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On Existence of Separable Contraction Metrics for Monotone Nonlinear Systems [★]

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Abstract: Finding separable certificates of stability is important for tractability of analysis methods for large-scale networked systems. In this paper we consider the question of when a nonlinear system which is contracting, i.e. all solutions are exponentially stable, can have that property verified by a separable metric. Making use of recent results in the theory of positive linear systems and separable Lyapunov functions, we prove several new results showing when this is possible, and discuss the application of to nonlinear distributed control design via convex optimization.

Keywords: Distributed control, Nonlinear systems, Convex optimization.

1. INTRODUCTION

Many emerging applications of system analysis and control involve large networks of interconnected nonlinear dynamic systems: smart-grid power systems, traffic management with autonomous vehicles, internet congestion control, and analysis of biological signalling networks, to name but a few. In order to make analysis methods scalable to large networks and robust to node dropouts or additions, it is crucial to be able to understand the overall network behaviour by way of conditions on just the local node dynamics and their interactions with immediate neighbours.

The traditional method for stability analysis makes use of a Lyapunov function: a positive-definite function of the system's state that decreases under flows of the system. Finding Lyapunov functions for arbitrary large-scale systems is generally intractable, but can be greatly simplified if one restricts the search to *separable* Lyapunov functions, in particular sum-separable, i.e. $V(x) = \sum_i V_i(x_i)$, or max-separable, i.e. $V(x) = \max_i V_i(x_i)$, where x_i denotes the state of the i^{th} subsystem, and x denotes the concatenation of all subsystem states (Dirr et al., 2015).

There has been substantial work recently on establishing when such separable Lyapunov functions should exist. In the case of linear positive systems, i.e. linear systems for which the non-negative orthant is flow-invariant, the Perron-Frobenius theory gives separable Lyapunov functions (Berman and Plemmons, 1994). More recently, several researchers have taken advantage of these properties to dramatically simplify problems of decentralized control design and system identification (Tanaka and Langbort, 2011), (Tanaka and Langbort, 2013), (Colombino et al., 2015), (Rantzer, 2015), (Umenberger and Manchester, 2016). In a recent paper, fundamental results about

separable Lyapunov functions for positive systems have been extended to linear time-varying systems (Khong and Rantzer, 2016).

For nonlinear systems. Recent work has focused on establishing conditions on existence of separable Lyapunov functions, especially in connection with input-to-state stability properties Dashkovskiy et al. (2010), Ito et al. (2012) and monotone systems Dirr et al. (2015), the natural nonlinear generalization of a linear positive system Smith (1995)

Contraction analysis generalizes techniques from linear systems to nonlinear systems. Roughly speaking, a system is contracting if the linearization along *every* solution is exponentially stable Lohmiller and Slotine (1998). This property is verified by the existence of a *metric*, which can be taken to be of Riemannian form though others are possible. This idea can be traced back to Lewis (1949) and has been explored in more detail recently by Forni and Sepulchre (2014) and extended to analysis of limit cycles by Manchester and Slotine (2013). Advantages of contraction analysis include the fact that questions of stability are decoupled from knowledge of a particular solution, unlike Lyapunov theory, and because the object of study is a family of linear time-varying systems, many familiar results can be extended to nonlinear systems.

A natural question to ask is when these contraction conditions can be verified in a scalable way. It was noted in Coogan (2016) that some commonly used metrics based on l^1 and l^∞ norm are naturally separable, a property which was used implicitly in several papers on networked system analysis and control, e.g. (Russo et al., 2011; Como et al., 2015). However, this does not answer the question when such a separable metric exists for systems which are known to be contracting with respect to a (not necessarily separable) metric, as in the case of linear positive systems.

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Recently, the technique of control contraction metrics has been introduced, which extends contraction analysis to constructive control design (Manchester and Slotine, 2017). In fact, synthesis conditions can be transformed to a convex problem: a set of pointwise linear matrix inequalities, which can be verified using sum-of-squares programming (see e.g. Aylward et al. (2008)). Despite being convex, these conditions do not generally scale to very large systems.

It was recently shown by Stein Shiromoto and Manchester (2016) that if one restricts the search to sum-separable control contraction metrics, then the problem of distributed control synthesis can be made convex. This can be considered an extension of the results of Tanaka and Langbort (2011) to a class of nonlinear systems.

Another motivating application is nonlinear system identification with guaranteed model stability, building on the results of Tobenkin et al. (2017). Recent results allow scalable computation for linear systems (Umenberger and Manchester, 2016). With separable contraction metrics, these could be extended to identification of large-scale nonlinear systems.

2. NOTATION AND PRELIMINARIES

For symmetric matrices A, B the notation $A \geq B$ ($A > B$) means that $A - B$ is positive semidefinite (positive definite), whereas for vectors $x, y \in \mathbb{R}^n$, the notation $x \geq y$ denotes element-wise inequality. The non-negative reals are denoted $\mathbb{R}^+ := [0, \infty)$, and the natural numbers from 1 to n are denoted $\mathbb{N}_{1,n}$. A smooth matrix function $M(x, t)$ is called *uniformly bounded* if there exists $\alpha_2 \geq \alpha_1 > 0$ such that $\alpha_1 I \leq M(x, t) \leq \alpha_2 I$ for all x, t . Given a vector field $v : \mathbb{R}^n \times \mathbb{R}^+ \rightarrow \mathbb{R}^n$ defined for $x \in \mathbb{R}^n, t \in \mathbb{R}^+$, we use the following notation for directional derivative of a matrix function: $\partial_v M := \sum_j \frac{\partial M}{\partial x_j} v_j$.

In this paper we consider time-varying nonlinear systems:

$$\dot{x} = f(x, t) \quad (1)$$

where $x(t) \in \mathbb{R}^n$ is the state vector at time $t \in \mathbb{R}^+$, and f is a smooth function of x and at least piecewise-continuous in t , though these can be relaxed somewhat. Note that this system representation can include systems with external control or disturbance inputs $\dot{x}(t) = f(x(t), u(t), w(t))$, where for our purposes we absorb these into the time-variation in (1).

A dynamical system is *monotone* if for any pair of solutions x^a and x^b , $x^a(0) \leq x^b(0)$ implies $x^a(t) \leq x^b(t)$ for all $t \geq 0$, where \leq denotes component-wise inequality. This property can be generalized to partial-orderings based on arbitrary cones, however the positive orthant is the natural cone for the purposes of studying separability. A differential characterization of monotonicity is that the off-diagonal elements of $\frac{\partial f}{\partial x}$ are non-negative – this is implied by the *Kamke-Müller conditions* (Smith, 1995).

Internally positive linear systems form an important subset of monotone systems. A continuous-time linear system

$$\dot{x} = A(t)x$$

is positive (and hence monotone) if $A_{ij}(t) \geq 0$ for all $i \neq j$ and for all t . In the case of time-invariant systems,

the following result is well-known (see e.g. Berman and Plemmons (1994), Rantzer (2015)).

Theorem 1. If A is positive and Hurwitz, then there exists $p_i > 0, q_i > 0$, and $d_i > 0, i = 1, 2, \dots, n$ such that the following functions

$$V_p(x) = \sum_i p_i |x_i|, \quad (2)$$

$$V_q(x) = \max_i \{q_i |x_i|\}, \quad (3)$$

$$V_d(x) = \sum_i d_i |x_i|^2, \quad (4)$$

are Lyapunov functions for the system $\dot{x} = Ax$. Moreover, one can take $d_i = p_i q_i$.

Note that $V_d(x)$ is a quadratic Lyapunov function $V_d(x) = x'Dx$ for which D is diagonal and $D_{ii} = d_i$, and V_p is linear on the non-negative orthant.

A recent paper partially extends these results to linear time-varying (LTV) systems:

Theorem 2. (Khong and Rantzer (2016)). A linear positive system $\dot{x} = A(t)x$, with $A(t)$ piecewise continuous and uniformly bounded for all $t \in \mathbb{R}^+$, is exponentially stable if and only if there exists a Lyapunov function $V(x, t) = x'P(t)x$ with P diagonal such that $\eta|x|^2 \leq V(x, t) \leq \rho|x|^2$ and $\dot{V}(x, t) \leq -\nu|x|^2$ for all $t \geq 0$ and some $\eta, \rho, \nu > 0$.

We utilize the following standard results of Riemannian geometry, see, e.g., Do Carmo (1992) for details. A Riemannian metric is a smoothly-varying inner product $\langle \cdot, \cdot \rangle_x$ on the tangent space of a state manifold \mathcal{X} ; this defines local notions of length, angle, and orthogonality. In this paper $\mathcal{X} = \mathbb{R}^n$ and the tangent space can also be identified with \mathbb{R}^n . We allow metrics to be smoothly time-varying, and use the following notation: $\langle \delta_1, \delta_2 \rangle_{x,t} = \delta_1' M(x, t) \delta_2$ and $\|\delta\|_{x,t} = \sqrt{\langle \delta, \delta \rangle_{x,t}}$. We call a metric *uniformly bounded* if $\exists \alpha_2 \geq \alpha_1 > 0$ such that $\alpha_1 I \leq M(x, t) \leq \alpha_2 I$ for all x, t . For a smooth curve $c : [0, 1] \rightarrow \mathbb{R}^n$ we use the notation $c_s(s) := \frac{\partial c(s)}{\partial s}$, and define the Riemannian length and energy functionals as

$$L(c, t) := \int_0^1 \|c_s\|_{c,t} ds, \quad E(c, t) := \int_0^1 \|c_s\|_{c,t}^2 ds,$$

respectively, with integration interpreted as the summation of integrals for each smooth piece. Let Γ be the set of piecewise-smooth curves $[0, 1] \rightarrow \mathbb{R}^n$, and for a pair of points $x, y \in \mathbb{R}^n$, let $\Gamma(x, y)$ be the subset of Γ connecting x to y , i.e. curves $c \in \Gamma(x, y)$ if $c \in \Gamma, c(0) = x$ and $c(1) = y$. A smooth curve $c(s)$ is *regular* if $\frac{\partial c}{\partial s} \neq 0$ for all $s \in [0, 1]$. The Riemannian distance $d(x, y, t) := \inf_{c \in \Gamma(x, y)} L(c, t)$, and we define $E(x, y, t) := d(x, y, t)^2$. Under the conditions of the Hopf-Rinow theorem a smooth, regular minimum-length curve (a geodesic) γ exists connecting every such pair, and the energy and length satisfy the following inequalities: $E(x, y, t) = E(\gamma, t) = L(\gamma, t)^2 \leq L(c, t)^2 \leq E(c, t)$ where c is any curve joining x and y . For time-varying paths $c(t, s)$, we also write $c(t) := c(t, \cdot) : [0, 1] \rightarrow \mathbb{R}^n$.

A nonlinear system (1) is called *contracting* if *all* solutions are exponentially stable. A central result of Lohmiller and Slotine (1998) is that if there exists a uniformly bounded metric $M(x, t)$ such that

$$\dot{M} + \frac{\partial f'}{\partial x} M + M \frac{\partial f}{\partial x} \leq -2\lambda M,$$

where $\dot{M} = \frac{\partial M}{\partial t} + \partial_f M$, then the system is contracting with rate λ . This inequality states that $\dot{V} \leq -2\lambda V$ where $V(x, \delta, t) = \delta' M(x, t) \delta$. From this, it is straightforward to establish that the Riemannian distance (and energy) between any pair of points decreases exponentially, and thus can serve as incremental Lyapunov functions.

Non-Riemannian contraction metrics, e.g. based on l^1 and l^∞ norms, can also be used (Lohmiller and Slotine, 1998) and offer simplified tests of contraction for certain applications, see, e.g. Russo et al. (2011).

3. PROBLEM STATEMENT

Let the state vector x of (1) be partitioned into N node states $x = [x'_1, x'_2, \dots, x'_N]$ where $x_i \in \mathbb{R}^{n_i}, i = 1, \dots, N$ and $n = \sum_{i=1, \dots, N} n_i$. The objective is to determine when contraction can be established using a set of local node metrics. In general, local nodes may have vector states although our main result will relate the the situation where $n_i = 1$ for all i , i.e. the state of each node is a scalar.

Following the terminology from Lyapunov functions (Dirr et al., 2015), a contraction metric $V(x, \delta)$ is called *sum-separable* if it can be decomposed as

$$V(x, \delta) = \sum_{i=1}^N \delta'_i M_i(x_i) \delta_i$$

and *max-separable* if

$$V(x, \delta) = \max_{i \in \mathbb{N}_{1, N}} \{ \delta'_i M_i(x_i) \delta_i \}$$

We will focus on the above (sum-type) notion of separability, but will also briefly discuss max-separable contraction metrics.

The following problem is, to the authors' knowledge, open: *Problem 3.* Suppose a dynamical system of the form (1) is both contracting and monotone. Characterize the additional conditions (if any) that imply the existence of a separable contraction metric.

For the case of linear systems, it is known that no further conditions are required beyond contraction (exponential stability) and monotonicity (positivity). In this paper, we do not solve this problem, but rather present some partial results towards this goal.

4. MOTIVATION: DISTRIBUTED CONTROL DESIGN

As well as producing scalable analysis conditions, a major motivation for separable contraction metrics is that they convexify the problem of distributed control design for nonlinear systems (Stein Shiromoto and Manchester, 2016). Here we briefly recap these results and provide new results.

Consider a control system of the form

$$\dot{x} = f(x) + B(x)u \quad (5)$$

In Manchester and Slotine (2017) it was shown that if a metric exists such that the following implication is true:

$$\delta' M B = 0 \implies \delta' (\dot{M} + A' M + M A + 2\lambda M) < 0 \quad (6)$$

then all solutions of (5) are exponentially stabilizable by state feedback. Furthermore, the Riemannian energy to a target solution is a control Lyapunov function which verifies this. By the formula for first variation, rate of change of Riemannian energy is

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} E = & - \langle \gamma_s(t, 1), f(x, t) + B(x, t)u \rangle_{x, t} - \langle \gamma_s(t, 1), \cdot \rangle_{x, t} \\ & + \langle \gamma_s(t, 0), \dot{x}^* \rangle_{x^*, t} + \frac{1}{2} \frac{\partial E}{\partial t} \end{aligned} \quad (7)$$

To be precise, (6) implies that the set of u for which (7) is satisfied is non-empty. Such a u can be found either by a path integral construction, or by solving e.g. a linear or quadratic program with the constraint (7).

As was demonstrated by Stein Shiromoto and Manchester (2016), when the metric is sum-separable the computation of the controller depends only on information from the local state information and neighbours. In this case, computation of the Riemannian energy (by way of finding a minimal geodesic between x^* and x) can be done a completely decentralized way, using only local node information, and a stabilizing feedback control can be constructed using only local and neighbor information.

For systems with vector nodes, i.e. $n_i > 0$ for at least one i , computing the minimal geodesic γ will require solving an online path optimization. Here we note in the case when nodes have scalar states this can be eliminated.

Proposition 4. Consider a system of the form (5). Suppose a sum-separable control contraction metric exists with $n_i = 1$ for all i . Then under the change of coordinates

$$z_i(x_i, t) = \int_0^{x_i} \theta_i(\xi, t) d\xi,$$

where $\theta_i(x_i, t) = \sqrt{m_i(x_i, t)}$, squared Euclidean distance to any target solution $|z - z^*|^2$ is a control Lyapunov function.

Proof. Under this coordinate change, tangent vectors transform as

$$\delta_{z, i} = \theta_i(x_i, t) \delta_i,$$

and hence

$$V(x, \delta) = \sum_{i \in \mathbb{N}_{1, n}} \delta'_i \theta_i(x_i, t) \theta_i(x_i, t) \delta_i = |\delta_{z, i}|^2$$

hence the CCM is Euclidean distance in z . Now, it was proven by Manchester and Slotine (2017) that the CCM conditions implying that energy is a CLF are invariant under smooth changes of coordinates, hence $|z - z^*|^2$ is a CLF.

Remark 5. In the general case of vector nodes, one can still construct the differential change of coordinate $\delta_{z, i} = \Theta_i(x_i, t) \delta_i$ by factoring $M_i(x_i, t) = \Theta(x_i, t)' \Theta(x_i, t)$ so that $|\delta_{z, i}|^2 = \delta' M(x, t) \delta$. However In general Θ is not integrable, so we may not be able to construct $z(x, t)$ with $\frac{\partial z}{\partial x} = \Theta(x, t)$.

5. MAIN RESULT

An important sub-class of nonlinear networked systems is those with nonlinear local (node) dynamics, but positive (cooperative) linear coupling between nodes. The main result of this paper is to extend the result of Khong and Rantzer (2016) to this class of systems.

To be precise, consider a system with state $x \in \mathbb{R}^n$ and dynamics that decomposes as

$$\dot{x}_i = f_i(x_i, t) = g_i(x_i, t) + \sum_{j=1}^n k_{ij}(t)x_j, \quad i = 1, 2, \dots, n, \quad (8)$$

where g_i, k_{ij} are piecewise-continuous in t , and $\frac{\partial g_i}{\partial x_i}$ is uniformly bounded above (in x and t) for each i , and $k_{ij} \geq 0$ for $i \neq j$.

Theorem 6. Suppose a system of the form (8) is contracting with respect to a metric of the form $V(x, \delta, t) = \delta' M(t) \delta$. Then it is contracting with respect to a sum-separable metric of the form

$$V_d(x, \delta, t) = \sum_i m_i(t) |\delta_i|^2.$$

Proof. For each $i \in \mathbb{N}_{1,n}$, define

$$\bar{G}_i(t) = \sup_{x_i} \frac{\partial g_i(x_i, t)}{\partial x_i}$$

which, by assumption on g_i is a piecewise-continuous function of t . Now, construct the diagonal matrix $\bar{G}(t) \in \mathbb{R}^{n \times n}$ with each $\bar{G}_i(t)$ as the i^{th} diagonal elements, and consider the linear time-varying system

$$\dot{z} = (\bar{G}(t) + K(t))z \quad (9)$$

where $K(t)$ is a matrix with i, j element given by $k_{i,j}(t)$.

By assumption that $k_{ij}(t) \geq 0$ for $i \neq j$, this system is positive. We will now show that it is also exponentially stable.

By definition, for each i and t there exists a sequence $\bar{x}_i^k(t)$ such that $\frac{\partial g_i}{\partial x_i}(\bar{x}_i^k(t), t) \rightarrow \bar{G}_i(t)$ as $k \rightarrow \infty$. Now, construct the sequence of states $x^k(t) = [x_1^k(t), x_2^k(t), \dots, x_n^k(t)]'$. By construction $\frac{\partial g}{\partial x}(x^k(t), t) \rightarrow \bar{G}(t)$ as $k \rightarrow \infty$.

Now, by assumption that the system is contracting, for every x and t and for some $\lambda > 0$ and some uniformly bounded $Q(t)$, the following inequality holds

$$\dot{Q}(t) + \frac{\partial f}{\partial x}(x, t)Q(t) + Q(t)\frac{\partial f}{\partial x}(x, t) \leq -2\lambda M(t).$$

By continuity of f , we therefore have the same condition with $\frac{\partial f}{\partial x}(x, t)$ replaced by $\bar{G}(t) + K(t)$. This implies that the LTV system (9) is exponentially stable and meets the conditions of Theorem 2, and therefore has a diagonal Lyapunov function.

Hence there exists a separable Lyapunov function $V(z, t) = z' M(t) z$ with $M(t) > 0$ diagonal and

$$\dot{M} + (\bar{G} + K)'M + M(\bar{G} + K) < -\nu I$$

for all t , and hence by uniform boundedness

$$(\bar{G} + K)'M + M(\bar{G} + K) \leq -2\lambda_d M$$

for some $\lambda_d > 0$.

Denote by m_i the i^{th} diagonal element of M . By construction of \bar{G} and the fact that $M_i > 0$ we have

$$M \frac{\partial f}{\partial x} - M(\bar{G} + K) = \sum_{i \in \mathbb{N}_{1,n}} M_i \left(\frac{\partial g_i}{\partial x_i} - \bar{G}_i \right) \leq 0,$$

from which it follows that

$$\dot{M}(t) + \frac{\partial f'}{\partial x}(x, t)M(t) + M(t)\frac{\partial f}{\partial x}(x, t) \leq 2\lambda_d M(t).$$

This completes the proof of the theorem.

6. ADDITIONAL RESULTS

In this section, we present some other conditions under which it is straightforward to show existence of a separable Lyapunov function.

6.1 Local Existence Along Trajectories

It is known that if a monotone system is exponentially stable at the origin (i.e. $\frac{\partial f}{\partial x}(0)$ is Hurwitz), then there exists sum and max separable Lyapunov functions in a neighbourhood of the origin (Dirr et al., 2015). These results extend easily to contraction metrics, and in fact the following stronger condition is available.

Theorem 7. Assume system (1) is contracting and monotone. Then in a neighbourhood of any bounded solution $x_0(t), t \in [0, \infty)$ of (1) there exists a sum-separable contraction metric.

Proof. For any bounded solution $x_0(t)$ we can construct a compact set \mathcal{X}_0 such that $x_0(t) \in \mathcal{X}_0$ for all t . Furthermore, by continuity inside this set the elements of $\frac{\partial f}{\partial x}$ are uniformly bounded.

Since the system is contracting and monotone, the LTV differential dynamics are exponentially stable and positive along any solution. Therefore, by Theorem 2 there exists a diagonal Lyapunov function $\delta' M(t) \delta$ for the differential dynamics, which therefore satisfies the condition

$$\dot{M}(t) + \frac{\partial f}{\partial x}(x_0(t), t)M(t) + M(t)\frac{\partial f}{\partial x}(x_0(t), t) \leq -2\lambda M(t)$$

for all $t \geq 0$.

Now, since M is uniformly bounded, by choosing some $\lambda_0 < \lambda$ we have that for all $t \geq 0$,

$$\dot{M}(t) + \frac{\partial f}{\partial x}(x, t)M(t) + M(t)\frac{\partial f}{\partial x}(x, t) \leq -2\lambda_0 M(t)$$

for all x in some neighbourhood of $x_0(t)$. This completes the proof of the theorem.

Remark 8. This does not imply that the neighbourhood on which the contraction condition holds is forward invariant, however as long as the diameter of the neighbourhood does not shrink to zero as time goes to infinity, such a set can be easily constructed.

Remark 9. Unlike the results of (Dirr et al., 2015), this theorem applies even to systems with no equilibria, e.g. systems with periodic forcing.

Remark 10. Note that by applying this result along all solutions we can construct lead to a *diagonal* metric of the form

$$V(x, \delta) = \sum_i m_i(x) \delta_i^2.$$

However, it is *not* guaranteed that $m_i(x)$ depends only on x_i , which is the condition for true sum-separability of the metric.

6.2 Networks with Weakly Nonlinear Coupling: Small Gain Condition

He we briefly remark that a well-known construction of separable Lyapunov functions based on small-gain type

coupling can be extended to the case of contraction and time-varying systems. The conditions we discuss date back to early results in decentralized control and vector Lyapunov functions (Moylan and Hill, 1978; Sandell et al., 1978), and have been substantially generalized recently, e.g. Ruffer (2010); Ito et al. (2012). A similar construction was considered in Russo et al. (2013).

For the results of this section, the node states can have arbitrary dimensions. Additionally, there is no assumption that the original system is monotone, though a linear time-varying comparison system is.

Suppose for some system of the form (1) with N nodes of dimension n_i , one constructs local differential storage functions $V(x_i, \delta_i, t)$, e.g. of the form $V(x_i, \delta_i, t) = \delta_i' M(x_i, t) \delta_i$. Suppose it can be verified that

$$\dot{V}_i \leq \sum_{j \in \mathcal{N}_i} \alpha_{ij} V_j$$

where $\alpha_{ii} < 0$ and $\alpha_{ij} \geq 0, i \neq j$. Note that unlike Russo et al. (2013) this allows nonlinear and time-varying gains between subsystems, due to the possible state-dependence of $M(x, t)$.

Now, consider the $N \times N$ matrix

$$H = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \dots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \dots & \alpha_{2n} \\ \vdots & & \ddots & \vdots \\ \alpha_{n1} & \alpha_{n2} & \dots & \alpha_{nn} \end{bmatrix}$$

and the the system

$$\dot{z} = Hz$$

is a positive LTI system. Hence, if this system is exponentially stable, there exists a linear Lyapunov function on the positive orthant

$$V_1(z) = \sum_i p_i z_i.$$

which verifies exponential decrease. So, constructing the metric

$$V(x, \delta) = \sum_i p_i V_i(x_i, \delta_i)$$

it is clear that this is a sum-separable contraction metric for the original system.

Remark 11. There also exists max-type Lyapunov function $V_2 = \max_i \{q_i z_i\}$ for the linear comparison system, which could be used to construct a max-separable contraction metric.

6.3 Networks with Weakly Nonlinear Coupling using the S-Procedure

A disadvantage of the results of the previous section is that they depend heavily on knowing a good construction of the local storage functions V_i . In this section, we give results that are based only on explicit bounds on the vector field.

The key result can be considered a differential version of the results in Colombino et al. (2015), based on the losslessness of the S-Procedure for positive systems. Related results were established in Tanaka and Langbort (2013).

In particular, consider systems with scalar nodes of the form

$$\dot{x}_i = f_i(x_i) = g_i(x_i) + \sum_{j=1}^{j=n} k_{ij} x_j + h_i(x, t), \quad i \in \mathbb{N}_{1,n}$$

where $k_{ij} \geq 0$ and $\frac{\partial g_i}{\partial x_i}$ is uniformly bounded, and the functions h_i satisfy

$$\delta_x' \frac{\partial h'}{\partial x} \frac{\partial h}{\partial x} \delta_x \leq \psi^2 \delta_x' H' H \delta_x \quad (10)$$

for all x, δ_x , for some non-negative matrix H and some $\psi \in \mathbb{R}$.

Theorem 12. Suppose the system is contracting for all nonlinear functions h_i satisfying (10) as verified by a metric $V(x, \delta) = \delta' M \delta$, then M can be taken to be diagonal.

Proof. We skip some details since they are similar to Theorem 6 and (Colombino et al., 2015, Theorem 4).

Construct the matrix \bar{G} as in the proof of Theorem 6 and consider the system

$$\dot{z} = \left(\bar{G} + K + \frac{\partial h}{\partial x}(x, t) \right) z$$

where h is a vector function with elements h_i . As in Theorem 6, a metric proving contraction for this system also proves contraction for the true nonlinear system.

Now, by (Colombino et al., 2015, Theorem 4), stability of this system for all h_i satisfying (10) is equivalent to the existence of a *diagonal* matrix M for which

$$\begin{bmatrix} \bar{G}' M + M \bar{G} & M \\ M & 0 \end{bmatrix} + \sum_i \theta_i \begin{bmatrix} \psi^2 H_i' H_i & \\ & -\Pi_i \end{bmatrix} < 0.$$

It is straightforward to show that this same M leads to a sum-separable contraction metric.

6.4 Stability verification via a virtual system

Given a system (1), a *virtual system* is another system driven by x :

$$\dot{y} = \bar{f}(y, x, t)$$

with the special property that $\bar{f}(x, x, t) = f(x, t)$ for all x, t . Analysing such systems is common in observer design and synchronisation problems, but can also be used to establish stability of other classes of systems (Wang and Slotine, 2005; Jouffroy and Slotine, 2004).

Let us detail a particular example of this application which can make use of positive systems theory. Many networked dynamical systems can be factored as

$$\dot{x} = N(x, t)x \quad (11)$$

where $N(x, t)$ is a positive matrix for all x, t . This system has the origin ($x(t) = 0 \forall t$) as a solution. To establish stability of this solution we can introduce the virtual system

$$\dot{y} = N(x(t), t)y \quad (12)$$

where $x(t)$ is the solution of the true system but considered as time-variation for the virtual system. Suppose this system is exponentially stable for all $x(t)$, and note that a particular solution of (12) is $y(t) = x(t)$ for all t . and hence solutions of (11) converge to zero exponentially.

Now, since (12) represents a family of positive LTV systems parameterized by the initial condition $x(0)$ for each

solution $x(t)$, it follows from Khong and Rantzer (2016) that there exists a family of diagonal Lyapunov function

$$V(y, t) = \sum m_i(t, x(0)) |y_i|^2$$

verifying stability of all solutions $y(t)$ of (12). Therefore, establishing stability by way of a positive virtual system is simplified compared to the general case.

7. CONCLUSIONS

In this paper we have investigated the problem of when a contracting system has a *separable* contraction metric. This problem is motivated by the need to find simplified stability conditions for large-scale nonlinear systems, and also be recent work that shows that nonlinear distributed control design problems can be made convex if a sum-separable contraction metric exists.

To our knowledge a complete characterisation remains unknown, however in this paper we have demonstrated a number of simple cases in which it can be established.

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