SOME ASPECTS OF THE EARLY HISTORY OF MATHEMATICS

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Submitted in Partial Fulfillment

of the Requirements for the

Degree of Bachelor of Science

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

June, 1956

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ABSTRACT

I have enumerated some of the reasons for studying the history of mathematics and have shown examples of the misconceptions about the subject which many people have. Then I investigated the definition of mathematics, and on the basis of that definition, I tried to list some facets of mathematics which can be measured to enable a historian to evaluate the degree of progress in its development.

To illustrate these features, I considered, briefly, the history of algebra from its origins to the end of the sixteenth century, showing how it progressed and retrogressed. Emphasis was given in proportion to the need thereof, additional detail being given where there was greater prevalence of erroneous ideas. I concluded with a short history of Hindu algebra paralleling the Western work.

TABLE OF CONTENTS

I. REASONS FOR THE STUDY OF THE HISTORY OF MATHEMATICS

INTRODUCTION

General history is studied for many reasons: to give a clearer understanding of the present situation and how it developed, to testify to the glory of the human species in general and a particular segment of humanity in particular, to understand the bases of various kinds of development in the hope that one will be able to apply the same or similar techniques to further progress, and to have a panoramic view of a scene far too complicated to study in its entirety in any great detail. Furthermore, one may study history critcally, first, because one is interested in the study, and second, because one soon discovers the enormous number of popular misconceptions which are firmly entrenched in men's minds and the unawareness or avoidance which many historians have of interesting and valuable information.

The above apply also to the study of the history of mathematics. The present situation in mathematics is comprehensible only to a highly intelligent person with a long specialized training. The pace at which mathematics is being expanded is startling, and the gap between what, on the one hand, laymen and school boys understand of it and its frontiers, on the other hand, is widening so greatly that most well-informed people other than professional mathematicians have no more than a vague notion as to what mathematics is.

As to mathematics being a testament to the achievements of humanity, I quote George Sarton:

> "The history of mathematics is exhilarating because it unfolds before us the vision of an endless series of victories of the human mind, victories without counterbalancing failures, that is, without dishonorable and humiliating ones, and without atrocities."[1]

The techniques of one era have been used in subsequent ones, and an awareness of what had already been done would have contracted the period of mathematical development from 1800 B.C.E. to 1600 C.E. to a fraction of the time between those dates. That this need for study of past events is much more dramatically illustrated by that period than in recent times is due to the fact that in the past several hundred years we have been much more aware of historical facts, and the accessibility of the knowledge of all eras has, in modern times, been much increased.

Perhaps an illustration would be in order. We quote Tobias Dantzig:

> "In the method of exhaustion, Archimedes possessed all the elements essential to an infinitesimal analysis... It is sufficient to say here that the idea of limit as conceived by Archimedes was adequate for the development of the calculus of Newton and Leibniz and that it remained practically unchanged until the days of Weierstrass and Cantor."[2]

Let us look at the dates of these men.[3] Archimedes lived from 287? to 212 B.C.E.; Leibniz from 1646 to 1716, and Newton from 1642 to 1727.

1. George Sarton, The Study of the History of Mathematics. p.13.
2. Tobias Dantzig, Number the Language of Science. p.130.sqq.
3. Webster's New Collegiate Dictionary, p.1011.sqq

For a final argument for the study of the history of mathematics, I quote again George Sarton:

> "The mathematical universe is already so large and diversified that it is hardly possible for a single mind to grasp it... So much energy would be needed for grasping it that there would be none left for creative research. A mathematical congress of today reminds one of the Tower of Babel... This illustrates in another way the increasing need for mathematical surveys, historical analyses, and philosophical elaborations."[1]

GENERAL MISCONCEPTIONS

The neglect of mathematics in general history books and the consequent distortions of the facts which are believed by a significant number of enlightened readers are very considerable, far more so than would be anticipated by students of the history of mathematics. If we consider the history of mathematics worth studying at all, considering this neglect and distortion, our project is, *a fortiori*, worthwhile. I want to demonstrate the validity of my complaint upon the general history books by two examples, one the work of a well known historian;[2] the other a very widely read book[3] representative in scope of the kind of history book ready by college students.

The following three discussions encompass the entire treatment of mathematics of the first text:

1. George Sarton, *The Study of the History of Mathematics*.p.14
2. H.G.Wells, *Outline of History*. p.194, p.629 sqq, p.377.
3. J.A.Rickard and Albert Hyma. College Outline Series. *Ancient, Medieval and Modern History*. p.8, p.11, p.13, p.36, p.48, p.180-1, p.268.

1. "Our modern numerals are Arabic; our arithmetic and algebra are essentially Semitic sciences."

2. Under the heading, "XXIII Science and Religion at Alexandria" the discussion of mathematics in its entirety is: "Particularly sound and good was the mathematical and geographical work. The names of Euclid, familiar to every schoolboy, Eratosthenes, who measured the size of the earth and came to within fifty miles of the true diameter, Apollonius who wrote on conic sections, stand out."

3. About the Moslems he says that they were influenced by the Nestorians, the Jews and the Indians. He concludes with: "... the zero it is stated, was unknown until the ninth century, when it was invented by a Moslim mathematician named Muhammad-Ibn-Musa [al Khoarizmi] who also was the first to use the decimal notation, and who gave the digits the value of position. This, however, is disputed by many Indian historians, who claim the zero and the decimal system as a distinctly Indian contribution.

In geometry the Arabs did not add much to Euclid, but algebra is practically their creation." With an anguished cry we interrupt. Firstly, the zero was introduced, not as Wells states in the ninth century, or even by the Indians whose first use of zero was probably no earlier than 500 C.E.[1] It was in "full use" by the Babylonians in 300 B.C.E.[2] Second, the decimal notation occurs in an Old Babylonian text at least 2500 years before al Khoarizmi first used it.[3] Third, credit for giving digits value

1. B.L. van der Waerden, Science Awakening, p.55 sqq. Other authors give later estimates.
2. Otto Neugebauer, The Exact Sciences in Antiquity. p.26
3. Neugebauer-Sachs, Mathematical Cuniform Texts. p.18.

of position also belongs to the Babylonians.[1] Subsequently I hope to show that "algebra is practically their creation" to be a very great hoax indeed.

Before considering the second text on General History, and to be entirely fair to the famed historian at whose expense I am trying to justify my own effort, I will state that he does mention the names of such men as Archimedes, Descartes, and Newton, but he never hints that they were productive or even interested in mathematics.

Preceding to the second example which is a less renowned but probably more often and more carefully perused history,[2] three sentences are devoted to the science and "contributions" of the Babylonians, no mention being made of their mathematics. Compare this with the following quotation about the Babylonians:[3] "But the spectacular ingenuity of their algebra - when we consider that nothing surpassing it was known in Europe till the sixteenth century A.D..." About the Greek contributions to mathematics, Pythagoras and Euclid are mentioned as being geometers. Rickard and Hyma conclude their remarks about mathematics and mathematicians[4] by noting that much progress was made in mathematics during the first half of the sixteenth century, that Newton was a professor of mathematics, that Descartes "was an expert in geometry and trigonometry," and that Simon Stevin introduced the decimal point.

Having tried to establish my contention that without looking very carefully, one will probably receive only false impressions of the

1. Otto Neugebauer, The Exact Science in Antiquity. p. 18.
2. J. A. Rickard and Albert Hyma. College Outline Series. Ancient, Medieval and Modern History. p.11 and p. 13.
3. E.T.Bell, The Development of Mathematics. p.37
4. Ibid: p.180 sqq.

history of mathematics from the majority of general histories, I would like to turn next to some of the problems involved in seeking out authentic information and then to illustrate by a consideration of one phase of this question. I do not want to elaborate here but merely state that one finds this same kind of mishandling of these subjects, though to a far lesser degree, in many histories of mathematics and in some mathematical texts, also.

II. THE NATURE OF THE PROBLEMS IN THE INVESTIGATION OF THE HISTORY OF MATHEMATICS

Perhaps the most important problems which confront the explorer of the history of mathematics are: deciding what mathematics really is, determining a method of measuring progress in mathematics, trying to interpolate accurately where there are huge gaps in the evidence, and interpreting the evidence which is available.

WHAT MATHEMATICS CONSISTS OF

In order to find a suitable measure for progress in mathematics, we must determine what mathematics is. Our description will have to be sufficiently flexible to be useful in the diverse stages from the earliest periods in the development of mathematics, or what has been called the era of premathematics, to more recent stages.

E. T. Bell states:[1] "Without the strictest deductive proof from admitted assumptions, explicitly stated as such, mathematics

1. E. T. Bell, The Development of Mathematics. p. 4.

does not exist." With this definition we would have to exclude from our consideration all the work much before Euclid. Indeed, the all important beginnings, the crucial embryonic development would be banished from the area of our scrutiny.

Bell himself mitigates the strictness of his definition with the following:[1] "... the validity of a proof is a function of time" and [2]"... criticizing our predecessors because they completely solved their problems within the limitations which they themselves imposed is as pointless as deploring our own inability to imagine the mathematics of seven thousand years hence." To illustrate, before the invention of negative numbers, the quadratic equation $x^2 = x + 30$ had only one root, namely $x = 6$, for the root $x = -5$ was non-existent. Thus, the mathematician who found the root $x = 6$ had solved the equation in its entirety.

We conclude with no pat definition. Mathematics from the sixth century B.C.E. can be adequately described by our first quotation from Bell. Before this period, we must admit arithmetic, algebraic and geometric manipulations as mathematics, albeit manipulation without proof.

MEASURING PROGRESS IN MATHEMATICS

In 1800 B.C.E. the Babylonians computed $\sqrt{2}$ to sixplace accuracy.[3] In 1800 C.E. the British offical accounting system used

1. E.T.Bell, The Development of Mathematics. p.10.
2. Ibid: p.11.
3. Otto Neugebauer, The Exact Science in Antiquity. p. 34.

wooden tallies as records of payment.[1] This dramatically illustrates the question, "How shall we measure progress in mathematics?"

One means of measuring mathematical progress is the generality with which a problem is treated and the relative emphasis placed on the method of solution, the justification of the method, and the "answer" itself.

For instance, the Egyptian mathematics, as we know it from the Moscow and Rhind papyri of about 1800 B.C.E., deals primarily with practical problems, its emphasis being particularly on calculations and achieving numerical dexterity.[2] The Babylonians, having passed this stage, have a much more algebraic approach and are primarily concerned with the mathematical relations between things.[3]

Another aspect of measuring mathematical progress is by the use of symbolism and choice of notation. Symbolism summarizes long chains of reasoning into simple mechanical processes requiring little thought. It condenses much effort and work and has, since it has come into use, freed the thinking ability of mathematicians to deal with problems far more complex than those which taxed the minds of the early Greeks. Furthermore, symbolic reasoning has suggested many generalizations. As an example, we quote Florian Cajori:

> "In one respect this and other Arabic algebras are inferior to both the Hindu and the Diophantine models: The Eastern Arabs use no symbols whatever. With respect to notation, algebras have been divided into three classes: 1) Rhetorical Algebras, in which no symbols are used, everything being written out in words. Under this head belong Arabic works (excepting those of the later Western Arabs),

1. Oystein Ore, Number Theory and Its History. p.8.
2. Otto Neugebauer, Vorlesungen uber Geschichte der Antiken Mathematishen Wisenschaften, Erster Band, IV Kapital, par.1, especially p.122
3. Ibid: pp 39 and 122.

the Greek works of Iamblichus and Thymaridas, and the works of the early Italian writers and of Regiomontanus. (2) Syncopated Algebras, in which, as in the first class, everything is written out in words, except that the abbreviations are used for certain frequently recurring operations and ideas. Such are the works of Diophantus, those of the later Western Arabs, and of the later European writers down to about the middle of the seventeenth century (excluding Vieta's)... (3) Symbolic Algebras, in which all forms and operations are represented by a fully developed symbolism... In this class maybe reckoned Hindu works as well as European since the middle of the seventeenth century."[1]

Another problem in studying mathematical history, especially the older history, is the evidence. First, there isn't enough of it. Second, it is often very difficult to interpret what evidence we do have. For instance, the simplest mathematical texts we have are Egyptian papyri which already have some sophistication. The earliest Babylonian texts are even more advanced (in Algebra) than these papyri. But what preceded these and how these were developed are problems are so intertwined that there is little wonder at the multiplicity of commonly held misconceptions and gross inaccuracies about mathematical history which are so prevalent.

III. SOME ASPECTS OF THE HISTORY OF ALGEBRA FROM THE BABYLONIANS TO THE ITALIANS AND FRENCH OF THE SIXTEENTH CENTURY.

THE BEGINNINGS

As an illustration and an amplification of what proceeds this chapter, I would like to examine some aspects of the history of algebra,

1. Florian Cajori, A History of Elementary Mathematics. p.108

looking at its origins, its early productivity in Mesopotamia, its decline and subsequent rise to where, in the sixteenth century, it first surpassed the work of the Babylonians who flourished so long before.

As was mentioned earlier, the evidence for what preceded the Egyptian papyri and Babylonian texts is scarce. We must glean what we can from the geometrical patterns of cultures whose mathematical development was arrested in the earliest stages, from the linguistic properties of languages of primitive peoples and from the etymological hints in, for example, the Indo-European languages.

The first of these sources need not concern us, for we shall restrict ourselves to algebraic considerations. An example of addition in its simplest stages can be observed by noticing the numbers of some Australian tribes e.g. Wiraduroi:[1]

1. numbai.
2. bula.
3. bul-numbai = 2-1.
4. bungu = many.
5. bungu-galan = very many.

Port Darwin:[2]

1. kulagook.
2. kalletillick.
3. kalletillick kulagook = 2-1.
4. kalletillick kalletillick = 2-2.

Perhaps etymological evidence is more recognizable. How our ancestors may first have reckoned can be guessed from such words as: calculate which is derived from calculus meaning a small stone in Latin, and tally which comes from talea, the Latin word for stick.

1. Levi Leonard Conant, The Number Concept. p.108
2. Ibid: p. 109.

That we count to the base ten is testimony to the importance of our fingers as early aids in calculation. Also as Dantzig points out,[1] one can compare the Sanskrit pantcha, meaning five, with pentcha, hand in Persian; similarly, the Russian piat, five, with piast, outstreched hand.

But let us return to the Egyptians.

EGYPTIAN ARITHMETIC AND ALGEBRA

The evidence we have concerning Egyptian mathematics is not very substantial. We know that they handled large numbers, in the hundreds of thousands in about 3500 B.C.E.[2] This is significant in comparison with the numerical limitations of the backward people of today (see above) to whom four is too large to be comprehensible. We also have two nearly complete mathematical papyri,[3] the Moscow papyrus containing somewhat more than twenty five examples and the Rhind papyrus containing over eighty examples, and numerous text fragments all of the Middle Kingdom and the Hyksos Period, between 1801 and 1580 B.C.E.[4]

Finally, the Akhmin papyrus[5] supposedly written between 500 and 900 C.E. shows no progress over that of Ahmes, author of the Rhind papyrus.

Hence, to the best of our knowledge, the Rhind and Moscow papyri represent the ultimate development of Egyptian mathematics and give us a latest bound on the time at which that development might have occurred.

1. Tobias Dantzig, Number the Language of Science. p.10.
2. E.T.Bell, The Development of Mathematics. p.32.
3. O. Neugebauer, Vorlesungen Über Geschichte Der Antiken Mathematishen Visenschaften. p.110.
4. T.Eric Peet, The Rhind Mathematical Papyrus. p.3.
5. Ibid: p.8.

I would like to point out that Ahmes,[1] writing in the period of the Hyksos, claims that his work is based on a text of the Middle Kingdom (no earlier than 2000 B.C.E.). I do not see the justification for the claims of several authors[2] that this indicates the nature of mathematics in Egypt a thousand years earlier.

Like all other developments in elementary mathematics, the scope of Egyptian algebra and arithmetic is greatly limited by the mode of numeration, notation and symbolism. The Egyptian numeration is decimal but lacking in place values. Fractions, excepting 2/3 were always written in the form 1/n, n integral. The processes of multiplication and division are additive consisting of successive duplications. To aid in these evaluations, there were tables of fractions to express $\frac{m}{r}$ as the sum of fractions with unit numerators. It is believed that the Egyptians knew no general technique for computing such tables and that the entries in these tables, having been discovered accidentally, were carefully saved. They handled examples of arithmetic and geometric progressions, solved simple equations in one unknown, and had symbols for addition, subtraction and equality.

The Egyptian algebra is not theoretical. Not only does it contain no theorems but hardly any general rules of procedure.[3] It is important, because only in the Egyptian papyri do we find this level of mathematical development, and it gives us some knowledge of the development of calculation with fractions.

1. T. Eric Peet, <u>The Rhind Mathematical Papyrus</u>. p.2.
2. e.g. Florian Cajori, <u>A History of Elementary Mathematics</u>, p.19 and S.R.K. Glanville, <u>The Legacy of Egypt</u>. p.174.
3. Florian Cajori, <u>A History of Elementary Mathematics</u>. p.20.

BABYLONIAN ALGEBRA

The sources which are available in this study are far more numerous than those of Egyptian mathematics. The texts which we have[1] are from two periods: the Old Babylonian period, of the time of the Hammurabi dynasty from about 1800 to 1600 B.C.E., and the "Seleucid" period, the last three centuries B.C.E. The texts of the Old Babylonian period are already quite advanced, but what preceded them is now unknown.

There is not much change[2] between the texts of the two periods except that, in the latter, the zero is used. In addition, some of the numerical tables are computed to a larger extent, but they do not involve new principles - Neugebauer suggests that this increased detail may be due to the development of astronomy in the Seleucid period.

Before describing the contents of the texts, it should be observed that the Babylonian scribes used a sexagesimal number system with place value notation. The place value notation gave the Babylonians the following advantages:

1) Use of the same simple techniques of computation whether they were dealing with integers, fractions, or mixed numbers;

2) a compact and readable notation; and

3) the ability to express arbitrarily large numbers by digits in the basic group. The Babylonians in fact were the first to use a positional system of numeration,[3] and this establishes them as the only people in antiquity with a satisfactory mode of calculation.

1. O. Neugebauer, The Exact Science in Antiquity. p.28.
2. Ibid: p. 28.
3. Oystein Ore, Number Theory and Its History. p.16.

In order to apply this system of numeration easily, the Babylonians compiled many mathematical tables. Indeed, their texts were divided into table-texts and problem texts.

The table texts[1] included tables of multiplication and tables of reciprocals which enabled the scribes to write all sexagesimal divisions in the form a/b within the range of the table. The tables have gaps, e.g. 1/7, where a recurrent sexagesimal fraction would be obtained, but even the Old Babylonian texts tell how to proceed in these irregular cases. There are also tables with complete sequences of consecutive numbers, both regular and irregular. The approximations of the reciprocals of the irregular numbers are to three or four places in the Old Babylonian texts and up to seventeen places in the Seleucid period. There are also other tables including some tabulating value of $n^3 + n^2$.

The nature of the problem texts is either the formulation or solution of mathematical problems. The Babylonians solved special cases of equations of the fourth and sixth orders[2] as well as shwoing a profound understanding of quadratics and cubics. The Babylonians displayed a knowledge of the Pythagorean theorem a thousand years before Pythagoras and also investigating satisfactorally the problem of generating "Pythagorean numbers."[3]

There seems to be little acknowledgment, except from specialists in that field, of the generality and theoretical emphasis of the Babylonian approach. For instance, algebraic texts are of two types.[4]

1. O. Neugebauer, The Exact Science in Antiquity. p.32 sqq.
2. Ibid: p.43.
3. This discussion and the various ways of interpreting their work in this field is fascinatingly described by Neugebauer, The Exact Sciences in Antiquity. pp.35,37 sqq and 39 sqq. With great reluctance, I am omitting that discussion.
4. Ibid: p.41 sqq.

The first states a problem and solves it, step by step, using the special numbers and then explaining the general procedure. When the problem involves multiplication by 1, the operation is performed just as it would be in the general case. The second type of problem text, consisting of a collection of problems, frequently has over two hundred problems on a small tablet. The problems are arranged in increasing difficulty. For example, they might all be quadratic equations beginning with those of the normal form and including very complicated problems all reducible to the normal form. Then one discovers that all the problems have the same solutions, say, $x = 3$, $x = 5$, showing that the numerical solution is not at all important. The Babylonians also solved cubic equations with numerical coefficients by reduction to the form $y^3 + y^2 = r$ and using the tabulated values of $n^3 + n^2$.

In conclusion,the claim that Babylonian Algebra was unsurpassed until the sixteenth century C.E. seems a just compliment to those ancient priestly scribes some of whose accomplishments are here briefly enumerated.

THE GREEKS

Of all the mathematicians of entiquity, the Greeks are most honored. This recognition is entirely justified, for mathematics, as we defined it earlier, did not exist before them and reached remarkable splendor in the works of Euclid. Though Greek mathematics' chief claim to fame is geometry, we are primarily interested in their algebra. There is a continuous tradition from Babylonian mathematics to the Greek work[1] contrary to the accepted beliefs.

1. O. Neugebauer, Exact Sciences in Antiquity. p.139 sqq.
   Also ably explained in van der Waerden... Chapter V especially p.124.

How the Greeks could demonstrate simultaneously such ingenuity in geometry and yet ignore the advances of the Babylonians in algebra and in numeration has puzzled mathematical historians for some time. Van der Waerden offers the first satisfactory explanation,[1] claiming that since the Greek definition of number encompassed only integers, problems involving non-integral rationals or irrational numbers could logically only be expressed geometrically by segments. This explanation is very appealing. But I feel that it cannot be stretched to explain the Greeks' systems of numeration all of which were quite unwieldy.[2] The alphabetic system of numeration which superseded the superior Attic system was derived from the Hebrew and Phoenician systems. This theory is generally accepted, though vehemently denied by Gow.[3]

The alphabetic system, so unsuggestive in calculating and unhelpful in arithmetic or algebraic manipulation, reflects the Greeks disdainful attitude toward calculations as examplified by Plato. Few Greek mathematicians attempted to revise the numerical symbolism. Among those who did were Apollonius and Archimedes, the latter in the Sand-reckoner. Evidence of much Greek computation (using the sexagesimal system) exists in, for example, Ptolemy and Theon, but these are of a much later period and in an astronomical context.

Euclid's Elements include a considerable amount of arithmetic, though it is clothed in geometric terms. The first ten propositions of the second Book are arithmetic, and the eleventh and fourteenth solve two forms of quadratic equations. The seventh, eight and ninth

1. Van der Waerden, Science Awakening p.125 sqq
2. James Gow, Short History of Greek Mathematics. p.41 sqq.
3. Ibid: p.43 sqq.

books deal with the theory of numbers and the tenth with irrationals. DeMorgan considers this last to be Euclid's most complete work.[1]

Subsequent Greek work in arithmetica included the sieve of Eratosthenes and Hypsicles' work on arithmetic progressions.

About 100 C.E., Nicomachus wrote a classificatory text about arithmetic which, in Boethius' translation, was the sole authoritative arithmetical text until the Moslems brought Hindu Mathematics to Europe. The notable change is that this is the first known Greek work which is inductive. He states a theorem about cube numbers:[2]

$$1 = 1$$
$$3 + 5 = 2^3$$
$$7 + 9 + 11 = 3^3$$

and in general:

$$[n(n-1) + 1] + \ldots + [n(n+1) - 1] = n^3.$$

Nicomachus is considered to be one of the Greeks' two notable algebraists. The more famous is Diophantus who wrote thirteen books on the integral solution of indeterminate equations. During the period of Diophantus there was a general renascence in algebraic interest as evidenced by many puzzles all analytic in nature and all involving the consideration of an unknown quantity.

Diophantus uses, for the first time in Greek works, a syncopated symbolism having letters denoting the unknown and powers of it (each power having a different symbol), and letters also for the operations subtraction and equality. Diophantus treats only positive solutions to equations and never states more than one solution, say,

1. James Gow, Short History of Greek Mathematics. p.83
2. Ivor Thomas, Greek Mathematical Works. Vol.I. p.105 sqq.

of a quadratic even when both roots are positive. (Recall that the Babylonians gave both solutions when they were positive.) Diophantus shows considerable ingenuity in his work with indeterminate equations, but he treats only special cases, and the distinct solutions of individual problems can not be generalized.

No notable contributions were made in Greek mathematics after Diophantus, and the era is generally considered to be closed with the assassination by the Romans of Hypatia, Theon of Alexandria's talented daughter, in 415.

THE ROMANS

The two Roman contributions to mathematics are negative: they killed Archimedes and murdered Hypatia. Their contribution to numeration is the Roman numerals, the multiplying, dividing and square rooting of which has baffled modern men until very recently.[1] Boethius wrote an Institutio Arithmetica but this is no more than Nicomachus in translation.

ARABIC ALGEBRA

From the second half of the eight century until the eleventh century nearly all the mathematical publications: translations, text books, treatises and commentaries were written in Arabic, and even until the end of the twelfth century the major mathematical works produced in Europe were in Arabic.[2]

The first comprehensive works in the Arab mathematical literature are those of Al Khoarizmi in the first half of the ninth century. Evaluations of this mathematician vary greatly, e.g.

1. Time, Weekly News Magazine, Vol.LXVII, No. 12, p.83
2. George Sarton, Introduction to the History of Science, Vol.I and II

"The greatest mathematician of the time, and, if one takes all circumstances into account, one of the greatest of all times was al Khwarizmi."[1] whereas Cajori states; "Alchwarizmi's algebra, like his arithmetic, contains nothing original."[2] His algebra, Hisab al-jabr wal-mugabala, contains analytic solutions of linear and quadratic equations as well as geometric solutions of quadratics.

The Moslem mathematicians and their non-Moslem associates began to produce many Arabic translations of Greek works and introduced Hindu notation into Europe. The first Hindu numerals and the Hindu notation for zero, a dot, appear in Arabic works in 874.[3]

About half a century later Abu Kamil perfected al Koharismi's algebra. He studied addition and subtraction of radicals, obtaining the formula $\sqrt{a} + \sqrt{b} = \sqrt{a + b = \sqrt{2ab}}$ and he could determine and construct the two (real)roots of a quadratic equation."[4]

Later in the tenth century Abu Jafar al Khazin solved al Mahani's cubic equation, and, c. 988, Al Kuhi studied Archimedian and Appolonian problems leading to equations of higher degree than quadratics some of which he solved geometrically.

Al Karkhi[5] was the greatest mathematician of the eleventh century, solving many Diophantine problems. He reduced equations of the type: $ax^{2p}+ bx^{p}=c$ to quadratics and was the first Arab to solve the series problems:

1. Goerge Sarton, Introduction to the History of Science, Vol.I.p.545.
2. Florian Cajori, A History of Elementary Mathematics. p.107.
3. George Sarton, Introduction to the History of Science, Vol.I. p.602.
4. Ibid: p.630
5. Ibid: p.718 sqq.

$$1^2 + 2^2 + \ldots + n^2 = \frac{2n+1}{3} (1 + 2 + \ldots + n)$$

and $$1^3 + 2^3 + \ldots + n^3 = (1 + 2 + \ldots + n)^2$$

using geometric proof. However, in the manner of many Moslem mathematicians, he carefully ignored the Hindus and wrote out all numerals in full.

With Omar Khayyam, in the 12th century, Arab mathematics simultaneously reached its highest level and the end of its dominance over the mathematical scene. He classified many equations, including thirteen kinds of cubics and gave partial geometric solutions to many of them. The Arabic scholars had spent themselves, and the immense activity in translating the mathematical works into Hebrew and Latin indicated the growing demand for such knowledge in the West. To quote Sarton: "At the beginning of the twelfth century it was impossible to become a full fledged mathematician and astronomer without a good knowledge of Arabic; by the end of the thirteenth century the situation was very different, for many classics of Greek science were already available in Latin and in Hebrew."[1]

1. George Sarton, <u>Introduction to the History of Science</u>, VOL.II. p.7.

EARLY ITALIAN AND FRENCH ALGEBRA

Some of the earliest mathematical works written by European were those of Leonardo of Pisa, also called Fibonacci. He travelled widely and published in Latin a rather thorough compilation (Liber abaci in 1202) of all the mathematics known to the Moslems. His work was influential, even causing the merchants to adopt the Hindu numerals and arithmetic techniques instead of the Roman numerals and the abacus. Leonardo was a mathematician in his own right as well as a translator and compiler. Although some of his work in diophantine systems is on a far lower level than Euclid, and although, failing to see the generality of a problem, he does not try to find all its solutions, still he is way ahead of his time, for instance, attempting to show the impossibility of solving an equation whose roots he could not find.[1]

More than one hundred years later, Nicole Oresme first suggested the idea of fractional powers.[2] But mathematical progress really gained impetus with the invention of the printing press.

The first comprehensive arithmetic to be printed was the Summa of Luca Pacioli, published in Venice in 1494.[3] This work explains the fundamental operations of arithmetic and extraction of square roots, but emphasizes commercial arithmetic with many examples. Many of the other early arithmetics were commercial in emphasis.

1. E.T. Bell, The Development of Mathematics. p.114 eqq.
2. Ibid: p.122
3. W.T. Sedgwick and H.W.Tyler, A Short History of Science. p.265 sqq.

Vieta,[1] a brilliant French lawyer who served as cryptanalysisist for Henry IV of France, was the first European to use a symbolic algebra. In his *In Artem Analyticam Isagoge*, knowns are denoted by consonents and unknowns by vowels and there is an exponential notation when the words *quadratus*, *cubus*, etc., follow the unknown instead of using separate symbols for distinct powers of the unknowns. Vieta gave rules for approximating the roots of equations which could not be directly solved in *De Numerosa Potestatum Resolutione* and *De Aequationum Recognitione et Emendatione*. Still he only admitted positive roots.

The first successful attack on cubics other than the forgotten Babylonian efforts and the geometric solutions of the Moslems was made by Scipio del Ferro.[2] He solved cubics of the form $x^3 + mx = n$ in about 1515. More general solutions were discovered by Nicola Fontana, known as Tartaglia, but were first published by Girolamo Cardan in 1545,[3] whom a great many recent elementary texts erroneously credit with this solution. Cardan, however, suggested $5+\sqrt{-15}$ and $5-\sqrt{-15}$ as answers to the problem[4] of dividing ten into two parts whose product is forty, a courageous step when other mathematicians accepted only positive real roots.

1. W.T.Sedgwick and H.W.Tyler, *A Short History of Science*. p.274 sqq. and A. Wolf, *A History of Science, Technology and Philosophy in the Sixteenth and Seventeenth Centuries*. pp. 189 and 599.
2. Florian Cajori, *A History of Elementary Mathematics*. p. 224.
3. A. Wolf, *A History of Science, Technology and Philosophy in the Sixteenth and Seventeenth Centuries*. p.190 sqq.
4. Florian Cajori, *A History of Elementary Mathematics*. p.227

One of the final triumphs of the sixteenth century was Lodovico Ferrari's solution of the quartic equation[1] wherein the first advance over the ancient Babylonians was made in Western Europe. Logarithems also appeared in this period but negative roots were not first used until the seventeenth century.

We leave the story here in the exciting era when men have just rediscovered mathematics and before the great giants Galileo, Descartes, Fermat, Newton and Leibniz appear. The first hard steps have been made, and after this period, the pace of progress is breathlessly swift.

There has been a gap in this narration. The Indian contributions to mathematics, some of which were known to the Moslems, have been omitted. This was done purposely, for the wonderful mystery of the Hindu development of algebra has only been recently discovered, and there are many aspects of it which preceded but did not influence the development of Western mathematics. This is why I have not discussed it earlier. I would like to conclude with the remarkable Indian Algebra.

HINDU ALGEBRA

Partly because it did not greatly influence the history of Western algebra and partly because only a fraction of it was available to Westerners, Hindu Algebra has been injudiciously neglected by most histories of mathematics. These books consider Indian mathematics primarily in the context of what the Moslems culled therefrom, and have been satisfied with that treatment alone.

1. Florian Cajori. A History of Elementary Mathematics. p. 226.

I have had a problem in studying Hindu algebra: the only thorough work in this field which was accessible to me is that written by Indians. They are very much aware of this Western neglect, and have put forth such emphatic retorts that one cannot help but believe that they are over-correcting this neglect. In fact one author[1] was so cutting and vehement in his attacks on Western mathematical historians that it is difficult to believe that his chauvinism does not bias his judgment of the evidence. Although I do not feel qualified to judge, I am taking one text[2] which appears to be both comprehensive and of scrupulous authorship as my primary guide in this field.

The Hindus developed an algebra at an early date and regarded it with reverence.[3] The algebra can be traced back to the period of the Sulba (800 - 500 B.C.E.) and the Brahmana (c.2000 B.C.E.) but at that stage it is mostly geometrical. In the Sulba is the geometric solution of a problem equivalent to the equation $ax = c^2$. A particular case of the equation $x^2 = n$ is found as early as 2000 B C.E. in the Satapatha Brahmana, the context being the problem of constructing a sacred temple to certain specifications!

It is interesting to observe some of the linguistic characteristics of powers of quantities (known and unknown) as they first occur in Hindu algebra as in the Uttaradhyayana-Sutra (c.300 B.C.E.). They remind one of the examples of the Australian tribes noted earlier, only they are here multiplicative:

1. L.V.Gurjar, Ancient Indian Mathematics and Vedha.
2. B. Datta and A.N.Singh. History of Hindu Mathematics. A Source Book. Parts I and II.
3. Ibid: p. 4. sqq.

second power is called: varga

third power: ghana

fourth power: varga-varga

sixth power: ghana-varga

twelfth power: ghana-varga-varga

The laws of signs, stated in full generality, do not seem to have been written out until the seventh century when they were written by Brahmagupta. However, in about 300 B.C.E. there is a Hindu classification of equations according to their degrees: simple, quadratic (varga), cubic (ghana) and biquadratic (varga-varga).

Linear equations in one unknown are solved by the Rule of False Position by Hindus from the beginning of the Christian Era until Aryabhatta I (499) who no longer uses it. Aryabhatta solves problems leading to a general equation.

Linear equations in several unknowns first appear in the Bakhshali treatise of c.200 C.E.[1] The Hindu mathematicians were apparently conversant with ways of human nature as well as in this phase of algebra as can be seen by the following illustrative example which Mahavira gave six hundred and fifty years after the Bakhshali treatise was written:

> "Four merchants were each asked separately by the customs officer about the total value of their commodities. The first merchant, leaving out his own investment, stated it to be 22; the second stated it to be 23; the third 24; and the fourth 27; each of them deducted his own amount in the investment. O friend, tell me separately the value of the commodity owned by each."

As early as 500 B.C.E. approximate solutions of quadratic equations are found in Katyayana, and exact solutions appear in the Bakhshali treatise of seven hundred years later. By the twelfth century, Bhaskara II recognizes two roots to quadratics but says, in the case of an equation with a

1. B.Datta and A.N.Singh. History of Hindu Mathematics. Part II.pp.47 & 297.

negative root, "But, in this case, the second (value) should not be accepted... People have no faith in the known becoming negative."[1]

The Hindus did not achieve great success in the solution of cubic and quartic equations, and in contrast to Western mathematicians, the Hindus found indeterminate analysis far simpler than determinate analysis.

Aryabbata I[2] (born 476) gives the earliest taeatment of the indeterminate equation of the first degree, and he gives general rules for their solution.

Later algebraists give rules for the solution of simultaneous linear indeterminate equations.

Brahmagupta, in the seventh century, discovered the lemmas known by his name:

if $N\alpha^2 + k = \beta^2$

and $N\alpha'^2 + k' = \beta'^2$

then $N(\alpha\beta' \pm \alpha'\beta)^2 + kk' = (\beta\beta' \pm N\alpha\alpha')^2$

These were rediscovered by Euler in 1764 and by Lagrange in 1768. Taking $\alpha' = \alpha$ and $\beta' = \beta$ and $k = k'$, we have

if $N\alpha^2 + k = \beta^2$

then $N(2\alpha\beta)^2 + k^2 = (\beta^2 + N\alpha^2)^2$

which is Brahmagupta's Corollary which was applied to the Square-nature equation: $Nx^2 + 1 = y^2$,

For N a non-square integer, this gives an infinite number of integral solutions. This result was known to the Hindu algebraists long before it was discovered by Fermat.

1. B.Datta and A.N.Singh. History of Hindu Mathematics. p.87.
2. Ibid: p.147 sqq.

The first known solution of the general indeterminate equation of the second degree is found in the Bijaganita of Bhaskara II[1] in the twelfth century. He shows how, having found one solution of the equation

$$ax^2 + bx + c = y^2$$

an infinite number of other solutions may be obtained. This solution was discovered by Euler in 1733 and its method improved (resulting in a method similar to Bhaskara's) in 1767. It also appears that Matsonago's solution of $z^2=x^2+by^2$ (c. 1735) and Desboves solution of the same equation (1879) are special cases of Bhaskara's treatment of $ax^2 + bxy + cy^2 = z^2$ Numerous similar examples may be found of Hindu solutions to problems long before the credited Western mathematician first discovered them.

Thus we see that the order of development of mathematics in the East differed greatly from that in the West, so that at the end of, say, the sixteenth century, each had extensive knowledge of which the other was ignorant.

Much remains to be done in reconstructing the history of early mathematics, and the popular treatments of the subject often lag very far behind the knowledge of the specialists in history. But when these problems of reconstruction and informing the laymen are solved, there is a far greater one to tackle, and in this field research has only begun. I am referring to the problem of discovering how this inventive process occurs, why it takes different directions among different races, why it's pace is accelerated in one era and retrogresses in another, what effect the sociological framework has on the mathematicians exposed to it, what inspires creativity and all the related problems. It seems to me that this problem will have two phases: a sociological one and a psychological one. I

1. B.Datta and AN. Singh, History of Hindu Mathematics. p.185. sqq.

believe the first phase will best be explored through the study of the development of ancient mathematics where we know little of the individuals who deserve the credit,and we will not be distracted by personalities. The study of the second phase obviously needs to focus on the more recent periods where we have much information about the people themselves.

This is not the place to go into more detail about the problems yet to be solved. I was motivated by interest in the history of mathematics, I was enraged by some of the careless treatment it has received, and I have learned about some of the problems involved and also a little of the history itself.

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