

The no-commitment model is not easy to analyze. This prevented us from deriving the path of the wage tax while capital income is taxed. A different framework might provide richer results.

It would also be worthwhile to permit the government to distinguish between old and new capital with an investment tax credit. The results of Judd (1987) and Goulder and Summers (1989), among others, suggest that a revenue-neutral, permanent increase in both the corporate profits tax and the investment tax credit might enhance welfare by shifting the tax burden to old capital.[13] In the right proportion, such an increase would be equivalent to a capital levy.

The threat of capital flight also constrains capital income taxes. Most prior work on optimal capital taxation in an open economy has used static or two-period models (Gordon 1991; Razin and Sadka 1989; Bond and Samuelson 1989; Feldstein and Hartman 1979). Such models do not capture intertemporal tradeoffs very well. The OLG simulation by Ha and Sibert (1992) does not suffer from this shortcoming, but

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[13] On the other hand, Fullerton and Henderson (1989, pp. 439-440) find that this policy might reduce efficiency by increasing the pre-existing tax preference for capital equipment.

lacks generality.

The model also assumes that government spending is unaffected by the timing of revenue. Results on the "flypaper effect" (Fisher 1982) suggest weakening this assumption.

APPENDIX A.

THE INDIVIDUAL'S PROBLEM

This section characterizes the behavior of the representative agent. Lemma 1 gives the first order conditions for a private optimum. Lemma 2 uses these conditions to compute the changes in consumption and labor supply that would result from small changes in r and w.

Suppose that the representative agent solves

$$\text{(A.1)} \qquad \max_{\{c_t,\ell_t:t\geq 0\}} \int_{t=0}^{\infty} e^{-\theta t} u(c_t,\ell_t)dt$$

subject to:

$$\text{(A.2)} \qquad k_0^P \text{ given}$$

$$\text{(A.3)} \qquad \dot{k}^P = rk^P + w\ell - c$$

$$\text{(A.4)} \qquad \lim_{t \to \infty} e^{-\int_0^t r_v dv} k_t^P = 0$$

$$\text{(A.5)} \qquad c \geq 0; \quad 0 \leq \ell \leq 1$$

Also suppose that the individual knows she will face wages and interest rates $\{w_t,r_t:t\geq 0\}$. Assume that this problem is not degenerate in the sense given in Seierstad and Sydsæter (1987, p. 85 and pp. 86-87, Note 5). Finally, assume that there is at least one path $\{(c_t,\ell_t): t \geq 0\}$ that satisfies (A.2-A.4) while also being an interior path: i.e., while also satisfying

$$\text{(A.6)} \qquad c_t > 0 \text{ and } 0 < \ell_t < 1 \text{ for all } t$$

Lemma 1

Under the above assumptions, necessary conditions for a solution to (A.1-5) include

$$\text{(A.7)} \qquad u_{ct} = u_{c0} e^{\int_0^t (\theta - r_v) dv}$$

and

$$\text{(A.8)} \qquad u_{\ell t} = -w_t u_{ct}$$

Proof

The proof that any interior optimal path satisfies (A.7) and (A.8) is a standard application of Pontryagin's Maximum Principle (Pontryagin <u>et al</u> 1962; Seierstad and Sydsæter 1987). By (2.1), (2.5) and (2.6), the existence of an interior path that satisfies (A.2-A.4) and (A.6) implies that any optimal path will also be interior (will also satisfy (A.6)) and thus be characterized by (A.7) and (A.8).■

Define $\mu = u_c$. Also let

$$\text{(A.9)} \qquad \Delta = u_{cc}u_{\ell\ell} - u_{c\ell}u_{\ell c}$$

Δ is positive by (2.7). Define $\hat{\mu} = d\mu/\mu$, and $\hat{w} = dw/w$.

Lemma 2

Suppose at time t the agent discovers that current and future interest rates and wages will change by $\{dr_x, dw_x: x \geq t\}$. The changes in consumption and labor, dc_x and $d\ell_x$, are:

$$(A.10) \qquad dc_x = -Y_x^c\left[\hat{\mu}_t - \int_{v=t}^{x}(dr_v)dv\right] + S_x^c\hat{w}_x$$

$$(A.11) \qquad d\ell_x = Y_x^\ell\left[\hat{\mu}_t - \int_{v=t}^{x}(dr_v)dv\right] + S_x^\ell\hat{w}_x$$

where Y^c, Y^ℓ, S^c, and S^ℓ are given by the following formulas.

$$(A.12) \qquad Y^c = \frac{u_{c\ell}u_\ell - u_{\ell\ell}u_c}{\Delta} > 0$$

$$(A.13) \qquad Y^\ell = \frac{u_{cc}u_\ell - u_{\ell c}u_c}{\Delta} > 0$$

$$(A.14) \qquad S^c = -\frac{u_{\ell c}u_\ell}{\Delta}$$

$$(A.15) \qquad S^\ell = \frac{u_{cc}u_\ell}{\Delta} > 0$$

The sign of S^c depends on whether leisure and consumption are net substitutes or net complements.

Proof

This proof is taken from Killingsworth (1988). From Lemma 1, the solution of the agent's problem includes the equation

$$(A.16) \qquad u_c(c_x,\ell_x) = \mu_t e^{\int_t^x(\theta-r_v)dv}$$

By (2.1) and (2.6), we may assume an interior solution for labor: $0 < \ell_x < 1$. Then from Lemma 1,

$$u_\ell(c_x,\ell_x) = -w_x\mu_t e^{\int_t^x(\theta-r_v)dv} \tag{A.17}$$

We totally differentiate these two equations to find the new paths of consumption and labor:

$$u_{ccx}dc_{tx} + u_{c\ell x}d\ell_{tx} = u_{cx}\left[\hat{\mu}_t - \int_{v=t}^{x}(dr_v)dv\right] \tag{A.18}$$

$$u_{\ell cx}dc_x + u_{\ell\ell x}d\ell_x = u_{\ell x}\left[\hat{\mu}_t + \hat{w}_x - \int_{v=t}^{x}(dr_v)dv\right] \tag{A.19}$$

This is a system of two equations in the two unknowns dc_x and $d\ell_x$. The solution of this system is the formulas given above. Y^c and Y^ℓ are positive by (2.7-2.9). The sign of S_x^ℓ follows by (2.7) and the assumptions that $u_{cc} < 0$, and $u_\ell < 0$. This completes the proof.■

APPENDIX B.

PROOFS OF THEOREMS 4.1-4.3

PROOF OF THEOREM 4.1

We can eliminate τ_r and τ_w from the government's problem by letting the government work with r and w directly. The government solves

$$(B.1) \qquad \max_{\{r_t, w_t : t \geq 0\}} \int_{t=0}^{\infty} e^{-\theta t} u(c_t, \ell_t) dt$$

subject to the constraints:

$$(B.2) \qquad k_0,\ k_0^P \text{ given}$$

$$(B.3) \qquad \lim_{t\to\infty} e^{-\int_0^t F_{kv} dv} k_t = 0$$

$$(B.4) \qquad \dot{\mu} = (\theta - r)\mu$$

where $\mu = u_c$;

$$(B.5) \qquad \dot{k} = F(k, \ell) - c - G$$

$$(B.6) \qquad \dot{k}^P = rk^P + w\ell - c$$

$$(B.7) \qquad c = c(\mu, w)$$

$$(B.8) \qquad \ell = \ell(\mu, w)$$

$$(B.9) \qquad r \geq 0$$

(B.4) and (B.7-B.8) are equivalent to (3.5-3.6). There is a one-to-one relation between consumption and labor supply, on the one hand, and μ and w, on the other. Normality of consumption and leisure implies that all income-expansion paths (Varian 1984, pp. 118-9) are strictly upward sloping in $(c, 1-\ell)$ space. The income-expansion path is uniquely determined by w. μ then picks out a unique consumption-leisure bundle on this path.

The present-value Hamiltonian for this problem is

$$\text{(B.10)} \quad H = e^{-\theta t}\Big[u(c,\ell) + \nu(F(k,\ell) - c - G) + \eta(rk^P + w\ell - c) + \zeta\mu(\theta - r) + \gamma r\Big]$$

We write c and ℓ for $c(\mu,w)$ and $\ell(\mu,w)$, respectively. First order conditions, all multiplied by $e^{\theta t}$, are

$$\text{(B.11)} \quad 0 = u_c c_w + u_\ell \ell_w + \eta(\ell + w\ell_w - c_w) + \nu(F_\ell \ell_w - c_w)$$

$$\text{(B.12)} \quad 0 = \eta k^P - \zeta\mu + \gamma$$

$$\text{(B.13)} \quad \dot{\eta} - \eta\theta = -\eta r$$

$$\text{(B.14)} \quad \dot{\nu} - \nu\theta = -\nu F_k$$

$$\text{(B.15)} \quad \dot{\zeta} - \zeta\theta = -u_c c_\mu - u_\ell \ell_\mu - \eta(w\ell_\mu - c_\mu) - \nu(F_\ell \ell_\mu - c_\mu) - \zeta(\theta - r)$$

(B.11) and (B.15) can be simplified. From (A.6) we find that, for any times t and x,

$$\text{(B.16)} \quad \hat{\mu}_x = \hat{\mu}_t - \int_{v=t}^{x} (dr_v)dv$$

Thus, we can rewrite (A.3) and (A.4) in Lemma 1 as

$$\text{(B.17)} \quad dc = -Y^c\hat{\mu} + S^c\hat{w}$$

and

$$\text{(B.18)} \quad d\ell = Y^\ell\hat{\mu} + S^\ell\hat{w}$$

where $\hat{\mu} = d\mu/\mu$ and $\hat{w} = dw/w$. All variables in (B.17-B.18) are measured at a given time.

Using these formulas, (B.11) can be written

$$(B.19) \quad 0 = -\mu(wS^{\ell} - S^{c}) + \eta(w\ell + wS^{\ell} - S^{c}) + \nu(F_{\ell}S^{\ell} - S^{c})$$

and (B.15) implies that

$$(B.20) \quad \dot{\zeta\mu} - \zeta\mu r = (\mu - \eta)(wY^{\ell} + Y^{c}) - \nu(F_{\ell}Y^{\ell} + Y^{c})$$

First we will explain why $r_0 = 0$ and why, if the path is convergent, $r = F_k = \theta$ in steady state. There is no way to satisfy a finite revenue requirement instantaneously, given the constraint on r and the assumption that $k_0^P > 0$. Thus, the initial shadow value of a marginal transfer from government to agent, η_0, must be strictly negative. However, $\zeta_0 = 0$ since μ_0 is free. Thus, by (B.12), $\gamma_0 > 0$ initially, which establishes that $r_0 = 0$. If the path is convergent, then in steady state we must also have $\dot{\eta} = \dot{\nu} = 0$, so $r = F_k = \theta$ by (B.13) and (B.14).[14] These observations are due to Chamley (1986).

We will now show that the wage tax is positive when $r > 0$.

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[14] This also requires $\eta_{\infty} \neq 0$ and $\nu_{\infty} \neq 0$, where the subscript "∞" denotes the limit as $t \to \infty$. Recall that ν is the shadow value of national capital, k. Since the path is convergent, $u_{c\infty} \neq 0$, so the value of consuming a marginal increase in national capital remains strictly positive in steady state. $\eta_{\infty} \neq 0$ because, by (B.13) and (B.4), η/u_c is constant, and because $\eta_0 \neq 0$ and $u_{c\infty} \neq 0$.

Claim

If $r > 0$,

(a) The wage tax can be expressed as

$$\text{(B.25)} \qquad \frac{\tau_w}{w} = -\frac{\eta_0}{\nu_0}\left[\frac{cY^\ell + \ell Y^c}{S^c Y^\ell + S^\ell Y^c}\right]\exp\left(\int_{v=0}^{t} \tau_{rv}\,dv\right)$$

(b) The wage tax is strictly positive.

Proof

We first show that (a) implies (b). By (A.11-A.14),

$$\text{(B.26)} \qquad S^c Y^\ell + S^\ell Y^c = -\frac{u_c u_\ell}{\Delta} > 0$$

Also, $\eta_0 < 0$, $\nu_0 > 0$, $Y^\ell > 0$, and $Y^c > 0$. Together with (a), these imply (b).

We now prove (a). When $r > 0$ in an interval of time, $\dot{\gamma} = \gamma = 0$ in the interior of that interval. But by (B.12), it always holds that

$$\text{(B.27)} \qquad \dot{\gamma} = \dot{\zeta}\mu + \zeta\dot{\mu} - \dot{\eta}k^P - \eta\dot{k}^P$$
$$= \theta\gamma + (\mu - \nu - \eta)(wY^\ell + Y^c) - \nu\tau_w Y^\ell - \eta(w\ell - c)$$

We have used (B.13), (B.20-21), and (B.6). Thus, when $\dot{\gamma} = \gamma = 0$, (B.27) implies that

$$\text{(B.28)} \qquad \tau_w = \frac{(\mu - \nu - \eta)(wY^\ell + Y^c) - \eta(w\ell - c)}{\nu Y^\ell}$$

But by (B.19), it always holds that

$$\text{(B.29)} \qquad \tau_w = \frac{(\mu - \nu - \eta)wY^\ell - \eta w\ell}{\nu S^\ell}$$

We have used the fact that

$$\text{(B.30)} \qquad wS^\ell - S^c = wY^\ell$$

which follows directly from (A.7) and (A.12-14).

Equate the right hand sides of (B.28) and (B.29) to obtain

$$(B.31) \qquad (\mu - \nu - \eta)\left[wY^{\ell}(Y^{\ell} - S^{\ell}) - Y^{c}S^{\ell}\right]$$

$$= \eta w\ell(Y^{\ell} - S^{\ell}) + \eta c S^{\ell}$$

Now use (B.30) to eliminate the $Y^{\ell} - S^{\ell}$ terms. We find that

$$(B.32) \qquad (\mu - \nu - \eta) = \frac{\eta(\ell S^{c} - cS^{\ell})}{Y^{\ell}S^{c} + Y^{c}S^{\ell}}$$

Substitute this expression into (B.29) to obtain

$$(B.33) \qquad \frac{\tau_w}{w} = -\frac{\eta}{\nu}\left[\frac{cY^{\ell} + \ell Y^{c}}{S^{c}Y^{\ell} + S^{\ell}Y^{c}}\right]$$

Then use (B.13-14) to obtain (B.25).■

END OF PROOF OF THEOREM 4.1

PROOF OF THEOREM 4.2

If we solve (B.19) for τ_w/w and then use (B.13-14) and (B.4) to express μ, η, and ν as exponential multiples of μ_0, η_0, and ν_0, we obtain:

$$(B.22) \qquad \frac{\tau_w}{w} = -\frac{Y^{\ell}}{S^{\ell}} + \left[\frac{\left(u_{c0}-\eta_0\right)Y^{\ell} - \eta_0\ell}{\nu_0 S^{\ell}}\right]\exp\left(\int_{v=0}^{t}\tau_{rv}dv\right)$$

By (4.1), $u_{c\ell} = 0$, so

$$(B.23) \qquad \frac{Y^{\ell}}{S^{\ell}} = 1$$

using (A.12) and (A.14). $\eta_0 < 0$ and $\nu_0 > 0$ from the proof of Theorem 4.1. $u_{c\ell} = 0$ implies that

$$(B.24) \qquad \frac{\ell}{S^{\ell}} = \frac{\ell u_{\ell\ell}}{u_{\ell}}$$

which is positive and constant if u is CRRA in labor supply. These facts prove (4.3) and verify that the quantity A in (4.3) is a positive constant.

END OF PROOF OF THEOREM 4.2

PROOF OF THEOREM 4.3

First, substitute the expression for τ_w in (B.29) into equation (B.27). This gives us

$$(B.34) \qquad S^{\ell}(\dot{\gamma} - \theta\gamma) = (\mu - \eta - \nu)(S^{c}Y^{\ell} + S^{\ell}Y^{c}) + \eta(cS^{\ell} - \ell S^{c})$$

In the proof of Theorem 4.1 we showed that, if $r > 0$ in a time interval, $\dot{\gamma} = \gamma = 0$ in the interior of the interval. Thus, when $r > 0$, (B.34) implies that

$$(B.35) \qquad \frac{\mu - \eta - \nu}{\eta} = \frac{S^{c}Y^{\ell} + S^{\ell}Y^{c}}{\ell S^{c} - cS^{\ell}} = -\frac{u_c}{cu_{cc} + \ell u_{\ell c}}$$

This uses (B.26) and (A.11-14).

Utility is separable in consumption and labor supply, so $u_{\ell c} = 0$. The right hand side of (B.35) equals $1/R(c)$, where $R(c)$ is the coefficient of relative risk aversion:

$$(B.36) \qquad R(c) = -\frac{cu_{cc}}{u_c}$$

Differentiate (B.35) with respect to time. By

(B.13-14) and (B.4) this gives

$$\text{(B.37)} \qquad \frac{\nu}{\eta}\left[F_k - r\right] = \frac{d}{dt}\left[\frac{1}{R(c)}\right]$$

Since $\nu/\eta < 0$, $\tau_r = F_k - r$ is positive, negative, or zero, depending on whether R(c) is respectively increasing, decreasing, or unchanging <u>over time</u> (not with respect to c). In particular, if R(c) is constant, then $\tau_r = 0$ (Chamley 1986).

Now divide (B.17) by dt and use (A.13) to eliminate S^c. We find that

$$\text{(B.38)} \qquad \dot{c} = -Y^c\left[\theta - r\right]$$

Consumption is increasing, decreasing, or constant if $r - \theta$ is respectively positive, negative, or zero.

Now suppose, as in the statement of Theorem 4.1, that u is IRRA ($R'(c) > 0$). If $F_k \neq \theta$, then there are six orders for r, F_k, and θ. We can rule out four of these.

a) $r \leq \theta < F_k$: then $\dot{c} \leq 0$, so $\dot{R} \leq 0$, so $\tau_r \leq 0$.

b) $\theta < F_k \leq r$: then $\dot{c} > 0$, so $\dot{R} > 0$, so $\tau_r > 0$.

c) $\theta < r < F_k$: then $\dot{c} > 0$, so $\dot{R} > 0$, to $\tau_r > 0$.

d) $r \leq F_k < \theta$: then $\dot{c} < 0$, so $\dot{R} < 0$, so $\tau_r < 0$.

e) $F_k < \theta \leq r$: then $\dot{c} \geq 0$, so $\dot{R} \geq 0$, so $\tau_r \geq 0$.

f) $F_k < r < \theta$: then $\dot{c} < 0$, so $\dot{R} < 0$, so $\tau_r < 0$.

Only (c) and (f) are internally consistent. If $F_k = \theta$, then $r = F_k = \theta$ is the only possibility. We have confirmed that $F_k - \theta$ and τ_r are of the same sign if $r > 0$.

The argument when u is DRRA in consumption is analogous, though less conclusive. There are six possible cases:

a) $r < \theta \leq F_k$: then $\dot{c} < 0$, so $\dot{R} > 0$, so $\tau_r > 0$.

b) $\theta \leq F_k < r$: then $\dot{c} > 0$, so $\dot{R} < 0$, so $\tau_r < 0$.

c) $\theta \leq r \leq F_k$: then $\dot{c} \geq 0$, so $\dot{R} \leq 0$, to $\tau_r \leq 0$.

d) $r < F_k < \theta$: then $\dot{c} < 0$, so $\dot{R} > 0$, so $\tau_r > 0$.

e) $F_k < \theta < r$: then $\dot{c} > 0$, so $\dot{R} < 0$, so $\tau_r < 0$.

f) $F_k \leq r \leq \theta$: then $\dot{c} \leq 0$, so $\dot{R} \geq 0$, so $\tau_r \geq 0$.

(c) and (f) are possible only if $F_k = r = \theta$. Thus, $\tau_r = 0$ only if $F_k = \theta$. This establishes what was claimed for the DRRA case.

END OF PROOF OF THEOREM 4.3

APPENDIX C.

PROOF OF THEOREM 5.1

As in the commitment case, we let the government set r and w directly. The problem in (3.12-3.21) becomes

(C.1) $$\max_{\{r_t, w_t : t \geq 0\}} \int_{t=0}^{\infty} e^{-\theta t} u(c_t, \ell_t) dt$$

subject to:

(C.2) k_0, k_0^P given,

(C.3) $$\lim_{t\to\infty} e^{-\int_0^t F_{kv} dv} k_t = 0$$

(C.4) $\dot{k} = F(k,\ell) - c - G$

(C.5) $\dot{k}^P = rk^P + w\ell - c$

(C.6) $c = c(\mu, w)$

(C.7) $\ell = \ell(\mu, w)$

(C.8) $\mu = \mu(k, k^P, r, w)$

(C.8a) $r \geq 0$

(C.6-C.8) are equivalent to (3.19-3.20). By the argument in the proof of Theorem 4.1, c and ℓ are determined uniquely by μ and w. However, w is under the government's direct control. Thus, only μ depends on k, k^P, and r, so we can rewrite (3.19) and (3.20) as (C.6-C.8).

We first show that $\mu(k,k^P,r,w)$ depends on k and k^P alone.

Claim C.1

For any $(k,k^P,r,w) \in (0,\infty) \times (0,\infty) \times [0,\infty) \times (-\infty,\infty)$,

$$\frac{\delta\mu(k,k^P,r,w)}{\delta r} = \frac{\delta\mu(k,k^P,r,w)}{\delta w} = 0 \tag{C.9}$$

Proof

Integrate (A.3) subject to (A.4) to obtain

$$k_t^P = \int_{s=t}^{\infty} \bar{\delta}_s^t \left[c_s - w_s \ell_s\right] ds \tag{C.10}$$

where we define

$$\bar{\delta}_s^t = \exp\left(- \int_{v=t}^{s} r_v dv\right) \tag{C.11}$$

Totally differentiate (C.10) to obtain

$$dk_t^P = \int_{s=t}^{\infty} \bar{\delta}_s^t \left[\left(- \int_{v=t}^{s} (dr_v) dv\right)\left(c_s - w_s \ell_s\right) + dc_s - w_s d\ell_s - \ell_s dw_s\right] ds \tag{C.12}$$

Reverse the order of integration and apply (C.10) to obtain

$$\int_{s=t}^{\infty} \bar{\delta}_s^t \left(- \int_{v=t}^{s} (dr_v) dv\right)\left(c_s - w_s \ell_s\right) ds = - \int_{v=t}^{\infty} \bar{\delta}_v^t k_v^P (dr_v) dv \tag{C.13}$$

Now apply (4.3-4.4) and (C.13) to (C.12) to obtain

$$dk_t^P = \int_{s=t}^{\infty} \bar{\delta}_s^t \left[-k_s^P dr_s - \ell_s dw_s\right. \tag{C.14}$$

$$- \left(w_s Y_s^{\ell} + Y_s^c\right)\left(\hat{\mu}_t - \int_{v=t}^{s} (dr_v)dv\right)$$

$$- \left(w_s S_s^{\ell} - S_s^c\right)\hat{w}_s\Bigg]ds$$

Consider a local perturbation of (r_t, w_t) to $(r_t + dr_t, w_t + dw_t)$. Since the change has zero duration, it has no effect on (k_s, k_s^P) for $s \geq t$. But by stationarity, the government's policy at time s is a function of (k_s, k_s^P) alone. The agent must anticipate that (r_s, w_s) will be unchanged for any $s > t$. Moreover, $w_s Y_s^{\ell} + Y_s^c > 0$ for $s > t$ by (A.12-A.13). (C.14) can only be satisfied by setting $\hat{\mu}_t = 0$.■

The present-value Hamiltonian for (C.1-C.8a) is

$$\text{(C.15)} \quad H = e^{-\theta t}\Big[u(c,\ell) + \nu(F(k,\ell) - c - G) + \eta(rk^P + w\ell - c) + \gamma r\Big]$$

Time subscripts are omitted. We now give the first order conditions for an optimum. For (C.16-C.17), we rely in part on Claim C.1.

$$\text{(C.16)} \quad 0 = u_c c_w + u_{\ell}\ell_w + \eta(\ell + w\ell_w - c_w) + \nu(F_{\ell}\ell_w - c_w)$$

$$\text{(C.17)} \quad 0 = \eta k^P + \gamma$$

$$\text{(C.18)} \quad \dot{\eta} - \eta\theta = -\Big[u_c c_{\mu} + u_{\ell}\ell_{\mu} + \nu(F_{\ell}\ell_{\mu} - c_{\mu}) + \eta(w\ell_{\mu} - c_{\mu})\Big]\frac{\partial\mu}{\partial k^P} - \eta r$$

(C.19) $\dot{\nu} - \nu\theta =$

$$-\left[u_c c_\mu + u_\ell \ell_\mu + \nu(F_\ell \ell_\mu - c_\mu) + \eta(w\ell_\mu - c_\mu)\right]\frac{\partial\mu}{\partial k}$$

$$- \nu F_k$$

η is the shadow value of a transfer from government to agent, also known as the *excess burden of taxation*.

<u>Claim</u> <u>C.2</u>

For any $t > 0$, either $r_t = 0$ or $\eta_t = 0$.

<u>Proof</u>

By complementary slackness, either $r_t = 0$ or $\gamma_t = 0$. But by (C.17) and the assumption that $k_t^P > 0$, $\gamma_t = 0$ if and only if $\eta_t = 0$.■

Since η_t is the excess burden of taxation, $\eta_t = 0$ if and only if the government has enough wealth at time t to fund all future expenditures. If this holds, The first best is attainable from time t on. There is no further taxation. Thus, Claim C.2 implies that, at any time t, either capital income is taxed at 100% ($r_t = 0$) or there is no taxation at times $t' \geq t$. This concludes the proof.

END OF PROOF OF THEOREM 5.1

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