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Quadratic Stability of Non-Linear Systems Modeled with Norm Bounded Linear Differential Inclusions

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Abstract: In this article we present an ordinary differential equation based technique to study the quadratic stability of non-linear dynamical systems. The non-linear dynamical systems are modeled with norm bounded linear differential inclusions. The proposed methodology reformulate non-linear differential inclusion to an equivalent non-linear system. Lyapunov function demonstrate the existence of a symmetric positive definite matrix to analyze the stability of non-linear dynamical systems. The proposed method allows us to construct a system of ordinary differential equations to localize the spectrum of perturbed system which guarantees the stability of non-linear dynamical system.

Keywords: quadratic stability; Lyapunov function; gradient system of ODE's; bounded linear differential inclusion

1. Introduction

The stability analysis of dynamical systems and its applications to control theory have attracted many researchers [1–8]. The contribution to some classical problems in Lyapunov stability, LQ control and controllability has given in eguation [2] by simply exploring theorems of alternative for linear time invariant systems and the determination of qualitative properties for the possible solutions corresponding to linear time invariant systems.

For linear time invariant systems, the computation of the spectrum of the system matrix gives sufficient information for stability conditions. However, the computation of spectrum of system matrix for linear time varying systems may not determine it's stability and instability. The exponential stability of linear time varying systems with $A(t)$ assumed to vary slowly [9,10]. The necessary and sufficient conditions for exponential stability of such systems are studied in [1,7,11–13].

In 1990, Lur'e, Postinkov and others in the Soviet Union applied lyapunov theory and had established methods based on Lyapunov theory to solve practical problems arising in control engineering, particularly the problems of stability of control system having non-linearity [14]. The control system was studied with the help of matrix inequality and the stability of control system was studied with linear matrix inequality techniques. These linear matrix inequalities were reduced into a class of polynomial inequalities and then solved by hand for a small control system. For a complete detail we refer on linear matrix inequalities [15] and the references therein.

In the early 1960's the next major breakthrough came when Yakuborich, Popov, Kalman and other researchers reduced the linear matrix inequalities in Lure's theory for simple graphical criterion by using positive real (PR) lemma [14]. The graphical criterion resulted in Popov criterion, Tsypkin criterion and circle criterion are useful to apply on higher order systems in control.

By 1970's several researchers knew methods to solve linear matrix inequalities and all of these methods are analytic or closed form solutions to solve linear matrix inequalities.

In 1976's paper, both Horisberger and Belarber [16] had showed the existence of the quadratic Lyapunov functions which uses linear matrix inequalities to check control systems stability analysis. The idea of making use of computer search for Lyapunov function appears in the paper of Schultz et al. [17].

The linear matrix inequalities acts an important tool to find the quadratic Lyapunov function. However, the standard interior point methods may become not much effective with the increase of modes. An interactive gradient descent algorithm is proposed in [18] which converges to quadratic Lyapunov function in finite steps. However, the convergence rate of gradient descent algorithm proposed in [18] could be imposed by introducing some randomness. The existence of Lyapunov function is the sufficient condition for the stability of linear time invariant system [19].

A various number of problems appearing in systems and control are reduceable to standard convex and quasi convex problems involving linear matrix inequalities [20]. These linear matrix inequalities problems possesses analytical solution up to a few special cases but fortunately such problems are solveable with existing numerical techniques. These inequalities appears in the form of Lyapunov or algebraic Riccati inequalities which signify the computational cost of control theory based on the top of solutions of algebraic Riccati equations to a theory based on the solution of Lyapunov inequalities.

Overview of the Article

Section 2 provides the preliminaries of our article. In particular, we present the definitions of positive definite and positive semi definite matrices, negative definite and negative semi definite matrices, matrix inequality and linear matrix inequality.

In Section 3 we provide an equivalent linear time invariant system with the bounded perturbation. Furthermore, in this section we discuss the quadratic stability of linear time invariant system. In Section 4 of this article, we present a gradient system of ordinary differential equations to relocate the smallest eigenvalue λ_1 from the spectrum of the perturbed matrix $(\hat{A} + D + \epsilon E(t)))$ with $\hat{A} = P + (A + B(I - D)^{-1}C^T$.

Section 5 of our article is devoted on the localization of eigenvalues λ_1, λ_2 from the spectrum of perturbed matrix. Furthermore an optimization problem is formulated and solved with the help of a gradient system of ordinary differential equations to discuss the quadratic stability.

In Section 6 we give the conclusion of our article.

2. Preliminaries

Definition 1. *A symmetric matrix $M^{n,n}$ is called a positive definite matrix if $X^T MX > 0,\ \forall X \neq 0 \in R^{n,1}$.*

Definition 2. *A symmetric matrix $M^{n,n}$ is called positive semi definite matrix if $X^T MX \geq 0,\ \forall X \in R^{n,1}$.*

Definition 3. *A symmetric matrix $M^{n,n}$ is called negative definite matrix if $X^T MX < 0,\ \forall x \neq 0 \in \mathbb{R}^{n,1}$.*

Definition 4. *A symmetric matrix $M^{n,n}$ is called negative semi-definite matrix if $X^T MX \leq 0,\ \forall X \in R^{n,1}$.*

Definition 5. *A matrix inequality $F : \mathbb{R}^m \to S^{n,n}$ in $X \in \mathbb{R}^{m,1}$ is defined as $F(X) := F_0 + \sum_{i=1}^{n} f_i(x)F_i \leq 0$, with $X = (x_1, x_2, x_3, ..., x_n)^T,\ F_0 \in S^{n,n},\ F_i \in \mathbb{R},\ i = 1 : n$.*

Definition 6. *A linear matrix inequality $F : \mathbb{R}^m \to S^{n,n}$ in the variable $X \in \mathbb{R}^{m,1}$ is defined as $F(X) := F_0 + \sum_{i=1}^{n} x_i F_i \leq 0$, with $X = (x_1, x_2, x_3, ..., x_n)^T,\ F_i \in S^{n,n},\ i = 0 : n$.*

Definition 7. *The solution $x^*(t)$ of dynamical system $x^*(t) = f(x,t)$ is stable if $\exists\ \delta(\epsilon, t_0)$ for $\epsilon > 0$ such that all solutions $x(t_0) - x^*(t_0) < \delta$ satisfies $x(t) - x^*(t) < \epsilon$, $\forall t \geq 0$.*

Definition 8. *The solution $x^*(t)$ of dynamical system is asymptotically stable (weak) if it is stable and $x(t) - x^*(0) \to 0$ as $t \to \infty$ and $x(t_0) - x^*(t_0)$ is small enough.*

Definition 9. *The dynamical system $x^*(t) = f(x,t)$ is asymptotically stable (strong) if $x(t) \to 0$ as $t \to \infty$ and $x(t_0)$ is small enough.*

Definition 10. *A square matrix $M \in \mathbb{R}^{n,n}$ is stable if $Re(\lambda_i(M)) < 0$ for all i and $\lambda_i(M)$ denotes the spectrum of the matrix M.*

Definition 11. *A linear time invariant system $x^*(t) = Ax(t)$ with $x(t_0) = x_0 \neq 0$ and $t > t_0$ is called asymptotically stale if the matrix M is a stability matrix.*

Definition 12. *A Lyapunov function $V(x)$ or generalized energy to a system $\frac{dx}{dt} = f(x)$ is a scalar valued function satisfying:*

(i) $V(x) : \mathbb{R}^{n,1} \to \mathbb{R}$, *a scalar function*

(ii) $V(x) > 0$, *positive definite*

(iii) $\frac{d}{dt}V(x) < 0$, *dissipativity.*

3. Linear Time Invariant System with Bounded Perturbation

Consider a linear time invariant system with a non-linear bounded perturbation $\Delta(t)$ as,

$$\begin{cases} \dot{x}(t) = Ax(t) + Bp, \quad x(0) = x_0 \\ q = Cx(t) + Dp, \\ p = \Delta(t)q, \ \|\Delta(t)\|_2 \leq 1. \end{cases} \tag{1}$$

In Equation (1), the perturbation $\Delta(t)$ is a time varying matrix so that the vector 2-norm of output vector p is bounded above by vector 2-norm of input vector q. If the 2-norm of perturbation matrix is unify, then both p and q are equal. In such case, we have following equivalent system to linear time invariant system described in Equation (1).

3.1. Equivalent System

The system equivalent to linear time invariant system in Equation (1) is of the form

$$\dot{x}(t) = (A + B(I-D)^{-1}C)x, \quad x(0) = x_0. \tag{2}$$

Next, we discuss the quadratic stability of linear time invariant system described in Equation (2).

3.2. Quadratic Stability

The linear time invariant system in Equation (2) is quadratically stable if there exists a positive definite matrix P such that $P = P^T > 0$ and satisfying the matrix inequality

$$(A + B(I-D)^{-1}C)P + P(A + B(I-D)^{-1}C)^T < 0,$$

$$\Longleftrightarrow$$

$$x^T\left((A + B(I-D)^{-1}C)P + P(A + B(I-D)^{-1}C)^T\right)x < 0,$$

$$\Longleftrightarrow$$

$$x^T\left((A+B(I-D)^{-1}C)P]x < -x^T[P(A+B(I-D)^{-1}C)\right)x. \quad (3)$$

In Equation (3), the expression $x^T\left((A+B(I-D)^{-1}C)P\right)x$ is negative if

$$(i) \quad x^T\left((A+B(I-D)^{-1}C)^T\right)x = 0,$$

$$(ii) \quad x^T\left((A+B(I-D)^{-1}C)^T\right)x > 0.$$

Unfortunately it's highly possible that

$$(iii) \quad x^T\left((A+B(I-D)^{-1}C)^T\right)x < 0.$$

In turn, this implies that the perturbed matrix $P(A+B(I-D)^{-1}C)^T < 0$.
Furthermore, all the eigenvalues of the matrix $P(A+B(I-D)^{-1})^T$ are negative, that is,

$$\lambda_i(P(A+B(I-D)^{-1}C)^T < 0 \quad \forall i.$$

Next, we perturbed the spectrum of $P(A+B(I-D)^{-1}C)^T$ by using an ordinary differential equation so that $\lambda_i(P+(A+B(I-D)^{-1}C)^T) > 0 \quad \forall i$.

4. A System of ODE's to Shift Smallest Eigenvalue λ_1

In this section, the main aim is to shift the smallest eigenvalues $\lambda_1(t)$ from the spectrum of perturbed matrix $((\hat{A}+D+\epsilon E(t))$ where $\hat{A} = (P+(A+B(I-D)^{-1}C)^T)$ and $\epsilon > 0$, a small positive fixed parameter and the matrix D is a diagonal matrix such that $(\hat{A}+D)$ possesses unit entries along it's main diagonal. The matrix $E(t)$ is such that it has all zero entries along it's main diagonal. Furthermore,

$$\|E(t)\|_F^2 = \sum_{ij} e_{ij}^2 \leq 1, \ \forall t$$

such that $\lambda_1(t)$ increases.

4.1. Formulation of Optimization Problem

We aim to compute the direction $Z = \dot{E}(t)$ such that $\lambda_1(t)$ has maximum growth that is $\frac{d}{dt}(\lambda_i(t)) > 0$. For this purpose, we need the matrix $\epsilon E(t)$ for $\epsilon > 0$. From the eigenvalue problem, we have that

$$(\hat{A}+D+\epsilon E(t))x(t) = \lambda(t)x(t), \quad (4)$$

where $x(t)$ is an eigenvector corresponding to $\lambda(t)$. Furthermore, $\|x(t)\|_2 \leq 1$.
This implies that

$$(x^*(\hat{A}+D+\epsilon E(t)) = \lambda(t)x^*(t). \quad (5)$$

Upon differentiating Equation (4) w.r.t. time t, we get

$$(\hat{A}+D+\epsilon E(t))\frac{d}{dt}x(t) + \epsilon\frac{d}{dt}(E(t))x(t) = \frac{d}{dt}(\lambda(t))x(t) + \lambda(t)\frac{d}{dt}(x(t)).$$

Multiplying with $x^*(t)$ throughout gives,

$$x^*(t)(\hat{A}+D+\epsilon E(t))\frac{d}{dt}x(t) + \epsilon x^*(t)\frac{d}{dt}(E(t))x(t) = \frac{d}{dt}(\lambda(t))x^*(t)x(t) + \lambda(t)x^*(t)\frac{d}{dt}(x(t)).$$

Since,

$$x^*(t)x(t) = \langle x(t), x(t)\rangle = \|x(t)\|_2^2 = 1,$$

This implies that,

$$x^*(t)(\hat{A}+D+\epsilon E(t))\frac{d}{dt}x(t)+\epsilon x^*(t)\frac{d}{dt}(E(t))x(t)=\frac{d}{dt}(\lambda(t))+\lambda(t)x^*(t)+\frac{d}{dt}(x(t)).$$

Thus,

$$\lambda(t)x^*(t)\frac{d}{dt}(x(t))+\epsilon x^*(t)\frac{d}{dt}(E(t)x(t))=\frac{d}{dt}(\lambda(t))+\lambda(t)x^*(t))\frac{d}{dt}(x(t)). \tag{6}$$

In views of Equation (5), we get

$$\lambda(t)x^*(t)=x^*(t)(\hat{A}+D+\epsilon E(t)).$$

Finally, Equation (6) takes the form

$$\frac{d}{dt}(\lambda(t))=\epsilon x^*(t)\frac{d}{dt}(E(t))x(t). \tag{7}$$

While taking $x^*(t)\frac{d}{dt}(x(t))=0$ in Equation (6), let $Z=\frac{d}{dt}(E(t))=\dot{E}(t)$, as optimization problem.

4.2. Optimization Problem

The following optimization problem allows us the direction $Z=\dot{E}(t)$ such that the solution of the system of ODE's obtained by solving optimization problem indicates the sufficient growth of the smallest eigenvalue $\lambda(t)$.

$$\begin{gathered} max(x_1^*(t)Zx_1(t)) \\ Subject\ to \\ \langle Z,E(t)\rangle=0 \\ diag(Z)=0. \end{gathered} \tag{8}$$

Here, $x_1(t)\in\mathbb{R}^{n,1}$ is an eigen vector associated to eigenvalue $\lambda_1(t)$. The symbol $*$ denotes the complex conjugate transpose for (complex) matrix. The solution to optimization problem presented in Equation (8) is obtained as following.

4.3. Lemma 4.2.1

Let $E(t)$ be a non-zero matrix with

$$\|E(t)\|_F\leq\sqrt{\sum_{ij}e_{ij}^2}\leq 1,$$

and let $x_1(t),x_1^*(t)$ be non-zero eigen vectors corresponding to eigen value $\lambda_1(t)$. The solution Z to optimization problem in Equation (8) is

$$Z=Proj(x_1(t)x_1^*(t))-\langle Proj(x_1(t)x_1^*(t),E(t)\rangle E(t), \tag{9}$$

with $Proj(\cdot)$ is the projection of Z onto manifold $E(t)$.

4.4. The System of ODE's

The solution

$$Z=Proj(x_1(t)x_1^*(t))-\langle Proj(x_1(t)x_1^*(t),E(t)\rangle E(t),$$

of optimization problem in Equation (8) suggest following system of ODE's on manifold $E(t)$,

$$\dot{E}(t) = Proj(x_1(t)x_1^*(t)) - \langle Proj(x_1(t)x_1^*(t), E(t)\rangle. \tag{10}$$

4.5. Characterization of ODE's

The solution of ODE's in Equation (10) has following properties:

$$(i) \quad \frac{d}{dt}(\lambda_1(t)) > 0,$$

$$(ii) \quad \dot{E}(t) = 0 \iff \frac{d}{dt}(\lambda_1(t)) = 0,$$

$$(iii) \quad \frac{d}{dt}(\lambda_1(t)) = 0 \iff E(t) \propto Proj(x_1(t)x_1^*(t)).$$

5. A System of ODE's to Shift $\lambda_1(t)$, $\lambda_2(t)$

In this section, we aim to shift simultaneously eigen values $\lambda_1(t)$ and $\lambda_2(t)$ from the spectrum of the perturbed matrix $(\hat{A} + D + \epsilon E(t))$ such that $\lambda_1(t) > 0$, $\lambda_2(t) > 0$. Here, $\hat{A} = P(A + B(I - D)^{-1})C)^T$ and $\epsilon > 0$. The matrix D is a diagonal matrix such that $(\hat{A} + D)$ possesses unit entries along it's main diagonal. The matrix $E(t)$ is such that it has all zero entries along it's main diagonal.

5.1. Optimization Problem

The following optimization problem allow us to compute the direction $Z = \dot{E}(t)$ such that the solution of the system of ODE's obtained after solving optimization problem gives maximum growth to increase $\lambda_1(t)$ and $\lambda_2(t)$. The optimization problem to increase both eigen values $\lambda_1(t)$ and $\lambda_2(t)$ is

$$\begin{gathered} max(x_1^*(t)Zx_1(t)) \\ Subject\ to \\ x_2^*(t)Zx_2(t) = x_1^*(t)Zx_1(t) \\ \langle Z, E(t)\rangle = 0 \\ diag(Z) = 0. \end{gathered} \tag{11}$$

Next, we give the solution to optimization problem in Equation (11).

5.2. System of ODE's

The solution to optimization in Equation (11) is given by the system of ODE's

$$\dot{E}(t) = (1-\mu)x_1^*(t)x_1(t) + \mu x_2^*(t)x_2(t) - \mu\{\langle x_1^*(t)x_1(t) - x_2^*(t)x_2(t), E(t)\rangle - \langle x_1^*(t)x_1(t), E(t)\rangle\}. \tag{12}$$

The system of ODE's in Equation (12) can be written as a function $C(t)$, where

$$C(t) = \begin{pmatrix} \frac{1-a_{11}}{\epsilon} & 0 & \dots & \dots & 0 \\ e_{21}(t) & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ e_{n1}(t) & \dots & \dots & e_{nn}(t) & \frac{1-a_{nn}}{\epsilon} \end{pmatrix}.$$

For ϵ sufficiently large enough, we have $diag(\hat{A} + D + \epsilon E(t)) = 1$ with

$$\hat{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}; \quad E(t) = \begin{pmatrix} e_{11}(t) & e_{12}(t) & \dots & e_{1n}(t) \\ e_{21}(t) & e_{22}(t) & \dots & e_{2n}(t) \\ \vdots & & \ddots & \vdots \\ e_{n1}(t) & e_{n2}(t) & \dots & e_{nn}(t) \end{pmatrix}.$$

For $\epsilon_0 \leq \epsilon$, we have that $\epsilon_0 = diag(\hat{A} - I) \leq \epsilon$. We fix $e_{11}(t), e_{12}(t), \dots, e_{nn}(t)$ and for $\epsilon >> \epsilon_0$, we have

$$\hat{A} + \epsilon E(0) = \begin{pmatrix} a_{11} + \epsilon e_{11}(t) & a_{12} + \epsilon e_{12}(t) & \dots & a_{1n} + \epsilon e_{1n}(t) \\ a_{21} + \epsilon e_{21}(t) & a_{22} + \epsilon e_{22}(t) & \dots & a_{2n} + \epsilon e_{2n}(t) \\ \vdots & & \ddots & \vdots \\ a_{n1} + \epsilon e_{n1}(t) & a_{n2} + \epsilon e_{n2}(t) & \dots & a_{nn} + \epsilon e_{nn}(t) \end{pmatrix}.$$

The perturbed matrix $\hat{A} + \epsilon E(0))$ has the form

$$\hat{A} + \epsilon E(0) = \begin{pmatrix} 1 & a_{12} + \epsilon e_{12}(t) & \dots & \dots & a_{1n} + \epsilon e_{1n}(t) \\ a_{21} + \epsilon e_{21}(t) & 1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & a_{nn} + \epsilon e_{nn}(t) \\ a_{n1} + \epsilon e_{n1}(t) & \dots & \dots & a_{nn} + \epsilon e_{nn}(t) & 1 \end{pmatrix}$$

Since, $e_{11}(t) = \frac{1-a_{11}}{\epsilon}, e_{12}(t) = \frac{1-a_{22}}{\epsilon}, \dots, e_{nn}(t) = \frac{1-a_{nn}}{\epsilon}$.
Thus the matrix $E(0)$ has the structure

$$E(0) = \begin{pmatrix} \frac{1-a_{11}}{\epsilon} & e_{12}(t) & \dots & \dots & e_{1n}(t) \\ e_{21}(t) & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & e_{nn}(t) \\ e_{n1}(t) & \dots & \dots & e_{nn}(t) & \frac{1-a_{nn}}{\epsilon} \end{pmatrix}.$$

The matrix $E(0)$ can be decomposed into the upper triangular matrix $C^T(0)$ and the lower triangle matrix $C(0)$ as,

$$C^T(0) = \begin{pmatrix} \frac{1-a_{11}}{\epsilon} & e_{12}(t) & \dots & \dots & e_{1n}(t) \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & e_{nn}(t) \\ 0 & \dots & \dots & 0 & \frac{1-a_{nn}}{\epsilon} \end{pmatrix}; \quad C(0) = \begin{pmatrix} \frac{1-a_{11}}{\epsilon} & 0 & \dots & \dots & 0 \\ e_{21}(t) & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ e_{n1}(t) & \dots & \dots & e_{nn}(t) & \frac{1-a_{nn}}{\epsilon} \end{pmatrix}.$$

Furthermore,

$$2\|C(0)\|_2^2 + \sum_i \frac{(1-a_{ii})^2}{\epsilon^2} = 1. \tag{13}$$

Similarly, the matrix $E(t)$ has the structure

$$E(t)=\begin{pmatrix} \frac{1-a_{11}}{\epsilon} & e_{12}(t) & \dots & \dots & e_{1n}(t) \\ e_{21}(t) & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & e_{nn}(t) \\ e_{n1}(t) & \dots & \dots & e_{nn}(t) & \frac{1-a_{nn}}{\epsilon} \end{pmatrix},$$

and

$$\|C(t)\| = \frac{1}{\sqrt{2}}\left(1-\sum_i \frac{(1-a_{ii})^2}{\epsilon^2}\right)^{\frac{1}{2}}. \tag{14}$$

Remark 1. *The Frobenius norm of* $C(0)$ *and* $C(t)$ *are exactly same, that is,*

$$\|C(0)\|_F = \|C(t)\|_F.$$

To minimize the eigen values $\lambda_1(t)$ and $\lambda_2(t)$, we compute matrix $\dot{E}(t)$ while taking projection of Z onto $E(t)$. This gives following result for $\dot{E}(t)$

$$E(t)=\begin{pmatrix} 0 & \dot{e}_{12}(t) & \dots & \dots & \dot{e}_{1n}(t) \\ \dot{e}_{21}(t) & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \dot{e}_{nn}(t) \\ \dot{e}_{n1}(t) & \dots & \dots & \dot{e}_{nn}(t) & 0 \end{pmatrix}.$$

The matrices $\dot{C}(t)$ and $\dot{C}^T(t)$ are obtained as

$$\dot{C}(t)=\begin{pmatrix} 0 & \dot{e}_{12}(t) & \dots & \dots & \dot{e}_{1n}(t) \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \dot{e}_{nn}(t) \\ 0 & \dots & \dots & 0 & 0 \end{pmatrix}; \quad \dot{C}^T(t)=\begin{pmatrix} 0 & 0 & \dots & \dots & 0 \\ \dot{e}_{12}(t) & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \dot{e}1n(t) & \dots & \dots & \dot{e}_{nn}(t) & 0 \end{pmatrix}.$$

The computation of $\dot{C}(t)$ gives the increase of $\lambda_{min}(t)$ in the following optimization problem,

$$\begin{array}{c} \frac{d}{dt}(\lambda_{min}(t)) = max(x^*(t)\dot{E}(t)x(t)) \\ subject\ to \\ \langle E(t), \dot{E}(t)\rangle = 0 \\ diag(Z) = 0. \end{array} \tag{15}$$

The solution of optimization problem in Equation (15) is given by ODE $\dot{E}(t)$ as,

$$\dot{E}(t) = Proj(x(t)x^*(t)) - \langle Proj(x(t)x^*(t), E(t)\rangle E(t),$$

with

$$Proj(x(t)x^*(t)) = x(t) - diag(x_1^2, \dots x_n^2)$$

and is given by

$$\begin{pmatrix} 0 & x_1x_2 & \dots & \dots & x_1x_n \\ x_2x_1 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & x_nx_n \\ x_nx_1 & \dots & \dots & x_nx_n & 0 \end{pmatrix}.$$

The above matrix can be decomposed into the upper triangular matrix $B(t)$ and the lower triangular matrix $B^T(t)$ as

$$B(t) = \begin{pmatrix} 0 & x_1x_2 & \dots & \dots & x_1x_n \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & x_nx_n \\ 0 & \dots & \dots & 0 & 0 \end{pmatrix},$$

$$B^T(t) = \begin{pmatrix} 0 & 0 & \dots & \dots & 0 \\ x_2x_1 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ x_nx_1 & \dots & \dots & x_nx_n & 0 \end{pmatrix}.$$

Remark 2. *The matrix* $B(t) = (\hat{A} + D + \epsilon E(t))$*. Finally, the solution of the optimization problem in Equation* (11) *takes the form*

$$\dot{E}(t) = (1-\mu)Proj(x_1(t)x_1^*(t)) + \mu Proj(x_2(t)x_2^*(t)) - \gamma E(t). \quad (16)$$

The solution $\dot{E}(t)$ in Equation (16) is given by Euler's method

$$E_{n+1} = E_n + h\dot{E}_n.$$

Thus, finally we compute all positive eigen values from the eigen value problem

$$(\hat{A} + D + \epsilon E(t))x(t) = \lambda(t)x(t).$$

5.3. Stability of Equivalent System

In this section, the aim is to show that the equivalent system in Equation (2) is stable and furthermore, the linear time invariant system in Equation (1) is also stable.

Since, $\lambda_i(\hat{A} + D + \epsilon E(t)) > 0 \quad \forall i$. This implies that the equivalent system in Equation (2) is stable if $\exists$ a positive definite matrix $P = P^T > 0$ satisfying the linear matrix inequality

$$x^T \left((A + B(I-D)^{-1}C)P \right) x < 0,$$

$$\Longleftrightarrow$$

$$(A + B(I-D)^{-1})P < 0,$$

$$\Longleftrightarrow$$

$$AP + B(I-D)^{-1}C)P < 0.$$

6. Conclusions

The non-linear differential inclusion is a strategy which uses the system model to control it and removes the gain scheduling and improves the performance of the non-linear systems under consideration. The non-linear differential inclusion controller is frequently used to achieve high maneuverability associated with high level accuracy. The proposed methodology based on ordinary differential equations:

- discuss the stability analysis of non-linear dynamical systems in control model with norm bounded linear differential inclusions.
- enable us to localize the spectrum which is hindrance in the stability such that the perturbed systems has spectrum in the left half plane which results in the stability of non-linear dynamical system.

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