Auctions of Digital Goods with Externalities

by

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Abstract

Data is increasingly important for firms, regulators, and researchers to develop accurate models for decision-making. Since data sets often need to be externally acquired, a systematic way to value and trade data is necessary. Moreover, buyers of data often interact with each other downstream, such as firms competing in a market. In this setting, an allocation of data may not only benefit the buying firm, but also impose negative externalities on the firm's competitors. The way data is allocated and sold should thus depend on the particulars of its downstream usage and the interaction between data buyers.

We capture the problem of valuing and selling data sets to buyers who interact downstream within the general framework of auctions of digital, or freely replicable, goods. We study the resulting single-item and multi-item mechanism design problems in the presence of additively separable, negative allocative externalities among bidders. Two settings of bidders' private types are considered, in which bidders either know the externalities that others exert on them or know the externalities that they exert on others. We obtain forms of the welfare-maximizing (efficient) and revenue-maximizing (optimal) auctions of single digital goods in both settings and highlight how the information structure affects the resulting mechanisms. We find that in all cases, the resulting allocation rules are deterministic single thresholding functions for each bidder. For auctions of multiple digital goods, we assume that bidders have independent, additive valuations over items and study the first setting of privately known incoming externalities. We show that the welfare-maximizing mechanism decomposes into multiple efficient single-item auctions using the Vickrey-Clarke-Groves mechanism. Under revenue-maximization, we show that selling items separately via optimal single-item auctions yields a guaranteed fraction of the optimal multi-item auction revenue. This allows us to construct approximately revenue-maximizing multi-item mechanisms using the aforementioned optimal single-item mechanisms.

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Contents

Chapter 1

Introduction

1.1 Motivation

Propelled by the digitization of services and the Internet of Things, a wide range of data has become increasingly crucial for firms and regulators to develop accurate models for production and planning decisions. As machine learning algorithms grow more standardized, the bottleneck for real-time modeling and decision-making lies largely in obtaining useful and up-to-date data sets. Such data frequently needs to be acquired from specialized external sources (e.g., information about consumers, satellite images, weather forecasts), which has led to the proliferation of data services selling such information. Moreover, buyers of data often interact with each other downstream, such as with firms competing in a market. In this case, an allocation of data may not only benefit the buying firm, but also impose negative externalities on the firm's competitors. The way data is allocated and sold should thus depend on the particulars of its downstream usage and the interaction between data buyers.

Meanwhile, the goal of data sellers is often either to maximize their revenue or to maximize social welfare, in the latter case ensuring the most efficient allocation of data amongst the buyers. To achieve either goal, one needs to be able to price collections of data sets, and to this end, to parameterize the value that data buyers have for an allocation of data. This task is shaped by two properties of data: (1) as a digital good, it is freely replicable and so there is no inherent scarcity of it, and (2) its value is intrinsically combinatorial, i.e., different data sets, such as training features for a machine learning model, often contain correlated information. Without additional structure, a buyer's valuation over a collection of data sets may require a prohibitively large number of parameters to express. However, under certain assumptions on the usage of data, we can reduce such combinatorial valuations to functions of a scalar allocation variable. (See Section 2.1.)

We are then led to the overarching question: how should a data seller allocate and price data sets to data buyers who may compete with each other downstream, in such a way that maximizes social welfare or the seller's expected revenue? We motivate the central model and mechanism design approach through which this thesis addresses this question with the following example.

1.1.1 Example: Allocative Externalities Arising from Competition Among Data Buyers

Overview. Consider the setting depicted in Figure 1-1, in which a monopolistic data seller sells data sets to firms who subsequently use them to train models for prediction tasks, and then use these models to make decisions, such as inventory management or task scheduling, in a downstream market.

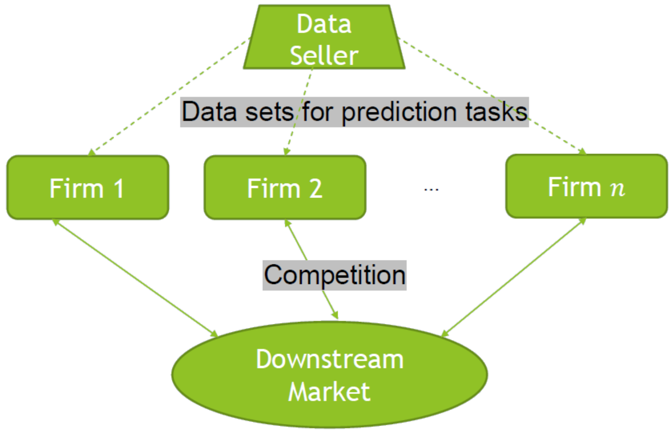

Figure 1-1: A monopolistic data seller sells data sets to firms who subsequently use them to train models for prediction tasks, and use these models to make decisions in a downstream market. These firms are in competition with each other downstream, so one firm's increase in prediction accuracy may hurt another's utility in the market.

Concretely, suppose several firms producing substitute goods are engaging in Cournot competition, and can purchase data from a third party. The usage of such data has the ultimate effect of increasing the buying firm's production efficiency, so more goods are produced for a given input investment. We assume there is a monotone increasing relationship between the quality or quantity of data purchased and the resulting increase in production efficiency.

By buying data, all else fixed, a firm will realize an increase in its equilibrium profit. However, since the firms are in competition with each other, a firm that implements data-driven improvements will also cause a negative externality on other firms' equilibrium profits. A higher degree of substitutability between two firms' products will magnify the negative effect of one firm's competitive advantage on the other.

The competitive interaction between data buyers can be modeled with varying levels of detail and complexity. We can start by considering first-order interactions between bidder's allocations, such that each bidder's utility is linear in the allocations to all bidders. That is, firm i always suffers a constant decrease in utility whenever another firm j is allocated the good (here, data sets), no matter what the allocations to other firms besides i and j are. More realistically, the negative externality a firm suffers may depend on more than just whether or not each competitor gets data, but also on which *groups* of other bidders get data, or on some nonlinear function of the number of other bidders who get data. However, under suitable conditions, a linear approximation of additively separable externalities may suffice, and such a model already demonstrates key ways in which externalities affect efficient and optimal mechanism design.

Formal Model. Consider two firms, indexed by $i \in \{1, 2\}$, producing perfect substitute goods. The firms each decide on a production input quantity $q_i \in [0, \infty)$. Each firm i's unit production cost is $c_i \in [0, \infty)$, such that the production cost incurred by each firm is $c_i q_i$. Meanwhile, firm i's production output, or yield, is given by $\alpha_i q_i$, where $\alpha_i \in [0, \infty)$ is called firm i's production efficiency.

Let $M \in \mathbb{R}_{\geq 0}$ be the market demand parameter, such that given the production input decisions q_1, q_2 and production efficiencies α_1, α_2 of the firms, the market price of the good is

$$\rho(q_1, q_2; \alpha_1, \alpha_2) = M - (\alpha_1 \cdot q_1 + \alpha_2 \cdot q_2).$$

Each firm i realizes market profits

$$\pi_i(q_1, q_2; \alpha_1, \alpha_2, c_1, c_2) = \rho(q_1, q_2; \alpha_1, \alpha_2) \cdot \alpha_i \cdot q_i - c_i \cdot q_i \tag{1.1.1}$$

Cournot Market Subgame. Let us gather all the market relevant parameters into the variable $\xi = (\alpha_1, \alpha_2, c_1, c_2, M)$. Both firms choose their production inputs q_i in order to maximize their market profits $\pi_i(q_1, q_2; \xi)$. We find the equilibrium production decisions $q_i^*(\xi)$ and profits $\pi_i^*(\xi)$ by simultaneously solving the firms' best response functions $\partial \pi_i / \partial q_i = 0$. We make assumptions on the parameters (mostly that the market demand M is large enough) so the ensuing equilibrium is an interior solution.

Thus the equilibrium production decision of firm 1 is

$$q_1^*(\xi) = \frac{\alpha_1 c_2 - 2\alpha_2 c_1 + M\alpha_1\alpha_2}{3{\alpha_1}^2\alpha_2},$$

and the equilibrium profit is given by

$$\pi_1^*(\xi) = \frac{\left(\alpha_1 c_2 - 2\alpha_2 c_1 + M\alpha_1\alpha_2\right)^2}{9{\alpha_1}^2{\alpha_2}^2} = (\alpha_1 q_1^*)^2.$$

By symmetry, we obtain similar expressions for firm 2, with the indices 1 and 2 swapped above.

Comparative Statics and Linearized Model. We Taylor expand the equilibrium profit functions around some initial parameter values $\xi_0 = (\alpha_{1,0}, \alpha_{2,0}, c_1, c_2)$. In our model, we are only interested in how changing the production efficiency α_i affects profits, so we do not account for perturbations in the parameter c_i.

$$\pi_i^*(\xi) - \pi_i^*(\xi_0) = \sum_{j=1}^{2} \frac{\partial \pi_i^*}{\partial \alpha_j}(\xi_0) \cdot \underbrace{(\alpha_j - \alpha_{j,0})}_{=:\Delta\alpha_j} + \frac{1}{2!}\sum_{j,k=1}^{2} \frac{\partial \pi_i^*}{\partial \alpha_j \alpha_k}(\xi_0) \cdot (\alpha_j - \alpha_{j,0})(\alpha_k - \alpha_{k,0}) + ...$$

The coefficients of the first order deviations give us comparative statics that show how changes in each firm's production parameters, with all other parameters held constant, affect the firms' equilibrium profits. In particular, for firm 1 they take the forms:

$$\frac{\partial \pi_1^*}{\partial \alpha_1}(\xi) = \frac{4c_1\left(\alpha_1 c_2 - 2\alpha_2 c_1 + M\alpha_1\alpha_2\right)}{9{\alpha_1}^3\alpha_2} = \frac{4c_1}{3\alpha_1}q_1^*$$
$$\frac{\partial \pi_1^*}{\partial \alpha_2}(\xi) = -\frac{2c_2\left(\alpha_1 c_2 - 2\alpha_2 c_1 + M\alpha_1\alpha_2\right)}{9\alpha_1{\alpha_2}^3} = -\frac{2\alpha_1 c_2}{3{\alpha_2}^2}q_1^*$$

Furthermore, the coefficients of the second order terms take the form:

$$\frac{\partial^2 \pi_1^*}{\partial \alpha_1^2} = -\frac{8c_1\left(\alpha_1 c_2 - 3\alpha_2 c_1 + M\alpha_1\alpha_2\right)}{9{\alpha_1}^4\alpha_2}$$
$$\frac{\partial^2 \pi_1^*}{\partial \alpha_1 \alpha_2} = \frac{\partial^2 \pi_1^*}{\partial \alpha_2 \alpha_1} = -\frac{4c_1c_2}{9{\alpha_1}^2{\alpha_2}^2}$$
$$\frac{\partial^2 \pi_1^*}{\partial \alpha_2^2} = \frac{2c_2\left(3\alpha_1 c_2 - 4\alpha_2 c_1 + 2M\alpha_1\alpha_2\right)}{9\alpha_1{\alpha_2}^4}$$

Depending on the magnitude of these higher order terms evaluated at ξ_0, a linear approximation to the equilibrium profit function may or may not be reasonable in the regime of values considered. Again, by symmetry, we obtain similar expressions for firm 2 by swapping the indices 1 and 2 above.

Restricting our attention to the first order Taylor approximation, we obtain the linear model of equilibrium profit for firm $i \in \{1, 2\}$ and $j \neq i$:

$$\pi_i^*(\xi) - \pi_i^*(\xi_0) \approx \frac{4c_i}{3\alpha_{i,0}} q_i^*(\xi_0) \cdot \Delta\alpha_i - \frac{2\alpha_{i,0}c_j}{3{\alpha_{j,0}}^2} q_i^*(\xi_0) \cdot \Delta\alpha_j$$

Labeling the coefficients

$$v_i := \frac{4c_i}{3\alpha_{i,0}} q_i^*(\xi_0)\,, \quad \eta_{i\leftarrow j} := \frac{2\alpha_{i,0}c_j}{3{\alpha_{j,0}}^2} q_i^*(\xi_0) \tag{1.1.2}$$

and the changes in production efficiency $x_i := \Delta\alpha_i$, we can re-express firm i's change in equilibrium profit as

$$\Delta\pi_i^*(x_1, x_2) := \pi_i^*(\xi) - \pi_i^*(\xi_0) \approx v_i \cdot x_i - \eta_{i\leftarrow j} \cdot x_j. \tag{1.1.3}$$

Note that v_i and $\eta_{i\leftarrow j}$ take nonnegative values. We can interpret v_i to be the value that firm i gets from an allocation of data that leads to an increase x_i of its production efficiency, while $\eta_{i\leftarrow j}$ is the negative externality caused by an allocation of data to firm j on firm i's market profits, which arises due to the Cournot competition between the firms.

Additively Separable Negative Externalities Now suppose there are n firms ($N = \{1, ..., n\}$) engaging in Cournot competition. Again, a third party offers data for sale which the firms can use to improve their production efficiency by an amount x_i which is monotonically increasing in the quality or quantity of the data allocated. We generalize the preceding derivation by Taylor expanding each firm's equilibrium profits with respect to the changes x_i. Keeping only the first-order terms in the expansion and ignoring higher-order effects, we get that each firm's change in equilibrium profit due to a given allocation of data inducing $(x_1, ..., x_n)$ can be approximated as

$$\Delta\pi_i^*(x_1, ..., x_n) \approx v_i \cdot x_i - \sum_{j \in N \setminus i} \eta_{i \leftarrow j} x_j$$

where v_i and $\eta_{i \leftarrow j}$ are appropriately defined nonnegative quantities.

Private values. Consider once more the case of 2 firms in Cournot competition. Suppose each firm privately knows its production cost c_i, while it shares a common prior (known to all firms and the data seller) on the distribution of all other firms' production costs. Further, suppose all initial production efficiencies $\alpha_{i,0}$ are common knowledge, as well as the initial equilibrium production decisions $q_i^*(\xi_0)$, which could have been observed in a previous season. Then the parameters v_i and $(\eta_{i \leftarrow j})_{j \in N \setminus i}$, in (1.1.2) are privately known to bidder i. We then let $t_i = v_i e_i - \sum_{j \in N \setminus i} \eta_{i \leftarrow j} e_j$ be the vector in $\mathbb{R}^n$ denoting firm i's private type, where e_i denotes the ith unit vector.

Auction Framework Though $x = (x_1, ..., x_n)$ captures the effects of an allocation of data among the n firms on their production efficiencies, we will also refer to x as the allocation itself. Let $p = (p_1, ..., p_n)$ be the vector of payments from the firms to the data seller. Each firm $i \in N$ then has utility function

$$u_i(x, p_i; t_i) = t_i \cdot x - p_i = v_i x_i - \sum_{j \in N \setminus i} \eta_{i \leftarrow j} x_j - p_i$$

with private type t_i.

Given this structure of utilities, how should the data seller allocate (i.e., choose which firms get data and at what level of quality) and price a collection of data sets to these n competing firms, in order to

(1) maximize social welfare, $\sum_{i \in N} t_i \cdot x$, or

(2) maximize the data seller's expected revenue $\mathbb{E}[\sum_{i \in N} p_i]$, where the expectation is taken with respect to the common prior on bidders' private types?

In this thesis, we answer these and related problems in an auction design framework.

1.2 Related Work

1.2.1 Single-Item Auctions

Both efficient, i.e., welfare-maximizing, and optimal, i.e., revenue-maximizing, auctions of a single nondivisible good to multiple bidders have been well characterized. The Vickrey-Clark-Groves (VCG) mechanism gives a family of payment rules for the welfare-maximizing allocation that satisfy incentive compatibility and individual rationality [37, 15, 20]. Optimal single-item auctions were studied in the seminal paper of [32], where the problem of maximizing revenue was essentially reduced to one of welfare maximization after transforming bidders' valuations into virtual valuations. This solution which relies on the assumption that the auction designer knows the prior distribution of bidders' valuations. A key result of this early work is that the efficient single-item auction with n bidders can be implemented as a second-price auction, while the corresponding optimal auction is a second-price auction with reserve prices determined by the bidders' distributions to extract more revenue. [29] provides a comprehensive introduction to auction theory.

The most relevant line of work in this field studies the question of designing auctions in the presence of externalities. Optimal single-item auctions with additive allocative externalities among bidders were studied in [27, 28]. They consider the same multidimensional, interdependent valuation setting as the one presented here, and in Chapter 3, we extend their results to the setting of digital goods auctions and additionally provide an optimal mechanism without the restriction to one-dimensional bids imposed in [28]. The survey [26] provides a useful reference. Many papers consider a similar additive model, but often assume that externality parameters are public [7, 12] or do not depend on the identity of the competitor [9], essentially reducing the auction to the single dimensional setting. Closest to our work is [18] which extended the setting of [27] to the situation where K copies of the same indivisible item are being sold. However, their focus was on quantifying the effect of changing the parameter K. Finally, we mention [21, 38] which consider single-dimensional non-additive models of externalities yielding tractable auctions.

1.2.2 Multi-Item Auctions

The welfare-maximizing auction for selling multiple items to multiple bidders can be derived using VCG mechanism. However, without additional assumptions, this method requires bidders to communicate their valuations on all subsets of items and must optimize the welfare over all possible allocations, which requires a exponentially large communication and computational complexity in the number of items.

Meanwhile, finding the optimal multi-item auction is difficult due to not only possibly combinatorial valuations, but also the vastly more complex structure of optimal auctions themselves. The non-intuitive properties of multi-item auctions, such as the nonmonotonicity of optimal revenue with respect to the distribution of bidders' valuations, are illustrated in [24]. Even in the setting of selling two goods to a single bidder, there is no simple characterization of incentive compatible optimal mechanisms– depending on the bidder's distributions of valuations, the optimal mechanism requires randomization or even an infinitely large menu, i.e., partition of the bidder type space based on allocation and payment rules [23].

In general, there are two lines of work regarding multi-item auctions, given the hardness [17] of finding the optimal such auction, and the generally unrealistic assumption that bidders' valuation distributions are common knowledge. The first line uses a duality-based framework to characterize the optimal auction, or find special conditions on the bidder distributions under which simple auctions are optimal

[14, 30, 16]. The second line of approach tries to approximate the optimal auction using simpler auctions, such as selling all the items separately, with guarantees on the fraction of optimal revenue that such simpler auctions achieve [22], [8]. [8] show that for a single buyer with additive and independent valuations over a set of items, the mechanism that chooses the more profitable of selling items individually or all together in a grand bundle, will recover a constant fraction (1/6) of the revenue of the optimal auction, which itself may be very complicated.

1.2.3 Sale of Information Goods

A key aspect of the problems we consider is the fact that buyers of goods or information may interact downstream, for example, through Cournot competition, which affects their valuation of the overall allocation of the goods. [33, 39] considered the related problem of *sharing* market-relevant information among competing oligopolists, and showed that the effect of such information sharing on the overall welfare of the firms depends on the type of competition in which they are engaged (e.g. Bertrand or Cournot competition), and the type of market-relevant parameters they are sharing (e.g. firms' individual production cost estimates or a common market demand parameter). In some cases it is not optimal for any firm to share information with the others, due to the overwhelming negative effects of increased competition on their downstream profit. These findings motivate the study of how different forms of interdependent valuation functions may affect the welfare-maximizing or revenue-maximizing allocation of data.

More recently, there has been a range of works modeling the *sale* of information, usually some noisy signal of a market-relevant parameter, to competing firms [11, 4]. Here, the information seller may add noise to the signal being sold, where such versioning is a unique feature of selling an information good, as well as restrict the set of firms who are offered the information. [11] shows how the optimal selling strategy depends on the form of downstream competition between the firms, but assumes that the competition structure and firms' resulting utility functions are known to the information seller.

In reality, data buyers may have private informational priors and valuations on data set allocations, which calls for the integration of an auction framework that incentivizes participation and truthful bidding by the buyers. A line of work studies mechanism design for the sale of data, in which the value of data is derived from its informativeness in a learning task. For procurement auctions, [19] consider a setting in which the buyer wishes to estimate a population statistic while the sellers experience a cost due to privacy loss. In [35], the authors consider a similar problem but assume a known prior on the sellers' costs. A budget-feasible regression problem is considered in [25] and [1] consider an online learning setting. [6] develops a two-sided market for selling and buying data, capturing the value of data through increases in prediction accuracy for buyer-specific machine learning models. In our work, we build on this model of valuation and study auctions of data in the presence of externalities.

Other recent works look specifically at the sale of consumer data to firms. [10, 36] study settings in which firms may use consumer data to set personalized prices. [2] study a form of externalities between data *sellers* who value their privacy. In their model, correlations between consumer signals yield equilibria where consumers sell their data for very cheap prices despite having high values for privacy. [3] provide a comprehensive review on the economic implications of collecting, using, and selling consumer data.

1.3 Contributions and Outline

In this thesis, we study welfare-maximizing and revenue-maximizing mechanisms for auctions of single digital goods and auctions of multiple heterogenous digital goods in the presence of additively separable, negative allocative externalities among bidders. Two scenarios are of interest: Setting 1 of privately known incoming externalities, and Setting 2 of privately known outgoing externalities. In the former, bidders privately observe their value(s) for the item(s) and the externalities that allocations to the other bidders would exert on them, and in the latter, bidders observe their item values and the externalities that they exert on other bidders. Building on characterizations of truthfulness and participation constraints from the literature, we solve for the efficient and optimal single-item auctions in both private type settings. Under revenue-maximization, we extend the results of [27] and [28] to the digital goods setting, and in the setting of privately known incoming externalities, provide an optimal mechanism with multidimensional bids under an independence assumption.

For auctions of multiple digital goods, we assume that bidders have additive valuations over the goods and study the setting of privately known incoming externalities. We obtain the form of the welfare-maximizing auctions using the VCG mechanism. For revenue-maximization, we prove that selling items separately via optimal single-item auctions yields a guaranteed fraction of the optimal multi-item auction revenue. To do this, we nontrivially extend the approximation technique of [22] to the current setting of interdependent valuations with endogenous participation constraints.

Organization of Thesis. The remainder of this thesis is organized as follows. Chapter 2 presents the model of digital goods auctions with externalities studied in this thesis, as well as a key motivating reduction of the problem of selling an arbitrary number of data sets used for g different prediction tasks to the problem of selling g digital goods. Chapter 3 studies welfare-maximizing and revenue-maximizing mechanisms for single digital goods with externalities in both settings of bidders' private types. Chapter 4 studies welfare-maximizing and revenue-maximizing mechanisms for multiple digital goods with externalities in the Setting 1 of private types. Finally, we conclude and discuss future work in Chapter 5.

Chapter 2

Model

In this chapter, we present the central model of digital goods auctions with externalities studied in this thesis. In Section 2.1, we capture the motivating problem of valuing and selling data sets for prediction tasks within the general framework of digital goods auctions. To do so, we reduce the task of selling an arbitrary number of data sets used for g different prediction tasks to one of selling g digital goods. Section 2.2 presents the formal model, including the form of bidder utilities, bidders' private types, the auction design problem, and definitions of truthfulness and participation constraints that are central to mechanism design.

2.1 Selling Data Sets through Digital Goods Auctions

At first glance, it seems natural to model the problem of selling data sets as one of selling multiple digital goods, with each good representing one data set. However, different data sets, e.g., training features for a machine learning model, often contain correlated information, so the value of data is inherently combinatorial. Without additional structural assumptions, multi-item auctions for data sets would have prohibitively large communication and computational requirements. Buyers may need to report their valuation for each possible subset of items (data sets) that can be allocated, which requires an exponentially large, in the number of items, set of parameters. Further, optimizing for social welfare or the seller's revenue could entail an intractable combinatorial optimization problem. However, this complexity can be bypassed if one assumes that data is only useful when it is actually used. That is, we assume that a buyer's valuation for data does not come from the specific data sets on sale, but rather from an increase in prediction accuracy of a quantity of interest.

Specifically, building on the model introduced in [6], we assume that buyers of data sets are interested in using the data to train machine learning models for prediction tasks, and that they derive an increase in utility from increases in prediction accuracy from the downstream use of their models. For example, a firm may want to predict consumer demand for a given product, and a more accurate prediction may increase the firm's net profit through better production decisions. Suppose $\mathcal{S}$ is the set of all available data sets, which could comprise $|\mathcal{S}|$ training features for a given machine learning task. Let G be the function mapping subsets of $\mathcal{S}$ to some quantity measuring the increase in prediction accuracy (e.g., based on the root-mean-squared error) that training on the subset provides. Assuming that the gain in prediction accuracy is monotone in the subsets of data sets used, let $G(\mathcal{S})$ be the maximal increase in prediction accuracy that a firm i can gain from the data seller. Letting $x_i(\mathcal{U}) = G(\mathcal{U})/G(\mathcal{S}) \in [0,1]$ be the fraction of this maximal prediction accuracy increase for subsets $\mathcal{U} \subseteq \mathcal{S}$, we can then represent a given allocation of data sets to a firm i with a single scalar value, $x_i \in [0,1]$. This reduction allows us to address the problem of optimally allocating and pricing data sets to the auction of a single digital, i.e., freely replicable, good, with allocation x_i to bidder i. Fractional values of x_i could be mapped back to allocations of subsets of the collection of all available data sets or simply interpreted as a fractional probability of getting allocated the entire collection of data, or some interpolation between the two.

Note that implicit in G are the particulars of the machine learning model used and the prediction task at hand. If data buyers seek to buy data for multiple, say g, different prediction tasks, we can model data sets for different contexts as g separate digital goods. For example, firms may be active in multiple countries, and thus seek relevant data sets for each market. Using the formulation above, we can reduce the sale of a collection of an arbitrary number of data sets used for g different prediction tasks to the sale of g heterogenous digital goods.

2.2 Model

We now present a general model of a digital goods auction with negative, additively separable externalities among bidders. The digital goods may represent allocations of data sets or other freely replicable goods. For simplicity, we introduce here the model of a single digital good for sale, and present the generalization to g digital goods in Chapter 4.

Let $N = [n]$ be the set of bidders interested in buying the digital good from the seller, or auctioneer. Let $x_i \in [0, 1]$ denote the probability of allocating the good to bidder $i \in N$. Then for a given allocation vector $x = (x_1, ..., x_n)$, each bidder i has the valuation

$$\nu_i(x) = v_i x_i - \sum_{j \in N \setminus i} \eta_{i \leftarrow j} x_j \, . \tag{2.2.1}$$

Here, $v_i \in \mathbb{R}_{\geq 0}$ is the value that bidder i derives from the good, and $\eta_{i \leftarrow j} \in \mathbb{R}_{\geq 0}$ is the magnitude of the negative externality that an allocation to bidder j has on bidder i's utility.

2.2.1 Private Types of Bidders

Note from (2.2.1) that each bidder i's valuation is a function of v_i and $(\eta_{i \leftarrow j})_{j \in N \setminus i}$. However in reality, depending on the particulars of the competition structure the bidders engage in, the private information a bidder has might differ. We call this private information the bidder's "type". We consider two natural settings:

Setting 1: Knowledge of Incoming Externalities. Bidder i's private type parameters are v_i and $(\eta_{i \leftarrow j})_{j \in N \setminus i}$. In this case, bidder i has knowledge of the externalities that other bidders cause on it.

Setting 2: Knowledge of Externalities Outgoing Externalities. Bidder i's private type parameters are v_i and $(\eta_{j\leftarrow i})_{j\in N\setminus i}$. In this case, bidder i has knowledge of the externalities that it causes on other bidders.

The difference in what defines the private type of a bidder, though subtle, crucially affects the form of the optimal allocation and payment functions.

Bidder Type Spaces and Bid Spaces. Let $t_i \in \Theta_i$ denote bidder i's private type vector, where Θ_i denotes the type space of bidder i. In Setting 1, we have $t_i \coloneqq v_i e_i - \sum_{j\in N\setminus i} \eta_{i\leftarrow j} e_j = v_i e_i - \eta_{i\leftarrow}$, where e_i denotes the i^{th} unit vector and $\eta_{i\leftarrow} := -\sum_{j\in N\setminus i} \eta_{i\leftarrow j} e_j$ is the vector of externalities imposed on bidder i. Similarly, in Setting 2, bidder i's type vector is $t_i \coloneqq v_i e_i - \sum_{j\in N\setminus i} \eta_{j\leftarrow i} e_j = v_i e_i - \eta_{\leftarrow i}$, where $\eta_{\leftarrow i} := -\sum_{j\in N\setminus i} \eta_{j\leftarrow i} e_j$ is the vector of externalities exerted by bidder i. With abuse of notation, we let t_i refer to both kinds of private types as its relevant definition will be clear from context. We further assume the type parameters lie in bounded ranges: $v_i \in [\underline{v}_i, \bar{v}_i]$ and $\eta_{ij} \in [\underline{\eta}_{ij}, \bar{\eta}_{ij}]$ for $i \in N, j \in N\setminus i$. The overall type space is $\Theta \coloneqq \prod_{i\in N} \Theta_i$. The collection of all the bidders' type vectors is denoted by $t = (t_1, ..., t_n) \in \Theta$. t_{-i} denotes the collection of type vectors of all bidders except bidder i.

We assume bidders are rational, selfish agents who act to maximize their utilities in a given auction setting. It is possible that participating in the auction, i.e., submitting a valid bid, receiving an allocation, and making a payment, may leave bidders worse off than simply not participating. To give bidders the option of non-participation, we define the bid spaces $B_i \coloneqq \Theta_i \cup \{\emptyset\}$ and $B \coloneqq \Pi_{i\in N} B_i$. Then a bidder can report any type in Θ_i, but can also choose to not participate in the auction by reporting $\emptyset$.

Throughout, we use the convention that a "hat" letter denotes a quantity reported by the bidders, as opposed to the "true" realization of the same quantity. For example, t_i denotes the (true) type of bidder i while $\hat{t}_i$ denotes its bid (i.e. reported type). Similarly, t_{-i} and $\hat{t}_{-i}$ denote respectively the true types and reported types of all bidders but bidder i.

Prior Distribution of Bidder Types. In certain settings we consider, making a distributional assumption on the private types of bidders will be necessary. For those settings, we let the bidders' private types t_i be drawn independently from commonly known distributions F_i on Θ_i. Let f_i be the corresponding density functions for F_i, $f = \prod_{i \in N} f_i$, and $F = \prod_{i \in N} F_i$ be the joint distribution function of t on Θ, likewise for the individual parameters v_i and $\eta_{i \leftarrow j}$, we denote the corresponding marginal density and distribution functions by f_{v_i}, $f_{\eta_{i \leftarrow j}}$, and $F_{v_i}, F_{\eta_{i \leftarrow j}}$, respectively.

2.2.2 Auction Design Setup

The auction design problem consists of designing the following two functions to maximize social welfare or the seller's revenue:

- an *allocation* function $x : B \to [0, 1]^n$;
- a *payment* function $p : B \to (\mathbb{R}_{\geq 0})^n$.

In short, given a vector of bids $\hat{t} \in B$ from the bidders, $x(\hat{t})$ is the resulting vector of allocations and $p(\hat{t})$ is the vector of payments required of the bidders. We abuse notation and let x denote both the vector of allocations and the function, which maps bids to this allocation vector. We similarly abuse notation for p.

We assume bidders have *quasilinear* net utility from participating in the auction. That is, given allocation and payment vectors x and p, respectively, and true types $t \in \Theta$, bidder i's utility is

$$u_i(x, p; t) \coloneqq \nu_i(x) - p_i = v_i x_i - \sum_{j \in N \setminus i} \eta_{i \leftarrow j} x_j - p_i \, .$$

Remark 2.2.1 (Key Difference From Standard Auction Set-Ups). The key difference from standard single-item auction setups is that for digital goods, such as data, there is no feasibility constraint on the allocation function $x(\cdot)$. In particular, we do not require that the sum of the allocations $(\sum_{i=1}^{n} x_i)$, is less than or equal to one. The absence of this feasibility constraint is key in obtaining a simple structure for the optimal auctions despite it being a multi-dimensional mechanism design problem (i.e., each bidder is parameterized by a n-dimensional vector).

Outside Option. When a bidder chooses not to participate in the auction, the auctioneer cannot charge the bidder any payment nor "dump" any goods on the bidder. That is, we have the restriction that $x_i(\hat{t}) = 0$ and $p_i(\hat{t}) = 0$ whenever $\hat{t}_i = \emptyset$. Note that even if a given bidder chooses not to participate in the auction, allocations to the other, participating bidders can still affect its utility through negative externalities.

Bidder i's utility when it does not participate and all remaining bidders $N\backslash i$ do participate depends only on others' bids and the true underlying types, and is called bidder i's "outside option". In standard auctions without externalities, the utility of the outside option is a constant usually set to 0, but in the present setting, it is endogenously determined by the mechanism's allocation rule and bidders' types. Explicitly, given a type vector $t \in \Theta$ and a vector of bids $\hat{t}_{-i}$ from other bidders, the utility of bidder i in its outside option is given by

$$u_i\big(x(\hat{t}_i = \emptyset, \hat{t}_{-i}), p(\hat{t}_i = \emptyset, \hat{t}_{-i}); t\big) = -\sum_{j \in N\backslash i} \eta_{i \leftarrow j} x_j(\hat{t}_i = \emptyset, \hat{t}_{-i})\,. \tag{2.2.2}$$

In general, fully specifying a mechanism involves defining $x(\hat{t})$ and $p(\hat{t})$ when any subset of bidders bids $\emptyset$ to not participate. However, since we are interested in designing Nash equilibria where all bidders participate (and bid truthfully), it suffices for us to only explicitly define the mechanism under single-bidder deviations from equilibrium and the equilibrium itself. That is, we seek allocation and payment rules $x(\hat{t})$ and $p(\hat{t})$ defined over all $\hat{t} \in B$ with at most one bid of $\emptyset$.

2.2.3 Truthfulness and Participation Constraints

A fundamental result in mechanism design known as the Revelation Principle [32] states that any dominant strategy or Bayes-Nash equilibrium outcome of an arbitrary mechanism can be implemented by a dominant strategy or Bayes-Nash, respectively, incentive compatible direct mechanism. In the present context, this implies that we can without loss of generality restrict our search for efficient and optimal auction mechanisms to be over incentive compatible (IC) and individually rational (IR) direct mechanisms where all bidders are incentivized to truthfully bid their true types. Suitable characterizations of IC conditions, which express "truthfulness constraints", and IR conditions, which express "participation constraints" allow one to express the efficient and optimal auction design problems as constrained optimization problems with objectives linear in the decision functions.

We now define ex-post and interim incentive compatibility and individual rationality conditions.

Ex-Post Constraints. We first consider ex-post truthfulness and participation constraints.

Definition 2.2.2 (Dominant Strategy Incentive Compatibility)**.** A mechanism (x, p) is *Dominant Strategy Incentive Compatible* (DSIC) if for all type vectors $t, \hat{t} \in \Theta$ and bidders $i \in N$

$$u_i\big(x(t_i, \hat{t}_{-i}), p_i(t_i, \hat{t}_{-i}); t\big) \geq u_i\big(x(\hat{t}), p_i(\hat{t}); t\big)\ .$$

Definition 2.2.3 (Ex-Post Individual Rationality)**.** A mechanism (x, p) is *ex-post Individually Rational* (ex-post IR) if for every type vector $t \in \Theta$ and bidders $i \in N$

$$u_i\big(x(t), p_i(t); t\big) \geq u_i\big(x(\emptyset, t_{-i}), p_i(\emptyset, t_{-i}); t\big)\ .$$

Dominant strategy incentive compatibility expresses that no matter what the true types are and what other players bid, a bidder cannot strictly increase its net utility by bidding untruthfully. Ex-post individual rationality expresses that no matter what the true types are, in a situation where all other bidders participate and bid truthfully, it is better for each bidder to report truthfully than to not participate. These two properties combined imply that participating and reporting truthfully is a dominant strategy equilibrium of the game induced by the mechanism.

Interim Constraints. In situations where types are drawn from a known prior distribution and bidders reason in expectation over other bidders' private types, conditioned on their own observed types, we consider *interim* relaxations of the IC and IR definitions.

To this end, define $V_i(\hat{t}_i; t_i) \coloneqq \mathbb{E}\big[u_i\big(x(\hat{t}_i, t_{-i}), p_i(\hat{t}_i, t_{-i}\big); t) \,\big|\, t_i\big]$ to be the interim expected utility of bidder $i \in N$ if it bids $\hat{t}_i \in B_i$ while having a true type $t_i \in \Theta_i$, and all other bidders bid their types truthfully. Note that the expectation is taken over a random realization $t \sim F$ conditioned on the event that bidder's i type is t_i.

Definition 2.2.4 (Bayes–Nash Incentive Compatibility)**.** A mechanism (x, p) is *Bayes–Nash Incentive Compatible* (BNIC) if for all types $t_i, \hat{t}_i \in \Theta_i$ and bidders $i \in N$, $V_i(t_i; t_i) \geq V_i(\hat{t}_i; t_i)$.

Definition 2.2.5 (Interim Individual Rationality)**.** A mechanism (x, p) satisfies *interim Individual Rationality* (interim IR) if for every type $t_i \in \Theta_i$ and bidders $i \in N$, $V_i(t_i; t_i) \geq V_i(\emptyset; t_i)$.

Given the setup here introduced, we derive and study welfare-maximizing and revenue-maximizing auction mechanisms in the subsequent chapters.

Chapter 3

Auctions of a Single Digital Good with Externalities

In this chapter, we study welfare-maximizing and revenue-maximizing mechanisms for single digital goods with externalities. We consider two scenarios of interest, which will be formalized below: Setting 1 of privately known incoming externalities, and Setting 2 of privately known outgoing externalities.

We build on characterizations of incentive compatible (IC) and individually rational (IR) mechanisms presented in [27] and [28], which in turn specialize more general characterization results [34] for mechanisms where bidders have utilities linear in their types. These results allow us to express the efficient and optimal auction design problems as constrained optimization problems with objectives linear in the decision functions. The appropriate characterizations and constraints depend on the form of bidders' private types, and we study Setting 1 and Setting 2 separately.

Section 3.1 presents relevant characterizations of truthfulness and participation constraints. Section 3.2 studies welfare-maximizing mechanisms, Section 3.3 studies revenue-maximizing mechanisms, and Section 3.4 provides a comparative discussion of the results.

3.1 Characterizations of IC and IR Mechanisms

An important step towards elucidating the solution structure of the welfare-maximizing and revenue-maximizing mechanisms is to obtain a characterization of the IC and IR constraints defined in 2.2.3. Since the present model of utilities has the same form as the one in [27, 28], we rely on the characterizations found in these papers, and state them below for completeness. These characterizations depend on the form of bidders' private types and thus are organized by private type setting.

3.1.1 Characterizations in Setting 1

We first consider the setting where private types are of the form $t_i = v_i e_i - \eta_{i\leftarrow}$. For ease of notation, we define the overall interim expected allocation function $y^{(i)}(\hat{t}_i) := \mathbb{E}[x(\hat{t}_i, t_{-i}) \,|\, t_i] = \mathbb{E}_{t_{-i}}[x(\hat{t}_i, t_{-i})]$ and the interim expected payment $q_i(\hat{t}_i) := \mathbb{E}[p_i(\hat{t}_i, t_{-i}) \,|\, t_i] = \mathbb{E}_{t_{-i}}[p_i(\hat{t}_i, t_{-i})]$ for each bidder $i \in N$, when i bids $\hat{t}_i \in B_i$ with true type $t_i \in \Theta_i$. Note that, under the given assumption of independent bidder types, the interim expected allocation and payment functions do not depend on bidder i's true type t_i. Also note that $y^{(i)}$ is a vector field mapping B_i to $[0,1]^n$. Under these definitions, we have that

$$V_i(\hat{t}_i; t_i) = t_i \cdot y^{(i)}(\hat{t}_i) - q_i(\hat{t}_i).$$

Finally, for each $i \in N$, we define the critical type $\mathring{t}_i = \underline{v}_i e_i - \underline{\eta}_{i\leftarrow}$, where $\underline{\eta}_{i\leftarrow} := \sum_{j \in N \setminus i} \underline{\eta}_{i\leftarrow j}$. $\mathring{t}_i$ is the vector in Θ_i closest to the origin and will feature in the following IC and IR characterizations.

Proposition 3.1.1 ([28, Proposition 1]). *Suppose bidders' private types are of the form $t_i = v_i e_i - \eta_{i\leftarrow}$ for each bidder $i \in N$. Then the mechanism (x, p) is BNIC if and only if for each bidder $i \in N$:*

(i) $y^{(i)}$ is conservative.

(ii) $y^{(i)}$ is monotone, that is $\langle s_i - t_i, y^{(i)}(s_i) - y^{(i)}(t_i) \rangle \geq 0$ for all $s_i, t_i \in \Theta_i$.

(iii) for each type $t_i \in \Theta_i$, the interim payment is given by

$$q_i(t_i) = \langle y^{(i)}(t_i), t_i \rangle - \int_{\mathring{t}_i}^{t_i} y^{(i)}(s_i) \cdot ds_i - C_i \,, \tag{3.1.1}$$

where C_i is an arbitrary integration constant whose value sets $V_i(\mathring{t}_i; \mathring{t}_i)$, the interim utility of bidder i when its type is $\mathring{t}_i = \underline{v}_i e_i - \underline{\eta}_{i\leftarrow}$.

We also provide the following characterization of interim IR for BNIC mechanisms that maximize revenue.

Proposition 3.1.2 (Adapted from [28, Proposition 3])**.** *Suppose private types are of the form $t_i = v_i e_i - \eta_{i\leftarrow}$ for each bidder $i \in N$. Then a revenue-maximizing BNIC mechanism satisfies the interim IR constraint $V_i(t_i; t_i) \geq V_i(\emptyset; t_i)$ if and only if this condition is satisfied for the critical type $\mathring{t}_i = \underline{v}_i e_i - \underline{\eta}_{i\leftarrow}$.*

Proof. We first show that the optimal outside option when bidder i does not participate allocates the digital good to all remaining participants $N \backslash i$. We then show that it suffices to check that interim IR is satisfied for the type $\mathring{t}_i$, and finally find the optimal value of the integration constant $V_i(\mathring{t}_i; \mathring{t}_i)$.

Optimal Outside Option. The interim IR constraint is essentially a constraint on the values that the constant $C_i = V_i(\mathring{t}_i; \mathring{t}_i)$ can take. That is, after plugging in the form of the payment rule (3.1.1), interim IR can be expressed as: for all $i \in N$ and $t_i \in \Theta_i$,

$$V_i(\mathring{t}_i; \mathring{t}_i) + \int_{\mathring{t}_i}^{t_i} y^{(i)}(s_i) \cdot ds_i \geq V_i(\emptyset; t_i).$$

Maximizing revenue corresponds to maximizing the expected sum of the interim payments $q_i(t_i)$ and thus of minimizing $V_i(\mathring{t}_i; \mathring{t}_i)$. Since for all $t_i \in \Theta_i, V_i(\emptyset; t_i) \geq -\sum_{j \in N \backslash i} \eta_{i \leftarrow j}$, we can maximize the feasible region for IR payments by setting $V_i(\emptyset; t_i) = -\sum_{j \in N \backslash i} \eta_{i \leftarrow j}$ with an outside option that allocates to all $j \in N \backslash i$ when i does not participate. That is, we set $x_j(\hat{t}_i = \emptyset, \hat{t}_{-i}) = \mathbb{1}\{i \neq j\}$ for all $i, j \in N$ and $t_{-i} \in \Theta_{-i}$.

Sufficiency of Checking Interim IR for Type $\mathring{t}_i$ If the interim IR constraint holds for all types t_i, then it clearly holds for the critical type $\mathring{t}_i$. Now suppose that $V_i(\mathring{t}_i; \mathring{t}_i) \geq V_i(\emptyset; \mathring{t}_i)$. Note that given the optimal outside option of allocating to all remaining bidders, we have that for every $t_i \in \Theta_i$,

$$V_i(\emptyset; t_i) = -\sum_{j \in N \setminus i} \eta_{i \leftarrow j}. \tag{3.1.2}$$

Then for every $t_i \in \Theta_i$

$$\begin{aligned}
V_i(t_i; t_i) &= V_i(\mathring{t}_i; \mathring{t}_i) + \int_{\mathring{t}_i}^{t_i} y^{(i)}(s_i) \cdot ds_i \\
&\geq V_i(\mathring{t}_i; \mathring{t}_i) + \sum_{j \in N \setminus i} (-\eta_{i \leftarrow j} - (-\underline{\eta}_{i \leftarrow j})) \\
&= V_i(\mathring{t}_i; \mathring{t}_i) + V_i(\emptyset; t_i) - V_i(\emptyset; \mathring{t}_i) \\
&\geq V_i(\emptyset; t_i)
\end{aligned}$$

where for the first inequality we used that $t_{i,i} = v_i \geq \underline{v}_i$, $t_{i,j} = -\eta_{i \leftarrow j} \leq -\underline{\eta}_{i \leftarrow j}$ and $y^i \geq 0$ as an allocation vector, the second equality follows from (3.1.2), and the last inequality follows from our assumption that $V_i(\mathring{t}_i; \mathring{t}_i) - V_i(\emptyset; \mathring{t}_i) \geq 0$. □

3.1.2 Characterizations in Setting 2

We now consider the case where the private types are of the form $t_i = v_i e_i - \eta_{\leftarrow i}$ for each bidder $i \in N$. Note that in this setting, bidder i's expected outside option utility $V_i(\emptyset; t_i)$ does not depend on t_i. For ease of notation, we define the interim expected allocation of each bidder i bidding $\hat{t}_i \in B_i$ with true type $t_i \in \Theta_i$ to be $y_i(\hat{t}_i) \coloneqq \mathbb{E}[x_i(\hat{t}_i, t_{-i}) \,|\, t_i] = \mathbb{E}_{t_{-i}}[x_i(\hat{t}_i, t_{-i})]$ and recall the definition of the interim expected payment $q_i(\hat{t}_i) = \mathbb{E}_{t_{-i}}[p_i(\hat{t}_i, t_{-i})]$. Again, note that under the given assumption of independent bidder types, the interim functions do not depend on bidder i's true type t_i. Under these definitions,

$$V_i(\hat{t}_i, t_i) = v_i y_i(\hat{t}_i) - \sum_{j \in N \setminus i} \mathbb{E}_{t_{-i}}[\eta_{i \leftarrow j} x_j(\hat{t}_i, t_{-i})] - q_i(\hat{t}_i).$$

Proposition 3.1.3 ([27, Proposition 2]). *Assume that private types are of the form $t_i = v_i e_i - \eta_{\leftarrow i}$ for each bidder $i \in N$. The mechanism (x, p) is BNIC if and only if for each bidder $i \in N$:*

(i) there exists a non-decreasing function $\widetilde{y}_i : [\underline{v}_i, \bar{v}_i] \to [0, 1]$ such that the interim allocation satisfies $y_i(v_i e_i - \eta_{\leftarrow i}) = \widetilde{y}_i(v_i)$ for almost all v_i and for all $\eta_{\leftarrow i} \in \prod_{j \in N \setminus i} [\underline{\eta}_{j \leftarrow i}, \bar{\eta}_{j \leftarrow i}]$.

(ii) the interim payment $q_i(t_i)$ for each type $t_i = v_i e_i - \eta_{\leftarrow i} \in \Theta_i$ is given by

$$q_i(t_i) = v_i \widetilde{y}_i(v_i) - \int_{\underline{v}_i}^{v_i} \widetilde{y}_i(v) dv - \sum_{j \in N \setminus i} \mathbb{E}[\eta_{i \leftarrow j} x_j(t_i, t_{-i}) \,|\, t_i] - C_i \,, \tag{3.1.3}$$

where C_i is an arbitrary integration constant.

Furthermore, if these conditions are satisfied, then $V_i(t_i; t_i)$ is constant with respect to $\eta_{\leftarrow i}$ for almost every $v_i \in [\underline{v}_i, \bar{v}_i]$, and $C_i = V_i(\underline{v}_i e_i - \eta_{\leftarrow i}; \underline{v}_i e_i - \eta_{\leftarrow i})$ for all $\eta_{\leftarrow i} \in \prod_{j \in N \setminus i} [\underline{\eta}_{j \leftarrow i}, \bar{\eta}_{j \leftarrow i}]$.

We provide the following alternative (and arguably simpler) proof of this proposition.

Proof. We first show the necessary implications of IC. Writing Definition 2.2.4 for $t_i = v_i e_i - \eta_{\leftarrow i}$ as the true type and $\hat{t}_i = \hat{v}_i e_i - \hat{\eta}_{\leftarrow i}$ as the reported type, and then vise versa, i.e.

$$\begin{aligned} V_i(t_i; t_i) &\geq V_i(\hat{t}_i; t_i) \\ V_i(\hat{t}_i; \hat{t}_i) &\geq V_i(t_i; \hat{t}_i). \end{aligned} \tag{3.1.4}$$

Combining the two inequalities yields

$$y_i(t_i)(v_i - \hat{v}_i) \geq y_i(\hat{t}_i)(v_i - \hat{v}_i).$$

By Lemma 3.1.4 below, this implies condition (1).

Note also that the inequality (3.1.4), by adding and subtracting the term $\hat{v}_i y_i(\hat{t}_i)$ on the right hand side and regrouping terms, can be written equivalently as

$$V_i(t_i; t_i) \geq V_i(\hat{t}_i; \hat{t}_i) + (v_i - \hat{v}_i)x_i(\hat{v}_i). \tag{3.1.5}$$

for all $t_i, \hat{t}_i \in \Theta_i$. Then plugging in $t_i = v_i e_i - \eta_{\leftarrow i}$ and $\hat{t}_i = \hat{v}_i e_i - \hat{\eta}_{\leftarrow i}$ into the preceding inequality yields $V_i(t_i; t_i) \geq V_i(\hat{t}_i; \hat{t}_i)$. Swapping the roles of t_i and $\hat{t}_i$ yields the inequality in the opposite direction, and we have that $V_i(t_i; t_i)$ is independent of $\eta_{\leftarrow i}$:

$$\forall v_i, \forall \eta_{\leftarrow i}, \forall \hat{\eta}_{\leftarrow i},\ V_i(v_i e_i - \eta_{\leftarrow i}) = V_i(v_i e_i - \hat{\eta}_{\leftarrow i}).$$

We henceforth write $\tilde{V}_i(v_i)$ to denote $V_i(v_i e_i - \eta_{\leftarrow i}; v_i e_i - \eta_{\leftarrow i})$ for any $\eta_{\leftarrow i}$, and likewise let $\tilde{y}_i(v_i) := y_i(v_i e_i - \eta_{\leftarrow i})$.

To prove (3), we first note that $V_i(t_i; t_i)$ is convex in v_i. (3.1.4) implies that

$$V_i(t_i; t_i) = \max_{\hat{t}_i \in \Theta_i} y_i(\hat{t}_i) v_i - \sum_{j \in N \setminus i} \mathbb{E}\big[\eta_{i \leftarrow j} x_j(\hat{t}_i, t_{-i}) \,\big|\, t_i\big] - q_i(\hat{t}_i).$$

Thus, $V_i(t_i; t_i)$ is the maximum of a family of linear functions of v_i and is thus convex in v_i. (3.1.5) implies that $y_i(t_i)$ is a subderivative of $\tilde{V}_i(v_i)$. In fact, since V_i is convex in v_i, it is differentiable almost everywhere and

$$\tilde{y}_i(v_i) = \frac{\partial V_i(t_i; t_i)}{\partial v_i} \text{ a.e.}$$

Further, this implies that

$$V_i(v_i e_i - \eta_{\leftarrow i}; v_i e_i - \eta_{\leftarrow i}) = \int_{\underline{v}_i}^{v_i} \tilde{y}_i(v) dv + \tilde{V}_i(\underline{v}_i) \tag{3.1.6}$$

where for the last term we used the fact that $V_i(\underline{v}_i e_i - \eta_{\leftarrow i}; \underline{v}_i e_i - \eta_{\leftarrow i}) = \tilde{V}_i(\underline{v}_i)$ and is independent of $\eta_{\leftarrow i}$. Now plugging in the following expression for V_i,

$$V_i(\hat{t}_i; t_i) = v_i y_i(\hat{t}_i) - \sum_{j \in N \setminus i} \mathbb{E}\big[\eta_{i \leftarrow j} x_j(\hat{t}_i, t_{-i}) \,\big|\, t_i\big] - q_i(\hat{t}_i)\,.$$

and solving for $q_i(t_i)$, we get

$$q_i(t_i) = v_i \tilde{y}_i(v_i) - \int_{\underline{v}_i}^{v_i} \tilde{y}_i(v) dv - \sum_{j \in N \setminus i} \mathbb{E}[\eta_{i \leftarrow j} x_j(t_i, t_{-i}) \,|\, t_i] - \tilde{V}_i(\underline{v}_i). \tag{3.1.7}$$

We next show the sufficiency of conditions (i) and (ii) for BNIC, by proving the equivalent condition for BNIC, (3.1.5). We have that for all $i \in N, t_i = v_i e_i - \eta_{\leftarrow i}$ and $\hat{t}_i = \hat{v}_i e_i - \hat{\eta}_{\leftarrow i}$,

$$\begin{aligned} V_i(t_i; t_i) - V_i(\hat{t}_i; \hat{t}_i) &= \int_{\hat{v}_i}^{v_i} \tilde{y}_i(v) dv \\ &\geq \tilde{y}_i(\hat{v}_i)(v_i - \hat{v}_i) \end{aligned}$$

where the first equality follows from (3.1.6) and the inequality follows from condition (i) that $\tilde{y}_i(v_i)$ is increasing in v_i. □

Lemma 3.1.4. *For $d \geq 1$, let $f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ be a bounded function such that*

$$f(x_2, y_2)(x_2 - x_1) \geq f(x_1, y_1)(x_2 - x_1), \quad (x_1, y_1), (x_2, y_2) \in \mathbb{R} \times \mathbb{R}^d\,.$$

Then, there exists a non-decreasing function $\tilde{f} : \mathbb{R} \to \mathbb{R}$ such that $f(x, y) = \tilde{f}(x)$ for all $y \in \mathbb{R}^d$ and for all but at most countably many values of $x \in \mathbb{R}$.

Proof. By a rescaling and shifting of f we assume without loss of generality that the range of f is contained in $[0, 1]$. Let us now define $S := \{x \in \mathbb{R} : \exists y_1, y_2 \in$

$\mathbb{R}^d$ s.t. $f(x, y_2) \neq f(x, y_1)\}$ and for $n \geq 1$, $S_n := \{x \in \mathbb{R} : \exists y_1, y_2 \in \mathbb{R}^d$ s.t. $f(x, y_2) - f(x, y_1) \geq 1/n\}$ and observe that $S = \cup_{n \geq 1} S_n$.

We now prove that $|S_n| \leq n$. Indeed, consider $x_1 < \cdots < x_m$, m distinct points in S_n, and for each $k \in [m]$, y_k^1, y_k^2 such that $f(x_k, y_k^2) - f(x_k, y_k^1) \geq 1/n$. Then

$$
\begin{aligned}
\frac{m}{n} &\leq \sum_{k=1}^{m} \left[f(x_k, y_k^2) - f(x_k, y_k^1) \right] \\
&= f(x_m, y_m^2) - \sum_{k=2}^{m} [f(x_k, y_k^1) - f(x_{k-1}, y_{k-1}^2)] - f(x_1, y_1^1) \\
&\leq f(x_m, y_m^2) - f(x_1, y_1^1) \leq 1 \,,
\end{aligned}
$$

where the first inequality uses the definition of S_n, the equality is summation by parts, the second inequality uses our assumption on f and the last inequality uses that the range of f is contained in $[0, 1]$. It then follows that $m \leq n$, i.e. that $|S_n| \leq n$, which in turn implies that S is countable.

Define $\tilde{f}$ by $\tilde{f}(x) = f(x, y)$ for $x \notin S$ (this definition does not depend on the choice of y by definition of S). Then our assumption on f immediately implies that $\tilde{f}$ is non-decreasing on $\mathbb{R} \backslash S$. We can thus extend $\tilde{f}$ to a non-decreasing function defined over all of $\mathbb{R}$ (for example by right continuity). The resulting $\tilde{f}$ satisfies the stated requirements. □

Finally, we have the following characterization of interim IR for BNIC mechanisms.

Proposition 3.1.5. *Suppose private types are of the form $t_i = v_i e_i - \eta_{\leftarrow i} \in \Theta_i$ for each bidder $i \in N$. Then a BNIC mechanism satisfies the interim IR constraint $V_i(t_i; t_i) \geq V_i(\emptyset; t_i)$ for all $t_i \in \Theta_i$, if and only if this condition is satisfied for some type of the form $\underline{v}_i e_i - \eta_{\leftarrow i}$, where $\eta_{\leftarrow i} \in \prod_{j \in N \backslash i} [\underline{\eta}_{j \leftarrow i}, \bar{\eta}_{j \leftarrow i}]$.*

Proof. Note that BNIC implies (3.1.6), and since the integrand $\tilde{y}_i \geq 0$, we have that $V_i(t_i; t_i) \geq \tilde{V}_i(\underline{v}_i)$ for all $t_i \in \Theta_i$. Since $V_i(\emptyset; t_i)$ is independent with respect to t_i, it is both necessary and sufficient for IR to hold that the IR condition holds for some type of the form $\underline{v}_i e_i - \eta_{\leftarrow i}$, for each bidder $i \in N$. □

3.2 Welfare Maximization

In this section, the seller's problem is to design allocation and payment functions, $x(\cdot)$ and $p(\cdot)$ that maximize the total social welfare, i.e. the sum of bidder valuations:

$$\mathrm{SW}(x;t) = \sum_{i\in N} \nu_i(x) = \sum_{i\in N} \Big(v_i x_i - \sum_{j\in N\setminus i} \eta_{i\leftarrow j} x_j \Big) \tag{3.2.1}$$

such that the auction: (i) is incentive compatible; (ii) satisfies individual rationality; (iii) has no positive transfers, i.e., the seller never pays a bidder to participate in the auction. We organize this section by the private types of the bidders according to the two settings described in Section 2.2.1.

3.2.1 Welfare Maximization in Setting 1

We first consider the case where the private type of bidder $i \in N$ takes the form $t_i = v_i e_i - \eta_{i\leftarrow}$, so each bidder observes the incoming allocative externalities that it suffers due to other bidders. Note that in this setting, a bidder i's valuation of a vector of allocations can be expressed as $\nu_i(x) = t_i \cdot x$. We instantiate the Vickrey–Clarke–Groves (VCG) mechanism for this setting and comment on the resulting allocation and payment functions.

We wish to maximize (3.2.1) subject to DSIC (Definition 2.2.2), ex-post IR (Definition 2.2.3), and the feasibility constraint that for all $i \in N, x_i \in [0,1]$ (Section 2.2.3). To define ex-post IR, recall that we need to instantiate the outside option, i.e. what occurs if bidder i chooses not to participate in the auction. Here, we choose the natural outside option, that is to run the welfare-maximizing auction with the remaining set $N\setminus i$ of bidders.

Efficient Allocation. Note that by rearranging terms, we can express the social welfare objective (3.2.1) as

$$\mathrm{SW}(x;t) = \sum_{i\in N}\left(v_i - \sum_{j\in N\setminus i}\eta_{j\leftarrow i}\right)x_i(t) = \sum_{i\in N} W_i(t)x_i(t) \tag{3.2.2}$$

where we let $W_i(t) \coloneqq v_i - \sum_{j\in N\setminus i}\eta_{j\leftarrow i}$ represent the "welfare contribution" of bidder i, that is, the net contribution to the social welfare, SW, if bidder i were allocated the good. As we shall see, a constant theme for the efficient and optimal mechanisms studied in this chapter is that W_i, or variants thereof, is the key quantity determining the allocation of bidder i. Since (3.2.2) is linear in the allocations x_i, it easily follows that the welfare-maximizing, or efficient, allocation under the above constraints is simply to allocate whenever $W_i(t)$ is nonnegative, i.e.

$$x_i(t) = \mathbb{1}\{W_i(t) \geq 0\} = \mathbb{1}\left\{v_i - \sum_{j\in N\setminus i}\eta_{j\leftarrow i} \geq 0\right\}. \tag{3.2.3}$$

As was alluded to in Section 2.2.2, although x_i is only constrained to be in $[0,1]$, the optimal allocation turns out to be one of two extremes: either allocate all data or none of it to a bidder.

IR and the Outside Option. To streamline presentation, let us define the welfare contribution of bidder j when (only) bidder i chooses to not participate in the auction to be, for $j \in N\setminus i$,

$$W_j^i(t_{-i}) \coloneqq v_j - \sum_{k\in N\setminus\{i,j\}}\eta_{k\leftarrow j}$$

Then following the same reasoning above, the welfare maximizing allocation of bidder j in the absence of bidder i is given by

$$x_j(t_i = \emptyset, t_{-i}) = \mathbb{1}\left\{W_j^i(t_{-i}) \geq 0\right\}. \tag{3.2.4}$$

and the value of bidder i's outside option utility is thus

$$u_i\big(x(\emptyset, t_{-i}), p_i(\emptyset, t_{-i}); t\big) = -\sum_{j \in N \setminus i} \eta_{i \leftarrow j} x_j(t_i = \emptyset, t_{-i}) = -\sum_{j \in N \setminus i} \eta_{i \leftarrow j} \mathbb{1}\big\{W_j^i(t_{-i}) \geq 0\big\}.$$

Note that the welfare-maximizing VCG mechanism does not guarantee that each bidder's net utility will be nonnegative, but rather no less than the utility if the bidder were to not participate in the auction, which could be negative due to externalities.

Remark 3.2.1. While we choose the outside option to be the welfare-maximizing auction with the remaining bidders, as is natural, we could instead have declared the ensuing auction to have any feasible allocation rule for the bidders $N \setminus i$ that does not depend on bidder i's bid. For instance, a feasible outside option is to allocate all data to every $j \in N \setminus i$ if bidder i does not participate, resulting in utility $u_i(\emptyset, t_{-i}; t_i, t_{-i}) = -\sum_{j \in N \setminus i} \eta_{i \leftarrow j}$. This is in fact the worst possible outside option for bidder i, which thereby increases the set of IR-satisfying mechanisms. Indeed, as discussed in Section 3.3, this worst-case outside option is the revenue-optimal one.

VCG Payment Rule. The payments associated with this allocation rule are for each bidder $i \in N$, and for all $t \in \Theta$,

$$\begin{aligned} p_i(t) &= \underbrace{\sum_{j \in N \setminus i} W_j^i(t_{-i}) x_j(t_i = \emptyset, t_{-i})}_{SW \text{ when } i \text{ is absent}} - \underbrace{\sum_{j \in N \setminus i} \left(v_j x_j - \sum_{k \in N \setminus j} \eta_{j \leftarrow k} x_k \right)}_{SW \text{ with } N \setminus i} \\ &= \sum_{j \in N \setminus i} \big(W_j^i(t_{-i})(x_j(t_i = \emptyset, t_{-i}) - x_j(t)) + \eta_{j \leftarrow i} x_i(t) \big) \\ &= \sum_{j \in N \setminus i} \big(W_j^i(t_{-i}) \big[\mathbb{1}\{W_j^i(t_{-i}) \geq 0\} - \mathbb{1}\{W_j(t) \geq 0\} \big] + \eta_{j \leftarrow i} \mathbb{1}\{W_i(t) \geq 0\} \big) \end{aligned} \tag{3.2.5}$$

Note that bidder i's payment is the sum of the change in welfare if it leaves the auction and the sum of externalities it induces in the current allocation.

Proposition 3.2.2 (Efficient Mechanism, Setting 1)**.** *The mechanism specified by* *allocation function* (3.2.3) *with outside option* (3.2.4) *and payment function* (3.2.5)*, maximizes social welfare among all DSIC and ex-post IR auctions, and has no positive transfer.*

A proof of these properties is given in Appendix A.1.1.

3.2.2 Welfare Maximization in Setting 2

We now consider the case where bidders know the externality that they would exert on other bidders if allocated the good, i.e., when the private type of each bidder $i \in N$, is $t_i = v_i e_i - \eta_{\leftarrow i}$.

Motivating Interim Constraints. Note that in this setting, bidder i cannot fully evaluate its valuation of a given allocation x, since it depends on the parameters $(\eta_{i \leftarrow j})_{j \in N \setminus i}$, which are part of the private types of the other bidders $N \setminus i$. Therefore, each bidder can only reason with its own realized type t_i and the commonly known priors on other bidders' types. It is more sensible, therefore, to impose interim versions of truthfulness (BNIC) and participation (interim IR) conditions (see Definitions 2.2.4 and 2.2.5 respectively).

Ex-Ante Welfare Optimality. As a first attempt toward a welfare-maximizing mechanism in this setting, one might try to use the previous welfare-maximizing allocation rule (3.2.3). Due to Proposition 3.1.3, however, this allocation violates BNIC when the private types are of the form $t_i = v_i e_i - \eta_{\leftarrow i}$, since the corresponding interim allocation $y_i(t_i) = \mathbb{1}\{W_i(t) \geq 0\} = \mathbb{1}\{v_i \geq \sum_{j \in N \setminus i} \eta_{j \leftarrow i}\}$ is not in general constant with respect to $\eta_{j \leftarrow i}$. Indeed, note that in this setting the welfare contribution of bidder i depends solely on t_i and can be expressed as $W_i(t) = t_i \cdot 1$.

In fact, any attempt to find such welfare-maximizing BNIC mechanisms will fail. It turns out that in general, no mechanism satisfying BNIC can be ex-post (pointwise) welfare-maximal over all types t, as stated next.

Proposition 3.2.3 (Impossibility of Ex-Post Optimality)**.** *Suppose bidders' private types are of the form $t_i = v_i e_i - \eta_{\leftarrow i}$ for each bidder $i \in N$. For any joint distribution*

F of types $t = (t_1, \ldots, t_n)$, let

$$\mathcal{X}_{BNIC}(F) \coloneqq \{x : \Theta \to [0,1]^n | \; \forall i \in N, y_i(t_i) = \tilde{y}_i(v_i) \text{ for some non-decreasing function } \tilde{y}_i\}$$

be the set of allocation functions that satisfy condition (i) in the BNIC characterization. Then there exists a distribution F of types on Θ s.t. for all $x \in \mathcal{X}_{BNIC}(F)$, there exists a $t^0 \in \Theta$ and $x' \in \mathcal{X}_{BNIC}(F)$ such that

$$\mathrm{SW}(x; t^0) < \mathrm{SW}(x'; t^0) \tag{3.2.6}$$

Proof. Consider the distribution of types F with probability mass 1/2 on each of two points: $t^a = (\hat{v}_1 e_1 - \eta^a_{\leftarrow 1}, \hat{t}_{-1})$ and $t^b = (\hat{v}_1 e_1 - \eta^b_{\leftarrow 1}, \hat{t}_{-1})$, where $\hat{v}_1 \in \mathbb{R} \geq 0$ and $\hat{t}_{-1} = (t_j : j \neq 1)$ take arbitrary, fixed values. Let $\eta^a_{j\leftarrow 1}$ and $\eta^b_{j\leftarrow 1}$, for $j \in N \backslash i$ be such that

$$\hat{v}_1 - \sum_{j \in N \backslash 1} \eta^a_{j \leftarrow 1} > 0 \tag{3.2.7}$$

$$\text{and } \hat{v}_1 - \sum_{j \in N \backslash 1} \eta^b_{j \leftarrow 1} < 0 \tag{3.2.8}$$

For instance, we can take each $\eta^a_{j\leftarrow 1} = 0$ and $\eta^b_{j\leftarrow 1} = 2\hat{v}_1$.

Note that for all $x(\cdot) \in \mathcal{X}_{BNIC}(F)$ and $t \in \Theta$, $x_1(t) = \mathbb{E}[x_1(t)|t_1] = \tilde{y}_i(v_i)$ for some increasing function $\tilde{y}_i(v_i)$. However, under distribution F, v_i only takes the single value $\hat{v}_i$, so the function $x_1(t)$ must be constant-valued.

Then for all $x(\cdot) \in \mathcal{X}_{BNIC}(F)$, and $t \in \Theta$, if $x_1(t) > 0$, let $y(\cdot)$ be such that $y_1(t) = 0$ and $y_j(t) = x_j(t)$ for all $j \neq 1$. We have that

$$\begin{aligned} &SW(x; t^b) - SW(y; t^b) \\ =&(\hat{v}_1 - \sum_{j \in N \backslash i} \eta^b_{j \leftarrow 1})(x_1(t^b) - y_1(t^b)) < 0 \end{aligned}$$

where the strict inequality follows from (3.2.8) and that for all $t, x_1(t) > 0 = y_1(t)$. Likewise, if $x_1(t) \leq 0$, let $y(\cdot)$ be such that $y_1(t) = 1$ and $y_j(t) = x_j(t)$ for all $j \neq 1$.

Then

$$\begin{aligned} & SW(x;t^a) - SW(y;t^a) \\ =& (\hat{v}_1 - \sum_{j \in N \setminus 1} \eta^a_{j \leftarrow 1})(x_1(t^a) - y_1(t^a)) < 0 \end{aligned}$$

where the strict inequality follows from (3.2.7) and that for all t, $x_1(t) \leq 0 < 1 = y_1(t)$.

Thus, we have shown that for any allocation rule in a BNIC mechanism, there is some type realization such that a different BNIC allocation rule yields a strictly greater social welfare, which is the statement of (3.2.6).

□

Since Proposition 3.2.3 implies that there are distributions in which no mechanism satisfying BNIC can also be welfare-maximizing over all type realizations, we relax the objective of finding a pointwise optimum to one of maximizing the *expected* social welfare, that is,

$$\mathbb{E}[\mathrm{SW}(x;t)] = \sum_{i \in N} \mathbb{E}\left[\left(v_i - \sum_{j \in N \setminus i} \eta_{j \leftarrow i}\right) x_i(t)\right]. \tag{3.2.9}$$

Proposition 3.2.4 (Welfare-Maximizing Allocation, Setting 2)**.** *Suppose that the map $v_i \mapsto v_i - \sum_{j \in N \setminus i} \mathbb{E}[\eta_{j \leftarrow i} | v_i]$ is non-decreasing for every bidder $i \in N$. Then the allocation rule maximizing the expected social welfare (3.2.9) under BNIC is*

$$x_i(t) = \mathbb{1}\left\{v_i \geq \sum_{j \in N \setminus i} \mathbb{E}[\eta_{j \leftarrow i} \,|\, v_i]\right\},\ i \in N\,. \tag{3.2.10}$$

Proof. To solve for the form of the expected welfare maximizing allocation function satisfying the IC constraints, we first express the objective in terms of the interim allocations $y_i(t_i)$. In terms of the welfare contribution $W_i(t) = v_i - \sum_{j \in N \setminus i} \eta_{j \leftarrow i}$, we have

$$\mathbb{E}\big[\mathrm{SW}(x;t)\big] = \sum_{i \in N} \mathbb{E}[W_i(t) x_i(t)] = \sum_{i \in N} \mathbb{E}\big[\mathbb{E}[W_i(t) x_i(t_i, t_{-i}) \,|\, t_i]\big] = \sum_{i \in N} \mathbb{E}[W_i(t) y_i(t_i)]\,.$$

As noted, in the present setting of private types, $W_i(t)$ depends only on t_i. Now, the BNIC characterization from Proposition 3.1.3 implies that there must exist functions $\widetilde{y}_i : [\underline{v}_i, \bar{v}_i] \to [0,1]$ such that $y_i(t_i) = \widetilde{y}_i(v_i)$ for almost all $t_i \in \Theta_i$. Plugging in this representation above, we get

$$\mathbb{E}[\mathrm{SW}(x;t)] = \sum_{i \in N} \mathbb{E}[W_i(t)\widetilde{y}_i(v_i)] = \sum_{i \in N} \mathbb{E}\big[\widetilde{y}_i(v_i)\, \mathbb{E}[W_i(t) \,|\, v_i]\big] .$$

Noting the linearity of the objective in $\widetilde{y}_i$, we find that the optimal allocation rule is

$$\widetilde{y}_i(v_i) = \mathbb{1}\big\{\mathbb{E}[W_i(t) \,|\, v_i] \geq 0\big\} = \mathbb{1}\Bigg\{v_i - \sum_{j \in N \setminus i} \mathbb{E}[\eta_{j \leftarrow i} \,|\, v_i] \geq 0\Bigg\} .$$

Under the given assumptions, $\widetilde{y}_i(v_i)$ is non-decreasing in v_i, so BNIC is satisfied. Finally, note that since we can express the objective function and constraints only in terms of the interim allocations y_i for $i \in N$, we can without loss of generality for all $i \in N$, set the allocation rule $x_i(t) = y_i(t_i) = \widetilde{y}_i(v_i)$. □

Remark 3.2.5. Note that if we were selling a non-replicable good rather than the digital good of our setting, the feasibility constraint $\sum_{i \in N} x_i \leq 1$ would couple the allocations and x_i would be a function of other bids v_j for $j \neq i$.

Proposition 3.2.6 (Payment Rule Associated with Welfare-Maximizing Allocation, Setting 2)**.** *Suppose that the map $v_i \mapsto v_i - \sum_{j \in N \setminus i} \mathbb{E}[\eta_{j \leftarrow i} | v_i]$ is non-decreasing for every bidder $i \in N$, and let $\tau_i := \inf\{v \in [\underline{v}_i, \bar{v}_i] \mid v \geq \sum_{j \in N \setminus i} \mathbb{E}[\eta_{j \leftarrow i} | v]\}$[1]. Consider the auction with the welfare-maximizing allocation rule described in Proposition 3.2.4, that also runs the welfare-maximizing allocation on the remaining set of bidders whenever some subset of bidders chooses not to participate in the auction. Then the BNIC payment rule with this allocation is given by*

$$p_i(t_i) = \sum_{j \in N \setminus i} \tau_i \cdot \mathbb{1}\{v_i \geq \tau_i\} - \sum_{j \in N \setminus i} \mathbb{E}\big[\eta_{i \leftarrow j} \cdot \mathbb{1}\{v_j \geq \tau_j\}\big] - C_i. \tag{3.2.11}$$

Furthermore, IR is satisfied whenever $C_i = V_i(t_i; t_i) \geq V_i(\emptyset; t_i)$, for some t_i of the

[1] Here and throughout this chapter, we use the convention that $0 \cdot \infty = 0$

form $\underline{v}_i e_i - \eta_{\leftarrow i} \in \Theta_i$. *In particular, if the maps* $v_j \mapsto v_j - \sum_{k\in N\setminus\{j,i\}} \mathbb{E}[\eta_{k\leftarrow j}|v_j]$ *are non-decreasing, for* $j \in N\setminus i$, *then* $V_i(\emptyset; t_i)$ *is given by*

$$\sum_{j\in N\setminus i} \mathbb{E}\left[\eta_{i\leftarrow j} \cdot \mathbb{1}\left\{v_j \geq \sum_{k\in N\setminus\{j,i\}} \mathbb{E}[\eta_{k\leftarrow j}\,|\,v_j]\right\}\right] \tag{3.2.12}$$

Proof. The induced interim payment rule $q_i(t_i) = \mathbb{E}[p_i(t)\,|\,t_i]$ associated with y_i as derived in Proposition 3.1.3, condition (ii) is

$$q_i(t_i) = v_i \cdot y_i(t_i) - \int_{\underline{v}_i}^{v_i} \widetilde{y}_i(v)dv - \sum_{j\in N\setminus i} \mathbb{E}\big[\eta_{i\leftarrow j} x_j(t_i, t_{-i})\,\big|\,t_i\big] - C_i\,. \tag{3.2.13}$$

Recall that the constant term C_i is set such that the payment function satisfies IR. By Proposition 3.1.5, it suffices to check IR for any type of the form $t_i = \underline{v}_i e_i - \eta_{\leftarrow i}$, for each $i \in N$. Here, bidder i's expected utility $V_i(\emptyset; t_i)$ if it doesn't participate is the sum of the externalities effects from the allocations $x_j(t_i = \emptyset, t_{-i})$ in the welfare-maximizing auction run with the remaining set $N\setminus i$ of bidders, and given the assumption of $v_j \mapsto v_j - \sum_{k\in N\setminus\{j,i\}} \mathbb{E}[\eta_{k\leftarrow j}|v_j]$ non-decreasing, we have

$$V_i(\emptyset; t_i) = \sum_{j\in N\setminus i} \mathbb{E}[\eta_{i\leftarrow j} x_j(t_i = \emptyset, t_{-i})(t_{-i})] = \sum_{j\in N\setminus i} \mathbb{E}\left[\eta_{i\leftarrow j}\, \mathbb{1}\left\{v_j \geq \sum_{k\in N\setminus\{j,i\}} \mathbb{E}[\eta_{k\leftarrow j}|v_j]\right\}\right] \tag{3.2.14}$$

Then any payment rule of the form (3.2.13) with the constant C_i set greater than or equal to $V_i(\emptyset; t_i)$ in (3.2.14) will give us an IR mechanism.

Finally, since the objective function and constraints can be expressed solely in terms of the interim payments q_i, we can set $p_i(t) \coloneqq q_i(t_i)$. Under the given assumption that $v_i \mapsto v_i - \sum_{j\in N\setminus i} \mathbb{E}[\eta_{j\leftarrow i}|v_i]$ is non-decreasing, we can re-express the allocation rule as

$$x_i(t) = y_i(t_i) = \mathbb{1}\left\{v_i \geq \sum_{j\in N\setminus i} \mathbb{E}[\eta_{j\leftarrow i}\,|\,v_i]\right\} \tag{3.2.15}$$

$$= \mathbb{1}\{v_i \geq \tau_i\} \tag{3.2.16}$$

The integral term in (3.2.13) then becomes

$$\int_{\underline{v}_i}^{v_i} \mathbb{1}\{v \geq \tau_i\} dv = (v_i - \tau_i) \cdot \mathbb{1}\{v_i \geq \tau_i\} = (v_i - \tau_i) \cdot y_i(t_i)$$

Plugging in the above expression, along with the form of the allocation (3.2.15), into (3.2.13) completes the proof. □

Proposition 3.2.7 (Efficient Mechanism, Setting 2). *The mechanism specified by allocation function* (3.2.10)*, and that runs the welfare-maximizing auction with the remaining bidders whenever a subset of bidders chooses not to participate, and uses payment function* (3.2.11) *maximizes expected social welfare among all BNIC, interim IR auctions.*

Proof. Immediate from Proposition 3.2.4 and 3.2.6. □

3.3 Revenue Maximization

In this section, we focus on the problem of designing auctions that achieve optimal revenue. Specifically, the goal is to design allocation and payment functions $x(\cdot)$ and $p(\cdot)$ to maximize the seller's expected revenue

$$\sum_{i \in N} \mathbb{E}\big[p_i(t)\big] \tag{3.3.1}$$

subject to BNIC and interim IR constraints. Note that we can also express the expected revenue as $\sum_{i \in N} \mathbb{E}\big[\mathbb{E}[p_i(t) \,|\, t_i]\big] = \sum_{i \in N} \mathbb{E}\big[q_i(t_i)\big]$ by the law of total expectation and the definition of the interim payments.

3.3.1 Revenue Maximization in Setting 1

We aim to maximize the seller's expected revenue subject to BNIC and IR constraints, where private types take the form $t_i = v_i e_i - \eta_{i\leftarrow}$ for each $i \in N$. Recall the notation for the interim expected allocation functions $y^{(i)}(\hat{t}_i) = \mathbb{E}_{t_{-i}}[x(\hat{t}_i, t_{-i})]$ and the interim expected payment functions $q_i(\hat{t}_i) := \mathbb{E}_{t_{-i}}[p_i(\hat{t}_i, t_{-i})]$, for each bidder $i \in N$. (See

Section 3.1.1). As is natural, $t_{i,k} = t_i \cdot e_k$ denotes the k^{th} component of the vector t_i, for $i, k \in N$.

Independence assumption. For this section, we make the simplifying assumption that the components of each bidder i's type are independent, so the probability distribution function of t_i factors as $f_i(t_i) = \Pi_{k \in N} f_{i,k}(t_{i,k})$, where we define the density functions $f_{i,k}(t_{i,k}) = f_{i,k}(-\eta_{i \leftarrow k}) := f_{\eta_{i \leftarrow k}}(\eta_{i \leftarrow k})$ for $i \neq k \in N$ and $f_{i,i}(t_{i,i}) = f_{v_i}(v_i)$ for $i \in N$. We further define the virtual value functions $\Phi_{i,k}(t_{i,k}) \coloneqq t_{i,k} - (1 - F_{i,k}(t_{i,k}))/f_{i,k}(t_{i,k})$.

Proposition 3.3.1 (Optimal Mechanism, Setting 1)**.** *Suppose bidders have private types of the form $t_i = v_i e_i - \eta_{i\leftarrow}$ for each bidder $i \in N$, and that $f_i(t_i)$ factorizes as $f_i(t_i) = \Pi_{k \in N} f_{i,k}(t_{i,k})$. Suppose also that the distribution F of bidder types is such that the virtual valuation functions $\Phi_{i,k}(t_{i,k})$ are nondecreasing. Then the mechanism with allocation rule*

$$x_i(t) = \mathbf{1}\Big\{ \sum_{k \in N} \Phi_{k,i}(t_{k,i}) \geq 0 \Big\}, \text{ for } i \in N,\ t \in \Theta \tag{3.3.2}$$

$$x_j(t_i = \emptyset, t_{-i}) = \mathbb{1}\{i \neq j\}, \text{ for } i, j \in N,\ t_{-i} \in \Theta_{-i} \tag{3.3.3}$$

and payment functions given in Proposition 3.1.1 condition (iii) with $C_i = -\sum_{j \in N \setminus i} \eta_{j \leftarrow i}$ is revenue-optimal among BNIC and interim IR auctions.

Proof. We present the proof in three parts: (1) we derive the optimal allocation rule, (2) we verify that the allocation and associated payment rules satisfy BNIC, and (3) we set the optimal constant term of the payment function subject to interim IR constraints.

Part 1: Deriving the Optimal Allocation. We first use the form of the interim payment functions from the BNIC characterization in Proposition 3.1.1 to express our objective solely in terms of interim allocation functions.

$$\begin{aligned}\mathbb{E}\left[\sum_{i\in N} p_i(t)\right] &= \sum_{i\in N}\mathbb{E}\left[\mathbb{E}\left[p_i(t)|t_i\right]\right] = \sum_{i\in N}\mathbb{E}\left[q_i(t_i)\right] \\ &= \sum_{i\in N}\mathbb{E}\left[y^{(i)}(t_i)\cdot t_i - \int_{\mathring{t}_i}^{t_i} y^{(i)}(s_i)\cdot ds_i - C_i\right] \\ &= \sum_{i\in N}\mathbb{E}\left[y^{(i)}(t_i)\cdot t_i - \int_{\underline{t}_i}^{t_i} y^{(i)}(s_i)\cdot ds_i - C_i'\right]\end{aligned}$$

In the last equality, we shifted the lower bound of integration from $\mathring{t}_i$ to $\underline{t}_i := \underline{v}_i e_i - \sum_{j\in N\setminus i}\bar{\eta}_{i\leftarrow j}e_j$ along with the corresponding constant of integration C_i to C_i'. The type $\underline{t}_i$ can be considered the "lowest" type of bidder i, as it yields the lowest valuation on any given allocation over all feasible types. Originally $C_i = V(\mathring{t}_i;\mathring{t}_i)$, and now the new constant of integration C_i' sets the value of $V(\underline{t}_i;\underline{t}_i)$. The constant term C_i' can be set independently of the allocation functions, and we defer finding the optimal such C_i' (and thus C_i) satisfying IR to the last part of this proof, after we have solved for the optimal allocation rules.

Expanding the inner product in the first term above and ignoring the constant C_i', we temporarily take our objective to be

$$\mathbb{E}\left[\sum_{i\in N}\left(\sum_{j\in N} y_j^{(i)}(t_i)t_{i,j}\right) - \int_{\underline{t}_i}^{t_i} y^{(i)}(s_i)\cdot ds_i\right] \tag{3.3.4}$$

Fix any $k\in N$. We now re-express the above integral term to be linear in $y_k^{(i)}(t_i)$. By Proposition 3.1.1, $y^{(i)}$ is a conservative vector field, so we can evaluate the line integral by taking any path from $\underline{t}_i = (\underline{t}_{i,1},...,\underline{t}_{i,n})$ to $t_i = (t_{i,1},...,t_{i,n})$. Let us take any path that first fixes the kth coordinate while moving all other coordinates to their final value at the point $(t_{i,1},...,t_{i,k-1},\underline{t}_{i,k},t_{i,k+1},...,t_{i,n})$ and then from there moves parallel to the kth coordinate axis to the endpoint t_i. That is, we evaluate the line integral as

$$\int_{\underline{t}_i}^{t_i} y^{(i)}(s_i)\cdot ds_i = \int_{(\underline{t}_{i,1},\ldots,\underline{t}_{i,n})}^{(t_{i,1},\ldots,t_{i,k-1},\underline{t}_{i,k},t_{i,k+1},\ldots,t_{i,n})} y^{(i)}(s_{i,1},\ldots,s_{i,k-1},\underline{t}_{i,k},s_{i,k+1},\ldots,s_{i,n})\cdot ds_i + \int_{\underline{t}_{i,k}}^{t_{i,k}} y_k^{(i)}(t_{i,1},\ldots,t_{i,k-1},s_{i,k},t_{i,k+1},\ldots,t_{i,n})ds_{i,k} \tag{3.3.5}$$

Note that the first integral term on the right hand side of (3.3.5) does not depend on the value of $t_{i,k}$. To emphasize this fact, we temporarily denote this quantity by

$$\xi_k(t_{i,1},\ldots,t_{i,k-1},t_{i,k+1},\ldots,t_{i,n}) := \int_{(\underline{t}_{i,1},\ldots,\underline{t}_{i,n})}^{(t_{i,1},\ldots,t_{i,k-1},\underline{t}_{i,k},t_{i,k+1},\ldots,t_{i,n})} y^{(i)}(s_{i,1},\ldots,s_{i,k-1},\underline{t}_{i,k},s_{i,k+1},\ldots,s_{i,n})\cdot ds_i. \tag{3.3.6}$$

Next, we use the assumption of independence of the components of t_i to evaluate the expectation of the second term on the right hand side of (3.3.5):

$$\begin{aligned}
&\mathbb{E}\left[\int_{\underline{t}_{i,k}}^{t_{i,k}} y_k^{(i)}(\underbrace{t_{i,1},\ldots,t_{i,k-1}}_{t_{i,k-}},s_{i,k},\underbrace{t_{i,k+1},\ldots,t_{i,n}}_{t_{i,k+}})\,ds_{i,k}\right]\\
&=\left(\prod_{j\in N\setminus k}\int_{\underline{t}_{i,j}}^{\bar{t}_{i,j}} dt_{i,j}f_{i,j}(t_{i,j})\right)\int_{\underline{t}_{i,k}}^{\bar{t}_{i,k}} dt_{i,k}f_{i,k}(t_{i,k})\int_{\underline{t}_{i,k}}^{t_{i,k}} ds_{i,k}y_k^{(i)}(t_{i,k-},s_{i,k},t_{i,k+})\\
&=\left(\prod_{j\in N\setminus k}\int_{\underline{t}_{i,j}}^{\bar{t}_{i,j}} dt_{i,j}f_{i,j}(t_{i,j})\right)\int_{\underline{t}_{i,k}}^{\bar{t}_{i,k}} ds_{i,k}y_k^{(i)}(t_{i,k-},s_{i,k},t_{i,k+})\int_{s_{i,k}}^{\bar{t}_k^i} dt_{i,k}f_{i,k}(t_{i,k})\\
&=\left(\prod_{j\in N\setminus k}\int_{\underline{t}_{i,j}}^{\bar{t}_{i,j}} dt_{i,j}f_{i,j}(t_{i,j})\right)\int_{\underline{t}_{i,k}}^{\bar{t}_{i,k}} ds_{i,k}f_{i,k}(s_{i,k})y_k^{(i)}(t_{i,k-},s_{i,k},t_{i,k+})\frac{1-F_{i,k}(s_{i,k})}{f_{i,k}(s_{i,k})}\\
&=\mathbb{E}\left[y_k^{(i)}(t_{i,1},\ldots,t_{i,k-1},t_{i,k},t_{i,k+1},\ldots,t_{i,n})\frac{1-F_{i,k}(t_{i,k})}{f_{i,k}(t_{i,k})}\right]\\
&=\mathbb{E}\left[y_k^{(i)}(t_i)\frac{1-F_{i,k}(t_{i,k})}{f_{i,k}(t_{i,k})}\right]
\end{aligned}$$

The first equality follows by definition of expectation and by the independence of the coordinates of t_i, the second by changing the order of integration, the third by definition of the CDF, and the fourth by the definition of expectation once again.

Plugging the last expression and (3.3.6) back into the objective (3.3.4), we get

$$\begin{aligned}
&\mathbb{E}\left[\sum_{i\in N}\left(\sum_{j\in N} y_j^{(i)}(t_i)t_{i,j}\right)\right.\\
&\qquad\qquad \left.-\left(\xi_k(t_{i,1},...,t_{i,k-1},t_{i,k+1},...,t_{i,n}) + y_k^{(i)}(t_i)\frac{1-F_{i,k}(t_{i,k})}{f_{i,k}(t_{i,k})}\right)\right]\\
=&\mathbb{E}\left[\sum_{i\in N} y_k^{(i)}(t_i)(t_{i,k} - \frac{1-F_{i,k}(t_{i,k})}{f_{i,k}(t_{i,k})})\right.\\
&\qquad\qquad \left.+\sum_{i\in N}\left(\sum_{j\in N\setminus k} y_j^{(i)}(t_i)t_{i,j} - \xi_k(t_{i,1},...,t_{i,k-1},t_{i,k+1},...,t_{i,n})\right)\right]\\
=&\mathbb{E}\left[\sum_{i\in N}\mathbb{E}\left[x_k(t)(t_{i,k} - \frac{1-F_{i,k}(t_{i,k})}{f_{i,k}(t_{i,k})})\middle| t_i\right]\right.\\
&\qquad\qquad \left.+\sum_{i\in N}\left(\sum_{j\in N\setminus k} y_j^{(i)}(t_i)t_{i,j} - \xi_k(t_{i,1},...,t_{i,k-1},t_{i,k+1},...,t_{i,n})\right)\right]\\
=&\mathbb{E}\left[x_k(t)\sum_{i\in N}\left(t_{i,k} - \frac{1-F_{i,k}(t_{i,k})}{f_{i,k}(t_{i,k})}\right)\right]\\
&\qquad\qquad +\mathbb{E}\left[\sum_{i\in N}\left(\sum_{j\in N\setminus k} y_j^{(i)}(t_i)t_{i,j} - \xi_k(t_{i,1},...,t_{i,k-1},t_{i,k+1},...,t_{i,n})\right)\right]
\end{aligned}$$

where we rearrange terms to arrive at the first equality, use the definition of interim allocation for the second, and the law of total expectation and linearity of expectation for the last equality. Note that neither the allocation function x_k nor any of the interim allocations $y_k^{(i)}$ to bidder k, for $i \in N$, feature in the second expectation term of the last expression. There are also no coupling constraints between the allocations x_k and x_j for $j \neq k$. Since we have expressed the objective as linear in $x_k(t)$, the optimal allocation rule $x_k(t)$ can thus be read off as

$$x_k(t) = \mathbb{1}\left\{\sum_{i\in N}\left(t_{i,k} - \frac{1-F_{i,k}(t_{i,k})}{f_{i,k}(t_{i,k})}\right) \geq 0\right\} \tag{3.3.7}$$
$$= \mathbb{1}\left\{\Phi_{k,k}(v_k) + \sum_{i\in N\setminus k}\Phi_{i,k}(-\eta_{i\leftarrow k}) \geq 0\right\}$$
$$= \mathbb{1}\left\{v_k - \frac{1-F_{k,k}(v_k)}{f_{k,k}(v_k)} + \sum_{i\in N\setminus k}\left(-\eta_{i\leftarrow k} - \frac{F_{\eta_{i\leftarrow k}}(\eta_{i\leftarrow k})}{f_{\eta_{i\leftarrow k}}(\eta_{i\leftarrow k})}\right) \geq 0\right\}$$

where $\Phi_{i,k}(t_{i,k}) = t_{i,k} - \frac{1-F_{i,k}(t_{i,k})}{f_{i,k}(t_{i,k})}$ is the virtual valuation of the parameter $t_{i,k}$ for $i,k \in N$, and we used the fact that $f_{i,k}(t_{i,k}) := f_{\eta_{i\leftarrow k}}(\eta_{i\leftarrow k})$ and correspondingly, $F_{i,k}(t_{i,k}) = 1 - F_{\eta_{i\leftarrow k}}(\eta_{i\leftarrow k})$. Since k was fixed arbitrarily, the form of the optimal allocation rule (3.3.7) holds for all $k \in N$.

Part 2: Verifying BNIC. We first show that the interim allocation vector fields $y^{(i)}$ are monotone. Note that for $k \in N$, the allocation functions $x_k(t)$ are only dependent on $t_{i,k}$, for all $i \in N$. Further, $y_k^{(i)}(t_i) = \mathbb{P}[\sum_{j\in N}(t_{j,k} - \frac{1-F_{j,k}(t_{j,k})}{f_{j,k}(t_k^j)}) \geq 0 | t_{i,k}]$ is increasing in $t_{i,k}$ since $\Phi_{i,k}(t_{i,k})$ is increasing in $t_{i,k}$, and $y_k^{(i)}(t_i)$ only depends on the parameter $t_{i,k}$. Given this, let $\tilde{y}_k^{(i)}$ be a single-parameter, increasing function such that $y_k^{(i)}(t_i) = \tilde{y}_k^{(i)}(t_{i,k})$. Then for all $s_i, t_i \in \Theta_i$,

$$(s_i - t_i)\cdot(y^{(i)}(s_i) - y^{(i)}(t_i))$$
$$= \sum_{k\in N}(s_{i,k} - t_{i,k})(\tilde{y}_k^{(i)}(s_{i,k}) - \tilde{y}_k^{(i)}(t_{i,k})) \geq 0.$$

The inequality holds because each term in the sum is nonnegative, since $\tilde{y}_k^{(i)}$ are increasing functions.

Next, note that the functions $\tilde{y}_k^{(i)}$ are integrable, and let $Y^{(i)}(t_i) = \sum_{k=1}^n \int_{\underline{t}_{i,k}}^{t_{i,k}} \tilde{y}_k^{(i)}(s_{i,k})ds_{i,k}$. It can be checked that $y^{(i)}$ is the gradient of potential function $Y^{(i)}$. Thus, the vector fields $y^{(i)}$ are conservative.

Part 3: IR and the Optimal Payment. Finally, we consider the interim IR constraint and the optimal constant term of the payment function. By Proposition

3.1.2, the optimal outside option is to allocate $x_j = 1$ to all bidders $j \in N\backslash i$ when bidder i does not participate and it suffices to check interim IR for the type $\mathring{t}_i = \underline{v}_i e_i - \underline{\eta}_{i\leftarrow}$. That is, interim IR given the optimal outside option is equivalent to having

$$C_i = V_i(\mathring{t}_i; \mathring{t}_i) \geq V_i(\emptyset; \mathring{t}_i) = -\sum_{j \in N\backslash i} \underline{\eta}_{i\leftarrow j}$$

Maximizing revenue corresponds to maximizing the expected sum of the interim payments $q_i(t_i)$ and thus of minimizing $V_i(\mathring{t}_i; \mathring{t}_i)$. Hence, in the revenue-maximizing auction, we set the constant C_i in the payment function to be $-\sum_{j\in N\backslash i} \underline{\eta}_{i\leftarrow j}$. Recall that we re-expressed our objective function in terms of the constant $C_i' = V_i(\underline{t}_i; \underline{t}_i)$, which is fully determined by the interim allocation rule $y^{(i)}$ given above and C_i. Thus the corresponding optimal constant C_i' that yields an IR mechanism is

$$C_i' = \int_{\underline{t}_i}^{\mathring{t}_i} y^{(i)}(s_i) \cdot ds_i + C_i = \int_{\underline{t}_i}^{\mathring{t}_i} y^{(i)}(s_i) \cdot ds_i - \sum_{j \in N\backslash i} \underline{\eta}_{i\leftarrow j}.$$

By construction, our payment rule satisfies the BNIC characterization of Proposition 3.1.1, so our overall mechanism is BNIC and interim IR.

□

Remark 3.3.2. We are able to prove this result despite the multidimensional nature of this auction due to two assumptions. The first one exploits the fact data is inherently a digital, freely replicable good and imposes no feasibility constraint on the allocation function besides $x_i \in [0, 1]$, allowing us to effectively decouple the allocations. The second, more restrictive, assumption is that the coordinates of t_i are independent. It is unclear whether it is necessary or simply an artefact of our proof technique.

Remark 3.3.3. Observe that the allocation rule given in Proposition 3.3.1 is similar in form to the threshold functions derived for the two social-welfare maximization cases (3.2.2) and (3.2.10) but where the virtual value functions (as introduced in [32]) now play the role of the relevant coordinates of the bidders' private types. As with standard revenue maximization settings, the optimal allocation is in general not efficient, i.e. welfare-maximizing, and allocates the digital good less often to bidders than the efficient allocation. An illustrative example is presented in Section 3.4.

3.3.2 Revenue Maximization in Setting 2

Recall in this case the private type of each bidder $i \in N$ is $t_i = v_i e_i - \eta_{\leftarrow i}$. Using the BNIC characterization of Proposition 3.1.3, Proposition 3.3.4 below shows that the problem of finding the revenue-optimal mechanism can be reduced to solving n distinct optimizations over single-variable functions. Throughout this section, we denote by F_{v_i} (resp. f_{v_i}) the cumulative (resp. probability) distribution function of the marginal distribution of v_i, for $i \in N$.

Proposition 3.3.4. *For each $i \in N$, let $y_i^\star$ be a solution to the maximization problem*

$$\sup_{y} \mathbb{E}\left[y(v_i)\left(v_i - \frac{1 - F_{v_i}(v_i)}{f_{v_i}(v_i)} - \sum_{j \in N \setminus i} \mathbb{E}[\eta_{j \leftarrow i} \mid v_i]\right)\right],$$

where the maximization is over the set of non-decreasing functions $y : [\underline{v}_i, \bar{v}_i] \to [0, 1]$. Then the mechanism with allocation functions $x_i(t) := y_i^\star(v_i)$ and $x_j(t_i = \emptyset, t_{-i}) = \mathbb{1}\{i \neq j\}$ for $i, j \in N, t \in T$, and payment function given by (3.1.3) with C_i set to $-\sum_{j \in N \setminus i} \mathbb{E}[\eta_{i \leftarrow j}]$ is revenue optimal among all BNIC and interim IR auctions.

Proof. We consider a mechanism (x, p) and use the BNIC characterization of Proposition 3.1.3. In particular, recall that there exists a non-decreasing function $\widetilde{y}_i$ such that $y_i(t_i) = \widetilde{y}_i(v_i)$, where y_i is the interim allocation. Plugging in the form of interim payments q_i given by (3.1.3) we get the expected revenue

$$\sum_{i \in N} \mathbb{E}[p_i(t)] = \sum_{i \in N} \mathbb{E}\left[v_i \widetilde{y}_i(v_i) - \int_{\underline{v}_i}^{v_i} \widetilde{y}_i(v) dv - \sum_{j \in N \setminus i} \mathbb{E}\left[\eta_{i \leftarrow j} \cdot x_j(t) \mid t_i\right] - C_i\right]. \tag{3.3.8}$$

Observe that the last term on the right-hand side is independent of the choice of (x, p) and can thus be ignored when searching for the revenue optimal auction.

For the second term, swapping the order of integration gives

$$\mathbb{E}\left[\int_{\underline{v}_i}^{v_i} \widetilde{y}_i(v)dv\right] = \int_{\underline{v}_i}^{\bar{v}_i} f_{v_i}(v_i)\left(\int_{\underline{v}_i}^{v_i} \widetilde{y}_i(v)dv\right)dv_i$$
$$= \int_{\underline{v}_i}^{\bar{v}_i} \big(1 - F_{v_i}(v_i)\big)\widetilde{y}_i(v_i)dv_i = \mathbb{E}\left[\frac{1 - F_{v_i}(v_i)}{f_{v_i}(v_i)} \cdot \widetilde{y}_i(v_i)\right].$$

For the third term, we write

$$\mathbb{E}\left[\sum_{i\in N}\sum_{j\in N\setminus i} \mathbb{E}\left[\eta_{i\leftarrow j} x_j(t) \,|\, t_i\right]\right] = \sum_{i\in N}\sum_{j\in N\setminus i} \mathbb{E}\left[\eta_{i\leftarrow j} \cdot x_j(t)\right]$$
$$= \sum_{i\in N}\sum_{j\in N\setminus i} \mathbb{E}\left[\eta_{j\leftarrow i} \cdot x_i(t)\right] = \sum_{i\in N}\sum_{j\in N\setminus i} \mathbb{E}\left[\eta_{j\leftarrow i} \cdot \mathbb{E}[x_i(t) \,|\, t_i]\right]$$
$$= \sum_{i\in N}\sum_{j\in N\setminus i} \mathbb{E}\left[\eta_{j\leftarrow i} \cdot \widetilde{y}_i(v_i)\right] = \sum_{i\in N}\sum_{j\in N\setminus i} \mathbb{E}\left[\widetilde{y}_i(v_i) \cdot \mathbb{E}[\eta_{j\leftarrow i} \,|\, v_i]\right],$$

where the first, third and last equality use the law of total expectation, the second equality is just a change of index and the penultimate is by definition of $\widetilde{y}_i$.

Combining the previous derivations, we get that the revenue maximizing problem is equivalent to maximizing

$$\sum_{i\in N} \mathbb{E}\left[\widetilde{y}_i(v_i)\left(v_i - \frac{1 - F_{v_i}(v_i)}{f_{v_i}(v_i)} - \sum_{j\in N\setminus i} \mathbb{E}[\eta_{j\leftarrow i} \,|\, v_i]\right)\right]. \tag{3.3.9}$$

where $\widetilde{y}_i$ is the interim allocation computed from x_i and must be non-decreasing by Proposition 3.1.3. Hence, we see that the objective function as well as the BNIC and IR constraints can be written solely in terms of the functions $(\widetilde{y}_i)_{i\in N}$. It is thus sufficient to optimize over them separately, under the constraint that $\widetilde{y}_i$ be non-decreasing and $[0, 1]$-valued. Given an optimal choice of $(y_i^\star)_{i\in N}$, we can then define x_i and p_i as in the proposition statement.

To complete the proof we need to choose the smallest constant of integration C_i in (3.3.8) such that interim IR is satisfied. By Proposition 3.1.5, it suffices to set C_i to be the lowest interim utility a bidder could get in any outside option, which is exactly $-\sum_{j\in N\setminus i}\mathbb{E}[\eta_{j\leftarrow i}]$. □

Remark 3.3.5. In contrast to Proposition 3.3.1, Proposition 3.3.4 does not make the assumption of independently coordinates for bidder i's type. However, it again crucially exploits that digital goods are freely replicable so there are no feasibility constraints coupling bidders' allocations.

As a corollary to Proposition 3.3.4, and similar to the single parameter setting [32], we obtain that under a certain regularity assumption, the optimal allocation rule takes a simple form: set a threshold value for each bidder above which the good is deterministically allocated and below which it is not. In other words, the optimization problem of Proposition 3.3.4 over single-variable functions further reduces to finding n parameters: the optimal threshold value of each bidder.

Corollary 3.3.6 (Optimal Mechanism, Setting 2)**.** *Define for $i \in N$, the virtual value function $\Phi_i(v_i) \coloneqq v_i - (1 - F_{v_i}(v_i))/f_{v_i}(v_i)$. Assume the function $\widetilde{\Phi}_i : v_i \mapsto \Phi_i(v_i) - \sum_{j\in N\setminus i}\mathbb{E}[\eta_{j\leftarrow i}\,|\,v_i]$ is non-decreasing and define $\tau_i \coloneqq \widetilde{\Phi}_i^{-1}(0)$. Then the mechanism given by allocation functions*

$$x_i(t) = \mathbb{1}\{v_i \geq \tau_i\} \quad \text{and} \quad x_j(t_i = \emptyset, t_{-i}) = \mathbb{1}\{i \neq j\}, \quad i, j \in N,\ t \in \Theta$$

and payment function

$$p_i(t) = \mathbb{1}\{v_i \geq \tau_i\} \cdot \tau_i + \sum_{j\in N\setminus i} \mathbb{E}\left[\eta_{i\leftarrow j}\,\mathbb{1}\{v_j < \tau_j\}\right], \quad i \in N,\ t \in \Theta$$

is revenue optimal among all BNIC and interim IR mechanisms.

Proof. Observe that the optimization problem in the statement of Proposition 3.3.4 can be written concisely in terms of $\widetilde{\Phi}_i(v_i)$ as $\sup_y \mathbb{E}[\widetilde{\Phi}_i(v_i)y(v_i)]$ where the optimization is over non-decreasing functions taking values in $[0, 1]$. Note that the pointwise optimal function y is given by $\mathbb{1}\{\widetilde{\Phi}_i(v_i) \geq 0\}$ and that this function is non-decreasing in v_i if $\widetilde{\Phi}_i$ is also non-decreasing. The result then follows from Proposition 3.3.4. □

Remark 3.3.7. Again, observe that $\widetilde{\Phi}_i$ is similar in form to the threshold functions derived for the two social-welfare maximization cases (3.2.2) and (3.2.10). In contrast to Proposition 3.3.1, the virtual function is only applied on the valuation v_i.

Example 3.3.8. If we further assume that $\eta_{j\leftarrow i}$ is independent of v_i, then $\mathbb{E}[\eta_{j\leftarrow i} \mid v_i] = \mathbb{E}[\eta_{j\leftarrow i}]$ and the last term in the definition of $\widetilde{\Phi}_i$ does not depend on v_i. In this case our assumption on $\widetilde{\Phi}_i$ is equivalent to the standard regularity assumption of the marginal distribution F_{v_i} of v_i. The payments also take the simpler form

$$p_i(v_i) = \mathbb{1}\{v_i \geq \tau_i\} \cdot \tau_i + \sum_{j \in N \setminus i} \mathbb{E}[\eta_{i\leftarrow j}]\mathbb{P}[v_j < \tau_j]\,.$$

3.4 Discussion

In this section, we discuss the techniques used and results for single digital good auctions. The related work of [28] and [27] reduce the optimal mechanism design problem down to be essentially one-dimensional: the first work, studying Setting 1 of private types, imposes symmetry assumptions on bidder types, and explicitly restricts bids to be single-dimensional, and the second, studying Setting 2, uses a characterization of incentive compatibility to show that bids are effectively one dimensional. In this chapter, we have extended these results to auctions with digital goods. In Setting 1, under an assumption of independent type parameters, we solved for the optimal mechanism with fully multidimensional bids and without requiring any symmetries. In Setting 2, the same IC characterization reducing the relevant information in a bid to a single parameter held in our setting.

We find that in both settings of private types and both objectives (welfare and revenue maximization), the prescribed allocation rules are thresholding functions which allocate the good to a bidder if its value for the good sufficiently outweighs the externalities it causes on other bidders, or else allocates nothing. The specific way in which the threshold is set depends on the situation considered and is summarized in Table 3.1.

We now provide some interpretation for Table 3.1. In Setting 1, we transition

$x_i(t) = \mathbb{1}\{\cdot\}$	**Setting 1** ($\eta_{i\leftarrow}$)	**Setting 2** ($\eta_{\leftarrow i}$)
Welfare Maximization	$v_i \geq \sum_{j\in N\setminus i} \eta_{j\leftarrow i}$	$v_i \geq \sum_{j\in N\setminus i} \mathbb{E}[\eta_{j\leftarrow i}\|v_i]$
Revenue Maximization	$\Phi_i(v_i) \geq \sum_{j\in N\setminus i} -\Phi_{j,i}(-\eta_{j\leftarrow i})$	$\Phi_i(v_i) \geq \sum_{j\in N\setminus i} \mathbb{E}[\eta_{j\leftarrow i}\|v_i]$

Table 3.1: Summary of efficient and optimal allocation rules $x_i(t)$ in the two settings of private types for the single-digital good auction considered in this chapter. Setting 1 is where each bidder knows the incoming externalities caused by others, and Setting 2 is where each bidder knows the outgoing externalities they cause on others. Here, $\phi_Z(z) := z - (1 - F_Z(z))/f_Z(z)$ denotes the virtual valuation function associated with a bounded random variable Z with distribution and density functions F_Z, f_Z, respectively.

from welfare maximization to revenue maximization simply by replacing the type parameters (both the value for the good and externalities) with virtual types. This parallels what happens in the standard result [32], where virtual values reduce the problem of maximizing revenue to maximizing welfare. In Setting 2, a similar reduction holds, but only the value v_i needs to be transformed via the virtual function. In this setting, the externality parameters reported by a bidder do not appear in the bidder's own utility function but rather other bidders' utilities. Thus, incentive compatible allocations ignore these reports and instead rely on the prior distribution on externalities.

For more intuition on efficient versus optimal allocations, we consider the special case of two bidders with uniformly distributed type parameters in Scenario 1. The revenue-maximizing allocation allocates to bidders less often than does the welfare-maximizing allocation and is in general not efficient. This is illustrated in Figure 3-1, where the welfare-maximizing and revenue-maximizing allocations are shown to partition the type space for t into the regions based on bidder 1's allocation. For details, see Appendix A.2.

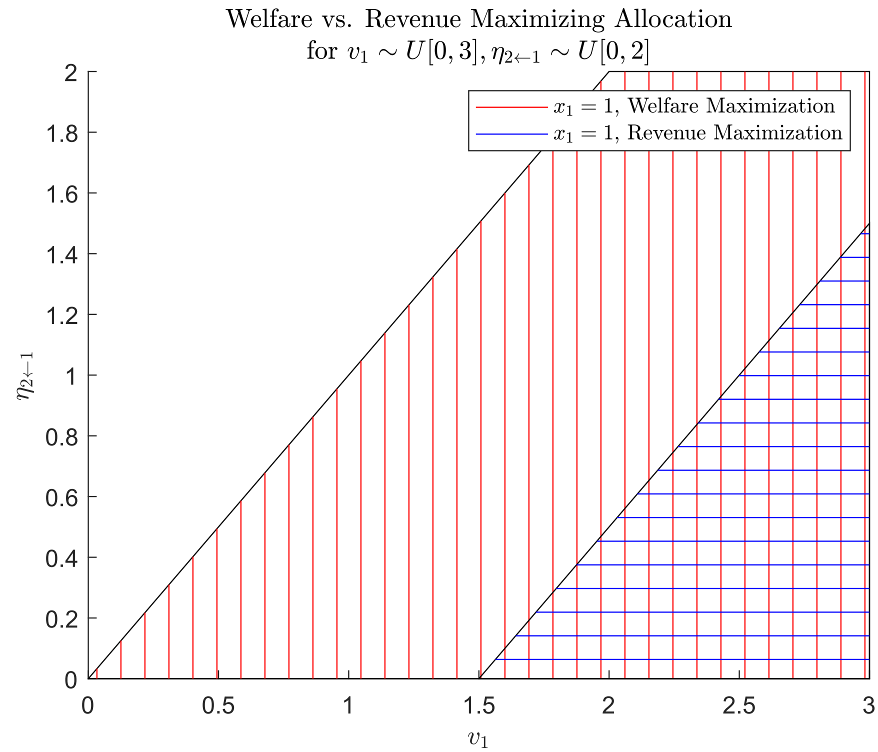

Figure 3-1: Partition of type space by welfare versus revenue maximizing allocations, assuming v_1 and $\eta_{2\leftarrow 1}$ are uniformly distributed on their respective domains $[0, 3]$ and $[0, 2]$. The shaded regions denote where bidder 1 is allocated the entire digital good ($x_1 = 1$) and the un-shaded regions correspond to the opposite case of $x_1 = 0$.

Finally, we revisit the problem of selling data sets for prediction tasks to buyers with downstream interactions. Given the reduction of this problem to one of selling a single digital good, as presented in Section 2.1, we interpret an allocation of $x_i \in [0, 1]$ to be either a probabilistic allocation of the entire collection of data to bidder i or the allocation of an appropriate subset of data. The findings in this chapter imply that all-or-nothing data allocations suffice: in both the welfare-maximizing and revenue-maximizing auctions, bidders either receive the entire collection of data or nothing at all, depending on the thresholding rules discussed above.

Chapter 4

Auctions of Multiple Digital Goods with Externalities

In this chapter, we consider auctions of multiple heterogeneous digital goods in the presence of negative externalities among bidders. We assume that bidders have additive valuations over the goods and study the setting of privately known incoming externalities, the analog of Setting 1 in Chapter 3. That is, bidders' private types include their valuations of each good and the negative externalities that allocations of goods to other bidders have on their utility.

Once again, we would like to find auction mechanisms that either maximize the social welfare or the seller's expected revenue. However, as will be explained in Section 4.3, optimal multi-item mechanisms are notoriously complicated and often not realistically implementable. To this end, we prove that simply selling each item in separate auctions can yield a guaranteed fraction of the optimal multi-item auction revenue. This allows us to construct approximately optimal multi-item mechanisms using the optimal single-item auctions derived in Chapter 3.

Section 4.1 presents the extension of the model in Chapter 2 to the setting of multiple digital goods. Section 4.2 studies welfare-maximizing mechanisms, and Section 4.3 studies (approximately) revenue-maximizing mechanisms.

4.1 Model

Let $G = [g]$ be the set of heterogeneous digital goods for sale, indexed by $k \in G$. Let $N = [n]$ be the set of bidders, usually indexed by $i, j \in N$. We assume independent items and independent bidders, i.e., valuations for items are independent across bidders and across items. As previously stated, we also assume that bidders' valuations are additive over the items. That is, each bidder's valuation of a set of items is simply the sum of the valuations of each item in the set.

We denote by T_i the private type of each bidder $i \in N$, where $T_i \in \mathbb{R}^{n \times g}$ is a matrix with $(j,k)^{\text{th}}$ entry

$$T_{i,j}^k = \begin{cases} v_i^k, \; j = i \\ -\eta_{i \leftarrow j}^k, \; j \neq i \end{cases}$$

Similar to the single-item case, $v_i^k \geq 0$ is bidder i's value of the kth good and $\eta_{i \leftarrow j}^k \geq 0$ is the magnitude of the negative externality that bidder j's allocation of item k has on i's utility. We collect all private types into $T = (T_1, ..., T_n)$, and let $T_{-i} = (T_j)_{j \neq i}$ contain all type matrices except for bidder i's. Let $\Theta = \prod_{i,j \in N} \prod_{k \in G} \Theta_{i,j}^k$ denote the type space, where $T_{i,j}^k \in \Theta_{i,j}^k = [-\bar{\eta}_{i \leftarrow j}^k, -\underline{\eta}_{i \leftarrow j}^k]$ for $j \neq i$ and $T_{i,i}^k \in \Theta_{ii}^k = [\underline{v}_i^k, \bar{v}_i^k]$ for $i, j \in N, k \in G$. For $i \in N$, define the critical type $\mathring{T}_i$ to be the type in Θ_i closest the origin, i.e., $\mathring{T}_{i,j}^k := -\underline{\eta}_{i,j}^k$ and $\mathring{T}_{i,i}^k := \underline{v}_i^k$ for $j \neq i \in N$ and $k \in G$. In the case of revenue maximization, we assume that the types T are distributed according to a distribution $\mathcal{F}$ on Θ. The assumptions of independent items and bidders translates to the type parameters $\{T_{i,j}^k : i, j \in N, k \in G\}$ being mutually independent.

We denote an allocation of items by the matrix $X \in \mathbb{R}^{n \times g}$ where the $(i,k)^{\text{th}}$ element $X_i^k \in [0,1]$ denotes the probability of allocating the kth good to bidder i. In our model, a bidder i with type T_i has a valuation $\nu_i(X; T_i)$ that is linear in the allocation X over both bidders and items, and can be expressed using the matrix inner product as

$$\nu_i(X; T_i) = T_i \cdot X \tag{4.1.1}$$

$$= \sum_{k \in G} \left(v_i^k X_i^k - \sum_{j \in N \setminus i} \eta_{i \leftarrow j}^k X_j^k \right) \tag{4.1.2}$$

Let $p = (p_1, ..., p_n) \in \mathbb{R}^n$ be the vector of payments from the bidders to the auctioneer. Bidders have quasilinear utilities $u_i(X, p_i; T_i) = \nu_i(X; T_i) - p_i$.

Auction Design As in the single-item setting (see Section 2.2.2), we augment the type spaces of bidders i with the element $\emptyset$ denoting a bid of non-participation, to obtain the space of feasible bids $B_i = T_i \cup \{\emptyset\}$. Let $B = \prod_{i \in N} B_i$ be the entire bid space. A auction mechanism $\mu = (X(\cdot), p(\cdot))$ consists of an allocation function $X(\cdot)$ and payment function $p(\cdot)$ mapping bids in B to allocations in $[0, 1]^{n \cdot g}$ and payment vectors in $\mathbb{R}^n$, respectively.

Two common goals of mechanism design are social welfare maximization and revenue maximization. Using the revelation principle, without loss of generality, we optimize for these two goals over direct mechanisms that are Bayes Nash incentive compatible (BNIC) and interim individually rational (interim IR), as defined in Section 2.2.3.

4.2 Welfare Maximization

To derive the class of welfare-maximizing auctions of multiple digital goods, as with the case of single goods, we instantiate the VCG mechanism. Because valuations are additive over items, the resulting efficient mechanism can be interpreted as running g VCG mechanism, one for selling each item separately.

To see this, we express the social welfare as

$$\begin{aligned}\mathrm{SW}(X;T) &= \sum_{i\in N} T_i \cdot X \\ &= \sum_{k\in G}\sum_{i\in N}\sum_{j\in N} T^k_{i,j}\cdot X^k_j = \sum_{k\in G}\sum_{i\in N}\Big(\sum_{j\in N} T^k_{j,i}\Big)\cdot X^k_i.\end{aligned}$$

Thus the welfare maximizing allocation rule is $X^k_i(T) = \mathbb{1}\{\sum_{j\in N} T^k_{j,i} \geq 0\} = \mathbb{1}\{v^k_i - \sum_{j\in N\setminus i}\eta_{j\leftarrow i} \geq 0\}$, which allocates item k to bidder i if the net contribution of such an allocation to the overall welfare is nonnegative. Because the social welfare decomposes as a sum over the items, the associated VCG payment rules also decompose as sums of VCG payments for the individual items. See Section 3.2.1, and specifically (3.2.5), for details on the single-item payment rules.

4.3 Revenue Maximization

We now consider the problem of revenue-maximizing auctions for multiple digital goods with externalities when bidders have additive valuations over the items. Despite the apparently simple condition of additive valuations, however, optimal multi-item mechanism design in this setting and more generally is notoriously complicated. With a single good for sale, the format of the optimal auction is the same regardless of bidders' type distributions. In the standard auction setting without externalities, for instance, the optimal mechanism may be implemented as a second price auction with reserve price. Though the reserve price depends on the type distribution, the fact that we have a single threshold function as the allocation rule remains the same. However, with multiple goods for sale, the *form* of the optimal auction itself depends on the type distribution, and may require randomness, uncountably large menus, and bundling (even when bidders have additive valuations over the items!) [16]. The optimal mechanisms can also exhibit unintuitive properties like being non-monotone in the distribution of bidders' valuations, i.e., even when bidders' values for items increase, the maximum expected revenue may decrease [24].

Because optimal multi-item mechanisms can be structurally complex and often not realistically implementable, we turn from trying to solve for exactly optimal mechanisms and instead seek mechanisms with a simple structure, such as selling items separately or as a grand bundle, that still perform reasonably well. In this section we prove that selling items separately via optimal single-item auctions (which are studied in Chapter 3) yields a guaranteed fraction of the optimal multi-item auction revenue. To do this, we nontrivially extend the approximation technique of [22] to the present setting of interdependent valuations with endogenous participation constraints.

4.3.1 Approximately Optimal Mechanisms

Once again using the Revelation Principle, we restrict the class of mechanisms $\mu = (X(\cdot), p(\cdot))$ considered to those that satisfy Bayes-Nash incentive compatibility (BNIC) and interim individual rationality (interim IR) (see Definitions 2.2.4, 2.2.5). Then for a given distribution of random valuations T, let $\mathrm{Rev}(T)$ denote the maximum expected revenue attainable by a BNIC, interim IR mechanism:

$$\mathrm{Rev}(T) := \sup_{\mu \text{ is IC, IR}} \mathbb{E}\left[\sum_{i \in N} p_i(T)\right]$$

This quantity will be the benchmark by which we measure the performance of classes of simple multi-item auctions.

Given that valuations are additive over items, it is natural to compare the sum of the revenues of single-item auctions for each of the goods with the optimal multi-item auction revenue. It turns out that under the running assumption of independent item and bidder valuations, selling items separately can guarantee a fraction of the optimal revenue. This is formalized in the following main result, which bounds the revenue from the optimal mechanism for selling 2 items in terms of the revenue from optimally selling each item separately.

Theorem 4.3.1. *Let T be the random matrix of types for n bidders and $g = 2$ items,* *distributed according to $\mathcal{F}$ on $\Theta = \Theta^1 \times \Theta^2$, such that the types $\{T^k_{i,j} : i, j \in N, k \in G\}$ are mutually independent. Further, suppose the average externalities are bounded by a constant factor $\gamma \geq 1$ times the smallest, such that $\mathbb{E}[T^k_{i,j}] \geq \gamma \mathring{T}^k_{i,j}$ for all $i \neq j \in N$, $k \in G$. Then*

$$\mathrm{Rev}(T^1) + \mathrm{Rev}(T^2) \leq \mathrm{Rev}(T) \leq (1 + n + \gamma) \cdot \big(\mathrm{Rev}(T^1) + \mathrm{Rev}(T^2)\big)$$

We first present a collection of relevant results that will be used in the proof of this theorem, but that also illustrate the nature of auctions of multiple digital goods with externalities. In particular, the fact that bidders have interdependent valuations and their outside option utility (i.e., utility under non-participation in the auction) is endogenously determined, foils attempts at a simple extension of existing constant-factor approximation results ([22]) that hold in settings without externalities.

Notation. Before moving on, we define the following useful quantities. Let $V_i(\hat{T}_i; T_i) := \mathbb{E}[u_i(X(\hat{T}_i, T_{-i}), p(\hat{T}_i, T_{\setminus i}); T_i)|T_i]$ be the interim expected utility of bidder i for reporting type $\hat{T}_i \in B_i$ given true type $T_i \in \Theta_i$, and assuming all other bidders report their types $T_{-i} \in \Theta_{-i}$ truthfully. Recall that a bid of $\emptyset$ represents non-participation in the auction, and we must have that $X^k_i(\hat{T}_i = \emptyset, T_{-i}) = 0$ for $k \in G$ and $p_i(\hat{T}_i = \emptyset, T_{-i}) = 0$, for all $i \in N$. Further, let $Y(\hat{T}_i) := \mathbb{E}[X(\hat{T}_i, T_{-i})|T_i] = \mathbb{E}_{T_{-i}}[X(\hat{T}_i, T_{-i})]$ denote the interim expected allocation from bidder i's perspective, when i bids $\hat{T}_i$ with true type T_i, and let $q_i(\hat{T}_i) := \mathbb{E}[p(\hat{T}_i, T_{-i})|T_i] = \mathbb{E}_{T_{-i}}[p(\hat{T}_i, T_{-i})]$ likewise denote the interim expected payment by bidder i. Note that, under the given assumption of independent bidder types, the interim expected allocation and payment functions do not depend on bidder i's true type T_i. Also note that $Y(\hat{T}_i) \in [0, 1]^{n \cdot g}$ is in general a different function for each bidder $i \in N$, which for notational simplicity we do not explicitly denote. We then have that

$$V_i(\hat{T}_i; T_i) = T_i \cdot Y(\hat{T}_i) - q_i(\hat{T}_i).$$

As in the single good case, the optimal allocation given a bid of non-participation $\hat{T}_i = \emptyset$ by bidder $i \in N$ is to allocate all goods to all bidders except i. For $i \in N$, let $Z_{(i)}$ denote this allocation matrix, such that $Z^k_{(i)j} = \mathbb{1}\{i \neq j\}$. Due to negative externalities, this allocation maximally depresses bidder i's utility of non-participation, and gives us the loosest IR constraint and thus the largest feasible set of mechanisms over which we maximize expected revenue. Then it is optimal to set $X(\hat{T}_i = \emptyset, T_{-i}) = Z_{(i)}$. Note that we only need to consider single bidder deviations (i.e., non-participation) from equilibrium under the Bayes-Nash solution concept.

The first of our results adapts the proof technique of [22, Proposition 6] to show that any optimal mechanism can be implemented with the **no positive transfer (NPT)** property. This property states that the auctioneer never has to pay a bidder to participate, i.e., $p(T) \geq 0$ for all $T \in \Theta$.

Proposition 4.3.2. *Let $\mu = (X(\cdot), p(\cdot))$ be an BNIC and interim IR mechanism on type space Θ. Then the following hold.*

(i) For all allocations $X \in [0,1]^{n \cdot g}$ and types $T_i \in \Theta_i$, $T_i \cdot (X - Z_{(i)}) \geq 0$.

(ii) $q_i(T_i) \geq \mathring{T}_i \cdot (Y(T_i) - Z_{(i)})$ for all $T_i \in \Theta_i$ if and only if $q_i(\mathring{T}_i) = \mathring{T}_i \cdot (Y(\mathring{T}_i) - Z_{(i)})$.

(iii) There exists a BNIC, interim IR, and NPT mechanism $\tilde{\mu} = (\tilde{X}(\cdot), \tilde{p}(\cdot))$ such that the allocation rule $\tilde{X}(T) = X(T)$ and $\mathbb{E}[\tilde{p}_i(T)] \geq \mathbb{E}[p_i(T)]$ for all $i \in N$ and $T \in \Theta$.

(iv) Let $\theta \subseteq \Theta$ be a subset of the type space. Then $\mathbb{E}\left[\sum_{i \in N} p_i(T)\, \mathbb{1}(T \in \theta)\right] \leq \mathrm{Rev}(T)$.

Proof. (i) Let $T_i \in \Theta_i, X \in [0,1]^{n \cdot g}$. Then

$$\begin{aligned} T_i \cdot X &= \sum_{k \in G} \left(\sum_{j \in N \setminus i} t_{i,j}^k x_j^k + t_{i,i}^k x_i^k \right) \\ &\geq \sum_{k \in G} \sum_{j \in N \setminus i} t_{i,j}^k \cdot 1 + 0 \\ &= T_i \cdot Z_{(i)} \end{aligned}$$

where we used that $x_j^k \in [0,1]$ for $j \in N, k \in G$ and $t_{i,j}^k \leq 0$ for $j \neq i$, and $t_{i,i}^k \geq 0$.

(ii) ($\Rightarrow$) Suppose $q_i(T_i) \geq \mathring{T}_i \cdot (Y(T_i) - Z_{(i)})$ for all $T_i \in \Theta_i$. In particular, this holds for the critical type $\mathring{T}_i$. Next, by the interim IR condition for $\mathring{T}_i$,

$$\begin{aligned} & \mathring{T}_i \cdot Y(\mathring{T}_i) - q_i(\mathring{T}_i) \geq \mathring{T}_i \cdot Y(\hat{T}_i = \emptyset) \\ \Rightarrow\ & \mathring{T}_i \cdot Y(\mathring{T}_i) - q_i(\mathring{T}_i) \geq \mathring{T}_i \cdot Z_{(i)} \\ \Rightarrow\ & q_i(\mathring{T}_i) \leq \mathring{T}_i \cdot (Y(\mathring{T}_i) - Z_{(i)}) \end{aligned}$$

where for the first implication we used the fact that the lowest utility from non-participation occurs when the allocation is $Z_{(i)}$, i.e., $\mathring{T}_i \cdot Y(\hat{T}_i = \emptyset) \geq \mathring{T}_i \cdot Z_{(i)}$ for any mechanism with interim allocation Y. Combined with the initial assumption, we get that equality must hold: $q_i(\mathring{T}_i) = \mathring{T}_i \cdot (Y(\mathring{T}_i) - Z_{(i)})$.

($\Leftarrow$) Suppose that $q_i(\mathring{T}_i) = \mathring{T}_i \cdot (Y(\mathring{T}_i) - Z_{(i)})$. By BNIC, for all $T_i \in \Theta_i$

$$\begin{aligned} & \mathring{T}_i \cdot Y(\mathring{T}_i) - q_i(\mathring{T}_i) \geq \mathring{T}_i \cdot Y(T_i) - q_i(T_i) \\ \Rightarrow\ & \mathring{T}_i \cdot Y(\mathring{T}_i) - (\mathring{T}_i \cdot Y(\mathring{T}_i) - \mathring{T}_i \cdot Z_{(i)}) \geq \mathring{T}_i \cdot Y(T_i) - q_i(T_i) \\ \Rightarrow\ & \mathring{T}_i \cdot (Z_{(i)} - Y(T_i)) \geq -q_i(T_i) \\ \Rightarrow\ & q_i(T_i) \geq \mathring{T}_i \cdot (Y(T_i) - Z_{(i)}) \end{aligned}$$

(iii) Define the new mechanism $\bar{\mu} = (\bar{X}(\cdot), \bar{p}(\cdot))$ with $\bar{X}(T) = X(T)$ and

$$\bar{p}_i(T) = p_i(T) + [-q_i(\mathring{T}_i) + \mathring{T}_i \cdot (Y(\mathring{T}_i) - Z_{(i)})]$$

for all $T \in \Theta, i \in N$. Here we let $q_i(\hat{T}_i)$ and $Y(\hat{T}_i)$ denote, respectively, the interim expected payment and allocation from bidder i's perspective for the mechanism μ, and accordingly let $\bar{q}_i(\hat{T}_i)$ and $\bar{Y}(\hat{T}_i)$ denote, respectively, the interim expected payment and allocations for the new mechanism $\bar{\mu}$. Then for all $T_i \in \Theta_i$, $\bar{Y} = Y$ and $\bar{q}_i(T_i) = q_i(T_i) - q_i(\mathring{T}_i) + \mathring{T}_i \cdot (Y(\mathring{T}_i) - Z)$. Let us further set the allocation in case of non-participation of a single bidder i to be the "worst-case" for i: $\bar{X}(\hat{T}_i = \emptyset, T_{-i}) = Z_{(i)}$.

Note that since μ is interim IR, the constant shift in bidder i's payments is nonnegative, i.e., $-q_i(\mathring{T}_i) + \mathring{T}_i \cdot (Y(\mathring{T}_i) - Z_{(i)}) \geq 0$, so $\bar{p}(T) \geq p(T)$. $\bar{\mu}$ satisfies BNIC since bidder i's utilities are uniformly shifted by a constant. Further, by Lemma 4.3.4, $\bar{\mu}$ is interim IR if and only if interim IR is satisfied for the types $\mathring{T}_i$, for $i \in N$. This condition indeed holds by construction:

$$\mathring{T}_i \cdot \bar{Y}(\mathring{T}_i) - \bar{q}_i(\mathring{T}_i) = \mathring{T}_i \cdot Y(\mathring{T}_i) - \mathring{T}_i \cdot (Y(\mathring{T}_i) - Z_{(i)}) = \mathring{T}_i \cdot Z_{(i)}.$$

Since $\bar{\mu}$ is BNIC, interim IR, and $\bar{q}_i(\mathring{T}_i) = \mathring{T}_i \cdot (Y(\mathring{T}_i) - Z_{(i)})$ for $i \in N$, statements (4.3.2.i) and (4.3.2.ii) imply that for all $T_i \in \Theta_i, \bar{q}_i(T_i) \geq 0$.

Now given $\bar{\mu}$, Lemma 4.3.5 gives us another BNIC and interim IR mechanism $\tilde{\mu} = (\tilde{X}(\cdot), \tilde{p}(\cdot))$ with $\tilde{p}_i(T) = \bar{q}_i(T_i) \geq 0$ for $i \in N$. Thus $\tilde{\mu}$ also satisfies NPT and for all $i \in N$, $\mathbb{E}[p_i(T)] \leq \mathbb{E}[\bar{p}_i(T)] = \mathbb{E}[\bar{q}_i(T_i)] = \mathbb{E}[\tilde{p}_i(T)]$.

(iv) Let $\tilde{\mu} = (\tilde{X}(\cdot), \tilde{p}(\cdot))$ be a BNIC, interim IR, NPT mechanism with $\mathbb{E}[\tilde{p}_i(T)] \geq \mathbb{E}[p_i(T)]$, as constructed in the proof of (4.3.2.iii). Then

$$\mathbb{E}[\sum_{i \in N} p_i(T)\, \mathbb{1}(T \in \theta)] \leq \mathbb{E}[\sum_{i \in N} \tilde{p}_i(T)\, \mathbb{1}(T \in \theta)] \leq \mathbb{E}[\sum_{i \in N} \tilde{p}_i(T)] \leq \mathrm{Rev}(T)$$

where we use that $\tilde{p} \geq 0$ by NPT, and the definition of $\mathrm{Rev}(T)$.

□

Remark 4.3.3. Note that the conditions in (4.3.2.ii) are stronger than NPT, by statement (4.3.2.i). In particular, we make use of the fact that the interim IR constraint is endogenously determined by the mechanism because of the externality effects, and the optimal such constraint allows the auctioneer to charge bidders against the "threat" of allocating goods to other bidders. By statement (4.3.2.iii), it suffices to consider BNIC, interim IR, and NPT mechanisms in the search for revenue-maximizing mechanisms.

The following two lemmas were used in the proof of Proposition 4.3.2. Lemma 4.3.4 is a characterization of interim IR similar to that of 3.1.2 and says that it suffices to check interim IR for the critical type $\mathring{T}_i \in \Theta_i$ for $i \in N$.

Lemma 4.3.4. *Suppose $\mu = (X(\cdot), p(\cdot))$ is a BNIC mechanism on type space Θ with the allocation rule $X(T_i = \emptyset, T_{-i}) = Z_{(i)}$ for $i \in N$. Then μ is interim IR if and only if interim IR holds for the critical types $\mathring{T}_i$, i.e., that*

$$\mathring{T}_i \cdot Y(\mathring{T}_i) - q(\mathring{T}_i) \geq \mathring{T}_i \cdot Z_{(i)}, \quad i \in N. \tag{4.3.1}$$

Proof. Suppose (4.3.1) holds. Then for all $T_i \in \Theta_i$,

$$\begin{aligned} T_i \cdot Y(T_i) - q_i(T_i) &\geq T_i \cdot Y(\mathring{T}_i) - q_i(\mathring{T}_i) \\ &= (T_i - \mathring{T}_i) \cdot Y(\mathring{T}_i) + (\mathring{T}_i \cdot Y(\mathring{T}_i)) - q_i(\mathring{T}_i)) \\ &\geq (T_i - \mathring{T}_i) \cdot Z_{(i)} + \mathring{T}_i \cdot Z_{(i)} \\ &= T_i \cdot Z_{(i)}, \end{aligned}$$

which is the desired interim IR condition. The first inequality is by BNIC and the second inequality is follows by (4.3.1) and from the fact that for any $X \in [0,1]^{n \cdot g}$, $(T_i - \mathring{T}_i) \cdot X \geq (T_i - \mathring{T}_i) \cdot Z_{(i)}$. We can see this by using the definition of the critical type $\mathring{T}_i$ to get

$$(T_i - \mathring{T}_i)_j^k = \begin{cases} v_i^k - \underline{v}_i^k \geq 0, \ i = j \\ -\eta_{i\leftarrow j}^k + \underline{\eta}_{i\leftarrow j}^k \leq 0, \ i \neq j. \end{cases}$$

For $k \neq G$, $X_i^k \geq Z_{(i)i}^k = 0$, and $X_j^k \leq Z_{(i)j}^k = 1$ for $j \neq i$, so that $(T_i - \mathring{T}_i) \cdot (X - Z_{(i)}) \geq 0$. Finally, the converse implication clearly holds. □

The next lemma allows us to, without loss of generality, choose the payment rule for each bidder i to be equal its interim payment rule. In particular, we can have that bidder i's payment only depends on i's bid T_i.

Lemma 4.3.5. *Suppose $\mu = (X(\cdot), p(\cdot))$ is a BNIC and interim IR mechanism on type space Θ. Then the mechanism $\tilde{\mu} = (\tilde{X}(\cdot), \tilde{p}(\cdot))$ with $\tilde{X} := X$ and $\tilde{p}_i(T) := q_i(T_i)$ is BNIC and interim IR, and $\mathbb{E}[\sum_{i\in N} p_i(T)] = \mathbb{E}[\sum_{i\in N} \tilde{p}_i(T)]$.*

Proof. Let $\tilde{X} := X$ and $\tilde{p}_i(T) := q_i(T_i) = \mathbb{E}[p_i(T)|T_i]$ for $i \in N, T \in \Theta$. The expected revenue remains unchanged by the law of total expectation and linearity of expectation: $\mathbb{E}[\sum_{i\in N} p_i(T)] = \mathbb{E}[\sum_{i\in N} \mathbb{E}[p_i(T)|T_i]] = \mathbb{E}[\sum_{i\in N} q_i(T_i)] = \mathbb{E}[\sum_{i\in N} \mathbb{E}[\tilde{p}_i(T)]$. The interim IR and BNIC conditions for μ only feature the interim expected payments for each bidder $i \in N$:

$$(IC) \qquad T_i \cdot \mathbb{E}[X(T_i, T_{-i})|T_i] - q_i(T_i) \geq T_i \cdot \mathbb{E}[X(\hat{T}_i, T_{-i})|T_i] - q_i(\hat{T}_i)$$

$$(IR) \qquad T_i \cdot \mathbb{E}[X(T_i, T_{-i})|T_i] - q_i(T_i) \geq T_i \cdot \mathbb{E}[X(T_i = \emptyset, T_{-i})|T_i]$$

Since $\tilde{q}_i(T_i) = q_i(T_i)$, BNIC and interim IR carry over to $\tilde{\mu}$. □

Lemmas 4.3.6, 4.3.8 and 4.3.9 will be used to bound various terms in the proof of Theorem 4.3.1. The first of these lemmas follows a proof technique similar to that used in [22] to derive a mechanism to sell one item given a two-item mechanism. However, in the current setting of interdependent valuations and endogenous IR constraints, extra care must be taken to ensure BNIC and interim IR of the induced mechanism.

Lemma 4.3.6. *Suppose $\mu = (X(\cdot), p(\cdot))$ is a BNIC and interim IR mechanism for 2 items on type space $\Theta_1 \times \Theta_2$. Fix some $\tau_2 \in \Theta_2$, and define the induced mechanism $\mu^{(1)} = (X^{(1)}(\cdot|\tau_2), p^{(1)}(\cdot|\tau_2))$ on Θ_1 by*

$$X^{(1)}(T^1|\tau_2) := X^1(T^1, \tau_2), \quad X^{(1)}(T_i^1 = \emptyset, T_{-i}^1|\tau_2) = Z_{(i)} \tag{4.3.2}$$

$$\begin{aligned} p_i^{(1)}(T^1|\tau_2) := {} & p_i(T^1, \tau_2) - \tau_i^2 \cdot \mathbb{E}[X^2(T^1, T^2)|T_i^1, T_i^2 = \tau_i^2] \\ & + T_i^1 \cdot \mathbb{E}[X^1(T^1, T^2)|T_i^1, T^2 = \tau^2] - T_i^1 \cdot \mathbb{E}[X^1(T^1, T^2)|T_i^1, T_i^2 = \tau_i^2] \qquad (4.3.3) \\ & + \tau_i^2 \cdot Z_{(i)}^2. \end{aligned}$$

Then $\mu^{(1)}$ is BNIC and interim IR.

An interpretation of the induced mechanism $\mu^{(1)}$ is as follows. Given a collection of bids T^1, the allocation of item 1 will be as the same as in the 2-item mechanism given bids τ^2 for item 2. We modify the bidder i's original payment rule by subtracting the effect that the allocation of item 2 would have had on i's utility. The third and fourth terms in (4.3.3) are corrections that will allow us to transfer BNIC properties from μ to $\mu^{(1)}$, and the final term will allow us to transfer interim IR properties.

Proof of Lemma 4.3.6. By Lemma 4.3.5, without loss of generality, assume that $p_i(T) = q_i(T_i)$ for all $T \in \Theta$. Under the induced mechanism, the interim expected utility of bidder i bidding $\hat{T}_i^1$, with true type T_i^1 is

$$\begin{aligned} V_i^{(1)}(\hat{T}_i^1; T_i^1) &= \mathbb{E}_{T^1}[T_i^1 \cdot X^{(1)}(\hat{T}_i^1, T_{-i}^1|\tau^2) - p_i^{(1)}(\hat{T}_i^1, T_{-i}^1|\tau^2) \,|\, T_i^1] \\ &= T_i^1 \cdot \mathbb{E}_{T^1}[X^1(\hat{T}_i^1, T_{-i}^1, \tau^2) \,|\, T_i^1] - q_i(\hat{T}_i^1, \tau_i^2) \\ &\quad + \tau_i^2 \cdot \mathbb{E}_{T^1, T_{-i}^2}[X^2(\hat{T}_i^1, T_{-i}^1, \tau_i^2, T_{-i}^2) \,|\, T_i^1] \\ &\quad - T_i^1 \cdot \mathbb{E}_{T^1}[X^1(\hat{T}_i^1, T_{-i}^1, \tau^2) \,|\, T_i^1] + T_i^1 \cdot \mathbb{E}_{T^1, T_{-i}^2}[X^1(\hat{T}_i^1, T_{-i}^1, \tau_i^2, T_{-i}^2) \,|\, T_i^1] \\ &\quad - \tau_i^2 \cdot Z_{(i)}^2, \end{aligned}$$

where we have substituted in the definition of $\mu^{(1)}$. After simplifying and re-arranging terms, we get

$$\begin{aligned} V_i^{(1)}(\hat{T}_i^1; T_i^1) = T_i^1 \cdot \mathbb{E}_T[X^1(\hat{T}_i^1, T_{-i}^1, T^2) \,|\, T_i^1, T_i^2 = \tau_i^2] \\ + \tau_i^2 \cdot \mathbb{E}_T[X^2(\hat{T}_i^1, T_{-i}^1, T^2) \,|\, T_i^1, T_i^2 = \tau_i^2] \\ - q_i(\hat{T}_i^1, \tau_i^2) - \tau_i^2 \cdot Z_{(i)}^2 \\ = V_i((\hat{T}_i^1, \tau_i^2); (T_i^1, \tau_i^2)) - \tau_i^2 \cdot Z_{(i)}^2. \end{aligned}$$

We thus see that $\mu^{(1)}$ inherits BNIC from μ since bidder i's interim utility function is only shifted by a constant:

$$\begin{aligned} V_i^{(1)}(\hat{T}_i^1; T_i^1) &= V_i((\hat{T}_i^1, \tau_i^2); (T_i^1, \tau_i^2)) - \tau_i^2 \cdot Z_{(i)}^2 \\ &\leq V_i((T_i^1, \tau_i^2); (T_i^1, \tau_i^2)) - \tau_i^2 \cdot Z_{(i)}^2 = V_i^{(1)}(T_i^1; T_i^1). \end{aligned}$$

Further, $\mu^{(1)}$ inherits interim IR as well:

$$\begin{aligned} V_i^{(1)}(T_i^1; T_i^1) &= V_i((T_i^1, \tau_i^2); (T_i^1, \tau_i^2)) - \tau_i^2 \cdot Z_{(i)}^2 \\ &\geq (T_i^1, \tau_i^2) \cdot Z_{(i)} - \tau_i^2 \cdot Z_{(i)}^2 \\ &= T_i^1 \cdot Z_{(i)}^1 = T_i^1 \cdot Y^{(1)}(T_i^1 = \emptyset, T_{-i}^1 | \tau_2). \end{aligned}$$

□

Remark 4.3.7. Unlike in the setting of [22], bidders here have interdependent valuations. This prevents us from being able to decouple bidders' allocations, for example by replacing the allocation function $X(T)$ with any of the bidders' interim expected allocations $Y(T_i)$. Further, we only inherit BNIC conditions from μ rather than stronger DSIC conditions that would easily transfer from μ to $\mu^{(1)}$, for all values of $\tau_i^2 \in \Theta_i^2$. To circumvent this difficulty, we add in the modification term in the penultimate line of (4.3.3).

Lemma 4.3.8. *Let $\pi \geq 0$ be a fixed constant. Let $\mu = (X(\cdot), p(\cdot))$ be the mechanism* *to sell one item to n bidders with type distribution $\mathcal{F}$ on type space $\Theta = \Theta^1$ given by*

$$
\begin{aligned}
&X_i(T) = \mathbb{1}\{w_i(T) \geq \pi\}, \quad T \in \Theta \\
&X_j(T_i = \emptyset, T_{-i}) = \mathbb{1}\{\sum_{k \in N \setminus i} T_{kj} \geq \pi\}, \quad T_{-i} \in \Theta_{-i} \\
&p_i(T) = \sum_{j \in N \setminus i} \left[\sum_{k \in N \setminus i} T_{kj} \left(X_j(T_i = \emptyset, T_{-i}) - X_j(T) \right) - T_{j,i} X_i(T) \right] + \pi X_i(T), \quad T \in \Theta
\end{aligned}
$$

where $w_i(T) := \sum_{j \in N} T_{j,i}$ for $i \in N$. Then μ is BNIC and interim IR with expected revenue

$$
\mathbb{E}\left[\sum_{i \in N} p_i(T)\right] \geq \mathbb{E}\left[\sum_{i \in N} \pi\, \mathbb{1}\{w_i(T) \geq \pi\}\right] \tag{4.3.4}
$$

Proof. We first extend μ to be a VCG mechanism to sell the single item to $n+1$ bidders, consisting of the original n bidders and a "phantom" bidder $n+1$ with type space $\tilde{\Theta}$ such that

$$
\begin{aligned}
&\tilde{\Theta}_{i,j} = \Theta_{i,j}, \quad i, j \in N \\
&\tilde{\Theta}_{i,n+1} = \{0\}, \quad \tilde{\Theta}_{n+1,i} = \{-\pi\}, \quad i \in N \\
&\tilde{\Theta}_{n+1,n+1} = \{0\},
\end{aligned}
$$

and with $\tilde{T}_{i,j}$ still distributed according to $\mathcal{F}_{i,j}$ for $i, j \in N$. Thus, each of the original bidders exerts a negative externality of magnitude π on the phantom if allocated the good, but the phantom has no value for and causes no externalities on others if allocated the good. Let $\tilde{\mu}$ be the VCG mechanism in the extended setting. Using Proposition 3.2.2 in Section 3.2.1, with the correspondence that $W_i = w_i(\tilde{T}) + \tilde{T}_{n+1,i}$, $W_j^i = w_i(\tilde{T}) + \tilde{T}_{n+1,i} - \tilde{T}_{i,j}$, and $W_{n+1} = W_{n+1}^i = 0$ for $i \neq j \in N$, we get the allocation and payment rules for $\tilde{\mu}$ on $\tilde{\Theta}$:

$$\tilde{X}_i(\tilde{T}) = \mathbb{1}\Big\{w_i(\tilde{T}) - \pi \geq 0\Big\}, \quad i \in N$$
$$\tilde{X}_j(\tilde{T}_i = \emptyset, \tilde{T}_{-i}) = \mathbb{1}\Bigg\{\sum_{k\in N\setminus i} \tilde{T}_{kj} - \pi \geq 0\Bigg\}, \quad j \neq i \in N$$
$$\tilde{p}_i(\tilde{T}) = \sum_{j\in N\setminus i}\Bigg[\Big(\sum_{k\in N\setminus i}\tilde{T}_{kj} - \pi\Big)(\tilde{X}_j(\tilde{T}_i = \emptyset, \tilde{T}_{-i}) - \tilde{X}_j(\tilde{T})) - \tilde{T}_{j,i}\tilde{X}_i(\tilde{T})\Bigg] + \pi\tilde{X}_i(\tilde{T}), \quad i \in N$$
$$\tilde{X}_{n+1}(\tilde{T}) = \mathbb{1}\{0 \geq 0\} = 1, \quad \tilde{p}_{n+1}(\tilde{T}) = 0.$$

These functions comprise a DSIC and ex-post IR mechanism, which also implies that $\tilde{\mu}$ is BNIC and interim IR. Note that the allocation and payment rules for μ are simply those for $\tilde{\mu}$ when restricted to bidders $i \in N$, and the expected interim utility functions of bidders $i \in N$ under $\tilde{\mu}$ and μ are equal since the phantom bidder causes no externalities. Thus, μ also satisfies BNIC and interim IR constraints.

Furthermore, note that $(\sum_{k\in N\setminus i}\tilde{T}_{kj} - \pi)(\tilde{X}_j(\tilde{T}_i = \emptyset, \tilde{T}_{-i}) - \tilde{X}_j(\tilde{T})) \geq 0$ and $\tilde{T}_{j,i} \leq 0$ for all $\tilde{T} \in \tilde{\Theta}, i \neq j \in N$. Then we immediately get that $\tilde{p}_i(\tilde{T}) \geq \pi\tilde{X}_i(\tilde{T})$ for $i \in N$, which implies (4.3.4). □

Lemma 4.3.9. *Given a set of goods G and of bidders N with types T distributed on type space Θ,*

$$\sum_{k\in G}\sum_{i\in N}\sum_{j\in N\setminus i} -\mathring{T}^k_{i,j} \leq \mathrm{Rev}(T).$$

Proof. Consider the mechanism $\mu = (X(\cdot), p(\cdot))$ with $X(T) = 0$ and $p_i(T) = -\mathring{T}_i \cdot Z_{(i)}$ for all $T \in \Theta$, and $X(T_i = \emptyset, T_{-i}) = Z_{(i)}$ for $i \in N, T_{-i} \in \Theta_{-i}$. That is, μ never allocates any item to any bidder but extracts the maximum payment possible under optimal participation constraints. This mechanism is BNIC (and DSIC) since bidders' utilities are constant with respect to their bids, and is interim IR (and ex-post IR) since for all $T_i \in \Theta_i$,

$$V_i(T_i; T_i) = -p_i(T) = \mathring{T}_i \cdot Z_{(i)} \geq T_i \cdot Z_{(i)}.$$

Thus, its expected revenue can be bounded by the optimal revenue: $\mathbb{E}\big[\sum_{i \in N} p_i(T)\big] = \sum_{k \in G} \sum_{i \in N} \sum_{j \in N \setminus i} -\mathring{T}^k_{i,j} \leq \mathrm{Rev}(T)$. □

We are now ready to prove our main theorem bounding the optimal revenue from selling two items in terms of the optimal revenue from selling the items separately.

Proof of Theorem 4.3.1. Suppose $\mu = (X(\cdot), p(\cdot))$ is a BNIC, interim IR, and NPT mechanism on Θ with types T distributed according to the distribution function $\mathcal{F}$. Let us first define the following quantities, which capture the effect of an allocation of item k to bidder i on the total welfare:

$$w_i^k(T^k) = \sum_{j \in N} T^k_{j,i} = v_i^k - \sum_{j \in N \setminus i} \eta^k_{j \leftarrow i}$$

and let $w^k_{(1)}(T^k) := \max_{i \in N} w^k_{(i)}$.

Since μ is NPT, $p_i(T) \geq 0$ and we partition the type space and bound the expected revenue from μ:

$$\begin{aligned} \mathbb{E}\left[\sum_{i \in N} p_i(T)\right] = & \, \mathbb{E}\left[\sum_{i \in N} p_i(T)\, \mathbb{1}\big\{w^1_{(1)}(T^1) \geq w^2_{(1)}(T^2)\big\}\right] \\ & + \mathbb{E}\left[\sum_{i \in N} p_i(T)\, \mathbb{1}\big\{w^2_{(1)}(T^2) \geq w^1_{(1)}(T^1)\big\}\right] \end{aligned} \tag{4.3.5}$$

Consider the first term. For a fixed $T^2 = \tau^2 \in \Theta^2$, let $\mu^{(1)} = (X^{(1)}(\cdot|\tau_2), p^{(1)}(\cdot|\tau_2))$ be the induced BNIC and interim IR mechanism on Θ_1 to sell item 1, as in Lemma 4.3.6. Then expressing $p_i(T^1, \tau^2)$ in terms of $\mu^{(1)}$ and given our independence assumptions, we write the following conditional expectation

$$
\begin{aligned}
&\mathbb{E}\left[\sum_{i\in N} p_i(T)\,\mathbb{1}\{w^1_{(1)}(T^1) \geq w^2_{(1)}(T^2)\}\,\middle|\, T^2=\tau^2\right] \\
&= \mathbb{E}_{T^1}\left[\sum_{i\in N}\big(p^{(1)}_i(T^1|\tau_2) + \tau^2_i\cdot\mathbb{E}[X^2(T^1,T^2)|T^1_i, T^2_i=\tau^2_i]\right.\\
&\qquad - T^1_i\cdot\mathbb{E}[X^1(T^1,T^2)|T^1_i, T^2=\tau^2] + T^1_i\cdot\mathbb{E}[X^1(T^1,T^2)|T^1_i, T^2_i=\tau^2_i] \\
&\qquad \left. - \tau^2_i\cdot Z^2_{(i)}\big)\,\mathbb{1}\{w^1_{(1)}(T^1)\geq w^2_{(1)}(\tau^2)\}\right].
\end{aligned}
\tag{4.3.6}
$$

We will bound (4.3.6) by attacking it in four parts.

Part 1. First, note that since $\mu^{(1)}$ is a BNIC and interim IR mechanism, statement (iv) of Proposition 4.3.2 bounds the first term by the optimal revenue from selling good 1 alone:

$$
\mathbb{E}\left[\sum_{i\in N} p^{(1)}_i(T^1|\tau_2)\,\mathbb{1}\{w^1_{(1)}(T^1)\geq w^2_{(1)}(\tau^2)\}\right] \leq \mathrm{Rev}(T^1).
$$

Part 2. Next, given $\tau^2\in\Theta^2$ and any allocation $X^2\in[0,1]^n$, we can write

$$
\begin{aligned}
\sum_{i\in N}\tau^2_i\cdot X^2 &= \sum_{i\in N}\sum_{j\in N}\tau^2_{i,j}X^2_j = \sum_{i\in N}(\sum_{j\in N}\tau^2_{j,i})X^2_i \\
&= \sum_{i\in N} w^2_i(\tau^2)X^2_i \leq \sum_{i\in N} w^2_{(1)}(\tau^2)\,\mathbb{1}\{w^2_{(1)}(\tau^2)\geq 0\}.
\end{aligned}
$$

Using this, we bound the second term in (4.3.6):

$$
\begin{aligned}
&\mathbb{E}\left[\sum_{i\in N}\tau^2_i\cdot\mathbb{E}[X^2(T^1,T^2)|T^1_i,T^2_i=\tau^2_i]\,\mathbb{1}\{w^1_{(1)}(T^1)\geq w^2_{(1)}(\tau^2)\}\right] \\
&\leq \sum_{i\in N}\mathbb{E}\big[w^2_{(1)}(\tau^2)\,\mathbb{1}\{w^2_{(1)}(\tau^2)\geq 0\}\,\mathbb{1}\{w^1_{(1)}(T^1)\geq w^2_{(1)}(\tau^2)\}\big] \\
&\leq n\,\mathbb{E}\left[\sum_{i\in N} w^2_{(1)}(\tau^2)\,\mathbb{1}\{w^2_{(1)}(\tau^2)\geq 0\}\,\mathbb{1}\{w^1_i(T^1)\geq w^2_{(1)}(\tau^2)\}\right].
\end{aligned}
\tag{4.3.7}
$$

By applying Lemma 4.3.8 with $\pi = w^2_{(1)}(\tau^2)\,\mathbb{1}\Big\{w^2_{(1)}(\tau^2)\geq 0\Big\}$, and in particular the inequality (4.3.4), we bound the quantity under the expectation in (4.3.7) by the expected revenue of some BNIC and interim IR mechanism. This expected revenue is in turn bounded above by the optimal revenue, $\mathrm{Rev}(T_1)$, over all BNIC and interim IR mechanisms. Thus the second term in (4.3.6) is bounded by $n\,\mathrm{Rev}(T_1)$.

Part 3. We show that the expectation of the second line in (4.3.6) vanishes. Temporarily let τ^2 denote a random variable independently and identically distributed with T^2, with distribution F^2 on Θ^2. Then

$$
\begin{aligned}
&\mathbb{E}\big[\mathbb{E}_{T^1}\left[-T_i^1 \cdot \mathbb{E}[X^1(T^1, T^2)|T_i^1, T^2 = \tau^2] + T_i^1 \cdot \mathbb{E}[X^1(T^1, T^2)|T_i^1, T_i^2 = \tau_i^2]\right]\big] \\
=&\mathbb{E}\big[\mathbb{E}_{T^1}\left[-T_i^1 \cdot \mathbb{E}[X^1(T^1, \tau^2)|T_i^1, \tau^2] + T_i^1 \cdot \mathbb{E}[X^1(T^1, \tau_i^2, T_{-i}^2)|T_i^1, \tau_i^2]\right]\big] \\
=&\mathbb{E}\big[\mathbb{E}\left[-T_i^1 \cdot X^1(T^1, \tau^2)\big|\tau^2\right]\big] + \mathbb{E}\big[\mathbb{E}\left[T_i^1 \cdot X^1(T^1, \tau_i^2, T_{-i}^2)\big|\tau^2\right]\big] \\
=&\mathbb{E}\big[-T_i^1 \cdot X^1(T^1, \tau^2)\big] + \mathbb{E}\big[-T_i^1 \cdot X^1(T^1, \tau^2)\big] \\
=&0,
\end{aligned}
$$

where we have made use of the independence of T_i^k's and the law of iterated expectations.

Part 4. Finally, we bound the contribution of the last term in (4.3.6) by using the assumption on the distributions of externality parameters, that $\mathbb{E}[T_{i,j}^k] \geq \gamma \mathring{T}_{i,j}^k$ for $i \neq j \in N$, $k \in G$. Then

$$
\begin{aligned}
&\mathbb{E}_T\left[\sum_{i\in N} \mathbb{E}\big[-T_i^2 \cdot Z_{(i)}^2 \,\mathbb{1}\{w_{(1)}^1(T^1) \geq w_{(1)}^2(T^2)\}\,\big|\, T^2\big]\right] \\
\leq&\mathbb{E}\left[\sum_{i\in N} -T_i^2 \cdot Z_{(i)}^2\right] = \mathbb{E}\Big[-\sum_{i\in N}\sum_{j\in N\setminus i} T_{i,j}^2\Big] \\
\leq&\gamma \sum_{i\in N}\sum_{j\in N\setminus i} -\mathring{T}_{i,j}^2 \leq \gamma\,\mathrm{Rev}(T^2)
\end{aligned}
$$

where the first inequality holds since $-T_i^2 \cdot Z_{(i)}^2 \geq 0$, and the second inequality by the distributional assumptions on $T_{i,j}$, and the final inequality by Lemma 4.3.9 applied with a single good, item 2.

Putting everything together, we have

$$\begin{aligned}&\mathbb{E}\left[\sum_{i\in N} p_i(T)\,\mathbb{1}\{w^1_{(1)}(T^1) \geq w^2_{(1)}(T^2)\}\right]\\ =& \operatorname{Rev}(T^1) + n\operatorname{Rev}(T^1) + \gamma\operatorname{Rev}(T^2).\end{aligned}$$

A similar argument holds for the second term in (4.3.5). Thus, given any BNIC, interim IR, and NPT mechanism μ, its expected revenue

$$\mathbb{E}\left[\sum_{i\in N} p_i(T)\right] \leq (1+n+\gamma)(\operatorname{Rev}(T^1) + \operatorname{Rev}(T^2)).$$

By statement (iii) of Proposition 4.3.2, $\operatorname{Rev}(T)$ can be taken as the optimal revenue over BNIC, interim IR, and NPT mechanisms. Finally, since the sum of the revenues from selling each of the two items in separate auctions is bounded by $\operatorname{Rev}(T)$, we conclude

$$\operatorname{Rev}(T^1) + \operatorname{Rev}(T^2) \leq \operatorname{Rev}(T) \leq (1+n+\gamma)(\operatorname{Rev}(T^1) + \operatorname{Rev}(T^2)).$$

□

Remark 4.3.10. While [22] prove that selling 2 goods separately provides a constant factor 2-approximation to the optimal revenue in the absence of externalities, we obtain a less desirable $(1 + n + \gamma)$ approximation factor. The dependence on the number n of bidders arises because the interdependent valuations couple bids, allocations, and utilities across bidders and thus restricts the set of BNIC mechanisms that we can use to bound the second term in (4.3.6). The γ term arises from the fact that the presence of negative externalities makes outside option utilities and thus IR constraints endogenously determined by bidders' private types. Reducing a two-item mechanism to a one-item mechanism tightens the IR constraints (since in the latter, one can only extract at most the magnitude of the externalities caused by a single item), a restriction which does not occur in standard auction settings.

Remark 4.3.11. To extend the result from auctions of 2 digital goods to auctions of g digital goods, one could follow the strategy used in [22], where an approximation bound similar to Theorem 4.3.1, but for 2 *bundles* of items, is used to prove the result inductively. However, this extension relies primarily on the common assumption of additive item valuations, while the 2-item setting already captures the key complications that externalities bring to the table.

Chapter 5

Conclusion

5.1 Summary

In this thesis, we set out to answer the question: how should a data seller allocate and price data sets to data buyers who may compete with each other downstream, in such a way that maximizes social welfare or the seller's expected revenue? Along the way, we captured the problem of valuing and selling data sets for prediction tasks within the more general framework of digital goods auctions, and motivated the model of additively separable negative externalities among the bidders.

Note that a multi-bidder (n) auction digital goods without externalities simply reduces to n single-bidder auctions, since the lack of a constraint in the supply of digital goods decouples the allocations of goods among bidders. In the presence of externalities, however, bidders' allocation functions are once again coupled, but this time through the interdependent valuations that determine their utility functions and affect strategic considerations like truthful bidding.

We studied two settings of the bidders' privately known information, or type: one in which bidders observe their value for each good sold and the externalities that other bidders exert on them, and one in which bidders observe their value for each good and the externalities that they exert on other bidders. The form of private types affects the characterization of incentive compatible and individually rational mechanisms, and thus the form of the efficient and optimal allocation functions.

Welfare-maximizing and revenue-maximizing mechanisms for auctions of a single digital good under both settings of private types were studied in Chapter 3. Welfare-maximizing and approximately revenue-maximizing mechanisms for auctions of multiple heterogenous digital goods with additive valuations over items were studied in Chapter 4.

A version of Chapters 2 and 3 of this thesis, on the sale of data and auctions of a single digital good, is published in [5].

5.2 Future Work

Beyond additively separable externalities. In the central model studied, bidders have utility functions linear in the allocations to all bidders. In particular, allocations of goods induce additively separable negative externalities among bidders, a setting which can capture first-order downstream interactions. However, many scenarios of interest may feature significant higher order, nonlinear interactions between the bidders. For example, a bidder's utility may depend on which *groups* of other bidders get an item or on the *number* of other bidders who get certain items. A natural next step would be to characterize and solve for efficient and optimal mechanisms given more general forms of externalities.

Learning. As we saw in Chapter 3, the optimal allocation rule for single digital good auctions with externalities takes the form of a thresholding function, with the specific threshold depending on the distribution of bidders' types. However, the standard assumption that this prior distribution is common knowledge is often unrealistic. Instead, an auction designer may need to *learn* either the distribution or directly, the optimal threshold to set, from either a single-shot auction with a large number of bidders or over multiple repeated rounds of auctions. [5] begins to study the sequential learning version of this problem, and it would be useful to explore whether one can exploit the structure of externalities among bidders to more efficiently learn the optimal auction. A related direction is to extend the line of work on learning simple auctions that approximate optimal multi-item auctions [31, 13] to the present setting of interdependent valuations arising from externalities.

Approximating Optimal Mechanisms. In Chapter 4, we showed that optimally selling 2 digital goods separately guarantees at least a $1/(1 + n + \gamma)$ fraction of the optimal revenue, where n is the number of bidders and γ captures a distributional assumption on the externality parameters. It is currently unknown how tight this lower bound is. In the setting without externalities, a constant factor 2-approximation to the optimal 2-item auction has been shown [22]. In the present setting, however, complications arise from the fact that bidders have interdependent valuations and that participation constraints are endogenously determined (i.e., a bidder's "outside option" utility depends on its externality parameters). A natural next step would be to explore is whether simple auctions, possibly including bundling, can guarantee a fraction of the optimal revenue that is in constant with respect to n or the distribution of externalities, or whether some dependence inherently necessary.

Appendix A

Appendix

A.1 Welfare Maximization

A.1.1 Proof of Proposition 3.2.2

Proof. We show that the specified VCG mechanism (1) satisfies DSIC, (2) ex-post IR, and (3) uses nonnegative payments. Recall that in this setting, private types are of the form $t_i = v_i e_i - \eta_{i\leftarrow}$, for $i \in N$.

1. For all $i \in N$ and all $t_i, \hat{t}_i \in \Theta_i, t_{-i}, \hat{t}_{-i} \in \Theta_{-i}$, let us temporarily define the following quantities for ease of notation. Note the only quantity varying in the following terms is bidder i's bid, while all other parameters are fixed.

$$x_i := x_i(t_i, \hat{t}_{-i}), \hat{x}_i := x_i(\hat{t}_i, \hat{t}_{-i})$$

$$x_j := x_j(t_i, \hat{t}_{-i}), \hat{x}_j := x_j(\hat{t}_i, \hat{t}_{-i}), x_j^i := x_j(t_i = \emptyset, t_{-i}), \text{ for } j \in N \backslash i$$

$$p_i := p_i(t_i, \hat{t}_{-i}), \hat{p}_i := p_i(\hat{t}_i, \hat{t}_{-i})$$

$$W_i := v_i - \sum_{j \in N \backslash i} \eta_{j \leftarrow i}, \hat{W}_i := \hat{v}_i - \sum_{j \in N \backslash i} \eta_{j \leftarrow i}$$

$$W_j = v_j - \sum_{k \in N \backslash j} \eta_{k \leftarrow j}, \hat{W}_j := v_j - \sum_{k \in N \backslash \{j,i\}} \eta_{k \leftarrow j} - \hat{\eta}_{i \leftarrow j}, \text{ for } j \in N \backslash i$$

We show that the following expression is nonnegative, which is precisely the statement of DSIC:

$$
\begin{aligned}
&u_i\big(x(t_i,\hat{t}_{-i}),p_i(t_i,\hat{t}_{-i});t\big)-u_i\big(x(\hat{t}),p_i(\hat{t});t\big)\\
&=(x_i-\hat{x}_i)v_i-\sum_{j\in N\setminus i}\eta_{i\leftarrow j}(x_j-\hat{x}_j)\\
&\qquad\qquad\qquad+\sum_{j\in N\setminus i}(-W_j^i(x_j^i-x_j)-\eta_{j\leftarrow i}x_i+W_j^i(x_j^i-\hat{x}_j)-\eta_{j\leftarrow i}\hat{x}_i)\\
&=(x_i-\hat{x}_i)(v_i-\sum_{j\in N\setminus i}\eta_{j\leftarrow i})+\sum_{j\in N\setminus i}(W_j^i-\eta_{i\leftarrow j})(x_j-\hat{x}_j)\\
&=(\mathbb{1}\{W_i\geq 0\}-\mathbb{1}\{\hat{W}_i\geq 0\})W_i+\sum_{j\in N\setminus i}W_j(\mathbb{1}\{W_j\geq 0\}-\mathbb{1}\{\hat{W}_j\geq 0\})\\
&\geq 0.
\end{aligned}
$$

For the first equality we used the second expression of the payment rule in (3.2.5), we regrouped terms and used the definitions of W_i, W_j for the second and third equalities. The final inequality holds because

$$
\mathbb{1}\{W_i\geq 0\}-\mathbb{1}\{\hat{W}_i\geq 0\}=\begin{cases}1 \text{ iff } W_i\geq 0 \text{ and } \hat{W}_i<0\\ -1 \text{ iff } W_i<0 \text{ and } \hat{W}_i\geq 0\\ 0 \text{ else.}\end{cases}
$$

and likewise for $\mathbb{1}\{W_j\geq 0\}-\mathbb{1}\{\hat{W}_j\geq 0\}$, implies that each term in the summation is nonnegative.

2. Let t be an arbitrary type realization. Showing ex-post IR is equivalent to showing

$$
v_ix_i-\sum_{j\in N\setminus i}\eta_{i\leftarrow j}x_j-p_i\geq u_i(x(\emptyset,t_{-i}),p_i(\emptyset,t_{-i});t)=-\sum_{j\in N\setminus i}\eta_{i\leftarrow j}x_j^i.
$$

Plugging in the payment rule, we get the equivalent inequality

$$v_i x_i - \sum_{j \in N \setminus i} \left(W_j^i (x_j^i - x_j) + \eta_{j \leftarrow i} x_i \right) \geq - \sum_{j \in N \setminus i} \eta_{i \leftarrow j} (x_j^i - x_j)$$

Rearranging and regrouping terms, we get that this is equivalent to

$$W_i x_i - \sum_{j \in N \setminus i} W_j (x_j^i - x_j) \geq 0$$

Since $x_i = \mathbb{1}\{W_i \geq 0\}$, the first term is always nonnegative. The terms in the summation are likewise nonnegative since

$$x_j^i - x_j = \begin{cases} 1 \text{ iff } W_j \geq 0 \text{ and } W_j^i < 0 \\ -1 \text{ iff } W_j < 0 \text{ and } W_j^i \geq 0 \\ 0 \text{ else.} \end{cases}$$

Thus, the IR constraint is satisfied for all types t.

3. Let t be an arbitrary type realization. Note that since $\eta_{j \leftarrow i} \geq 0$, we can bound the payments given by (3.2.5) by

$$p_i \geq \sum_{j \in N \setminus i} W_j^i (\mathbb{1}\{W_j^i \geq 0\} - \mathbb{1}\{W_j \geq 0\})$$

We have that

$$\mathbb{1}\{W_j^i \geq 0\} - \mathbb{1}\{W_j \geq 0\} = \begin{cases} 1 \text{ iff } W_j^i \geq 0 \text{ and } W_j < 0 \\ -1 \text{ iff } W_j^i < 0 \text{ and } W_j \geq 0 \\ 0 \text{ else} \end{cases}$$

Matching up the cases, we get that $p_i \geq 0$, so payments are nonnegative.

□

A.2 Comparing Efficient and Optimal Allocations

We consider the special case of two bidders with uniformly distributed type parameters in Setting 1 and compare the welfare-maximizing and revenue-maximizing allocation functions.

As stated in Section 3.3.1, given a distribution function $F_{\eta_{i\leftarrow j}}$ and corresponding density function $f_{\eta_{i\leftarrow j}}$ for $\eta_{i\leftarrow j}$ on $[\underline{\eta}_{i\leftarrow j}, \bar{\eta}_{i\leftarrow j}]$, for $i \neq j \in N$, we define the distribution of $t_{i,j}$ on $[-\bar{\eta}_{i\leftarrow j}, -\underline{\eta}_{i\leftarrow j}]$ by the distribution and density functions

$$F_{i,j}(t_{i,j}) = 1 - F_{\eta_{i\leftarrow j}}(-t_{i,j}) = 1 - F_{\eta_{i\leftarrow j}}(\eta_{i\leftarrow j})$$
$$f_{i,j}(t_{i,j}) = f_{\eta_{i\leftarrow j}}(-t_{i,j}) = f_{\eta_{i\leftarrow j}}(\eta_{i\leftarrow j}).$$

Further, for all $i, j \in N$, we define the virtual value functions $\Phi_{i,j}(t_{i,j}) := t_{i,j} - (1 - F_{i,j}(t_{i,j}))/f_{i,j}(t_{i,j})$. Then for all $i \in N$ and $j \in N \backslash i$, we can express the virtual functions as

$$\Phi_{i,i}(t_{i,i}) = v_i - (1 - F_{v_i}(v_i))/f_{v_i}(v_i)$$
$$\Phi_{i,j}(t_{i,j}) = -\eta_{i\leftarrow j} - F_{\eta_{i\leftarrow j}}(\eta_{i\leftarrow j})/f_{\eta_{i\leftarrow j}}(\eta_{i\leftarrow j})$$

Suppose all the parameters v_i and $\eta_{i\leftarrow j}$, for $i \neq j \in N$ are uniformly distributed on their respective domains. The virtual value functions take the forms

$$\Phi_{i,i}(t_{i,i}) = 2v_i - \bar{v}_i$$
$$\Phi_{i,j}(t_{i,j}) = -2\eta_{i\leftarrow j} + \underline{\eta}_{i\leftarrow j}$$

The optimal allocation rule 3.3.2 then becomes

$$
\begin{aligned}
x_k(t) &= \mathbb{1}\left\{\sum_{i \in N} \phi_{i,k}(t_{i,k}) \geq 0\right\} \\
&= \mathbb{1}\left\{(2v_k - \bar{v}_k) + \sum_{i \in N \setminus k} (-2\eta_{i \leftarrow k} + \underline{\eta}_{i \leftarrow k}) \geq 0\right\} \\
&= \mathbb{1}\left\{v_k - \sum_{i \in N \setminus k} \eta_{i \leftarrow k} \geq \frac{\bar{v}_k - \sum_{i \in N \setminus k} \underline{\eta}_{i \leftarrow k}}{2}\right\}
\end{aligned}
$$

In the case of $n = 2$ bidders, bidder 1's allocation is

$$x_1(t) = \mathbb{1}\left\{v_1 - \eta_{2 \leftarrow 1} \geq \frac{\bar{v}_1 - \underline{\eta}_{2 \leftarrow 1}}{2}\right\}$$

Meanwhile, the welfare-maximizing allocation rule for bidder 1 is

$$x_1(t) = \mathbb{1}\{v_1 - \eta_{2 \leftarrow 1} \geq 0\}.$$

Thus, the revenue-maximizing allocation allocates to bidders less often than does the welfare-maximizing allocation. The optimal mechanism therefore is not in general efficient. This is illustrated in Figure 3-1, where the welfare-maximizing and revenue-maximizing allocations are shown to partition the type space for t into the regions based on bidder 1's allocation.

Bibliography

[1] Jacob Abernethy, Yiling Chen, Chien-Ju Ho, and Bo Waggoner. Low-cost learning via active data procurement. In *Proceedings of the Sixteenth ACM Conference on Economics and Computation*, EC '15, page 619–636, New York, NY, USA, 2015. Association for Computing Machinery.

[2] Daron Acemoglu, Ali Makhdoumi, Azarakhsh Malekian, and Asuman Ozdaglar. Too much data: Prices and inefficiencies in data markets. Working Paper 26296, National Bureau of Economic Research, September 2019.

[3] Alessandro Acquisti, Curtis Taylor, and Liad Wagman. The economics of privacy. *Journal of economic Literature*, 54(2):442–92, 2016.

[4] Anat R Admati and Paul Pfleiderer. Selling and trading on information in financial markets. *The American Economic Review*, 78(2):96–103, 1988.

[5] Anish Agarwal, Munther Dahleh, Thibaut Horel, and Maryann Rui. Towards data auctions with externalities. *arXiv preprint arXiv:2003.08345*, 2020.

[6] Anish Agarwal, Munther Dahleh, and Tuhin Sarkar. A marketplace for data: An algorithmic solution. In *Proceedings of the 2019 ACM Conference on Economics and Computation*, EC '19, page 701–726, New York, NY, USA, 2019. Association for Computing Machinery.

[7] Jorge Aseff and Hector Chade. An optimal auction with identity-dependent externalities. *The RAND Journal of Economics*, 39(3):731–746, 2008.

[8] Moshe Babaioff, Nicole Immorlica, Brendan Lucier, and S Matthew Weinberg. A simple and approximately optimal mechanism for an additive buyer. In *2014 IEEE 55th Annual Symposium on Foundations of Computer Science*, pages 21–30. IEEE, 2014.

[9] Alexandre Belloni, Changrong Deng, and Saša Pekeč. Mechanism and network design with private negative externalities. *Operations Research*, 65(3):577–594, 2017.

[10] Dirk Bergemann and Alessandro Bonatti. Markets for information: An introduction. *Annual Review of Economics*, 11:85–107, 2019.

[11] Kostas Bimpikis, Davide Crapis, and Alireza Tahbaz-Salehi. Information sale and competition. *Management Science*, 65(6):2646–2664, 2019.

[12] Isabelle Brocas. Optimal allocation mechanisms with type-dependent negative externalities. *Theory and Decision*, 75(3):359–387, September 2013.

[13] Yang Cai and Constantinos Daskalakis. Learning multi-item auctions with (or without) samples. In *2017 IEEE 58th Annual Symposium on Foundations of Computer Science (FOCS)*, pages 516–527. IEEE, 2017.

[14] Yang Cai, Nikhil R Devanur, and S Matthew Weinberg. A duality-based unified approach to bayesian mechanism design. *SIAM Journal on Computing*, (0):STOC16–160, 2019.

[15] Edward H Clarke. Multipart pricing of public goods. *Public choice*, pages 17–33, 1971.

[16] Constantinos Daskalakis. Multi-item auctions defying intuition? *ACM SIGecom Exchanges*, 14(1):41–75, 2015.

[17] Constantinos Daskalakis, Alan Deckelbaum, and Christos Tzamos. The complexity of optimal mechanism design. In *Proceedings of the twenty-fifth annual ACM-SIAM symposium on Discrete algorithms*, pages 1302–1318. SIAM, 2014.

[18] Changrong Deng and Sasa Pekec. Money for nothing: Exploiting negative externalities. In *Proceedings of the 12th ACM Conference on Electronic Commerce*, EC '11, page 361–370, New York, NY, USA, 2011. Association for Computing Machinery.

[19] Arpita Ghosh and Aaron Roth. Selling privacy at auction. In *Proceedings of the 12th ACM Conference on Electronic Commerce*, EC '11, page 199–208, New York, NY, USA, 2011. Association for Computing Machinery.

[20] Theodore Groves. Incentives in teams. *Econometrica: Journal of the Econometric Society*, pages 617–631, 1973.

[21] Nima Haghpanah, Nicole Immorlica, Vahab Mirrokni, and Kamesh Munagala. Optimal auctions with positive network externalities. *ACM Trans. Econ. Comput.*, 1(2), May 2013.

[22] Sergiu Hart and Noam Nisan. Approximate revenue maximization with multiple items. *Journal of Economic Theory*, 172:313–347, 2017.

[23] Sergiu Hart and Noam Nisan. Selling multiple correlated goods: Revenue maximization and menu-size complexity. *Journal of Economic Theory*, 183:991–1029, 2019.

[24] Sergiu Hart and Philip J Reny. Maximal revenue with multiple goods: Nonmonotonicity and other observations. *Theoretical Economics*, 10(3):893–922, 2015.

[25] Thibaut Horel, Stratis Ioannidis, and S. Muthukrishnan. Budget feasible mechanisms for experimental design. In Alberto Pardo and Alfredo Viola, editors, *LATIN 2014: Theoretical Informatics*, pages 719–730, Berlin, Heidelberg, 2014. Springer Berlin Heidelberg.

[26] Philippe Jehiel and Benny Moldovanu. *Allocative and Informational Externalities in Auctions and Related Mechanisms*, volume 1 of *Econometric Society Monographs*, page 102–135. Cambridge University Press, 2006.

[27] Philippe Jehiel, Benny Moldovanu, and Ennio Stacchetti. How (not) to sell nuclear weapons. *The American Economic Review*, 86(4):814–829, 1996.

[28] Philippe Jehiel, Benny Moldovanu, and Ennio Stacchetti. Multidimensional mechanism design for auctions with externalities. *Journal of Economic Theory*, 85(2):258 – 293, 1999.

[29] Vijay Krishna. *Auction theory.* Academic press, 2009.

[30] Alejandro Manelli and Daniel R Vincent. Multidimensional mechanism design: Revenue maximization and the multiple-good monopoly. 2004.

[31] Jamie Morgenstern and Tim Roughgarden. Learning simple auctions. In *Conference on Learning Theory*, pages 1298–1318, 2016.

[32] Roger B. Myerson. Optimal auction design. *Mathematics of Operations Research*, 6(1):58–73, 1981.

[33] Michael Raith. A general model of information sharing in oligopoly. *Journal of Economic Theory*, 71(1):260–288, 1996.

[34] Jean-Charles Rochet. A necessary and sufficient condition for rationalizability in a quasi-linear context. *Journal of mathematical Economics*, 16(2):191–200, 1987.

[35] Aaron Roth and Grant Schoenebeck. Conducting truthful surveys, cheaply. In *Proceedings of the 13th ACM Conference on Electronic Commerce*, EC '12, page 826–843, New York, NY, USA, 2012. Association for Computing Machinery.

[36] Curtis Taylor and Liad Wagman. Consumer privacy in oligopolistic markets: Winners, losers, and welfare. *International Journal of Industrial Organization*, 34:80–84, 2014.

[37] William Vickrey. Counterspeculation, auctions, and competitive sealed tenders. *The Journal of finance*, 16(1):8–37, 1961.

[38] Chaoli Zhang, Xiang Wang, Fan Wu, and Xiaohui Bei. Efficient auctions with identity-dependent negative externalities. In *Proceedings of the 17th International Conference on Autonomous Agents and MultiAgent Systems*, AAMAS '18, page 2156–2158, Richland, SC, 2018. International Foundation for Autonomous Agents and Multiagent Systems.

[39] Amir Ziv. Information sharing in oligopoly: The truth-telling problem. *The RAND Journal of Economics*, pages 455–465, 1993.