ON THE HYDRODYNAMICS OF SUPERFLUID HELIUM

by

ALFRED CLARK, JR.

B.S., Purdue University

(1958)

SUBMITTED IN PARTIAL FULFILLMENT

OF THE REQUIREMENTS FOR THE

DEGREE OF DOCTOR OF

PHILOSOPHY

at the

MASSACHUSETTS INSTITUTE OF

TECHNOLOGY

September, 1963

Signature of Author Signature redacted
Department of Mathematics, August 16, 1963

Certified by Signature redacted
Thesis Supervisor

Accepted by Signature redacted
Chairman, Departmental Committee
on Graduate Students

ABSTRACT

On the Hydrodynamics of Superfluid Helium

by

Alfred Clark, Jr.

Submitted to the Department of Mathematics on August 16, 1963 in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

The present work is concerned with continuum theories of the hydrodynamics of superfluid helium. A comparative study has been made of some of the existing macroscopic theories of the flow of superfluid helium. The work includes (i) a detailed critical study of the derivations of the hydrodynamic equations in the various theories, (ii) alternative derivations of the equations in some cases and (iii) a discussion of some possible generalizations of some of the existing theories. The theories under consideration include the equations originally proposed by Landau for the hydrodynamics of the two-fluid model, the hydrodynamic equations proposed by Lin for his one-fluid model, the theory of dissipative processes for the two-fluid model proposed by Lin, and the theory of Bekarevich and Khalatnikov.

Thesis Supervisor: C. C. Lin

Title: Professor of Mathematics

ACKNOWLEDGEMENTS

I am greatly indebted to Prof. C. C. Lin for continued guidance and encouragement during the execution of this work. I wish also to acknowledge the valuable assistance of Miss Mary Beth Andre and my wife in the preparation of the final manuscript. I am grateful for the financial support given me during the course of this work by the Massachusetts Institute of Technology and by the National Science Foundation.

TABLE OF CONTENTS

I INTRODUCTION

In the problem of superfluid helium, one may distinguish two main aspects - the development of a molecular theory and the development of a hydrodynamic theory. Ultimately, one would hope to develop the macroscopic hydrodynamic theory from the molecular theory. However, as Lin [25] has emphasized recently, the development of the hydrodynamic theory need not await the resolution of the extremely difficult problems of the molecular theory - one can take advantage of the possibility of developing the macroscopic theory in a more or less phenomenological manner. Of course, the microscopic theory often serves as a guide in the development of macroscopic theories, but the connection at present between the two types of theories is still largely a qualitative one.

In the present work, we consider the problem of developing a macroscopic theory of the hydrodynamics of helium II. In spite of great effort, this problem remains an open question, and a variety of hydrodynamic equations have been proposed. Since this is still a controversial subject, we consider here - on an equal basis - some of the hydrodynamic equations which have been proposed. More specifically, the present work contains (i) a review and criticism of some of the current hydrodynamic theories, with the intent of making more explicit the principles underlying the derivations, (ii) a discussion of some possible generalizations of these theories

and (iii) a presentation of alternative derivations of the hydrodynamic equations for some of the theories, with the intent of furnishing some new view points. In the remainder of this section, we give a brief review of the hydrodynamic theories to be discussed in more detail later, and a brief discussion of the general features of the problem of determining a set of hydrodynamic equations to describe the flow of helium II.

In obtaining a set of hydrodynamic equations for helium II, one is faced with two fundamental interrelated problems: (i) the selection of a set of (local) macroscopic quantities (e.g., density, velocity, etc.) which will adequately describe the flow of helium II and (ii) the derivation of a set of equations describing the change with time of these quantities. For an ordinary fluid, the problem (i) above presents no difficulties - the macroscopic configuration may be adequately described by mass density, the macroscopic velocity and an additional thermodynamic variable such as the specific entropy. For liquid helium II, however, there is abundant evidence that additional macroscopic quantities are needed to describe the the various flow phenomena; thus the selection of a set of macroscopic quantities to describe the flow of helium II is a matter of some difficulty and importance. Most of the current continuum theories of the flow of helium II take as their starting point some form of the two-fluid model, originated and developed by Landau, London, and Tisza. The qualitative features of this model have been deduced by Landau [20] from

his theory of phonons and rotons, and also by London [28] from the theory of Bose-Einstein condensation. These alternative points of view have in common that a definite molecular picture is used as a guide in the determination of an appropriate set of macroscopic variables. The recent work of Lin [26,25, 24] has shown, however, that the two-fluid model is by no means the only starting point for the development of a continuum theory of helium II, and, also, that even the selection of an appropriate set of macroscopic variables may be accomplished in a phenomenological way without recourse to a definite molecular picture.

Even after the choice of macroscopic quantities has been made, there remains the problem of determining a correct thermodynamic description of the helium II. Again, there is abundant evidence that the helium II system requires more thermodynamic variables for its description than does an ordinary fluid. Here, also, a definite molecular picture can be used as a guide in setting down the fundamental thermodynamic equation.

Finally, there is the problem of determining the hydrodynamic equations. Although the approach to this problem depends to some extent on the choice of macroscopic variables, there are certain general principles which have been used, with some success, to obtain a set of hydrodynamic equations. Among these, we may mention (i) variational principles (for perfect fluid theories) and (ii) imposition of macroscopic conservation laws. Of course, analogy with ordinary hydrodynamics is an important underlying guiding principle in all

approaches to the continuum theory of helium II.

Although there are many problems in the development of a hydrodynamic theory of helium II, there are two issues in particular which are central; the first of these is the question of the rotation of the superfluid component. This is not a question which can be resolved within a given hydrodynamic theory; rather, one must take a definite point of view on this question in order to develop a hydrodynamic theory. In fact, the current theories may be roughly divided into three types with respect to the question of superfluid rotation: (i) theories in which $\mathrm{curl}\, \underline{v}_s = 0$ always (Landau [20], Lifshitz and Khalatnikov [22]), (ii) theories in which $\mathrm{curl}\, \underline{v}_s$ plays a special role (e.g., the quantized vortex line theory of Feynman [7] and Hall and Vinen [12, 13, 14, 38], or the continuum theory of Bekarevich and Khalatnikov [3] in which $|\mathrm{curl}\, \underline{v}_s|$ is a thermodynamic variable) and (iii) theories in which no special assumptions are made about the nature of $\mathrm{curl}\, \underline{v}_s$ (Lin [26,25,24]). A theoretical explanation of the problem of superfluid rotation must eventually come from a microscopic theory; one can, however, compare the results of experiments with the predictions of the various hydrodynamic theories and thus gain indirect evidence relevant to the problem. The second central issue is the question of mutual friction (volume momentum exchange) between the two components. Again, one must take a definite point of view on this question in order to develop a hydrodynamic theory.

It is convenient to consider the perfect fluid theories

first, before considering the more general theories including dissipative processes. The principal reason for this is that most of the theories of dissipative processes have as their starting point some form of the perfect fluid equations, so that the development of a perfect fluid theory is a necessary preliminary.

For reversible flows of helium II, the equations proposed by Landau [20, 22] have been well-verified experimentally. As Lifshitz and Khalatnikov [22] have shown, it is possible to deduce Landau's equations by (i) assuming the two-fluid model, (ii) imposing the macroscopic conservation laws and (iii) assuming that the superfluid component must move irrotationally (i.e., curl $\underline{v}_s \equiv 0$). An analysis of their work in detail is given in II-A-1; in particular, it is shown there that the assumption curl $\underline{v}_s \equiv 0$ is a crucial one, in that it is no longer possible to deduce a unique set of equations once it is dropped.

If one takes the point of view that curl $\underline{v}_s$ may vanish for a certain class of solutions, but that it does not necessarily vanish, then the imposition of the conservation laws is not sufficient to determine uniquely the perfect fluid equations for the two-fluid model. However, one may make use of an idea advanced by Landau [20] - namely, that there be no volume momentum exchange between the two components (other than that due to normal fluid-superfluid transitions). In order to make use of this idea, however, one must interpret the two-fluid model in a rather literal fashion and assume that it

is possible to give a separate thermodynamic and hydrodynamic description for each component; the only coupling between the components is through the equilibrium condition for the normal fluid--superfluid transitions. A derivation of the perfect fluid equations along these lines is given in section II-A-2. It is shown there that a perfectly definite set of hydrodynamic equations may be obtained in this manner, and that the equations are identical in form with the Landau equations without, however, the restrictive condition $\operatorname{curl} \underline{v}_s \equiv 0$.

Whenever we are considering a perfect fluid theory, we may expect that some sort of variational principle will be valid. As Lin has shown (see Serrin [33] , p. 148, for a discussion of this), there are difficulties with a variational principle even for ordinary hydrodynamics which however may be overcome by taking into account the Lagrangian nature of the system. In the case of the two-fluid model (which does not admit a Lagrangian description), it is not so clear that the difficulties can be resolved. Zilsel [40] has obtained a set of hydrodynamic equations for the two-fluid model by means of a variational principle. His variational principle leads to the Landau equations (with the restrictive condition $\operatorname{curl} \underline{v}_s \equiv 0$); however, his equations also entail a restriction on the quantity $\operatorname{curl} \underline{v}_n$. A detailed discussion of Zilsel's work is given in II-A-3.

A variational principle has also been used by Lin [25,24] to obtain the perfect fluid equations for his one fluid model. The equations obtained by Lin from his variational principle

are similar to the Landau equations. In Lin's theory, however, the condition $\text{curl}\, \underline{v}_s = 0$ is characteristic of a particular class of solutions, and for this class his equations are identical with Landau's. However, Lin's equations also admit more general solutions for which $\text{curl}\, \underline{v}_s \neq 0$. The variational principle and the equations obtained by Lin are discussed in sections II-B-1 and II-B-2. Since there are at present still some difficulties associated with this variational principle, the possibility of obtaining the perfect fluid equations for the one-fluid model from the conservation laws has been examined. It is shown in section II-B-3 that this method does not lead to a unique set of equations (this, of course, is to be compared with a similar result for the two-fluid model when the restriction $\text{curl}\, \underline{v}_s \equiv 0$ is not imposed).

In Chapters III and IV of the present work, a discussion is given of the hydrodynamic equations including dissipative processes, as proposed by various authors. As in the case of the perfect fluid theories, the form of the final equations depends greatly on the role assigned to the quantity $\text{curl}\, \underline{v}_s$. Even so, the method for deriving the hydrodynamic equations is essentially the same--the basic principles are the macroscopic conservation laws and the principle of increase of entropy. Of course, these principles must be supplemented by other special considerations in each case.

Lifshitz and Khalatnikov [22] have presented an extension of Landau's theory to include dissipative processes. In the

development of their theory, the restriction $\text{curl}\, \underline{v}_s \equiv 0$ is imposed. Their equations, along with a possible generalization, are discussed in section III-B.

Since it is known that, under some conditions, the superfluid component seems to rotate in some manner, theories in which $\text{curl}\, \underline{v}_s \equiv 0$ always are not expected to be of universal validity. Indeed, the experimental discovery that the superfluid can rotate, in conjunction with Landau's idea that $\text{curl}\, \underline{v}_s$ must vanish, has been one of the factors in the development of the quantized vortex line theory. Lin [25,24,23] however, has advanced the idea that under certain conditions the superfluid component can rotate in bulk like an ordinary fluid, since at present there seems to be no compelling evidence to the contrary. In the theory of dissipative processes developed by Lin, there is a momentum transfer associated with the superfluid rate of strain as well as the normal fluid rate of strain, and the stress--rate of strain relations are characterized by four shear viscosity coefficients; the nonlinear boundary condition proposed by Lin allows a slip of the superfluid component at a solid wall, while the normal component is assumed to adhere to the wall. A derivation and discussion of Lin's equations is given in section III-C.

The current prevailing theories of the rotation of superfluid helium are based on the Onsager-Feynman theory of quantized vortex lines. Hall and Vinen [12,13,14,38] have developed a set of hydrodynamic equations on the basis of the two-fluid model and the Onsager-Feynman theory. Bekarevich and

Khalatnikov [3] have presented a theory based on continuum principles which, in contrast to the theory of Hall and Vinen, does not depend on the specific features of the quantized vortex line theory. Bekarevich and Khalatnikov based their derivation on the two-fluid model and the single additional assumption that the thermodynamic internal energy of the helium II depends on the magnitude of the superfluid vorticity, as well as the usual thermodynamic variables. They were able to obtain the same final hydrodynamic equations as those obtained by Hall and Vinen. A detailed discussion and criticism of their derivation is given in section IV-B-1. An alternative development of their equations is offered in section IV-B-2. On one hand the theory of Bekarevich and Khalatnikov has the advantage of not resting on a specific molecular picture; on the other hand, its possible relevance to the quantized vortex line theory is not clear. This point is discussed in detail in section IV-B-3. Finally, in section IV-C a possible alternative approach to constructing a hydrodynamic theory including quantized vortex lines is discussed; the equations given by Hall [13] to describe "vortex waves" are obtained there.

In Chapter V, a summary of the various results obtained is given in the form of a unified mathematical scheme which includes the various theories as special cases. Some proposals for further work are also given there.

II PERFECT FLUID THEORIES

A. Two-Fluid Model

1. Equations from conservation laws

Although Tisza presented a set of approximate (linear) hydrodynamic equations for the two-fluid model of helium II in 1938 [35], the first full set of hydrodynamic equations was presented by Landau in 1941 [20]. Landau's method is briefly desribed in [20], and given in more detail by Lifshitz and Khalatnikov [22]. We give in the following a detailed presentation of the derivation of [22], along with some elaboration of the argument at various points.

According to the two-fluid model, we may (formally) regard helium II as a "mixture" of two fluids - the normal fluid and the superfluid. Each fluid has its own macroscopic velocity and its own mass density. Thus the total mass density ρ is given by $\rho = \rho_n + \rho_s$, where ρ_n and ρ_s are the normal and superfluid densities, and the momentum per unit volume $\underline{j}$ is given by $\underline{j} = \rho_n \underline{v}_n + \rho_s \underline{v}_s$, where $\underline{v}_n$, $\underline{v}_s$ are the normal and superfluid velocities, and the entropy per unit volume is given by ρs, where s is the entropy per unit mass. Landau [20] has given a microscopic theory of helium II which leads to the two-fluid model in a natural way. In this theory one thinks of the normal fluid as a "gas" of thermal excitations (phonons and rotons), while the superfluid component is the

"inert background" in which the phonons and rotons move. It is clear from this picture that the macroscopic theory must allow for the possibility of conversion of normal fluid into superfluid (and vice versa).

Then the laws of conservation of mass, momentum and entropy will have the forms

$$\frac{\partial \rho}{\partial t} + \mathrm{div}(\underline{\mathcal{J}}_\rho) = 0 , \qquad (1)$$

$$\frac{\partial j_i}{\partial t} + \frac{\partial}{\partial x_j}(\pi_{ij}) = 0 , \qquad (2)$$

$$\frac{\partial}{\partial t}(\rho s) + \mathrm{div}(\underline{\mathcal{J}}_s) = 0 , \qquad (3)$$

where $\underline{\mathcal{J}}_\rho$ is the mass flux vector, $\underline{\mathcal{J}}_s$ the entropy flux vector and π_{ij} the momentum flux tensor. We must also require the conservation of energy, which is expressed by

$$\frac{\partial E}{\partial t} + \mathrm{div}\,\underline{Q} = 0 , \qquad (4)$$

where E is the total energy per unit volume and $\underline{Q}$ is the energy flux vector. To complete the set of equations, one additional (vector) equation is needed, and we may take this to be the equation for the superfluid velocity $\underline{v_s}$. This may be written as

$$\frac{\partial \underline{v_s}}{\partial t} + \underline{v_s} \cdot \nabla \underline{v_s} = \underline{f} . \qquad (5)$$

Equations (1) - (5) would be a complete set of hydrodynamic equations if the dependence of the quantities $\underline{\mathcal{J}}_\rho$, $\underline{\mathcal{J}}_s$, $\underline{Q}$, π_{ij} and $\underline{f}$ on the quantities ρ_n, ρ_s, S, $\underline{v}_n$ and $\underline{v}_s$ were known, and if the dependence of the total energy E on ρ_n, ρ_s, $\underline{v}_n$, $\underline{v}_s$ and S were known. These equations represent 9 scalar equations for the 9 scalar quantities ρ_n, ρ_s, S, v_{ni}, v_{si} and it would seem that any choice of $\underline{\mathcal{J}}_\rho$, $\underline{\mathcal{J}}_s$, $\underline{Q}$, π_{ik} and $\underline{f}$ (consistent with the Galilean transformation laws) would lead to a set of hydrodynamic equations. This is not the case, however, because of the following essential feature of the two-fluid model: the 9 quantities ρ_n, ρ_s, S, v_{ni}, v_{si} are not independent, but satisfy a sort of equilibrium relation. This conclusion is a consequence of the possibility of normal fluid - superfluid transitions mentioned above. Thus we must require that the 9 scalar equations (1) - (5) for the 8 independent quantities be consistent, and this leads to a restriction on the "fluxes" $\underline{\mathcal{J}}_\rho$, π_{ij}, $\underline{\mathcal{J}}_s$, $\underline{Q}$ and $\underline{f}$. Preliminary to exploiting this fact, however, we must examine the dependence of the energy E on the quantities ρ_n, ρ_s, S, v_{ni}, v_{si} and also the nature of the equilibrium relation.

The energy E should be expressible as a function of 8 independent quantities. It is convenient to choose the quantities ρ, $\underline{j}$, S and $\underline{v}_s$ as the independent ones. It is to be expected that ρ_n (for example) will be a definite function of these 8 independent quantities. (Although it would seem that any choice of 8 quantities could be used, the discussion

of part II-A-2 will make clear the significance of this particular choice). By means of the Galilean transformation for energy, we may express the energy E in terms of the energy E_o as measured in the superfluid rest frame; thus

$$E(\rho, s, \underline{j}, \underline{v}_s) = E_o(\rho, s, \underline{j}_o) + \underline{j}_o \cdot \underline{v}_s + \frac{\rho v_s^2}{2} \qquad (6)$$

where $\underline{j}_o$ is the momentum per unit volume as measured in the superfluid rest frame, and $E_o(\rho, s, \underline{j}_o) = E(\rho, s, \underline{j}_o, 0)$. The momentum $\underline{j}_o$ is related to $\underline{j}$ by the Galilean transformation formula

$$\underline{j} = \rho \underline{v}_s + \underline{j}_o . \qquad (7)$$

E_o may depend on $\underline{j}_o$ only through the scalar quantity $\frac{1}{2} j_o^2$, so that

$$E_o = f(\rho, s, \tfrac{1}{2} j_o^2). \qquad (8)$$

The function f cannot be determined by continuum principles, but should (in principle) follow from a detailed microscopic theory. According to Landau's view of a gas of excitations (the normal fluid) moving in an inert background (the superfluid), the observer moving with the inert background (i.e., in the superfluid rest frame) should see something quite similar to the motion of an ordinary gas with a drift velocity $\underline{w} = \underline{v}_n - \underline{v}_s$. This qualitative picture is a basis for the assumption that the intensities corresponding to the quantities ρ, s, $\underline{j}_o$ are the same as those for an ordinary fluid. For an

ordinary fluid of energy E per unit volume, density ρ, entropy ρs per unit volume and momentum $\underline{j}$ per unit volume, the differential of E is given by $dE = \underline{v}\cdot d\underline{j} + T d(\rho s) + \Phi d\rho$, where $\underline{v}$ is the velocity of motion, T the temperature and Φ, the thermodynamic potential is related to the pressure p by $\rho\Phi = E - \underline{v}\cdot\underline{j} - T\rho s + p$. Thus we assume that the differential of E_o has the form

$$dE_o(\rho, s, \underline{j}_o) = \Phi d\rho + T d(\rho s) + \underline{w}\cdot d\underline{j}_o \tag{9}$$

where

$$\rho\Phi = E_o - \underline{w}\cdot\underline{j}_o - T\rho s + p. \tag{10}$$

From (8), we have that $\left(\frac{\partial E_o}{\partial \underline{j}_o}\right)_{\rho,s} = \underline{j}_o \left(\frac{\partial f}{\partial[\frac{1}{2}j_o^2]}\right)_{\rho,s}$; since $\underline{j}_o = \rho_n \underline{w}$, we see that the normal fluid density is given by

$$\rho_n = \frac{1}{\left(\frac{\partial f}{\partial[\frac{1}{2}j_o^2]}\right)_{\rho,s}}, \tag{11}$$

and we may solve (11) to obtain ρ_n as a function of ρ, s and $\frac{1}{2}w^2$. Thus the specification of the function $E_o(\rho, s\underline{j}_o)$ determines both the dependence of the total energy on the independent macroscopic quantities, and the dependence of the normal fluid density on ρ, s, $\frac{1}{2}w^2$.

For purposes of comparison with other formulations of the two-fluid thermodynamics, it is convenient to introduce the internal energy per unit mass e, defined by

$$E = \tfrac{1}{2}\rho_n v_n^2 + \tfrac{1}{2}\rho_s v_s^2 + \rho e. \tag{12}$$

Then one may show from (6), (9) and (10) that

$$\rho e = E_o - \tfrac{1}{2}\rho_n w^2, \tag{13}$$

and

$$de = \frac{p}{\rho^2}d\rho + Tds + \tfrac{1}{2}w^2 dx, \tag{14}$$

where

$$x = \frac{\rho_n}{\rho} \tag{15}$$

is the normal fluid concentration. Thus the natural independent variables for e are ρ, s and x (instead of ρ, s, $\frac{1}{2}j_o^2$), and the relation analogous to (11) is

$$\left(\frac{\partial e}{\partial x}\right)_{\rho,s} = \tfrac{1}{2}w^2. \tag{16}$$

Some comment about equation (10) (which may be regarded as a definition of the pressure p) is perhaps in order. First, one may show from (6) - (10) that the derivative of the total energy in a volume v with respect to v (at fixed total momentum, entropy and mass, and fixed superfluid velocity) is $-p$; second, we shall see that the quantity p does indeed play the role of a normal stress in the hydrodynamic equations.

Although the arguments leading to equation (9) (and, in particular, to the relation $\left(\frac{\partial E_o}{\partial \underline{j}_o}\right)_{\rho,s} = \underline{w}$ are perhaps not compelling, it will be shown later (sections II-A-2 and II-A-3) that different approaches to the determination of the thermodynamic description of helium II lead to the same results.

The remaining problem in the derivation of the hydrodynamic equations is the determination of the quantities $\underline{\mathcal{J}}_\rho$, π_{ij}, $\underline{\mathcal{J}}_s$, $\underline{Q}$ and $\underline{f}$.

By analogy with ordinary hydrodynamics, we expect that the mass flux $\underline{\mathcal{J}}_\rho$ may be identified with the momentum density $\underline{j}$; thus we assume

$$\underline{\mathcal{J}}_\rho = \underline{j} = \rho_n \underline{v}_n + \rho_s \underline{v}_s \,. \tag{17}$$

This, of course, is consistent with a very literal interpretation of the two-fluid model. However, there does remain the possibility that $\underline{\mathcal{J}}_\rho \neq \underline{j}$; in fact, Lee and Yang [21] have derived, for a dilute system of hard sphere bosons, a set of transport equations which exhibit this possibility. Since much of the "physical feeling" for the macroscopic quantities of the two-fluid model is rooted in the concepts of ordinary hydrodynamics, the introduction of a mass flux $\underline{\mathcal{J}}_\rho \neq \underline{j}$ would require a critical re-evaluation of the physical significance of the quantities ρ_n , ρ_s , $\underline{v}_n$ and $\underline{v}_s$. In the following we will always assume (17) to hold.

It is usually assumed in the two-fluid model that the total entropy resides in the normal fluid; in view of the microscopic picture of the normal fluid as a gas of thermal excitations, this is a natural assumption to make. In the perfect fluid theory for the two-fluid model, the only mechanism for entropy transport is convection; thus we take the entropy flux $\underline{\mathcal{J}}_s$ to be

$$\underline{\mathcal{J}}_s = \rho s \underline{v}_n \,. \tag{18}$$

We now consider the determination of the fluxes π_{ij} and

$\underline{Q}$. We note first that the conservation of angular momentum (assuming that the angular momentum density is $\underline{r} \times \underline{j}$) requires that π_{ij} be symmetric. Since the conservation equations must be invariant in form under Galilean transformations, and since the transformation properties of E and $\underline{j}$ under such a transformation are known (cf. (6) and (7)), the transformation properties of the fluxes π_{ij} and $\underline{Q}$ may be easily deduced. We may use these transformation formulae to express π_{ij} and $\underline{Q}$ in terms of π_{ij}°, the momentum flux as seen in the superfluid rest frame, and $\underline{Q}^{\circ}$, the energy flux as seen in the superfluid rest frame. Thus

$$\pi_{ij} = \rho v_{si} v_{sj} + v_{si} j_{oj} + v_{sj} j_{oi} + \pi_{ij}^{\circ} \tag{19}$$

$$\underline{Q} = \left(\frac{\rho v_s^2}{2} + \underline{j}_o \cdot \underline{v}_s + E_o\right)\underline{v}_s + \frac{v_s^2}{2}\underline{j}_o + \left(\underline{\underline{\pi}}^{\circ} \cdot \underline{v}_s\right) + \underline{Q}^{\circ}, \tag{20}$$

where $\underline{\underline{\pi}}^{\circ} \cdot \underline{v}_s$ denotes the vector with components $\pi_{ij}^{\circ} v_{sj}$. The advantage of this is that π_{ij}° and $\underline{Q}^{\circ}$, being Galilean invariants, can depend on $\underline{v}_n$ and $\underline{v}_s$ only through the difference $\underline{w} = \underline{v}_n - \underline{v}_s$. Thus the problem is to determine π_{ij}°, $\underline{Q}^{\circ}$ and the function $\underline{f}$ in equation (5).

The range of possibilities for the function $\underline{f}$ depends greatly on whether or not the superfluid can rotate. According to Landau's original view [20], the superfluid flow must be irrotational, which means that $\underline{f}$ must have the form

$$\underline{f} = \nabla \phi. \tag{21}$$

According to the later work of Feynman [7] and Hall and Vinen

[12,13,14,38], the superfluid remains point-wise irrotational but imitates bulk rotation by means of quantized vortex lines. In the theories of Hall and Vinen, the rotation is held to be closely connected with the dissipative phenomenon of mutual friction. According to Lin's theory [26, 25,24] , irrotational motion of the superfluid is a possible solution of the perfect fluid equations, but the equations also allow more general solutions for which $\operatorname{curl} \underline{v}_s \neq 0$. A detailed discussion of the question of superfluid rotation has been given by Lin [25]. Since this is still an unsettled question, we consider both possibilities here.

We examine first the case when the superfluid flow must be irrotational; then equation (21) must hold, and the problem is reduced to the determination of the quantities Φ, π_{ij}° and $\underline{Q}^{\circ}$. As mentioned earlier, the 9 (scalar) equations represented by (1) - (5) are not independent, and we must require that they be consistent. To impose this requirement, we first calculate $\frac{\partial E}{\partial t}$ in terms of $\frac{\partial \rho}{\partial t}$, $\frac{\partial s\rho}{\partial t}$, $\frac{\partial \underline{v}_s}{\partial t}$ and $\frac{\partial \underline{j}}{\partial t}$ by means of (6) and (7); then we may express these last time derivatives in terms of the fluxes by means of (1) - (3) and (5). Finally we substitute the expression for $\frac{\partial E}{\partial t}$ into (4), and the resulting equation must be an identity, since it contains no time derivatives. It is convenient to express the final result in terms of $\pi_{ij}^{\circ\prime}$, $\underline{Q}'$ and Φ', where

$$\pi_{ij}^{\circ} = p\delta_{ij} + j_{oi} w_j + \pi_{ij}^{\circ\prime}, \tag{22}$$

$$\underline{Q}_o = \Phi \underline{j}_o + T\rho s \underline{w} + \underline{j}_o w^2 + \underline{Q}', \tag{23}$$

and $$\Phi = -\Phi + \Phi' . \qquad (24)$$

The result is

$$\operatorname{div}\left\{ Q' - \underline{\underline{\pi}}^{o\prime} \cdot \underline{w} \right\} + \pi_{ik}^{o\prime} \frac{\partial v_{ni}}{\partial x_k} + \left(\underline{j} - \rho \underline{v}_n \right) \cdot \nabla \Phi' \equiv 0 . \qquad (25)$$

One obvious "solution" of (25) is $\underline{Q}' \equiv 0$, $\underline{\underline{\pi}}^{o\prime} \equiv 0$ and $\Phi' \equiv 0$; however, one can also find non-zero solutions, so some further information is needed in order to obtain a unique final answer. In principle, the quantities $\pi_{ij}^{o\prime}$, $\underline{Q}'$ and Φ' may depend on ρ , ρs and $\underline{w}$, as well as spatial derivatives of all of the independent macroscopic variables. In a perfect fluid theory, however, one might expect that the fluxes will not depend on the gradients of the macroscopic variables, although there seems to be no compelling reason to believe this. In any case, the "simplest" set of hydrodynamic equations will correspond to this case, so we simply assume that π_{ij}^{o} , $\underline{Q}_o$ and Φ (and consequently $\pi_{ij}^{o\prime}$, Q_o' and Φ') depend only on the relative velocity $\underline{w}$ (by Galilean invariance, these quantities cannot depend on $\underline{v}_n$ and $\underline{v}_s$ separately) and on the thermodynamic quantities, and not on the spatial derivatives of these quantities. Then, from the first and third terms in (25), only gradients of w_i - and not of v_{ni} alone - can arise; since $\pi_{ik}^{o\prime}$ is to be independent of these gradients, it must be that $\underline{\underline{\pi}}^{o\prime} \equiv 0$. Then (25) takes the form

$$\operatorname{div} \underline{Q}' + \left(\underline{j} - \rho \underline{v}_n \right) \cdot \nabla \Phi' \equiv 0 ,$$

or

$$\operatorname{div} \underline{Q}' - \rho_s \underline{w} \cdot \nabla \Phi' \equiv 0. \tag{26}$$

Since $\underline{w}$ is the only vector on which $\underline{Q}'$ may depend, $\underline{Q}' = \lambda \underline{w}$, where λ is a Galilean invariant scalar function. A straightforward analysis then shows that $\lambda \equiv 0$, and $\Phi' \equiv$ constant (which we may take as zero). Thus under the assumptions made, we obtain a unique set of hydrodynamic equations. The final expressions for the fluxes may be written as

$$\pi_{ij} = \rho_n v_{ni} v_{nj} + \rho_s v_{si} v_{sj} + p\delta_{ij}, \tag{27}$$

$$\underline{Q} = \left(\Phi + \tfrac{1}{2} v_s^2\right) \underline{j} + T\rho s \underline{v_n} + \rho_n \underline{v_n} \left(\underline{w} \cdot \underline{v_n}\right), \tag{28}$$

$$\underline{f} = -\nabla \Phi. \tag{29}$$

For purposes of comparison with other derivations, we write out the final equations here; for convenience, the internal energy e (cf. (13)) is used in place of E_0. The equations are

mass:
$$\frac{\partial \rho}{\partial t} + \operatorname{div}\left(\rho_n \underline{v_n} + \rho_s \underline{v_s}\right) = 0, \tag{30}$$

momentum:
$$\frac{\partial}{\partial t}\left(\rho_n \underline{v_n} + \rho_s \underline{v_s}\right) + \operatorname{div}\left(\rho_n \underline{v_n} v_n + \rho_s \underline{v_s} v_s\right) =$$

$$= -\nabla p, \tag{31}$$

entropy: $$\frac{\partial}{\partial t}(\rho s) + \operatorname{div}(\rho s \underline{v}_n) = 0, \tag{32}$$

and the superfluid equation: $$\frac{\partial \underline{v}_s}{\partial t} = -\nabla\left(\tfrac{1}{2} v_s^2 + \Phi\right),$$

or

$$\frac{\partial \underline{v}_s}{\partial t} + \underline{v}_s \nabla \underline{v}_s = s\nabla T - \frac{\nabla p}{\rho} + x \nabla \tfrac{1}{2} w,^2 \tag{33}$$

and the restrictive equation $\operatorname{curl} \underline{v}_s \equiv 0$.
These equations imply the energy equation, which may be written as

$$\frac{\partial}{\partial t}\left(\tfrac{1}{2}\rho_n v_n^2 + \tfrac{1}{2}\rho_s v_s^2 + \rho e\right) + \operatorname{div}\left\{\tfrac{1}{2}\rho_n v_n^2 \underline{v}_n + \right. \tag{35}$$

$$\left. + \tfrac{1}{2}\rho_s v_s^2 \underline{v}_s + (\rho e + p)\underline{v} + T\rho s(\underline{v}_n - \underline{v}) + \rho x w^2 (\underline{v}_n - \underline{v})\right\} = 0,$$

where we have introduced the mass velocity $\underline{v} = x\underline{v}_n + (1-x)\underline{v}_s = \frac{\underline{j}}{\rho}$.
The energy equation may be put into another form which clearly shows the physical significance of the various terms (Zilsel [40]; London [29]):

$$\frac{\partial}{\partial t}\left(\tfrac{1}{2}\rho_n v_n^2 + \tfrac{1}{2}\rho_s v_s^2 + \rho e\right) + \operatorname{div}\left\{\tfrac{1}{2}\rho_n v_n^2 \underline{v}_n + \tfrac{1}{2}\rho_s v_s^2 \underline{v}_s + \rho e \underline{v}\right. \tag{36}$$

$$\left. + \rho s\left(\frac{\partial e}{\partial s}\right)_{\rho,x}(\underline{v}_n - \underline{v}) + \rho x\left(\frac{\partial e}{\partial x}\right)_{\rho,s}(\underline{v}_n - \underline{v})\right\} = -\operatorname{div}(p\underline{v}).$$

The first three terms in the energy flux $(\frac{1}{2}\rho_n v_n^2 \underline{v}_n + \frac{1}{2}\rho_s v_s^2 \underline{v}_s + \rho e \underline{v})$

represent the convection of energy by the motion; the last two terms give a correction to the term $\rho e \underline{v}$, this correction being needed because the entropy and normal fluid concentration follow the motion of the normal fluid rather than the bulk motion. Finally, we may obtain an equation for the normal fluid by combining the mass, momentum and superfluid equations. The result is

$$\frac{\partial \underline{v}_n}{\partial t} + \underline{v}_n \cdot \nabla \underline{v}_n = -\frac{1}{\rho} \nabla p - \frac{1-x}{x} s \nabla T - \frac{1-x}{2} \nabla w^2 - \underline{w} \frac{\Gamma}{\rho_n}, \quad (37)$$

where

$$\Gamma = \frac{\partial \rho_n}{\partial t} + \operatorname{div}(\rho_n \underline{v}_n) \quad (38)$$

and is the volume rate of conversion. (Γ of course cannot be independently specified, but is determined by the spatial distribution of the independent quantities; in fact one can obtain an explicit (though not very enlightening) expression for Γ).

We consider now the boundary conditions to be satisfied at a solid-helium II interface. For convenience, we assume that the solid is at rest (which we may accomplish by an appropriate Galilean transformation). First of all, we must require that there be no mass flux across the interface; thus

$$\underline{j} \cdot \underline{n}\Big|_{\text{wall}} = \rho_n (\underline{v}_n \cdot \underline{n}) + \rho_s (\underline{v}_s \cdot \underline{n})\Big|_{\text{wall}} = 0 \quad (39)$$

or

$$x (\underline{v}_n \cdot \underline{n}) = -(1-x)(\underline{v}_s \cdot \underline{n}) \qquad \text{at the wall.}$$

If there is no energy transport across the wall, then we must also have $\underline{Q}\cdot\underline{n}=0$ at the wall. (Equation (39)) of course holds whether or not there is an energy transport.) This, in conjunction with (39), leads to $\underline{v_n}\cdot\underline{n}=\underline{v_s}\cdot\underline{n}=0$. Finally, it is expected that the temperature will be continuous in this case. Thus

$$\underline{v_n}\cdot\underline{n}\Big|_{\text{wall}}=0, \qquad \text{(case of no energy transport)}$$

$$\underline{v_s}\cdot\underline{n}\Big|_{\text{wall}}=0, \qquad (40)$$

and

$$T_{\text{fluid}}=T_{\text{wall}}.$$

The discussion of the boundary conditions for the case of non-zero energy transport is complicated by a peculiar theoretical difficulty which is best illustrated by an example. Suppose there is a uniform heat flux of magnitude H from the solid to the helium II (see Fig. 1). Then by conservation of energy, we have

$$\underline{Q}\cdot\underline{n}=\underline{H}\cdot\underline{n}$$

which, by using (39), we may write as

$$(\underline{v_n}\cdot\underline{n})\left\{\frac{\rho x}{1-x}(\underline{v_n}\cdot\underline{n})^2+\rho s T_{\text{fluid}}\right\}=\underline{H}\cdot\underline{n}, \qquad (41)$$

where for simplicity, we have assumed that the tangential component of $\underline{v_n}$ vanishes. Since $\underline{H}\cdot\underline{n}>0$ and since the quantity in brackets is always positive, we have $\underline{v_n}\cdot\underline{n}>0$. Now consider

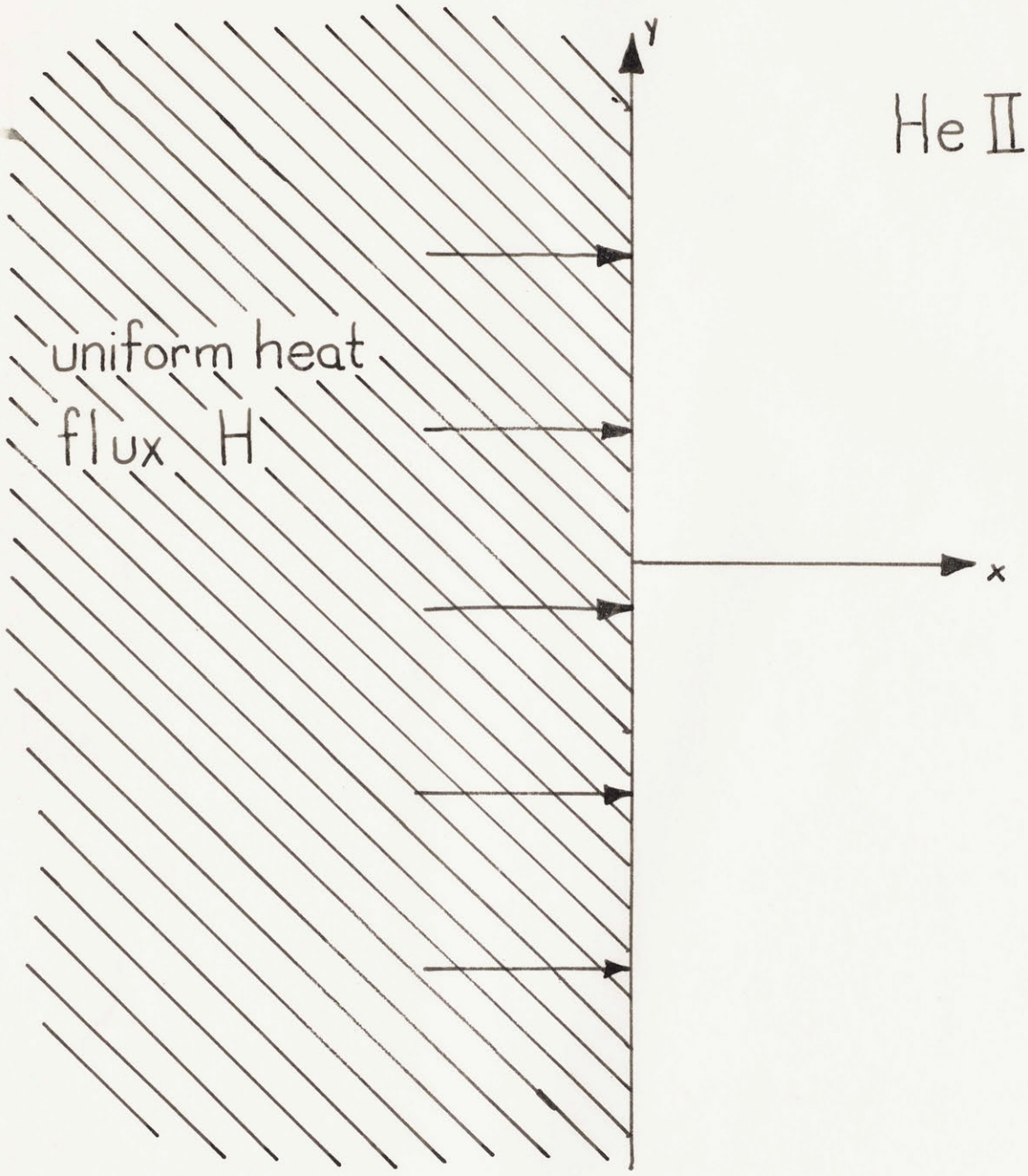

Figure 1

the entropy flux in the solid; this is simply $\frac{(\underline{\underline{H}}\cdot\underline{n})}{T_{wall}}$. The entropy flux in the fluid is given by $\rho s(\underline{v}_n\cdot\underline{n})$. By the second law of thermodynamics, the entropy flux in the fluid must be at least as large as that in the solid; hence

$$\rho s(\underline{v}_n\cdot\underline{n}) \geq \frac{\underline{\underline{H}}\cdot\underline{n}}{T_{wall}} = \frac{(\underline{v}_n\cdot\underline{n})}{T_{wall}}\left\{\frac{\rho x}{1-x}(\underline{v}_n\cdot\underline{n})^2 + \rho s T_{fluid}\right\},$$

or, since $\underline{v}_n\cdot\underline{n}>0$, we may write this as

$$T_{wall} - T_{fluid} \geq \frac{x}{(1-x)s}(\underline{v}_n\cdot\underline{n})^2 \cdot \begin{pmatrix}\text{net energy flux}\\ \text{directed into}\\ \text{fluid}\end{pmatrix} \qquad (42)$$

It would appear that the temperature of the wall must be greater than that of the fluid. The reason for (41) is clear: some of the thermal energy in the solid is converted into what is mechanical energy in the helium; if the temperatures were equal, we would have a decrease of entropy. (It should be pointed out that the actual magnitude of the effect is extremely small; if, for example, $T_{fluid} = 1.8^\circ K$ and $\underline{v}_n\cdot\underline{n} = 10\,cm/sec$, then $x \simeq \frac{1}{3}$, $s \simeq 5.35\cdot 10^6 \frac{ergs}{gm.deg.}$, and (42) gives $T_{wall} - T_{fluid} \geq \sim 10^{-5}{}^\circ K$). If the net energy transport is from the fluid into the solid, then the second law requires only that

$$T_{wall} - T_{fluid} \leq \frac{x}{(1-x)s}(\underline{v}_n\cdot\underline{n})^2 \begin{pmatrix}\text{net energy}\\ \text{flux directed}\\ \text{into solid}\end{pmatrix} \qquad (43)$$

and, in particular, we may have $T_{wall} = T_{fluid}$. The result (42) means that we cannot specify $T_{wall} = T_{fluid}$ in the case of a heat flux into the fluid, and in fact, it is not clear on the basis

of the perfect fluid theory what the boundary conditions on T_{fluid} should be. However, it is possible to resolve this question by means of the more general theories which include dissipative processes, and this analysis will be given in detail in part III. For now we only note that in almost all cases of practical interest, the kinetic energy terms in the energy flux are negligible, and we may use a set of effective boundary conditions of the form

$$x(\underline{v}_n \cdot \underline{n}) + (1-x)(\underline{v}_s \cdot \underline{n}) = 0,$$

$$\rho s T_{fluid} (\underline{v}_n \cdot \underline{n}) = \underline{H} \cdot \underline{n}, \tag{44}$$

$$(T_{wall} - T_{fluid}) = A(\underline{H} \cdot \underline{n}),$$

where the thermal resistance A depends on the fluid temperature T_{fluid}.

In the perfect fluid theory, there are, of course, no conditions imposed on the tangential components of $\underline{v}_n$ and $\underline{v}_s$.

This completes the presentation of Landau's derivation of the hydrodynamic equations for the two-fluid model. Although the method can be extended to the case of irreversible processes, a discussion of this will be deferred to part III. It is perhaps well to explicitly list the assumptions and principles which were necessary in order to obtain a definite set of hydrodynamic equations: (i) the two-fluid model (with densities ρ_n, ρ_s and velocities $\underline{v}_n$, $\underline{v}_s$, together with the (usual) assumption that the superfluid component carries no

entropy), (ii) the imposition of conservation laws and the Galilean relativity principle, (iii) an assumption about the form of the thermodynamic energy function E_0, (iv) the assumption that $\mathrm{curl}\, \underline{v}_s \equiv 0$ always, and (v) the assumption that the fluxes do not depend on the gradients of the fundamental macroscopic variables. If the assumption $\mathrm{curl}\, \underline{v}_s = 0$ is dropped, then it is no longer possible to obtain a definite set of hydrodynamic equations. We give a brief discussion of the results in this case below.

If it is no longer required that $\mathrm{curl}\, \underline{v}_s$ must be zero, then equation (21) no longer is necessarily true. It is still convenient to introduce the quantities $\pi_{ij}^{0\prime}$ and $\underline{Q}'$ (equations (22) and (23)), and it is convenient to introduce

$$\underline{f} = -\nabla \Phi + \underline{f}'. \tag{45}$$

The calculations are exactly as before, and the restrictive equation corresponding to (25) is

$$\mathrm{div}\left\{\underline{Q}' - \underline{\underline{\pi}}^{0\prime} \cdot \underline{v}_n\right\} + \pi_{ij}^{0\prime} \frac{\partial v_{ni}}{\partial x_j} + (\underline{j} - \rho \underline{v}_n) \cdot \left(\underline{f}' + \left[\mathrm{curl}\, \underline{v}_s\right] \times \underline{w}\right) \equiv 0. \tag{46}$$

Since $(-\underline{j} + \rho \underline{v}_n) = \rho_s \underline{w}$, this may be written as

$$\mathrm{div}\left\{\underline{Q}' - \underline{\underline{\pi}}^{0\prime} \cdot \underline{v}_n\right\} + \pi_{ij}^{0\prime} \frac{\partial v_{ni}}{\partial x_j} - \rho_s \underline{w} \cdot \underline{f}' \equiv 0. \tag{47}$$

In the case of $\mathrm{curl}\, \underline{v}_s = 0$, we have $\underline{f} = -\nabla \Phi$, where Φ depends only on thermodynamic quantities (and the relative velocity $\underline{w}$); $\underline{f}$, however, depends on the spatial derivatives of

these quantities so we must expect that, in the case $\mathrm{curl}\,\underline{v}_s \neq 0$ the "force" $\underline{f}'$ in general will depend on the gradients of the macroscopic variables.

The equation (41) allows a wide variety of "solutions". In particular, we see that it is satisfied for $\pi_{ij}^{o'} \equiv 0$, $\underline{Q}' \equiv 0$ and $\underline{f}' = \underline{w} \times \underline{c}$, where $\underline{c}$ is an arbitrary (Galilean invariant) vector. (If we choose $\underline{c} = \times \mathrm{curl}\,\underline{v}_s$ so that $\underline{f}' = \times \underline{w} \times \mathrm{curl}\,\underline{v}_s$, the resulting equations are identical with those obtained by Lin [25,24,26]; these equations will be discussed in section II-B). The appearance of only the normal fluid rate of strain tensor in (41) is somewhat misleading, since it is merely a consequence of the choice of notation. In fact, if we decompose the (as yet unknown) functions $\underline{\underline{\pi}}^{o'}$ and $\underline{f}'$ as follows:

$$\pi_{ij}^{o'} = \pi_{ij}^{n'} + \pi_{ij}^{s'},$$

$$\rho_s \underline{f}' = \underline{F} - \mathrm{div}\,\underline{\underline{\pi}}^{s'}, \tag{48}$$

then equation (47) may be written in the more symmetric form

$$\mathrm{div}\left\{ \underline{Q}' - \underline{\underline{\pi}}^{n'} \cdot \underline{v}_n - \underline{\underline{\pi}}^{s'} \cdot \underline{v}_s \right\} + \pi_{ij}^{n'} \frac{\partial v_{ni}}{\partial x_j} + \pi_{ij}^{s'} \frac{\partial v_{si}}{\partial x_j} - \underline{w} \cdot \underline{F} \equiv 0. \tag{49}$$

By way of example, we give the following specification of $\underline{Q}'$, $\underline{\underline{\pi}}^{n'}$, $\underline{\underline{\pi}}^{s'}$ and $\underline{F}$, which satisfies (49):

$$\underline{Q}' = \underline{\underline{\pi}}^{n'} \cdot \underline{v}_n + \underline{\underline{\pi}}^{s'} \cdot \underline{v}_s,$$

$$\pi_{ij}^{n'} = \beta_n w_i w_j, \quad \pi_{ij}^{s'} = \beta_s w_i w_j, \tag{50}$$

and $\underline{F} = \beta_n \underline{w} \cdot \nabla \underline{v}_n + \beta_s \underline{w} \cdot \nabla \underline{v}_s$

(where β_n , β_s are arbitrary scalar functions of the thermodynamic variables and of w^2).

The above examples show clearly that one needs some additional information in the case $\text{curl}\, \underline{v}_s \neq 0$ in order to obtain a definite set of hydrodynamic equations. Actually, it is not so surprising that there should be some difficulty in deducing a unique set of hydrodynamic equations from conservation laws, because we cannot really expect to obtain any information about the momentum transfer between the two components from the fact that the total momentum is conserved.

Although it was a part of Landau's original theory [20] that there should be no momentum exchange between the two components (other than that due to $\rho_n \rightleftarrows \rho_s$ transitions), he did not need to appeal to this principle in his development of the hydrodynamic equations, since the conservation laws plus the requirement $\text{curl}\, \underline{v}_s = 0$ uniquely determined these equations. In the next section we will show that if the requirement $\text{curl}\, \underline{v}_s = 0$ is dropped, one may still obtain a unique set of hydrodynamic equations by making use of this principle of no momentum exchange. Furthermore, the resultling equations are identical in form with Landau' equations (without the restriction $\text{curl}\, \underline{v}_s = 0$).

2. Equations from formal analogy with ordinary hydrodynamics

In this section we will give an alternative derivation of the perfect fluid equations; the final equations obtained are the Landau equations without the restriction $\mathrm{curl}\, \underline{v}_s = 0$. In this way the Landau equations--which have been well-verified experimentally--are freed from the assumption that the the superfluid cannot rotate. The derivation is based primarily on analogy with ordinary hydrodynamics, and also the principle that there should be no momentum exchange between the two components; appeal is also made to certain qualitative ideas from the microscopic theory. It will be seen that the point of view taken here affords a particularly simple characterization of Landau's hydrodynamic equations.

The starting point for the derivation is the two-fluid model; we assume that the helium may be regarded as a "mixture" of two fluids; the normal fluid has density ρ_n , velocity $\underline{v}_n$, and entropy $\rho_n s_n$ per unit volume, while the superfluid has density ρ_s , velocity $\underline{v}_s$ and no entropy (the possibility of superfluid entropy is easily included in the following theory; however, there are no experiments as yet to indicate such a generalization). We assume that each of the fluids has a thermodynamic internal energy function, so that if ϵ_n , ϵ_s are the total energies per unit volume of the normal fluid and superfluid respectively, we have

$$\epsilon_n = \frac{1}{2} \rho_n v_n^2 + \rho_n e_n$$

and $$\epsilon_s = \frac{1}{2}\rho_s v_s^2 + \rho_s e_s \,, \tag{51}$$

where e_n is the internal energy of the normal fluid, per unit mass of normal fluid, and similarly for e_s. We expect that e_n will be a function of ρ_n, s_n, and that the normal fluid pressure p_n and the normal fluid temperature T_n are given by

$$de_n(\rho_n, s_n) = \frac{p_n}{\rho_n^2}\, d\rho_n + T_n ds_n \,. \tag{52}$$

Since the superfluid component has no entropy, e_s will be a function of ρ_s only, and the superfluid pressure p_s is given by

$$de_s(\rho_s) = \frac{p_s}{\rho_s^2}\, d\rho_s \tag{53}$$

The existence of the functions e_n, e_s and the equations (52), (53) are simply assumed on the basis of analogy with the thermodynamics of ordinary substances.

Suppose now we consider the total energy $\mathcal{E}$ in a fixed volume V ; $\mathcal{E}$ will be given by

$$\mathcal{E} = (\epsilon_n + \epsilon_s)V = \epsilon V \tag{54}$$

The total momentum $\underline{J}$, mass M and entropy S in V are given by

$$\underline{J} = V(\underline{j}_n + \underline{j}_s), \quad (\underline{j}_n = \rho_n \underline{v}_n, \underline{j}_s = \rho_s \underline{v}_s), \quad M = V(\rho_n + \rho_s)$$

and $$S = V\rho_n s_n \,. \tag{55}$$

If the two fluids are in equilibrium, we would expect that the distribution of the total momentum and the total mass over the two components would be given by an energy minimum principle, namely

$$d\epsilon\big|_{\underline{J},M,S,V} = 0 \,. \tag{56}$$

This equation leads to the results

$$\underline{v}_n = \underline{v}_s \,, \tag{57}$$

and

$$\Phi_n = \Phi_s \,, \tag{58}$$

where we have introduced the thermodynamic potentials Φ_n and Φ_s, defined by

$$\Phi_n = e_n - TS_n + \frac{p_n}{\rho_n} \,, \text{ and } \Phi_s = e_s + \frac{p_s}{\rho_s} \,. \tag{59}$$

Equation (57) would seem to say that the two fluids must move together when in equilibrium. In fact, Lee and Yang [21] have emphasized the point of view that the states for which $\underline{v}_n \neq \underline{v}_s$ are in reality <u>quasi-equilibrium</u> states with a macroscopic lifetime. The problem, then, is to modify the equilibrium condition (56) so as to allow for such states; to do this, we borrow some qualitative ideas from the microscopic theory ([29] and [21]). The key point is that the superfluid velocity $\underline{v}_s$ is not an average over a thermal distribution, and a change in $\underline{v}_s$ requires a coherent change in the state of a large number of particles; thus we may take $\underline{v}_s$ to be an invariant of a microscopic collision. The collisions of course also conserve momentum, mass and energy, so that the natural

thermodynamic description is to regard the entropy as a function of the momentum, mass, energy and superfluid velocity $\underline{v_s}$, or, equivalently, to regard the energy as a function of the entropy, mass, momentum and superfluid velocity. From this point of view, then, it is natural to take $\underline{j}$, ρ, s, $\underline{v_s}$, as the eight independent macroscopic quantities describing the motion. We may then expect that the modified equilibrium principle takes the form

$$d\epsilon\Big|_{\underline{j},s,\rho,\underline{v_s}} = 0 , \tag{60}$$

and this leads to the single equilibrium condition

$$\Phi_n - \Phi_s = \tfrac{1}{2} w^2 . \tag{61}$$

If we again introduce the total internal energy per unit total mass, e, by

$$\epsilon_n + \epsilon_s = \tfrac{1}{2}\rho_n v_n^2 + \tfrac{1}{2}\rho_s v_s^2 + \rho e \tag{62}$$

or

$$\rho e = \rho_n e_n + \rho_s e_s ,$$

and the entropy s per unit total mass by

$$\rho_n s_n = \rho s \tag{63}$$

then, from (52), (53) and (61), we have

$$de = T ds + \left(\frac{P_n + P_s}{\rho^2}\right) d\rho + \tfrac{1}{2} w^2 dx , \tag{64}$$

and, if we introduce the total pressure P by

$$P = P_n + P_s , \tag{65}$$

(64) becomes

$$de = Tds + \frac{P}{\rho^2} d\rho + \tfrac{1}{2} w^2 dx . \tag{66}$$

Thus we have arrived at the same thermodynamic description as that given in the last section; however, the above derivation emphasizes the nature of the equilibrium relation and the distinctive nature of the superfluid velocity $\underline{v_s}$. For later convenience, we note here that the differentials of the partial pressures P_n , P_s are given by

$$\begin{aligned} \frac{dP_n}{P_n} &= \frac{dP}{\rho} + \frac{(1-x)}{x} s dT + (1-x) d\left(\tfrac{1}{2} w^2\right) , \\ \frac{dP_s}{P_s} &= \frac{dP}{\rho} - s dT - x\, d\left(\tfrac{1}{2} w^2\right) . \end{aligned} \tag{67}$$

It should be pointed out that the development of the thermodynamics to this point could also have been used in the last section, since, in general, the thermodynamic description of the system is something which must be decided on before obtaining a set of hydrodynamic equations.

It is perhaps of interest to pause here to consider the relation between the thermodynamic argument given here and the argument given by Landau (presented in the preceding section). According to Landau, the observer moving with the superfluid

sees something quite similar to the motion of an ordinary fluid with a total energy E_0 (cf. equation (6)) per unit volume, momentum $\underline{j}_0 = \rho_n \underline{w}$ per unit volume and an entropy ρs per unit volume. Thus if we consider a volume V moving with the superfluid, the total energy is (equations (51) and (54) evaluated in the rest frame of the superfluid)

$$\varepsilon_0 = V E_0 = V\left\{\tfrac{1}{2}\rho_n w^2 + \rho_n e_n + \rho_s e_s\right\},$$

the total momentum is $J_0 = V \rho_n \underline{w}$, the total entropy is $S = \rho s V = \rho_n s_n V$, and the total mass is $M = V(\rho_n + \rho_s)$. On the basis of Landau's argument and the energy minimum principle of thermodynamics, we might expect the equilibrium to be determined by

$$d\varepsilon_0\Big|_{\underline{J}_0, V, M, S} = 0 \qquad (68)$$

and, in fact, (68) yields the condition (61) ($\Phi_n - \Phi_s = \tfrac{1}{2} w^2$).

We now consider the derivation of the hydrodynamic equations for reversible processes. The main principles to be used here are (i) analogy with ordinary hydrodynamics, and (ii) the assumption that there is no coupling between the fluids except the $\rho_n \rightleftarrows \rho_s$ transitions.

First of all, there will be two continutity equations which, however, must take into account the $\rho_n \rightleftarrows \rho_s$ transitions which conserve only the total mass. Thus if we let Γ be the volume rate of conversion of superfluid into normal fluid, we may write the continuity equations as

$$\frac{\partial \rho_n}{\partial t} + \operatorname{div}\left(\rho_n \underline{v}_n\right) = \Gamma \,, \tag{69}$$

and

$$\frac{\partial \rho_s}{\partial t} + \operatorname{div}\left(\rho_s \underline{v}_s\right) = -\Gamma .$$

(Again, Γ is not independently specified, but is determined by the flow conditions.)

To obtain the momentum equations, we assume that each fluid simply satisfies an Euler equation (with the normal fluid pressure p_n acting only on the normal fluid and the superfluid pressure p_s acting on the superfluid) with the momentum exchange due to the transitions $\rho_n \rightleftarrows \rho_s$ included. Since the transitions conserve total momentum, we may write the two momentum equations as

$$\frac{\partial}{\partial t}\left(\rho_n \underline{v}_n\right) + \operatorname{div}\left(\rho_n \underline{v}_n \underline{v}_n\right) = -\nabla p_n + \underline{\mathcal{F}} \,,$$

$$\frac{\partial}{\partial t}\left(\rho_s \underline{v}_s\right) + \operatorname{div}\left(\rho_s \underline{v}_s \underline{v}_s\right) = -\nabla p_s - \underline{\mathcal{F}} \,, \tag{70}$$

where $\underline{\mathcal{F}}$ represents the effect of transitions. $\underline{\mathcal{F}}$ may be expressed in terms of the transition rate Γ as follows: in a unit volume, there is superfluid momentum of amount $\rho_s \underline{v}_s$; the transitions do not change $\underline{v}_s$, and the change in ρ_s in time δt due to transitions is $-\Gamma \delta t$; thus the rate of change of $\rho_s \underline{v}_s$ due to transitions is $-\Gamma \underline{v}_s$, and, since the transitions conserve the total momentum, the rate of change of the

normal fluid momentum is

$$\underline{\mathcal{J}} = \Gamma \underline{v_s} . \tag{71}$$

By using (69) and (71), we may write (70) as

$$\frac{\partial \underline{v_n}}{\partial t} + \underline{v_n} \cdot \nabla \underline{v_n} = -\frac{\nabla p_n}{\rho_n} - \frac{\Gamma}{\rho_n} \underline{w}, \tag{72}$$

and

$$\frac{\partial \underline{v_s}}{\partial t} + \underline{v_s} \cdot \nabla \underline{v_s} = -\frac{\nabla p_s}{\rho_s} , \tag{73}$$

or, by (67),

$$\frac{\partial \underline{v_n}}{\partial t} + \underline{v_n} \cdot \nabla \underline{v_n} = -\frac{\nabla p}{\rho} - \frac{(1-x)}{x} S \nabla T - (1-x) \nabla \tfrac{1}{2} w^2, \tag{74}$$

$$\frac{\partial \underline{v_s}}{\partial t} + \underline{v_s} \cdot \nabla \underline{v_s} = -\frac{\nabla p}{\rho} + S \nabla T + x \nabla \tfrac{1}{2} w^2, \tag{75}$$

and these are the same as the equations originally obtained by Landau, without, however, the restriction $\operatorname{curl} \underline{v_s} = 0$.

Finally, we have the equation expressing the conservation of entropy, which we take as

$$\frac{\partial}{\partial t}(\rho s) + \operatorname{div}(\rho s \underline{v_n}) = 0. \tag{76}$$

It is easy to show that the above equations lead to the same equations for $\underline{j}$ and E as were obtained by means of the conservation laws. In terms of the partial pressures p_n , p_s the energy equation may be written as

$$\frac{\partial}{\partial t}\left\{\tfrac{1}{2}\rho_n v_n^2 + \rho_n e_n + \tfrac{1}{2}\rho_s v_s^2 + \rho_s e_s\right\} + \operatorname{div}\left\{\left[p_n + \rho_n e_n + \tfrac{1}{2}\rho_n v_n^2\right]\underline{v}_n + \left[p_s + \rho_s e_s + \tfrac{1}{2}\rho_s v_s^2\right]\underline{v}_s\right\} = 0 , \tag{77}$$

and the energy equations for each component read

$$\frac{\partial}{\partial t}\left\{\tfrac{1}{2}\rho_n v_n^2 + \rho_n e_n\right\} + \operatorname{div}\left\{\left[p_n + \rho_n e_n + \tfrac{1}{2}\rho_n v_n^2\right]\underline{v}_n\right\} = \Gamma\left[\Phi_n - \tfrac{1}{2}w^2 + \tfrac{1}{2}v_s^2\right], \tag{78}$$

$$\frac{\partial}{\partial t}\left\{\tfrac{1}{2}\rho_s v_s^2 + \rho_s e_s\right\} + \operatorname{div}\left\{\left[p_s + \rho_s e_s + \tfrac{1}{2}\rho_s v_s^2\right]\underline{v}_s\right\} = -\Gamma\left[\Phi_s + \tfrac{1}{2}v_s^2\right], \tag{79}$$

where the terms on the right-hand sides give the energy exchange rate due to transitions (as may be verified by a calculation similar to the one used to obtain an expression for $\underline{\mathcal{F}}$).

Thus we see that from the present point of view, each fluid satisfies the perfect fluid equations of ordinary hydrodynamics when suitable provisions are made for the $\rho_n \rightleftarrows \rho_s$ transitions. In this sense, then, Landau's equations--without the restriction $\operatorname{curl} \underline{v}_s = 0$ --are the "simplest" possible equations for the hydrodynamics of the two-fluid model.

This derivation shows that if one combines the laws of ordinary thermodynamics and hydrodynamics with a quite literal interpretation of the two-fluid model, the Landau equations are obtained in an unambiguous manner. Although the final equations are consistent with the condition $\operatorname{curl} \underline{v}_s = 0$, they do not require it and they admit more general solutions for which $\operatorname{curl} \underline{v}_s \neq 0$.

3. Variational principle

In attempting to derive the hydrodynamic equations describing reversible processes in helium II, one might expect some sort of variational principle to hold. The first attempt to obtain the equations for helium II by means of a variational principle was made by Tisza in 1947 [36], but Tisza's final equations were valid only in a linear approximation. In 1950, Zilsel [40] derived Landau's equations by means of a variational principle; this derivation (also given in London's book [29]) will be discussed in detail below. First, however, there are some general difficulties with the variational principle to be discussed.

These difficulties stem from the fact that the variational principles of mechanics apply to a Lagrangian description of the motion, whereas a Lagrangian description of the motion of helium II within the framework of the two-fluid model is manifestly impossible. One would still hope to get some sort of "Eulerian" variational principle for the two-fluid model, but there are difficulties with such a variational principle even in the case of ordinary hydrodynamics. Of course, one may use a Lagrangian description for discussing the motion of an ordinary fluid and obtain the hydrodynamic equations in a straightforward manner from a variational principle (Herivel, [16]). However, we wish to obtain a variational principle directly in terms of the Eulerian description of the motion, since this is the only sort of variational

principle which we may hope to generalize to include the two-fluid model. As a preliminary to the discussion of the variational principle for the two-fluid model, we consider the case of an ordinary fluid.

As a first approach, we consider the Lagrangian function

$$L = \int_V d^3\underline{r} \left\{ \tfrac{1}{2}\rho v^2 - \rho e \right\}, \tag{79}$$

where $\underline{v}$ is the velocity, ρ the density and $e(\rho,s)$ is the specific internal energy, s being the specific entropy. The conditions that mass and entropy be conserved are to be incorporated by means of Lagrangian multipliers, so that the modified Lagrangian is given by

$$L' = \int_V d^3\underline{r} \left\{ \tfrac{1}{2}\rho v^2 - \rho e - \alpha\left[\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \underline{v})\right] - \beta\left[\frac{\partial}{\partial t}(\rho s) + \operatorname{div}(\rho s \underline{v})\right]\right\}, \tag{80}$$

where α , β are the Lagrangian multipliers. The variational principle is then simply $\delta \int dt\, L' = 0$, with ρ , s and $\underline{v}$ varied independently. The three variational equations are

$$\delta \underline{v}: \qquad \rho\{\underline{v} + \nabla\alpha + s\nabla\beta\} = 0, \tag{81}$$

$$\delta\rho: \quad \tfrac{1}{2}v^2 - e - \frac{p}{\rho} + \frac{D\alpha}{Dt} + s\frac{D\beta}{Dt} = 0, \tag{82}$$

$$\text{and } \delta s: \quad \frac{D\beta}{Dt} = T, \tag{83}$$

where p , T are given by

$$de = \frac{p}{\rho^2} d\rho + T\,ds, \tag{84}$$

and $\frac{D}{Dt} = \underline{v}\cdot\nabla + \frac{\partial}{\partial t}$. Elimination of α and β leads to the usual

momentum equation, $\rho \frac{\partial \underline{v}}{\partial t} + \rho \underline{v} \cdot \nabla \underline{v} = -\nabla p$, but, from (81), we see that $\text{curl}\, \underline{v} = \nabla s \times \nabla \beta$ vanishes whenever s is uniform in space. Thus this Eulerian variational principle (apparently due to Eckart [5] originally) yields the hydrodynamic equations, but also restricts the class of possible solutions. However, Lin [33] has shown how one may modify the Eulerian variational principle in such a way as to make it equivalent to the general hydrodynamic equations. According to Lin one must take into account the fact that the velocity field $\underline{v}$ is actually the (substantial) time derivative of a displacement field; in order to take this into account in the variational principle, one introduces the Lagrangian coordinates $\underline{X}(\underline{x},t)$ of the fluid particles and the constraints

$$\frac{D\underline{X}}{Dt} = 0. \tag{85}$$

The constraints may be introduced by a Lagrange multiplier, $\underline{\gamma}$, and the Lagrangian function is then taken as

$$L' = \int_v d^3\underline{r} \left\{ \tfrac{1}{2}\rho v^2 - \rho e - \alpha \left[\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \underline{v}) \right] - \beta \left[\frac{\partial}{\partial t}(\rho s) + \operatorname{div}(\rho s \underline{v}) \right] - \rho \underline{\gamma} \cdot \frac{D\underline{X}}{Dt} \right\}. \tag{86}$$

The variational equations are obtained by independently varying $\underline{v}$, $\underline{X}$, ρ and s ; these equations can be written as

$$\delta v: \quad v + \nabla \alpha + s \nabla \beta - \gamma_j \nabla X_j = 0, \tag{87}$$

$$\delta \rho: \quad \tfrac{1}{2} v^2 - e - \frac{p}{\rho} + \frac{D\alpha}{Dt} + \frac{D\beta}{Dt} = 0, \tag{88}$$

$$\delta s: \quad \frac{D\beta}{Dt} = T, \tag{89}$$

and

$$\delta \underline{X}: \qquad \frac{D\underline{\gamma}}{Dt} = 0. \tag{90}$$

Elimination of the multipliers α, β and $\underline{\gamma}$ leads to the usual hydrodynamic equation, $\rho \frac{D\underline{v}}{Dt} = -\nabla p$, but this time $\mathrm{curl}\, \underline{v}$ does not necessarily vanish, even when s is uniform in space. Serrin [33] has completed the equivalence proof by showing that every solution of the hydrodynamic equations is also an extremal for the variational principle.

Although the above modification works for ordinary hydrodynamics (and in fact was used by Lin [24] to obtain the equations for helium II within the framework of his one-fluid model), it does not help in the search for a variational principle for the two-fluid model, since, in introducing the constraint (85), we have had to appeal directly to the Lagrangian nature of the system. Of course one can formally introduce constraints of the form (85) for each of the velocity fields, but the physical significance of such a device for the two-fluid model is not clear. Whitlock [39] has considered some extensions of Zilsel's work along these lines, and this work will be briefly discussed below.

We now consider Zilsel's derivation of the hydrodynamic equations for the two-fluid model. The starting point is the usual two-fluid model with velocities $\underline{v}_n$, $\underline{v}_s$, density ρ, specific entropy s and normal fluid concentration $x = \rho_n/\rho$. The laws of conservation of mass and entropy,

$$\frac{\partial \rho}{\partial t} + \operatorname{div}\left\{\rho\left[x\underline{v}_n + (1-x)\underline{v}_s\right]\right\} = 0, \tag{91}$$

$$\frac{\partial}{\partial t}(\rho s) + \operatorname{div}\left\{\rho s \underline{v}_n\right\} = 0, \tag{92}$$

are to be treated as constraints in the variational principle. The Lagrangian density is taken as $\frac{1}{2}\rho x v_n^2 + \frac{1}{2}\rho(1-x)v_s^2 - \rho e$, where e is the specific internal energy and is assumed to be a function of ρ, s and x. Then the variational principle takes the form

$$\delta\int_{t_0}^{t_1} dt \int_V d^3\underline{r}\left\{\frac{1}{2}x v_n^2 + \frac{1}{2}\rho(1-x)v_s^2 - \rho e - \alpha\left[\frac{\partial \rho}{\partial t} + \operatorname{div}\left[\rho\left(x\underline{v}_n + (1-x)\underline{v}_s\right)\right]\right.\right.$$
$$\left.\left. - \beta\left[\frac{\partial}{\partial t}(\rho s) + \operatorname{div}(\rho s \underline{v}_n)\right]\right\} = 0, \tag{93}$$

where α and β are Lagrange multipliers. In the variation (93), the nine quantities ρ, s, x, $\underline{v}_n$, $\underline{v}_s$ are varied independently although, as mentioned earlier, the number of independent quantities in the two-fluid model is eight; it turns out that the variational principle gives the thermodynamic equilibrium relation, as well as the dynamical equations. The variational equations are

$$\delta\rho: \ \frac{1}{2}\left[x v_n^2 + (1-x)v_s^2\right] - e - \frac{p}{\rho} + \frac{\partial \alpha}{\partial t} + \left[x\underline{v}_n + (1-x)\underline{v}_s\right]\cdot\nabla\alpha \tag{94}$$
$$+ s\left(\frac{\partial \beta}{\partial t} + \underline{v}_n \cdot \nabla\beta\right) = 0,$$

$$\delta s: \quad \frac{\partial \beta}{\partial t} + v_n \cdot \nabla\beta = T, \tag{95}$$

$$\delta x: \quad \frac{1}{2}\left(v_n^2 - v_s^2\right) + (\underline{v}_n - \underline{v}_s)\cdot\nabla\alpha = \left(\frac{\partial e}{\partial x}\right)_{\rho,s}, \tag{96}$$

$$\delta \underline{v}_n : \qquad \underline{v}_n + \nabla\alpha + \frac{s}{x}\nabla\beta = 0, \tag{97}$$

and

$$\delta \underline{v}_s : \qquad \underline{v}_s + \nabla\alpha = 0, \tag{98}$$

where

$$T \equiv \left(\frac{\partial e}{\partial s}\right)_{\rho,x} \quad , \qquad \frac{P}{\rho^2} \equiv \left(\frac{\partial e}{\partial \rho}\right)_{s,x}. \tag{99}$$

The elimination of the multipliers α and β leads to the equations

$$\frac{\partial \underline{v}_s}{\partial t} + \underline{v}_s \cdot \nabla \underline{v}_s = -\frac{1}{\rho}\nabla P + s\nabla T + \frac{x}{2}\nabla w^2, \tag{100}$$

$$\frac{\partial \underline{v}_n}{\partial t} + \underline{v}_n \cdot \nabla \underline{v}_n = -\frac{1}{\rho}\nabla P - \frac{1-x}{x} s\nabla T - \frac{1-x}{2}\nabla w^2 - \frac{\underline{w}\,\Gamma}{\rho x} \tag{101}$$

(where $\Gamma = \frac{\partial \rho_n}{\partial t} + \mathrm{div}(\rho_n \underline{v}_n)$) (102)

and

$$\left(\frac{\partial e}{\partial x}\right)_{\rho,s} = \frac{1}{2} w^2. \tag{103}$$

Equations (100) -(103) along with (91) and (92) are exactly the Landau equations discussed in the previous sections. However, the variational equations also imply the restrictive equations

$$\mathrm{curl}\ \underline{v}_s = 0 \tag{104}$$

and $$\text{curl}\, \underline{v}_n = \nabla s_n \times \nabla \beta \qquad (s_n = S/x). \tag{105}$$

It is perhaps worth noting that if a superfluid entropy S_s had been included in the formulation, then the restrictive equations would have taken the form

$$\text{curl}\, \underline{v}_s = \nabla S_s \times \nabla \beta \tag{106}$$

and

$$\text{curl}\, v_n = \nabla S_n \times \nabla \beta, \tag{107}$$

and these equations are very similar to the restrictive equations obtained from the unmodified variational principle for an ordinary fluid. Although there is no reason to believe that the superfluid entropy is not zero, the equations (106) and (107) lend support to the idea that the restrictive equations (104) and (105) stem from a defect in the variational principle and do not have physical content. (The fact that the variational principle yields equation (104) is often used as an argument in support of the physical statement that the superfluid cannot rotate; the restrictive equation (105), which also follows from the variational principle, is, however, usually ignored.)

In an attempt to remove the restrictions on the velocity fields, Whitlock [39] has discussed the effect of including constraints of the type (85) in Zilsel's variational approach. He found that if one introduces constraints of the type (85) for both the superfluid and normal fluid velocities, it is difficult to eliminate the Lagrange multipliers without going

to higher order equations of motion. He also found that if one introduces a constraint only on the normal fluid velocity, (i) the resulting equations of motion are the same as those obtained above ((100) - (103)), (ii) there is no restricitve equation of the type (105) on the normal fluid velocity, and (iii) there is still the restriction that $\text{curl}\, \underline{v}_s = 0$; thus in this way the Landau equations (and the constraint $\text{curl}\, \underline{v}_s = 0$) are equivalent to a modified variational principle. However, the physical significance of introducing the "Lagrangian co-ordinates" of the normal fluid is not clear, both because of the possibility of $\rho_n \rightleftarrows \rho_s$ transitions (as Whitlock points out), but, even more so, because of the impossibility of identifying a given fluid element with the normal fluid or superfluid component.

Although Lin was able to resolve the difficulties with the Eulerian variational principle in ordinary hydrodynamics, it is significant that his resolution involved a direct appeal to the Lagrangian nature of the fluid. The two-fluid model, however, defies a Lagrangian description, and a complete resolution of the difficulties with the two-fluid variational principle is yet to be given. The difficulties in the two-fluid model associated with the lack of a Lagrangian description are fundamental ones, and, to quote London ([29], p. 127), "they appear so intrinsic that we might question whether the two-fluid concept is actually compatible with the principles of classical particle-mechanics." One way around these difficulties is to abandon the two-fluid model. In the

next section we give a detailed discussion of the hydrodynamics of helium II on the basis of Lin's one-fluid model [24].

B. One-Fluid Model

1. Introduction

Although the two-fluid model has served as the starting point for most discussions of the hydrodynamics of helium II, Lin([26,25,24]) has shown that a consistent hydrodynamic theory for helium II may be developed in a natural way by a suitable generalization of the concepts of ordinary hydrodynamics. This section will be devoted to a detailed discussion of Lin's one-fluid theory for reversible processes.

Lin's theory is a direct generalization of the ordinary hydrodynamics of a single fluid. The starting point for the generalization is the experimental observation that helium II at rest can transmit heat in a reversible manner. This fact is accounted for in Lin's theory by introducing, as a new thermodynamic variable, the entropy flux $\underline{S}$ relative to the fluid; then one may define a velocity of internal convection $\underline{c}$ by the relation

$$\underline{S} = \rho s \underline{c}, \tag{108}$$

where ρ is the density and s the specific entropy. The other macroscopic variable needed for the description of the helium II motion is the macroscopic velocity $\underline{v}$. The central problem, then, is to determine the hydrodynamic equations governing the behavior of the quantities ρ, s, $\underline{v}$ and $\underline{c}$. Two equations may be immediately written down describing the laws of conservation of mass and conservation of entropy;

these are

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \underline{v}) = 0, \tag{109}$$

and

$$\frac{\partial}{\partial t}(\rho s) + \operatorname{div}\{\rho s(\underline{v} + \underline{c})\} = 0. \tag{110}$$

Two further (vector)equations are needed --one each for $\underline{v}$ and $\underline{c}$. The derivation of these equations will be given in section II-B-2 (variational principle) and section II-B-3 (conservation laws). As in the case of the two-fluid model, some discussion of the thermodynamics of the system is a necessary preliminary to the derivation of the hydrodynamic equations.

It is assumed that there exists an internal energy function e , such that the total energy per unit volume is given by

$$E = \frac{1}{2}\rho v^2 + \rho e. \tag{111}$$

The function e will depend on the thermodynamic variable $\underline{c}$, as well as the quantities ρ and s ; thus in general,

$$e = e(\rho, s, S), \qquad S = \frac{1}{2}c^2. \tag{112}$$

Equation (112) does not give a complete specification of the thermodynamics of the system, however, because one must know the proper extensive variable associated with the quantity $\underline{c}$ in order to identify the pressure and temperature with the derivatives of e with respect to ρ and s . This is a

difficult point and one which cannot be resolved without some further information on the physical role of the quantity $\underline{c}$; however, the derivation of the hydrodynamic equations in section II-B-2 gives rise to a situation in which there is a "natural simplest" choice. Further discussion of this point is then deferred to the next section.

2. Variational principle

For reversible processes, one expects that the equations may be obtained from a variational principle, and, as a natural generalization of ordinary hydrodynamics, it is assumed that the equations for both $\underline{v}$ and $\underline{c}$ may be obtained from a variational principle. In writing down a Lagrangian density for the system, one must know how to split the energy (111) into a potential part and a kinetic part. For an ordinary fluid, the correct variational principle is obtained by including the thermodynamic internal energy with the potential energy of the system; in the present case, however, one must allow for the possibility that some of the thermodynamic energy e (112) is actually kinetic in nature. Thus we write the energy e as

$$e = e_P + e_K \tag{113}$$

where e_P is the potential part, e_K the kinetic part. Then it is to be expected that the Lagrangian density will be given by

$$\mathcal{L} = \rho\left[\tfrac{1}{2}v^2 - e_L\right], \tag{114}$$

where

$$e_L = e_P - e_K . \tag{115}$$

At this point, e_P and e_K are not known; however, we shall see that the correct splitting (113) may be deduced from the equations of motion..

The variation is to be carried out with the equations (109) and (110) incorporated as constraints; in accordance with the discussion of II-A-3, we introduce the Lagrangian coordinates $\underline{\underline{X}}$, and the constraint

$$\frac{D\underline{\underline{X}}}{Dt} = 0. \tag{116}$$

The variational principle is then

$$\delta\int_{t_1}^{t_2}dt\int_V d^3\underline{x}\left\{\rho\left(\tfrac{1}{2}v^2 - e_L\right) - \alpha\left[\frac{\partial\rho}{\partial t} + \operatorname{div}(\rho\underline{v})\right] - \beta\left[\frac{\partial}{\partial t}(\rho s) + \operatorname{div}\{\rho s(\underline{v}+\underline{c})\}\right]\right.$$

$$\left. - \underline{\lambda}\,\rho\cdot\frac{D\underline{\underline{X}}}{Dt}\right\} = 0 \tag{117}$$

where ρ, s, $\underline{v}$, $\underline{c}$, $\underline{\underline{X}}$ are to be varied independently, and α, β, $\underline{\lambda}$ are Lagrangian multipliers. The variational equations are

$$\delta\rho: \qquad -e_L - \frac{P_L}{\rho} + \frac{1}{\lambda}v^2 + \frac{D\alpha}{Dt} + s\frac{D\beta}{Dt} + s\underline{c}\cdot\nabla\beta = 0, \tag{118}$$

$$\delta s: \qquad \frac{D\beta}{Dt} + \underline{c}\cdot\nabla\beta = T_L, \tag{119}$$

$$\delta\underline{v}: \qquad \underline{v} + \nabla\alpha + s\nabla\beta = \sum_{i=1}^{3}\lambda_i\nabla X_i, \tag{120}$$

$$\delta\underline{c}: \qquad s\nabla\beta = z\underline{c}, \tag{121}$$

and

$$\delta \underline{\underline{X}}: \qquad \frac{D\underline{\underline{\lambda}}}{Dt} = 0, \tag{122}$$

where we have introduced the derivatives of e_L, namely,

$$de_L = T_L\, dS + \frac{P_L}{\rho^2}\, d\rho + z\, d\varsigma . \tag{123}$$

The final hydrodynamic equations are obtained by eliminating the Lagrange multipliers, and they may be written as

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \underline{v}) = 0, \tag{124}$$

$$\frac{\partial}{\partial t}(\rho s) + \operatorname{div}\left\{\rho s(\underline{v} + \underline{c})\right\} = 0, \tag{125}$$

$$\rho \frac{D\underline{v}}{Dt} = \operatorname{div} \underline{\underline{\tau}}\,, \quad \underline{\underline{\tau}} = -P_L \underline{\underline{I}} + z\rho \underline{c}\,\underline{c}\,, \tag{126}$$

and

$$\frac{\partial}{\partial t}\left(\frac{z\underline{c}}{s}\right) + \nabla\left\{\frac{z}{s}\,\underline{c}\cdot(\underline{v} + \underline{c}) - T_L\right\} = 0, \tag{127}$$

or

$$\frac{\partial}{\partial t}(\rho z c_i) + \frac{\partial}{\partial x_j}\left\{(v_j + c_j)\rho z c_i\right\} = \rho s \frac{\partial T_L}{\partial x_i} - \rho z c_j \frac{\partial}{\partial x_i}(v_j + c_j).$$

Thus the reversible heat transfer gives rise to a momentum transfer, as is evident from (126). We may also write (126) in a form which expresses the law of conservation of momentum, namely,

$$\frac{\partial}{\partial t}(\rho \underline{v}) + \operatorname{div} \underline{\underline{\Pi}} = 0, \tag{128}$$

where the momentum flux tensor $\underline{\underline{\Pi}}$ is given by

$$\Pi_{ij} = \rho v_i v_j + p_L \delta_{ij} - \rho z c_i c_j . \tag{129}$$

There are several points to discuss in completing the derivation. We first consider the problem of determining the energy parts e_p and e_K. The basic principle to be used is the fact that, in the perfect fluid theory, the hydrodynamic equations (124) - (127) must imply the conservation of total energy. The energy density is $E = \frac{1}{2}\rho v^2 + \rho e$, and the energy equation must have the form

$$\frac{\partial}{\partial t}\left[\frac{1}{2}\rho v^2 + \rho e\right] + \operatorname{div} \underline{Q} = 0, \tag{130}$$

where $\underline{Q}$ is the (as yet undetermined) energy flux vector. Although $\underline{Q}$ is unknown, we may gain some information about the form of $\underline{Q}$ from the Galilean transformation formulae. If $\underline{Q}_0$ denotes $\underline{Q}$ as measured in the rest frame of the fluid, then $\underline{Q}$ and $\underline{Q}_0$ are related by

$$\underline{Q} = E\underline{v} + (\underline{\underline{\Pi}}_0 \cdot \underline{v}) + \underline{Q}_0 \tag{131}$$

where $\underline{\underline{\Pi}}_0$ is the momentum flux tensor as measured in the fluid rest frame - i.e., (from (124))

$$\Pi_{0ij} = p_L \delta_{ij} - \rho z c_i c_j \quad ; \tag{132}$$

thus

$$\underline{Q} = E\underline{v} + p_L \underline{v} - \rho z \underline{c}(\underline{c} \cdot \underline{v}) + \underline{Q}_0 .$$

Since we are dealing with a perfect fluid theory, it is plausible to assume that $\underline{Q}_0$ does not depend on any of the spatial derivatives of the quantities ρ , s , $\underline{v}$ and $\underline{c}$; since $\underline{Q}_0$ is a Galilean invariant, it cannot depend on $\underline{v}$, so that we have

$$\underline{Q}_0 = \lambda \underline{c}, \tag{133}$$

where, in general, $\lambda = \lambda(\rho, s, \varsigma)$. In the calculations, to follow, it is convenient to replace the unknown scalar λ by another unknown scalar function λ' , given by

$$\lambda = \rho s T - \rho z c^2 + \lambda'. \tag{134}$$

Then the energy flux is given by

$$\underline{Q} = \left(\tfrac{1}{2}\rho v^2 + \rho e + p_L\right)\underline{v} - \rho z \underline{c}(\underline{c}\cdot\underline{v}) + \rho s T \underline{c} - \rho z c^2 \underline{c} + \lambda' \underline{c} . \tag{135}$$

The calculations proceed as follows: from the hydrodynamic equations (124) - (127), we may calculate $\frac{\partial E}{\partial t}$ directly in terms of ρ , s , $\underline{v}$, $\underline{c}$ and their spatial derivatives; if the result of this calculation and the expression (135) for $\underline{Q}$ are substituted into the energy equation (130), the result is

$$\rho \frac{De'}{Dt} + \operatorname{div} \lambda' \underline{c} = 0, \tag{136}$$

where

$$e' = e - e_L + 2 z s. \tag{137}$$

In the calculation of $\rho \frac{De'}{Dt}$, it is convenient to regard e' as a function of ρ, s and $\kappa = \frac{1}{2}\left(\frac{z\underline{c}}{s}\right)^2$; then

$$\rho \frac{De'}{Dt} = \rho \frac{\partial e'}{\partial \rho}\frac{D\rho}{Dt} + \rho \frac{\partial e'}{\partial s}\frac{Ds}{Dt} + \rho \frac{\partial e'}{\partial \kappa}\frac{z\underline{c}}{s}\cdot\frac{D}{Dt}\left(\frac{z\underline{c}}{s}\right),$$

and we may use the hydrodynamic equations (124) - (127) and equation (130) to obtain

$$-\left\{\delta_{ij}\frac{\partial e'}{\partial \rho} + \rho\frac{\partial e'}{\partial \kappa}\frac{zc_i}{s}\frac{zc_j}{s}\right\}\frac{\partial v_i}{\partial x_j} - \frac{\partial e'}{\partial s}\frac{\partial}{\partial x_j}\left(\rho s c_j\right) - \rho\frac{\partial e'}{\partial \kappa}\frac{zc_i}{c}\frac{\partial}{\partial x_i}\left(\frac{z}{s}c^2 - T_L\right)$$

$$+\frac{\partial}{\partial x_j}\left(\lambda' c_j\right) = 0. \qquad (138)$$

Since no time derivatives appear in (138), it must be an identity; in particular the coefficient of $\frac{\partial v_i}{\partial x_j}$ must vanish, so that

$$\delta_{ij}\frac{\partial e'}{\partial \rho} + \rho\frac{\partial e'}{\partial \kappa}\frac{zc_i}{s}\frac{zc_j}{s} \equiv 0. \qquad (139)$$

Equation (139) must hold for all values of $\underline{c}$, and it is easy to show that

$$\frac{\partial e'}{\partial \rho} \equiv 0, \qquad \frac{\partial e'}{\partial \kappa} \equiv 0$$

so that $e' = e'(s)$. Then (139) becomes

$$-\frac{de'}{ds}\,\mathrm{div}\left(\rho s\underline{c}\right) + \mathrm{div}\left(\lambda'\underline{c}\right) \equiv 0. \qquad (140)$$

In (140), we may regard λ' as a function of ρ, s and $\frac{1}{2}c^2 = \varepsilon$. Then (140) becomes

$$\left\{c_i c_j \frac{\partial \lambda'}{\partial s} + \lambda' \delta_{ij} - \rho s \frac{de'}{ds}\delta_{ij}\right\}\frac{\partial c_i}{\partial x_j} + \left\{c_i \frac{\partial \lambda'}{\partial \rho} - c_i s \frac{de'}{ds}\right\}\frac{\partial \rho}{\partial x_i} +$$

$$+\left\{c_i \frac{\partial \lambda'}{\partial s} - c_i \rho \frac{de'}{ds}\right\}\frac{\partial s}{\partial x_i} \equiv 0,$$

and from here it is easy to show that

$$\frac{d^2 e'}{ds^2} = 0, \tag{141}$$

and

$$\lambda' = \rho s \frac{de'}{ds}. \tag{142}$$

Thus the most general form for e' allowed is $e' = \gamma_1 s + \gamma_2$, where γ_1 , γ_2 are constants (not depending on any of the thermodynamic or flow variables). Then from (137), we obtain

$$e_L = e + 2zs - e'$$

$$= e + 2zs - \gamma_1 s - \gamma_2.$$

It is clear that the terms $\gamma_1 s + \gamma_2$ contribute nothing to the variational equations, because the integral of $\rho\gamma_1 s$ over the volume V is proportional to the total entropy and the integral of $\rho\gamma_2$ is proportional to the total mass, both of which are conserved. Thus we may take $\gamma_1 = \gamma_2 = 0$ without affecting the variational principle; then we have the final results

$$e_L = e + 2zs,$$

$$e_K = -zs, \tag{143}$$

and

$$e_p = e + zs.$$

We may also note that a definite expression for the energy flux vector has been obtained from this calculation, namely

$$\underline{Q} = \left(\tfrac{1}{2}\rho v^2 + \rho e + P_L\right)\underline{v} - \rho z \underline{c}(\underline{c}\cdot\underline{v}) + \rho s T_L - \rho z c^2 \underline{c}. \quad (144)$$

As mentioned in the preceding section, the theory is not really complete until we have identified the extensive variable associated with $\underline{c}$, as we may not obtain the pressure and temperature in terms of the derivatives of the energy function until this third extensive variable is known. At present, there seem to be no simple arguments within the framework of the one-fluid theory which would furnish a definite answer to this question. Thus in the absence of more definite information, we simply assume (on the basis of "maximum simplicity" of the hydrodynamic equations) that the quantities P_L and T_L may be identified with the thermodynamic pressure and temperature; that is,

$$P = P_L \quad , T = T_L \; ; \quad (145)$$

then the differentials of the thermodynamic quantities e, e_L and e_ρ become

$$de_L = \frac{P}{\rho^2} d\rho + T dS + z ds, \quad (146)$$

$$de = \frac{P}{\rho^2} d\rho + T dS - \underline{c}\cdot d z \underline{c}, \quad (147)$$

and

$$de_\rho = \frac{P}{\rho^2} d\rho + T dS - S dz. \quad (148)$$

In section II-B-4, where a discussion of the relation of the present theory with the two-fluid theory is given, it will be shown that the quantity $z\underline{c}$ has a definite interpretation as an extensive variable(per unit mass) within the framework of the two-fluid theory.

With this identification of $z\underline{c}$ as the third extensive variable, it is possible to give a physical interpretation to the terms in the energy flux $\underline{Q}$. By using (130), (144) and (147) we may write the energy equation (130) in the form

$$\frac{\partial}{\partial t}\left\{\tfrac{1}{2}\rho v^2 + \rho e\right\} + \operatorname{div}\left\{\tfrac{1}{2}\rho v^2 \underline{v} + \rho e \underline{v} + \rho\left(\frac{\partial e}{\partial s}\right)_{\rho, z\underline{c}} s(\underline{v}+\underline{c}-\underline{v}) + \right.$$

$$\left. + \rho\left(\frac{\partial e}{\partial z c_i}\right)_{\rho,s} z c_i (\underline{v}+\underline{c}-\underline{v})\right\} = \operatorname{div}\left(\underline{v}\cdot\underline{\underline{T}}\right), \tag{149}$$

$$\left(\text{where} \quad \underline{\underline{T}} = -p\underline{\underline{I}} + \rho z \underline{c}\underline{c}\right).$$

The term on the right gives the rate of working of the stresses; in the divergence on the left, we have the convection of kinetic energy, $\frac{1}{2}\rho v^2 \underline{v}$, and convection of internal energy, $\rho e \underline{v}$; the last two terms are corrections to the convection of internal energy, these terms occuring as a consequence of the fact that s and $z\underline{c}$ are convected with velocity $\underline{v}+\underline{c}$, rather than $\underline{v}$.

At a stationary solid wall, the mass, entropy and energy fluxes must vanish (assuming there is no energy input to the fluid through the wall), so that

$$\begin{aligned} \underline{v}\cdot\underline{n} &= 0, \\ \underline{c}\cdot\underline{n} &= 0. \end{aligned} \tag{150}$$

(As mentioned before, it is necessary to consider irreversible processes in the discussion of the boundary conditions when there is a net energy flux between the wall and the fluid).

Lin [24] has presented some simple examples which show that the present theory predicts thermal waves (second sound) as well as ordinary sound waves.

There is one rather serious difficulty with the present derivation which should be discussed, and that is the fact that the variational equation (121) predicts that

$$\operatorname{curl} \underline{c} = \nabla\left(\frac{s}{z}\right) \times \nabla\beta, \tag{151}$$

or, alternatively,

$$\operatorname{curl}\left(\frac{z\,\underline{c}}{s}\right) = 0,$$

so that curl $\underline{c}$ vanishes whenever s/z is uniform in space. At this stage, one could accept (151) as having physical significance; however, there are several reasons for regarding (151) as an unphysical restrictive equation. First, equation (151) is very reminiscent of the restrictive equations obtained from the unmodified variational principle for an ordinary fluid and from Zilsel's variational principle for the two-fluid model, and, in these cases, at least some of the restrictive equations are due to defects in the variational principles. Second, we may note that a restrictive equation of the form (151) always arises whenever the Lagrangian density depends on a vector $\underline{c}$ only through a term of the type $e_L(\rho, s, \frac{1}{2}c^2)$ and a constraint term of the type $\beta\,\mathrm{div}(\rho s \underline{c})$. Finally we may note that, in

terms of the two-fluid model (cf. II-B-4), equation (151) predicts that $\operatorname{curl}(\underline{v}_n - \underline{v}_s) = 0$ whenever s and $x = \rho_n/\rho$ are uniform in space, and it is known that this equation is too restrictive to be in accord with some of the experimental results.

One might hope to remove the restriction (151) by modifying the variational principle in some way. However, the Lagrangian nature of the system has already been taken into account by means of the constraints (116) expressing the "conservation of particle identity", so it is not obvious as to what sort of modifications one should try. Formally, one can introduce "Lagrangian coordinates" for the velocity $\underline{c}$, and constraints similar to (116); the physical significance of such a device is not clear, however, and the resulting equations for $\underline{v}$ and $\underline{c}$ (after elimination of the Lagrange multipliers) are of higher order than one would expect.

There is one important difference between the present case and the case of an ordinary fluid which may be a clue to the source of the difficulty. This is the fact that, for an ordinary fluid, the (local) conservation of entropy is equivalent to the statement that the entropy of each fluid particle stays constant, whereas in the one-fluid model for helium II, the entropy of each fluid particle does not stay constant (in general), even though the entropy is conserved. Thus in the present case, there is energy transfer of an essentially thermal nature (even though the transfer process is thermodynamically reversible), so that it is not entirely

clear that one can expect a variational principle of the form $\delta \int dt\,(T-V) = 0$ to hold.

Since the variational principle is being used here to deduce the equations (and not to give an alternative derivation of the equations which are already known), it would seem to be important to resolve this difficulty; however, this has not been done as yet. In the absence of a more satisfactory derivation of the equations, we simply take the equations of motion to be those given by the above variational principle, without, however, retaining the restrictive equation (151). This procedure is somewhat arbitrary, since the various forms of the equations for $\underline{c}$ (cf. equation (127)) are no longer equivalent when the restrictive equation (151) is dropped. Of all the possible forms for the equation for $\underline{c}$, one can perhaps justify a preference for the second of equations (127), since this is an equation for the time rate of change of the extensive quantity (per unit volume) $\rho \underline{c}$. Thus we tentatively take the complete set of hydrodynamic equations in the following form:

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \underline{v}) = 0,$$

$$\frac{\partial}{\partial t}(\rho s) + \operatorname{div}\{\rho s(\underline{v}+\underline{c})\} = 0, \qquad (152)$$

$$\frac{\partial}{\partial t}(\rho v_i) + \frac{\partial}{\partial x_j}(\rho v_i v_j) = \frac{\partial}{\partial x_j}\left(-p\delta_{ij} + \Sigma \rho c_i c_j\right),$$

and

$$\frac{\partial}{\partial t}(\rho z c_i) + \frac{\partial}{\partial x_j}\left\{(v_j + c_j)\rho z c_i\right\} = \rho s \frac{\partial T}{\partial x_i} - \rho z c_j \frac{\partial}{\partial x_i}(v_j + c_j).$$

The thermodynamic relation (147) is the same as before, and the energy equation (which follows from (152)) is still given by (149).

Because of the difficulties in the derivation leading to (152), it would be very desirable to obtain the equations by another method. In the next section, we examine the possibility of deriving the equations from the conservation laws and the Galilean transformation formulae. It will be shown there that, although the equations (152) are consistent with the conservation laws and the Galilean transformation formulae, one cannot deduce a unique set of equations from that method.

3. Conservation laws

Landau's work (discussed in II-A-1) has shown that the imposition of the conservation laws plus the requirements of Galilean invariance are sufficient to determine a unique set of hydrodynamic equations (for reversible processes) for the two-fluid model, provided that it is assumed that always $\text{curl}\, \underline{v}_s = 0$. It was also shown in II-A-1 that it is no longer possible to deduce a unique set of hydrodynamic equations when the requirement $\text{curl}\, \underline{v}_s = 0$, is dropped. Thus it is no surprise that when the method is applied to Lin's one-fluid model, it does not lead to a unique set of hydrodynamic equations. The method does, however, limit the possible form of the hydrodynamic equations, and in particular, the equations (152) are consistent with the restriction imposed by the conservation laws and the requirements of Galilean invarience.

Since the method does not lead to a definite set of hydrodynamic equations, it is of limited interest, and the calculations will be described only very briefly here.

It is assumed that the macroscopic state is characterized by the values of ρ, s, $\underline{v}$ and $\underline{c}$; the mass flux is taken to be $\rho\underline{v}$, the entropy flux to be $\rho s(\underline{v}+\underline{c})$. Then the laws of conservation of mass and entropy are the same as before,

$$\frac{\partial \rho}{\partial t} + \text{div}\,(\rho \underline{v}) = 0, \tag{153}$$

and

$$\frac{\partial}{\partial t}(\rho s) + \text{div}\{\rho s\,(\underline{v}+\underline{c})\} = 0. \tag{154}$$

The conservation of momentum is expressed by an equation of the form

$$\frac{\partial}{\partial t}(\rho v_i) + \frac{\partial}{\partial x_j}(\pi_{ij}) = 0, \tag{155}$$

where the momentum flux tensor π_{ij} is as yet undetermined. The total energy per unit volume is taken to be

$$E = \frac{1}{2}\rho v^2 + \rho e, \tag{156}$$

where $e = e(\rho, s, z\underline{c})$ is the specific internal energy, and

$$de = \frac{p}{\rho^2} d\rho + T dS - \underline{c} \cdot d z\underline{c} \tag{157}$$

defines the pressure p and the temperature T. (The thermodynamic description (157), obtained in the preceding section, is assumed to be valid here.) Then the conservation of energy is expressed by an equation of the form

$$\frac{\partial E}{\partial t} + \operatorname{div} \underline{Q} = 0, \tag{158}$$

where the energy flux vector $\underline{Q}$ is as yet undetermined. Finally, there will be an equation for the convection velocity $\underline{c}$, and it is convenient to write this equation in the form

$$\frac{\partial}{\partial t}(\rho z c_i) + \frac{\partial}{\partial x_j}\left\{(v_j + c_j)\rho z c_i\right\} = F_i, \tag{159}$$

where F_i is as yet undetermined. Although the quantities π_{ij}, $\underline{Q}$ and $\underline{F}$ are not known, we may use the Galilean

transformation formulae to express $\underline{Q}$ and π_{ij} in terms of their values, $\underline{Q}_0$ and π_{ij}^0, in the rest frame of the fluid, and we may also deduce that $\underline{F}$ must be a Galilean invariant. Thus

$$\pi_{ij} = \rho v_i v_j + \pi_{ij}^0,$$

and

$$\underline{Q} = E\underline{v} + (\underline{\underline{\Pi}}^0 \cdot \underline{v}) + \underline{Q}_0 .$$

For convenience in the calculations to follow, we introduce new unknown quantities π'_{ij}, $\underline{Q}'$ and $\underline{F}'$ by the equations

$$\pi_{ij}^0 = p\delta_{ij} - \rho z c_i c_j + \pi'_{ij},$$

$$\underline{Q}_0 = \rho s T \underline{c} - \rho z c^2 \underline{c} + \underline{Q}',$$

and

$$F_i = \rho s \frac{\partial T}{\partial x_i} - \rho z c_j \frac{\partial}{\partial x_i}(v_j + c_j) + F_i'.$$

The quantities π'_{ij}, $\underline{Q}'$ and $\underline{F}'$ are all Galilean invariants, and the problem of determining the hydrodynamic equations is reduced to the problem of determining π'_{ij}, $\underline{Q}'$ and $\underline{F}'$.

The procedure is the same as before; we have 8 independent quantities (ρ , s , $\underline{v}$, $\underline{c}$) and 9 (scalar) equations for these quantities. We may obtain an equation which the quantities π'_{ij}, $\underline{Q}'$ and $\underline{F}'$ must satisfy in the following manner: we first calculate $\frac{\partial E}{\partial t}$ in terms of the macroscopic variables (and their spatial derivatives) directly from the hydrodynamic equations; then this expression for $\frac{\partial E}{\partial t}$ and the above expression for $\underline{Q}$ are substituted into the energy equation (158). The final result of the calculation is the

equation

$$0 = \pi'_{ij} \frac{\partial v_i}{\partial x_j} + \operatorname{div} \underline{Q}' - \underline{c} \cdot \underline{F}' . \qquad (160)$$

Since this equation contains no time derivatives, it must be an identity, and it thus restricts the possible choices of π'_{ij}, $\underline{Q}'$ and $\underline{F}'$. (We may note that the equations (152) of the preceding section correspond to the choice $\pi'_{ij} \equiv 0$, $\underline{Q}' \equiv 0$ and $\underline{F}' \equiv 0$). It is easy to show from (160) that the most general expression for $\underline{F}'$ (for arbitrary π'_{ij} and $\underline{Q}'$) is given by

$$\underline{F}' = \frac{1}{c^2} \underline{c} \left\{ \pi'_{ij} \frac{\partial v_i}{\partial x_j} + \operatorname{div} \underline{Q}' \right\} + \underline{\alpha} \times \underline{c} , \qquad (161)$$

where $\underline{\alpha}$ is an arbitrary (Galilean invariant) vector. The fluxes π'_{ij} and $\underline{Q}'$ are Galilean invariant; thus if one makes the plausible assumption that these fluxes do not depend on the gradients of the macroscopic quantities, then one can show that the most general expressions for π'_{ij} and $\underline{Q}'$ are

$$\pi'_{ij} = \lambda_1 c_i c_j + \lambda_2 c^2 \delta_{ij}$$

and

$$\underline{Q}' = \lambda_3 \underline{c}$$

where λ_1, λ_2, λ_3 are scalar functions of ρ, s and $\frac{1}{2}c^2$. Even with this simplification, there are still 3 arbitrary scalar functions (λ_1, λ_2, λ_3) and an arbitrary (Galilean invariant) vector $\underline{\alpha}$. Thus this method by no means yields a definite set of hydrodynamic equations.

One can simplify the above results a little by giving a physical argument to determine the energy flux vector. On the basis of the discussion in the preceding section, we expect that the convective terms in the energy flux are of the form

$$\tfrac{1}{2}\rho v^2 \underline{v} + \rho e \underline{v} + \rho s T \underline{c} - \rho z c^2 \underline{c},$$

and that the terms accounting for the rate of working of the fluid stresses are of the form

$$(\underline{\underline{\pi}}^{\circ} \cdot \underline{v}),$$

so that the total energy flux is then

$$\underline{Q} = E\underline{v} + \underline{\underline{\pi}}^{\circ} \cdot \underline{v} + \rho s T \underline{c} - \rho z c^2 \underline{c}.$$

This corresponds to $\underline{Q}' \equiv 0$, so that (161) now reads

$$\underline{F}' = \frac{1}{c^2} \underline{c}\, \pi'_{ij} \frac{\partial v_i}{\partial x_j} + \underline{\alpha} \times \underline{c}. \tag{162}$$

There does not seem to be any simple argument which would allow an unambiguous determination of the stress tensor (and consequently of π'_{ij}), but even if we accept the form of the stress tensor as given by the variational principle (corresponding to $\pi'_{ij} \equiv 0$), we still have in $\underline{F}$ an arbitrary vector $(\underline{\alpha} \times \underline{c})$ perpendicular to $\underline{c}$. If we agree to accept the momentum flux tensor and the energy flux vector as obtained from the variational principle (i.e., we take $\underline{Q}' \equiv 0$, $\pi'_{ij} \equiv 0$), then the most general set of equations consistent with the

the conservation laws and the requirements of Galilean invariance is

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \underline{v}) = 0,$$

$$\frac{\partial}{\partial t}(\rho s) + \operatorname{div}\{\rho s(\underline{v} + \underline{c})\} = 0,$$

$$\frac{\partial}{\partial t}(\rho v_i) + \frac{\partial}{\partial x_j}(\rho v_i v_j) = \frac{\partial}{\partial x_j}(-p\delta_{ij} + \rho z c_i c_j), \tag{163}$$

and

$$\frac{\partial}{\partial t}(\rho z c_i) + \frac{\partial}{\partial x_j}\{(v_j + c_j)\rho z c_i\} = \rho s \frac{\partial T}{\partial x_i}$$

$$- \rho z c_j \frac{\partial}{\partial x_i}(v_j + c_j) + F_i',$$

where F_i' is an arbitrary Galilean invariant vector perpendicular to $\underline{c}$. These are the same as the equations (152), except for the term $\underline{F}'$ in the equation for $\underline{c}$.

In summary: the method yields a restrictive equation which the momentum flux tensor, the energy flux vector and the "force" $\underline{F}$ must satisfy, but does not yield a unique set of hydrodynamic equations. The equations (152) obtained from the variational principle are consistent with this restrictive equation.

4. Relation with the two-fluid theory

As Lin ([25, 24]) has shown, the hydrodynamic equations of his one-fluid theory are essentially equivalent to Landau's equations for the two-fluid model. In fact, the equations for the two-fluid model may be obtained from the equations of the one-fluid model by means of a simple mathematical transformation.

In the one-fluid model, the state of the fluid is completely described by the quantities ρ , s , $\underline{v}$ and $\underline{c}$ (and the internal energy function); the basic variables of the two-fluid model may be taken as ρ , s , $\underline{v}_n$, $\underline{v}_s$ and $x = \frac{\rho_n}{\rho}$ (where x is not independent, but may be regarded as a function of the other two-fluid variables). Thus if we specify $\underline{v}_n$ and $\underline{v}_s$ in terms of the one-fluid variables, the equations (152) (or (163)) of the one-fluid model will yield a set of equations for ρ , s , $\underline{v}_n$, $\underline{v}_s$ which are then to be compared with Landau's equations. In both models, we have definite expressions for the densities and fluxes of mass, entropy, momentum and energy; it turns out that the corresponding quantities in the two models are all equal, if we have

$$\underline{v}_n = \underline{v} + \underline{c}\,, \qquad (164)$$

$$\underline{v}_s = \underline{v} + z\underline{c}\,,$$

$$\frac{\rho_n}{\rho} = x = \frac{z}{z-1} \quad \left(\text{or } z = \frac{-x}{1-x}\right).$$

Of course, if we regard the one-fluid model as basic, then (164) may be regarded as simply a mathematical definition of the quantities $\underline{v}_n$, $\underline{v}_s$ and x. Then we may further define $\rho_n = x\rho = \frac{z\rho}{z-1}$, $\rho_s = (1-x)\rho = \frac{\rho}{1-z}$. By using the relations (164), we may (after some calculations) write the equations (163) in the following form:

$$\frac{\partial \rho}{\partial t} + \operatorname{div}\left\{\rho_n \underline{v}_n + \rho_s \underline{v}_s\right\} = 0,$$

$$\frac{\partial}{\partial t}(\rho s) + \operatorname{div}\left\{\rho s \underline{v}_n\right\} = 0,$$

and

$$\begin{cases} \frac{\partial}{\partial t}\left(\rho_n v_{ni} + \rho_s v_{si}\right) + \frac{\partial}{\partial x_j}\left(\rho_n v_{ni} v_{nj} + \rho_s v_{si} v_{sj}\right) = -\frac{\partial P}{\partial x_i}, \\ \frac{\partial \underline{v}_s}{\partial t} + \underline{v}_s \cdot \nabla \underline{v}_s = -\frac{1}{\rho} \nabla p + s\nabla T + \frac{x}{2} \nabla w^2 + x \underline{w} \times (\operatorname{curl} \underline{v}_s) \\ + \frac{\underline{F}'}{\rho}, \end{cases}$$

(where $\underline{w} = \underline{v}_n - \underline{v}_s$)

or

$$\begin{cases} \frac{\partial \underline{v}_n}{\partial t} + \underline{v}_n \cdot \nabla \underline{v}_n = -\frac{1}{\rho} \nabla p - \frac{1-x}{x} s\nabla T - \frac{1-x}{2} \nabla w^2 \\ -(1-x)\underline{w} \times (\operatorname{curl} \underline{v}_s) - \frac{(1-x)}{\rho x} \underline{F}' - \frac{\underline{w}\Gamma}{\rho_n}, \\ \\ \frac{\partial \underline{v}_s}{\partial t} + \underline{v}_s \cdot \nabla \underline{v}_s = -\frac{1}{\rho} \nabla p + s\nabla T + \frac{x}{2} \nabla w^2 + \\ + x \underline{w} \times (\operatorname{curl} \underline{v}_s) + \frac{\underline{F}'}{\rho}, \end{cases}$$

(where $\Gamma = \frac{\partial \rho_n}{\partial t} + \operatorname{div}(\rho_n \underline{v}_n)$).

If we accept the equations as given by the variational principle (equations (152)), then $\underline{F}' \equiv 0$, and the equations are the same as Landau's, except for the terms in $\underline{w} \times \operatorname{curl} \underline{v}_s$. The equation for $\underline{v}_s$ may be put into the form (still assuming $\underline{F}' \equiv 0$)

$$\frac{\partial \underline{v}_s}{\partial t} = -\nabla\left\{ e + \frac{p}{\rho} - sT + \frac{1}{2} v_s^2 - x\left(1 - \tfrac{1}{2}x\right) w^2 \right\} + \underline{v} \times \operatorname{curl} \underline{v}_s ,$$

and this clearly shows that the equations permit the class of solutions with $\operatorname{curl} \underline{v}_s = 0$; for this class of solutions, the equations are identical with Landau's equations. However, the equations also allow more general solutions for which $\operatorname{curl} \underline{v}_s \neq 0$.

As we saw in the preceding section, any choice of $\underline{F}'$ such that $\underline{c} \cdot \underline{F}' = 0$ gave a set of hydrodynamic equations ((163)) which were consistent with the conservation laws and the requirements of Galilean invariance. It is of particular interest to note that the choice

$$\underline{F}' = \rho z \underline{c} \times \operatorname{curl}(\underline{v} + z\underline{c}), \qquad (166)$$

or

$$\underline{F}' = -\rho_n \underline{w} \times \operatorname{curl} \underline{v}_s ,$$

gives the equations

$$\frac{\partial \underline{v}_n}{\partial t} + \underline{v}_n \cdot \nabla \underline{v}_n = -\frac{1}{\rho} \nabla p - \frac{1-x}{x} s \nabla T - \frac{1-x}{2} \nabla w^2 - \frac{\underline{w} \Gamma}{\rho_n} , \qquad (167)$$

$$\frac{\partial \underline{v}_s}{\partial t} + \underline{v}_s \cdot \nabla \underline{v}_s = -\frac{1}{\rho} \nabla p + s \nabla T + \frac{x}{2} \nabla w^2,$$

which are the Landau equations (again, the equations (167) permit solutions with $\operatorname{curl} \underline{v}_s = 0$, but also allow more general solutions).

Although we have obtained the two-fluid equations from the one-fluid equations and the relations (164), we have yet to compare the thermodynamic descriptions. According to the one-fluid theory, the total energy per unit volume is given by $E = \frac{1}{2}\rho v^2 + \rho e$, where $de = \frac{p}{\rho^2} d\rho + T dS - c \cdot d\underline{z}\, \underline{c}$. In the two-fluid model, the total energy is $E = \frac{1}{2}\rho_n v_n^2 + \frac{1}{2}\rho_s v_s^2 + \rho \tilde{e}$, which defines the specific internal energy $\tilde{e}$ of the two-fluid model. By equating the two expressions for the energy and making use of (164), we find that

$$\tilde{e} = e - \frac{1}{2} x(1-x) w^2 = e + \frac{1}{2} z c^2, \tag{168}$$

or

$$\tilde{e} = e_p \tag{169}$$

Thus

$$d\tilde{e} = d\left(e + \tfrac{1}{2} z c^2\right)$$

$$= \frac{p}{\rho^2} d\rho + T dS - \frac{1}{2} c^2 dz$$

$$= \frac{p}{\rho^2} d\rho + T dS + \frac{1}{2} w^2 dx, \tag{170}$$

so that the thermodynamic description obtained by transformation from the one-fluid model is the same as that given

originally for the two-fluid model (equation (14) or equation (64)). We may also note now that the extensive variable $\rho \underline{\zeta}$ in the one-fluid model may be written as

$$-\rho \underline{\zeta} = \rho_n \underline{w}$$

and thus in the two-fluid model may be interpreted as the (negative of) the momentum per unit volume of the fluid in the superfluid rest frame.

From the above, it is seen that the equations of the one-fluid model are in substantial agreement with Landau's equations for the two-fluid model. The equations (165) (with $\underline{F}' \equiv 0$) are those given by Lin [24]. These equations differ from Landau's equations by the extra terms in $\underline{w} \times \text{curl } \underline{v}_s$, but it would be difficult to decide from experiments whether the perfect fluid equations should contain such terms since it is usually true that dissipative processes are important in flows for which $\text{curl } \underline{v}_s \neq 0$. Also, because of the difficulty with the variational principle, it is perhaps slightly preferable to retain the more general equations (165) with $\underline{F}' \neq 0$ (which include the Landau equations (167) as a special case). Perhaps it is well to emphasize again that Landau's method for deriving the two-fluid equations yields a unique result only when the restrictive equation $\text{curl } \underline{v}_s = 0$ is imposed. As we saw in section II-A-1, Landau's method fails to yield a unique set of equations if the restrictive equation $\text{curl } \underline{v}_s = 0$ is dropped. Thus if we do not insist that $\text{curl } \underline{v}_s$ must vanish, then the difficulties in obtaining a definite set of

hydrodynamic equations are present in Landau's theory to the same extent as in Lin's one-fluid theory.

C. Discussion

In this section, we give a brief discussion and comparison of the two-fluid model and the one-fluid model, and the various methods used to obtain the hydrodynamic equations for reversible flows of helium II. We consider first the two-fluid model.

The derivation of the Landau equations for the two-fluid model from the conservation laws (as discussed in II-A-1) has the advantage that it is based on general principles of universal validity (i.e., conservation laws and Galilean invariance). However, the possibility of unambiguously obtaining the Landau equations in this manner depends in an essential way on the additional special requirement that $\mathrm{curl}\ \underline{v}_s$ must vanish. If this requirement is not imposed, then the conservation laws and the Galilean relativity principle are not sufficient to determine the hydrodynamic equations uniquely, and some additional principle is needed. In his 1941 paper [20], Landau gave such an additional principle--namely, that there be no momentum exchange between the two components (other than that due to $\rho_n \rightleftarrows \rho_s$ transitions). In a sense, this principle is really the foundation of the two-fluid model because it is the one principle which distinguishes the helium two-fluid system from a mixture of two ordinary substances in which a chemical reaction (or dissociation) is taking place. In order to formulate this principle precisely and use it in the derivation of the hydrodynamic equations, however, it is

clear that a necessary prerequisite is a complete thermodynamic and hydrodynamic description for each of the components of the two-fluid system. The derivation of the hydrodynamic equations given in II-A-2 is based on these ideas, and it was shown there that one may deduce the Landau equations without the restriction $\operatorname{curl} \underline{v}_s = 0$ in an unambiguous manner. Finally, in section II-A-3, a discussion of Zilsel's variational principle for the two-fluid model was given. Although this method yields the Landau equations, it entails certain restrictions on the quantities $\operatorname{curl} \underline{v}_n$ and $\operatorname{curl} \underline{v}_s$. It is probable that these restrictive conditions do not have physical significance, and that they arise from defects in the variational principle connected with the fact that one cannot introduce a Lagrangian description of the two-fluid system.

For Lin's one-fluid model, two derivations of the hydrodynamic equations were discussed. The first of these, the variational principle discussed in II-B-2, leads to a definite set of hydrodynamic equations, but also implies a restriction on the quantity $\operatorname{curl} \underline{\zeta}$. It is possible that this difficulty stems from the fact that the Hamilton's principle of mechanics was used to obtain the equations of motion, whereas we know that even for reversible flows of helium II energy transfer processes of an essentially thermal nature may take place. Lin* has suggested that a more thorough study of the thermostatics of the one-fluid model may reveal a way in which the variational principle should be modified to take these thermal

---

* private communication

processes into account. The other derivation of the equations for the one-fluid model (based on the conservation laws and the Galilean relativity principle, section II-B-3) did not yield a unique set of equations (this, of course, is to be compared with an exactly similar result for the two-fluid model when the restriction $\mathrm{curl}\, \underline{v}_s = 0$ is not imposed). The class of equations consistent with the conservation laws includes the Landau equations and also the equations originally proposed by Lin [25,24].

As discussed earlier (II-B-4), the equations of the one-fluid model are equivalent in all essential respects with the Landau equations for the two-fluid model. Although the macroscopic equations are essentially the same for the two models, the underlying physical ideas are very different. The origins of the two-fluid model are the various microscopic theories of liquid helium II, while the one-fluid model represents a generalization of ordinary hydrodynamics suggested by macroscopic observations. Since the relationship between the various microscopic theories and the actual molecular structure of helium II is not clear at present, it is reassuring that one may obtain hydrodynamic equations equivalent to those of the two-fluid model from the one-fluid model which has its foundations directly in macroscopic observations.

III DISSIPATIVE PROCESSES - I

A. Introduction

In the preceding chapter, several independent derivations of the hydrodynamic equations for reversible processes were discussed. Although the physical bases for the derivations were quite different, and although the condition $\mathrm{curl}\ \underline{v}_s = 0$ was an independent requirement in one of the theories and merely characteristic of a particular class of solutions in the other theories, the form of the hydrodynamic equations obtained was essentially the same in all cases. In particular, for reversible motions starting from rest, the two-fluid equations as derived in II-A-2 and the one-fluid equations as given by Lin both predict $\mathrm{curl}\ \underline{v}_s = 0$ and consequently reduce to Landau's equations.

When we consider dissipative processes, however, the question of superfluid rotation can no longer be avoided. In fact, the present hydrodynamic theories (including dissipative processes) may be roughly divided into three types; (i) theories in which $\mathrm{curl}\ \underline{v}_s = 0$ always [22], (ii) theories in which $\mathrm{curl}\ \underline{v}_s$ plays a special role (e.g., the quantized vortex line theories of Feynman [7] and Hall and Vinen [12,13,14,38], or the continuum theory of Bekarevich and Khalatnikov [3] in which $|\mathrm{curl}\ \underline{v}_s|$ is a thermodynamic variables) and (iii) theories in which no special assumptions are made about the nature of $\mathrm{curl}\ \underline{v}_s$ (Lin [26,25,24]). The remainder of this chapter will be devoted to

a detailed discussion of some theories of types (i) and (iii) above*. (The two-fluid model will be taken as the starting point in all of the following discussions; although the theory of dissipative processes has not yet been developed directly in terms of the one-fluid model, the results for the two-fluid model may be easily expressed in terms of the one-fluid model by means of the transformation formulae given in the preceding chapter.) Before entering into a detailed discussion of the various theories, there are a number of general considerations relevant to all of the theories to be discussed first.

Since present knowledge of the microscopic structure of superfluid helium is rather uncertain, it is desirable to base the derivation of the hydrodynamic equations on general continuum principles, rather than a specific molecular picture. From this point of view, then, the terms in the equations representing dissipative processes are to be obtained in a phenomenological manner. It is important, however, that the processes considered be consistent with the essential features of the two-fluid model. Thus, for example, if we include mut mutual friction terms in the hydrodynamic equations, we must be prepared to give a plausible, qualitative description of a microscopic process which (i) leads to a macroscopic momentum exchange between the two components and (ii) does not destroy the two-fluid nature of the system by rapidly producing a state in which $\underline{v}_n = \underline{v}_s$. Since the normal fluid behaves in some respects like a gas of thermal excitations, it is clear that

* Chapter IV is devoted to a discussion of some theories of type (ii)

processes such as thermal conduction and normal fluid molecular momentum transfer require no special discussion. However, the introduction of a superfluid viscosity or a mutual friction force really requires some discussion of the compatibility of these concepts with the two-fluid model. Some attempt will be made to discuss these points as they arise in the derivations of sections III-B and III-C.

The general method to be used in deriving the equations is the same in all cases, so it is perhaps appropriate to outline the method here. (It is essentially the standard method of the theory of non-equilibrium thermodynamics as given, for example, in [4]). The starting point for all of the derivations is the two-fluid model. It is assumed in each case that the correct equations of the perfect fluid theory are known. The independent perfect fluid equations may be taken as the equations expressing conservation of mass, momentum and energy, and an equation for the superfluid velocity $\underline{v}_s$. When dissipative processes are considered, the mass, momentum and energy equations will have the same form, (i.e., the form of a conservation equation) but, in general, the fluxes will contain additional terms (which are to be determined). The equation for $\underline{v}_s$ will also contain additional terms, but since this equation is not a conservation law, the form of the terms to be added must be decided upon (that is, we could add to the equation for $\underline{v}_s$ a contribution from a dissipative stress tensor, $\frac{1}{\rho}\frac{\partial \tau_{ij}^{(s)}}{\partial x_j}$, or a mutual friction term, $\frac{1}{\rho_s} F_i$, or both). This is a point which must be settled for each of the theories individually, so that further discussion of this is deferred

to sections III-B and III-C. Then the independent equations describing dissipative processes will be the conservation equations (mass, momentum and energy) and the equation for $\underline{v}_s$. All of these equations will contain terms describing the effects of dissipative processes (which we may call the dissipative fluxes) which are to be determined. These dissipative fluxes are determined by the requirement that the rate of creation of entropy be positive. To carry this out, one calculates the rate of entropy creation (from the hydrodynamic equations); the resulting expression is a bilinear form in the dissipative fluxes and the gradients of intensive quantities ($\frac{\partial T}{\partial x_i}$, $\frac{\partial v_{ni}}{\partial x_j}$, $\frac{\partial v_{si}}{\partial x_j}$, etc.). Then it is assumed that the dissipative fluxes are linear functions of the gradients of the intensive quantities. (This is the usual procedure in developing a theory of irreversible processes for a given continuum system; in any case, it is a plausible first approximation, although, until more is known about the microscopic structure of helium II, one cannot exclude the possibility that some of the important dissipative processes are inherently nonlinear). The phenomenological coefficients connecting the dissipative fluxes with the gradients of the intensities will in general depend on the (Galilean invariant) independent macroscopic quantities (e.g., we may take the phenomenological coefficients to be functions of ρ, s and $\underline{w} = \underline{v}_n - \underline{v}_s$). The expression for the rate of entropy creation then becomes a quadratic form in the gradients of the intensities; the requirement that this form be positive definite places restrictions on the phenomenological coefficients. In principle, each of the dissipa-

dissipative fluxes may depend on the gradients of all of the independent intensities. In the case of an isotropic fluid, the phenomenological coefficients can only depend on scalars such as ρ , s ; it is possible to show in this case that a given flux depends only on intensity gradients of the same tensorial rank (see, for example, [4]). In the present case, however, the phenomenological coefficients may depend on the vector $\underline{w}$, as well as the scalar thermodynamic variables, and in the most general dissipative equations, there can be coupling between processes of different tensorial rank. In the discussions given in sections III-B and III-C, we will generally assume that the phenomenological coefficients do not depend on $\underline{w}$, since this entails a significant simplification of the resulting equations; some discussion of the more general case will be given, however, in order to illustrate the types of terms which can conceivably appear in the most general phenomenological equations.

We may note here that some experiments (e.g., experiments on the form of the free surface of helium II in a rotating cylinder (Osbourne [31]) or the rotating bucket experiments of Reppy and Lane [32]) indicate that the superfluid is definitely rotating in some manner; thus the condition $\mathrm{curl}\, \underline{v}_s = 0$ is not of universal validity, and, consequently, theories in which $\mathrm{curl}\, \underline{v}_s$ must always vanish are of somewhat limited interest. The question as to how the superfluid component rotates (that is, whether it can rotate in bulk like an ordinary fluid, or whether it merely imitates bulk rotation by sustaining

quantized vortex lines) is one for which the experimental evidence is as yet inconclusive. Lin [25] has given a detailed discussion of the relevance of the various experiments to the question of the existence of quantized vortex lines or quantized circulation.

It is perhaps also appropriate to include in this general discussion of dissipative processes a few remarks about the concept of mutual friction. The general discussion given above is adequate for dealing with those dissipative processes which arise as a result of a non-uniform spatial distribution of the various intensive quantities. In some theories of liquid helium II, however, mutual friction forces depending on the relative velocity $\underline{w} = \underline{v}_n - \underline{v}_s$ are introduced [8]. This sort of mutual friction is clearly a different sort of dissipative process than those considered above, and really represents a partial breakdown of the two-fluid model. Usually there is a critical velocity associated with such forces, below which the friction force is absent. (It is by no means clear that it is necessary to introduce such friction forces in order to explain the various experimental results. Lin [25] has shown that many of the results from channel flow experiments may be satisfactorily explained on the basis of his theory of dissipative processes (to be discussed in detail in section III-C). Further, Staas and Taconis [34] have shown that some results from channel flow experiments may be explained quantitatively by invoking known results from the theory of turbulent flow of an ordinary fluid.) Since this sort of mutual friction

force does not depend on the gradients of the intensive quantities, it is not necessary to decide whether or not such forces are present when carrying out the sort of derivation outlined above. Hall and Vinen [12, 13, 14, 38] have given a theory which includes mutual friction terms arising from the scattering of the normal fluid excitations by quantized vortex lines "in the superfluid component". Bekarevich and Khalatnikov [3] have obtained from a continuum theory the same results as Hall and Vinen; in their theory (to be discussed in detail in Chapter IV), the mutual friction is a dissipative process depending partly on the gradients of intensive quantities, but the general theory outlined above must be supplemented by a number of special considerations in order to obtain the hydrodynamic equations of their theory.

B. The Case curl $\underline{v}_s = 0$

1. Introduction

In the present section, we give a derivation of the hydrodynamic equations describing the flow of helium II including dissipative processes, following the work of Landau, and Lifshitz and Khalatnikov [22]. As mentioned in the introduction, this theory is of limited interest, since it is now known that the superfluid can rotate in some manner. Nevertheless, the theory to be presented here (being the simplest theory dealing with dissipative processes in helium II) is a good starting point to illustrate the application of the general method described in the introduction. Furthermore, the equations obtained in [22] are not the most general equations possible, so it is perhaps of some interest to indicate the nature of the most general equations consistent with the general principles underlying the derivation.

One of the fundamental assumptions of the present theory is that curl $\underline{v}_s$ must vanish always. The equations describing reversible flows are the Landau equations as discussed in II-A-2. For convenience, these equations, together with the expressions for the various densities and fluxes, are written out below.

mass: $$\frac{\partial \rho}{\partial t} + \operatorname{div} \underline{j} = 0, \quad \rho = \rho_n + \rho_s, \quad \underline{j} = \rho_n \underline{v}_n + \rho_s \underline{v}_s$$

momentum: $$\frac{\partial j_i}{\partial t} + \frac{\partial}{\partial x_j}\left(\pi^{\circ}_{ij}\right) = 0, \quad \pi^{\circ}_{ij} = \rho_n v_{ni} v_{nj} + \rho_s v_{si} v_{sj} + p\delta_{ij}$$

energy: $\frac{\partial E}{\partial t} + \operatorname{div} \underline{Q}^{\circ} = 0$, $\quad E = \frac{1}{2}\rho_n v_n^2 + \frac{1}{2}\rho_s v_s^2 + \rho e$,

$$de = \frac{P}{\rho^2} d\rho + T\,dS + \frac{1}{2} w^2 dx, \tag{171}$$

$$\underline{Q}^{\circ} = \left(\Phi + \frac{1}{2} v_s^2\right)\underline{j} + T\rho S \underline{v_n} + \rho_n \underline{v_n}\,(\underline{w} \cdot \underline{v_n}),$$

where

$$\Phi = e + \frac{P}{\rho} - ST - \frac{1}{2} x w^2,$$

superfluid: $\frac{\partial \underline{v_s}}{\partial t} + \underline{v_s} \cdot \nabla \underline{v_s} + \nabla \Phi = 0$,

and $\qquad \operatorname{curl} \underline{v_s} = 0.$

2. Equations for dissipative processes

We now derive the equations including dissipative processes according to the general method outlined in the introduction. The equations for mass, momentum and energy must still have the form of conservation laws. Thus these three equations will have the form

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \underline{j} = 0, \tag{172}$$

$$\frac{\partial j_i}{\partial t} + \frac{\partial}{\partial x_j}\left(\pi^{\circ}_{ij} + \pi'_{ij}\right) = 0, \tag{173}$$

and

$$\frac{\partial E}{\partial t} + \operatorname{div}\left(\underline{Q}_{\circ} + \underline{Q}'\right) = 0, \tag{174}$$

where the increments π'_{ij} and $\underline{Q}'$ due to the dissipative processes are assumed to be linear functions of a set of independent gradients (such a set would be for example, $\frac{\partial v_{ni}}{\partial x_j}$, $\frac{\partial v_{si}}{\partial x_j}$ $\frac{\partial P}{\partial x_i}$ and $\frac{\partial T}{\partial x_i}$). The coefficients connecting the fluxes π'_{ij} and $\underline{Q}'$ with the gradients will in general depend on $\underline{v_n}$, $\underline{v_s}$ and the thermodynamic variables (subject to the restrictions imposed by the Galilean relativity principle). The equation for the superfluid is not a conservation law; however, since we are imposing the restriction $\operatorname{curl} \underline{v_s} = 0$, the superfluid equation must have the form

$$\frac{\partial \underline{v_s}}{\partial t} + \underline{v_s} \cdot \nabla \underline{v_s} + \nabla\left(\Phi + \Phi'\right) = 0, \tag{175}$$

where Φ' is also taken to be a linear function of the independent gradients.

In the case of the perfect fluid theory, the equations (171) imply the entropy equation $\frac{\partial}{\partial t}(\rho s) + \operatorname{div}(\rho s \underline{v}_n) = 0$. In the present case, the equations (172) - (175) will also yield an equation for the entropy which, however, has the form

$$\frac{\partial}{\partial t}(\rho s) + \operatorname{div}(\rho s \underline{v}_n + \underline{\mathcal{J}}_s') = \frac{R}{T} , \tag{176}$$

where $\underline{\mathcal{J}}_s'$ represents the effect of the dissipative processes on the entropy flux (and is assumed to be a linear function of independent gradients), and R/T , the rate of entropy production, is to be positive definite. Equation (176) is not an independent equation, but is a consequence of the equations (172) - (175); in fact, it is a matter of straightforward calculation to show from (172) - (175) that

$$T\left\{\frac{\partial}{\partial t}(\rho s) + \operatorname{div}(\rho s \underline{v}_n)\right\} = -\operatorname{div}\underline{Q}' - \rho_s \underline{w} \cdot \nabla \Phi' + v_{ni} \frac{\partial \pi_{ij}'}{\partial x_j} .$$

Then, after a simple transformation and use of (176), we have

$$R = -\operatorname{div}\left\{\underline{Q}' - \underline{v}_n \cdot \underline{\underline{\pi}}' + \rho_s \underline{w} \Phi' - T\underline{\mathcal{J}}_s'\right\} - \pi_{ij}' \frac{\partial v_{ni}}{\partial x_j} + \Phi' \operatorname{div} \rho_s \underline{w} - \underline{\mathcal{J}}_s' \cdot \nabla T \tag{177}$$

Equation (177) has the form $R = -\operatorname{div} \underline{F} + R_1$, where $\underline{F}\,(= \underline{Q}' - T\underline{\mathcal{J}}_s' - \underline{v}_n \cdot \underline{\underline{\pi}}' + \rho_s \underline{w} \Phi')$ is a linear vector function of the independent gradients, and R_1 is a homogeneous quadratic function of the independent gradients. If $\underline{F}$ were not identically zero, then the local rate of entropy production R would contain terms

linear in the second derivatives of the intensities, as well as terms quadratic in the first derivatives; in this case, we could not possibly make R positive definite by any choice of phenomenological coefficients. Therefore $\underline{F} \equiv 0$, and we have*

$$\underline{Q}' = \underline{v}_n \cdot \underline{\underline{\Pi}}' - \rho_s \underline{w} \Phi' + T \underline{\mathcal{J}}_s', \tag{178}$$

$$R = -\Pi'_{ij} \frac{\partial v_{ni}}{\partial x_j} + \Phi' \operatorname{div} \rho \underline{w} - \underline{\mathcal{J}}_s' \cdot \nabla T. \tag{179}$$

Thus the quantities Π'_{ij}, Φ' and $\underline{\mathcal{J}}_s'$ --all linear functions of the independent gradients--are to be determined so as to make R positive definite. The requirements of Galilean relativity are satisfied provided Π'_{ij}, Φ' and $\underline{\mathcal{J}}_s'$ are all Galilean invariants. Thus the coefficients connecting these quantities with the independent gradients must then be Galilean invariant; this means that (in general) these phenomenological coefficients will depend on $\underline{w}(= \underline{v}_n - \underline{v}_s)$ and any two independent thermodynamic variables (such as ρ and s or p and T).

From the expression (179) for the entropy production, we see that the gradients conjugate to the fluxes Π'_{ij}, Φ' and $\underline{\mathcal{J}}_s'$ are, respectively, $-\frac{\partial v_{ni}}{\partial x_j}$, $\operatorname{div} \rho_s \underline{w}$ and $-\nabla T$. Although in principle we may express the fluxes Π'_{ij}, Φ' and $\underline{\mathcal{J}}_s'$ in terms of any set of independent gradients, the conjugate gradients

* A careful discussion of this point requires a much more detailed argument; a more complete discussion is given in the Appendix to Chapter III.

furnished by the expression for the entropy production are the natural choice. Since π'_{ij} is to be symmetric, the gradient conjugate to π'_{ij} may be taken as

$$e_{ij}^{(n)} = \frac{1}{2}\left\{ \frac{\partial v_{ni}}{\partial x_j} + \frac{\partial v_{nj}}{\partial x_i} \right\} \tag{180}$$

Thus the phenomenological relations (to be determined) will relate the fluxes to the gradients

$$e_{ij}^{(n)}, \quad \operatorname{div} \rho_s \underline{w} \quad \text{and} \quad \nabla T. \tag{181}$$

The procedure for finding the general expressions for the fluxes is straightforward but laborious. By way of example we consider the problem of finding the most general expression for the entropy flux $\underline{\mathcal{J}}'_s$. Since $\underline{\mathcal{J}}'_s$ is a linear function of the quantities (181), we have

$$\mathcal{J}'_{si} = \alpha_i \operatorname{div} \rho_s \underline{w} + \beta_i e_{kk}^{(n)} + \gamma_{ij} \frac{\partial T}{\partial x_j} + \eta_{ijk} \tilde{e}_{jk},$$

where

$$\tilde{e}_{jk} = e_{jk}^{(n)} - \frac{1}{3} \delta_{jk} e_{ll}^{(n)}. \tag{182}$$

It is clear that the coefficients α_i, β_i, etc. are tensors (with rank equal to the number of subscripts). These tensors will depend on the vector $\underline{w}$ and on the independent Galilean invariant scalars (e.g., on p, T and $\frac{1}{2}w^2$). We may (without loss of generality) assume that $\eta_{ijk} = \eta_{ikj}$, and $\eta_{ijj} = 0$. Then the most general expressions for the tensor coefficients are

$$\alpha_i = a_1 w_i , \quad \beta_i = a_2 w_i ,$$

$$\gamma_{ij} = a_3 w_i w_j + a_4 \delta_{ij} , \tag{183}$$

$$\eta_{ijk} = a_5 \left\{ w_i w_j w_k - \tfrac{1}{2} w^2 (w_j \delta_{ik} + w_k \delta_{ij}) \right\} ,$$

where the a's are all Galilean invariant scalars. Then $\underline{\mathcal{J}}_s'$ has the form

$$\mathcal{J}_{si}' = a_1 w_i \operatorname{div} \rho_s \underline{w} + a_2 w_i e_{kk}^{(n)} + a_3 w_i (w \cdot \nabla T) + a_4 \frac{\partial T}{\partial x_i} \tag{184}$$
$$+ a_5 \left\{ w_i w_j w_k - \tfrac{1}{2} w^2 (w_j \delta_{ik} + w_k \delta_{ij}) \right\} \tilde{e}_{jk} .$$

The calculations for the other fluxes are similar; the most general expressions for the remaining fluxes are given by

$$\Phi' = a_6 e_{kk}^{(n)} + a_7 \operatorname{div} \rho_s \underline{w} + a_8 \underline{w} \cdot \nabla T + a_9 w_i w_j \tilde{e}_{ij} , \tag{185}$$

$$\pi_{kk}' = a_{10} e_{kk}^{(n)} + a_{11} \operatorname{div} \rho_s \underline{w} + a_{12} \underline{w} \cdot \nabla T + a_{13} w_i w_j \tilde{e}_{ij} , \tag{186}$$

and

$$\pi_{ij}' - \tfrac{1}{3} \pi_{kk}' \delta_{ij} = a_{14} \left\{ w_i w_j - \tfrac{1}{3} w^2 \delta_{ij} \right\} e_{kk}^{(n)}$$
$$+ a_{15} \left\{ w_i w_j - \tfrac{1}{3} w^2 \delta_{ij} \right\} \operatorname{div} \rho_s \underline{w} + a_{16} \left\{ w_i w_j - \tfrac{1}{3} w^2 \delta_{ij} \right\} \underline{w} \cdot \nabla T$$
$$+ a_{17} \left\{ w_i \frac{\partial T}{\partial x_j} + w_j \frac{\partial T}{\partial x_i} - \frac{2}{3} \delta_{ij} \underline{w} \cdot \nabla T \right\}$$
$$+ \left\{ a_{18} w_i w_j + a_{19} \delta_{ij} \right\} w_k w_l \tilde{e}_{kl} + a_{20} \tilde{e}_{ij} \tag{187}$$
$$- \tfrac{1}{4} \left\{ w^2 a_{18} + 3 a_{19} \right\} \left\{ w_j w_k \delta_{li} + w_i w_k \delta_{lj} + w_l w_i \delta_{jk} \right.$$
$$\left. + w_l w_j \delta_{ik} \right\} \tilde{e}_{kl} .$$

Thus the most general dissipative fluxes may be expressed in terms of 20 scalar phenomenological coefficients. The condition that the entropy production be positive definite imposes certain restrictions on these scalar coefficients, although it would clearly be a formidable task to obtain explicit conditions.

The complexity of the general expressions (184) - (187) for the fluxes stems from the fact that the phenomenological coefficients in general may depend on the direction of the vector $\underline{w}$. We wish now to consider a much simpler special case--we assume that the phenomenological coefficients connecting the fluxes π_{ij}', Φ' and $\underline{\mathcal{J}}_s'$ with their conjugate gradients (181) depend on $\underline{w}$ only through w^2 . Then the equations (184) -(187) are replaced by the much simpler equations

$$\mathcal{J}_{si}' = -\frac{K}{T}(\nabla T)_i , \tag{188}$$

$$\Phi' = \lambda_1 e_{kk}^{(n)} + \lambda_2 \operatorname{div} \rho_s \underline{w} , \tag{189}$$

or

$$\Phi' = \lambda_1 \operatorname{div} \underline{v}_n + \lambda_2 \operatorname{div} \rho_s \underline{w} ,$$

$$\pi_{kk}' = -3\lambda_3 \operatorname{div} \underline{v}_n - 3\lambda_4 \operatorname{div} \rho_s \underline{w} , \tag{190}$$

and

$$\pi_{ij}' = \frac{1}{3}\pi_{kk}' \delta_{ij} = -2\mu_{nn} \tilde{e}_{ij} , \tag{191}$$

where there are now only 6 scalar phenomenological coefficients, K , λ_1 , λ_2 , λ_3 , λ_4 and μ_{nn}. The rate of entropy production R/T is then given by

$$R = 2\mu_{nn}\tilde{e}_{ij}\tilde{e}_{ij} + \lambda_3(\operatorname{div}\underline{v}_n)^2 + (\lambda_1+\lambda_4)(\operatorname{div}\underline{v}_n)(\operatorname{div}\rho_s\underline{w})$$
$$+ \lambda_2(\operatorname{div}\rho_s\underline{w})^2 + \frac{\kappa}{T}(\nabla T)^2. \qquad (192)$$

For R to be positive definite, it is necessary and sufficient that we have

$$\mu_{nn} \geq 0, \quad \lambda_3 \geq 0, \quad \lambda_2 \geq 0, \quad \kappa \geq 0$$

and

$$(\lambda_1+\lambda_4)^2 \leq 4\lambda_3\lambda_2. \qquad (193)$$

With the fluxes given by (188) - (191), the equations obtained here agree with those given by Lifshitz and Khalatnikov [22], although these equations are clearly a special case and not (as claimed in [22]) the most general equations. Thus to obtain the equations as given in [22], we must make the additional assumption that the phenomenological coefficients are independent of the direction of the vector $\underline{w}$.

No attempt has been made to give a physical interpretation to the many terms in the most general expressions for the fluxes (equations (184) - (187)); in fact the main reason for obtaining the general equations was simply to illustrate the great variety of dissipative terms consistent with the present form of the two-fluid model and with the general theory of linear irreversible processes. For the remainder of this section, we will only consider the special equations for the fluxes given by Lifshitz and Khalatnikov [22] (equation (188) - (191)).

It is desirable to give a physical interpretation to the processes represented by the 6 scalar phenomenological coeffi-

It is desirable to give a physical interpretation to the processes represented by the 6 scalar phenomenological coefficients κ , μ_{nn}, λ_1 , λ_2 , λ_3 and λ_4. It is clear from the expressions for the fluxes that κ and μ_{nn} are the thermal conductivity and the normal fluid viscosity respectively. The other 4 coefficients (λ_1 , λ_2 , λ_3 , λ_4) are analogous to coefficients of bulk viscosity and may be regarded as a phenomenological representation of certain relaxation processes (representing, for example, the effect of the finite relaxation time of the processes which tend to maintain the normal fluid concentration at its equilibrium value).

The superfluid equation, the expressions for the entropy production and the dissipative part of the energy flux may be rewritten in a manner which allows an interpretation of the dissipative terms in terms of stress tensors and mutual friction. To this end we introduce

$$\tau_{ij}^{(n)\prime} = \rho_s \Phi' \delta_{ij} - \pi'_{ij} = 2\mu_{nn}\tilde{e}_{ij} + (\rho_s\lambda_1 + \lambda_3)\,\mathrm{div}\,\underline{v}_n\,\delta_{ij} + (\rho_s\lambda_2 + \lambda_4)\,\mathrm{div}\,\rho_s\underline{w}\,\delta_{ij}, \tag{199}$$

$$\tau_{ij}^{(s)\prime} = -\rho_s\Phi'\delta_{ij} = -\rho_s\lambda_1\,\mathrm{div}\,\underline{v}_n\,\delta_{ij} - \rho_s\lambda_2\,\mathrm{div}\,\rho_s\underline{w}\,\delta_{ij},$$

and

$$\underline{F} = \Phi'\nabla\rho_s = (\lambda_1\,\mathrm{div}\,\underline{v}_n + \lambda_2\,\mathrm{div}\,\rho_s\underline{w})\,\nabla\rho_s.$$

Then the superfluid equation may be written as

$$\frac{\partial v_{si}}{\partial t} + \underline{v}_s\cdot\nabla v_{si} + \frac{\partial\Phi}{\partial x_i} = \frac{1}{\rho_s}\frac{\partial\tau_{ij}^{(s)\prime}}{\partial x_j} + \frac{1}{\rho_s}F_i. \tag{200}$$

The dissipative part of the energy flux is (from (178))

$$Q_i' = -K \frac{\partial T}{\partial x_i} - \tau_{ij}^{(n)\prime} v_{nj} - \tau_{ij}^{(s)\prime} v_{sj} , \tag{201}$$

and the rate of entropy production R/T is given by

$$R = \frac{\partial v_{ni}}{\partial x_j} \tau_{ij}^{(n)\prime} + \frac{\partial v_{si}}{\partial x_j} \tau_{ij}^{(s)\prime} + \underline{w} \cdot \underline{F} + \frac{K}{T} (\nabla T)^2 . \tag{202}$$

Thus the quantities $\tau_{ij}^{(n)}$, $\tau_{ij}^{(s)}$, and $\underline{F}$ as defined by (199) play the role of a dissipative normal fluid stress, a dissipative superfluid stress--and a mutual friction, respectively. This is perhaps best regarded as a formal interpretation, however, since, in the present case, the form of the superfluid equation is dictated by the physical requirement $\operatorname{curl} \underline{v}_s = 0$.

3. Boundary conditions

Finally, we must consider the boundary conditions to be satisfied at a helium II - solid interface. We consider first the case when the solid is at rest; the results for the more general case of a solid in motion will be obtained by means of a Galilean transformation. From the picture of the normal fluid as a gas of thermal excitations, we may conclude that it is plausible to require the tangential component of $\underline{v}_n$ to vanish at the wall; thus

$$\underline{v}_n \times \underline{n}\Big|_{\text{wall}} = 0, \tag{203}$$

where $\underline{n}$ is the unit normal to the wall. One condition on the components $\underline{v}_n \cdot \underline{n}$ and $\underline{v}_s \cdot \underline{n}$ is immediately obtained from the requirement that the mass flux $\underline{j} \cdot \underline{n}$ vanish; thus

$$x\, \underline{v}_n \cdot \underline{n} + (1-x)\, \underline{v}_s \cdot \underline{n}\Big|_{\text{wall}} = 0. \tag{204}$$

The other condition on the quantities $\underline{v}_n \cdot \underline{n}$, $\underline{v}_s \cdot \underline{n}$ is usually obtained by equating $\rho s T \underline{v}_n \cdot \underline{n}$ to the heat current in the wall. However, the discussion in II-A-1 shows that this boundary condition leads to some difficulties in principle, so that a more careful discussion of the flow in the immediate vicinity of a heated wall is required. The boundary condition $\rho s T \underline{v}_n \cdot \underline{n} =$ =(heat current in the wall) formally corresponds to a flux of normal fluid through the wall; in actuality, we know that the heat supplied to the fluid through the wall acts to convert superfluid into normal fluid, and in fact we might expect that

there will be a thin boundary layer near the heated wall within which superfluid- normal fluid conversion is an important process. It seems plausible to suppose that the normal fluid excitations created very near the wall will be very nearly at rest, and that the usual acceleration mechanisms (temperature and pressure gradients) will bring the excitations away from the wall. With these crude arguments as a guide, we might then conjecture that the correct boundary conditions on $\underline{v}_n \cdot \underline{n}$ and $\underline{v}_s \cdot \underline{n}$ are simply that

$$\underline{v}_n \cdot \underline{n}\Big|_{wall} = 0, \tag{205}$$

and

$$\underline{v}_s \cdot \underline{n}\Big|_{wall} = 0.$$

A further boundary condition is obtained from the fact that the energy flux must be continuous across the wall; thus if $\underline{H}$ is the heat flux vector in the solid, we have

$$\underline{Q} \cdot \underline{n}\Big|_{wall} = \underline{H} \cdot \underline{n}\,,$$

or

$$-K \nabla T \cdot \underline{n}\Big|_{wall} = \underline{H} \cdot \underline{n}\,. \tag{206}$$

We assume for now that the temperature is continuous across the interface (a brief discussion of the modification of the boundary conditions to take into account the so-called Kapitza effect will be given below; since the Kapitza effect is not peculiar to superfluid helium, it was felt desirable to discuss the consequences of the above boundary conditions before including a further complication). Thus the complete boundary

conditions in the case of a solid at rest are

$$\underline{v}_n\Big|_{wall} = 0 \ , \tag{207}$$

$$\underline{n}\cdot\underline{v}_s\Big|_{wall} = 0 \ , \tag{208}$$

$$-K\nabla T\cdot\underline{n}\Big|_{wall} = \underline{H}\cdot\underline{n} \ , \tag{209}$$

and

$$T_{helium} = T_{wall} \ . \tag{210}$$

Since these are not the usual boundary conditions for the two-fluid model, some discussion of the consequences of these boundary conditions is called for. First, of all, we see from the boundary condition (209) that there will be appreciable temperature gradients in the helium in the immediate vicinity of the wall. Since even a small temperature gradient tends to excite a large internal convection heat current in helium II, we may expect the large temperature gradients to be confined to a thin layer near the wall. In this layer, the process of normal fluid-superfluid conversion will be important, as well as the dissipative viscous and thermal conduction processes. We now verify this qualitative picture by a more detailed analysis of a specific problem. Namely, we consider the steady flow induced in a half-space filled with helium II by a uniform heat flux $\underline{H}$ through the boundary wall (cf., Fig. 1, p.29). At a great distance from the wall, we know the flow will be a uniform counterflow; near the wall, we expect a thin boundary layer in which the gradients of the various quantities are

large. Within the steady boundary layer, a balance is achieved among the pressure and temperature gradients, the viscous dissipative terms (including the bulk viscosity terms involving λ_1, λ_2, λ_3 and λ_4) and the nonlinear acceleration terms ($\underline{v}_s \cdot \nabla \underline{v}_s$, etc.). For sufficiently small heat input H, we may expect the nonlinear terms to be negligible; we will verify this from the solution obtained on that assumption (and also make more exact the criterion that H be sufficiently small). We may also anticipate that (for H sufficiently small) the total change in temperature, pressure, normal fluid concentration, etc. through the boundary layer will be small; thus we may treat the thermodynamic quantities and the phenomenological coefficients as constants wherever these quantities appear in the coefficients of differentiated terms. When these approximations are made, the differential equations describing the flow reduce to ordinary linear differential equations with constant coefficients. The solution is easily obtained, and we have

$$v_n = \frac{H}{T\rho s}\left\{1 - e^{-x/\delta}\right\},$$

$$v_s = -\frac{\rho_n}{\rho_s} v_n, \qquad (211)$$

$$T = T_\infty + \frac{H\delta}{K} e^{-x/\delta},$$

$$P = P_\infty + \frac{(\lambda_3 + \rho\lambda_4)}{T\rho s\delta} H e^{-x/\delta}$$

where

$$\delta^2 = \frac{K(\lambda_1 + \lambda_2\rho + \lambda_3/\rho + \lambda_4 + \frac{4}{3}\mu_{nn}/\rho)}{T\rho s^2},$$

and p_∞, T_∞ denote the (constant) values of p, T for $x \to \infty$. The total temperature drop through the boundary layer is

$$\Delta T = T_{wall} - T_\infty = \frac{H\delta}{K}. \tag{212}$$

Although little is known about the values of the bulk viscosities, for purposes of order of magnitude estimates we simply assume that the quantities $\rho\lambda_1$, $\lambda_2\rho^2$, λ_3 and $\rho\lambda_4$ are all of the same order of magnitude as the normal fluid viscosity μ_{nn}. Then

$$\delta^2 \sim \frac{K\,\mu_{nn}}{T\rho^2 s^2}, \tag{213}$$

$$\frac{\Delta T}{T_\infty} \sim \frac{H}{\sqrt{\frac{\rho^2 s^2 T^3 K}{\mu_{nn}}}}. \tag{214}$$

The relative magnitude of the nonlinear convective terms which were neglected may be shown to be

$$\frac{H}{\left(\frac{\mu_{nn} sT}{\delta}\right)} \sim \frac{H}{\left(\frac{\mu_{nn} T^3 s^4}{K}\right)^{1/2}}. \tag{215}$$

Thus the conditions for the analysis to be valid are

$$H << \left(\frac{\rho^2 s^2 T^3 K}{\mu_{nn}}\right)^{1/2} \tag{216}$$

and

$$H << \left(\frac{\mu_{nn} T^3 s^4 \rho^2}{K}\right)^{1/2},$$

or, in terms of $v_{n\infty}$, the conditions are

$$v_{n\infty} << \left(\frac{TK}{\mu_{nn}}\right)^{1/2}, \tag{217}$$

$$v_{n\infty} << \left(\frac{\mu_{nn} T s^2}{K}\right)^{1/2}. \tag{218}$$

By way of example, for $T = 1.8^\circ K$, $\mu_{nn} \sim 12 \cdot 10^{-6}$ poise , $s \sim 5.35 \cdot 10^{6}$ ergs/gm-deg. $K \sim 10^{4}$ ergs/sec-cm-deg, $\rho \sim .146$ gm/cm^3 , and we have

$$\delta \sim 3 \cdot 10^{-7} \text{ cm} ; \tag{219}$$

the conditions (217) and (218) are, respectively,

$$v_{n\infty} << 4 \cdot 10^{4} \text{ cm/sec}, \tag{220}$$

$$v_{n\infty} << 2.5 \cdot 10^{2} \text{ cm/sec}. \tag{221}$$

Thus for heat fluxes small enough to induce moderate velocities, the above analysis is adequate. (However, the smallness of the estimate (219) for δ casts doubt on the validity of a quantitative continuum analysis of the boundary layer.)

Several comments on the conditions (207) - (210) and the above analysis are perhaps in order here. First of all, the temperature change across the boundary layer will generally be extremely small in situations of practical interest (for example: if the heat flux is such as to induce a normal fluid velocity ~ 10 cm/sec, then for $T \simeq 1.8^\circ K$, $\Delta T/T < 10^{-3}$); in such

situations, we may often ignore the heat conduction and bulk viscosity terms in the equations, and replace the boundary conditions (207) - (210) by the "effective boundary condition"

$$\rho s T \underline{v}_n \cdot \underline{n} = \underline{H} \cdot \underline{n} ,$$

$$x \underline{v}_n \cdot \underline{n} + (1-x) \underline{v}_s \cdot \underline{n} = 0, \qquad (222)$$

$$\underline{v}_n \times \underline{n} = 0$$

and

$$T_{wall} - T_{helium} = A \underline{H} \cdot \underline{n},$$

where the thermal resistance of the boundary layer is given by

$$A = \frac{K}{\delta} . \qquad (223)$$

The boundary conditions (222) are equivalent to the results of the approximate boundary layer analysis given above. For somewhat larger heat currents, one would have to take into account the nonlinear convective terms in the equations, and the equations (222) would probably no longer give a satisfactory approximation to the actual flow.

Finally, we should mention that analyses quite similar to the above have been given by Gorter [9] and Kronig [19], although their stated purpose was to explain the Kapitza boundary effect (the name given to a boundary effect --first observed by Kapitza in 1941 [18]--in which a temperature discontinuity occurs at the solid-liquid interface when heat flows from the solid into the liquid). Since this effect has

also been observed in liquid He_3 [6], it seems unlikely that a theoretical explanation entirely in terms of the two-fluid hydrodynamics is possible (or even desirable). In fact, according to some of the theoretical explanations (e.g., Little [27]), such an effect should occur whenever a heat current passes through the interface separating two dissimilar materials. Both the theoretical and experimental results indicate that the quantitative features of the effect are described by a relation of the form

$$\Delta T \equiv (T_{wall} - T_{fluid}) = A_K (\underline{H} \cdot \underline{n}), \tag{224}$$

where the thermal resistance A_K depends on the temperature. The theoretical results indicate that the temperature jump actually takes place at the boundary and is not distributed over a thin region of fluid.

It is of some interest that the condition (224) is of the same form as the effective boundary condition (222) derived from the boundary layer analysis; however, experimental results indicate that for temperatures not too low, the measured resistance A is much larger than the value given by (223) (subject to the reservation that we have had to guess the order of magnitude of the bulk viscosities λ_1, λ_2, λ_3 and λ_4). For this reason, and the theoretical reasons given above, it is reasonable to conclude that the Kapitza boundary resistance and the boundary layer resistance are independent phenomena. Then we would expect the actual boundary conditions (including the Kapitza effect) to be

$$\underline{v}_n \Big|_{wall} = 0, \tag{225}$$

$$\underline{n} \cdot \underline{v}_s \Big|_{wall} = 0, \tag{226}$$

$$-\kappa \nabla T \cdot \underline{n} \Big|_{wall} = \underline{H} \cdot \underline{n}, \tag{227}$$

and

$$T_{wall} - T_{helium} \Big|_{wall} = A_K (\underline{H} \cdot \underline{n}), \tag{228}$$

where $\underline{n}$ is the unit normal pointing into the fluid. For moderate heat currents, the approximate boundary layer analysis will still apply, and the effective boundary conditions will be

$$\underline{v}_n \times \underline{n} \Big|_{wall} = 0, \tag{229}$$

$$\rho s T \underline{v}_n \cdot \underline{n} \Big|_{wall} = \underline{H} \cdot \underline{n}, \tag{230}$$

$$x \underline{v}_n \cdot \underline{n} + (1-x) \underline{v}_s \cdot \underline{n} \Big|_{wall} = 0, \tag{231}$$

$$T_{wall} - T_{helium} \Big|_{wall} = (A + A_K)(\underline{H} \cdot \underline{n}), \tag{232}$$

with A still given by (223).

In the case of a wall moving with velocity $\underline{u}$, the above boundary conditions are still valid provided we replace $\underline{v}_n$ and $\underline{v}_s$ by $\underline{v}_n - \underline{u}$ and $\underline{v}_s - \underline{u}$.

Much of the above discussion of the boundary conditions is not dependent on the fact that we have assumed $\operatorname{curl} \underline{v}_s = 0$, and some of the results will be taken over directly in the

discussion of the boundary conditions in the other theories to be discussed.

C. Lin's Theory of Dissipative Processes

1. Introduction

In this section, we give a discussion of the theory of dissipative processes in helium II proposed by Lin [26,25,23]. The starting point for this theory is the two-fluid model; however, no restrictions are imposed on the quantity $\mathbf{curl}\ \underline{v}_s$. The equations are to be obtained by means of the general method outlined in the introduction (III-A). In order to apply this method, one must know the correct perfect fluid equations. In the present case, however, we are requiring that $\mathbf{curl}\ \underline{v}_s$ vanish; thus (as discussed in detail in Chapter II) there is still some doubt as to the exact form of the perfect fluid equations. Most of the uncertainty is centered around momentum exchange terms which are perpendicular to the relative velocity. It is easy to show that such terms contribute nothing to the expression for the rate of entropy production. Thus as far as the present discussion of irreversible processes is concerned, the more general perfect fluid equations (such as equations(165)) will lead to exactly the same theory of irreversible processes as the Landau equations. With this justification, then, we take the perfect fluid equations to be the Landau equations (as given by (171)) without the restriction $\mathbf{curl}\ \underline{v}_s = 0$. (The derivation of II-A-2 also lends weight to this choice.)

Although it is the purpose of this section to give a derivation and discussion of the dissipative equations proposed by Lin, the method of derivation leads to somewhat more

general equations, and some discussion of these more general equations is also included. In this way, we obtain some idea of the scope of the most general equations consistent with the general principles underlying the derivation; the discussion also indicates some of the ways in which Lin's theory may be generalized, should experimental results call for a generalization.

2. Equations for dissipative processes

We now derive the equations including dissipative processes. The equations for mass, momentum and energy must still have the form of conservation laws. Thus

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \underline{j} = 0, \tag{233}$$

$$\frac{\partial j_i}{\partial t} + \frac{\partial}{\partial x_j}\left(\pi^{\circ}_{ij} + \pi'_{ij}\right) = 0, \tag{234}$$

and

$$\frac{\partial E}{\partial t} + \operatorname{div}\left(\underline{Q}_{\circ} + \underline{Q}'\right) = 0, \tag{235}$$

where π°_{ij} and $\underline{Q}^{\circ}$ are given by (171); π'_{ij} and $\underline{Q}'$ are the dissipative contributions to the momentum flux tensor and the energy flux vector. The entropy equation will have the form

$$\frac{\partial}{\partial t}(\rho s) + \operatorname{div}\left\{\rho s \underline{v}_n + \underline{\mathcal{J}}_s'\right\} = \frac{R}{T}, \tag{236}$$

where $\underline{\mathcal{J}}_s'$ is the dissipative contribution to the entropy flux, and R/T, the volume rate of entropy production, is to be positive definite. In accordance with the general theory, the quantities $\underline{\mathcal{J}}_s'$, π'_{ij} and $\underline{Q}'$ are assumed to be functions of a set of independent gradients. For reasons to be discussed below, we deviate from the general theory outlined, in that we do not yet require that these quantities be linear functions of the gradients; for now we only require that the

dissipative terms $\mathcal{F}_s'$, π_{ij}' and $\underline{Q}'$ vanish whenever all of the independent gradients vanish.

Finally, we need an equation for the superfluid component. In the preceding section the form of the superfluid equations was fixed by the requirement $\text{curl}\ \underline{v}_s = 0$. In the present case, however it is by no means clear what form to assume for the superfluid equation (that is, whether to add a dissipative stress tensor, a mutual friction or both). Physically this means that we must have some idea of what sort of dissipative processes may occur, and how they affect the superfluid momentum, since the phenomenological theory cannot tell what sort of dissipative terms are present. In general, the superfluid momentum could be affected by a dissipative volume force (mutual friction) and a dissipative stress tensor. Thus we assume for the superfluid component an equation of the form

$$\frac{\partial v_{si}}{\partial t} + \underline{v}_s \cdot \nabla v_{si} + \frac{\partial \Phi}{\partial x_i} = \frac{1}{\rho_s} F_i' + \frac{1}{\rho_s} \frac{\partial \tau_{ji}^{(s)'}}{\partial x_j}, \qquad (237)$$

where the dissipative tensor $\tau_{ji}^{(s)'}$ and the dissipative volume force F_i' are functions of the independent gradients (and, of course, the thermodynamic variables and the relative velocity $\underline{w}$). (We are not considering at present the possibility that there is a mutual friction force dependent only on the thermodynamic variables and the relative velocity $\underline{w}$; as mentioned earlier, such a mutual friction force is a special sort of dissipative process whose contribution to the expression for the entropy production is entirely independent of the contribution from the gradient-dependent terms. In any case,

we may add such a term to the final equations, since the determination of the gradient-dependent dissipative terms is not affected in any way by the inclusion of this special kind of mutual friction.)

The equations (233) - (237) are not independent, as the hydrodynamic equations imply entropy equation. By a straightforward calculation, one may show that the entropy production R/T and the dissipative fluxes satisfy the equation

$$R = \tau_{ji}^{(n)'} \frac{\partial v_{ni}}{\partial x_j} + \tau_{ji}^{(s)'} \frac{\partial v_{si}}{\partial x_j} + \underline{F}' \cdot \underline{w} - \underline{\mathcal{F}}_s' \cdot \nabla T \qquad (238)$$

$$- \frac{\partial}{\partial x_j} \left\{ Q_j' + \tau_{ji}^{(n)'} v_{ni} + \tau_{ji}^{(s)'} v_{si} - T \mathcal{F}_{sj}' \right\},$$

where we have introduced $\tau_{ij}^{(n)'}$, defined by

$$\pi_{ij}' = -\tau_{ij}^{(n)'} - \tau_{ij}^{(s)'}. \qquad (239)$$

Although conservation of angular momentum requires that π_{ij}' be symmetric, in general $\tau_{ij}^{(n)'}$ and $\tau_{ij}^{(s)'}$ may have anti-symmetric parts which add to zero. It is then convenient to write $\tau_{ij}^{(n)'}$ and $\tau_{ij}^{(s)'}$ as

$$\tau_{ij}^{(n)'} = S_{ij}^{(n)'} + A_{ij}', \qquad (240)$$

and

$$\tau_{ij}^{(s)'} = S_{ij}^{(s)'} - A_{ij}', \qquad (241)$$

where $S_{ij}^{(n)'}$, $S_{ij}^{(s)'}$ are symmetric and A_{ij}' is anti-symmetric. Then (238) may be written as

$$R = S_{ij}^{(n)\prime} e_{ij}^{(n)} + S_{ij}^{(s)\prime} e_{ij}^{(s)} + A_{ji}' (\omega_{ij}^{(n)} - \omega_{ij}^{(s)}) - \mathcal{J}_s' \cdot \nabla T + \underline{F}' \cdot \underline{w}$$

$$- \frac{\partial}{\partial x_j} \left\{ Q_j' + \tau_{ji}^{(n)\prime} v_{ni} + \tau_{ji}^{(s)\prime} v_{si} - T \mathcal{J}_{sj} \right\},$$

where (242)

$$e_{ij}^{(n)} = \frac{1}{2} \left\{ \frac{\partial v_{ni}}{\partial x_j} + \frac{\partial v_{nj}}{\partial x_i} \right\}, \quad e_{ij}^{(s)} = \frac{1}{2} \left\{ \frac{\partial v_{si}}{\partial x_j} + \frac{\partial v_{sj}}{\partial x_i} \right\},$$

and

$$\omega_{ij}^{(n)} = \frac{1}{2} \left\{ \frac{\partial v_{ni}}{\partial x_j} - \frac{\partial v_{nj}}{\partial x_i} \right\}, \quad \omega_{ij}^{(s)} = \frac{1}{2} \left\{ \frac{\partial v_{si}}{\partial x_j} - \frac{\partial v_{sj}}{\partial x_i} \right\}. \quad (243)$$

From the assumption that the dissipative terms depend only on the first derivatives of the intensities, and from the requirement that R be positive definite, one may show that

$$R = S_{ij}^{(n)\prime} e_{ij}^{(n)} + S_{ij}^{(s)\prime} e_{ij}^{(s)} + A_{ji}' (\omega_{ij}^{(n)} - \omega_{ij}^{(s)}) - \underline{\mathcal{J}}_s' \cdot \nabla T + \underline{F}' \cdot \underline{w}, \quad (244)$$

and

$$Q_j' = T \mathcal{J}_{sj}' - \tau_{ji}^{(n)\prime} v_{ni} - \tau_{ji}^{(s)\prime} v_{si}. \quad (245)$$

(The argument for this is discussed in detail in the Appendix to Chapter III.)

So far, we have not assumed that the dependence of the dissipative fluxes on the gradients is linear. (We have, however, assumed that the fluxes do not depend on the second or higher derivatives of the intensities, and this assumption is crucial in going from (242) to (244) and (245).) We now assume that the fluxes $S_{ij}^{(n)\prime}$, $S_{ij}^{(s)\prime}$, A_{ij}' and $\underline{\mathcal{J}}_s'$ depend linearly on the independent gradients. The gradient-dependent volume force $\underline{F}'$, however, requires special consideration. If $\underline{F}'$ were linear in the gradients, then there would be a

contribution $\underline{w}\cdot\underline{F}$ to the quantity R which would be linear in the gradients, whereas the remaining terms in R are homogeneous quadratic in the gradients; since R is to be positive definite, it must be that $\underline{w}\cdot\underline{F}=0$ (still assuming $\underline{F}'$ is linear in the gradients). In this case then, $\underline{F}'$ would not be a dissipative term at all, but would be exactly the sort of perpendicular momentum exchange term which some of the perfect fluid theories allow and which we have agreed to neglect. Thus it must be that the first approximation to the gradient-dependent volume force $\underline{F}'$ is quadratic in the gradients of the intensities. We may note that the equations of Landau, Lifshitz and Khalatnikov (as discussed in III-B) for the case $\operatorname{curl} v_s = 0$ correspond to the special choices

$$S_{ij}^{(n)\prime} = 2\mu_{nn}\tilde{e}_{ij}^{(n)} + (\rho_s\lambda_1+\lambda_3)\operatorname{div}\underline{v}_n\,\delta_{ij} + (\rho_s\lambda_2+\lambda_4)\cdot \operatorname{div}\rho_s\underline{w}\,\delta_{ij}$$

$$S_{ij}^{(s)\prime} = -(\rho_s\lambda_1\operatorname{div}\underline{v}_n + \rho_s\lambda_2\operatorname{div}\rho_s\underline{w})\delta_{ij}$$

$$A_{ij}' \equiv 0$$

and

$$\underline{F}' = (\lambda_1\operatorname{div}\underline{v}_n + \lambda_2\operatorname{div}\rho_s\underline{w})\nabla\rho_s,$$

so that their theory is included as a special case in the present general theory, as one would expect.

The special arguments necessary for the volume force are not particulary convincing; in fact, they more or less serve to show that the volume mutual friction force is a concept which does not fit comfortably into the framework of the

present general theory (with one possible exception: from (244), we see that the quantity conjugate to $\underline{F}'$ in the entropy production is not a gradient at all, but is the relative velocity $\underline{w}$; thus mutual friction forces dependent only on the scalar thermodynamic variables and the relative velocity $\underline{w}$ can easily be accomodated in the present theory; as mentioned earlier, the determination of the gradient-dependent terms is not affected in any way by the inclusion of such a mutual friction). As mentioned earlier, a volume mutual friction force really represents a partial breakdown of the two-fluid model, so that it is not surprising that we meet some difficulties when we attempt to treat it on an equal footing with the more usual dissipative processes.

We may also note that, within the context of Lin's theory, there is no obvious mechanism which could give rise to a volume mutual friction force. For this reason and the reasons discussed above, we will assume from here on that the volume mutual friction force $\underline{F}'$ is identically zero. Then the expression for the rate of the entropy production is given by

$$R = S_{ij}^{(n)\prime} e_{ij}^{(n)} + S_{ij}^{(s)\prime} e_{ij}^{(s)} + A_{ij}' (\omega_{ij}^{(n)} - \omega_{ij}^{(s)}) - \underline{\mathcal{J}}_s' \cdot \nabla T. \qquad (246)$$

Thus the gradients conjugate to the fluxes $S_{ij}^{(n)\prime}$, $S_{ij}^{(s)\prime}$, A_{ij}', and $\underline{\mathcal{J}}_s'$ are respectively,

$$e_{ij}^{(n)}, \quad e_{ij}^{(s)}, \quad \omega_{ij} \equiv \omega_{ij}^{(n)} - \omega_{ij}^{(s)} \text{ and } -\nabla T, \qquad (247)$$

and the phenomenological relations connecting the fluxes with

the gradients (247) are to be determined so that R is positive definite.

As in the preceding section, the most general relations connecting the fluxes with the gradients are very complicated, since the coefficients in general may depend on the direction of the vector $\underline{w}$. Although it is possible to obtain the general phenomenological relations for the present case, they will not be considered in detail so we do not write them out here. (We may note, however, that the most general relations are characterized by 44 scalar coefficients, subject to the restrictions imposed by the condition $R \geq 0$.) We now assume that the coefficients connecting the fluxes with the conjugate gradients (247) do not depend on the direction of the vector $\underline{w}$. The phenomenological relations are then greatly simplified, and we may write them in the form

$$
\begin{aligned}
S_{ij}^{(n)\prime} &= 2\mu_{nn}\tilde{e}_{ij}^{(n)} + 2\mu_{ns}\tilde{e}_{ij}^{(s)} + \lambda_{nn} e_{kk}^{(n)}\delta_{ij} + \lambda_{ns} e_{kk}^{(s)}\delta_{ij}, \\
S_{ij}^{(s)} &= 2\mu_{sn}\tilde{e}_{ij}^{(n)} + 2\mu_{ss}\tilde{e}_{ij}^{(s)} + \lambda_{sn} e_{kk}^{(n)}\delta_{ij} + \lambda_{ss} e_{kk}^{(s)}\delta_{ij}, \\
A_{ij}' &= -2\gamma\omega_{ij}, \qquad \underline{J}_s' = -\frac{K}{T}\nabla T,
\end{aligned}
\tag{249}
$$

where

$$
\tilde{e}_{ij}^{(n)} = e_{ij}^{(n)} - \tfrac{1}{3}\delta_{ij} e_{kk}^{(n)}, \qquad \tilde{e}_{ij}^{(s)} = e_{ij}^{(s)} - \tfrac{1}{3} e_{kk}^{(s)}\delta_{ij}.
$$

Thus the phenomenological relations are characterized by 10 scalar coefficients. The rate of entropy production R/T is then given by

$$R = 2\mu_{nn}\tilde{e}_{ij}^{(n)}\tilde{e}_{ij}^{(n)} + 2(\mu_{ns}+\mu_{sn})\tilde{e}_{ij}^{(n)}\tilde{e}_{ij}^{(s)} + 2\mu_{ss}\tilde{e}_{ij}^{(s)}\tilde{e}_{ij}^{(s)}$$
$$+ \lambda_{nn}(\operatorname{div}\underline{v}_n)^2 + (\lambda_{ns}+\lambda_{sn})(\operatorname{div}\underline{v}_n)(\operatorname{div}\underline{v}_s) + \lambda_{ss}(\operatorname{div}\underline{v}_s)^2$$
$$+ \gamma(\operatorname{curl}\underline{w})^2 + \frac{K}{T}(\nabla T)^2.$$

The necessary and sufficient conditions for R to be positive definite are

$$\mu_{nn} \geq 0, \quad \mu_{ss} \geq 0, \quad 4\mu_{nn}\mu_{ss} \geq (\mu_{ns}+\mu_{sn})^2,$$
$$\lambda_{nn} \geq 0, \quad \lambda_{ss} \geq 0 \quad 4\lambda_{nn}\lambda_{ss} \geq (\lambda_{ns}+\lambda_{sn})^2, \tag{251}$$
$$\gamma \geq 0 \quad \text{and} \quad K \geq 0.$$

We now consider the physical interpretation of the processes represented by the phenomenological equations (249). From the picture of the normal fluid as a gas of thermal excitations, we may expect that there will be some transport of energy and momentum by molecular collision processes; thus we may identify K as the coefficient of thermal conductivity and μ_{nn} as the normal fluid shear viscosity. The quantities μ_{ss}, μ_{ns}, μ_{sn} are likewise interpreted as shear viscosities, μ_{ss} being the superfluid viscosity and μ_{ns}, μ_{sn} being shear exchange coefficients. It would be desirable to give some interpretations in terms of the microscopic theory for the processes represented by μ_{ss}, μ_{ns} and μ_{sn}; however, a clear picture of the sort of microscopic processes occuring in flowing helium II is still lacking, so that such an interpretation must await further development of the microscopic theory.

(Of course, one may make conjectures on the basis of the qualitative microscopic theories now available. For example, from the picture of the superfluid component as a kind of single coherent quantum state, we might suspect that any non-uniformity in the superfluid velocity field would be resisted by some sort of quantum-mechanical exchange forces; it is conceivable that such forces could be represented phenomenologically in terms of a "viscosity" coefficient μ_{ss}). It is perhaps appropriate to mention here that comparisons of Lin's theory with experiments indicate that μ_{ss} is of the same order of magnitude as μ_{nn}, and that $\mu_{ns} = -\mu_{ss}$.

The quantities λ_{nn}, λ_{ns}, λ_{sn}, λ_{ss} are analogous to the coefficient of bulk viscosity for an ordinary fluid; they may be regarded as phenomenological representations of certain relaxation processes (representing, for example, the effect of the finite relaxation time of the processes which tend to maintain the normal fluid concentration at its equilibrium value). At present, little more can be said about the nature of the processes represented by these coefficients (or about the order of magnitude of the coefficients).

Until now, nothing has been said about the possible physical significance of the anti-symmetric part of the stress tensors, A_{ij}'. The physical interpretation of this term is most easily seen from the equations for the angular momenta of the two components. From the momentum equation and the superfluid equation, one may easily derive the following equations for the angular momenta:

$$\frac{\partial}{\partial t}\iiint_V d\tau\,(\underline{r}\times\rho_n\underline{v}_n)_i = -\iint_S d\sigma\cdot\underline{n}\cdot\underline{v}_n(\underline{r}\times\rho_n\underline{v}_n)_i + \iint_S d\sigma\,\epsilon_{ijk}r_j n_l(\tau_{lk}^{(n)\prime} - p_n\delta_{lk})$$

$$+\iiint_V d\tau\,\Gamma(\underline{r}\times\underline{v}_s)_i - \iiint_V d\tau\,2\gamma(\mathrm{curl}\,\underline{w})_i\,,$$

and

$$\frac{\partial}{\partial t}\iiint_V d\tau\,(r\times\rho_s\underline{v}_s)_i = -\iint_S d\sigma\,(\underline{n}\cdot\underline{v}_s)(\underline{r}\times\rho_s\underline{v}_s)_i + \iint_S d\sigma\,\epsilon_{ijk}r_j n_l(\tau_{lk}^{(s)\prime} - p_s\delta_{lk})$$

$$-\iiint_V d\tau\,\Gamma(\underline{r}\times\underline{v}_s)_i + \iiint_V d\tau\,2\gamma(\mathrm{curl}\,\underline{w})_i\,,$$

where V is a volume fixed in space bounded by the closed surface S (with unit normal $\underline{n}$), and where $\Gamma = \frac{\partial\rho_n}{\partial t} + \mathrm{div}\,\rho_n\underline{v}_n$ is the rate of superfluid-normal fluid conversion, and p_n, p_s are the normal fluid and superfluid pressures (defined in II-A-2). Thus the angular momentum balance for each component includes four types of terms: (i) a convective term (the first term on the right-hand side of each equation), (ii) a torque term (the second term on the right-hand side), (iii) a (reversible) volume rate of exchange due to the $\rho_n \rightleftarrows \rho_s$ transitions (third term) and (iv) an irreversible volume rate of exchange of angular momentum (the last term on the right-hand side of each equation). Thus the physical process associated with the anti-symmetric parts of the stress tensors is a sort of "angular momentum mutual friction" which tends to resist a (local) relative rotation of the two components. In the development of the present theory, we have assumed that the only momentum exchange between the two components is that due to $\rho_n \rightleftarrows \rho_s$ transitions (i.e., we have assumed that there is no mutual

friction); this assumption, however, does not exclude the possibility of coupling by angular momentum exchange, and, as we have seen, such coupling terms arise in a natural way in the development of the general phenomenological theory. From the angular momentum equation (252), we see that there is an effective body couple, $-2\gamma \operatorname{curl} \underline{w}$, acting on the normal fluid component and an effective body couple, $+2\gamma \operatorname{curl} \underline{w}$, acting on the superfluid component. The local relative angular velocity is $\frac{1}{2} \operatorname{curl} \underline{w}$, and the contribution of this process to the rate of energy dissipation is (from equation (250)), as we might expect,

$$\tfrac{1}{2} \operatorname{curl} \underline{w} \cdot 2\gamma \operatorname{curl} \underline{w} = \gamma (\operatorname{curl} \underline{w})^2.$$

Although this is a useful interpretation, it is not strictly correct, since the terms in question are really surface effects, as they come from a part of the stress tensors. We may also see this from the fact that there is a contribution to the energy flux (cf. equation (253)) from these terms, whereas a true "mutual body couple" would not contribute anything to the energy flux (just as a mutual friction effecting momentum exchange does not contribute to the energy flux).

$$Q_i' = T \mathcal{J}_{si}' - S_{ij}^{(n)\prime} v_{nj} - S_{ij}^{(s)\prime} v_{sj} - A_{ij}' w_j. \qquad (253)$$

The question of whether or not such extra terms should be included in the hydrodynamic equations for helium II can only be answered by comparison of the theory with experiment.

Since it is known from various experiments that the rotation of helium II seems to bring into play some sort of rotation-dependent dissipative processes, the possibility of including the additional terms discussed above should be considered more seriously. For the remainder of this section however, we will simply assume that the anti-symmetric terms in the stress tensors are zero; this corresponds to the physical statement that there is no volume angular momentum exchange between the two components (other than that due to $\rho_n \rightleftarrows \rho_s$ transitions). A decisive discussion of this point must await a careful study of the equations including the anti-symmetric terms in the stress tensors.

The complete system of hydrodynamic equations may be easily written down by substituting the expressions found for the dissipative fluxes into the conservation equations (233) - (235) and the superfluid equation (237). For convenience of reference, all of the relevant formulae are written out below. The equations are essentially those as given by Lin [25].

continuity equation: $$\frac{\partial \rho}{\partial t} + \operatorname{div}\left(\rho_n \underline{v}_n + \rho_s \underline{v}_s\right) = 0,$$

momentum equation: $$\frac{\partial j_i}{\partial t} + \operatorname{div}\left(\underline{\underline{\pi}}^{\circ} + \underline{\underline{\pi}}'\right) = 0,$$

where $$\underline{j} = \rho_n \underline{v}_n + \rho_s \underline{v}_s$$

$$\pi_{ij}^{\circ} = \rho_n v_{ni} v_{nj} + \rho_s v_{si} v_{sj} + p\delta_{ij},$$

$$\pi_{ij}' = -2(\mu_{nn} + \mu_{sn})\tilde{e}_{ij}^{(n)} - 2(\mu_{ns} + \mu_{ss})\tilde{e}_{ij}^{(s)} - (\lambda_{nn} + \lambda_{sn})\tilde{e}_{kk}^{(n)}\delta_{ij} - (\lambda_{ns} + \lambda_{ss})\tilde{e}_{kk}^{(s)}\delta_{ij}$$

$$e_{ij}^{(n)} = \frac{1}{2}\left\{\frac{\partial v_{ni}}{\partial x_j} + \frac{\partial v_{nj}}{\partial x_i}\right\}, \quad \tilde{e}_{ij}^{(n)} = e_{ij}^{(n)} - \frac{1}{3}\delta_{ij} e_{kk}^{(n)},$$

$$e_{ij}^{(s)} = \frac{1}{2}\left\{\frac{\partial v_{si}}{\partial x_j} + \frac{\partial v_{sj}}{\partial x_i}\right\}, \quad \tilde{e}_{ij}^{(s)} = e_{ij}^{(s)} - \frac{1}{3}\delta_{ij} e_{kk}^{(s)},$$

super fluid equation :

$$\frac{\partial \underline{v}_s}{\partial t} + \underline{v}_s \cdot \nabla \underline{v}_s = S\nabla T - \frac{1}{\rho}\nabla p + X \nabla \frac{1}{2} w^2 + \frac{1}{\rho_s} \operatorname{div} \underline{\underline{\tau}}^{(s)'}, \quad (255)$$

where:

$$\tau_{ij}^{(s)'} = 2\mu_{sn}\tilde{e}_{ij}^{(n)} + 2\mu_{ss}\tilde{e}_{ij}^{(s)} + \lambda_{sn} e_{kk}^{(n)}\delta_{ij} + \lambda_{ss} e_{kk}^{(s)}\delta_{ij}$$

normal fluid equation :

$$\frac{\partial \underline{v}_n}{\partial t} + \underline{v}_n \cdot \nabla \underline{v}_n = -\frac{1}{\rho}\nabla p - \frac{(1-X)}{X} S\nabla T - \frac{(1-X)}{2}\nabla w^2 - \frac{w}{\rho_n}\Gamma + \frac{1}{\rho_n} \operatorname{div} \underline{\underline{\tau}}^{(n)'}, \quad (256)$$

where

$$\tau_{ij}^{(n)'} = 2\mu_{nn}\tilde{e}_{ij}^{(n)} + 2\mu_{ns}\tilde{e}_{ij}^{(s)} + \lambda_{nn} e_{kk}^{(n)}\delta_{ij} + \lambda_{ns} e_{kk}^{(s)}\delta_{ij},$$

and $\Gamma = \frac{\partial \rho_n}{\partial t} + \operatorname{div}(\rho_n \underline{v}_n)$;

energy equation :

$$\frac{\partial E}{\partial t} + \operatorname{div}(\underline{Q}_0 + \underline{Q}') = 0, \quad (257)$$

where

$$E = \frac{1}{2}\rho_n v_n^2 + \frac{1}{2}\rho_s v_s^2 + \rho e,$$

$$e = e(\rho, s, x), \quad de = \frac{p}{\rho^2} d\rho + T ds + \frac{1}{2} w^2 dx,$$

(the equilibrium condition being $\left(\frac{\partial e}{\partial x}\right)_{\rho,s} = \frac{1}{2} w^2$)

$$\underline{Q}_0 = \left(\Phi + \tfrac{1}{2} v_s^2 \right) \underline{j} + T \rho s \underline{v}_n + \rho_n \underline{v}_n (\underline{w} \cdot \underline{v}_n),$$

$$\Phi = e + \frac{P}{\rho^2} - TS - \tfrac{1}{2} x w^2,$$

$$\underline{Q}' = - \underline{\underline{\tau}}^{(n)\prime} \cdot \underline{v}_n - \underline{\underline{\tau}}^{(s)\prime} \cdot \underline{v}_s - K \nabla T ;$$

Entropy equation:

$$\frac{\partial}{\partial t}(\rho s) + \operatorname{div}\left(\rho s \underline{v}_n - \frac{K}{T} \nabla T \right) = \frac{R}{T}, \qquad (258)$$

where

$$R = \tau_{ij}^{(n)\prime} e_{ij}^{(n)} + \tau_{ij}^{(s)\prime} e_{ij}^{(s)} + \frac{K}{T} (\nabla T)^2$$

$$= 2 \mu_{nn} \tilde{e}_{ij}^{(n)} \tilde{e}_{ij}^{(n)} + 2 (\mu_{ns} + \mu_{sn}) \tilde{e}_{ij}^{(n)} \tilde{e}_{ij}^{(s)} + 2 \mu_{ss} \tilde{e}_{ij}^{(s)} \tilde{e}_{ij}^{(s)}$$
$$+ \lambda_{nn} (\operatorname{div} \underline{v}_n)^2 + (\lambda_{ns} + \lambda_{sn})(\operatorname{div} \underline{v}_n)(\operatorname{div} \underline{v}_s)$$
$$+ \lambda_{ss} (\operatorname{div} \underline{v}_s)^2 + \frac{K}{T} (\nabla T)^2 ;$$

the phenomenological coefficients must satisfy the conditions

$$\mu_{nn} \geq 0, \quad \mu_{ss} \geq 0, \quad 4 \mu_{nn} \mu_{ss} \geq (\mu_{ns} + \mu_{sn})^2,$$

$$\lambda_{nn} \geq 0, \quad \lambda_{ss} \geq 0, \quad 4 \lambda_{nn} \lambda_{ss} \geq (\lambda_{ns} + \lambda_{sn})^2, \text{ and } K \geq 0.$$

Since in the present theory the hydrodynamic equations have been deduced from (more or less) general principles, it is perhaps well to explicitly list the various principles and assumptions used in the deduction and then to give a brief analysis of the dependence of the final equations on each of

the principles. These principles and assumptions are:

(i) The Landau equations (without the restriction $\operatorname{curl} \underline{v}_s = 0$) for reversible flows.

(ii) No momentum exchange between the two components other than that due to $\rho_n \rightleftarrows \rho_s$ transitions (i.e., no mutual friction).

(iii) No angular momentum exchange between the two components other than that due to $\rho_n \rightleftarrows \rho_s$ transitions.

(iv) The dissipative terms in the equations are linear functions of the gradients of the independent macroscopic variables.

(v) The phenomenological coefficients connecting the dissipative fluxes with the conjugate set of gradients are independent of the direction of the vector $\overline{w}$.

These are the five assumptions which we will discuss individually. In the deduction of the equations, we have also used the conservation laws for mass, energy and momentum, the Galilean relativity principle, and the law of increase of entropy, none of which need any discussion. We have also assumed that the equilibrium thermodynamic description of the two-fluid system remains valid for dissipative flows; as in the case of ordinary hydrodynamics, we expect this to be a valid approximation as long as the macroscopic length and time scales remain much greater than the corresponding microscopic scales.

(i) We have assumed that the Landau equations are the correct equations for describing reversible flows. As we have

seen earlier, however, some of the derivations of the hydrodynamic equations do not lead in an unambiguous manner to the Landau equations, the ambiguity being centered mainly around momentum exchange terms which are perpendicular to the relative velocity $\underline{w}$. Since such terms contribute nothing to the dissipation function R , it is clear that the determination of the dissipative fluxes is completely unaffected by the presence of such terms. The equations originally proposed by Lin [25] differ from the equations (254) - (258) by just such a term.

(ii) We have assumed that there is no momentum exchange between the components (other than that due to $\rho_n \rightleftarrows \rho_s$ transitions). If this assumption is dropped, then we must take into account the possibility that there exists a volume mutual friction. The quantity conjugate to the mutual friction in the expression for the dissipation function is the relative velocity $\underline{w}$. Thus we may easily include in the theory a mutual friction force dependent on $\underline{w}$ and the scalar thermodynamic variables; furthermore, since the contribution of such a mutual friction to the dissipation function R is independently positive definite (because the remaining terms in R depend on the gradients of the macroscopic variables), it follows that the determination of the gradient-dependent terms is unaffected by the presence of such a mutual friction. Thus we may include an arbitrary mutual friction (independent of the gradients) by simply adding $\underline{F}/\rho_s$ to the right-hand side of (255), $-\underline{F}/\rho_n$ to the right-hand side of (256), and $\underline{w}\cdot\underline{F}$ to

the expression (258) for R ($\underline{F}$ must satisfy $\underline{w}\cdot\underline{F}>0$).

(iii) The assumption that there be no exchange of angular momentum between the two components was made in order to exclude the possibility of the stress tensors having antisymmetric parts. When this assumption is dropped, the general equations then contain terms which lead to a (dissipative) volume exchange of angular momentum between the two components.

(iv) We have also made the general assumption that the dissipative fluxes are linear functions of the independent gradients. In the derivation of the expression (244) for the dissipation function R , it was only assumed that the dissipative fluxes did not depend on second or higher spatial derivatives of the macroscopic quantities; it was not necessary to assume at this stage that the dependence of the fluxes on the first derivatives was linear. We may then distinguish two cases: (1) if the first non-vanishing approximation to a given dissipative flux is linear in the gradients, then, in general, we must expect the second approximation* to depend on higher order derivatives as well as quadratic terms, so that the validity of equation (244) for the entropy production will be limited to the case when we consider only the linear first approximation to the fluxes; (2) it may happen, however, that the first approximation to a given dissipative term is a non-linear function of the conjugate gradient, in which case equation (244) for the entropy production will still be valid.

We do not expect this second alternative to be relevant in

* We are using "first approximation" and "second approximation" here to refer to the results of an imagined approximate calculation of the fluxes from a detailed microscopic theory.

the case of the viscous stresses or the heat flux vector. However, mutual friction forces of the form $\underline{F} = Aw^2\underline{w}$ (being cubic in the conjugate quantity $\underline{w}$) have sometimes been included in the hydrodynamic equations for helium II [8] ; also it is conceivable that the anti-symmetric parts of the stress tensors discussed earlier could be nonlinear functions of the relative vorticity curl $\underline{w}$. In any case, we may consistently include such nonlinear terms in the present theory.

(v) Finally, we have assumed that the phenomenological coefficients connecting the dissipative fluxes with the conjugate set of gradients are independent of the direction of the vector $\underline{w}$. Roughly speaking, then, we are assuming that a drift velocity of the normal fluid excitation gas (with respect to the superfluid background) has no effect on the molecular transport processes. (Strictly speaking, we have still allowed the possibility that the phenomenological coefficients depend on $\underline{w}$ through w^2 ; in practice, we expect this dependence to be very slight.) If this assumption is dropped, then the most general equations (characterized by 44 scalar phenomenological coefficients) may still be obtained in a straightforward manner, but they are very complicated.

This completes the derivation of the hydrodynamic equations for dissipative processes as proposed by Lin [25]. In the next section, we give a discussion of the boundary conditions that are to be satisfied at a solid-liquid helium interface.

3. Boundary conditions

We consider first the case of a solid wall at rest; the more general case of a wall in motion will be obtained by means of a Galilean transformation.

In general, there will be an energy flux between the fluid and the wall; we let $\underline{H}$ be the heat flux vector in the wall. For the quantities $\underline{n}\cdot\underline{v}_n$ and $\underline{n}\cdot\underline{v}_s$, and for the temperature T, the discussion of III-B-3 applies unchanged, and we have the boundary conditions

$$\left.\begin{aligned} &\underline{n}\cdot\underline{v}_n = 0 \\ &\underline{n}\cdot\underline{v}_s = 0 \\ \text{and}\quad &T_{wall} - T_{helium} = A_K(\underline{H}\cdot\underline{n}), \end{aligned}\right\} \text{at the wall} \tag{259}$$

where A_K is the Kapitza thermal resistance and $\underline{n}$ is the unit normal pointing into the fluid.

In addition to the boundary conditions (259), one also needs conditions on the tangential components of $\underline{v}_n$, $\underline{v}_s$. For these quantities, Lin [25,23] has proposed the following boundary conditions:

$$\begin{aligned} &\underline{n}\times\underline{v}_n = 0, \\ \text{and}\quad &\left\{\tau_{ij}^{(s)} n_i - \beta v_s^2 v_{sj}\right\}_{\text{tangential}} = 0, \end{aligned} \tag{260}$$

where β-is an additional phenomenological coefficient which

in general will depend on the thermodynamic quantities.

A final boundary condition is obtained from the physical requirement that the energy flux be continuous across the interface; thus

$$(\underline{Q}_o + \underline{Q}' - \underline{H}) \cdot \underline{n} = 0.$$

By use of (257), (259) and (260), we may write this as

$$\beta v_s^4 + \underline{n} \cdot K \nabla T + \underline{n} \cdot \underline{H} = 0.$$

In the case of a solid wall moving with a velocity $\underline{U}_{wall}$, the complete set of boundary conditions are

$$\begin{aligned} &\underline{v}_n - \underline{U}_{wall} = 0, \\ &\underline{n} \cdot (\underline{v}_s - U_{wall}) = 0, \\ &\left\{ \tau_{ij}^{(s)} n_i - \beta \left| \underline{v}_s - \underline{U}_{wall} \right|^2 (\underline{v}_s - \underline{U}_{wall})_j \right\}_{tangential} = 0, \\ &(\underline{H} + K \nabla T) \cdot \underline{n} + \beta \left| \underline{v}_s - \underline{U}_{wall} \right|^4 = 0, \end{aligned} \tag{261}$$

and

$$T_{wall} - T_{helium} = A_K (\underline{H} \cdot \underline{n}).$$

It is of interest to examine the entropy flux. In the solid, the entropy flux is simply $\underline{\mathcal{J}}_{solid} = \frac{\underline{H}}{T_{wall}}$, and the entropy flux in the fluid is $\underline{\mathcal{J}}_{helium} = -K \nabla T / T_{helium}$. Then from (261) it may be shown that

$$\underline{n} \cdot \left\{ \underline{\mathcal{J}}_{helium} - \underline{\mathcal{J}}_{solid} \right\} = -\underline{n} \cdot \left\{ \frac{K \nabla T}{T_{helium}} + \frac{\underline{H}}{T_{wall}} \right\}$$

$$\underline{n}\cdot\left\{\underline{\mathcal{J}}_{helium}-\underline{\mathcal{J}}_{solid}\right\}=\frac{(\underline{n}\cdot\underline{H})^2 A_K}{(T_{wall})(T_{helium})}+\beta\frac{|\underline{v}_s-\underline{U}_{wall}|^4}{T_{helium}}$$

$$=\frac{(\underline{n}\cdot\underline{H})^2 A_K}{(T_{wall})(T_{helium})}+\frac{1}{T_{helium}}\frac{1}{\beta^{1/3}}\left(\underline{e}_s\cdot\underline{\underline{\tau}}^{(s)}\cdot\underline{n}\right)^{4/3}, \qquad (262)$$

where $\underline{e}_s$ is a unit vector in the direction of $(\underline{v}_s-\underline{U}_{wall})$. The expression (262) shows that there is intensive dissipation associated with the Kapitza boundary effect and with the superfluid slip at the boundary. The last of the expressions (262) shows that in the limit $A_K\to 0$ (no Kapitza resistance) and $\beta\to\infty$ (no superfluid slip), the entropy flux is continuous. It is interesting to note that even in the absence of a heat flux through the boundary ($\underline{H}=0$), there is a finite temperature gradient in the helium, given by (cf. (261)) $K\nabla T\cdot\underline{n}=-\beta|\underline{v}_s-\underline{U}_{wall}|^4$. The heat current associated with this gradient is of course exactly equal to the rate at which mechanical energy is being dissipated at the interface as a consequence of the slip condition. (We may note that the heat current induced by the "slip-dissipation" is extremely small in cases of practical interest; we have $|K\nabla T\cdot\underline{n}|=\beta|\underline{v}_s-\underline{U}_{wall}|^4$, and, since it is believed that $\beta\sim 10^{-2}\frac{dynes-sec^3}{cm^5}$, we have, even for $|\underline{v}_s-\underline{U}_{wall}|\sim 10$ cm/sec, $|K\nabla T\cdot\underline{n}|\sim 10^2\ ergs/sec\text{-}cm^2$ and a heat current of this size is converted by the thermal boundary layer into an internal convection current in which the fluid velocities are only of the order of $\sim 10^{-4}$ cm/sec.)

The physical picture associated with the boundary condition on the tangential component of $\underline{v}_s$ is perhaps most easily

discussed in terms of a special case; namely, we consider the boundary condition to be satisfied at a solid wall moving parallel to itself when $\underline{v}_n$, $\underline{v}_s$ have only components in the direction of the wall motion (Fig. 2). Then the boundary condition takes the form

$$\mu_{sn} \frac{\partial v_n}{\partial y} + \mu_{ss} \frac{\partial v_s}{\partial y} = \beta (v_s - U_{wall})^3, \text{ at the wall.}$$

In any given particular flow, there will be a characteristic velocity U (such as the wall velocity), a characteristic length L (such as the viscous penetration depth in an oscillatory flow), and a characteristic viscosity μ. In dimensionless form, the above boundary condition is (where primes denote dimensionless quantities)

$$\frac{\mu}{\beta L U^2} \left\{ \mu'_{sn} \frac{\partial v_n'}{\partial y'} + \mu'_{ss} \frac{\partial v_s'}{\partial y'} \right\} = (v_s' - U'_{wall})^3. \qquad (263)$$

Thus in the "low-speed" limit ($LU^2 \ll \frac{\mu}{\beta}$), the boundary condition is approximately $\mu_{sn} \frac{\partial v_n}{\partial y} + \mu_{ss} \frac{\partial v_s}{\partial y} = 0$; in the "high-speed" limit ($LU^2 \gg \mu/\beta$), the condition is approximately $v_s = U_{wall}$. In general, the boundary condition allows a slip of the superfluid relative to the wall, the amount of slip being related to the superfluid shear stress at the wall. Some further discussion of this boundary condition will be given in the next section, III-C-4.

Finally, we wish to discuss the thermal boundary layer, and the relation between the boundary conditions (261) and the more usual boundary conditions on the quantities $\underline{n} \cdot \underline{v}_n$ and $\underline{n} \cdot \underline{v}_s$.

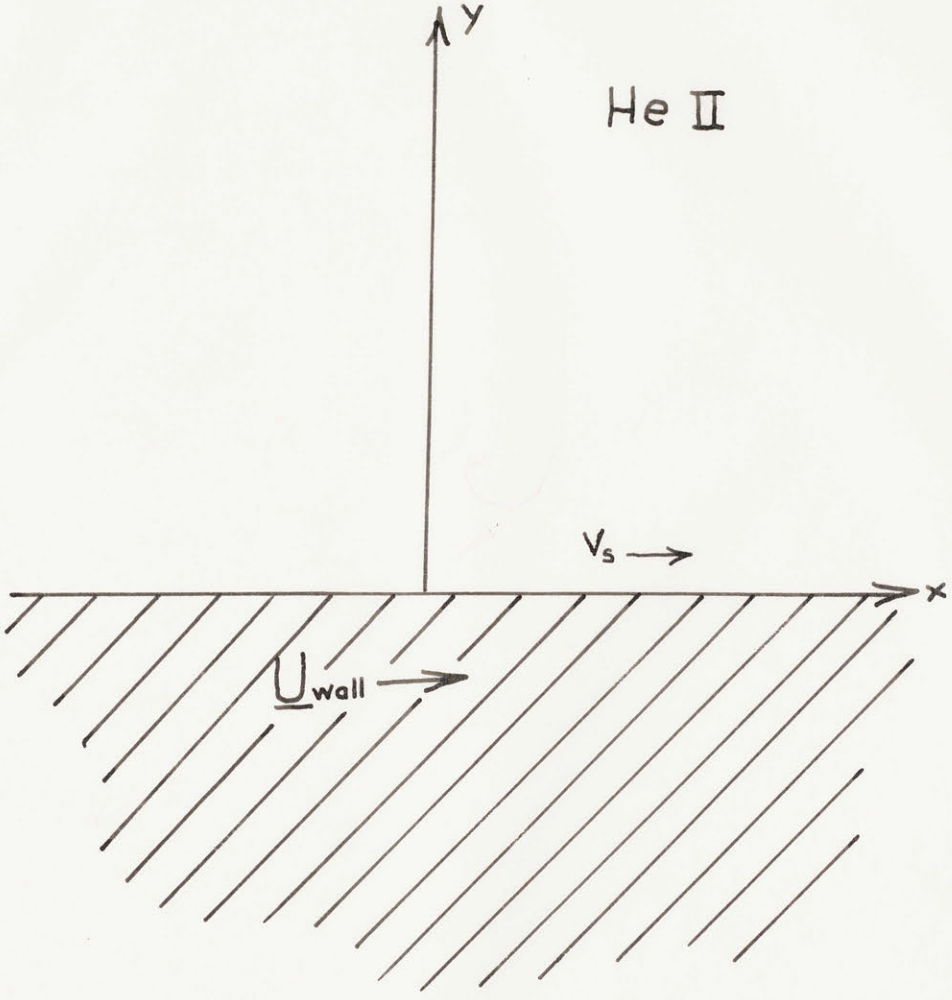

Figure 2

The discussion is given only very briefly here, since it exactly parallels the more detailed discussion of III-B-3.

We consider the flow induced by a steady (uniform) heat current through a solid wall (see Fig. 1, p. 29). Far away from the wall, there will be a uniform counterflow of the two components. In a very thin layer near the wall, the effects of viscosity, ordinary thermal conduction and superfluid-normal fluid transitions are all important. If the heat current is not too large, the total change in the temperature through the boundary layer will be small; thus we may treat the thermodynamic quantities and the phenomenological coefficients as constants wherever these quantities appear in the coefficients of the differentiated terms. Also, we may expect the nonlinear terms in the equations to be negligible provided H is sufficiently small. With these approximations, the equations are easily solved, and we obtain the results

$$T = T_\infty + \frac{H\delta}{\kappa} e^{-x/\delta} ,$$

$$v_n = \frac{H}{T\rho s}\left\{1 - e^{-x/\delta}\right\},$$

$$v_s = -\frac{\rho_n}{\rho_s} v_n , \qquad (264)$$

and

$$p = p_\infty + \frac{H e^{-x/\delta}}{T\rho s \delta}\left\{\frac{4}{3}(\mu_{nn}+\mu_{sn}) + (\lambda_{nn}+\lambda_{sn}) - \frac{4}{3}\frac{\rho_n}{\rho_s}(\mu_{ns}+\mu_{ss}) - \frac{\rho_n}{\rho_s}(\lambda_{ns}+\lambda_{ss})\right\}$$

where p_∞, T_∞ denote the (constant) values of p, T for $x \to \infty$.

The thickness of the boundary layer δ is given by

$$\delta^2 = \frac{\kappa}{T s^2 \rho_s^2}\left\{x^2\left(\tfrac{4}{3}\mu_{ss}+\lambda_{ss}\right)+(1-x)^2\left(\tfrac{4}{3}\mu_{nn}+\lambda_{nn}\right)-x(1-x)\left[\tfrac{4}{3}\left(\mu_{ns}+\mu_{sn}\right)\right.\right.$$

$$\text{or}\quad \left.\left.+\lambda_{ns}+\lambda_{sn}\right]\right\} \tag{265}$$

$$\delta^2 = \frac{\kappa}{T s^2 \rho_s^2}\left\{\left[\tfrac{4}{3}\left[x\sqrt{\mu_{ss}}-(1-x)\sqrt{\mu_{nn}}\right]^2+\tfrac{4}{3}x(1-x)\left[\sqrt{4\mu_{nn}\mu_{ss}}-\left(\mu_{ns}+\mu_{sn}\right)\right]\right.\right.$$

$$\left.\left.+\left[x\sqrt{\lambda_{ss}}-(1-x)\sqrt{\lambda_{nn}}\right]^2+x(1-x)\left[\sqrt{4\lambda_{nn}\lambda_{ss}}-\left(\lambda_{ns}+\lambda_{sn}\right)\right]\right\}$$

The conditions on the phenomenological coefficients (cf. (258)) insure that $\delta^2 \geq 0$ always. The total change in temperature through the boundary layer is given by

$$T(0) - T_\infty = \frac{H\delta}{\kappa},$$

so that the thermal resistance A of the boundary layer is given by

$$A = \frac{\delta}{\kappa}. \tag{266}$$

Although almost nothing is known about the magnitude of the "bulk viscosities" λ_{nn}, λ_{ns}, λ_{sn} and λ_{ss}, for purposes of order of magnitude estimates, we assume that they are of the same order of magnitude as μ_{nn}. As already mentioned, there is evidence that $\mu_{ns}+\mu_{ss} \simeq 0$ and $\mu_{nn} \sim \mu_{ss}$. With these estimates then, the sample numbers given in III-B-3 are relevant here. (Namely, that for $T = 1.8^\circ K$, $\delta \sim 3 \cdot 10^{-7}$ cm.; also that the total changes in temperature are negligible for H such that $v_{n\infty} \ll 4 \cdot 10^{4}$ cm/sec, and that the nonlinear terms are

negligible for $v_{n\infty} << 2.5 \cdot 10^2 \frac{cm.}{sec.}$)

This completes the discussion of the boundary conditions. In the next section, we give (following the work of Lin [25]) a brief discussion of the present equations and boundary conditions.

4. Discussion

The basic hydrodynamic equations for Lin's theory of dissipative processes are given by equations (254) - (258). (Again we note that both the perfect fluid and dissipative equations originally proposed by Lin contained additional terms proportional to $\underline{w} \times \text{curl}\, \underline{v}_s$, representing a non-dissipative momentum exchange between the two components). The foundations and some possible generalizations of these equations have been discussed in detail in III-C-2. The boundary conditions to be satisfied at the interface between a solid wall and the liquid helium II are given by (261). Before considering some particular solutions of the equations, there are two related features of Lin's theory which we wish to discuss--the superfluid viscosity and the nonlinear boundary condition on the tangential component of the superfluid velocity.

It has been argued that the phenomenon of superfluidity precludes the possibility of superfluid viscosity and superfluid rotation. Lin [25] has given a detailed discussion of this matter, so we only note the following points here: (i) there is no compelling evidence from the microscopic theory to force us to the conclusion that $\text{curl}\, \underline{v}_s = 0$ or that there is no superfluid viscosity, (ii) the possibility of superfluid viscosity is entirely consistent with a phenomenological theory of helium II, as the derivation of the preceding section shows, and (iii) the experimental evidence relevant to this question (e.g., the Andonikashvili experiment and the

frictionless flows through very narrow channels) may be explained in terms of a lack of shear interaction between the superfluid component and the solid wall--that is, in terms of the nonlinear boundary condition proposed by Lin. We may also note that on the basis of Lin's theory, one may explain the experimental results for flows in "wide" channels (where the mean velocity is a nonlinear function of the pressure gradient, Lin [23]) , and the results of various damping experiments at finite amplitude (Gribben [11]).

Lin [25,23] has given a detailed discussion of the nonlinear boundary condition (261) on the tangential component of the superfluid velocity. Briefly, the idea is as follows: Many experiments with oscillating systems in helium II exhibit the phenomenon of what we might call ampltude-dependent superfluid entrainment. Also, experimental channel flows are characterized by a nonlinear relation between the mean rates of flow and the driving forces (pressure and temperature gradients). These both suggest that the shear interaction of the superfluid with a solid wall is "amplitude dependent". A general boundary condition with this property is

$$\tau_{ij}^{(s)} n_j t_i^{(\alpha)} = f(q^2)\, q_i t_i^{(\alpha)} ,$$

where n_i is the unit normal to the wall, $t_i^{(\alpha)}$ $(\alpha=1,2)$ is a tangent vector to the wall and $\underline{q} = \underline{v}_s - \underline{U}_{wall}$ is the relative velocity of the superfluid component and the wall. The boundary condition for an ordinary viscous fluid is recovered in the limit $f \to \infty$, corresponding to a strong interaction of the fluid with the

wall; thus if we take the first two terms in an expansion of f,

$$f(q^2) = \alpha + \beta q^2 + \cdots ,$$

the case of the ordinary fluid corresponds to the limit $\alpha \to \infty$. For the superfluid component of helium II, however, there is ample experimental evidence that the interaction with the wall is weak (as well as being amplitude-dependent). Thus we may expect that in the present case, $\alpha \to 0$; then the first approximation to the boundary condition is

$$\tau_{ij}^{(s)} n_j t_i^{(\alpha)} = \beta \left| \underline{v}_s - \underline{U}_{wall} \right|^2 (\underline{v}_s - \underline{U}_{wall}) \cdot \underline{t}^{(\alpha)} ,$$

which is the boundary condition (261). The results of preliminary analysis indicate that $\beta \sim 10^{-2} \frac{\text{dynes-sec}^3}{\text{cm}^5}$. Further analysis of existing experimental data is needed to obtain a more accurate estimate and to determine the dependence of β on the temperature.

In many flow situations, the velocities are small (compared with the first and second sound velocities), and the temperature variations throughout the flow are small. In such situations, the quantities ρ_n, ρ_s and S, as well as the phenomenological coefficients, may be taken as constant throughout the flow (we are also at present excluding flows with high frequency time dependence, such as first and second sound waves). The temperature and pressure then become hydrodynamic variables (in the same sense that the pressure is a

hydrodynamic variable in the incompressible flow of an ordinary fluid). Also, the rate of entropy production and the energy transport by thermal conduction are usually negligible in such flows. With these approxomations, the hydrodynamic equations may be written in the form

$$\frac{\partial \underline{v}_n}{\partial t} + \underline{v}_n \cdot \nabla \underline{v}_n = -\frac{1}{\rho} \nabla p - \frac{(1-x)}{x} s \nabla T - \frac{(1-x)}{2} \nabla w^2 + \frac{\mu_{nn}}{\rho_n} \nabla^2 \underline{v}_n + \frac{\mu_{ns}}{\rho_n} \nabla^2 \underline{v}_s , \tag{267}$$

$$\frac{\partial \underline{v}_s}{\partial t} + \underline{v}_s \cdot \nabla \underline{v}_s = -\frac{1}{\rho} \nabla p + s \nabla T + x \nabla \frac{1}{2} w^2 + \frac{\mu_{sn}}{\rho_s} \nabla^2 \underline{v}_n + \frac{\mu_{ss}}{\rho_s} \nabla^2 \underline{v}_s \tag{268}$$

and

$$\operatorname{div} \underline{v}_n = 0, \quad \operatorname{div} \underline{v}_s = 0. \tag{269}$$

The equation for the total momentum (obtained from (267) and (268)) may be written as

$$\frac{\partial}{\partial t}\left\{\rho_n \underline{v}_n + \rho_s \underline{v}_s\right\} + \operatorname{div}\left\{\rho_n \underline{v}_n \underline{v}_n + \rho_s \underline{v}_s \underline{v}_s\right\} = -\nabla p + \left\{\mu_{nn} + \mu_{sn}\right\} \nabla^2 \underline{v}_n + \left\{\mu_{ns} + \mu_{ss}\right\} \nabla^2 \underline{v}_s . \tag{270}$$

The boundary conditions (261) may also be simplified within the present approximation, provided the heat input through the boundary walls is not too large. In this case, the effective boundary conditions may be obtained from the thermal boundary layer analysis (as is discussed in detail in III-B-3). (The conditions on the tangential components of $\underline{v}_n$, $\underline{v}_s$ are unchanged.) These effective boundary conditions are

$$\underline{n} \times (\underline{v}_n - \underline{U}_{wall}) = 0$$

$$\rho s T (\underline{v}_n - \underline{U}_{wall}) \cdot \underline{n} = \underline{H} \cdot \underline{n}$$

$$\left\{ \tau_{ij}^{(s)\prime} n_i - \beta \left| \underline{v}_s - \underline{U}_{wall} \right|^2_{tangential} (\underline{v}_s - \underline{U}_{wall})_j \right\}_{tangential} = 0,$$

and (271)

$$x(\underline{v}_n - \underline{U}_{wall}) \cdot \underline{n} + (1-x)(\underline{v}_s - \underline{U}_{wall}) \cdot \underline{n} = 0$$

with $$T_{wall} - T_{helium} = (A_K + A)(\underline{H} \cdot \underline{n}),$$

where $$\tau_{ij}^{(s)\prime} = 2\mu_{sn} e_{ij}^{(n)} + 2\mu_{ss} e_{ij}^{(s)},$$

where $\underline{n}$ is the unit normal pointing into the fluid, A_K is the Kapitza resistance and A, representing the effect of the thermal boundary layer, is given by (266).

As Lin has shown [25], the comparison of the present theory with experimental results for simple channel flows (i.e., uni-directional flows induced in an infinitely long channel by constant temperature and pressure gradients) yields some information about the 4 viscosity coefficients μ_{nn}, μ_{ns}, μ_{sn} and μ_{ss}. It is a well-verified experimental fact that the heat current in a channel flow is accurately proportional to the pressure gradient (even when the relation between the heat current and the temperature gradient is a markedly nonlinear one); one can show that the present theory predicts this result, provided

$$\mu_{ns} + \mu_{ss} = 0. \qquad (272)$$

One may see this from equation (270) and the fact that the heat current is proportional to the mean normal fluid velocity. (The analysis is given in detail in the Appendix.) We may also note that if (272) holds, the present theory predicts a linear relationship between the torque and angular velocity in a rotating cylinder viscometer, in agreement with the experiment results. (Strictly speaking, the experimental results can only lead to some condition such as $\mu_{ns} + \mu_{ss} \ll \mu_{nn}$; in what follows, we will assume (272) to hold.) The physical content of (272) is that there is no contribution from the superfluid component to the total momentum transfer by shear forces.

From the view that the motion of th superfluid "background" should be relatively unaffected by the motion of the normal fluid "excitations", one might argue that the condition

$$\mu_{sn} = 0 \tag{273}$$

should be satisfied. At present, however, the experimental evidence for (273) in inconclusive. An alternative conjecture is that the Onsager reciprocity relations are relevant here, so that

$$\mu_{ns} = \mu_{sn}. \tag{274}$$

Strictly speaking, however, Onsager's theorem is applicable only when the fluxes are time derivatives of thermodynamic state variables. The extension of Onsager's theorem to continuous systems in which the fluxes are not time derivatives of the thermodynamic state variables requires special

consideration for each particular system (see, for example, [4], Chapter IV), and it is not at all clear that one can expect the Onsager relations to be valid in the present case. It is of course possible that μ_{sn} satisfies neither of the conditions (273) or (274). At present, this is still an open question, although preliminary evidence from experiments seems to favor the choice $\mu_{ns} = \mu_{sn}$.

It is possible to give a satisfactory explanation of a variety of experimental results on the basis of Lin's theory. Thus for channel flows the theory (i) predicts that the heat current is proportional to the pressure gradient (provided $\mu_{ns} + \mu_{ss} = 0$) (ii) predicts the observed nonlinear relation between the pressure gradient and mean flow velocity for isothermal flows in "wide" channels [23], (iii) predicts the London equations ($\nabla p = \rho s \nabla T$) for the fountain pressure in the limit of very narrow channels. None of these predictions depends on which of the conditions (273), (274) (if either) μ_{sn} satisfies. A detailed analysis of channel and pipe flows is given in the Appendix. Other experiments which receive a satisfactory explanation on the basis of Lin's theory are (i) the Andronikashvili disc-pile experiments (for which the present theory predicts that only the normal fluid will take part in the motion, provided the frequency of oscillation is sufficiently low) and (ii) experiments on the damping of an oscillating disc (for which the theory predicts the observed amplitude-dependence of the damping, and the two critical velocities - Gribben [11]). A brief analysis of the

Andronikashvili experiment is given in the Appendix.

At present, it is difficult to give any accurate numerical estimate of the viscosity coefficients and the boundary constant in Lin's theory. There are several reasons for this. First of all, because of the extremely small viscosity of helium II ($\sim 10^{-5}$ poise), there is a definite possibility of turbulence in many flow experiments. (Indeed, Staas et al. [34] have shown that the experimentally observed deviations from the linear relation between heat current and pressure gradient in pipe flows at higher velocities may be accounted for quantitatively by means of an empirical formula analogous to the Blasius resistance law for the turbulent flow of an ordinary fluid.) Also, one cannot discount the possibility of a critical velocity marking the threshold of a volume mutual friction force. A further difficulty is that in many cases, the agreement of theory with experiment does not seem to be critically dependent on the numerical values of the dissipative parameters--thus, for example, Gribben [11] found that the agreement between theory and experiment for the damping of an oscillating disc was tolerably good for a wide range of values of the ratio μ_{ss}/μ_{nn}. Also, preliminary analyses of channel flow data have shown this same feature. A more detailed discussion of this problem--along with some preliminary estimates for the various coefficients--is given in the Appendix.

APPENDIX TO CHAPTER III

We consider here the problem of obtaining unambiguously expressions for the dissipative energy flux and the dissipation function separately from the single equation relating them. From the complete set of macroscopic variables, we single out the total energy per unit volume E, and the entropy per unit volume S. Then the differential of the energy will have the form

$$dE = T\,dS + \sum_{\alpha=1}^{n} X_\alpha\, dY_\alpha , \tag{A-1}$$

where $\{Y_\alpha\}$ are the remaining independent extensive quantities (per unit volume). (The Y_α may include, for example, the Cartesian components of a vector quantity.) We assume that the equations for reversible processes are known, and that they may be written as

$$\frac{\partial E}{\partial t} + \operatorname{div} \underline{Q}_0 = 0,$$

$$\frac{\partial S}{\partial t} + \operatorname{div} (\underline{J}_S^0) = 0, \tag{A-2}$$

and

$$\frac{\partial Y_\alpha}{\partial t} + f_\alpha = 0.$$

(In many cases, the equations for the Y_α will be in the form

of conservation laws; however, we do not need to make this restrictive assumption in order to carry out the derivation here.) Since the equations A-2 are to form a consistent set of equations for reversible processes, it follows from A-1 and A-2 that

$$\operatorname{div} \underline{Q}_0 - T \operatorname{div} \underline{\mathcal{J}}_s^0 - \sum_\alpha \mathcal{X}_\alpha f_\alpha \equiv 0. \tag{A-3}$$

We now assume that the effects of dissipative processes may be taken into account by including dissipative fluxes in the equations; thus we take the hydrodynamic equations in the following form:

$$\frac{\partial E}{\partial t} + \operatorname{div}(\underline{Q}_0 + \underline{Q}') = 0,$$

$$\frac{\partial S}{\partial t} + \operatorname{div}(\underline{\mathcal{J}}_s^0 + \underline{\mathcal{J}}_s') = \frac{R}{T} \tag{A-4}$$

and

$$\frac{\partial Y_\alpha}{\partial t} + f_\alpha + \operatorname{div} \underline{\mathcal{J}}_\alpha' = 0.$$

(In particular, then, we are excluding the case of "volume" dissipative effects such a mutual friction; the above formalism, however, includes the theory of section III-B (the case $\operatorname{curl} \underline{v}_s = 0$) and also Lin's theory of dissipative processes without mutual friction.) It is assumed that the fluxes $\underline{Q}'$, $\underline{\mathcal{J}}_s'$ and $\underline{\mathcal{J}}_\alpha'$ are functions of an independent set of gradients (they need not be linear functions; however, they must not depend on spatial derivatives of higher order, and they must vanish whenever

all of the independent gradients vanish). The dissipation function R is to be positive definite. From (A-1) and (A-4), we may obtain the following equation relating R and the dissipative fluxes:

$$R = -\operatorname{div}\left\{\underline{Q}' - T\underline{J}_s' - \sum_\alpha X_\alpha \underline{J}_\alpha'\right\} - \underline{J}_s' \cdot \nabla T - \sum_\alpha \underline{J}_\alpha' \cdot \nabla X_\alpha \qquad \text{(A-5)}$$

In the calculations of Chapter III, we have satisfied equations of the form (A-5) by taking

$$\underline{Q}' = T\underline{J}_s' + \sum_\alpha X_\alpha \underline{J}_\alpha' \qquad \text{(A-6)}$$

and

$$R = -\underline{J}_s' \cdot \nabla T - \sum_\alpha \underline{J}_\alpha' \cdot \nabla X_\alpha \; ; \qquad \text{(A-7)}$$

then the fluxes $\underline{J}_s'$, $\underline{J}_\alpha'$ were determined by finding the most general expressions which make R positive definite. Let us denote the general expressions obtained for $\underline{Q}'$, $\underline{J}_s'$ and $\underline{J}_\alpha'$ in this way by $\hat{\underline{Q}}$, $\hat{\underline{J}}_s'$, $\hat{\underline{J}}_\alpha'$. We now wish to examine the equation (A-5) in order to determine if there are more general solutions for $\underline{Q}'$, $\underline{J}_s'$ and $\underline{J}_\alpha'$ than those of the class $\hat{\underline{Q}}$, $\hat{\underline{J}}_s$, $\hat{\underline{J}}_\alpha$. We introduce $\tilde{\underline{Q}}$ by

$$\tilde{\underline{Q}} = \underline{Q}' - T\underline{J}_s' - \sum_\alpha X_\alpha \underline{J}_\alpha' \; ; \qquad \text{(A-8)}$$

then

$$R = -\operatorname{div} \tilde{\underline{Q}} - \underline{J}_s' \cdot \nabla T - \sum_\alpha \underline{J}_\alpha' \cdot \nabla X_\alpha \qquad \text{(A-9)}$$

By hypothesis, $\tilde{\underline{Q}}$ depends on $\{X_\alpha\}$ and $\{\nabla X_\alpha\}$ (where, for convenience we now include T and ∇T in the sets $\{X_\alpha\}, \{\nabla X_\alpha\}$). Then $\operatorname{div} \tilde{\underline{Q}}$ is given by

$$\operatorname{div} \underline{\tilde{Q}} = \sum_i \frac{\partial \tilde{Q}_i}{\partial x_i} = \sum_{\alpha,i} \frac{\partial Q_i}{\partial \underline{X}_\alpha} \frac{\partial \underline{X}_\alpha}{\partial x_i} + \sum_{\alpha,i,j} \frac{\partial \tilde{Q}_i}{\partial \left(\frac{\partial \underline{X}_\alpha}{\partial x_j} \right)} \frac{\partial^2 \underline{X}_\alpha}{\partial x_i \partial x_j} .$$

The terms in $\frac{\partial^2 \underline{X}_\alpha}{\partial x_i \partial x_j}$ would give rise to terms in R which are linear in the second derivatives of $\underline{X}_\alpha$; since R is to be positive definite, these terms must vanish, so that we have the conditions

$$\frac{\partial \tilde{Q}_i}{\partial \left(\frac{\partial \underline{X}_\alpha}{\partial x_j} \right)} + \frac{\partial \tilde{Q}_j}{\partial \left(\frac{\partial \underline{X}_\alpha}{\partial x_i} \right)} = 0, \quad \text{all } i, j, \alpha, \tag{A-10}$$

on the vector $\underline{\tilde{Q}}$. It is possible to show (see below for a proof) that the most general solution of the system of partial differential equations (A-10) is given by

$$\underline{\tilde{Q}} = \sum_\alpha \underline{\omega}^{(\alpha)} \times \nabla \underline{X}_\alpha + \sum_{\alpha,\beta} \omega^{(\alpha\beta)} \left\{ \nabla \underline{X}_\alpha \times \nabla \underline{X}_\beta \right\}, \tag{A-11}$$

where $\underline{\omega}^{(\alpha)}$, $\omega^{(\alpha\beta)}$ are independent of the gradients $\frac{\partial \underline{X}_\alpha}{\partial x_i}$ (but in general will depend on the $\underline{X}_\alpha$). We may write (A-11) as

$$\underline{\tilde{Q}} = \sum_\beta \underline{\Omega}^{(\beta)} \times \nabla \underline{X}_\beta ,$$

where $$\underline{\Omega}^{(\beta)} = \underline{\omega}^{(\beta)} + \sum_\alpha \omega^{(\alpha\beta)} \nabla \underline{X}_\alpha ;$$

then we have

$$\operatorname{div} \underline{\tilde{Q}} = \sum_\beta \nabla \underline{X}_\beta \cdot \operatorname{curl} \underline{\Omega}^{(\beta)},$$

so that

$$R = - \left\{ \underline{J}_s' + \operatorname{curl} \underline{\Omega}^{(s)} \right\} \cdot \nabla T - \sum_\alpha \left\{ \underline{J}_\alpha' + \operatorname{curl} \underline{\Omega}^{(\alpha)} \right\} \cdot \nabla \underline{X}_\alpha . \tag{A-12}$$

Since $\mathcal{R}$ is to be positive definite, it must be that

$$\underline{\mathcal{J}}_s' = -\operatorname{curl} \underline{\Omega}^{(s)} + \hat{\underline{\mathcal{J}}}_s ,$$

$$\underline{\mathcal{J}}_\alpha' = -\operatorname{curl} \underline{\Omega}^{(\alpha)} + \hat{\underline{\mathcal{J}}}_\alpha , \tag{A-13}$$

and $$\underline{\mathcal{Q}}' = T \underline{\mathcal{J}}_s' + \sum_\alpha \underline{\mathcal{X}}_\alpha \underline{\mathcal{J}}_\alpha' + \tilde{\underline{\mathcal{Q}}} ,$$

or

$$\underline{\mathcal{Q}}' = T \hat{\underline{\mathcal{J}}}_s + \sum_\alpha \underline{\mathcal{X}}_\alpha \hat{\underline{\mathcal{J}}}_\alpha - T \operatorname{curl} \underline{\Omega}^{(s)} - \sum_\alpha \underline{\mathcal{X}}_\alpha \operatorname{curl} \underline{\Omega}^{(\alpha)}$$

$$+ \sum_\alpha \underline{\Omega}^{(\alpha)} \times \nabla \underline{\mathcal{X}}_\alpha + \underline{\Omega}^{(s)} \times \nabla T$$

or

$$\underline{\mathcal{Q}}' = \hat{\underline{\mathcal{Q}}} - \operatorname{curl} \{ T \underline{\Omega}^{(s)} + \sum_\alpha \underline{\mathcal{X}}_\alpha \underline{\Omega}^{(\alpha)} \} . \tag{A-14}$$

Thus the class of most general solutions $\underline{\mathcal{J}}_s'$, $\underline{\mathcal{J}}_\alpha'$, $\underline{\mathcal{Q}}'$ differ from the class $\hat{\underline{\mathcal{J}}}_s$, $\hat{\underline{\mathcal{J}}}_\alpha$ and $\hat{\underline{\mathcal{Q}}}$ only by the presence of a curl in each flux. However, only the divergence of a given flux has physical significance, so that the curl terms have no significance, and we may thus assume that equations (A-6) and (A-7) hold. We note again that the proof of this result does not require that the fluxes be linear functions of the gradients--it requires only that the fluxes vanish whenever all of the gradients vanish and that the fluxes do not depend on the higher spatial derivatives of the gradients.

We now give a proof that (A-11) is the general solution of (A-10). For convenience, we change the notation somewhat. We are given that $\underline{Q}$ is a vector valued function of n vectors $\underline{r}^{(\alpha)}$, $\alpha = 1, \cdots, n$ (where we are using $\underline{r}^{(\alpha)}$ instead of $\nabla \underline{\mathcal{X}}_\alpha$), and

that

$$\frac{\partial Q_i}{\partial r_j^{(\alpha)}} + \frac{\partial Q_j}{\partial r_i^{(\alpha)}} = 0 \; ; \quad (i,j = 1,2,3 \; , \; \alpha = 1, \ldots, n)$$

we are also given that $\underline{Q}$ vanishes when all the $\underline{r}'s$ are zero. We then wish to show that the most general solution is of the form

$$\underline{Q} = \sum_{\alpha} \underline{\omega}^{(\alpha)} \times \underline{r}^{(\alpha)} + \sum_{\alpha,\beta} \omega^{(\alpha\beta)} (\underline{r}^{(\alpha)} \times \underline{r}^{(\beta)}) \qquad \text{(A-15)}$$

where $\omega^{(\alpha)}, \omega^{(\alpha\beta)}$ are independent of the $\underline{r}'s$. The proof is by induction on n , the number of vectors on which $\underline{Q}$ depends. For $n=1$, the mathematical problem is the same as that of finding the most general displacement which is such that the strain tensor vanishes identically; the well-known solution is

$$\underline{Q} = \underline{\omega}^{(1)} \times \underline{r}^{(1)} + \underline{Q}_0 \, ,$$

where $\underline{\omega}^{(1)}$, $\underline{Q}_0$ are independent of $\underline{r}^{(1)}$. Since $\underline{Q}$ must vanish for $\underline{r}^{(1)} = 0$, we have

$$n=1 \quad : \quad \underline{Q} = \underline{\omega}^{(1)} \times \underline{r}^{(1)} .$$

For $n=2$, we have from

$$\frac{\partial Q_i}{\partial r_j^{(1)}} + \frac{\partial Q_j}{\partial x_i^{(1)}} = 0 ,$$

$$\underline{Q}^{(1)} = \underline{\omega}^{(1)} \times \underline{r}^{(1)} + \underline{Q}_0$$

where $\underline{Q}_0$ and $\underline{\omega}^{(1)}$ may now depend on $\underline{r}^{(2)}$. From $\frac{\partial Q_i}{\partial r_j^{(2)}} + \frac{\partial Q_j}{\partial r_i^{(2)}} = 0$, one may then show that

$$n=2: \quad \underline{Q}^{(2)} = \lambda\{\underline{r}^{(1)} \times \underline{r}^{(2)}\} + \underline{\omega}^{(1)} \times \underline{r}^{(1)} + \underline{\omega}^{(2)} \times \underline{r}^{(2)},$$

where λ, $\underline{\omega}^{(1)}$, $\underline{\omega}^{(2)}$ are all independent of $\underline{r}^{(1)}$, $\underline{r}^{(2)}$. Now suppose the theorem true for $n=k$, and consider $\underline{Q}^{(k+1)}$, a function of $\underline{r}^{(1)} \cdots \underline{r}^{(k+1)}$. Regarding $\underline{Q}^{(k+1)}$ for the moment as a function of $\underline{r}^{(k)}$ and $\underline{r}^{(k+1)}$, we get

$$\underline{Q}^{(k+1)} = \lambda(\underline{r}^{(1)}, \cdots, \underline{r}^{(k-1)})\{\underline{r}^{(k)} \times \underline{r}^{(k+1)}\} + \omega^{(k+1)}(\underline{r}^{(1)}, \cdots, \underline{r}^{(k-1)}) \times \underline{r}^{(k+1)} + \underline{Q}(\underline{r}^{(1)}, \cdots, \underline{r}^{(k)})$$

or

$$\underline{Q}^{(k+1)} = \{\lambda \underline{r}^{(k)} + \underline{\omega}^{(k+1)}\} \times \underline{r}^{(k+1)} + Q^{(k)}(\underline{r}^{(1)}, \cdots, \underline{r}^{(k)})$$

From the equation

$$\frac{\partial Q_i^{(k+1)}}{\partial r_j^{(s)}} + \frac{\partial Q_j^{(k+1)}}{\partial x_i^{(s)}} = 0 \qquad \text{(A-16)}$$

evaluated for $s \neq k+1$ and for $\underline{r}^{(k+1)} = 0$, we get

$$\frac{\partial Q_i^{(k)}}{\partial r_j^{(s)}} + \frac{\partial Q_j^{(k)}}{\partial r_i^{(s)}} = 0, \qquad s = 1, \cdots, k.$$

Thus the induction hypothesis applies to $\underline{Q}^{(k)}$ so that $\underline{Q}^{(k)}$ has the proper form. From (A-16) evaluated for $s \neq k$ and for $\underline{r}^{(k)} = 0$, we may conclude that the induction hypothesis also applies to the term

$$\underline{\omega}^{(k+1)}(\underline{r}^{(1)}, \cdots, \underline{r}^{(k-1)}) \times \underline{r}^{(k+1)}$$

(which is a function of only K vectors), so that this term is of the proper form. Finally, from (A-16) evaluated for $s \neq K$, $s \neq K+1$, we may show that λ is independent of all the vectors $\underline{r}^{(\alpha)}$. Hence $\underline{Q}^{(K+1)}$ also has the form (A-15) and this completes the proof.

CHAPTER IV - DISSIPATIVE PROCESSES - II

A. Introduction

In this chapter, we will consider some theories of liquid helium II in which the quantity $\text{curl}\, \underline{v}_s$ plays a special role. We will be concerned primarily with the continuum theory proposed by Bekarevich and Khalatnikov [3] in which the the quantity $|\text{curl}\, \underline{v}_s|$ is a thermodynamic variable. The derivation of the hydrodynamic equations proposed by Hall and Vinen [12,13, 14,38] for the motion of helium II with quantized vortex lines will not be discussed in detail here. However in section IV-C we will discuss briefly an alternative continuum approach to the hydrodynamics of helium II with quantized vortex lines. It is perhaps helpful to recall here the essential features of these theories before beginning a detailed discussion.

In his 1941 paper [20], Landau advanced the idea that the flow of the superfluid component must be potential flow--that is, that $\text{curl}\, \underline{v}_s = 0$. The experiments of Andronikashvili [1] with an oscillating disc pile seemed to provide a striking experimental verification of this hypothesis. Later experiments, however, indicated that the superfluid component could rotate in some manner (e.g., the free surface experiments of Osbourne [31]). Onsager [30] and later, independently, Feynman [7] suggested that, although the superfluid component cannot rotate in bulk, it can support line singularities

analogous to vortex lines in ordinary hydrodynamics. Feynman gave some qualitative quantum-mechanical arguments to support the view that the strength of the vortices is quantized according to the formula

$$\oint \underline{v} \cdot d\underline{s} = 2\pi n \frac{\hbar}{m} .$$

Hall and Vinen [12,13,14,38] then developed a hydrodynamic theory for helium II on the basis of the Onsager-Feynman theory of quantized vortex lines. They were able to obtain a definite set of hydrodynamic equations (a comprehensive review of their work--theoretical and experimental--is given in [14,38]). However, the derivation of Hall and Vinen is dependent in an essential manner on some special additional assumptions about the nature of the vortex motion in helium II (for example, they assume that the force on a vortex line and the velocity of translation of the line relative to the mean superfluid velocity are related via the classical Magnus effect formula). In an attempt to obtain a hydrodynamic theory of vortex motion in helium II which is independent of specific assumptions about the nature of the vortex motion, Bekarevich and Khalatnikov [3] have developed a hydrodynamic theory based only on general continuum principles (conservation laws, increase of entropy, etc.) and the single additional assumption that the thermodynamic internal energy of the helium II depends on $|\mathrm{curl}\, \underline{v}_s|$, as well as the usual thermodynamic variables. It is noteworthy that their hydrodynamic equations agree with those presented earlier by Hall [14].

Since the form of the hydrodynamic equations for helium II is still a controversial matter, it is desirable that any new phenomenological theory, such as the one proposed by Bekarevich and Khalatnikov, include (i) a careful mathematical development of the hydrodynamic equations based on the general assumptions of the theory and (ii) a discussion of the physical basis of the phenomenological theory. In the presentation of Bekarevich and Khalatnikov [3], however, (i) the mathematical development of the theory seems obscure in places, and (ii) almost no discussion is given of the physical basis of their theory. (More specifically, there are a number of questions connected with their phenomenolgical theory which deserve discussion; for example: (i) on the basis of the Onsager-Feynman theory of quantized vortex lines, should we expect to be able to develop a continuum theory for describing the motion of helium II, and, if so, is the inclusion of the superfluid vorticity in the thermodynamic variables a plausible starting point for the development of such a theory? (ii) are there other plausible pictures of the microscopic structure of helium II which would also lead to a continuum theory of the form proposed by Bekarevich and Khalatnikov [3]?)

In section IV-B-1, we give a detailed presentation, together with some criticisms, of the derivation of the hydrodynamic equations as presented by Bekarevich and Khalatnikov [3]. In section IV-B-2, we give an alternative mathematical development of their theory which seems in some respects to be more satisfactory. In IV-B-3, we give some discussion of

the physical basis of their theory. Finally in IV-C, we give a brief outline of an alternative approach to the problem of building a hydrodynamic theory of helium II on the basis of the Onsager-Feynman vortex line theory.

B. Theory of Bekarevich and Khalatnikov

1. Bekarevich's and Khalatnikov's derivation of the hydrodynamic equations

According to Bekarevich and Khalatnikov, the fundamental distinction between motions for which $\text{curl}\, \underline{v}_s = 0$ and those for which $\text{curl}\, \underline{v}_s \neq 0$ is that, in the latter case, the thermodynamic internal energy of the helium II depends on $|\text{curl}\, \underline{v}_s|$. They have used this idea, together with the usual conservation laws and invariance principles, to obtain a set of hydrodynamic equations for helium II. In this section we present a review of their derivation along with detailed discussions of some points in their work which seem obscure.

The method of derivation used by Bekarevich and Khalatnikov is essentially the same as the general method discussed in Chapter III. Briefly, it is as follows: one writes down the equations expressing the conservation of mass, energy and momentum (where the energy and momentum fluxes are to be determined), and also an equation for the superfluid velocity $\underline{v}_s$; these equations must yield an equation for the entropy which satisfies the law of increase of entropy, and this requirement leads to a single equation relating the energy flux, momentum flux and the dissipation function to known quantities; from this single equation, one determines the energy flux, momentum flux and dissipation function separately. It is clear that, in order to carry out this procedure (especially the last step), one must have some knowledge of what quantities the unknown fluxes may depend on. In Chapter III, this procedure

was carried out systematically by starting out with the perfect fluid equations, and then assuming that the increments to the various fluxes were linear functions of the independent gradients. In the present case, although the procedure cannot be carried out in quite such a straightforward manner, we would still expect to first develop a perfect fluid theory and then a theory of dissipative processes (this will be carried out in IV-B-2). Bekarevich and Khalatnikov, however, have (apparently) identified rotational flow ($\mathrm{curl}\, \underline{v}_s \neq 0$) with dissipative flow. In their theory, they start from the Landau equations for reversible processes, rather than developing a theory of reversible flows for the case in which the internal energy depends on $|\mathrm{curl}\, \underline{v}_s|$; the Landau equations, however, refer to a different physical system (one in which the internal energy does not depend on the vorticity) so that their procedure does not seem entirely satisfactory.

We now proceed to the details of their derivation. (We have changed the notation somewhat to conform with that used in the present work.) According to the Galilean transformation formula, the total energy per unit volume E may be expressed in terms of the total energy per unit volume as measured in the superfluid rest frame, E_0, as

$$E = \frac{1}{2}\rho v_s^2 + \underline{v}_s \cdot \underline{j}_0 + E_0, \tag{273}$$

where ρ is the total density, $\underline{v}_s$ the superfluid velocity and $\underline{j}_0$ is the momentum per unit volume in the superfluid rest frame. The total momentum per unit volume $\underline{j}$ is related to $\underline{j}_0$

by

$$\underline{j} = \underline{j}_0 + \rho \underline{v}_s . \tag{274}$$

In the usual version of the two-fluid model, the differential of the energy E_0 is given by $dE_0 = \Phi d\rho + T d(\rho s) + \underline{w} \cdot d\underline{j}_0$, (cf. equation (9)) where $\rho \Phi = E_0 - \underline{w} \cdot \underline{j}_0 - T\rho s + p$ (cf. equation (10)). In the present case, there will be an additional term expressing the dependence of the energy on the superfluid vorticity $\underline{\omega} \equiv \operatorname{curl} \underline{v}_s$. Thus

$$dE_0 = \Phi d\rho + T d(\rho s) + \underline{w} \cdot d\underline{j}_0 + \underline{\lambda} \cdot d\underline{\omega} . \tag{275}$$

Bekarevich and Khalatnikow make the plausible assumption that the differential coefficient $\underline{\lambda}$ depends only on the direction of the vector $\underline{\omega}$ (conceivably, it could depend on the direction of the the vector $\underline{w}$ as well). Then, for

$$\underline{\lambda} = \lambda \underline{\nu} , \quad \underline{\nu} = \frac{\underline{\omega}}{|\underline{\omega}|} , \tag{276}$$

we have

$$dE_0 = \Phi d\rho + T d(\rho s) + \underline{w} \cdot d\underline{j}_0 + \lambda d\omega . \tag{277}$$

(According to the Onsager-Feynman theory of quantized vortex lines, the coefficient λ is given approximately by $\lambda = \rho_s \frac{h}{m} \ln \frac{R}{a}$, where R/a is the ratio of the distance between vortices to the effective core radius of the vortex. Thus λ is expected to be virtually independent of the vorticity ω .) The relation between the pressure p and the potential Φ is given by Bekarevich and Khalatnikov (without discussion) as

$$p = -E_o + T\rho s + \Phi\rho + \underline{w}\cdot\underline{j}_o . \qquad (278)$$

This is the same as equation (10) relating p and Φ for the ordinary two-fluid model. If the vorticity $\underline{\omega}$ is taken as an extensive quantity per unit volume* (as the formula (275) would imply), one would expect that p and Φ are related by $p = -E_o + T\rho s + \Phi\rho + \underline{w}\cdot\underline{j}_o + \lambda\omega$, rather than by (278). In fact, Bekarevich and Khalatnikov find it convenient at a later point in their derivation to introduce a "renormalized" pressure, $p_o = p + \lambda\omega$ (with p given by (278)). In section IV-B-2, we will give a more detailed discussion of the thermodynamics of the present system; for now, we accept (277) and (278), and continue with the derivation of Bekarevich and Khalatnikov.

The equations expressing the conservation of mass, energy and momentum are

$$\frac{\partial\rho}{\partial t} + \operatorname{div}\underline{j} = 0, \qquad (279)$$

$$\frac{\partial E}{\partial t} + \operatorname{div}(\underline{Q}_o + \underline{q}) = 0, \qquad (280)$$

and

$$\frac{\partial \underline{j}}{\partial t} + \operatorname{div}(\underline{\underline{\pi}}^o + \underline{\underline{\Pi}}) = 0, \qquad (281)$$

where $\underline{Q}_o$ and $\underline{\underline{\pi}}^o$ are the energy flux vector and momentum flux tensor of the Landau theory for reversible processes, namely

$$\underline{Q}_o = (\Phi + \tfrac{1}{2}v_s^2)\,\underline{j} + \rho s T \underline{v}_n + \underline{v}_n(\underline{v}_n\cdot\underline{j}_o) \qquad (282)$$

---

* According to the Onsager-Feynman theory, the total length of vortex line in a volume V is given by $\int_V d\tau\ \omega/\kappa$, where $\kappa = h/m$, is the circulation around each line; thus ω is proportional to the total length of vortex line per unit volume.

and

$$\pi^{\circ}_{ij} = \rho v_{si} v_{sj} + v_{si} j_{oj} + v_{sj} j_{oi} + p\delta_{ij} , \tag{283}$$

or

$$\pi^{\circ}_{ij} = \rho_n v_{ni} v_{nj} + \rho_s v_{si} v_{sj} + p\delta_{ij} ; \tag{284}$$

according to Bekarevich and Khalatnikov, the increments $\underline{q}$ and $\underline{\underline{\pi}}$ represent the effects of the dissipative processes (whereby they apparently mean the effects of the dissipative processes and the effect on the structure of the equations of including the vorticity in the thermodynamic variables). There will also be an equation for the superfluid velocity $\underline{v_s}$; this equation may be written as

$$\frac{\partial \underline{v_s}}{\partial t} + \underline{v_s} \cdot \nabla \underline{v_s} + \nabla \Phi = \underline{f} , \tag{285}$$

where the quantity $\underline{f}$ is to be determined. Finally, there will be an equation for the entropy ρs of the form

$$\frac{\partial}{\partial t}(\rho s) + \operatorname{div} \rho s \underline{v_n} = \frac{R}{T} , \tag{286}$$

where the dissipation function R is to be positive definite. (One should really include a contribution to the entropy flux from dissipative processes; as in the theories discussed in Chapter III, such a term is associated with the process of thermal conduction which, as Bekarevich and Khalatnikov point out, is easily included in their theory. They also point out that their theory is easily modified to include dissipative gradient terms in the superfluid equation such as those

discussed in section III-B-2 (cf. equation (175) and (189).) The entropy equation (286) is not independent, but is a consequence of the hydrodynamic equations. By eliminating the time derivatives in the usual manner, we may obtain the following equation relating the quantities $\underline{f}$, $\underline{\underline{\Pi}}$, $\underline{q}$ and R :

$$\operatorname{div}\left\{-\underline{q} + \underline{\underline{\Pi}} \cdot \underline{v}_n + \lambda \underline{\nabla} \times [\underline{f} + \underline{\omega} \times \underline{w}]\right\} = R + \left(\Pi_{ik} - \lambda \omega \delta_{ik} + \lambda \frac{\omega_i \omega_k}{\omega}\right) \frac{\partial v_{ni}}{\partial x_k}$$

$$+ (\underline{f} + \underline{\omega} \times \underline{w}) \cdot (\operatorname{curl} \lambda \underline{\nabla} - \rho_s \underline{w}). \qquad (287)$$

At this point, the authors conclude (without giving any justification) that

$$\underline{q} = \underline{\underline{\Pi}} \cdot \underline{v}_n + \lambda \underline{\nabla} \times (\underline{f} + \underline{\omega} \times \underline{w}), \qquad (288)$$

and

$$R = -\left[\Pi_{ik} - \frac{\lambda \omega_i \omega_k}{\omega} - \lambda \omega \delta_{ik}\right] \frac{\partial v_{ni}}{\partial x_k} - (\underline{f} + \underline{\omega} \times \underline{w}) \cdot (\operatorname{curl} \lambda \underline{\nabla} - \rho_s \underline{w}).$$

In order to obtain the results (228) and (289) from (287), one would have to give careful arguments concerning the dependence of the (as yet unknown) quantities $\underline{f}$, $\underline{\underline{\Pi}}$ and $\underline{q}$ on the macroscopic variables and their gradients. It is easy enough to find special choices of $\underline{f}$, $\underline{\underline{\Pi}}$ and $\underline{q}$ which satisfy (287) without satisfying (288) and (289). Rather than trying to find principles which would allow one to obtain (288) and (289) from (287), we prefer to leave this point for the more systematic derivation of the equations given in the next section; thus we accept (288) and (289), and continue with the derivation of Bekarevich and Khalatnikov.

The dissipation function R is to be positive definite.

Bekarevich and Khalatnikov, however, assume that each of the two terms in R are separately positive definite; again, they give no justification and counter-examples may be easily found. If we make this assumption, then we must have

$$-(\underline{f} + \underline{\omega} \times \underline{w}) \cdot (\operatorname{curl} \lambda \underline{v} - \rho_s \underline{w}) \geq 0, \tag{290}$$

and

$$-(\pi_{ik} - \lambda \omega \delta_{ik} + \frac{\lambda \omega_i \omega_k}{\omega}) \frac{\partial v_{ni}}{\partial x_k} \geq 0. \tag{291}$$

From (290), Bekarevich and Khalatnikov conclude (without justification) that the most general form for $\underline{f}$ is

$$\underline{f} = -\underline{\omega} \times \underline{w} + \alpha \underline{\omega} \times (\operatorname{curl} \lambda \underline{v} - \rho_s \underline{w}) + \beta \underline{v} \times [\underline{\omega} \times (\operatorname{curl} \lambda \underline{v} - \rho_s \underline{w})]$$
$$- \gamma \underline{v} [\underline{\omega} \cdot (\operatorname{curl} \lambda \underline{v} - \rho_s \underline{w})], \tag{292}$$

with β , $\gamma \geq 0$. Here, we may make a more definite criticism of their reasoning. First of all, it is clear that we can say almost nothing about the most general form for $\underline{f}$ unless we know what vectors $\underline{f}$ may depend on. However, it would seem reasonable to suppose that $\underline{f}$ may depend on $\underline{w}$, $\underline{\omega}$ and $\operatorname{curl} \lambda \underline{v}$ (as well as any Galilean invariant scalars). Then it is a straightforward matter to show that the most general $\underline{f}$ satisfying (290) and depending on $\underline{w}$, $\underline{\omega}$ and $\operatorname{curl} \lambda \underline{v}$ is given by (where $\underline{p} = \operatorname{curl} \lambda \underline{v} - \rho_s \underline{w}$)

$$\underline{f} = -\underline{\omega} \times \underline{w} - c_0 \underline{p} + \underline{p} \times \{\underline{p} \times [c_1 \underline{w} + c_2 \underline{\omega} \times \underline{w} + c_3 \underline{\omega} \times \underline{p} + c_4 \underline{\omega} \times (\underline{\omega} \times \underline{w})$$
$$+ c_5 \underline{\omega} \times (\underline{\omega} \times \underline{p})]\}, \tag{293}$$

where $c_0 \geq 0$, $c_i (i=1,\cdots,5)$ are arbitrary, and where the c's in general will depend on all of the scalar invariants of the vectors $\underline{\omega}$, $\underline{w}$ and $\mathrm{curl}\,\lambda\underline{\nu}$. The expression (293) for $\underline{f}$ is clearly much more general than the expression (292). If we assume further that the sum $\underline{f} + \underline{\omega}\times\underline{w}$ does not depend on the vector $\underline{w}$, then the most general $\underline{f}$ such that $\underline{f}+\underline{\omega}\times\underline{w}$ depends only on $\underline{p}$ and $\underline{\omega}$ and such that (290) is satisfied is given by

$$\underline{f}+\underline{\omega}\times\underline{w} = -c_0\underline{p} + \underline{p}\times\left[\underline{p}\times\left\{c_3\,\underline{\omega}\times\underline{p} + c_5\,\underline{\omega}\times(\underline{\omega}\times\underline{p})\right\}\right], \qquad (294)$$

where $c_0 \geq 0$, c_3 and c_5 are functions of the scalar invariants of $\underline{p}$ and $\underline{\omega}$ (and of the thermodynamic variables). If now we choose c_0 , c_3 and c_5 as

$$c_0 = \frac{\gamma}{\omega p^2}(\underline{\omega}\cdot\underline{p})^2 + \frac{\beta}{\omega p^2}(\underline{\omega}\times\underline{p})^2,$$

$$c_3 = \frac{\alpha}{p^2}\,, \text{ and } c_5 = \frac{\gamma-\beta}{\omega p^2}$$

where α , β , γ are scalar functions and $\beta \geq 0$, $\gamma \geq 0$, then (294) reduces to

$$\underline{f} = -\underline{\omega}\times\underline{w} + \alpha\,\underline{\omega}\times\underline{p} + \beta\,\underline{\nu}\times\{\underline{\omega}\times\underline{p}\} - \gamma\,\underline{\nu}(\underline{\omega}\cdot\underline{p}), \qquad (295)$$

which is the expression (292) given by Bekarevich and Khalatnikov. Thus (292) is a special case of (294); we may note that (292) is truly a special case since not every vector of the type (294) may be written in the form (292) with $\beta \geq 0$, $\gamma \geq 0$. A more serious criticism is that we have had to assume that the combination $\underline{f} + \underline{\omega}\times\underline{w}$ does not depend on $\underline{w}$;

this would seem to be very unnatural, since $\underline{f}$ rather than $\underline{f} + \underline{\omega} \times \underline{w}$ is the quantity of physical significance. Although Bekarevich and Khalatnikov may have had reasons for choosing the special form (292), it is clear from the above discussion that some justification for this choice is called for.

Now consider the tensor terms; if we introduce τ_{ik} by

$$\Pi_{ik} = \tilde{\Pi}_{ik} - \tau_{ik},$$

where

$$\tilde{\Pi}_{ik} = \lambda \omega \delta_{ik} - \lambda \frac{\omega_i \omega_k}{\omega},$$

then the condition (291) is

$$\tau_{ik} \frac{\partial v_{ni}}{\partial x_k} \geq 0.$$

Thus τ_{ik} must have the form

$$\tau_{ik} = \mu_{iklm} \frac{\partial v_{nl}}{\partial x_m};$$

in principle, the viscosity tensor μ_{iklm} could depend on $\underline{\omega}$ (as pointed out by Bekarevich and Khalatnikov) and on the relative velocity $\underline{w}$ as well. As a first approximation, however, we may take μ_{iklm} to be isotropic so that the stress tensor τ_{ik} is given by

$$\tau_{ik} = 2\mu_n \left\{ e_{ij}^{(n)} - \tfrac{1}{3} \delta_{ij} e_{kk}^{(n)} \right\} + \lambda_n e_{kk}^{(n)} \delta_{ij}. \qquad (296)$$

This completes the derivation of the hydrodynamic equations

as given by Bekarevich and Khalatnikov. The final equations for the normal fluid component, the superfluid component and the momentum are (where $\beta' \rho_s = 1 + \alpha \rho_s$)

$$\frac{\partial \underline{v}_n}{\partial t} + \underline{v}_n \cdot \nabla \underline{v}_n = -\frac{\nabla p_0}{\rho} - \frac{(1-x)}{x} S \nabla T - (1-x) \nabla \tfrac{1}{2} w^2 + \frac{\omega}{\rho} \nabla \lambda - \frac{\Gamma}{\rho_n} \underline{w} + \operatorname{div} \underline{\underline{\tau}}$$
$$+ \frac{\beta' \rho_s^2}{\rho_n} \left\{ \underline{\omega} \times \left(\underline{w} - \frac{\operatorname{curl} \lambda \underline{\nu}}{\rho_s} \right) \right\} + \frac{\beta \rho_s^2}{\rho_n} \underline{\nu} \times \left\{ \underline{\omega} \times \left(\underline{w} - \frac{\operatorname{curl} \lambda \underline{\nu}}{\rho_s} \right) \right\}$$
$$- \frac{\gamma \rho_s^2}{\rho_n} \underline{\nu} \left\{ \underline{\omega} \cdot \left(\underline{w} - \frac{\operatorname{curl} \lambda \underline{\nu}}{\rho_s} \right) \right\}, \qquad (298)$$

$$\frac{\partial \underline{v}_s}{\partial t} + \underline{v}_s \cdot \nabla \underline{v}_s = -\frac{\nabla p_0}{\rho} + S \nabla T + \times \nabla \tfrac{1}{2} w^2 + \frac{\omega}{\rho} \nabla \lambda - \frac{1}{\rho_s} \underline{\omega} \times \operatorname{curl} \lambda \underline{\nu}$$
$$- \beta' \rho_s \left\{ \underline{\omega} \times \left(\underline{w} - \frac{\operatorname{curl} \lambda \underline{\nu}}{\rho_s} \right) - \beta \rho_s \underline{\nu} \times \left\{ \underline{\omega} \times \left(\underline{w} - \frac{\operatorname{curl} \lambda \underline{\nu}}{\rho_s} \right) \right\} \right.$$
$$+ \gamma \rho_s \underline{\nu} \left\{ \underline{\omega} \cdot \left(w - \frac{\operatorname{curl} \lambda \underline{\nu}}{\rho_s} \right) \right\}, \qquad (299)$$

and

$$\frac{\partial j_i}{\partial t} + \frac{\partial}{\partial x_j} \left\{ \rho_n v_{ni} v_{nj} + \rho_s v_{si} v_{sj} + p_0 \delta_{ij} - \frac{\lambda \omega_i \omega_j}{\omega} - \tau_{ij} \right\} = 0, \qquad (300)$$

where

$$p_0 = p + \lambda \omega.$$

There are several additional points which we would like to note here. First of all, we see from the expression (288) for the increment to the energy flux that the additional stresses due to the vortex motion (namely the terms $\tilde{\pi}_{ij} = \lambda \omega \delta_{ij} - \frac{\lambda \omega_i \omega_j}{\omega}$ in π_{ij}) are a part of the normal fluid stress

tensor. This is somewhat surprising, since it is the vorticity of the superfluid component which plays a special role here (we shall see in the next section that this part of the stress tensor actually should be associated with the superfluid component). Another hint that something is amiss is the fact that Bekarevich and Khalatnikov have given a physical interpretation of the term $\tilde{\underline{\underline{\Pi}}} \cdot \underline{v}_n + \lambda \underline{\nu} \times (\underline{f} + \underline{\omega} \times \underline{w})$ in the energy flux in terms of the transport of energy by the vortex motion. However, the term $\tilde{\underline{\underline{\Pi}}} \cdot \underline{v}_n$ already has an interpretation as the rate of working of the stress tensor $\tilde{\underline{\underline{\Pi}}}$; thus what is needed is a physical interpretation of the term $\lambda \underline{\nu} \times (\underline{f} + \underline{\omega} \times \underline{w})$ alone. This point will be discussed in more detail in the next section.

In situations of practical interest, there are a number of approximations which can be made. First of all, for flows at moderate speeds we may take the fluid to be incompressible; the quantities ρ_n, ρ_s, S are then all approximately constant, and P , T become hydrodynamic variables. Also, on the basis of the microscopic theory (the Onsager-Feynman theory) the coefficient λ should be approximately constant. According to Bekarevich and Khalatnikov, the term in the mutual friction having the coefficient γ represents the effect of deviations of the direction of individual vortex filaments from the direction of the mean curl of the velocity $\underline{v}_s$; they state that the effect is very small so that γ is negligible in comparison with β , β'. A further simplification is that we may neglect the energy dissipation for flows

of moderate velocities. When these approximations are made, the equations take the form

$$\frac{\partial \underline{v}_n}{\partial t} + \underline{v}_n \cdot \nabla \underline{v}_n = -\frac{\nabla p_0}{\rho} - \frac{(1-x)}{x} s\nabla T + \frac{1}{2}\frac{\rho_s}{\rho} B' \left\{ \underline{\omega} \times \left(\underline{w} - \frac{\operatorname{curl} \lambda \underline{\nu}}{\rho_s} \right) \right\}$$

$$+ \mu_n \nabla^2 \underline{v}_n + \frac{1}{2}\frac{\rho_s}{\rho} B \underline{\nu} \times \left\{ \underline{\omega} \times \left(\underline{w} - \frac{\operatorname{curl} \lambda \underline{\nu}}{\rho_s} \right) \right\}, \qquad (301)$$

$$\frac{\partial \underline{v}_s}{\partial t} + \underline{v}_s \cdot \nabla \underline{v}_s = -\frac{\nabla p_0}{\rho} + s\nabla T + x \nabla \frac{1}{2} w^2 - \frac{1}{\rho_s} \underline{\omega} \times \operatorname{curl} \lambda \underline{\nu}$$

$$- \frac{1}{2}\frac{\rho_n}{\rho} B' \left\{ \underline{\omega} \times \left(w - \frac{\operatorname{curl} \lambda \underline{\nu}}{\rho_s} \right) \right\} - \frac{1}{2}\frac{\rho_n}{\rho} B \left\{ \underline{\omega} \times \left(\underline{w} - \frac{\operatorname{curl} \lambda \underline{\nu}}{\rho_s} \right) \right\}, \qquad (302)$$

and

$$\operatorname{div} \underline{v}_n = 0, \qquad \operatorname{div} \underline{v}_s = 0, \qquad (303)$$

where

$$p_0 = p + \lambda\omega, \quad \beta = \frac{1}{2}\frac{B \rho_n}{\rho \rho_s}, \quad \beta' = \frac{1}{2}\frac{B' \rho^n}{\rho \rho_s}.$$

The equation for the total momentum is

$$\frac{\partial}{\partial t}\left\{ \rho_n v_{ni} + \rho_s v_{si} \right\} + \frac{\partial}{\partial x_j}\left\{ \rho_n v_{ni} v_{nj} + \rho_s v_{si} v_{sj} + p_0 \delta_{ij} - \lambda \frac{\omega_i \omega_j}{\omega} \right.$$

$$\left. - 2\mu_n e_{ij}^{(n)} \right\} = 0. \qquad (304)$$

The equations (301) and (303) are in agreement with the equations obtained by Hall [14] (there is apparently a computational error in equation (51) of reference [14]; the results there are in agreement with the equations given by Bekarevich and Khalatnikov provided this is taken into account). The equation for the superfluid vorticity is also of interest;

when the coefficient γ is taken to be zero, this equation may be written as

$$\frac{\partial \underline{\omega}}{\partial t} = \operatorname{curl}\left\{\underline{f} + \underline{v}_s \times \underline{\omega}\right\} = \operatorname{curl}\left\{\underline{v}_L \times \underline{\omega}\right\}, \tag{305}$$

where $$\underline{v}_L = \underline{v}_s + \frac{1}{\rho_s}\operatorname{curl}\lambda\underline{\nu} + \frac{1}{2}B'\frac{\rho_n}{\rho}\left\{\underline{w} - \frac{1}{\rho_s}\operatorname{curl}\lambda\underline{\nu}\right\}$$

$$+ \frac{1}{2}B\frac{\rho_n}{\rho}\underline{\nu} \times \left(\underline{w} - \frac{1}{\rho_s}\operatorname{curl}\lambda\underline{\nu}\right). \tag{306}$$

Equation (305) has the form of a transport equation with the velocity of transport $\underline{v}_L$ being given by (306).

Finally, Bekarevich and Khalatnikov have given a discussion of the boundary conditions which we review briefly here. The condition on $\underline{v}_n$ at a solid surface is taken to be the usual one; thus at a wall moving with velocity $\underline{U}_{wall}$, we have

$$\underline{v}_n = \underline{U}_{wall}. \tag{307}$$

(The present discussion is limited to the case in which there is no heat current through the wall, so that $\underline{v}_n$ and $\underline{v}_s$ satisfy $(\underline{v}_n - \underline{U}_{wall})\cdot\underline{n} = 0$, $(\underline{v}_s - \underline{U}_{wall})\cdot\underline{n} = 0$ and so that the thermal conduction term in the energy flux is unimportant). Since the equation for $\underline{v}_s$ contains second space derivatives of $\underline{v}_s$, a further boundary condition is needed. Bekarevich and Khalatnikov have shown how to obtain the general form of this boundary condition from the requirement that the rate of energy dissipation be positive definite. Let $\underline{n}$ be the unit

normal pointing from the fluid into the solid. Then the force (per unit area) exerted on the solid by the fluid is

$$F_i = (\pi_{ij} + p\delta_{ij})n_j .$$

The rate of working of the stresses is then (taking $U_{wall} \cdot \underline{n} = 0$)

$$\underline{F} \cdot \underline{U}_{wall} = \pi_{ij} U_{wall\,i}\, n_j$$

The rate at which energy flows (per unit area) from the fluid to the solid is

$$\underline{q} \cdot \underline{n} = \pi_{ij} n_i v_{nj} + \lambda \underline{n} \cdot \{\underline{v} \times (\underline{f} + \underline{\omega} \times \underline{w})\}.$$

Thus the rate of energy dissipation is given by

$$D = \underline{q} \cdot \underline{n} - \underline{F} \cdot \underline{U}_{wall} ,$$

or since

$$\underline{v}_n = \underline{U}_{wall} , \quad D = \lambda \underline{n} \cdot \{\underline{v} \times (\underline{f} + \underline{\omega} \times \underline{w})\} \tag{308}$$

or

$$D = \lambda (\underline{n} \times \underline{v}) \cdot (\underline{f} + \underline{\omega} \times \underline{w}).$$

From the condition $D > 0$, the authors conclude that

$$\underline{f} + \underline{\omega} \times \underline{w} \Big|_{wall} = \varsigma (\underline{n} \times \underline{\omega}) + \varsigma' \{\underline{\omega} \times (\underline{v} \times \underline{n})\} \tag{309}$$

where (presumably) ς , ς' are material constants and $\varsigma \geq 0$. We may attempt to supply the reasoning as follows: the quantity $(\underline{f} + \underline{\omega} \times \underline{w})\big|_{wall}$ must be such as to insure that D is positive definite; if we assume that $(\underline{f} + \underline{\omega} \times \underline{w})\big|_{wall}$ depends

only on the vectors $\underline{n}$ and $\underline{\omega}_{wall}$, then the general form for $\underline{f} + \underline{\omega} \times \underline{w}$ may be written as

$$\underline{f} + \underline{\omega} \times \underline{w}\Big|_{wall} = \varepsilon(\underline{n} \times \underline{\omega}) + \varepsilon'\{\underline{\omega} \times (\underline{\nu} \times \underline{n})\} + \varepsilon'' \underline{n}, \qquad (310)$$

where ε, ε' and ε'' may depend on the invariant $\underline{\nu} \cdot \underline{n}$ as well as on the scalar thermodynamic variables. (There is no reason why the quantity $\underline{f} + \underline{\omega} \times \underline{w}|_{wall}$ should be independent of the vector $\underline{w}$, as Bekarevich and Khalatnikov have apparently assumed; if we allow such a dependence, the expression (310) is replaced by a much more general one.) The condition $D \geq 0$ requires that $\varepsilon \geq 0$. We may obtain the result of Bekarevich and Khalatnikov by assuming that $\varepsilon'' = 0$ and that ε, ε' are independent of $\underline{\nu} \cdot \underline{n}$. The boundary conditions (309) may be written in terms of the vortex velocity $\underline{v}_L$ (in the case $\gamma = 0$); thus

$$(\underline{v}_L - \underline{U}_{wall}) \times \underline{\omega}\,|_{wall} = \varepsilon(\underline{n} \times \underline{\omega}) + \varepsilon'(\underline{\omega} \times [\underline{\nu} \times \underline{n}]),$$

or, introducing

$$(\underline{v}_L - \underline{U}_{wall})_{transverse} = \underline{\nu} \times ([\underline{v}_L - \underline{U}_{wall}] \times \underline{\nu}),$$

we have

$$(\underline{v}_L - \underline{U}_{wall})_{transverse}\,|_{wall} = \varepsilon \underline{\nu} \times (\underline{n} \times \underline{\nu}) + \varepsilon'(\underline{n} \times \underline{\nu}). \qquad (311)$$

According to Bekarevich and Khalatnikov, the limiting case $\varepsilon, \varepsilon' \to 0$ corresponds to an absolutely rough surface. (Thus in their theory, the parameters ε, ε' will presumably

depend on the condition of the wall surface.)

This completes the review of the derivation given by Bekarevich and Khalatnikov. In the next section, we will give an alternative derivation of the hydrodynamic equations which provides tentative answers to some of the questions raised here.

2. An alternative development of the equations

In this section we give an alternative derivation of the hydrodynamic equations for helium II. The starting point is the same as in the theory of Bekarevich and Khalatnikov--namely, the two-fluid model with the additional feature that the thermodynamic internal energy depends on the superfluid vorticity. As pointed out in the preceding section, Bekarevich and Khalatnikov have identified rotational flow with dissipative flow. We prefer to distinguish clearly between rotational flow and dissipative flow; thus we begin the derivation by first obtaining the equations for reversible processes in the case when the internal energy depends on $|\mathrm{curl}\ \underline{v}_s|$.

As a helpful preliminary, we consider first the case of an ordinary fluid in which the internal energy depends on the vorticity. The state of the system is described by the mass per unit volume ρ, the entropy per unit mass, S, the fluid velocity $\underline{v}$ and the vorticity $\underline{\omega} = \mathrm{curl}\,\underline{v}$. The total energy per unit volume E is given by

$$E = \frac{1}{2}\rho v^2 + \rho e, \tag{312}$$

where the internal energy e is a function of ρ, S and $\underline{\omega}$. The pressure p and temperature T are then given by

$$de = \frac{p}{\rho^2}\,d\rho + T\,dS + \underline{\lambda}\cdot d\left(\frac{\underline{\omega}}{\rho}\right). \tag{313}$$

Thus, as a thermodynamic variable, $\underline{\omega}$ is taken to be an

extensive quantity per unit volume. Since the energy e is a scalar, we must have

$$\underline{\lambda} = \beta \underline{\omega} \tag{314}$$

where β is a scalar function of ρ, s and ω^2. The hydrodynamic equations must include the conservation laws for mass, momentum and energy; thus

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \rho \underline{v} = 0, \tag{315}$$

$$\frac{\partial E}{\partial t} + \operatorname{div} \operatorname{div} \underline{Q} = 0, \tag{316}$$

and

$$\frac{\partial}{\partial t}(\rho \underline{v}) + \operatorname{div}(\rho \underline{v} \underline{v}) = \operatorname{div} \underline{\underline{\sigma}}, \tag{317}$$

where $\underline{Q}$ is the energy flux vector and $\underline{\underline{\sigma}}$ is the stress tensor. Since we are considering only reversible processes, the entropy must be conserved, so we have

$$\frac{\partial}{\partial t}(\rho s) + \operatorname{div}(\rho s \underline{v}) = 0. \tag{318}$$

In the determination of $\underline{\underline{\sigma}}$ and $\underline{Q}$, it is convenient to introduce the quantities $\underline{Q}_0$ and $\underline{\underline{\tau}}$ by

$$\underline{Q} = E \underline{v} - \underline{\underline{\sigma}} \cdot \underline{v} + \left[\frac{\omega}{\rho} \times \operatorname{curl} \underline{\lambda}\right] \times \underline{\lambda} + \underline{Q}_0, \tag{319}$$

and

$$\underline{\underline{\sigma}} = \beta \underline{\omega}\, \underline{\omega} - p \underline{\underline{I}} + \underline{\underline{\tau}}. \tag{320}$$

The invariance of the equations under Galilean transformations requires that $\underline{Q}_o$ and $\underline{\underline{\tau}}$ be Galilean invariant. The equations (315) - (318) are not all independent, and the requirement that they be consistent may be written in the form (after some calculations)

$$\operatorname{div} \underline{Q}_o + \underline{\lambda} \cdot \operatorname{curl}\left(\frac{1}{\rho} \operatorname{div} \underline{\underline{\tau}}\right) - \underline{\underline{\tau}} : \operatorname{grad} \underline{v} + \underline{\lambda} \cdot (\nabla T \times \nabla S) \equiv 0. \quad (321)$$

We now consider the special case in which the entropy is uniform in space (this includes the problem of eventual interest, since the superfluid component of helium II has no entropy). Then (321) becomes

$$\operatorname{div} \underline{Q}_o + \underline{\lambda} \cdot \operatorname{curl}\left(\frac{1}{\rho} \operatorname{div} \underline{\underline{\tau}}\right) - \underline{\underline{\tau}} : \operatorname{grad} \underline{v} \equiv 0. \quad (322)$$

The simplest "solution" of (322) is to take $\underline{\underline{\tau}} \equiv 0$, $\underline{Q}_o \equiv 0$, and we will do this here (it is conceivable that other choices of $\underline{Q}_o$ and $\underline{\underline{\tau}}$ will satisfy (322); however, the physical interpretation given below lends weight to the choice made here). Then the complete set of hydrodynamic equations is

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \rho \underline{v} = 0,$$

$$\frac{\partial}{\partial t}(\rho \underline{v}) + \operatorname{div}(\rho \underline{v} \underline{v}) = \operatorname{div}(\underline{\lambda} \underline{\omega} - p \underline{\underline{I}}), \quad (323)$$

or

$$\rho \frac{D \underline{v}}{D t} = -\nabla p + \underline{\omega} \cdot \nabla \underline{\lambda} \quad (324)$$

or

$$\frac{\partial \underline{v}}{\partial t} + \underline{v}\cdot\nabla\underline{v} = -\nabla\Phi - \frac{\underline{\omega}}{\rho}\times \operatorname{curl}\underline{\lambda},$$

where

$$\Phi = e + \frac{p}{\rho} - \underline{\lambda}\cdot\frac{\underline{\omega}}{\rho}, \tag{325}$$

$$s = \text{constant}, \tag{326}$$

and

$$\frac{\partial E}{\partial t} + \operatorname{div}\left\{E\underline{v} - \left(\underline{\lambda}\,\underline{\omega} - p\underline{\underline{I}}\right)\cdot\underline{v} + \left(\frac{\underline{\omega}}{\rho}\times\operatorname{curl}\underline{\lambda}\right)\times\underline{\lambda}\right\} = 0. \tag{327}$$

The equation for the vorticity is

$$\frac{\partial \underline{\omega}}{\partial t} = \operatorname{curl}\left\{\underline{v}\times\underline{\omega} + \frac{1}{\rho}\operatorname{curl}\underline{\lambda}\times\underline{\omega}\right\}, \tag{328}$$

or

$$\frac{\partial \underline{\omega}}{\partial t} = \operatorname{curl}\left\{\underline{v}_L\times\underline{\omega}\right\},$$

where

$$\underline{v}_L = \underline{v} + \frac{1}{\rho}\operatorname{curl}\underline{\lambda}. \tag{329}$$

Thus we see that there is an additional term in the stress tensor, $\underline{\lambda}\,\underline{\omega}$, and that the vorticity satisfies a transport equation with a transport velocity $\underline{v}_L$ given by (329). In the energy flux, there is a convection term, $E\underline{v}$, a term giving the rate of working of the stresses, $-\underline{\underline{\sigma}}\cdot\underline{v}$, and an additional term, $\left(\frac{\underline{\omega}}{\rho}\times\operatorname{curl}\underline{\lambda}\right)\times\underline{\lambda}$. We may give a physical interpretation of this term as follows: from the vorticity equation, we have

$$\frac{d}{dt}\int_V \underline{\omega}\,d\tau = \int_S dS\ \underline{n}\times(\underline{v}_L\times\underline{\omega}),$$

where v is a volume fixed in space, bounded by S ; thus the rate at which vorticity crosses S per unit area is $-\underline{n}\times(\underline{v}_\perp\times\underline{\omega})$, so the transport of energy from this source is

$$-\underline{\lambda}\cdot\underline{n}\times(\underline{v}_\perp\times\underline{\omega}) = \underline{n}\cdot\underline{\lambda}\times(\underline{v}_\perp\times\underline{\omega}).$$

The flux of energy relative to the moving fluid is then given by

$$\underline{n}\cdot\underline{\lambda}\times\left[(\underline{v}_\perp-\underline{v})\times\underline{\omega}\right] = \underline{n}\cdot\left\{\frac{1}{\rho}\,\mathrm{curl}\,\underline{\lambda}\times\underline{\omega}\right\}\times\underline{\lambda}$$

in agreement with $\underline{n}\cdot\underline{Q}\,|_{\text{fluid rest frame}}$. This completes the discussion of the case of an ordinary fluid, and we now consider the application of these results to the two-fluid model.

To derive the perfect fluid equations, we will use a modification of the method discussed in II-A-2, since this method yields a unique set of equations even in the case when $\mathrm{curl}\,\underline{v}_s \neq 0$. Since the method is discussed in detail there, we give only a brief discussion here. As a starting point for the derivation, we assume that each of the two components has a complete thermodynamic description and that the only coupling between them is due to $\rho_n \rightleftarrows \rho_s$ transitions. Thus the total energies per unit volume of the superfluid and normal fluid components are given by

$$\epsilon_n = \tfrac{1}{2}\rho_n v_n^2 + \rho_n e_n,$$

and

$$\epsilon_s = \tfrac{1}{2}\rho_s v_s^2 + \rho_s e_s. \qquad (330$$

The specific internal energy e_n is a function of ρ_n and s_n, and

$$de_n = \frac{p_n}{\rho_n^2} d\rho_n + T_n ds_n, \tag{331}$$

which defines the normal fluid pressure p_n and temperature T_n. The energy e_s is a function of $\underline{\omega} = \text{curl } \underline{v}_s$ and ρ_s (we are assuming the entropy of the superfluid component to be zero, as usual), and

$$de_s = \frac{p_s}{\rho_s^2} d\rho_s + \underline{\lambda} \cdot d\left(\frac{\underline{\omega}}{\rho_s}\right), \tag{332}$$

which defines superfluid pressure p_s ; since e_s is a scalar function of ρ_s , $\underline{\omega}$, we must have

$$\underline{\lambda} = \lambda \underline{\nu} \tag{333}$$

where

$$\underline{\nu} = \frac{\underline{\omega}}{|\underline{\omega}|}, \tag{334}$$

and λ is a scalar function of ρ_s and $\frac{1}{2}\omega^2$.

The only coupling between the two components is (by assumption) that due to $\rho_n \rightleftarrows \rho_s$ transitions, and we assume further that the system is in equilibrium with respect to this process. The equilibrium condition is obtained from an energy minimum principle. If we consider a fixed volume V , the total energy is

$$\varepsilon = V(\epsilon_n + \epsilon_s),$$

the total momentum is $\underline{J} = V(\rho_n \underline{v}_n + \rho_s \underline{v}_s)$, the total entropy is $S = V(\rho_n S_n)$, and the total mass is $M = V(\rho_n + \rho_s)$. As discussed in Chapter II, the equilibrium is also to be taken at constant $\underline{v}_s$; for the same reasons, we also take the equilibrium at constant $\underline{\omega}$. Thus the equilibrium condition is given by

$$d\mathcal{E}\Big|_{V, \underline{J}, M, S, \underline{v}_s, \underline{\omega}} = 0, \tag{335}$$

and this yields the condition

$$\Phi_n - \Phi_s = \tfrac{1}{2} w^2, \tag{336}$$

where

$$\Phi_n = e_n + \frac{p_n}{\rho_n} - T_n s_n ,$$

and

$$\Phi_s = e_s + \frac{p_s}{\rho_s} - \frac{\underline{\lambda} \cdot \underline{\omega}}{\rho_s} . \tag{337}$$

If we introduce the entropy per unit total mass s , $(\rho s = \rho_n s_n)$, and internal energy e per unit mass $(\rho e = \rho_n e_n + \rho_s e_s$), then

$$de = T_n ds + \frac{p_n + p_s}{\rho^2} d\rho + (\Phi_n - \Phi_s) dx + \underline{\lambda} \cdot d\left(\frac{\underline{\omega}}{\rho}\right). \tag{338}$$

If we introduce the total pressure p by

$$p = p_n + p_s , \tag{339}$$

and the temperature $T = T_n$, then (338) becomes

$$de = T ds + \frac{p}{\rho^2} d\rho + \tfrac{1}{2} w^2 dx + \underline{\lambda} \cdot d\left(\frac{\underline{\omega}}{\rho}\right). \tag{340}$$

For later convenience, we note here that the differentials of the partial pressures p_n and p_s are given by

$$\frac{dp_n}{p_n} = \frac{dp}{\rho} + (1-x)\, d\tfrac{1}{2}w^2 + \frac{(1-x)}{x}\, s\,dT - (1-x)\,\frac{\omega}{p_s}\cdot d\underline{\lambda}, \qquad (341)$$

and

$$\frac{dp_s}{p_s} = \frac{dp}{\rho} - x\, d\tfrac{1}{2}w^2 - s\,dT + x\,\frac{\omega}{p_s}\cdot d\underline{\lambda}. \qquad (342)$$

We now consider the hydrodynamic equations for reversible processes. First of all, there will be two continuity equations which, however, must take into account the $p_n \rightleftarrows p_s$ transitions which conserve only the total mass. Thus

$$\frac{\partial p_n}{\partial t} + \operatorname{div}(p_n \underline{v}_n) = \Gamma,$$

$$\text{and} \qquad (343)$$

$$\frac{\partial p_s}{\partial t} + \operatorname{div}(p_s \underline{v}_s) = -\Gamma,$$

where the volume rate of conversion Γ is not to be independently specified, but is determined by the flow conditions and the equilibrium condition (336). Since we are considering only reversible processes, the entropy is conserved and we have

$$\frac{\partial}{\partial t}(p_n s_n) + \operatorname{div}(p_n s_n \underline{v}_n) = 0. \qquad (344)$$

To obtain the momentum equations, we simply assume that each component satisfies an appropriate perfect fluid equation,

with, however, the momentum transfer due to the $\rho_n \rightleftarrows \rho_s$ transitions taken into account. Thus for the normal fluid component, we assume an Euler equation, so that

$$\frac{\partial}{\partial t}(\rho_n \underline{v}_n) + \mathrm{div}(\rho_n \underline{v}_n \underline{v}_n) = -\nabla p_n + \Gamma \underline{v}_s , \qquad (345)$$

where the term in $\Gamma \underline{v}_s$ represents the volume rate of increase of normal fluid momentum due to transitions. For the superfluid component, we make use of the results obtained earlier in this section to write the momentum equation as

$$\frac{\partial}{\partial t}(\rho_s \underline{v}_s) + \mathrm{div}(\rho_s \underline{v}_s \underline{v}_s) = -\rho_s \nabla \Phi_s - \underline{\omega} \times \mathrm{curl}\, \underline{\lambda} - \Gamma \underline{v}_s ,$$

or

$$\frac{\partial}{\partial t}(\rho_s \underline{v}_s) + \mathrm{div}(\rho_s \underline{v}_s \underline{v}_s) = \mathrm{div}(\underline{\lambda}\,\underline{\omega} - p_s \underline{\underline{I}}) - \Gamma \underline{v}_s , \qquad (346)$$

or

$$\frac{\partial \underline{v}_s}{\partial t} + \underline{v}_s \cdot \nabla \underline{v}_s = -\nabla \Phi_s - \frac{\underline{\omega}}{\rho_s} \times \mathrm{curl}\, \underline{\lambda} .$$

(the three forms are all equivalent). The equations (343) - (346) form a complete set of hydrodynamic equations for reversible processes. From these equations, we may derive the two energy equations

$$\frac{\partial}{\partial t}\left\{\tfrac{1}{2}\rho_n v_n^2 + \rho_n e_n\right\} + \mathrm{div}\left\{\left(\tfrac{1}{2}\rho_n v_n^2 + \rho_n e_n + p_n\right)\underline{v}_n\right\} = \Gamma\left(\Phi_n - \tfrac{1}{2}v_n^2 + \underline{v}_n \cdot \underline{v}_s\right), \qquad (347)$$

and

$$\frac{\partial}{\partial t}\left\{\tfrac{1}{2}\rho_s v_s^2 + \rho_s e_s\right\} + \mathrm{div}\left\{\left(\tfrac{1}{2}\rho_s v_s^2 + \rho_s e_s + p_s\right)\underline{v}_s\right\} - \left(\underline{\lambda}\,\underline{\omega} - p_s \underline{\underline{I}}\right) \cdot \underline{v}_s +$$

$$+\left(\frac{\omega}{\rho_s} \times \operatorname{curl} \underline{\lambda}\right) \times \underline{\lambda} \Big\} = -\Gamma \left(\Phi_s + \frac{1}{2} v_s^2\right). \qquad (348)$$

The terms on the right-hand sides give the rate of energy exchange due to $\rho_n \rightleftarrows \rho_s$ transitions (as may be easily verified by a direct calculation of this energy exchange). For purposes of comparison with the results of Bekarevich and Khalatnikov, we may write the equations in the following form:

$$\frac{\partial \underline{v}_n}{\partial t} + \underline{v}_n \cdot \nabla \underline{v}_n = -\frac{\nabla P}{\rho} - \frac{(1-x)}{x} s \nabla T - (1-x) \nabla \frac{1}{2} w^2 + \frac{\omega}{\rho} \nabla \lambda - \frac{\Gamma \underline{w}}{\rho_n}, \qquad (349)$$

$$\frac{\partial \underline{v}_s}{\partial t} + \underline{v}_s \cdot \nabla \underline{v}_s = -\frac{\nabla P}{\rho} + x \nabla \frac{1}{2} w^2 + s \nabla T + \frac{\omega}{\rho} \nabla \lambda - \frac{\underline{\omega}}{\rho_s} \times \operatorname{curl} \underline{\lambda}, \qquad (350)$$

and

$$\frac{\partial \underline{j}}{\partial t} + \operatorname{div} \left\{ \rho_n \underline{v}_n \underline{v}_n + \rho_s \underline{v}_s \underline{v}_s + P \underline{\underline{I}} - \frac{\lambda \underline{\omega} \underline{\omega}}{\omega} \right\} = 0. \qquad (351)$$

These equations are in agreement with the equations of Bekarevich and Khalatnikov in the special case when the coefficients β, β' and γ of their theory (cf. equations (298) - (300)) are all zero. In the present case, however, the term $\frac{\omega_s}{\rho_s} \times \operatorname{curl} \underline{\lambda}$ in the superfluid equation is not a dissipative term but comes from the stress tensor of the perfect fluid theory. Furthermore, the extra term in the stress tensor $\underline{\lambda} \underline{\omega}$ is here associated with the superfluid component and not with the normal fluid component as in their theory.

We now consider the extension of the present theory to include dissipative processes. As mentioned earlier, the systematic procedure used in Chapter III for deriving the equations for dissipative processes cannot be applied in a straightforward manner to the present theory(mainly because we are here interested in a volume dissipative process--mutual friction--which does not fit easily into the framework of the general theory of Chapter III). For this reason, we do not attempt to consider the most general dissipative equations; rather, we restrict attention to the type of dissipative terms considered by Bekarevich and Khalatnikov. More specifically, we consider two dissipative processes--(i) a mutual friction force effecting momentum exchange between the components and (ii) a dissipative stress tensor acting on the normal fluid component. (In the present theory, as in the theory, as in the theory of Bekarevich and Khalatnikov, we may easily include thermal conduction and the dissipative gradient terms in the superfluid equation (cf. II-B-2); however, for purposes of comparison with their equations wewill omit such terms here.) The independent equations will be the conservation equations for mass, momentum and energy, and the equation for the superfluid velocity. These equations may be written as

$$\frac{\partial \rho}{\partial t} + \operatorname{div}\left(\rho_n \underline{v}_n + \rho_s \underline{v}_s\right) = 0, \tag{352}$$

$$\frac{\partial \underline{j}}{\partial t} + \operatorname{div}\left\{\rho_n \underline{v}_n \underline{v}_n + \rho_s \underline{v}_s \underline{v}_s + p\underline{\underline{I}} - \lambda \frac{\underline{\omega}\,\underline{\omega}}{\omega} - \underline{\underline{\tau}}\right\} = 0, \tag{353}$$

$$\frac{\partial E}{\partial t} + \operatorname{div}\left\{\underline{Q}_0 + \underline{Q}'\right\} = 0, \tag{354}$$

where

$$\underline{Q}_o = \left[\tfrac{1}{2}\rho_n v_n^2 + \rho_n e_n + p_n\right]\underline{v}_n + \left[\tfrac{1}{2}\rho_s v_s^2 + \rho_s e_s\right]\underline{v}_s - \left[\lambda \frac{\underline{\omega}\,\underline{\omega}}{\omega} - p_s \underline{\underline{I}}\right]\cdot \underline{v}_s$$

$$+ \left[\frac{\underline{\omega}}{\rho_s} \times \mathrm{curl}\,\underline{\lambda}\right] \times \underline{\lambda}, \qquad (355)$$

and

$$\frac{\partial \underline{v}_s}{\partial t} + \underline{v}_s \cdot \nabla \underline{v}_s = -\nabla \Phi_s - \frac{\underline{\omega}}{\rho_s} \times \mathrm{curl}\,\underline{\lambda} + \underline{f},$$

where $\underline{\underline{\tau}}$ is the dissipative stress tensor, $\rho_s \underline{f}$ is the mutual friction force per unit volume and $\underline{Q}'$ is the contribution to the energy flux from the dissipative processes. We may obtain the form of $\underline{Q}'$ by plausible arguments. First of all, there will be a term $-\underline{\underline{\tau}} \cdot \underline{v}_n$ giving the rate of working of the stress tensor. $\underline{\underline{\tau}}$. We may expect that the term $\underline{f}$ in the superfluid equation will affect the transport of vorticity and will thus lead to an additional term in the energy flux. The vorticity equation is

$$\frac{\partial \underline{\omega}}{\partial t} = \mathrm{curl}\left\{\left(\underline{v}_s + \frac{1}{\rho_s}\mathrm{curl}\,\underline{\lambda}\right) \times \underline{\omega} + \underline{f}\right\}; \qquad (356)$$

thus the rate of transport of vorticity across a surface with normal $\underline{n}$ is

$$-\underline{n} \times \left\{\left(\underline{v}_s + \frac{1}{\rho_s}\mathrm{curl}\,\underline{\lambda}\right) \times \underline{\omega} + \underline{f}\right\}. \qquad (357)$$

Then the rate of energy transport by this process in the rest frame of the superfluid is

$$-\underline{\lambda}\cdot\underline{n}\times\left\{\left(\frac{1}{\rho_s}\,\mathrm{curl}\,\underline{\lambda}\right)\times\underline{\omega}+\underline{f}\right\}=\underline{n}\cdot\left\{\underline{\lambda}\times\underline{f}+\left(\frac{\underline{\omega}}{\rho_s}\times\mathrm{curl}\,\underline{\lambda}\right)\times\underline{\lambda}\right\},\quad(358)$$

so that we would expect a term $\underline{\lambda}\times\underline{f}$ in the dissipative energy flux $\underline{Q}'$. Thus we assume

$$\underline{Q}'=\underline{\lambda}\times\underline{f}\;-\underline{\underline{\tau}}\cdot\underline{v}_n\,.\qquad(359)$$

For the entropy, we expect an equation of the form

$$\frac{\partial}{\partial t}(\rho s)+\mathrm{div}(\rho s\underline{v}_n)=\frac{R}{T}\,,\qquad(360)$$

with R positive definite. Equation (360) is not independent, and, from the hydrodynamic equations (352) - (355), we may show that

$$R=\underline{\underline{\tau}}:\mathrm{grad}\,\underline{v}_n+\underline{f}\cdot\left\{\rho_s\underline{w}-\mathrm{curl}\,\underline{\lambda}\right\}.\qquad(361)$$

We now assume (as did Bekarevich and Khalatnikov) that the dissipative processes represented by $\underline{\underline{\tau}}$ and $\underline{f}$ are independent in the sense that their contributions to the dissipation function are independently positive definite. Then $\underline{\underline{\tau}}$ must have the form

$$\tau_{ik}=\mu_{iklm}\frac{\partial v_{nl}}{\partial x_m}\;;$$

if we assume that the viscosity tensor is isotropic, then this reduces to

$$\tau_{ik} = 2\mu_n \left\{ e_{ij}^{(n)} - \tfrac{1}{3}\delta_{ij} e_{kk}^{(n)} \right\} + \lambda_n e_{kk}^{(n)} \delta_{ij}.$$

The mutual friction $\underline{f}$ must be determined so that

$$-\underline{f} \cdot \underline{P} \geq 0$$

where

$$\underline{P} = \operatorname{curl} \underline{\lambda} - \rho_s \underline{w}. \tag{362}$$

We may also express $\underline{P}$ in terms of the vorticity transport velocity of the perfect fluid theory, $\underline{v}_L^{(0)}$, where (cf. equation (329))

$$\underline{v}_L^{(0)} = \underline{v}_s + \frac{1}{\rho_s} \operatorname{curl} \underline{\lambda} \; ; \tag{363}$$

then

$$\underline{P} = \rho_s (\underline{v}_L^{(0)} - \underline{v}_n) \; ;$$

thus the dissipative mutual friction force is closely connected with the transport of superfluid vorticity relative to the normal fluid. As in the discussion of the preceding section, little more can be said about $\underline{f}$ until we have determined what quantities $\underline{f}$ may depend on. We may note, however, that the choice

$$\underline{f} = \beta' \underline{\omega} \times \underline{P} + \beta \underline{\nu} \times \{ \underline{\omega} \times \underline{P} \} - \gamma \underline{\nu} (\underline{\omega} \cdot \underline{P}),$$

where

$$\underline{\nu} = \frac{\underline{\omega}}{|\underline{\omega}|} \quad \text{and} \quad \beta \geq 0, \; \gamma \geq 0, \tag{364}$$

satisfies (362) and leads to exactly the equations for $\underline{v}_n$, $\underline{v}_s$

that were obtained by Bekarevich and Khalatnikov (equations (298) - (300)). The justification of the choice (364), though, is no easier here than in the derivation of Bekarevich and Khalatnikov. We may easily write down the most general $\underline{f}$ depending only on $\underline{\omega}$ and $\underline{w}, \underline{p}$ and satisfying (362); the result (cf. equation (293)) involves 6 scalar coefficients (one of which must be positive) and includes (364) as a special case. Even if we assume that $\underline{f}$ depends only on $\underline{p}$ and $\underline{\omega}$, the resulting expression for $\underline{f}$ is still more general than (364). In the case that $\gamma = 0$ (which, according to Bekarevich and Khalatnikov, is the case of practical interest), we can arrive at the form (364) by using the microscopic (Onsager-Feynman) theory as a qualitative guide. In this theory (as developed by Hall and Vinen), the mechanism responsible for mutual friction is the scattering of the normal fluid excitations by the quantized vortex lines. From this picture, it is reasonable to suppose that the mutual friction force is perpendicular to the superfluid vorticity $\underline{\omega}$; it is also reasonable to suppose that the relevant relative velocity is $\underline{v}_n - \underline{v}_L^{(0)}$, rather than $\underline{v}_n - \underline{v}_s$. Thus we assume that $\underline{f}$ depends only on $\underline{\omega}$ and $\underline{p} = \rho_s(\underline{v}_L^{(0)} - \underline{v}_n)$ (and the scalar thermodynamic variables), and that $\underline{f} \cdot \underline{\omega} = 0$. Then it is easy to show that the most general $\underline{f}$ (also satisfying (362)) is given by

$$\underline{f} = \beta' \underline{\omega} \times \underline{p} + \beta \underline{\nabla} \times \{\underline{\omega} \times \underline{p}\},$$

where

$$\beta \geq 0,$$

so that in this case, the present equations for $\underline{v}_n$ and $\underline{v}_s$ agree exactly with those of Bekarevich and Khalatnikov in the case $\gamma=0$.

This completes the alternative derivation of the equations of Bekarevich and Khalatnikov. Although the final equations for $\underline{v}_n$ and $\underline{v}_s$ are the same in both derivations, it is hoped that the present derivation brings out more clearly the physical significance of the various terms and also the nature of the arguments needed to obtain the final equations.

3. Physical basis of the theory

As mentioned earlier, the theory of Bekarevich and Khalatnikov is based on the two-fluid model, the usual conservation laws and invariance principles and the single additional assumption that the thermodynamic internal energy depends on the superfluid vorticity, as well as the usual thermodynamic variables of the two-fluid theory. It is this last assumption which is the new feature of their theory.and which deserves careful discussion. In the preceding two sections, we have considered the mathematical development of their theory. In the present section, we make some attempt to discuss the physical basis of the theory; in particular we wish to (i) examine the theory from the point of view of the Onsager-Feynman theory of quantized vortex lines and (ii) discuss the possibility that the theory of Bekarevich and Khalatnikov might be relevant to some microscopic picture other than the Onsager-Feynman theory. We consider (i) first.

According to the Onsager-Feynman theory the rotation of helium II gives rise to line singularities--quantized vortex lines--in the the superfluid component. It is supposed that these vortex lines consist of a small core around which the circulation of the macroscopic velocity field $\underline{v}_s$ is quantized according to the formula

$$\oint \underline{v}_s \cdot d\underline{s} = \frac{2\pi n \hbar}{m} . \qquad (365)$$

The effective core diameter is believed to be of the order of 10^{-8} cm. (there is, however, some evidence that it may be much larger [37]). Thus, according to the prevailing theories, the vortex line is an "excitation" which is in some respects microscopic (the very small core) and in some respects macroscopic (because of the associated macroscopic velocity field). In the development of a hydrodynamic theory from these ideas, it is clearly desirable to try and determine to what extent the vortex lines are to be treated as microscopic excitations and to what extent they are be treated as hydrodynamic structures. We may note first of all that in a typical flow problem (e.g., the steady flow in a cylinder rotating with an angular velocity of 1 rad/sec.), the average distance between vortex lines is of the order of 10^{-2} cm. Thus even though the vortices are fairly close together, their average separation is still much greater than any (presently known) microscopic length scales. The energy per unit length of a vortex line is usually taken to be

$$\pi \rho_s \frac{\hbar^2}{m^2} \ln \frac{R}{a},$$

where R is of the order of the distance between vortices and a is the effective core radius. This energy (which is obtained by simply integrating the kinetic energy density of a hydrodynamic vortex field) is the energy which Bekarevich and Khalatnikov have included in the thermodynamic internal energy. However, this extra energy is associated primarily

with the organized, macroscopic motion induced by the vortex; thus we may ask why it is not already included in the ordinary kinetic energy of motion. To be sure, if we consider a spatial average, $\langle \underline{v}_s \rangle$, of the superfluid velocity field over a region of space containing many vortex lines, then, for

$$\underline{v}_s = \langle \underline{v}_s \rangle + \underline{v}_s'$$

$$\left\langle \tfrac{1}{2} \rho_s v_s^2 \right\rangle = \tfrac{1}{2} \rho_s \langle \underline{v}_s \rangle^2 + \tfrac{1}{2} \rho_s \langle \underline{v}_s'^2 \rangle \neq \tfrac{1}{2} \rho_s \langle \underline{v}_s \rangle^2 ;$$

however, in this case, the extra energy is analogous to the extra energy of turbulent fluctuations in the turbulent motion of an ordinary fluid--it is in no way associated with the thermodynamic internal energy, since it appears as a consequence of averaging a macroscopic velocity field and not as a consequence of averaging over a thermal distribution. The essence of the argument here is that the vortex motion will appear regular and macroscopic in character when viewed on a sufficiently small ($\sim 10^{-2}$ cm)--but still macroscopic--length scale. One would expect some sort of equilibrium thermodynamics to hold for volume elements whose linear extent is even smaller than 10^{-2} cm; thus it is not clear that the continuum model of Bekarevich and Khalatnikov is relevant to the Onsager-Feynman theory of quantized vortex lines. The turbulence analogy, however, does suggest an alternative approach to constructing a hydrodynamic theory on the basis of the two-fluid model and the Onsager-Feynman theory. This will be considered in detail in the next section (IV-C).

We may also consider the possibility that the theory of Bekarevich and Khalatnikov is an adequate continuum representation of some microscopic theory other than the Onsager-Feynman theory. To discuss this we may first ask for a description in general terms of the physical picture associated with the inclusion of a vorticity dependent term in the internal energy function. The presence of an extra term dependent on the vorticity in the energy means that there is some internal structure in the fluid which tends to resist rotation (it is not really necessary to assume that this structuring is produced by the rotation; as Lin* has pointed out, the excess energy may be that which is required to align structures already present in the fluid). The fact that this extra energy is included in the thermodynamic internal energy would seem to correspond to a situation in which the internal structures are microscopic in size (e.g., as Lin [25] has suggested, the structures may be small vortex rings of sub-macroscopic diameter). Thus we have a plausible, general sort of microscopic picture to which the theory of Bekarevich and Khalatnikov would seem to be applicable. However, there is a further point regarding the development of such a theory which we discuss by means of an example--namely, we consider the development of the hydrodynamic equations for an ordinary fluid in which there is an internal angular momentum in addition to the usual $\underline{r} \times \rho \underline{v}$ angular momentum density (the extra angular momentum is associated with the rotation of the constituent molecules of the fluid about their centers of mass; this theory

* private communication

has been developed systematically by Grad [10], and a concise presentation is given in [4, p.304]). The essential features of this theory relevant to the discussion here are (i) there is an extra term in the energy dependent on a local internal angular velocity $\underline{\Omega}$, $\underline{\Omega}$ being the local mean angular velocity of the molecules, (ii) the rate of energy dissipation from the processes which tend to produce rotational equilibrium is proportional to the square of the difference $\underline{\Omega} - \frac{1}{2}\,\mathrm{curl}\,\underline{v}$, and (iii) if the relaxation time for these processes is much less than the relevant macroscopic time scales, then $\underline{\Omega} \simeq \frac{1}{2}\mathrm{curl}\,\underline{v}$. Thus in the case of helium II, we might expect that there would be some internal parameter such as $\underline{\Omega}$ above to describe the mean state of the internal structures; then the extra energy would depend on $\underline{\Omega}$ and the dissipation would depend on $\underline{\Omega} - \frac{1}{2}\,\mathrm{curl}\,\underline{v_s}$. For steady flows (or flows with slow time variations) we would have $\underline{\Omega} \simeq \frac{1}{2}\,\mathrm{curl}\,\underline{v_s}$; thus the quantity $\underline{\omega} \equiv \mathrm{curl}\,\underline{v_s}$ would enter into the thermodynamics through a sort of equilibrium condition and not as a fundamental thermodynamic variable. The detailed development of a theory along the lines described here may well differ considerably from the development given by Bekarevich and Khalatnikov for their theory.

In summary: there are definite arguments which indicate that the theory of Bekarevich and Khalatnikov may not be relevant to the two-fluid model with quantized vortex lines; although their theory may have some relevance to other microscopic pictures, it is possible that the detailed development of such theories should proceed along somewhat different lines.

C. An Alternative Approach to the Hydrodynamic Theory with Quantized Vortex Lines

In the theory of Bekarevich and Khalatnikov, the hydrodynamic equations obtained are a generalization of Landau's equations (as discussed in III-B); also, their equations are (ostensibly) equations for local macroscopic variables ($\underline{v}_s$, $\underline{v}_n$ etc.). Hall and Vinen [12,13,14,38] have taken an alternative approach--namely, they attempt to obtain equations for averages over regions containing many vortex lines of the local variables. Thus in their theory, the basic equations for the local quantities are essentially the same as the Landau equations; however, the particular solutions of interest--flows involving a large number of vortex lines--are too complicated to calculate in detail, so that the development of equations for average quantities is essential. In carrying out this development of the equations for average quantities, Hall and Vinen find it necessary to make a number of additional specific assumptions about the nature of the vortex flows. In view of our present uncertain knowledge regarding the detailed structure and properties of quantized vortex lines, it seems desirable to base the derivation of the hydrodynamic equations on as few assumptions about the specific features of vortex flows as possible. In the present section, then, we offer an alternative derivation of the equation proposed by Hall [13,14] to describe "vortex waves". We consider only the case in which mutual friction is absent (or unimportant, as is the case for lower temperatures), mainly because it is not clear how to include mutual friction in the present

theory. (In this connection, however, we may note that the theory of mutual friction given by Hall and Vinen is not free of conceptual difficulties. For example (i) it is not clear why the Magnus effect should be relevant, since one would expect the vortices to move with the local superfluid velocity, (ii) they do not distinguish clearly between mean and local variables in their derivation, and (iii) in their treatment of roton-vortex line scattering, they (apparently) regard the force as acting on the vortex core, rather than being distributed throughout the fluid; however, the cross-section is calculated from an interaction term of the form $\underline{P} \cdot \underline{v}_s$, where $\underline{P}$ is the roton momentum, and $\underline{v}_s$ is the macroscopic vortex velocity field.)

For now, we wish only to consider those flows for which the detailed structure of the vortex core is unimportant. Thus flows in which the processes of vortex line creation and destruction and mutual friction are important are outside the scope of the present discussion. We are then considering only the hydrodynamic aspects of the vortex lines, and we may thus expect that the continuum hydrodynamic equations will be adequate for describing the flow. We are still assuming that the superfluid component is pointwise irrotational, so that the appropriate hydrodynamic equations are the Landau equations as discussed in II-A (we will not be concerned with dissipative processes in the present discussion, so we use the simpler perfect fluid equations of Chapter II). The equation for $\underline{v}_s$ may be written as (cf. equation (33))

$$\frac{\partial \underline{v}_s}{\partial t} + \underline{v}_s \cdot \nabla \underline{v}_s = -\nabla \Phi \qquad (366)$$

(where $\nabla \Phi = s \nabla T - \frac{\nabla P}{\rho} + x \nabla \frac{1}{2} w^2$).

In the case of moderate velocities and small variations of temperature in the flow, we also have

$$\text{div } \underline{v}_s = 0. \qquad (367)$$

Equations (366) and (367) are formally the same as the equations for ordinary incompressible perfect fluids. Thus possible solutions are flows in which the vorticity is concentrated in line singularities, and we are interested in solutions of this type where the number of vortex lines is large. (In accordance with the quantized vortex line theory, we are assuming here that the response of the superfluid to rotation is an array of vortex lines.) Here, as in the case of the turbulent flow of an ordinary fluid, we may expect that the detailed velocity distribution is irregular and fluctuating, even when the external conditions are steady. Since it is obviously not practicable to attempt a detailed calculation of such a velocity field, we wish to consider some sort of average velocity. Thus we assume that the motion can be separated into a mean flow $\langle \underline{v}_s \rangle$ and a fluctuating part $\underline{v}_s'$, and we assume that the averaging operation commutes with space and time differentiation (in some special flows, $\langle \underline{v}_s \rangle$ may be regarded as a space or time average; in more general flows,

we may consider $\langle \underline{v}_s \rangle$ to be a statistical average). Then, upon averaging, we obtain from (366) and (367) the equations

$$\frac{\partial \langle \underline{v}_s \rangle}{\partial t} + \langle \underline{v}_s \rangle \cdot \nabla \langle \underline{v}_s \rangle = -\nabla \langle \Phi \rangle - \operatorname{div} \{ \langle \underline{v}_s' \underline{v}_s' \rangle \} \qquad (368)$$

and

$$\operatorname{div} \langle \underline{v}_s \rangle = 0.$$

We may note first of all that $\operatorname{curl} \langle \underline{v}_s \rangle \neq 0$ in general; in fact, if $N(c)$ denotes the number of vortex lines passing through a circuit C, we have

$$\oint_C \underline{v}_s \cdot d\underline{s} = N(c) \cdot \frac{h}{m}$$

so that

$$\oint_C \langle \underline{v}_s \rangle \cdot d\underline{s} = \frac{h}{m} \langle N(c) \rangle$$

and

$$\iint_A \operatorname{curl} \langle \underline{v}_s \rangle \cdot \underline{n} \, d\sigma = \frac{h}{m} \langle N(c) \rangle. \qquad (370)$$

We now consider the correlation term $\langle \underline{v}_s' \underline{v}_s' \rangle$ appearing in equations (368) for the mean flow. This term is analogous to the Reynolds' stress in the equations for the turbulent flow of an ordinary fluid. However, the detailed structure of the vortex flows under consideration here would seem to bear little resemblance to the structure of turbulent flow. What is needed, in fact, is a statistical theory of vortex line flows; such a theory has not been developed, however, and, at present, we can only make a few conjectures about the nature of these flows. From the point of view of applications, it

would be very desirable to express the correlation $\langle \underline{v}_s' \underline{v}_s' \rangle$ in terms of mean flow quantities. If the vortex line array is very chaotic and if there is little correlation between the directions of neighboring filaments*, then, on the basis of the ordinary theory of turbulence, we might be skeptical about the possibility of relating the correlations to the mean flow in a universal manner. However, for more regular vortex flows, we might hope to relate $\langle \underline{v}_s' \underline{v}_s' \rangle$ to the mean flow in some simple manner. (An example of what we mean by a regular vortex flow" is perhaps helpful here. Hall and Vinen have conjectured that, for a helium filled circular cylinder in steady rotation, the vortex lines are parallel to the axis of rotation, the array is (nearly) uniform in spacing, and the lines are at rest when viewed in a reference frame rotating with the cylinder. In the present discussion, we would only conjecture that, on the average, the array has these properties; it seems likely that at any given instant there would be small, irregular deviations.) If we suppose that we are dealing with a regular vortex flow, in the sense described above, the neighboring vortex filaments will be approximately parallel, and will have the direction of the mean vorticity, $\mathrm{curl} \langle \underline{v}_s \rangle$. Roughly speaking then, the averaging implied in $\langle \underline{v}_s \rangle$ will in this case be over the possible positions--and not the possible directions--of the vortex filaments. Since the velocity fields induced by the vortices are transverse to the filament direction, we make the plausible assumption that

the correlation tensor $R_{ij} = \langle v_{si}' v_{sj}' \rangle$ is transverse to $\underline{\omega} = \mathrm{curl} \langle \underline{v}_s \rangle$ -

* Vinen [38] has considered such flows; he uses the terminology "superfluid turbulence" to describe such a flow.

that is, we assume that

$$\omega_i R_{ij} \equiv 0. \tag{371}$$

If R_{ij} is to satisfy (371), it must depend on ω_i. The simplest case is when R_{ij} depends only on ω_i (and possibly scalar quantities such as ρ_s, etc.); then R_{ij} must have the form

$$R_{ij} = \lambda \{ \omega^2 \delta_{ij} - \omega_i \omega_j \}, \tag{372}$$

and we assume this form here. We may relate the scalar λ to ϵ, the energy per unit length of a vortex line, as follows: consider a vortex tube (i.e., a vortex tube defined by the mean vorticity - curl $\langle \underline{v}_s \rangle$) of length δl, cross-section δA; the energy in this tube associated with the fluctuations is

$$\tfrac{1}{2} \rho_s R_{ii} \cdot \delta A \delta l = \tfrac{1}{2} \rho_s 2 \lambda \omega^2 \delta A \delta l ;$$

since the mean total length of vortex line in the tube is $\delta l \cdot \frac{\omega}{\kappa} \delta A$ (where $\kappa = \frac{h}{m}$ is the circulation around a single vortex), this energy may also be written as

$$\epsilon \, \delta l \cdot \delta A \cdot \frac{\omega}{\kappa} ,$$

so that we obtain

$$\lambda = \frac{\epsilon}{\rho_s \omega \kappa} . \tag{373}$$

If we introduce $\nu = \epsilon / \rho_s \omega \kappa$, then the equation for $\langle \underline{v}_s \rangle$ may be written as

$$\frac{\partial \langle \underline{v}_s \rangle}{\partial t} + \langle \underline{v}_s \rangle \cdot \nabla \langle \underline{v}_s \rangle = - \nabla \{ \Phi + \nu \omega \} + \mathrm{div} \{ \nu \frac{\underline{\omega}\,\underline{\omega}}{\omega} \}, \qquad (374)$$

and this is the equation given by Hall [13] to describe vortex waves.

At present, the above theory must be regarded as simply a conjecture; however, it is at least free of specific assumptions concerning the nature of the vortex flows (such as Hall and Vinen's assumption that the Magnus effect is relevant). As mentioned earlier, what is needed is a systematic development of a statistical theory of flows containing a large number of vortex lines, and the justification of the conjectures here must await the development of such a theory.

CHAPTER V - SUMMARY

A. Introduction

In the preceding chapters we have considered several theories of the hydrodynamics of liquid helium. At present, the experimental evidence is not sufficient to choose among (or reject all of) these theories. A conservative but definite statement concerning the comparison of theory with experiment is that (i) the Landau equations for reversible flows have been well verified experimentally for flows in which dissipative effects are unimportant and (ii) the Landau equations are inadequate to describe all of the experimental results. The central problem in the hydrodynamics of helium II is to determine how the Landau equations are to be modified in order to describe dissipative flows, and the theories discussed in the preceding chapters represent some of the current approaches to this problem. In this chapter, we present a summary of these theories in the form of a unified mathematical scheme which includes the theories discussed earlier as special cases.

In part B, the general thermodynamic and hydrodynamic equations applicable to all cases are given. In part C, the various special cases are discussed with the aid of the general framework of part B.

B. General Equations

1. Thermodynamics

In the discussion of the general equations, it is convenient to use the separate thermodynamic description for each component. Since this description has been discussed in detail in sections II-A-2 and IV-B-2, we only summarize the relevant formulae here. It is supposed that the total energy per unit volume E may be split into two parts--the superfluid part E_s and the normal fluid part E_n. Thus

$$E = E_n + E_s$$

where $$E_n = \frac{1}{2}\rho_n v_n^2 + \rho_n e_n, \tag{375}$$

and

$$E_s = \frac{1}{2}\rho_s v_s^2 + \rho_s e_s.$$

The specific internal energy e_n is supposed to be a function of ρ_n and s_n (s_n being the entropy per unit mass of normal fluid), and the specific internal energy e_s is supposed to be a function of ρ_s and $\omega \equiv |\text{curl } \underline{v}_s|$. The partial pressures P_n, P_s and the temperature T are defined by

$$de_n = \frac{P_n}{\rho_n^2} d\rho_n + T ds_n, \tag{376}$$

and

$$de_s = \frac{P_s}{\rho_s^2} d\rho_s + \lambda d\left(\frac{\omega}{\rho_s}\right).$$

The condition for equilibrium with respect to $\rho_n \rightleftarrows \rho_s$ transitions

is given by

$$\Phi_n - \Phi_s = \tfrac{1}{2} w^2, \qquad (\underline{w} = \underline{v}_n - \underline{v}_s)$$

where $$\Phi_n = e_n + \frac{p_n}{\rho_n} - T s_n \tag{377}$$

and

$$\Phi_s = e_s + \frac{p_s}{\rho_s} - \frac{\lambda \omega}{\rho_s} .$$

The thermodynamic quantities for the composite system are then given by

$$e = x e_n + (1-x) e_s ,$$

$$s = x s_n , \qquad p = p_n + p_s , \tag{378}$$

and the differential of e is given by

$$de = \frac{p}{\rho^2} d\rho + T ds + \lambda d\left(\frac{\omega}{\rho}\right) + \tfrac{1}{2} w^2 dx . \tag{379}$$

The differentials of the partial pressures may be expressed in terms of the thermodynamic variables of the composite system as follows:

$$\frac{d p_n}{p_n} = \frac{dp}{\rho} + \frac{(1-x)}{x} s dT + (1-x)\, d\tfrac{1}{2} w^2 - (1-x) \frac{\omega}{\rho_s} d\lambda ,$$

and (380)

$$\frac{d p_s}{p_s} = \frac{dp}{\rho} - s dT - x\, d\tfrac{1}{2} w^2 + x \frac{\omega}{\rho_s} d\lambda .$$

In the case that the internal energy does not depend on ω ,

we simply take $\lambda = 0$ in the above formulae.

It should be pointed out that the splitting of the energy by (375) and the thermodynamic formulae (376) are really additional assumptions concerning the nature of the two-fluid model. (Although these assumptions are consistent with the view of helium II as a "mixture" of two interpenetrating fluids which can, in the first approximation, move freely through one another, it is not clear at present that we have the right to ascribe an independent existence to each fluid to the extent of giving a separate thermodynamic description for each).

2. Hydrodynamic equations

The basic equations for reversible flows of helium II (in the case that the energy does not depend on the superfluid vorticity ($\lambda = 0$)) are the Landau equations. In more general theories, these equations are supplemented by some combination of the following: (i) additional stress tensors $\underline{\underline{\tau}}^{(n)}$, $\underline{\underline{\tau}}^{(s)}$ acting on the normal and supercomponents respectively, (ii) a mutual friction force per unit volume, $\underline{F}$, (iii) dependence of the thermodynamic internal energy on the superfluid vorticity $\omega = |\mathrm{curl}\ \underline{v}_s|$, (iv) an additional term in the entropy flux, $\underline{J}_s$* and (v) an additional term in the energy flux, $\underline{Q}'$. The general hydrodynamic equations may then be written in the following form:

Mass:

normal fluid: $\dfrac{\partial \rho_n}{\partial t} + \mathrm{div}\,(\rho_n \underline{v}_n) = \Gamma$

superfluid: $\dfrac{\partial \rho_s}{\partial t} + \mathrm{div}\,(\rho_s \underline{v}_s) = -\Gamma$

(Γ is the volume rate of conversion, and is determined by the flow conditions and the equilibrium condition (377).)

(381)

total: $\dfrac{\partial \rho}{\partial t} + \mathrm{div}\,(\rho_n \underline{v}_n + \rho_s \underline{v}_s) = 0$

Momentum:

normal fluid:
$$\frac{\partial}{\partial t}(\rho_n \underline{v}_n) + \mathrm{div}(\rho_n \underline{v}_n \underline{v}_n) + \nabla p_n - \Gamma \underline{v}_s = \mathrm{div}\,\underline{\underline{\tau}}^{(n)} - \underline{F} \qquad (382)$$

superfluid:
$$\frac{\partial}{\partial t}(\rho_s \underline{v}_s) + \mathrm{div}\,(\rho_s \underline{v}_s \underline{v}_s) + \nabla p_s + \Gamma \underline{v}_s = \mathrm{div}\left\{\frac{\lambda}{\omega}\,\underline{\omega}\underline{\omega}\right\} + \mathrm{div}\,\underline{\underline{\tau}}^{(s)} + \underline{F} \qquad (383)$$

total:
$$\frac{\partial}{\partial t}(\rho_n \underline{v}_n + \rho_s \underline{v}_s) + \mathrm{div}(\rho_n \underline{v}_n \underline{v}_n + \rho_s \underline{v}_s \underline{v}_s) + \nabla p = \mathrm{div}\left\{\underline{\underline{\tau}}^{(n)} + \frac{\lambda\,\underline{\omega}\underline{\omega}}{\omega} + \underline{\underline{\tau}}^{(s)}\right\} \qquad (384)$$

where for reasons which will become apparent, we have chosen

* Corresponding to ordinary thermal conduction

to write the total additional stress tensor in the supercomponent in the form $\frac{\lambda}{\omega}\underline{\omega}\,\underline{\omega} + \underline{\underline{\tau}}^{(s)}$. The equation for the superfluid vorticity may be written as

$$\text{vorticity:}\quad \frac{\partial \underline{\omega}}{\partial t} = \operatorname{curl}(\underline{v}_L \times \underline{\omega}) + \operatorname{curl} \underline{f}\,, \tag{385}$$

where

$$\underline{v}_L = \underline{v}_s + \frac{1}{\rho_s}\operatorname{curl}\lambda\underline{\nu}\,, \qquad \underline{\nu} = \frac{1}{\omega}\underline{\omega} \tag{386}$$

and

$$\underline{f} = \frac{\underline{F}}{\rho_s} + \frac{1}{\rho_s}\operatorname{div}\underline{\underline{\tau}}^{(s)}. \tag{387}$$

The scalar ω satisfies the equation

$$\text{scalar vorticity:}\quad \frac{\partial \omega}{\partial t} + \operatorname{div}\{\underline{v}_L\,\omega + \underline{\nu}\times\underline{f}\} = \omega\,\underline{\nu}\,\underline{\nu} : \operatorname{grad}\underline{v}_L + \underline{f}\cdot\operatorname{curl}\underline{\nu}. \tag{388}$$

The energy equation is

$$\text{energy:}\quad \frac{\partial}{\partial t}\{E_n + E_s\} + \operatorname{div}(\underline{Q}_o) = -\operatorname{div}\underline{Q}', \tag{389}$$

where E_n , E_s are the total energies per unit volume of the normal and supercomponents (cf. (375)), and where $\underline{Q}_o$, the energy flux vector of the Landau equations, is given by

$$\underline{Q}_o = (E_n + P_n)\underline{v}_n + (E_s + P_s)\underline{v}_s\,. \tag{390}$$

The hydrodynamic equations must imply an equation for the entropy of the form

$$\text{entropy:}\quad \frac{\partial}{\partial t}(\rho s) + \operatorname{div}(\rho s \underline{v}_n) = -\operatorname{div}\{\underline{\mathcal{F}}_s\} + \frac{R}{T}\,, \tag{391}$$

where R , the (local) volume rate of energy dissipation, is to be positive definite. From the equations (381)-(391) one may show that

$$R = \underline{\underline{\tau}}^{(n)} : \operatorname{grad} \underline{v}_n + \underline{\underline{\tau}}^{(s)} : \operatorname{grad} \underline{v}_L + \underline{F} \cdot (\underline{v}_n - \underline{v}_L) - \underline{\mathcal{J}}_s \cdot \nabla T -$$

$$- \operatorname{div} \left\{ \underline{Q}' - \left[-\underline{\underline{\tau}}^{(n)} \cdot \underline{v}_n - \left(\frac{\lambda}{\omega} \underline{\omega}\,\underline{\omega} + \underline{\underline{\tau}}^{(s)} \right) \cdot \underline{v}_L + \lambda \omega (\underline{v}_L - \underline{v}_s) + \right.\right.$$

$$\left.\left. + \lambda \omega \left(\frac{\underline{v} \times \underline{f}}{\omega} \right) + T \underline{\mathcal{J}}_s \right] \right\} . \qquad (392)$$

From (392), it is plausible to take

$$R = \underline{\underline{\tau}}^{(n)} : \operatorname{grad} \underline{v}_n + \underline{\underline{\tau}}^{(s)} : \operatorname{grad} \underline{v}_L + \underline{F} \cdot (\underline{v}_n - \underline{v}_L) - \underline{\mathcal{J}}_s \cdot \nabla T , \qquad (393)$$

and

$$\underline{Q}' = - \underline{\underline{\tau}}^{(n)} \cdot \underline{v}_n - \left(\frac{\lambda}{\omega} \underline{\omega}\,\underline{\omega} + \underline{\underline{\tau}}^{(s)} \right) \cdot \underline{v}_L + \lambda \omega (\underline{v}_L - \underline{v}_s) + \lambda \omega \left(\frac{\underline{v} \times \underline{f}}{\omega} \right) +$$

$$+ T \underline{\mathcal{J}}_s . \qquad (394)$$

In justification of the splitting of (392) into (393) and (394), we may note that (i) the equations (393) and (394) give the correct expressions for R and $\underline{Q}'$ for all of the various theories discussed in the previous chapters and (ii) it is possible to give a physical interpretation of the various terms in the expression (394) for $\underline{Q}'$. Of course, the fact that R must be positive definite--in conjunction with (392)--severely limits the possible forms of R and $\underline{Q}'$. In particular, in the case $\lambda = 0$ and $\underline{F} \equiv 0$, the splitting of (392) into (393) and (394) may be proved under certain mild conditions on $\underline{\underline{\tau}}^{(n)}$, $\underline{\underline{\tau}}^{(s)}$ and $\underline{\mathcal{J}}_s$ (cf. Appendix to Chapter III). In the general case,

however, we must rely on physical interpretation to assure us that $\underline{Q}'$ as given by (394) is the correct expression for the additional energy flux, and, hence, that (393) is the correct expression for the dissipation function.

The reason for writing the total additional superfluid stress tensor as $\frac{\lambda \underline{\omega}\underline{\omega}}{\omega} + \underline{\underline{\tau}}^{(s)}$ is clear from (393): only the part $\underline{\underline{\tau}}^{(s)}$ is a dissipative stress.

The hydrodynamic equations given above provide a general framework for a discussion of the various theories, and this will be taken up in the next section. Although the question of boundary conditions will be discussed in conjunction with each particular case in the next section, there is one general point regarding boundary conditions which we wish to make here. Suppose we consider a helium II--solid boundary; let $\underline{Q}|_{\text{rest frame}}$ denote the total energy flux in the helium, evaluated in the rest frame of the solid. Then let

$$\underline{Q}|_{\text{rest frame}} = \underline{Q}_1 + \underline{J}_s T \; , \tag{395}$$

that is, $\underline{Q}_1$ is everything in the energy flux except the entropy flux term. In the solid, there will be a heat current $\underline{H}$, and an associated entropy flux $\underline{H}/T$. The conservation of energy requires that

$$\underline{Q}_1 \cdot \underline{n} + T \underline{J}_s \cdot \underline{n} = \underline{H} \cdot \underline{n} \; , \tag{396}$$

where, for definiteness, we take $\underline{n}$ to be the unit normal pointing into the fluid. If $\underline{Q}_1 \cdot \underline{n} = 0$, then the entropy flux is continuous; if $\underline{Q}_1 \cdot \underline{n} \neq 0$, however, then the entropy flux is

not continous, and the interface acts (formally) as an entropy source--that is, there is extensive dissipation at the boundary. The net production of entropy at the interface must be positive; the entropy flowing away from the interface into the fluid is $\underline{J}_s \cdot \underline{n}$, and the entropy flowing away from the interface into the solid is $-\underline{H} \cdot \underline{n}/T$, thus we must require

$$\underline{J}_s \cdot \underline{n} - \frac{\underline{H} \cdot \underline{n}}{T} \geq 0 \qquad (397)$$

or

$$\underline{Q}_1 \cdot \underline{n} \leq 0.$$

Thus whatever the boundary conditions imposed, they must be such that (397) is satisfied. In the case of an ordinary fluid (with the usual no-slip boundary conditions), the quantity $\underline{Q}_1 \cdot \underline{n} = 0$; in several of the hydrodynamic theories of helium however, the quantity $\underline{Q}_1 \cdot \underline{n}$ does not vanish, and the restriction (397) must be satisfied (and, in fact, in some of the theories, one may use the condition (397) to determine an appropriate boundary condition). Further discussion of this point will be given in the next section in connection with each particular case. (In the above analysis, we have tacitly assumed that the temperature is continuous at the interface. If we include the possibility of a temperature jump (Kapitza effect), then (397) must be replaced by

$$\underline{Q}_1 \cdot \underline{n} \leq \frac{T_{wall} - T_{He}}{T_{wall}} \underline{H} \cdot \underline{n} ;$$

according to the Kapitza effect relation, $T_{wall} - T_{He} = A_K (\underline{H} \cdot \underline{n})$,

where $A_K > 0$ is the thermal resistance of the boundary, and the condition becomes

$$A_K \frac{(\underline{H}\cdot\underline{n})^2}{T_{wall}} - \underline{Q}_1 \cdot \underline{n} \geq 0.$$

If we assume that the Kapitza effect and the dissipative effects included in $\underline{Q}_1$ contribute independently to the dissipation, then we again require $\underline{Q}_1 \cdot \underline{n} \leq 0$; in particular, this will be true if the boundary conditions are such that $\underline{Q}_1 \cdot \underline{n}$ is independent of the magnitude of the heat current, $\underline{H}\cdot\underline{n}$.)

C. Equations for Particular Theories

1. Reversible flows in the case $\lambda = 0$.

In the case that the internal energy doesn't depend on the superfluid vorticity, the Landau equations are recovered for $\underline{\underline{\tau}}^{(n)}$, $\underline{\underline{\tau}}^{(s)}$, $\underline{F}$ and $\underline{\mathcal{F}}_s$ all zero. In the present general scheme, the Landau equations appear as only a possible set of perfect fluid equations. However, one may deduce the Landau equations from the conservation laws and the requirement that $\operatorname{curl} \underline{v}_s \equiv 0$ (cf. II-A-1; we may also note that one may deduce the Landau equations from the conservation laws and the less stringent requirement that the superfluid vorticity must move with the superfluid velocity--that is, that $\frac{\partial}{\partial t} \operatorname{curl} \underline{v}_s = \operatorname{curl}\{\underline{v}_s \times \operatorname{curl} \underline{v}_s\}$). An alternative approach is given in section II-A-2, where it is shown that one may obtain the Landau equations (without the restriction $\operatorname{curl} \underline{v}_s = 0$) from the assumption that the two components are two independent perfect fluids coupled only by the $\rho_n \rightleftarrows \rho_s$ transitions, with the transitions taking place at constant $\underline{v}_s$. However, these two deductions of the Landau equations depend on assumptions whose validity (even in the perfect fluid theory) is not evident. In particular, we see from equation (393) for the dissipation function, that a mutual friction force $\underline{F}$ which is perpendicular to $\underline{w} = \underline{v}_n - \underline{v}_s$ does not lead to any dissipation ($\underline{v}_\perp \equiv \underline{v}_s$ in the case $\lambda = 0$) and may thus be included in the perfect fluid equations. Indeed, the equations proposed by Lin (cf. II-B-4) differ from the Landau equations by just a term (namely, $\underline{F} = \rho x(1-x) \underline{w} \times \operatorname{curl} \underline{v}_s$).

In general, then the Landau equations may be supplemented by a (non-dissipative) mutual friction force perpendicular to $\underline{w}$ *. The boundary conditions to be satisfied at a wall moving with velocity $\underline{U}$ are simply that $\underline{n} \cdot (\underline{v}_n - \underline{U}) = 0$ and $\underline{n} \cdot (\underline{v}_s - \underline{U}) = 0$, where $\underline{n}$ is the normal to the wall.

* It is natural to ask if the addition of a mutual friction force perpendicular to $\underline{w}$ is the most general modification of the Landau equations allowed within the framework of the perfect fluid theory. The answer is yes, provided it is assumed that (i) the stress tensors $\underline{\underline{\tau}}^{(n)}$, $\underline{\underline{\tau}}^{(s)}$ are independent of the gradients of the macroscopic quantities, and (ii) the terms $\underline{\underline{\tau}}^{(n)} : \mathrm{grad}\, \underline{v}_n + \underline{\underline{\tau}}^{(s)} : \mathrm{grad}\, \underline{v}_s$ and $\underline{F} \cdot \underline{w}$ in the dissipation function are separately zero. The assumption (i) is a reasonable assumption in a perfect fluid theory, and the assumption (ii) (corresponding to the physical statement that the volume momentum transfer processes represented by $\underline{F}$ are independent of the processes represented by $\underline{\underline{\tau}}^{(n)}$, $\underline{\underline{\tau}}^{(s)}$) excludes the possibility of stress tensors $\underline{\underline{\tau}}^{(n)}$, $\underline{\underline{\tau}}^{(s)}$ depending on $\underline{w}$ and a related mutual friction $\underline{F}$ specified in such a way that $R = 0$ (cf. eq. (50), II-A-1, for an example).

2. Dissipative flows in the case $\lambda = 0$

The equations for dissipative processes are obtained by determining the quantities $\underline{\underline{\tau}}^{(n)}$, $\underline{\underline{\tau}}^{(s)}$, $\underline{F}$ and $\underline{\mathcal{J}}_s$ so that the dissipation function (cf. (393)) is positive definite. Under the reasonable assumption that the contributions of $\underline{F}$ and the group $\{\underline{\underline{\tau}}^{(n)}, \underline{\underline{\tau}}^{(s)}, \underline{\mathcal{J}}_s\}$ to the dissipation function are independently positive definite, one may study separately the transport terms $\underline{\underline{\tau}}^{(n)}$, $\underline{\underline{\tau}}^{(s)}$, $\underline{\mathcal{J}}_s$ and the mutual friction $\underline{F}$.

We consider the quantities $\underline{\underline{\tau}}^{(n)}$, $\underline{\underline{\tau}}^{(s)}$ and $\underline{\mathcal{J}}_s$ first. In the simplest case, these quantities are taken to be linear functions of the rate of strain tensors $e_{ij}^{(n)}$, $e_{ij}^{(s)}$ and the temperature gradient ∇T, and it is assumed that the helium II is isotropic with respect to the transport processes; then the resulting equations are those of Lin's theory (III-C-2)--that is, $\underline{\mathcal{J}}_s = -\frac{K}{T}\nabla T$ (thermal conduction) and the stress-rate of strain relations are characterized by four shear exchange coefficients (and also four "bulk viscosity" coefficients in the general case). There are several ways in which the theory may be generalized. First, we may note that the conservation of angular momentum requires only that the sum $\underline{\underline{\tau}}^{(n)} + \underline{\underline{\tau}}^{(s)}$ be symmetric; thus $\underline{\underline{\tau}}^{(n)}$, $\underline{\underline{\tau}}^{(s)}$ could have anti-symmetric parts adding to zero, and the effect of including such terms has been considered in Chapter III (III-C-2). The effective body force associated with the anti-symmetric stress is proportional to $\mathrm{curl}\,(\underline{v}_n - \underline{v}_s)$, and there is a (dissipative) volume exchange of angular momentum between the two components induced by the

the stress. Another way in which the theory may be generalized is to take into account the possibility that, because of the relative motion of the two fluids, the transport processes may not be isotropic (or, to put it another way, the viscosity tensors and the thermal conductivity tensor may depend on the direction of the relative velocity $\underline{w}$); the resulting general equations are very complicated and have not been studied in detail.

The mutual friction $\underline{F}$ must be such that $\underline{w}\cdot\underline{F} \geq 0$, and we can say little more about $\underline{F}$ until we know what vectors $\underline{F}$ may depend on. In the simplest case, $\underline{F}$ depends on $\underline{w}$ alone, and we have $\underline{F} = C_0\underline{w}$, (where $C_0 > 0$ may depend on w^2 as well as the scalar thermodynamic variables) and this includes the mutual friction originally proposed by Gorter and Mellink [8] in 1949. Another case of interest is when $\underline{F}$ depends on $\underline{w}$ and $\underline{\omega} = \mathrm{curl}\, \underline{v}_s$; then the most general form is $F = C_0\underline{w} + C_1\, \underline{w}\times\underline{\omega} + C_2\, \underline{w}\times[\underline{w}\times(\underline{\omega}\times\underline{w})]$, (where $C_0 > 0$ and C_0 , C_1 , C_2 may depend on the scalars w^2 , ω^2 and $(\underline{w}\cdot\underline{\omega})^2$) and this includes both the Gorter-Mellink mutual friction and the mutual friction proposed by Hall and Vinen [12] (in the case of "straight vortex lines") as special cases.

The boundary conditions to be satisfied are of some interest. When there is a heat current between a bounding wall and the helium II, the condition $\rho ST\, \underline{v}_n\cdot\underline{n} = \underline{H}\cdot\underline{n}$, where $\underline{n}$ is the normal to the wall, is usually imposed (with $\underline{v}_s\cdot\underline{n}$ being determined from the requirement that the total mass flux vanish). However, it can be shown that in principle this

condition leads to a violation of the law of increase of entropy (cf. (397) and preceding discussion), and that the boundary conditions (even in the case of a heat current through the wall) should be $\underline{n}\cdot\underline{v}_n=0$ and $\underline{n}\cdot\underline{v}_s=0$. (In practice, the boundary condition $\rho s T \underline{v}_n\cdot\underline{n} = \underline{H}\cdot\underline{n}$ is an excellent approximation; the internal convection mode of heat transfer in helium II in conjunction with the boundary conditions $\underline{n}\cdot\underline{v}_n=0$, $\underline{n}\cdot\underline{v}_s=0$ leads to a very thin boundary layer through which $\underline{v}_n\cdot\underline{n}$ rapidly changes from zero to $\underline{H}\cdot\underline{n}/\rho s T$ (cf. III-B-3; also II-A-1 and III-C-4).) Since the equations for $\underline{v}_n$, $\underline{v}_s$ now contain second spatial derivatives, one also needs conditions on the tangential components of $\underline{v}_n$, $\underline{v}_s$. The condition on $\underline{v}_n$ is usually taken to be $(\underline{v}_n-\underline{U})_{\text{tangential}}=0$ ($\underline{U}$ is the wall velocity). For the quantity $(\underline{v}_s-\underline{U})_{\text{tangential}}$, Lin has proposed the condition

$$(\underline{n}\cdot\underline{\underline{\tau}}^{(s)})_{\text{tangential}} = \beta\,|\underline{v}_s-\underline{U}|^2(\underline{v}_s-\underline{U})_{\text{tangential}} \qquad (398)$$

where β is independent of $(\underline{v}_s-\underline{U})$ but may depend on the thermodynamic variables (cf. III-C-3 and III-C-4 for a discussion of this boundary condition). (From the general requirement (397), one may deduce the condition $\underline{n}\cdot\underline{\underline{\tau}}^{(s)}\cdot(\underline{v}_s-\underline{U})\geq 0$; thus in the event of superfluid slip, the general form of the condition (398)--i.e., a relation between $\underline{n}\cdot\underline{\underline{\tau}}^{(s)}\big|_{\text{tangential}}$ and $(\underline{v}_s-\underline{U})_{\text{tangential}}$ --is already determined by the law of increase of entropy. The rate of energy dissipation at the interface associated with the superfluid slip is $\beta\,|\underline{v}_s-\underline{U}|^4$ per unit area.)

3. Reversible flows for $\lambda \neq 0$

When the internal energy depends on the superfluid vorticity, the simplest hydrodynamic equations are obtained from the general equations by taking $\underline{\underline{\tau}}^{(n)}$, $\underline{\underline{\tau}}^{(s)}$, $\underline{F}$ and $\underline{\mathcal{F}}_s$ all zero. We will limit the discussion of this section to that case.

From equation (383) for the superfluid momentum, we see that there is an additional stress $\frac{\lambda}{\omega}\,\underline{\omega}\,\underline{\omega}$ acting on the supercomponent. From equation (385), it is evident that the superfluid vorticity no longer moves with the superfluid, but moves with a velocity $\underline{v}_L = \underline{v}_s + \frac{1}{\rho_s}\,\mathrm{curl}\,\lambda\underline{\gamma}$. (In the present case, $\underline{f} = \frac{\underline{F}}{\rho_s} + \frac{1}{\rho_s}\,\mathrm{div}\,\underline{\underline{\tau}}^{(s)} \equiv 0$.) Equation (388) for the scalar vorticity $\omega = |\mathrm{curl}\,\underline{v}_s|$ also illustrates this; the term $\omega\,\underline{\gamma}\,\underline{\gamma} : \mathrm{grad}\,\underline{v}_L$ on the right-hand side of (388) represents the production of vorticity by stretching of the vortex lines. The extra terms in the energy flux (eg. (394)) are

$$\underline{Q}' = -\left\{\frac{\lambda}{\omega}\,\underline{\omega}\,\underline{\omega}\right\}\cdot\underline{v}_L + \lambda\omega(\underline{v}_L - \underline{v}_s). \tag{399}$$

We may interpret the first term in (399) as the rate of working of the extra stresses, provided we assume that the stresses do work on the vorticity velocity $\underline{v}_L$ rather than $\underline{v}_s$; the second term is a correction to the convection term $\rho_s e_s \underline{v}_s$, expressing the fact that the vorticity moves with velocity $\underline{v}_L$ rather than $\underline{v}_s$. Alternatively, we may write (399) as

$$\underline{Q}' = -\left\{\frac{\lambda}{\omega}\,\underline{\omega}\,\underline{\omega}\right\}\cdot\underline{v}_s + \lambda\omega(\underline{v}_L - \underline{v}_s)_{\text{transverse}}. \tag{400}$$

where

$$(\underline{v}_L - \underline{v}_s)_{\text{transverse}} = \underline{\nabla} \times \{(\underline{v}_L - \underline{v}_s) \times \underline{\nabla}\}; \tag{401}$$

thus we may also regard the vorticity as convected with the velocity $\underline{v}_s + (\underline{v}_L - \underline{v}_s)_{\text{transverse}}$ rather than $\underline{v}_L$, and the extra stresses as working on $\underline{v}_s$ rather than $\underline{v}_L$.

The boundary conditions at a wall moving with velocity $\underline{U}$ may be taken as $(\underline{v}_s - \underline{U}) \cdot \underline{n} = 0$ and $(\underline{v}_n - \underline{U}) \cdot \underline{n} = 0$. Since the equation for $\underline{v}_s$ now contains second space derivatives, further boundary conditions are needed. Although it is not obvious what further boundary conditions to impose, the general criterion (397) yields in the present case (of no dissipation)

$$\lambda \omega \underline{n} \cdot (\underline{v}_L - \underline{U})_{\text{transverse}} = 0, \tag{402}$$

where

$$(\underline{v}_L - \underline{U})_{\text{transverse}} = \underline{\nabla} \times \{(\underline{v}_L - \underline{U}) \times \underline{\nabla}\}.$$

Since two further boundary conditions are needed, the simplest choice (satisfying (402)) is

$$(\underline{v}_L - \underline{U})_{\text{transverse}} = 0. \tag{403}$$

If one regards the vortex lines as moving with velocity $\underline{v}_L$, the condition (403) is equivalent to the statement that the point of attachment of a given line to the boundary remains fixed relative to the boundary.

4. Dissipative flows for $\lambda \neq 0$

In the general equations, there will be dissipative terms $\underline{\underline{\tau}}^{(n)}$, $\underline{\mathcal{I}}_s$ and $\underline{F}$. (It has not been found possible to include a dissipative stress tensor $\underline{\underline{\tau}}^{(s)}$ in the supercomponent in the case $\lambda \neq 0$; the reasons for this will be discussed below. For now we assume that $\underline{\underline{\tau}}^{(s)} = 0$.) Again, we make the plausible assumption that the contributions to the dissipation function from the volume mutual friction $\underline{F}$ and the transport terms $\underline{\mathcal{I}}_s$, $\underline{\underline{\tau}}^{(n)}$ are independently positive definite.

The quantities $\underline{\underline{\tau}}^{(n)}$ and $\underline{\mathcal{I}}_s$ are to be determined from the condition (cf. (393))

$$\underline{\underline{\tau}}^{(n)} : \operatorname{grad} \underline{v}_n - \underline{\mathcal{I}}_s \cdot \nabla T \geq 0.$$

In the simplest case, $\underline{\mathcal{I}}_s = -\frac{K}{T} \nabla T$ (K being the thermal conductivity) and $\underline{\underline{\tau}}^{(n)}$ is expressed in terms of the normal fluid rate of strain tensor through a shear viscosity and a bulk viscosity. As in the discussion of section 2, we may consider the more general case in which the viscosity tensor and the thermal conductivity tensor depend on the direction of the vector $\underline{w}$. (Because of the distinctive role of the quantity $\underline{\omega} = \operatorname{curl} \underline{v}_s$, one could also consider the possibility that the viscosity and conductivity tensor depend on the direction of $\underline{\omega}$ as well.)

The mutual friction $\underline{F}$ must be determined so that $\underline{F} \cdot (\underline{v}_n - \underline{v}_L) \geq 0$; again, we can say little more about $\underline{F}$ until we know what vectors $\underline{F}$ may depend on. Certainly $\underline{F}$ will depend on $\underline{v}_n - \underline{v}_L$; it

also seems reasonable to suppose that $\underline{F}$ depends on $\underline{\omega} = \text{curl}\, \underline{v}_s$ and $\underline{w} = \underline{v}_n - \underline{v}_s$ as well. The most general $\underline{F}$ depending on these three vectors and satisfying $\underline{F} \cdot (\underline{v}_n - \underline{v}_L) \geq 0$ is given by

$$\underline{F} = C_0 (\underline{v}_n - \underline{v}_L) + (\underline{v}_n - \underline{v}_L) \times \Big\{ (\underline{v}_n - \underline{v}_L) \times \Big[C_1 \underline{w} + C_2 \underline{\omega} \times \underline{w} + C_3 \underline{\omega} \times (\underline{v}_n - \underline{v}_L) + C_4 \underline{\omega} \times (\underline{\omega} \times \underline{w}) + C_5 \underline{\omega} \times (\underline{\omega} \times [\underline{v}_n - \underline{v}_L]) \Big] \Big\}. \tag{404}$$

C_0 must be positive, while the other 5 scalar coefficients are arbitrary. (In general, the C's will depend on the scalar invariants of $\underline{v}_n - \underline{v}_L$, $\underline{w}$ and $\underline{\omega}$, as well as the thermodynamic variables.) The expression (404) includes--as special cases--the mutual friction given by Bekarevich and Khalatnikov [3] (cf. IV-B-1) and the mutual friction given by Hall [14]. The mutual friction leads to an extra term in the energy flux (cf. (394)), namely

$$\lambda \omega \left\{ \frac{\underline{\nu} \times \underline{F}}{\omega \rho_s} \right\}.$$

From the equation (388) for the vorticity $\omega = |\text{curl}\, \underline{v}_s|$, we see that the mutual friction affects the transport of vorticity and thus leads to an additional term in the energy flux. In fact, it is convenient to define a modified vorticity velocity $\tilde{\underline{v}}_L$ by

$$\tilde{\underline{v}}_L = \underline{v}_L + \frac{\underline{\nu} \times \underline{F}}{\omega \rho_s}. \tag{405}$$

Then the energy flux (394) may be written as

$$\underline{Q}' = -\underline{\underline{\tau}}^{(n)} \cdot \underline{v}_n - \frac{\Lambda}{\omega} \underline{\omega}\, \underline{\omega} \cdot \tilde{\underline{v}}_L + \lambda \omega (\tilde{\underline{v}}_L - \underline{v}_s) + T \underline{\mathcal{J}}_s. \tag{406}$$

For the boundary conditions to be satisfied at a solid boundary, we have, as before, $\underline{n}\cdot(\underline{v}_s - \underline{U})$ and $\underline{v}_n - \underline{U} = 0$. Since the equation for $\underline{v}_s$ contains second spatial derivatives, two further conditions are needed. The general criterion (397) yields the condition

$$\lambda \omega \underline{n} \cdot (\tilde{\underline{v}}_L - \underline{U})_{transverse} \leq 0. \tag{407}$$

One possible boundary condition is

$$(\tilde{\underline{v}}_L - \underline{U})_{transverse} = 0 \; ; \tag{408}$$

more generally, we may allow a slip provided (407) is satisfied. The form of the boundary condition in case of a slip depends on what vectors $(\tilde{\underline{v}}_L - \underline{U})_{transverse}$ may depend on. Thus if we assume that $(\tilde{\underline{v}}_L - \underline{U})_{transverse}$ (at the wall) may depend on $\underline{\nabla}$ and $\underline{n}$, the most general boundary condition satisfying (407) is

$$(\tilde{\underline{v}}_L - \underline{U})_{transverse} = \varepsilon_1 \underline{\nabla} \times (\underline{\nabla} \times \underline{n}) + \varepsilon_2 \underline{n} \times \underline{\nabla} \tag{409}$$

where $\varepsilon_1 \geq 0$, and ε_1 , ε_2 may depend on $(\underline{\nabla}\cdot\underline{n})^2$ as well as scalar thermodynamic variables. One may obtain more general slip boundary conditions by allowing the quantity $(\tilde{\underline{v}}_L - \underline{U})_{transverse}$ (evaluated at the wall) to depend on $\underline{w}$, as well as $\underline{\nabla}$ and $\underline{n}$.

Finally, we discuss briefly the case when $\underline{\underline{\tau}}^{(s)} \neq 0$. In this case, it is possible to develop a set of hydrodynamic equations similar to Lin's equations for dissipative processes; however, from (393) we see that the relevant rate of strain tensors are those formed from $\underline{v}_n$ and $\underline{v}_L$, rather than $\underline{v}_n$ and

$\underline{v}_s$. The difficulty occurs with the boundary conditions; from the general criterion (397), we must have (taking $\underline{v}_n - \underline{U} = 0$, $\underline{n} \cdot (\underline{v}_s - \underline{n}) = 0$)

$$-\underline{n} \cdot \underline{\underline{\tau}}^{(s)} \cdot (\underline{v}_\perp - \underline{U}) + \underline{n} \cdot \lambda\omega (\underline{v}_\perp - \underline{U})_{\text{transverse}} + \underline{n} \cdot \lambda\omega \left(\frac{\underline{\nu} \times \underline{f}}{\omega} \right) \leq 0; \quad (410)$$

in the present case, however, the quantity $\underline{f} = \frac{\underline{F}}{\rho_s} + \frac{1}{\rho_s} \operatorname{div} \underline{\underline{\tau}}^{(s)}$ contains fourth derivatives of $\underline{v}_s$ (since $\underline{v}_\perp$ already contains second derivatives of $\underline{v}_s$). Because of the condition (410), the boundary conditions must necessarily involve the term $\lambda\omega \left(\frac{\underline{\nu} \times \underline{f}}{\omega} \right)$, and thus must involve the fourth derivative of $\underline{v}_s$. Thus although the equations may be consistently developed in this case, the difficulty with the boundary conditions seems to preclude the possibility of simultaneously having $\underline{\underline{\tau}}^{(s)} \neq 0$, $\lambda \neq 0$.

D. Further Work

Although there are a number of points connected with the present work which deserve further investigation, there are three in particular which we wish to mention briefly here.

The first of these is the difficulty with the variational principle in Lin's one-fluid model (cf. II-B-2; the variational principle entails a restriction on the quantity $\operatorname{curl} \underline{c}$, or, in terms of the two-fluid model, on $\operatorname{curl}(\underline{v}_n - \underline{v}_s)$). It would be highly desirable to resolve this difficulty so that a unique set of equations could be obtained from the one-fluid model. As discussed in Chapter II (II-B-2; II-C), it is possible that the difficulty stems from the fact that the Hamilton's principle of mechanics was used to obtain the equations of motion, whereas we know that even for reversible flows of helium II energy transfers of an essentially thermal nature may take place.

The second point is concerned with the problem of rotation of helium II. As discussed in Chapter IV (IV-B-3), it is possible that it is an oversimplification to allow the internal energy of helium II to depend directly on the vorticity $\underline{\omega} = \operatorname{curl} \underline{v}_s$; one might expect that there will be an internal parameter describing the microscopic rotational state of helium II and that for steady (or slowly varying) flows this internal paramenter is approximately equal to $\underline{\omega} = \operatorname{curl} \underline{v}_s$. Such a theory has been developed for an ordinary fluid (cf. e.g. [4, p.304]), and it would be of great interest to see whether it is capable of explaining some the peculiar experimental results

for helium II in rotation.

The third point is also concerned with the rotation of helium II, and is simply that it might be of some interest to develop a statistical theory to describe flows containing a large number of vortex filaments (cf. IV-C for some discussion of this).

CONCLUSION

One of the central issues at present in the theory of liquid helium II is the question of superfluid rotation. This is a question which cannot be resolved by continuum principles; rather, one must take a definite view point on this matter in order to develop a hydrodynamic theory. Indeed, the principal point on which the theories discussed in the preceding chapters differ is in their treatment of the superfluid rotation. Ultimately, the choice among the various theories must be made on the basis of comparison with experiment. However, it is not entirely a straightforward matter to obtain a definite set of hydrodynamic equations on the basis of qualitative assumptions about the nature of the superfluid rotation; because of this, it is sometimes difficult to gauge the relevance to the problem of superfluid rotation of particular experimental evidence. A principal aim of the present work has been to attempt to clarify the steps leading from the fundamental assumptions to the hydrodynamic equations in the various theories, both by critically examining the various derivationa of the hydrodynamic equations and, in some cases, by offering alternative derivations.

APPENDIX--APPLICATIONS OF LIN'S THEORY OF DISSIPATIVE PROCESSES

We give here an analysis of some simple, specific flow problems on the basis of Lin's theory of dissipative processes (as discussed in section III-C). These examples serve to illustrate some of the features of Lin's theory; also, the comparison of theoretical results with experiment leads to some information about the viscosity coefficients and the coefficient β in the nonlinear boundary condition. We use the approximate form of the hydrodynamic equations appropriate for moderate velocities and small temperature variations throughout the flow. These equations are

$$\frac{\partial \underline{v}_n}{\partial t} + \underline{v}_n \cdot \nabla \underline{v}_n = -\frac{1}{\rho} \nabla p - \frac{(1-x)}{x} s \nabla T - \frac{(1-x)}{2} \nabla w^2 + \frac{\mu_{nn}}{\rho_n} \nabla^2 \underline{v}_n + \frac{\mu_{ns}}{\rho_n} \nabla^2 \underline{v}_s \quad (A\text{-}1)$$

$$\frac{\partial \underline{v}_s}{\partial t} + \underline{v}_s \cdot \nabla \underline{v}_s = -\frac{1}{\rho} \nabla p + s \nabla T + x \nabla \frac{1}{2} w^2 + \frac{\mu_{sn}}{\rho_s} \nabla^2 \underline{v}_n + \frac{\mu_{ss}}{\rho_s} \nabla^2 \underline{v}_s , \quad (A\text{-}2)$$

and

$$\operatorname{div} \underline{v}_n = 0 , \quad \operatorname{div} \underline{v}_s = 0. \quad (A\text{-}3)$$

In this approximation, ρ , x , s , μ_{nn}, μ_{ns}, μ_{sn}, μ_{ss} are all taken as constant throughout the flow, and the pressure p and temperature T are then "hydrodynamic" variables. In the problems discussed here, it will not be necessary to consider the case of a heat flux through a boundary wall; thus the boundary conditions (271) may be written as

$$\underline{v}_n = \underline{U}_{wall},$$

$$\underline{n} \cdot (\underline{v}_s - \underline{U}_{wall}) = 0,$$

$$\left\{ \tau_{ij}^{(s)'} n_i - \beta \left| \underline{v}_s - \underline{U}_{wall} \right|^2 (\underline{v}_s - \underline{U}_{wall})_j \right\}_{tangential} = 0, \qquad (A-4)$$

where

$$\tau_{ij}^{(s)'} = 2\mu_{sn} e_{ij}^{(n)} + 2\mu_{ss} e_{ij}^{(s)},$$

and $\underline{n}$ is the unit normal pointing into the fluid. In section A we consider simple channel flows, in section B we consider the steady flow between rotating cylinders and in section C we consider the Andronikashvili experiment. In section D, some preliminary results on the numerical values of the dissipative coefficients are discussed.

A. Simple Channel Flows

We consider in some detail here the flow induced in an infinitely long channel by constant temperature and pressure gradients along the channel. The height of the channel is 2h (cf. Fig. 3), and it is assumed that the depth is great enough so that the flow may be taken as two-dimensional. Thus we assume that the pressure and temperature are given by $p = -\bar{p}z + p_o$, $T = -\bar{T}z + T_o$, where $\bar{p}$, $\bar{T}$ are the constant gradients, and we look for solutions of the form $\underline{v}_n = v_n(y)\,\underline{e}_z$ and $\underline{v}_s = v_s(y)\,\underline{e}_z$. From the equations (A-1), (A-2), it follows that $v_n(y)$, $v_s(y)$ are quadratic functions of y. The boundary conditions to be satisfied in this case are

$$v_n(\pm h) = 0,$$

and

$$\left\{\mu_{sn}\frac{dv_n}{dy} + \mu_{ss}\frac{dv_s}{dy}\right\}_{y=\pm h} = \mp\beta\left\{v_s(\pm h)\right\}^3.$$

The velocity profiles which satisfy these boundary conditions and the equations are given by

$$v_n(y) = C_n(h^2 - y^2) \tag{A-5}$$

and

$$v_s(y) = C_s(h^2 - y^2) + D_s, \tag{A-6}$$

where

$$C_n = \frac{1}{2\Delta\mu}\left\{\bar{p}\left[x\mu_{ss} - (1-x)\mu_{ns}\right] + \rho s(1-x)\bar{T}\left[\mu_{ns} + \mu_{ss}\right]\right\}, \tag{A-7}$$

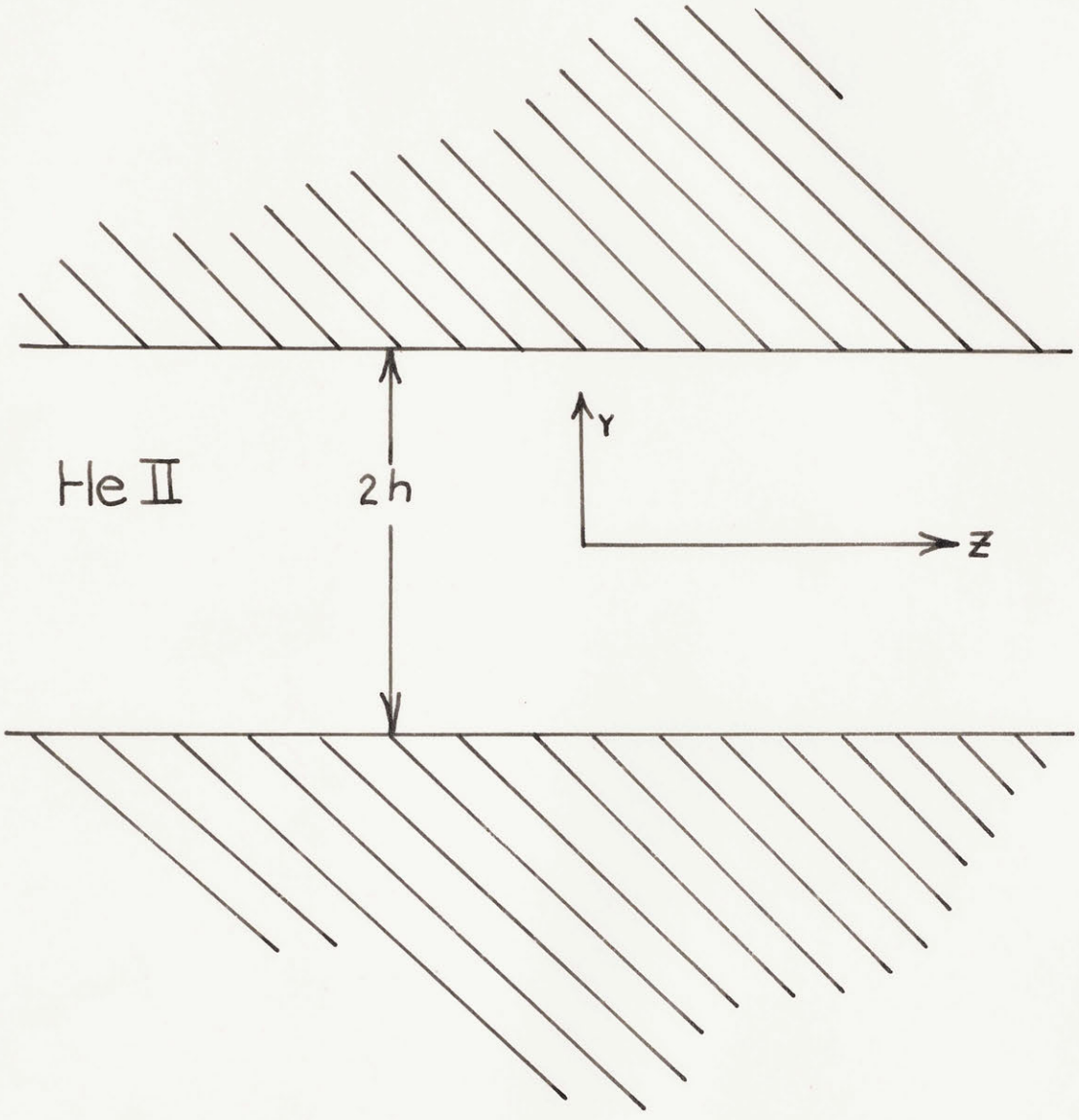

Figure 3

$$C_s = \frac{1}{2\Delta\mu}\left\{\bar{P}\left[(1-x)\mu_{nn} - x\mu_{sn}\right] - \rho s(1-x)\bar{T}\left[\mu_{sn}+\mu_{nn}\right]\right\}, \qquad (A\text{-}8)$$

$$D_s = \left\{\frac{h(1-x)(\bar{P} - \rho s\bar{T})}{\beta}\right\}^{1/3}, \qquad (A\text{-}9)$$

where

$$\Delta\mu = \mu_{nn}\mu_{ss} - \mu_{ns}\mu_{sn}. \qquad (A\text{-}10)$$

(We should note that, strictly speaking, the temperature must also contain a y--dependent term; from the equations, one may show that $T = T_o - \bar{T}z + T'(y)$, where $T'(y) = -\frac{x}{s}\frac{1}{2}w^2(y)$. However, even for velocities of the order of $10\frac{cm}{sec}$, the total variation of T across the channel is only of the order $10^{-5}{}^{o}K$. In most cases of practical interest, this term will be unimportant, and we shall ignore it from here on). The mean velocities are given by

$$\bar{v}_n = \frac{2}{3}h^2 C_n, \qquad (A\text{-}11)$$

and

$$\bar{v}_s = \frac{2}{3}h^2 C_s + D_s, \qquad (A\text{-}12)$$

the heat current is

$$W = \rho s T \bar{v}_n = \frac{2}{3}h^2 \rho s T C_n, \qquad (A\text{-}13)$$

and the mass flux is

$$M = \frac{4}{3} h^3 \rho \left(x C_n + (1-x) C_s \right) + 2h\rho (1-x) D_s . \qquad \text{(A-14)}$$

It is clear from (A-6) and (A-5) that the nonlinear boundary condition allows a slip of the superfluid at the boundary; the magnitude of the slip velocity is determined by the "thermomechanical head" $\bar{p} - \rho s \bar{T}$.

It is well-verifed experimentally that the heat current in channel flows is proportional to the pressure gradient. It is clear from (A-7) and (A-13) that the present theory predicts this provided

$$\mu_{ns} + \mu_{ss} = 0, \qquad \text{(A-15)}$$

and in the remainder of the Appendix, we will assume (A-15) to hold. In this case, C_n is given by

$$C_n = \frac{\bar{p}\,\mu_{ss}}{2\Delta\mu} = \frac{\bar{p}}{2(\mu_{nn} + \mu_{sn})}, \qquad \text{(A-16)}$$

while C_s and D_s are still given by (A-8) and (A-9). Thus the pressure gradient alone is the driving force for the normal fluid, whereas both $\bar{p}$ and $\bar{T}$ are driving forces for the superfluid. For the combination $v_s + \frac{\mu_{sn}}{\mu_{ss}} v_n$, however, we have

$$v_s + \frac{\mu_{sn}}{\mu_{ss}} v_n = \frac{\beta}{h\mu_{ss}} \frac{1}{2} D_s^{\frac{3}{2}} (h^2 - y^2) + D_s, \qquad \text{(A-17)}$$

so that the "thermomechanical head" $\bar{p} - \rho s \bar{T}$ alone is the driving force for $v_s + \frac{\mu_{sn}}{\mu_{ss}} v_n$.

It is of interest to consider the special case when there is no net mass flux (internal convection); then the (nonlinear) relation between the pressure gradient $\bar{P}$ and the temperature gradient $\bar{T}$ is given (implicitly) by

$$\frac{2}{3} x h^2 C_n + \frac{2}{3} (1-x) h^2 C_s + (1-x) D_s = 0. \tag{A-18}$$

For $h \to 0$, one may show from (A-18) that

$$\bar{P} = \rho_s \bar{T} + O(h^5) \tag{A-19}$$

(or, more explicitly,

$$\bar{P} \simeq \rho_s \bar{T} \left\{ 1 - \frac{\beta (\rho_s \bar{T})^2}{27 \mu_{ss}^3 (1-x)^4} \left(\frac{x \mu_{ss} - (1-x) \mu_{sn}}{\mu_{nn} + \mu_{sn}} \right)^3 h^5 \right\} \Bigg),$$

so that the London equation is recovered in the limit of $h \to 0$. For larger values of h, the expression for the heat current as a function of the temperature gradient may be written as

$$W = \frac{h^2 \rho_s T}{3(\mu_{nn} + \mu_{sn})} \rho_s \bar{T} \left\{ 1 - \frac{\lambda}{3} \frac{\hat{S}^3}{\hat{T}} \right\}, \tag{A-20}$$

where

$$\lambda = \frac{\frac{x}{1-x} - \frac{\mu_{sn}}{\mu_{ss}}}{\left\{ \frac{x}{1-x} + \frac{(1-x)(\mu_{nn} + \mu_{sn}) - \mu_{sn}}{\mu_{ss}} \right\}}, \tag{A-21}$$

where $\hat{T}$, a dimensionless measure of the temperature gradient, is given by

$$\hat{T} = \rho_s \bar{T} / \mathcal{R}$$

where

$$\frac{1}{\mathcal{R}} = \frac{\left(\frac{x}{1-x} - \frac{\mu_{sn}}{\mu_{ss}}\right)\left(\frac{x}{1-x} + \frac{(1-x)(\mu_{nn}+\mu_{sn})}{\mu_{ss}} - \frac{\mu_{sn}}{\mu_{ss}}\right)^{1/2} \beta^{1/2} h^{5/2}}{3\sqrt{1-x}\,(\mu_{nn}+\mu_{sn})^{3/2}}, \tag{A-22}$$

and where $\hat{S}$ is the (unique) root of

$$\frac{1}{3}\hat{S}^3 + \hat{S} = \hat{T}. \tag{A-23}$$

As a function of $\hat{T}$, the quantity $\frac{\lambda \hat{S}^3}{3\hat{T}}$ increases monotonically from zero to λ as $\hat{T}$ increases from zero to infinity. (From the restrictions (258) on the viscosity coefficients, one may show that $\lambda<1$, so that $W/\bar{T}$ is always positive.) If we introduce

$$W_0 = \frac{h^2 \rho s T \mathcal{R}}{3(\mu_{nn}+\mu_{sn})}, \tag{A-24}$$

then

$$W = W_0\left\{\hat{T} - \frac{\lambda}{3}\hat{S}^3\right\}, \tag{A-25}$$

or

$$W = W_0\{(1-\lambda)\hat{T} + \lambda\hat{S}\},$$

and

$$W = W_0\hat{T}\{1 + O(\hat{T}^2)\}, \quad \hat{T}\to 0 \tag{A-26}$$

$$W = W_0(1-\lambda)\hat{T}\{1 + O(\hat{T}^{-2/3})\}, \quad \hat{T}\to\infty. \tag{A-27}$$

Another case of interest is the isothermal flow in a channel under a pressure gradient. In this case, the relation between the mean flow velocity and the pressure gradient is

$$\bar{v} = x\bar{v}_n + (1-x)\bar{v}_s = \frac{h^2\bar{P}}{3(\mu_{nn}+\mu_{sn})}\left\{x + (1-x)^2\left(\frac{\mu_{nn}+\mu_{sn}}{\mu_{ss}}\right) - (1-x)\frac{\mu_{sn}}{\mu_{ss}}\right\}$$

$$+ (1-x)\left\{\frac{(1-x)h}{\beta}\right\}^{1/3}\bar{P}^{1/3}. \qquad \text{(A-28)}$$

The calculations for the flow in an infinitely long circular pipe of radius a are entirely similar, and we only quote some of the results here. The profiles and mean velocities are (we are assuming $\mu_{ns}+\mu_{ss}=0$ still)

$$v_n(r) = C_n(a^2 - r^2), \qquad \text{(A-29)}$$

$$v_s(r) = C_s(a^2 - r^2) + D_s, \qquad \text{(A-30)}$$

$$C_n = \frac{\bar{P}}{4(\mu_{nn}+\mu_{sn})}, \qquad \text{(A-31)}$$

$$C_s = \frac{1}{4\mu_{ss}(\mu_{nn}+\mu_{sn})}\left\{\bar{P}\left[(1-x)\mu_{nn} - x\mu_{sn}\right] - (1-x)\rho s\bar{T}(\mu_{nn}+\mu_{sn})\right\}; \qquad \text{(A-32)}$$

$$D_s = \left\{\frac{a(1-x)(\bar{P}-\rho s\bar{T})}{2\beta}\right\}^{1/3} \qquad \text{(A-33)}$$

$$\bar{v}_n = \frac{1}{2}C_n a^2, \quad \bar{v}_s = \frac{1}{2}C_s a^2 + D_s. \qquad \text{(A-34)}$$

For internal convection, the relation between the heat current and the temperature gradient is

$$\bar{W} = \frac{a^2\rho s T}{8(\mu_{nn}+\mu_{sn})}\rho s\bar{T}\left\{1 - \frac{\lambda}{4}\frac{\tilde{S}^3}{\tilde{T}}\right\}, \qquad \text{(A-35)}$$

where

$$\lambda = \frac{\frac{x}{1-x} - \frac{\mu_{sn}}{\mu_{ss}}}{\left\{\frac{x}{1-x} - \frac{\mu_{sn}}{\mu_{ss}} + \frac{(1-x)(\mu_{nn}+\mu_{sn})}{\mu_{ss}}\right\}}, \qquad \text{(A-36)}$$

where

$$\hat{T} = \rho s \bar{T} / \tilde{\mathcal{R}}$$

with

$$\tilde{\mathcal{R}} = \frac{8\sqrt{1-x}\,(\mu_{nn}+\mu_{sn})^{3/2}}{\left(\frac{x}{1-x} - \frac{\mu_{sn}}{\mu_{ss}}\right)\left(\frac{x}{1-x} - \frac{\mu_{sn}}{\mu_{ss}} + \frac{(1-x)(\mu_{nn}+\mu_{sn})}{\mu_{ss}}\right)^{1/2} a^{5/2}\beta^{1/2}}, \qquad \text{(A-37)}$$

and where $\tilde{S}$ is the root of

$$\frac{1}{4}\tilde{S}^3 + \tilde{S} = \hat{T}. \qquad \text{(A-38)}$$

For isothermal flow under pressure gradient, the mean velocity is given by

$$\bar{V} = x\bar{V}_n + (1-x)\bar{V}_s = \frac{a^2\bar{P}}{8(\mu_{nn}+\mu_{sn})}\left\{x + (1-x)^2\left(\frac{\mu_{nn}+\mu_{sn}}{\mu_{ss}}\right) - (1-x)\frac{\mu_{sn}}{\mu_{ss}}\right\}$$

$$+ (1-x)\left\{\frac{(1-x)a}{2\beta}\right\}^{1/3}\bar{P}^{1/3}. \qquad \text{(A-39)}$$

B. Flow Between Rotating Cylinders

We now consider the steady flow between two concentric cylinders, the inner cylinder being of radius a and having angular velocity Ω_a, and the outer cylinder being of radius b and having angular velocity Ω_b (cf. Fig. 4). We look for solutions of the form $\underline{v}_s = v_s(r)\,\underline{e}_\theta$, $\underline{v}_n = v_n(r)\,\underline{e}_\theta$; it follows from the equations that

$$v_n(r) = A_n r - \frac{B_n}{r}, \tag{A-40}$$

and

$$v_s(r) = A_s r - \frac{B_s}{r}. \tag{A-41}$$

$v_n(r)$ must satisfy

$$v_n(a) = a\,\Omega_a, \qquad v_n(b) = b\,\Omega_b, \tag{A-42}$$

so that A_n, B_n are given by

$$A_n = \Omega_a + \frac{\gamma^2\,\Omega_{rel}}{\gamma^2 - 1} \tag{A-43}$$

$$B_n = \frac{\gamma^2 a^2\,\Omega_{rel}}{\gamma^2 - 1}, \tag{A-44}$$

where

$$\gamma = \frac{b}{a}, \quad \Omega_{rel} = \Omega_b - \Omega_a. \tag{A-45}$$

The boundary conditions for $v_s(r)$ are

$$\left\{\mu_{sn}\, r \frac{d}{dr}\left(\frac{v_n}{r}\right) + \mu_{ss}\, r \frac{d}{dr}\left(\frac{v_s}{r}\right)\right\}_{r=a} = \beta\left\{v_s(a) - a\,\Omega_a\right\}^3, \tag{A-46}$$

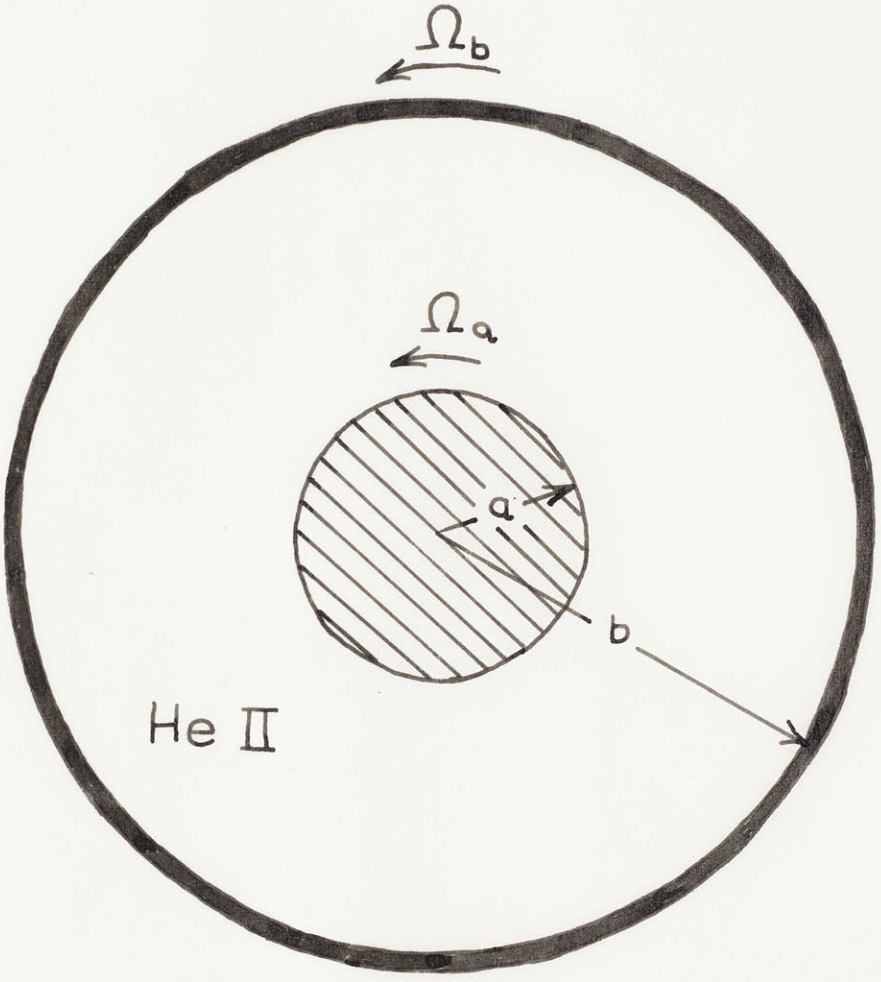

Figure 4

and
$$\left\{\mu_{sn} r \frac{d}{dr}\left(\frac{v_n}{r}\right) + \mu_{ss} r \frac{d}{dr}\left(\frac{v_s}{r}\right)\right\}_{r=b} = -\beta\left\{v_s(b) - b\Omega_b\right\}^3, \tag{A-47}$$

so that A_s , B_s satisfy

$$\frac{2}{a^2}\left\{\mu_{sn} B_n + \mu_{ss} B_s\right\} = \beta\left\{A_s a - \frac{B_s}{a} - a\Omega_a\right\}^3, \tag{A-48}$$

and
$$\frac{2}{b^2}\left\{\mu_{sn} B_n + \mu_{ss} B_s\right\} = -\beta\left\{A_s b - \frac{B_s}{b} - b\Omega_b\right\}^3. \tag{A-49}$$

The solutions for A_s , B_s may be written as

$$A_s = A_n - \frac{\gamma^2(\gamma^{1/3}+1)}{(\gamma^2-1)(\gamma^2+\gamma^{1/3})}\left(\frac{\mu_{sn}+\mu_{ss}}{\mu_{ss}}\right)\Omega_{rel}(1-q), \tag{A-50}$$

$$B_s = B_n\left\{1 - \left(\frac{\mu_{sn}+\mu_{ss}}{\mu_{ss}}\right)(1-q)\right\}, \tag{A-51}$$

where q is the (unique) root of

$$2q = \hat{\Omega}^2(1-q)^3, \tag{A-52}$$

where

$$\hat{\Omega}^2 = \frac{\beta\left(\frac{\mu_{sn}+\mu_{ss}}{\mu_{ss}}\right)^2 a^3 \gamma^4 \Omega_{rel}^2 (\gamma^2-1)}{\mu_{ss}(\gamma^2+\gamma^{1/3})^3}. \tag{A-53}$$

The parameter q varies from 0 to 1 as $\hat{\Omega}^2$ varies from 0 to ∞. For $\Omega_{rel}=0$, q vanishes, and in this case the normal fluid and the superfluid move together in a rigid body rotation. In general, there is a slip of the superfluid at the boundary;

thus

$$a\Omega_a - v_s(a) = \left(\frac{\mu_{sn}+\mu_{ss}}{\mu_{ss}}\right)\frac{a^2\gamma^2\Omega_{rel}(1-\eta)}{\gamma^2+\gamma^{1/3}}, \qquad (A\text{-}54)$$

and

$$b\Omega_b - v_s(b) = \left(\frac{\mu_{sn}+\mu_{ss}}{\mu_{ss}}\right)\frac{a\gamma\,\Omega_{rel}(1-\eta)}{\gamma^{5/3}+1}, \qquad (A\text{-}55)$$

so that, also,

$$\left\{a\Omega_a - v_s(a)\right\} = -\gamma^{2/3}\left\{b\Omega_b - v_s(b)\right\}. \qquad (A\text{-}56)$$

Thus the velocity of the superfluid relative to the wall is always of opposite sign for the two boundaries, and the magnitude of the slip is greater (by a factor of $\gamma^{2/3}$) at the inner wall.

A particular case of interest is the flow in a rotating cylinder viscometer; in this case we have $\Omega_a = 0$, and we wish to calculate the torque (per unit length) on the inner cylinder as a function of $\Omega = \Omega_b$. The calculation is easily made from the above results, and the torque is given by

$$T = \frac{4\pi\gamma^2a^2\Omega}{\gamma^2-1}\left(\mu_{nn}\mu_{ss}-\mu_{ns}\mu_{sn}\right)\left\{1+\eta\frac{(\mu_{ss}+\mu_{sn})(\mu_{ss}+\mu_{ns})}{(\mu_{nn}\mu_{ss}-\mu_{ns}\mu_{sn})}\right\}. \qquad (A\text{-}57)$$

From the channel flow experiments, we have concluded that

$$\mu_{ns}+\mu_{ss}=0; \qquad (A\text{-}58)$$

it then follows that

$$T = \frac{4\pi\gamma^2a^2\Omega}{\gamma^2-1}\left(\mu_{nn}+\mu_{sn}\right). \qquad (A\text{-}59)$$

Thus the torque is a linear function of Ω (in agreement with the experimental results - [15]), and the viscosity measured by such an experiment is the combination $\mu_{nn}+\mu_{sn}$.

If we also impose the Onsager relation

$$\mu_{ns} = \mu_{sn} , \tag{A-60}$$

so that

$$\mu_{sn} = - \mu_{ss} ,$$

then it can be shown that in all cases of steady flow between rotating cylinders, the superfluid component moves with the normal fluid. By way of contrast, we may note that in the case $\mu_{sn}=0$, the superfluid and normal fluid in general do not move together.

For convenience of reference, we note here that the [illegible] angular momentum (per unit length) of a flow between rotating cylinders is given by

$$L = \frac{\pi \rho a^4 (\gamma^2-1)}{2} \left\{ (\gamma^2+1)\Omega_a + \gamma^2 \Omega_{rel} - (1-X)\left(\frac{\mu_{sn}+\mu_{ss}}{\mu_{ss}}\right) \cdot \right.$$

$$\left. \cdot (1-q)\,\Omega_{rel}\,\frac{\gamma^2(\gamma^{1/3}-1)}{\gamma^2+\gamma^{1/3}} \right\} . \tag{A-61}$$

C. Andronikashvili Experiment

Only the simplest aspects of the problem will be treated here--no consideration is given to edge effects or nonlinear convective terms, the oscillating surfaces are taken as infinite in extent, and the nonlinear boundary condition is treated only in the low-speed approximation (for which it is a linear boundary condition).

In the problem to be considered, Lin's hydrodynamic equations may be reduced to the system

$$\rho_n \frac{\partial v_n}{\partial t} = \mu_{nn} \frac{\partial^2 v_n}{\partial z^2} + \mu_{ns} \frac{\partial^2 v_s}{\partial z^2},$$

and (A-62)

$$\rho_s \frac{\partial v_s}{\partial t} = \mu_{sn} \frac{\partial^2 v_n}{\partial z^2} + \mu_{ss} \frac{\partial^2 v_s}{\partial z^2},$$

where v_n , v_s are velocity components parallel to the oscillating surfaces and z is the space coordinate perpendicular to the surfaces. Solutions of (A-62) that are periodic in time may be found in the form $e^{i\omega t} e^{pz}$. It is convenient to introduce the kinematic viscosities

$$\nu_{nn} = \frac{\mu_{nn}}{\rho_n} , \quad \nu_{ss} = \frac{\mu_{ss}}{\rho_s} \tag{A-63}$$

$$\nu_{ns} = \frac{\mu_{ns}}{\sqrt{\rho_n \rho_s}} , \text{ and } \quad \nu_{sn} = \frac{\mu_{sn}}{\sqrt{\rho_n \rho_s}} ;$$

it is also convenient to introduce

$$\Delta_\nu = \nu_{nn}\nu_{ss} - \nu_{ns}\nu_{sn}, \tag{A-64}$$

and

$$\nu = \sqrt{(\nu_{nn} + \nu_{ss})^2 - 4\Delta\nu}. \tag{A-65}$$

(We assume here that ν is real; in particular, this is true if $\mu_{ns}\,\mu_{sn} \geq 0$, which includes the cases $\mu_{sn} = \mu_{ns}$, $\mu_{sn} = 0$.) Then the solutions of (A-62) are the form

$$\begin{pmatrix} \sqrt{\rho_n} & v_n \\ \sqrt{\rho_s} & v_s \end{pmatrix} = e^{i\omega t}\left\{(ae^{P_+ z} + be^{-P_+ z})\begin{pmatrix} 2\nu_{ns} \\ \nu_{ss} - \nu_{nn} - \nu \end{pmatrix}\right.$$

$$\left. + (ce^{P_- z} + de^{-P_- z})\begin{pmatrix} 2\nu_{ns} \\ \nu_{ss} - \nu_{nn} + \nu \end{pmatrix}\right\}, \tag{A-66}$$

where

$$P_+ = \frac{1}{2}(1+i)\sqrt{\frac{\omega}{\Delta\nu}}\sqrt{\nu_{nn} + \nu_{ss} + \nu}, \tag{A-67}$$

and

$$P_- = \frac{1}{2}(1+i)\sqrt{\frac{\omega}{\Delta\nu}}\sqrt{\nu_{nn} + \nu_{ss} - \nu}.$$

Thus in the present case there are two viscous penetration depths given by

$$\delta_\pm = \frac{1}{\mathrm{Re}(P_\pm)} = 2\sqrt{\frac{\Delta\nu}{\omega(\nu_{nn} + \nu_{ss} \pm \nu)}} \tag{A-68}$$

The nonlinear boundary condition to be satisfied at a wall is

$$\mu_{sn}\frac{\partial v_n}{\partial z} + \mu_{ss}\frac{\partial v_s}{\partial z} = \beta(v_s - U_{wall})^3; \tag{A-69}$$

For sufficiently small wall velocities, this condition is, approximately,

$$\mu_{sn} \frac{\partial v_n}{\partial z} + \mu_{ss} \frac{\partial v_s}{\partial z} = 0, \qquad \text{(low-speed)} \qquad \text{(A-70)}$$

whereas for sufficiently high wall velocities, the condition is

$$v_s = U_{wall} \qquad \text{(high-speed)}. \qquad \text{(A-71)}$$

We now consider the Andronikashvili experiment, taking the idealized situation of the flow between two oscillating parallel infinite discs. The discs are located at $z = \pm h$, and the boundary velocity is taken to be

$$\underline{v}_{disc} = \underline{e}_\theta \, A r \omega \cos\omega t,$$

where r is the polar radius measured from the axis of rotation. We look for solutions of the form

$$\underline{v}_n(r,z,t) = \underline{e}_\theta \, r v_n(z,t) \qquad \text{(A-72)}$$

$$\underline{v}_s(r,z,t) = \underline{e}_\theta \, r v_s(z,t).$$

Then v_n, v_s must satisfy the equations(A-62). We consider here only the low-speed limit of the boundary conditions, so that

$$v_n(\pm h,t) = A\omega\cos\omega t = \mathrm{Re}\, A\omega e^{i\omega t}$$

$$\mu_{sn}\frac{\partial v_n}{\partial z}(\pm h,t)+\mu_{ss}\frac{\partial v_s}{\partial z}(\pm h,t)=0. \qquad (A-73)$$

The solutions for v_n , v_s may be written in the form

$$\begin{pmatrix}\sqrt{p_n} & v_n\\ \sqrt{p_s} & v_s\end{pmatrix}=\frac{\sqrt{p_n}\,A\omega e^{i\omega t}}{2\nu_{ns}}\left\{\frac{\cosh p_+ z}{\cosh p_+ h}\begin{pmatrix}2\nu_{ns}\\ \nu_{ss}-\nu_{nn}-\nu\end{pmatrix}+\left[1-\frac{2\nu_{sn}\nu_{ns}+\nu_{ss}(\nu_{ss}-\nu_{nn}+\nu)}{2\nu_{sn}\nu_{ns}+\nu_{ss}(\nu_{ss}-\nu_{nn}-\nu)}\cdot\right.\right.$$

$$\left.\left.\cdot\frac{p_-\tanh p_- h}{p_+\tanh p_+ h}\right]^{-1}\cdot\left[\frac{\cosh p_- z}{\cosh p_- h}\begin{pmatrix}2\nu_{ns}\\ \nu_{ss}-\nu_{nn}+\nu\end{pmatrix}-\frac{\cosh p_+ z}{\cosh p_+ h}\begin{pmatrix}2\nu_{ns}\\ \nu_{ss}-\nu_{nn}-\nu\end{pmatrix}\right]\right\}, \qquad (A-74)$$

The shears are given by

$$\left.\frac{\partial v_n}{\partial z}\right|_h=\frac{2A\omega e^{i\omega t}\,\nu\nu_{ss}\,p_+\tanh p_+ h\;p_-\tanh p_- h}{p_-\tanh p_- h\left[2\nu_{sn}\nu_{ns}+\nu_{ss}(\nu_{ss}-\nu_{nn}+\nu)\right]-p_+\tanh p_+ h\left[2\nu_{sn}\nu_{ns}+\nu_{ss}(\nu_{ss}-\nu_{nn}-\nu)\right]} \qquad (A-75)$$

and

$$\left.\frac{\partial v_s}{\partial z}\right|_h=-\frac{\mu_{sn}}{\mu_{ss}}\left.\frac{\partial v_n}{\partial z}\right|_h.$$

A particular case of interest is when the penetration depths are large compared with the disc spacing h; then, approximately,

$$\frac{\partial v_n}{\partial z}\simeq\frac{A\omega^2 e^{i\omega t}\,i\,h\,\nu_{ss}}{\Delta_\nu}, \qquad (\text{for } |p_\pm|h\ll 1) \qquad (A-76)$$

and the order of magnitude condition for the validity of the low-speed approximation to the boundary condition may be written as

$$\omega\ll\frac{\rho h}{\beta R^2 A^2}, \qquad (A-77)$$

where R is the radius of the oscillating disc.

The angular momentum of the fluid contained between the two discs of radius R, separation 2h is given by

$$L = \frac{\pi R^4}{i\omega} \rho_n \frac{\Delta v}{v_{ss}} \left. \frac{\partial v_n}{\partial z} \right|_h ; \qquad (A-78)$$

in the limit $|p_\pm| h \ll 1$, we have

$$L = \pi R^4 \rho_n A \omega h e^{i\omega t} . \qquad (A-79)$$

If the space between the discs were filled with a solid of density ρ_n , it is easy to show that the angular momentum would also be given by (A-79). Thus in the limit of large penetration depths and low speeds, only the normal fluid is carried by the discs-- in agreement with experiment--and this conclusion is independent of any restrictive assumptions on the viscosity coefficients (such as $\mu_{sn} = 0$, or $\mu_{ns} = \mu_{sn}$).

D. Comparison with Experiment

Ultimately, one would hope to obtain values for the quantities μ_{nn}, μ_{ns}, μ_{sn}, μ_{ss} and β as functions of temperature from the analysis of various experiments. In the present section, we give some preliminary results along these lines. Although the experimental evidence for the relation $\mu_{ns}+\mu_{ss}=0$ is convincing, it would be very desirable to obtain some experimental evidence relevant to deciding whether $\mu_{ns}=\mu_{sn}$ or $\mu_{sn}=0$. (Of course, it is possible that neither of $\mu_{ns}=\mu_{sn}$ or $\mu_{sn}=0$ holds; since there are, however, arguments in favor of both of these relations, we will not in the present section consider the more general case.) The preliminary results given here tend to favor the case $\mu_{ns}=\mu_{sn}$.

We consider first experiments designed to measure the viscosity of helium II--the damping of an oscillating disc and the rotating cylinder viscometer. According to Lin's theory, the rotating cylinder viscometer measures the combination $\mu_{nn}+\mu_{sn}$ (cf. section B of this Appendix). For an ordinary fluid, the damping of an oscillating disc serves as a measure of the product $\rho\mu$. In the usual interpretation of the experimental results for helium II, it is assumed that the relevant density is ρ_n and the relevant viscosity is μ_n. The oscillating disc experiment may be analyzed on the basis of Lin's theory, and the quantity μ_n expressed in terms of the parameters of Lin's theory. Table 1 shows which quantities in Lin's theory play the role of the effective viscosity. Thus it is clear that if $\mu_{sn}=0$, the effective viscosities

Experiments	For $\mu_{sn}=0$	For $\mu_{sn}=\mu_{ns}$
Rotating cylinder viscometer	μ_{nn}	$\mu_{nn}(1-r)=\mu_{nn}-\mu_{ss}$
Oscillating disc (low-speed approximation to boundary condition)	μ_{nn}	$\mu_{nn}\left\{\frac{1+\frac{rx}{1-x}+2\sqrt{\frac{rx(1-r)}{1-x}}}{1+\frac{rx}{(1-x)(1-r)}+2\sqrt{\frac{rx}{(1-x)(1-r)}}}\right\}$

Note: $x=\frac{\rho_n}{\rho}$, $r=\frac{\mu_{ss}}{\mu_{nn}}$

Table 1. Effective Normal Fluid Viscosity From Lin's Theory

should be the same; however, the measured results are not the same (see, e.g., Atkins [2], p. 105), the discrepancy being particulary great at low temperatures, where the viscosity from the disc experiments is significantly larger. In the case $\mu_{sn} = \mu_{ns}$, Lin's theory predicts that the effective viscosities will be different; in fact, for low temperatures ($x \to 0$), the effective viscosity in the disc experiment tends to μ_{nn}, whereas the effective viscosity in the rotating cylinder viscometer is $\mu_{nn} - \mu_{ss}$. For high temperatures ($x \to 1$), both effective viscosities tend to $\mu_{nn} - \mu_{ss}$, and the experimental results show that the discrepancy between the measured viscosities is indeed smallest near the λ - point. The experimental results then seem to indicate the choice $\mu_{ns} = \mu_{sn}$. However, there are several reasons for regarding this as only a tentative conclusion:(i) in order to calculate the effective viscosity from the disc experiments, one must know the normal fluid density ρ_n, and it is difficult to accurately determine the quantity at the lower temperatures for which the discrepancy between the two effective viscosities is greatest, (ii) an accurate measurement of the damping of the disc is more difficult at lower temperatures, and (iii) the theory (in the case $\mu_{ns} = \mu_{sn}$) predicts that the disc effective viscosity should always be greater than the rotating cylinder viscosity, whereas the experimentally determined values of the disc viscosity seem to be somewhat smaller than the rotating cylinder viscosity for temperatures above $1.6\,^{\circ}K$. In the case $\mu_{sn} = \mu_{ns}$, the experimental results ([2], p. 105) indicate that (i) at lower temperatures ($T \simeq 1.4\,^{\circ}K$) $\mu_{nn} \sim 20 \cdot 10^{-6}$ poise and

$\mu_{ss} \sim 5 \cdot 10^{-6}$ poise (ii) at higher temperatures (near the λ - point) $\mu_{nn} - \mu_{ss} \sim 20 \cdot 10^{-6}$ poise and (iii) at intermediate temperatures ($T \sim 1.8^{\circ}K$) $\mu_{nn} - \mu_{ss} \sim 12 \cdot 10^{-6}$ poise.

Another possible experimental test to choose between and $\mu_{ns} = \mu_{sn}$ is suggested by the equation (A-61) for the angular momentum of the flow between rotating cylinders. In particular, for the choice $\Omega_a = -\gamma^2 \Omega_b$, the angular momenta for the two cases are

$$\mu_{ns} = \mu_{sn} : \quad L = 0 \qquad \text{(A-80)}$$

and

$$\mu_{sn} = 0 : \quad L = -\frac{\pi \rho a^4}{2} (1-x)(1-q) \Omega_b \frac{(\gamma^4 - 1)\gamma^2 (\gamma^{1/3} - 1)}{\gamma^2 + \gamma^{1/3}}. \qquad \text{(A-81)}$$

Thus it should be possible to obtain some evidence relevant to this question provided the angular momentum can be measured with an accuracy sufficient to discern (A-80) from (A-81). Although Reppy and Lane [32] have made measurements of the angular momentum of helium II contained in a single rotating cylinder, it would presumably be much more difficult in the case of fluid contained between two rotating cylinders. The situation is further complicated by the fact that if Ω_b is too large, the parameter q (cf. (A-52), (A-53)) will be close to unity; on the other hand, Ω_b must be large enough so that the measurement can choose between (A-80) and (A-81).

As discussed in part C of this Appendix, the Andronikashvili experiment does not choose between $\mu_{ns} = \mu_{sn}$ and $\mu_{sn} = 0$, since the theory predicts that (for sufficiently large

penetration depths and sufficiently small amplitudes of oscillation) only the normal component moves with the discs. However, it is possible to obtain some order of magnitude estimates of the boundary constant β from some oscillating disc-pile experiments of Hollis-Hallet [17]. He found that above a certain critical amplitude, the period of oscillation was observed to increase, indicating that the discs were carrying a fraction of the total liquid greater than $\frac{\rho_n}{\rho}$. This is agreement with Lin's theory, in which--because of the nonlinear boundary condition-- increasing entrainment of the superfluid component occurs with increasing amplitude of oscillation. From Hollis-Hallett's data some (very crude) estimates of β were made. In the temperature range $1.17^\circ K$ to $2.14^\circ K$, values of β in the range $4 \cdot 10^{-3} - 5 \cdot 10^{-2}$ (cgs) were found; although the estimates were far too crude to determine the temperature dependence of β, there was some indication of a maximum in the vicinity of $1.8^\circ K$. For temperatures above $1.5^\circ K$, all of the β values were in the range $2 \cdot 10^{-2} - 5 \cdot 10^{-2}$ (cgs). These estimates are at best only order of magnitude.

Further evidence for the choice $\mu_{sn} = \mu_{ns}$ may be obtained from the results of some pipe flow experiments by Staas, Taconis and van Alphen [34]. They have studied both laminar and turbulent flows in a pipe in the particular case when the temperature and pressure gradients are related by $\bar{P} = \rho s \bar{T}$. The onset of turbulence was determined experimentally by noting when the heat current ceased to be proportional to the pressure gradient. They found that over a wide range of

temperatures (and for pipes of three different diameters) the onset of turbulence corresponded to a value of 1200 for the Reynolds' number based on the pipe diameter, the mean normal fluid velocity, the normal fluid viscosity μ_n and the total fluid density. The analysis of this experiment on the basis of Lin's theory yields the following results: (i) for $\mu_{sn}=0$, only the normal component moves, so that the density of the moving fluid is ρ_n ; (ii) for $\mu_{ns}=\mu_{sn}$, the normal and super components move together, so that the density of the moving fluid is ρ . Since the experimentally determined critical Reynolds' number based on the total density ρ is constant over a wide range of temperatures, this would seem to be further evidence for the choice $\mu_{ns}=\mu_{sn}$.

In principle, the experimental results for various channel and pipe flows afford the best opportunity for obtaining accurate numerical values for the viscosity coefficients and the boundary constant β . In practice, however, one must be sure that the flows analyzed are laminar. The work of Staas et al. described above has given a criterion for turbulence in the case that $\bar{p}=\rho s\bar{T}$. However, in their experiments, the normal and supercomponents move together (in the case $\mu_{ns}=\mu_{sn}$), so that the critical Reynolds' number--which was based on the mean fluid velocity--could equally well have been based on the mean mass velocity or the mean superfluid velocity. In more general pipe flows, there may be two critical Reynolds' numbers; even so, the criterion furnished by Staas et al. should be a valuable guide in the interpretation of these more

general flows. Of course, even for laminar flows one cannot discount entirely the possibility of a volume mutual friction force and an associated critical velocity. In spite of these difficulties, the pipe and channel flows still seem to be the most likely source of accurate values of the viscosities and the boundary constant β .

CITED REFERENCES

[1] Andronikashvili, E. L., J. Phys. Moscow, 10, 201 (1946).

[2] Atkins, K. R., Liquid Helium, Cambridge (1959).

[3] Bekarevich, I. L. and I. M. Khalatnikov, Sov. Phys. JETP 13, 643 (1961).

[4] de Groot, S. R. and P. Mazur, Non-Equilibrium Thermodynamics, North Holland Publishing Co., Amsterdam (1962).

[5] Eckart, C., Phys. Rev., 54, 920 (1938).

[6] Fairbank, H. A. and D. M. Lee, Symposium on Liquid and Solid He_3, p. 26,32, Ohio State University (1957).

[7] Feynman, R. P., Progress in Low Temperature Physics, vol. I(C. J. Gorter, editor), Chapter I. North Holland Publishing Co., Amsterdam (1955).

[8] Gorter, C. J. and J. H. Mellink, Physica 15, 285 (1949).

[9] Gorter, C. J., Proc. International Conference on Low Temperature Physics, p. 971 Oxford (1951).

[10] Grad, H., Comm. Pure Appl. Math, 5, 455 (1952).

[11] Gribben, R. J., J. Math and Physics, 40, 177, 189 (1961).

[12] Hall, H. E. and W. F. Vinen, Proc. Roy. Soc. A, 238, 204, 215 (1956).

[13] Hall, H. E., Proc. Roy. Soc. A, 245, 546 (1958).

[14] Hall, H. E., Adv. Phys. 9, 89 (1960).

[15] Heikkila, W. J. and A. C. Hollis-Hallett, Can. J. Phys. 33, 420 (1955).

[16] Herivel, J. W., Proc. Camb. Phil. Soc. 51, 344 (1955).

[17] Hollis-Hallett, A. C., Proc. Roy. Soc. A, 210, 404 (1952).

[18] Kapitza, P., J. Phys. Moscow, 4, 181 (1941).

[19] Kronig, R., Proc. International Conference on Low Temperature Physics, p. 99, Oxford (1951).

[20] Landau, L. D., J. Phys., Moscow, 5, 71 (1941).

[21] Lee, T. D. and C. N. Yang, Phys. Rev. 113, 1406 (1959).

[22] Lifshitz, E. M. and I. M. Khalatnikov, Nuovo Cimento, Suppl. 3, 735 (1956).

[23] Lin, C. C., Phys. Rev. Letters 2, 245 (1959).

[24] Lin, C. C., "On the Hydrodynamics of Liquid Helium II, Part I: Perfect Fluid Theory", Preprint, Institute for Advanced Study, Princeton, N. J. (1960).

[25] Lin, C. C., Hydrodynamics of Helium II, notes for Enrico Fermi International School of Physics, Varenna, Italy, July 3-15, 1961.

[26] Lin, C. C., Nuovo Cimento (to be published)

[27] Little, W. A., Can. J. Phys., 37, 334 (1959).

[28] London, F., Nature, 141, 643 (1938).

[29] London, F., Superfluids, vol. II, John Wiley and Sons, New York (1954).

[30] Onsager, L., Nuovo Cimento 6, Suppl. 2, 249 (1949).

[31] Osbourne, D. V., Proc. Phys. Soc. A, 63, 909 (1950).

[32] Reppy, J. D. and C. T. Lane, Proc. 7th International Conference on Low Temperature Physics, p. 443 Toronto (1960).

[33] Serrin, J. Mathematical Principles of Classical Fluid Mechanics, Handbuch der Physik, vol. VIII, Springer, Berlin (1959).

[34] Staas, F. A., K. W. Taconis and W. M. Van Alphen, Physica 27, 893 (1961).

[35] Tisza, L., Nature, 141, 913 (1938); Compt. Rend., 207, 1035, 1186 (1938).

[36] Tisza, L., Phys. Rev., 72, 838 (1947).

[37] Vinen, W. F., Proc. Roy. Soc. A, 210, 404, (1952).

[38] Vinen, W. F., Progress in Low Temperature Physics, vol. III (C. J. Gorter, editor), Chapter I. North Holland Publishing Co., Amsterdam (1960).

[39] Whitlock, R. T., "On the Significance of Lin's Velocity Constraint in Two-Fluid Hydrodynamics", Dept. Army Project No. 5B99-01-004, U. S. AROD Project No. 2430, Contract No. DA-33-019-ORD-3003, Interim Technical Report No. 2. (1961).

[40] Zilsel, P. R. , Phys. Rev. 79, 309 (1950).

BIOGRAPHICAL NOTE

The author was born May 5, 1936 at Elizabethton, Tennessee. The author attended Purdue University (Sept., 1954 -- June, 1958) and was awarded the B.S, degree in Engineering Sciences in June, 1958. The author has been in attendance at M.I.T. from Sept., 1958 until the present, with the exception of one summer (1962) with the International Business Machines Corporation at Yorktown Heights, New York. While at M.I.T., the author was a research assistant (Sept. 15, 1958 -- Sept. 15, 1961; Sept 15, 1962 -- present) and also held an NSF Cooperative Graduate Fellowship (Sept. 15, 1961 -- June 15, 1962).