PROBLEM SET 9
Post date: Thursday, April 27th
Due date: Thursday, May 4th

1. Time scale for variation in microlensing

   Consider the microlensing scenario described in the April 20 lecture. ("Microlensing" is the limit in which the images cannot be separated; one just examines the total intensity produced by the sum of the two images.) Imagine that the source is a star in the Large Magellanic Cloud (about 100 kiloparsecs away; 1 pc = 3.26 light years = 3.1×10^{13} km); imagine that the lens is an unseen object of about one solar mass in the galaxy, roughly 10 kiloparsecs away. If the lens is orbiting in the galactic potential with a typical galactic orbit speed $v_{\mathrm{orb}} \sim 200$ km/sec, what is the timescale over which you expect the light curve to vary? In other words, roughly how long will it take for the light curve to brighten and then dim?

   For students who could not attend that lecture, see Hartle's textbook, Chapter 11.

2. Proper motion distance

   Using the definition of proper motion distance given in Carroll [Eq. (8.124)], compute $d_M(z)$. Your final result should be similar in form to Eq. (8.123) of Carroll and should confirm the rule that

   $$d_L(z) = (1+z)d_M(z) = (1+z)^2 d_A(z) \, ,$$

   where $d_L(z)$ is luminosity distance and $d_A(z)$ is angular diameter distance.

3. Particle horizons in the early universe

   [Note: Carroll Sec. 8.8 provides important background to this and the next two problems. A very readable and even more comprehensive discussion is given in Chapter 3 of *Cosmological Inflation and Large Scale Structure*, by Andrew R. Liddle and David H. Lyth (Cambridge University Press, 2000), and in Chapter 13 of *An Introduction to Modern Cosmology*, by Andrew R. Liddle (Wiley, 2003).]

   The early universe was so hot and dense that all matter existed in the form of a plasma. As such, photons did not stream freely, but rather scattered repeatedly off the ionized charges in the plasma. "Recombination" is the name given to the point at which the universe cooled sufficiently that electrons and protons could bind into atoms; at this point, photons can stream freely. The cosmic microwave background (CMB) dates from the recombination epoch, which is now known to have occurred at a redshift $z_r \simeq 1200$.

Consider a $k = 0$ FRW cosmology which is radiation dominated prior to recombination and is matter dominated afterwards[1]. Let $r_H(t_r)$ be the maximum coordinate distance that a photon can travel from the big bang to the epoch of recombination; this is the "comoving horizon radius" at recombination. (More properly, it is the comoving *particle* horizon radius, to be contrasted with the *event* horizon we will define in about two weeks.) Only points that are within each other's particle horizons are in causal contact.

Let $r_{\text{obs}}(t_r)$ be the coordinate distance from recombination to an observer on earth; thus $2 \times r_{\text{obs}}(t_r)$ is the coordinate separation between patches of the CMB on opposite parts of the sky.

(a) Compute $r_H(t_r)$ and $r_{\text{obs}}(t_r)$.

The CMB is extremely isotropic; its temperature varies by roughly a part in 10^5 over the sky.

(b) Given the isotropy of the CMB, why is the smallness of the ratio r_H/r_{obs} so puzzling? (If you find that this ratio is not small, you might want to revise your answer to part a!) This is known as the "horizon problem".

(c) Compute the angular scale θ_c such that sources separated by $\theta < \theta_c$ as seen today were in causal contact prior to $t = t_r$, but sources separated by $\theta > \theta_c$ were not.

Some background: For the remaining two problems, we repeatedly need to refer to temperature and energy scales commonly used in early universe cosmology. Since this subject grew out of particle physics, the units of particle physics describe its relevant quantities — e.g., temperatures are usually given in MeV or GeV. This means that temperatures are really being reported in the combination $k_B T$, where k_B is Boltzmann's constant.

The following conversions and definitions will prove useful in problems 4 and 5:

$$
\begin{aligned}
T_0 &\equiv \text{Mean temperature of universe today (CMB temperature)} = 2.73 \, \text{K} \\
k_B &\equiv \text{Boltzmann's constant} = 8.6 \times 10^{-11} \, \text{MeV/K} \\
\hbar &= 6.6 \times 10^{-22} \, \text{MeV sec} \\
\sigma &\equiv \text{Stefan-Boltzmann constant} = \frac{\pi^2 k_{\text{B}}^4}{60 \hbar^3 c^2} \, .
\end{aligned}
$$

The last constant comes in handy when converting between temperature and energy density for radiation: A gas of photons[2] at temperature T has an energy density

$$\rho = 4(\sigma/c)T^4 \, .$$

Finally, don't forget that the temperature of radiation redshifts: Radiation whose temperature is T today was at temperature $a(t)T/a(0) = (1+z)T$ at an earlier time t.

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[1]In fact, the universe became matter dominated somewhat prior to recombination: The redshift at which matter and radiation contribute equal to the mean density of the universe is $z_{\text{eq}} \simeq 3000$. The error we make in approximating $z_{\text{eq}} \simeq z_r$ is small enough for the purposes of this and the next two problems that we may safely consider the universe to be radiation dominated prior to recombination, and matter dominated afterwards.

[2]If you read the texts by Liddle, you will discover that we are neglecting neutrinos and other relativistic particles in this analysis. Our conclusions will thus be off by a factor of order unity; fine for our purposes.

4. The horizon problem resolved: The inflationary universe

Suppose that at some very early epoch, $t_1 \leq t \leq t_2$ (where $t_1 \ll t_2 \ll t_r$), the universe resides in a "false vacuum" state: A scalar field ϕ fills spacetime and provides an effective vacuum energy $\rho_{\text{vac}} \simeq V(\phi)$. (For more details about how a scalar field of this type is characterized and evolves, see Carroll pp. 369 – 370.) This vacuum energy acts like a cosmological constant $\Lambda = 8\pi\rho_{\text{vac}}$. Up to factors of order unity, we may write $\rho_{\text{vac}} \simeq (4\sigma/c)T_{\text{vac}}^4$, where $k_B T_{\text{vac}} \sim 10^{15}\,\text{GeV}$. (Note: *Don't* take this formula for density as a function of temperature too seriously! The vacuum doesn't actually have a temperature, and it is certainly not a gas of photons. This formula is designed to give this scalar field an energy density that is comparable to the scale of grand unified theories. We are recycling the photon energy density formula because it gives us a nice way of understanding the energy scales at which this field operates.)

As the universe expands, the potential slowly evolves. Around time t_2, the scalar field decays into standard model particles, and the stress energy tensor is no longer dominated by $V(\phi)$. This particles provide the matter and radiation content for our universe; it is then radiation dominated until recombination, and matter dominated thereafter.

We will take "slowly evolving" to mean that we can regard $V(\phi)$ as a constant from $t_1 \leq t \leq t_2$, and assume that $V(\phi) \simeq 0$ for $t > t_2$. (See Carroll Sec. 8.8, Liddle and Lyth Chapter 3, or Liddle Chapter 13 for further discussion.)

(a) By inserting appropriate factors of G and c, and using the fact that 1 MeV $= 1.6 \times 10^{-12}$ gm cm^2/sec^2, compute the value of the effective cosmological constant Λ in inverse seconds squared.

(b) Repeat the calculation in Problem 3, part (a). Find the value of $N_e \equiv \Delta t(\Lambda/3)^{1/2}$ (where $\Delta t = t_2 - t_1$) such that $r_{\text{obs}}(t_r)/r_H(t_r) = 1$. ($N_e$ is often referred to as the number of "e-foldings" of expansion for reasons that are hopefully obvious.) Estimate t_2. (You may approximate $t_1 \simeq 0$ for this estimate).

(c) What is the spatial expansion factor, $a(t_2)/a(t_1)$, during this "inflationary" epoch?

(d) Recalculate the angular scale θ_c in this inflationary universe.

5. The flatness problem

(a) Use the Friedmann equations to derive a general expression relating values of Ω at different times — i.e., Ω_1 at t_1 (corresponding to scale factor a_1) and Ω_2 at t_2 (corresponding to a_2). Do this calculation for a matter dominated universe, and for a radiation dominated universe. Express your answer in terms of $\Omega - 1$ and the scale factor. Do not take into account the inflationary physics of Problem 4.

(b) Observations today tell us that the universe is flat with about 1% accuracy; i.e., $\Omega = 1 \pm 0.01$. Continuing to neglect inflation, estimate the value of $\Omega - 1$ at the epoch of nucleosynthesis ($T_N \sim 1$ MeV), and at the presumed epoch of grand unification ($T_{\text{GUT}} \sim 10^{15}$ GeV). Given that the universe is flat with at least 1% accuracy today, how close to flat must it have been in these earlier epochs?

The incredible precision with which the universe must have been flat is called the "flatness" problem. The early universe's initial conditions must either have been tuned

with incredible precision to give the universe we see today, or some mechanism must exist to drive the universe into a nearly flat state.

(c) Repeat part (a) using the inflationary universe of Problem 4.

(d) How does inflation resolve the flatness problem? If the universe passes through enough "*e*-foldings" to cure the horizon problem of Problem 4, is it likely to leave the universe flat enough to account for the observed spatial curvature today?