The Method of Moments in Convolved Random Matrix Models and Discrete Analogues

by

Andrew Ahn

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Abstract

We study the global and local asymptotics of Macdonald processes, its degenerations, and related models using the method of difference operators. We focus on three applications. First, we consider random plane partitions with interactions, arising from Macdonald processes with periodic weighting. For these models, the Macdonald parameters q, t become interaction parameters for the underlying dimer model. We establish global limit shape and fluctuation theorems in the limit as the mesh size goes to 0 and the interaction parameters tend to 1. Second, we consider a particle system obtained by generalizing the notion of squared singular values of products of truncated orthogonal, unitary, and symplectic matrices to a one-parameter family of deformed models. This procedure is analogous to the extension of classical real, complex, and quaternion matrix ensembles to β-ensembles. A discrete time Markov chain is obtained by considering iterative multiplication of matrix factors and its appropriate generalization. We show global limit shapes and fluctuations when time is of constant order and the number of particles tend to infinity. We also establish local limit theorems at the right edge when time increases with the number of particles. Third, we discuss new developments in the method of difference operators to models beyond Macdonald processes. We apply this technique to obtain moments formulas for eigenvalues of sums of unitarily invariant random matrices and for measures derived from tensor products of representations of the unitary group.

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Contents

Chapter 1

Introduction

1.1 Background and Methods

The purpose of this thesis is to discuss several methods and results about random particle systems from random matrices, statistical mechanics, and asymptotic representation theory, with an eye toward their asymptotic behavior as the number of particles tends to infinity. The particle systems we study are derived from a two-parameter family of special functions known as the Macdonald symmetric functions. Our methods are rooted in the exploitation of this connection with symmetric function theory.

To illustrate the types of results and methods studied in this thesis, consider the following example. Let A, B be independent $N \times N$ complex random matrices with random squared singular values $\boldsymbol{a} := (a_1 \geq \cdots \geq a_N)$ and $\boldsymbol{b} := (b_1 \geq \cdots \geq b_N)$ respectively, and let $\boldsymbol{c} := (c_1 \geq \cdots \geq c_N)$ be the squared singular values of the product AB. Additionally, assume that the distributions of A and B are invariant under right multiplication by unitary matrices — the significance of this symmetry assumption will be relevant for our discussion on results and methods later. We may ask:

Question. *If the empirical measures $\frac{1}{N}\sum_{i=1}^{N}\delta_{a_i}$, $\frac{1}{N}\sum_{i=1}^{N}\delta_{b_i}$ converge weakly in probability to some deterministic limiting probability measures μ_1 and μ_2 respectively, how do the squared singular values of AB behave as $N \to \infty$ at the global and local scales.*

Global scaling refers to the scaling under which the size of the spectrum of $(AB)^*(AB)$ is non-vanishing and of constant order, whereas local scaling focuses on a microscopic region of the spectrum such that individual eigenvalues of $(AB)^*(AB)$ (equivalently squared singular values of AB) remain distinguishable from one another.

The asymptotics of $\boldsymbol{c}$ are well-understood at the global scale. Under a variety of hypotheses on A and B, it is known that the empirical measures $\frac{1}{N}\sum_{i=1}^{N}\delta_{c_i}$ converge weakly in probability to a deterministic measure as $N \to \infty$ known as the *multiplicative free convolution of μ_1 and μ_2* (see [66]), depending only on the limits μ_1 and μ_2. Furthermore, the centered measures $\sum_{i=1}^{N}\delta_{c_i} - \mathbb{E}\sum_{i=1}^{N}\delta_{c_i}$ converge to a Gaussian limit as $N \to \infty$, see e.g. [31] and references therein. The former result is known as a *law of large numbers* and the latter a *central limit theorem* by analogy with the homonymous results from classical probability.

For an example of a local scale result, let us focus our attention on the eigenvalues at the rightmost boundary of the spectrum of $(AB)^*(AB)$; that is, the largest squared singular values of AB. We refer to this boundary as the *right edge*. It is known that if A and B are independent truncated Haar unitary matrices — submatrices of random Haar distributed unitary matrices — then as the sizes of the matrices tend to infinity in a suitable manner, the limiting point process appearing at the right edge is given by the *Airy point process*. The Airy point process is a universal limiting point process which appears at the boundary of many interacting particle systems, further details are given within the body of this thesis. This result can be obtained from theorems of [11,40] and standard correlation kernel methods. Far less can be said under more general assumptions on A and B; see Chapter 4 for progress in this direction.

In this thesis, we consider analogous global and local asymptotic questions in a family of models subsuming this example. In Chapter 3, we study the global asymptotics of $\boldsymbol{c}$ for matrices A and B whose distributions exhibit different symmetries than that of right unitary invariance. Specifically, we take A and B to be both truncated orthogonal or truncated symplectic matrices. More generally, our asymptotic methods apply to a family of deformed models parametrized by $\beta > 0$ where $\beta = 1, 2, 4$ correspond to products of truncated orthogonal, unitary, and symplectic matrices respectively. We note that this deformation is consistent with the standard extrapolation of the Dyson index for β-ensembles in random matrices, see e.g. [25]. We also study local asymptotics in the setting where the number of matrix factors tends to infinity along with matrix size.

Naturally, one can consider sums instead of products of random matrices. Again, in this setup, the symmetries of the matrix distributions are important. Along this line, we study the eigenvalues of a matrix $A + B$ where A, B are Hermitian matrices whose distributions are invariant under conjugation by unitary matrices, see Chapter 4. Specifically, we discuss some structural results which indicate future directions for asymptotic analysis.

There are several discrete analogues of these random matrix models which are also studied in this thesis. In one direction, we can replace matrices with representations of the unitary group and eigenvalues with signatures, which index the irreducible representations of the unitary group. In this setting, random matrix sums correspond to tensor products of representations. We study this model in Chapter 4. In another direction, we obtain global asymptotic results for *plane partitions with interactions*. These are $2d$-stepped surface models which recover the β-deformed models above by restricting to certain marginals and taking suitable limits.

In our discussion below, we further elucidate the connection between the models in this thesis. A more detailed summary of the models and results in this thesis is provided in Section 1.2.

The methods in this thesis are rooted in symmetric function theory. The key functions in our methods are analogous to the characteristic and moment generating functions from classical probability. To give an idea of the methods, we continue in the setting of our example of right unitarily invariant matrix products. The unitary symmetry implies that the distributions of $\boldsymbol{a}$ and $\boldsymbol{b}$ determine the distribution of $\boldsymbol{c}$. Under suitable hypotheses on the distributions of $\boldsymbol{a}$ and $\boldsymbol{b}$, e.g. compact supports, we have

$$\mathbb{E}_{\boldsymbol{a}} \frac{s_\lambda(a_1, \ldots, a_N)}{s_\lambda(1, \ldots, 1)} \cdot \mathbb{E}_{\boldsymbol{b}} \frac{s_\lambda(b_1, \ldots, b_N)}{s_\lambda(1, \ldots, 1)} = \mathbb{E}_{\boldsymbol{c}} \frac{s_\lambda(c_1, \ldots, c_N)}{s_\lambda(1, \ldots, 1)} \tag{1.1.1}$$

for any partition λ of length at most N, where

$$s_\lambda(x_1,\dots,x_N) = \frac{\det\left(x_i^{\lambda_j+N-j}\right)_{1\le i,j\le N}}{\det\left(x_i^{N-j}\right)_{1\le i,j\le N}}$$

is the Schur function in N variables indexed by λ, see [29, Proposition 2.2], [43, Chapter VII]. This relation follows from the fact that the zonal spherical functions of the Gelfand pair $(\mathrm{GL}_N(\mathbb{C}), \mathrm{U}(N))$ are given by the normalized Schur functions. For $N = 1$, this reduces to the factorization of moments in products of independent random variables.

Just as the moments can be lifted to the moment generating function, we can analytically continue the map

$$F_{\boldsymbol{x}}(\lambda) : \lambda \mapsto \frac{s_\lambda(x_1,\dots,x_N)}{s_\lambda(1,\dots,1)}$$

in $\lambda = (\lambda_1 \ge \dots \ge \lambda_N) \in \mathbb{Z}^N$ to all of $\mathbb{C}^N$, which gives the generating function

$$\mathbb{C}^N \ni \boldsymbol{z} = (z_1,\dots,z_N) \mapsto \mathbb{E}_{\boldsymbol{c}} F_{\boldsymbol{c}}(\boldsymbol{z}).$$

Then, our multiplication identity lifts to

$$\mathbb{E}_{\boldsymbol{a}} F_{\boldsymbol{a}}(\boldsymbol{z}) \cdot \mathbb{E}_{\boldsymbol{b}} F_{\boldsymbol{b}}(\boldsymbol{z}) = \mathbb{E}_{\boldsymbol{c}} F_{\boldsymbol{c}}(\boldsymbol{z}), \qquad \boldsymbol{z} \in \mathbb{C}^N. \tag{1.1.2}$$

Moreover, there exists a family of differential operators $\{D_k\}_{k>0}$ acting on functions in $\boldsymbol{z}$ such that

$$D_k F_{\boldsymbol{x}}(\boldsymbol{z}) = \left(\sum_{i=1}^N x_i^k\right) F_{\boldsymbol{x}}(\boldsymbol{z}), \qquad k > 0, \quad x_1 \ge \dots \ge x_N \tag{1.1.3}$$

Thus, just as differentiation of the Laplace transform recovers the moments of a random variable, the action of D_k on $\mathbb{E}_{\boldsymbol{c}} F_{\boldsymbol{c}}(\boldsymbol{z})$ recovers the moments of the measure $\sum_{i=1}^N \delta_{c_i}$. Applying several D_k's gives the joint moments.

The upshot of the discussion above is that if the functions $\mathbb{E}_{\boldsymbol{a}} F_{\boldsymbol{a}}(\boldsymbol{z})$, $\mathbb{E}_{\boldsymbol{b}} F_{\boldsymbol{b}}(\boldsymbol{z})$ have a suitably nice form, then the moments of $\boldsymbol{c}$ may be analyzed via the action of D_k and the convolution relation (1.1.2). Notice that we can further iterate (1.1.2), extending the study of products AB to that of a time-dependent process $A_1 \cdots A_T$ in discrete time $T \ge 1$ — actually, this process is Markovian due to the unitary symmetry in the distribution.

A similar structure arises if we replace complex matrices and unitary symmetry with real matrices and orthogonal symmetry, or quaternion matrices and quaternionic unitary symmetry. In this case (1.1.1) continues to hold if we replace s_λ with the zonal spherical functions of the Gelfand pairs $(\mathrm{GL}_N(\mathbb{R}), \mathrm{O}(N))$ and $(\mathrm{GL}_N(\mathbb{H}), \mathrm{U}(N,\mathbb{H}))$ respectively, where $\mathbb{H}$ denotes the skew field of quaternions and $\mathrm{U}(N,\mathbb{H})$ denotes the quaternionic unitary group. We can similarly extend these zonal spherical functions to the *Heckman-Opdam hypergeometric functions* $F_{\boldsymbol{x}}(\boldsymbol{z};\theta)$ where $\theta = 1/2, 1, 2$ correspond to the real, complex, and quaternion models. Like the complex case, for each $\theta > 0$, the Heckman-Opdam hypergeometric functions are equipped with a family of differential operators

$\{D_k^\theta\}$ which satisfy analogous eigenrelations as that of (1.1.3).

The fact that the Heckman-Opdam hypergeometric functions and associated differential operators come as a one-parameter family in $\theta > 0$ beyond $\theta = 1/2, 1, 2$ suggests that we can study models generalizing the real, complex, and quaternion random matrix models. This idea is explored in Chapter 3 and leads to the β-deformed models mentioned earlier which generalize squared singular values of truncated Haar orthogonal, unitary, and symplectic matrices. We note that β and θ are related by $\beta = 2\theta$.

The Heckman-Opdam hypergeometric functions are themselves degenerations of the Macdonald symmetric functions $\{P_\lambda(\boldsymbol{z})\}_\lambda$, indexed by signatures $\lambda = (\lambda_1 \geq \cdots \geq \lambda_N) \in \mathbb{Z}^N$ and parametrized by $0 < q, t < 1$. As before, the Macdonald symmetric functions are eigenfunctions of a family of difference operators $\{D_k^{q,t}\}_{k>0}$. This structure leads to a vast collection of probability measures on partitions $\mu = (\mu_1, \mu_2, \ldots, \mu_N)$, known as *Macdonald measures*, whose moments are analyzable by the action of $D_k^{q,t}$ on the generating function $\mathbb{E}\frac{P_\mu(\boldsymbol{z})}{P_\mu(\boldsymbol{y})}$ where $\boldsymbol{y}$ is a fixed choice of normalizing specialization. Similarly, these measures extend to stochastic processes by multiplication of these generating functions, as with (1.1.2), and related procedures. This leads to the *Macdonald processes*. Suitable degenerations of these processes give rise to the aforementioned matrix models and their generalizations. In Chapter 2, we study the global asymptotics of Macdonald processes under a periodicity assumption.

The application of difference operators to Macdonald processes first appeared in the pioneering work of Borodin and Corwin [6] in the context of directed polymer models. Since then, the method has been used for various degenerations of Macdonald processes, e.g. [8], [22]. While the application of operators is a standard procedure in representation theory, such as the action of Casimirs on characters, [6] succeeded in adapting this representation theoretic technique to a probabilistic framework. For our applications, we are especially interested in the adaption of these representation theoretic tools to the method of moments in our models. Many works preceding [6] have also ported representation theoretic ideas into probabilistic method of moments, e.g. [50]. However, this thesis focuses on further developments on the approach introduced in [6]. In particular, Chapters 2 and 3 utilize a different family of operators from that of [6], discovered by Negut in [49], which was initially used in [32] for the β-Jacobi corners process. In Chapter 4, we consider an extension of this method for models beyond Macdonald processes derived from tensor products of representations of the unitary group and sums of random matrices with unitary symmetry.

1.2 Chapter Summaries

We now provide more detailed summaries of the subsequent chapters. We note that each chapter is self-contained.

In Chapter 2, we study scaling limits of periodically weighted skew plane partitions with semilocal interactions and general boundary conditions. These random plane partitions are Macdonald processes and the semilocal interactions correspond to the parameters q, t for the Macdonald symmetric functions. We consider the limit as $q, t \to 1$ and the mesh size goes to 0 where the boundary of the skew plane partitions converges to piecewise linear limits. The periodicity enforces conditions on the boundary, and we characterize the family of piecewise linear limits which are achievable under this limit transition. We also show that, under this limit, the height functions converge to a deterministic limit shape and that the global fluctuations are given by the 2-dimensional Gaussian free field. Specializing to the noninteracting case, this verifies Kenyon and Okounkov's conjecture, on the fluctuations of periodic, bipartite dimer models, for the case of the r^{volume} measure and periodic variations under general boundary conditions. Chapter 2 is based on [2].

In Chapter 3, we study Markov chains formed by squared singular values of products of truncated orthogonal, unitary, symplectic matrices (corresponding to the Dyson index $\beta = 1, 2, 4$ respectively) where time is the number of terms in the product. More generally, we consider the β-Jacobi product process obtained by extrapolating to arbitrary $\beta > 0$. When the time scaling is preserved, we show that the global fluctuations are jointly Gaussian with explicit covariances. For time growing linearly with matrix size, we show convergence of moments after suitable rescaling. When $\beta = 2$, our results imply that the right edge converges to a process which interpolates between the Airy point process and a deterministic configuration. This process connects a time-parametrized family of point processes appearing in the works of Akemann-Burda-Kieburg [4] and Liu-Wang-Wang [42] across time. In the arbitrary $\beta > 0$ case, our results show tightness of the particles near the right edge. The limiting moment formulas correspond to expressions for the Laplace transform of a conjectural β-generalization of the interpolating process. Chapter 3 is based on [1].

In Chapter 4, we describe a new approach extending the method of differential operators to models which are not Macdonald processes. We do not answer any asymptotic questions, but provide the foundation for future work in the direction of asymptotics. The main results are moment formulas for two types of measures. The first type consists of spectral measures of random matrices with the form

$$U_1 A_1 U_1^* + \cdots + U_k A_k U_k^* \tag{1.2.1}$$

where $A_1, \ldots, A_k$ are arbitrary deterministic diagonal matrices and $U_1, \ldots, U_k$ are independent Haar unitary matrices. The second type consists of probability measures derived from representations of the unitary group. We express the moments of these measures in terms of certain lifts of Schur functions, and obtain contour integral representations for these lifts.

Chapter 2

Random Macdonald Plane Partitions

2.1 Introduction

Given Young diagrams $\mu \subset \lambda$, a *skew plane partition* supported in the skew diagram λ/μ is an array of nonnegative numbers $(\pi_{i,j})_{(i,j)\in\lambda/\mu}$ weakly decreasing in each index. For this chapter, we set $\lambda = N^M = (\underbrace{N, \ldots, N}_{M \text{ times}})$. By viewing $\pi_{i,j}$ as the number of unit cubes on (i,j), we may interpret a skew plane partition as a discrete, stepped surface in $\mathbb{R}^3$, see Figure 2.1. The *volume* of a skew plane partition is the number of unit cubes, that is $\sum_{i,j}\pi_{i,j}$. The projected image of this stepped surface further admits the interpretation of a skew plane partition as a lozenge tiling; a tiling of the triangular lattice by rhombi of three types. Yet another perspective is that a skew plane partition is a dimer covering of the honeycomb lattice.

			4	**2**
	4	**2**	**2**	**1**
3	**3**	**2**	**0**	**0**

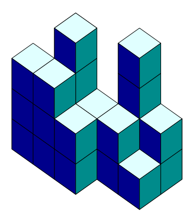
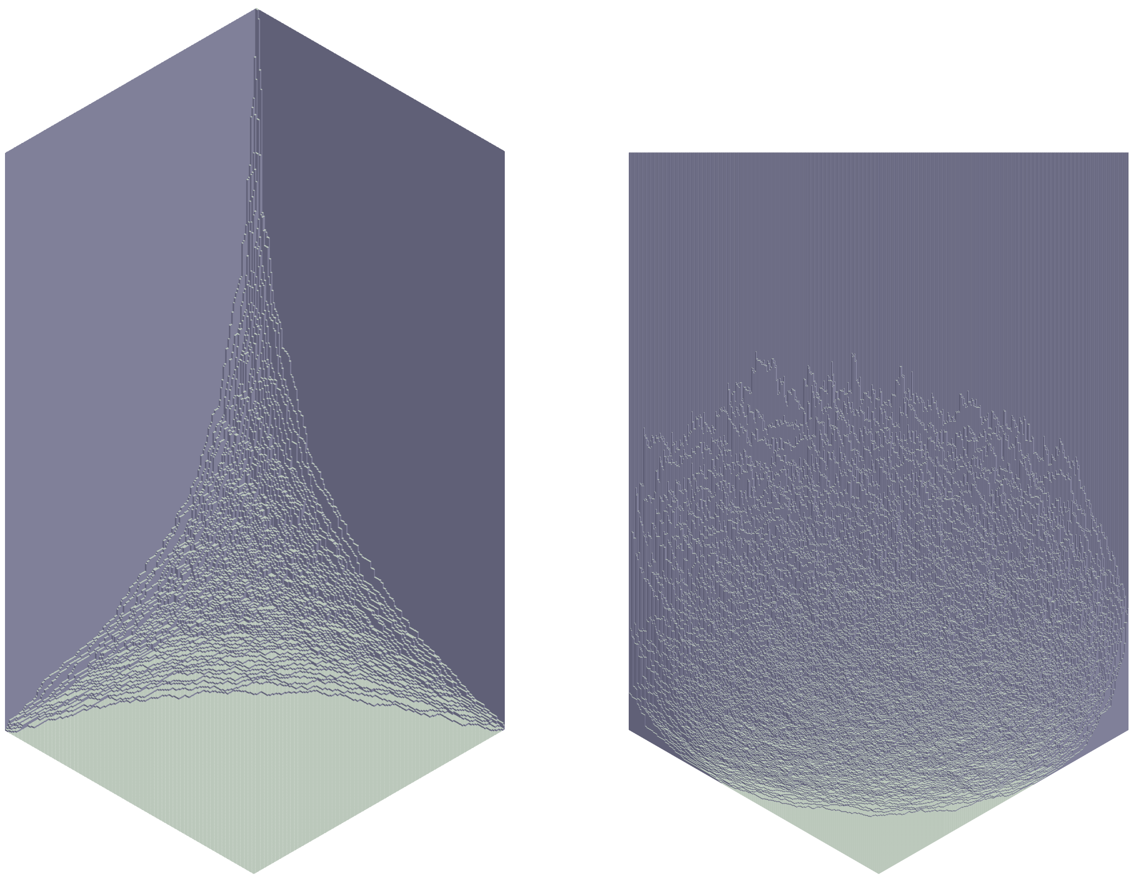


Figure 2.1: Skew plane partition with support $(5,5,5)/(3,1,0)$.

The central objects of this chapter are *Macdonald plane partitions*, a broad class of measures on skew plane partitions which are also *Macdonald processes*, we defer precise definitions of Macdonald processes and Macdonald plane partitions to Sections 2.2 and 2.3. The distribution of a Macdonald plane partition is determined by its domain N^M/μ, interaction parameters $0 < q, t < 1$, and a set of diagonal weights. We obtain general moment formulas for the height function on Macdonald plane partitions. Specializing to periodic diagonal weights, we establish a limit shape theorem and Gaussian free field fluctuations as the mesh size goes to 0 and $q, t \to 1$, see Section 2.2 for a definition of the Gaussian free field and Figure 2.2 for some large samples which demonstrate the limit shape phenomenon. This is consistent with Kenyon and Okounkov's conjecture [38, Section 1.5, page 15] that the fluctuations of bipartite, periodic dimer models are given by the Gaussian free field as the mesh size goes to 0; we note that their conjecture was stated in the context of non-interacting models. The periodic weighting places constraints on the types of boundaries we can consider, and we classify all possible limiting piecewise linear domain boundaries given these constraints. For this classification, we require a notion of *singular points*; distinguished points where the model exhibits unbounded and singular behavior. The asymptotic analysis of these singular points is one of the main technical difficulties in this chapter.



(a) Limit shape for r^{volume} ordinary partition.

(b) Limit shape for 6-periodic model

Figure 2.2: Samples of large plane partitions from the non-interacting case.

Macdonald processes were first studied by Borodin and Corwin in [6] who introduced the method of difference operators. As mentioned in the introduction to this thesis, the Macdonald processes and their degenerations form a rich family of models which intersect directed polymer, random matrix, and dimer models to name a few, e.g. [6], [8], [22]. We use a different family of operators from that of [6] which directly accesses moments of the height function for Macdonald plane partitions. These operators were discovered by Negut [49] and applied to the β-Jacobi corners process in [32] and [28, Appendix 1] by Borodin, Gorin, and Zhang.

We comment on the interaction parameters (q, t) and their relation to other models. For the

Macdonald plane partitions, the parameter pair (q, t) modulates the interaction between lozenges through the *interaction type* $\log q/\log t$. The interaction type determines the strength of the interaction where 1 corresponds to no interaction and greater deviations from 1 correspond to stronger interactions. We obtain the r^{volume} measure on plane partitions, introduced by Vershik in [65], if there is no interaction and one-periodicity in the diagonal weights. The global and local limits of the r^{volume} measure have been well-researched (see [14, 45, 51, 52]). If we set $q = 0$, we obtain the Hall-Littlewood plane partitions, parametrized by t, which were studied by Vuletić in [68] and Dimitrov in [22]. Under a limit transition which preserves the interaction type, the Macdonald plane partitions with one-periodic diagonal weights degenerate to the β-Jacobi product process where $\beta = 2\log q/\log t$, studied in Chapter 3.

The remainder of the chapter is organized as follows. Section 2.2 provides a more detailed background on random skew plane partitions, introduces the Macdonald plane partitions, and states the main results of this chapter: limit shape and global fluctuation theorems. In Section 2.3, we extend the difference operators of Negut to formal equalities for joint moments of general Macdonald processes, then specialize to obtain contour integral formulas for the joint moments of random skew plane partitions. We derive conditions for which the Macdonald plane partitions converge to well-defined *non-singular* limits and describe this notion of non-singularity in Section 2.4.1. In Section 2.5, we perform asymptotics on the contour integral formulas for the joint moments. Section 2.7 proves the main results on the limit shape and Gaussian free field fluctuations, relying on properties of (the complex structure on) the liquid region and frozen boundary obtained in Section 2.6.

Notation. Let $\mathbf{i}$ denote the imaginary unit, i.e. the square root of -1 in the upper half plane. Given an interval $[a, b] \subset \mathbb{R}$, we write $[[a, b]] := [a, b] \cap \mathbb{Z}$. Given a set $K \subset \mathbb{C}$, we denote the interior of K by $\text{int}(K)$ and the closure of K by $\text{cl}(K)$. Let $\mathbb{R}_{>0}$ ($\mathbb{R}_{\geq 0}$) denote the positive (nonnegative) real numbers and $\mathbb{Z}_{>0}$ ($\mathbb{Z}_{\geq 0}$) denote the positive (nonnegative) integers.

2.2 Model and Results

We now introduce our models and results with greater detail. For clarity, we begin by introducing the non-interacting models and results, corresponding to Subsections 2.2.1 and 2.2.2. In Subsection 2.2.3, we parallel the preceding discussion for more general interacting models.

2.2.1 Plane Partitions and Lozenge Tilings

We interchangeably say Young diagrams and partitions. Let $\mu \subset N^M = (\underbrace{N, \ldots, N}_{M \text{ times}})$ be a Young diagram. By the *back wall* of N^M/μ we mean the upper boundary of the skew diagram N^M/μ (see Figure 2.3). Let $\pi = (\pi_{i,j})$ be a skew plane partition with support N^M/μ. For $-M < v < N$, the diagonal section $\pi^v = (\pi_{a,v+a}, \pi_{a+1,v+a+1}, \ldots)$ is an ordinary partition, where a is the least integer such that $(a, v + a)$ is a box in N^M/μ.

A skew plane partition can be viewed as a 3-dimensional object by stacking $\pi_{i,j}$ cubes above the box (i, j) as in Figure 2.3. The resulting (projected) image is a tiling of lozenges ◇, ⬡, ⬡. For our purposes, we transform the lozenges by the affine transformation taking ◇ ↦ ▱, ⬡ ↦ □,

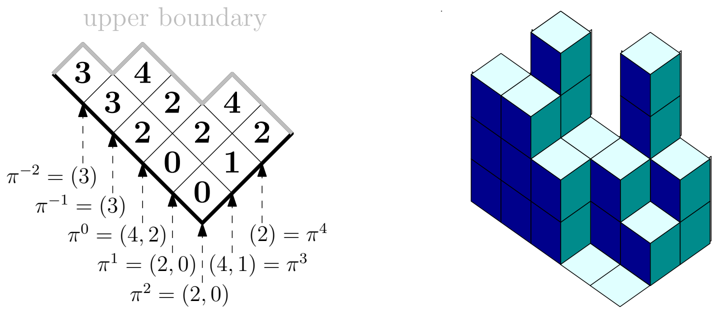


Figure 2.3: (Left) A skew plane partition with support $5^3/(3, 1, 0)$. The grey upper boundary is the back wall. We label the partitions along the diagonal sections from $v = -2$ to 3 from left to right. (Right) The skew plane partition as a 3-dimensional object.

▱ $\mapsto$ ▯. Take the standard basis of $\mathbb{R}^2$ for the resulting image with lengths so that the transformed lozenge ▫ is the unit square, see Figure 2.4. This gives a unique *projected coordinate* system for the tiling, up to the choice of origin. The back wall is then the graph of some function $B : I \to \mathbb{R}$ which is piecewise linear with slopes 0 or 1. The domain I of B is an interval of length $M + N$. For convenience, choose the origin in projected coordinates so that $I = (-M, N)$. Then the centers of the projected horizontal lozenges ▱ corresponding to the diagonal section π^v have x-coordinate v, see Figure 2.4. Denote by $\mathcal{P}_B$ the set of plane partitions with back wall B. We may also consider semi-infinite or infinite back walls by taking M or N to ∞.

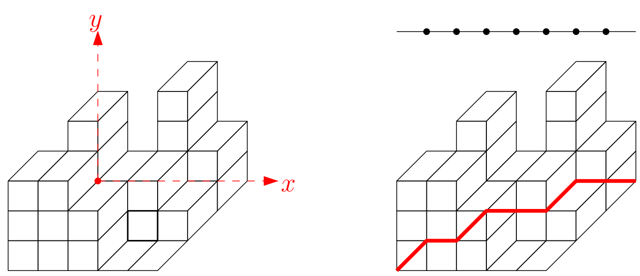


Figure 2.4: The skew plane partition in Figure 2.3 after the affine transformation. (Left) The projected coordinate axes in red. (Right) The back wall in red, the line above represents the domain $(-3, 5)$ of the back wall, and the dots correspond to coordinates of the diagonal sections.

Fix a skew plane partition $\pi \in \mathcal{P}_B$. We define the height function which takes a point (x, y) and gives the height at that point. More precisely, the *height function* $h : (I \cap \mathbb{Z}) \times \mathbb{R} \to \mathbb{R}$ is the piecewise linear function which reports the total length of vertical line segments below the point (x, y) in projected coordinates, see Figure 2.5.

For a partition $\lambda = (\lambda_1, \lambda_2, \ldots)$ define $|\lambda| = \sum_{i \geq 1} \lambda_i$. Consider the random (skew) plane partition (RPP) with probability distribution on $\mathcal{P}_B$ defined by

$$\mathbb{P}(\pi) \propto \prod_{-M<v<N} r_v^{|\pi^v|} \tag{2.2.1}$$

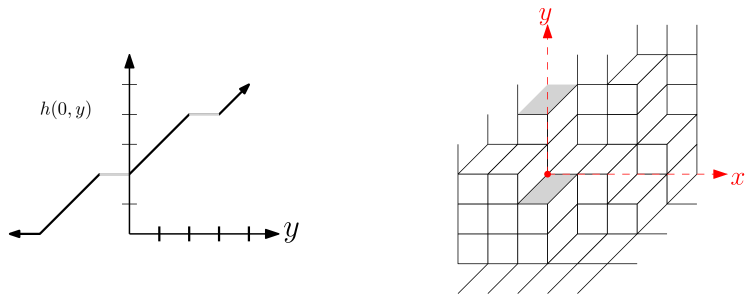


Figure 2.5: Graph of the height function at $x = 0$. The gray filled tiles correspond to the flat, gray parts of the graph.

for a sequence of weights $r_v > 0$ such that the weights above are summable. When r_v is constant in v, this is the r^{vol} measure studied in [14, 45, 51, 52].

Definition 2.2.1. Let $\boldsymbol{s} = (\ldots, s_{-1}, s_0, s_1, \ldots)$ be a p-periodic, bi-infinite sequence of positive numbers. Denote by $\mathbb{P}^{B,r,\boldsymbol{s}}$ the probability measure on $\mathcal{P}_B$ defined by (2.2.1) where

$$r_v = s_v r,$$

given that the weights are summable.

We note that local limits for a specific class of back walls B are studied for $p = 2$ in [46].

2.2.2 Results

Our main result is an explicit description of the global fluctuations, in terms of the Gaussian free field, of the measures $\mathbb{P}^{B,r,\boldsymbol{s}}$ as $r \to 1$ and B converges to some limiting $\mathcal{B}$ after rescaling. More precisely, we consider the following limit regime.

Limit Conditions. Fix a p-periodic, bi-infinite sequence $\boldsymbol{s} = (\cdots, s_{-1}, s_0, s_1, \cdots) \in \mathbb{R}^{\infty}_{>0}$ such that $s_0 \cdots s_{p-1} = 1$. Let $\mathbb{P}^{B,r,\boldsymbol{s}}$ be parametrized by a small parameter $\varepsilon > 0$ where $B : I^{\varepsilon} \to \mathbb{R}$ and $r := e^{-\varepsilon}$ vary with ε such that

1. there exist integers
$$\inf I^{\varepsilon} = v_0(\varepsilon) < \cdots < v_n(\varepsilon) = \sup I^{\varepsilon}$$
such that for each $1 \le \ell \le n$, B' is p-periodic on $(v_{\ell-1}, v_\ell) \cap (\mathbb{Z} + \frac{1}{2})$;

2. there exists an interval $I \subset \mathbb{R}$ and a piecewise linear $\mathcal{B} : I \to \mathbb{R}$ with non-differentiable points
$$\inf I = V_0 < \cdots < V_n = \sup I$$

such that

$$\varepsilon v_\ell(\varepsilon) \to V_\ell \quad (0 \le \ell \le n), \quad \varepsilon B^\varepsilon(x/\varepsilon) \to \mathcal{B}(x)$$

as $\varepsilon \to 0$, where the latter convergence is uniform over any compact subset of I.

Remark 1. The condition $s_0 \cdots s_{p-1} = 1$ is to ensure the existence of a non-trivial limit shape. If $s_0 \cdots s_{p-1} < 1$, then the limit shape becomes trivial; 0-volume upon rescaling. If $s_0 \cdots s_{p-1} > 1$, then for r close to 1 the weights of $\mathbb{P}^{B,r,\boldsymbol{s}}$ are no longer summable.

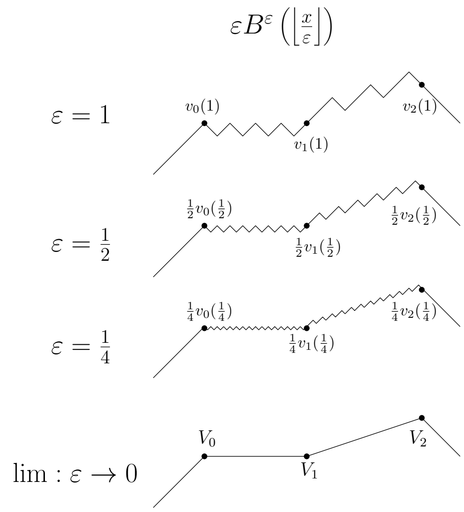


Figure 2.6: An example Limit Condition (2) illustrated by the graphs of $\varepsilon B^\varepsilon(\left(\left\lfloor \frac{x}{\varepsilon} \right\rfloor\right)$ at $\varepsilon = 1, \frac{1}{2}, \frac{1}{4}$ and the limit $\varepsilon \to 0$, along with transition points $\varepsilon v_i(\varepsilon)$.

In Section 2.4.1, we classify the set $\mathfrak{B}(\boldsymbol{s})$ of possible limits of back walls $\mathcal{B}$ attained by $\mathbb{P}^{B,r,\boldsymbol{s}}$ satisfying the Limit Conditions. Our law of large numbers and fluctuations results are restricted to a dense subset $\mathfrak{B}^\Delta(\boldsymbol{s}) \subset \mathfrak{B}(\boldsymbol{s})$, defined in Section 2.4.1. The reason for this restriction is related to the presence of *singular points*; a concept further explained in Section 2.4.1. Elements $\mathcal{B} \in \mathfrak{B}^\Delta(\boldsymbol{s})$ correspond to RPP limits with only finitely many singular points whereas $\mathcal{B} \in \mathfrak{B}(\boldsymbol{s}) \setminus \mathfrak{B}^\Delta(\boldsymbol{s})$ correspond to RPP limits with a continuum's worth of singular points. Our methods in general are limited to accessing models with finitely many singular points, thus this restriction is necessary.

Before proceeding to the main result, it is convenient to state the following limit shape result under our limit regime. Let h denote the random height function of $\mathbb{P}^{B,r,\boldsymbol{s}}$.

Theorem 2.2.2. *Suppose $\mathbb{P}^{B,r,\boldsymbol{s}}$ satisfies the Limit Conditions such that $\mathcal{B} \in \mathfrak{B}^\Delta(\boldsymbol{s})$. Then there exists a deterministic Lipschitz 1 function $\mathcal{H} : I \times \mathbb{R} \to \mathbb{R}$ such that we have the convergence*

$$\varepsilon h\left(\left\lfloor \frac{x}{\varepsilon} \right\rfloor, \frac{y}{\varepsilon}\right) \to \mathcal{H}(x,y)$$

of measures on $y \in \mathbb{R}$, weakly in probability as $\varepsilon \to 0$ for all $x \in I$. An explicit description of this height function is given in Section 2.7 in terms of its exponential moments.

Remark 2. We note that there exists an approach to Theorem 2.2.2 through the variational principle [17, 38, 39]. Our approach is different with the benefit of giving explicit formulas for exponential moments and being generalizable to the Macdonald plane partitions introduced in Section 2.2.3.

Let $p_{\diamond}, p_{\lozenge}, p_{\square}$ denote the local proportions of the subscripted lozenges, if they exist. Given the deterministic limit $\mathcal{H}$, the local proportions of lozenges at $(x, y) \in I \times \mathbb{R}$ *are well-defined* and given by

$$\nabla \mathcal{H}(x, y) = (1 - p_{\diamond}, -p_{\lozenge})$$
$$p_{\diamond} + p_{\lozenge} + p_{\square} = 1$$

It is convenient to encode the local proportions by a complex parameter $z \in \overline{\mathbb{H}}$ so that

$$p_{\diamond} = \arg z, \quad p_{\lozenge} = \frac{1}{p} \sum_{i=0}^{p-1} \arg(1 - s_0 \cdots s_i z) \tag{2.2.2}$$

where the argument is chosen to be 0 on the positive reals. There is a unique such choice of $z \in \overline{\mathbb{H}}$ for any given triple $(p_{\diamond}, p_{\lozenge}, p_{\square})$. In the case where the period is 1 the parameter z admits a nice geometric interpretation: the triangle $(0, 1, z)$ has angles $\pi(p_{\diamond}, p_{\lozenge}, p_{\square})$, see Figure 2.7. For higher periods p, the author is unaware of a simple geometric alternative to (2.2.2).

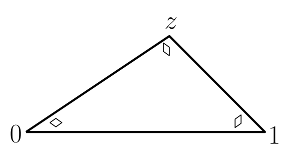


Figure 2.7: Geometric description for parameter z in 1-periodic case.

We briefly recall the pullback of the Gaussian free field. Detailed discussions of the 2-dimensional Gaussian free field can be found in [24, Section 4; 60].

Definition 2.2.3. The Gaussian free field $\mathfrak{H}$ (with Dirichlet boundary conditions) on $\mathbb{H}$ is defined to be the generalized centered Gaussian field on $\mathbb{H}$ with covariance

$$\mathbb{E}\, \mathfrak{H}(z)\mathfrak{H}(w) = -\frac{1}{2\pi} \log \left| \frac{z - w}{z - \bar{w}} \right|$$

Given a domain D and a homeomorphism $\Omega : D \to \mathbb{H}$, the Ω-pullback of the Gaussian free field $\mathfrak{H} \circ \Omega$ is a generalized centered Gaussian field on D with covariance

$$\mathbb{E}\, \mathfrak{H}(\Omega(u))\mathfrak{H}(\Omega(v)) = -\frac{1}{2\pi} \log \left| \frac{\Omega(u) - \Omega(v)}{\Omega(u) - \overline{\Omega}(v)} \right|.$$

Definition 2.2.4. Let $\mathcal{I}$ be some indexing set and some family $\{\xi_i\}_{i \in \mathcal{I}}$ of random variables. Moreover, for each $\varepsilon > 0$, define a family of random variables $\{\xi_i^\varepsilon\}_{i\in\mathcal{I}}$. We say that $\{\xi_i^\varepsilon\}_{i\in\mathcal{I}} \to \{\xi_i\}_{i\in\mathcal{I}}$ as $\varepsilon \to 0$ in distribution if for any finite collection $i_1, \dots, i_k \in I$ the random vector $(\xi_{i_1}^\varepsilon, \dots, \xi_{i_k}^\varepsilon)$ converges in distribution to $(\xi_{i_1}, \dots, \xi_{i_k})$.

Let $\mathfrak{H}$ denote the Gaussian free field with Dirichlet boundary conditions on $\mathbb{H}$, and denote by

$$\overline{h}(x, y) = h(x, y) - \mathbb{E}h(x, y)$$

the centered height function. Let the *liquid region* be defined to be the set of (x, y) such that all the local proportions $p_{\square}, p_{\square}, p_{\diamond}$ are positive. We are now ready to state the main result for the "Schur case".

Theorem 2.2.5. *Suppose $\mathbb{P}^{B,r,\boldsymbol{s}}$ satisfies the Limit Conditions such that $\mathcal{B} \in \mathfrak{B}^{\Delta}(\boldsymbol{s})$. The map $\zeta(x, y) = e^x z(x, y)$, where z is defined by (2.2.2), is a homeomorphism from the liquid region to $\mathbb{H}$. Moreover, the centered, rescaled height function $\sqrt{\pi}\,\overline{h}\left(\lfloor \frac{x}{\varepsilon} \rfloor, \frac{y}{\varepsilon}\right)$ converges to the ζ-pullback of the GFF in the sense that we have the following convergence in distribution*

$$\left\{\sqrt{\pi}\int \overline{h}\left(\left\lfloor \frac{x}{\varepsilon}\right\rfloor, \frac{y}{\varepsilon}\right) e^{-ky}\, dy\right\}_{x\in I, k\in\mathbb{Z}_{>0}} \to \left\{\int \mathfrak{H}(\zeta(x, y))e^{-ky}\, dy\right\}_{x\in I, k\in\mathbb{Z}_{>0}}.$$

In [38], Kenyon and Okounkov conjectured that the fluctuations of the height function for $\mathbb{Z}\times\mathbb{Z}$ periodic, bipartite dimer models are given by the Gaussian free field. Theorem 2.2.5 confirms this conjecture for periodically weighted skew plane partitions.

Remark 3. We note that modifying the Limit Conditions so that $r = \exp(-c\varepsilon)$ for some constant c amounts to scaling the coordinates of the plane partition. For this reason, we consider $c = 1$ to reduce the number of parameters. In this case $\zeta(x, y) = e^x z(x, y)$.

Remark 4. We emphasize that in the uniform lozenge tiling models studied in previous works, the uniformization map from $\mathscr{L}$ onto $\mathbb{H}$ is *not* given by the complex slope $\zeta(x, y)$, e.g. in [15, 16, 57]. For the uniform models, the parameter c in the remark above is taken to be 0 so that $\zeta(x, y) = z(x, y)$, and this gives a covering map from $\mathscr{L}$ onto $\mathbb{H}$ with degree > 1. As an example, $z(x, y)$ gives a 2-sheeted covering for the uniform lozenge tilings of a regular hexagon due to rotational symmetry. The reason $\zeta(x, y)$ gives a uniformization map for our models can be related to the fact that there is only one connected component for the frozen region corresponding to $\diamond$, which is a consequence of our models having no "ceiling".

Remark 5. The map ζ depends continuously on the back wall $\mathcal{B}$. Thus Theorem 2.2.5 provides a continuous family of GFFs parametrized by $\mathcal{B}$ corresponding to asymptotic RPPs.

2.2.3 Macdonald Plane Partitions

We now introduce a two-parameter family of deformation for the RPP defined by (2.2.1). These deformations correspond to the (q, t)-parameter family of Macdonald symmetric functions with (2.2.1) corresponding to the Schur case $q = t$. Instead of q, we will take a parameter $\alpha > 0$ so that

$q = t^\alpha$. Fix $0 < t < 1$, $\alpha > 0$ and define the corresponding probability distribution on $\mathcal{P}_B$

$$\mathbb{P}(\pi) = w_{\alpha,t}(\pi) \prod_{-M<v<N} r_v^{|\pi^v|} \tag{2.2.3}$$

where $w_{\alpha,t}$ is an r-independent (α, t)-*Macdonald weight*, and the summability of the weights (2.2.3) coincides with the summability of the weights (2.2.1). The Macdonald weights $w_{\alpha,t}(\pi)$ can be described in terms of semilocal contributions of the plane partition. We explain this in greater detail below, after introducing the coordinate system.

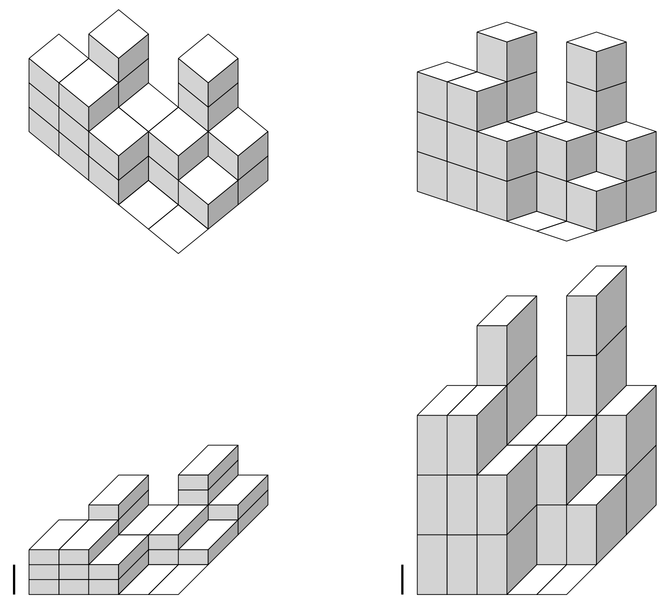


Figure 2.8: Left: $\alpha = 1/2$; Right: $\alpha = 2$. Top: 3-d partition projected onto $(1, \alpha, \alpha)$-plane. Bottom: Transformed tiling with scaling so that the line segments by the lower left corner denote unit lengths.

For the Macdonald plane partitions, it will be convenient to consider a different tiling and set of coordinates which we call the α-*coordinates*. We transform ◇, ◫, ▱ to ▱, □, ▱ where the widths of □, ▱ are 1, the height of □ is α, and the height of ▱ is 1. This transformation is not affine since the height of ▱ does not scale by α for $\alpha \neq 1$, see Figure 2.8. To understand where the coordinates come from, consider the 3-dimensional plane partition. If the height corresponds to the third coordinate, then the projection in Figure 2.3 is onto the $(1, 1, 1)$-plane. If instead we project onto the $(\alpha, \alpha, 1)$-plane, then after choosing the basis parallel to the edges of the ◫ lozenge we obtain the α-coordinates (up to sign of direction), see top row of Figure 2.8.

Fix a plane partition $\pi \in \mathcal{P}_B$. We define the height function as before which gives the height at (x, y). More precisely, the *height function* $h : (I \cap \mathbb{Z}) \times \mathbb{R}$ is defined as $\frac{1}{\alpha}$ times the total length of vertical line segments beneath a point (x, y) in α-coordinates, see Figure 2.10. The $\frac{1}{\alpha}$ term is included because the α-coordinates *contract* the height from $\mathbb{R}^3$ by α.

Consider further these vertical segments which are formed by intersections of an adjacent pair of lozenges ⬯, ▫. We say that the vertical segment formed by the intersection of such a pair of lozenges is a *turn*. If the pair goes from ⬯ to ▫ (▫ to ⬯) from left to right, then we call it an *internal turn* (*external turn*), see Figure 2.9.

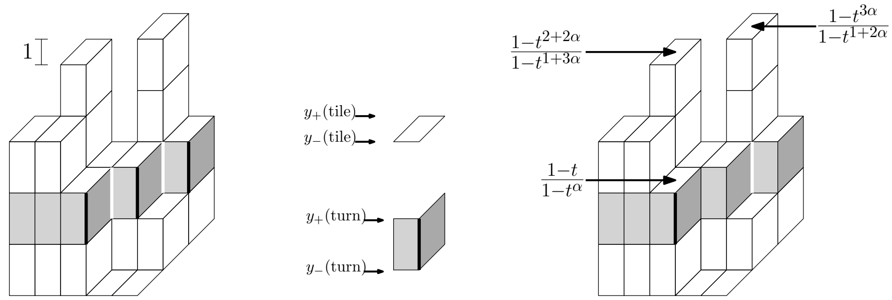


Figure 2.9: Left: We mark external and internal turns black and white resp. along the grey band. Middle: y_+, y_- indicated for tiles and turns. Right: Weights contributed by the flat tiles above the highlighted turns.

Denote the set of turns of π by $\mathcal{T}(\pi)$. A turn T is a vertical segment $\{x_0\} \times [y_0, y_0 + \alpha]$ and we set $x(T) = x_0$, $y_-(T) = y_0$, $y_+(T) = y_0 + \alpha$. Similarly, given a lozenge ▱ along the diagonal section $x = v$ it spans a set of y-coordinates of the form $[y_1, y_1 + 1]$ in which case we let $x(▱) = v$, $y_-(▱) = y_1$, $y_+(▱) = y_1 + 1$. We now introduce an interaction between a turn $T \in \mathcal{T}(\pi)$ and horizontal lozenges ▱ which lie directly above it, given by the weight

$$v_{\alpha,t}(\pi, T) = \prod_{\substack{▱ : x(▱)=x(T) \\ y_-(▱) \geq y_+(T)}} \frac{1 - t^{y_+(▱) - y_+(T)}}{1 - t^{y_-(▱) - y_-(T)}}. \tag{2.2.4}$$

Note that if $\alpha = 1$, then the weight is identically 1. If $\alpha < 1$ (> 1), then each fraction in (2.2.4) is > 1 (< 1). With this setup, we now define the Macdonald weight:

$$w_{\alpha,t}(\pi) = \prod_{\substack{T \in \mathcal{T}(\pi) \\ T \text{ is external}}} v_{\alpha,t}(\pi, T) \prod_{\substack{T \in \mathcal{T}(\pi) \\ T \text{ is internal}}} v_{\alpha,t}(\pi, T)^{-1}.$$

In words, if $\alpha < 1$ then the weight $w_{\alpha,t}$ favors external turns over internal turns, and the strength of the preference is amplified by the presence of horizontal lozenges directly above the turn. Decreasing α further exaggerates this interaction. For $\alpha > 1$, this preference is reversed for internal and external turns, and increasing α exaggerates the interaction.

Analogues of Theorems 2.2.2 and 2.2.5 exist for the Macdonald RPPs.

Definition 2.2.6. Let $\boldsymbol{s} = (\ldots, s_{-1}, s_0, s_1, \ldots)$ be a p-periodic, bi-infinite sequence of positive numbers such that $s_0 \cdots s_p = 1$. Denote by $\mathbb{P}^{B,r,\boldsymbol{s}}_{\alpha,\mathrm{t}}$ the probability measure on $\mathcal{P}_B$ defined by (2.2.3) where

$$r_v = s_v r, \quad t = r^{\mathrm{t}}, \quad q = t^{\alpha}$$

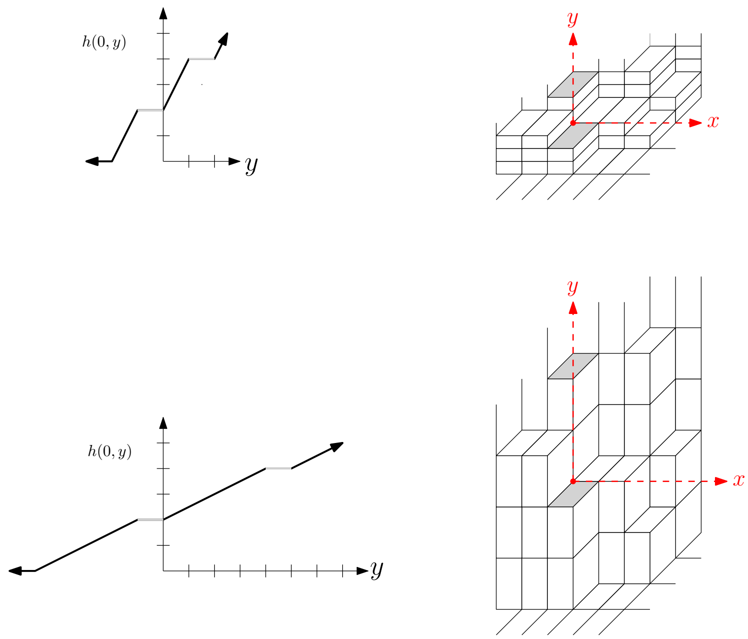


Figure 2.10: For $\alpha = 1/2$ (top) and $\alpha = 2$ (bottom), the graph of the height function at $x = 0$ and the associated tiling where the gray filled tiles correspond to the flat, gray parts of the graph.

given that the weights are summable.

This generalizes the family $\mathbb{P}^{B,r,\boldsymbol{s}}$ defined earlier which corresponds to $\alpha = 1$ (in which case the value of $\mathfrak{t}$ is immaterial). The limit regime we consider is a generalization of the Limit Conditions for $\mathbb{P}^{B,r,\boldsymbol{s}}_{\alpha,\mathfrak{t}}$ where we fix $\boldsymbol{s}, \alpha, \mathfrak{t}$.

Theorem 2.2.7. *Suppose $\mathbb{P}^{B,r,\boldsymbol{s}}_{\alpha,\mathfrak{t}}$ satisfy Limit Conditions with fixed $\boldsymbol{s}, \alpha, \mathfrak{t}$ such that $\mathcal{B} \in \mathfrak{B}^{\Delta}(\boldsymbol{s})$. Then there is a deterministic Lipschitz 1 function $\mathcal{H} : I \times \mathbb{R} \to \mathbb{R}$* independent of $\alpha, \mathfrak{t}$ *such that we have the convergence*

$$\varepsilon h \left(\left\lfloor \frac{x}{\varepsilon} \right\rfloor, \frac{y}{\varepsilon} \right) \to \frac{1}{\alpha} \mathcal{H}(x, y)$$

of measures on $y \in \mathbb{R}$, weakly in probability as $\varepsilon \to 0$ for all $x \in I \setminus \{V_\ell\}_{\ell=0}^{n}$ (recall these are the differentiable points of the continuous, piecewise linear limit $\mathcal{B}$ of back walls). An explicit description of this height function is given in Section 2.7.

Theorem 2.2.8. *Suppose $\mathbb{P}^{B,r,\boldsymbol{s}}_{\alpha,\mathfrak{t}}$ satisfy Limit Conditions with fixed $\boldsymbol{s}, \alpha, \mathfrak{t}$ such that $\mathcal{B} \in \mathfrak{B}^{\Delta}(\boldsymbol{s})$. Then the map $\zeta(x, y) = e^x z(x, y)$, where z is defined by (2.2.2), is a homeomorphism from the liquid region to $\mathbb{H}$* independent of $\alpha, \mathfrak{t}$. *Moreover, the centered, rescaled height function $\sqrt{\alpha\pi}\, \overline{h} \left(\lfloor \frac{x}{\varepsilon} \rfloor, \frac{y}{\varepsilon} \right)$ converges to the ζ-pullback of the GFF in the sense that we have the following convergence in distribution of the random family*

$$\left\{ \sqrt{\alpha\pi} \int \overline{h} \left(\left\lfloor \frac{x}{\varepsilon} \right\rfloor, \frac{y}{\varepsilon} \right) e^{-k\mathfrak{t}y}\, dy \right\}_{x \in I, k \in \mathbb{Z}_{>0}} \to \left\{ \int \mathfrak{H}(\zeta(x, y)) e^{-k\mathfrak{t}y}\, dy \right\}_{x \in I, k \in \mathbb{Z}_{>0}}$$

for all $x \in I \setminus \{V_\ell\}_{\ell=0}^n$.

Remark 6. We prove stronger statements (see Theorems 2.7.2 and 2.7.4) which remove the restriction $x \in I \setminus \{V_\ell\}_{\ell=0}^n$ in exchange for a microscopic separation condition. In these improved theorems, we replace $\lfloor x/\varepsilon \rfloor$ with some sequence $x(\varepsilon)$ such that $\varepsilon x(\varepsilon) \to x$ for *any* $x \in I$ with the caveat that certain $\varepsilon x(\varepsilon)$ need to be separated by some microscopic distance from certain *singular points*, see Definition 2.4.6. This separation condition can be removed for the $\alpha = 1$ case, and is also unnecessary whenever $\mathsf{t} = k$ for any positive integer $k > 0$. We expect that the statement of Theorem 2.2.8 should still hold in the absence of this condition. Due to technical complications, we did not pursue this refinement.

Notation. Let $B : I \to \mathbb{R}$ be a back wall for some RPP. Denote $I_V = I \cap \mathbb{Z}$ and $I_E = I \cap (\mathbb{Z} + \frac{1}{2})$. For back walls B^a denoted by superscripts, we denote the corresponding sets with superscripts: I^a (domain of B^a), $I_V^a = I^a \cap \mathbb{Z}$, $I_E^a = I^a \cap (\mathbb{Z} + \frac{1}{2})$.

2.3 Joint Expectations of Observables

The main goal of this section is to obtain formulas for expectations associated to the height function. Consider

$$\wp_k(\lambda; q, t) = (1 - t^{-k}) \sum_{i=1}^{\ell(\lambda)} q^{k\lambda_i} t^{k(-i+1)} + t^{-k\ell(\lambda)}$$

where $\lambda = (\lambda_1, \lambda_2, \ldots)$ is a partition, $\ell(\lambda)$ denotes the number of indices i such that $\lambda_i \neq 0$, and $k \in \mathbb{Z}_{\geq 0}$. The following proposition gives a connection between $\wp_k$ and height functions.

Proposition 2.3.1. *Consider a plane partition* $\pi \in \mathcal{P}_B$. *Fix* $\alpha > 0$, *and let* h *be the height function. Then*

$$\int_{-\infty}^{\infty} h(x, y) t^{ky}\, dy = \frac{t^{kB(x)}}{\alpha k^2 (\log t)^2} \cdot \wp_k(\pi^x; t^\alpha, t).$$

The main result of this section (stated in Theorem 2.3.17) is a formula for the joint expectation

$$\mathbb{E}[\wp_{k_1}(\pi^{x_1}) \cdots \wp_{k_m}(\pi^{x_m})], \tag{2.3.1}$$

where $k_1, \ldots, k_m \in \mathbb{Z}_{\geq 0}$, $x_1, \ldots, x_m \in I$, and $(\pi^x)_{x \in I}$ are the diagonals of $\pi \sim \mathbb{P}_{\alpha,\mathsf{t}}^{B,r,\boldsymbol{s}}$. This gives us an expression whose asymptotics are accessible, and the aforementioned proposition provides the link to interpret these asymptotics in terms of the height function.

To arrive at a formula for (2.3.1), we establish a more general expression for observables of *formal* Macdonald processes in Section 2.3.1 (Theorem 2.3.12 and Corollary 2.3.13). Here, we combine and generalize the approaches of [6], [7] and [32]. In Section 2.3.2, we specialize these formal expressions to the case of $\mathbb{P}_{\alpha,\mathsf{t}}^{B,r,\boldsymbol{s}}$ to prove Theorem 2.3.17; our formula for (2.3.1). We note that the formal expressions obtained in Section 2.3.1 are applicable to a much more general setting than ours.

Before proceeding, we prove Proposition 2.3.1.

Proof of Proposition 2.3.1. As defined in Section 2.2, the height function at (x, y) is the total length of vertical line segments beneath a point (x, y). Let Y_i denote the y_+ coordinate of the ith highest $\varparallelogram$ lozenge along the x-diagonal section. Then $Y_i = \alpha\pi_i^x - i + 1 + B(x)$. We claim that the height function is given by the formula

$$h(x, y) = \frac{1}{\alpha}\left(y - B(x) + \int_{-1}^{0} |\{i \geq 1 : Y_i + u \geq y\}|\, du\right). \tag{2.3.2}$$

To see how to obtain (2.3.2), note that the integral counts the total number of $\varparallelogram$ lozenges lying above (x, y), counting non-integer amounts of $\varparallelogram$ if (x, y) lies on the lozenge by the vertical distance from (x, y) to the top of $\varparallelogram$. For y large, the height function is just $\frac{1}{\alpha}(y - B(x))$. As we decrease y, the integral term in (2.3.2) enters since no vertical segments are added when passsing through a $\varparallelogram$ lozenge. This proves the claim.

Let $N = \ell(\pi^x)$, and note that $\partial_y h = \frac{1}{\alpha}\left(1 - \sum_{i=1}^{\infty} \mathbf{1}[Y_i - 1, Y_i]\right)$ which is 0 on $(-\infty, Y_{N+1}) = (-\infty, -N + B(x))$. Then

$$\begin{aligned}
\int_{-\infty}^{\infty} h(x, y) t^{ky}\, dy &= -\frac{1}{k \log t} \int_{-\infty}^{\infty} (\partial_y h)(x, y) t^{ky}\, dy \\
&= -\frac{1}{\alpha k \log t} \int_{Y_{N+1}}^{\infty} \left(1 - \sum_{i=1}^{N} \mathbf{1}[Y_i - 1, Y_i](y)\right) t^{ky}\, dy \\
&= \frac{1}{\alpha k^2 (\log t)^2} \left(t^{kY_{N+1}} + (1 - t^{-k}) \sum_{i=1}^{N} t^{kY_i}\right) \\
&= \frac{t^{kB(x)}}{\alpha k^2 (\log t)^2} \cdot \wp_k(\pi^x; t^\alpha, t).
\end{aligned}$$

□

2.3.1 Formal Expectations

In this subsection, we obtain formal expressions for observables of formal Macdonald processes. In Sections 2.3.1, 2.3.1, 2.3.1, we provide some background on symmetric functions and notions to give rigorous meaning to the formal expressions we work with. In Section 2.3.1, we define the formal Macdonald process and associated objects. In Section 2.3.1, we give a formal expression for single cut observables of formal Macdonald processes, originally obtained in [32]. In Section 2.3.1, we extend these formulas to multicut observables of formal Macdonald processes.

Symmetric Functions

The following background on symmetric functions and additional details can be found in [43, Chapters I and VI].

Let $\mathbb{Y}$ denote the set of partitions. Recall that we represent $\lambda \in \mathbb{Y}$ as the nondecreasing sequence $(\lambda_1, \lambda_2, \ldots)$ of its parts and denote by $\ell(\lambda)$ the number indices i such that $\lambda_i \neq 0$. Given $\mu, \lambda \in \mathbb{Y}$, we write $\mu \prec \lambda$ if $\ell(\mu), \ell(\lambda) \le N$ and

$$\lambda_1 \ge \mu_1 \ge \lambda_2 \ge \mu_2 \ge \cdots \ge \lambda_N \ge \mu_N.$$

Given a countably infinite set $X = (X_1, X_2, \ldots)$ of variables, let Λ_X denote the algebra of symmetric functions on X over $\mathbb{C}$. For sets $X^{(1)}, \ldots, X^{(n)}$ of variables, let $\Lambda_{(X^{(1)}, \ldots, X^{(n)})}$ denote the algebra of symmetric functions on the disjoint union of these sets.

Recall the power symmetric functions $p_0(X) = 1$ and

$$p_k(X) = \sum_{i \ge 1} X_i^k, \quad k \in \mathbb{Z}_{>0}.$$

These symmetric functions are generators of the algebra Λ_X. For each $\lambda \in \mathbb{Y}$, define

$$p_\lambda(X) = \prod_{i=1}^{\ell(\lambda)} p_{\lambda_i}(X).$$

Then $\{p_\lambda(X)\}_{\lambda \in \mathbb{Y}}$ forms a linear basis of Λ_X. Fixing $0 < q, t < 1$, we have the scalar product

$$\langle p_\lambda, p_\mu \rangle = \delta_{\lambda\mu} \prod_{i=1}^{\ell(\lambda)} \frac{1 - q^{\lambda_i}}{1 - t^{\lambda_i}} \prod_{i=1}^{\infty} i^{m_i(\lambda)} m_i(\lambda)!$$

where $m_i(\lambda)$ is the multiplicity of i in λ.

The *Macdonald symmetric functions* $\{P_\lambda(X; q, t)\}_{\lambda \in \mathbb{Y}}$ are the unique (homogeneous) symmetric functions satisfying

$$\langle P_\lambda(X; q, t), P_\mu(X; q, t) \rangle = 0$$

for $\lambda \ne \mu$ and with leading monomial $X_1^{\lambda_1} X_2^{\lambda_2} \cdots$ with respect to lexicographical ordering of the powers $(\lambda_1, \lambda_2, \ldots)$. This implies that $\{P_\lambda(X; q, t)\}_{\lambda \in \mathbb{Y}}$ forms a linear basis for Λ_X. Let $Q_\lambda(X; q, t)$ represent the multiple of $P_\lambda(X; q, t)$ satisfying

$$\langle P_\lambda(X; q, t), Q_\lambda(X; q, t) \rangle = 1.$$

For $\lambda, \mu \in \mathbb{Y}$, the *skew Macdonald symmetric functions* $P_{\lambda/\mu}(X; q, t)$, $Q_{\lambda/\mu}(X; q, t)$ are uniquely defined by

$$P_\lambda(X, Y) = \sum_{\mu \in \mathbb{Y}} P_{\lambda/\mu}(X) P_\mu(Y),$$
$$Q_\lambda(X, Y) = \sum_{\mu \in \mathbb{Y}} Q_{\lambda/\mu}(X) Q_\mu(Y).$$

For a single variable x, we have the following expressions for skew Macdonald symmetric functions.

Let $f(u) = \frac{(tu;q)_\infty}{(qu;q)_\infty}$ with $(u;q)_\infty := \prod_{i\geq 0}(1-uq^i)$,

$$P_{\lambda/\mu}(x) = \delta_{\mu\prec\lambda}\psi_{\lambda/\mu}(q,t)x^{|\lambda|-|\mu|} \qquad \text{and} \qquad Q_{\lambda/\mu}(x) = \delta_{\mu\prec\lambda}\phi_{\lambda/\mu}(q,t)x^{|\lambda|-|\mu|} \tag{2.3.3}$$

where the coefficients are

$$\psi_{\lambda/\mu}(q,t) = \prod_{1\leq i\leq j\leq \ell(\lambda)} \frac{f(q^{\mu_i-\mu_j}t^{j-i})f(q^{\lambda_i-\lambda_{j+1}}t^{j-i})}{f(q^{\lambda_i-\mu_j}t^{j-i})f(q^{\mu_i-\lambda_{j+1}}t^{j-i})}, \tag{2.3.4}$$

$$\phi_{\lambda/\mu}(q,t) = \prod_{1\leq i\leq j\leq \ell(\mu)} \frac{f(q^{\lambda_i-\lambda_j}t^{j-i})f(q^{\mu_i-\mu_{j+1}}t^{j-i})}{f(q^{\lambda_i-\mu_j}t^{j-i})f(q^{\mu_i-\lambda_{j+1}}t^{j-i})}. \tag{2.3.5}$$

The skew Macdonald symmetric functions satisfy the branching rule:

$$P_{\lambda/\nu}(X,Y) = \sum_{\mu\in\mathbb{Y}} P_{\lambda/\mu}(X)P_{\mu/\nu}(Y), \quad Q_{\lambda/\nu}(X,Y) = \sum_{\mu\in\mathbb{Y}} Q_{\lambda/\mu}(X)Q_{\mu/\nu}(Y) \tag{2.3.6}$$

for any $\lambda, \nu \in \mathbb{Y}$.

We say that a unital algebra homomorphism $\rho : \Lambda_X \to \mathbb{C}$ is a *specialization*. Given a specialization ρ and $f \in \Lambda_X$, we write $f(\rho)$ instead of $\rho(f)$ in view of the special case of function evaluation. The specializations we are interested in will have the following form. Take a sequence $\{a_i\}_{i=1}^\infty$ of nonnegative real numbers such that $a_1 \geq a_2 \geq \cdots$ and $\sum_{i=1}^\infty a_i < \infty$, define ρ by

$$p_n(\rho) = \sum_{i=1}^\infty a_i^n$$

for $n > 0$. This uniquely determines the specialization ρ because the power symmetric functions generate the algebra of symmetric functions. For such specializations, we may write $\rho = (a_1, a_2, a_3, \ldots)$. If the only nonzero members of the sequence are $a_1, \ldots, a_N$, we may write $\rho = (a_1, \ldots, a_N)$.

A specialization ρ is (q,t)-*Macdonald-positive* if $P_\lambda(\rho; q, t) \geq 0$ for all partitions λ. The aforementioned specialization $\rho = (a_1, a_2, \ldots)$ with $a_i \geq 0$ for all $i \geq 1$ is Macdonald positive, as follows from the nonnegativity of (2.3.3), (2.3.4), (2.3.5).

Graded Topology

Let F be a field and $\mathcal{A}$ be a $(\mathbb{Z}_{\geq 0}$-)graded algebra over F. Let $\mathcal{A}_n$ denote the nth homogeneous component of $\mathcal{A}$. Throughout this section, let us assume that all of our graded algebras have $\dim \mathcal{A}_n < \infty$ for every $n \geq 0$.

Definition 2.3.2. Given $a \in \mathcal{A}$, define $\operatorname{ldeg}(a)$ to be the minimum degree among the homogeneous components of a. The *graded topology* is the topology on $\mathcal{A}$ where a sequence $a_n \in \mathcal{A}$ converges to $a \in \mathcal{A}$ if and only if

$$\operatorname{ldeg}(a_n - a) \to \infty$$

as $n \to \infty$. Denote the completion of $\mathcal{A}$ under this topology by $\widehat{\mathcal{A}}$.

The completion $\widehat{\mathcal{A}}$ consists of formal sums $\sum_{n=1}^{\infty} a_n$ where $a_n \in \mathcal{A}_n$. Given two graded algebras $\mathcal{A}$ and $\mathcal{A}'$ over F, we give the following grading to $\mathcal{A} \otimes_F \mathcal{A}'$. If $a \in \mathcal{A}_m$ and $a' \in \mathcal{A}'_n$, then $a \otimes a' \in (\mathcal{A} \otimes_F \mathcal{A}')_{m+n}$.

For a field $F \supset \mathbb{C}$ and a graded algebra $\mathcal{A}$ over $\mathbb{C}$, denote by $\mathcal{A}[F]$ the graded algebra $\mathcal{A} \otimes_{\mathbb{C}} F$ over F; i.e. the extension of scalars from $\mathbb{C}$ to F. Given graded algebras $\mathcal{A}^{(1)}, \ldots, \mathcal{A}^{(k)}$ over $\mathbb{C}$, we denote the completion of $(\mathcal{A}^{(1)} \otimes \cdots \otimes \mathcal{A}^{(n)})[F]$ under the graded topology by

$$\mathcal{A}^{(1)} \,\widehat{\otimes}\, \cdots \,\widehat{\otimes}\, \mathcal{A}^{(k)}[F] \quad \text{or} \quad \widehat{\bigotimes}_{i=1}^{k} \mathcal{A}^{(i)}[F].$$

Let $\Lambda_X[F]$ denote the F-algebra of symmetric functions in $X = \{x_1, x_2, \ldots\}$, a set of variables, with coefficients in F. Take the natural grading on $\Lambda_X[F]$ in which $(\Lambda_X[F])_n$ is spanned by monomials of total degree n. Given disjoint ordered sets of variables $Z_1, \ldots, Z_n$ with $Z_i = (z_{i,1}, \ldots, z_{i,k_i})$, let $\mathcal{L}(Z_1, \ldots, Z_n)$ denote the field of formal Laurent series in the variables

$$\bigcup_{i=1}^{n} \left\{ \frac{z_{i,1}}{z_{i,2}}, \ldots, \frac{z_{i,k_i-1}}{z_{i,k_i}}, z_{i,k_i} \right\}.$$

The space $\widehat{\bigotimes}_{i=1}^{k} \Lambda_{X^i}[F]$ consists of formal sums

$$\sum_{\lambda^1, \ldots, \lambda^N \in \mathbb{Y}} c_{\lambda^1, \ldots, \lambda^N} P_{\lambda^1}(X^1) \cdots P_{\lambda^N}(X^N)$$

where $c_{\lambda^1, \ldots, \lambda^N} \in F$.

For fields $\mathbb{C} \subset F_1 \subset F_2$, $1 \le k \le N$, there is the natural inclusion map

$$\widehat{\bigotimes}_{i=1}^{k} \Lambda_{X^i}[F_1] \hookrightarrow \widehat{\bigotimes}_{i=1}^{k} \Lambda_{X^i}[F_2]. \tag{2.3.7}$$

We also have consistency

$$\Lambda_{X^1}[F] \otimes_F \cdots \otimes_F \Lambda_{X^N}[F] \cong \Lambda_{X^1} \otimes \cdots \otimes \Lambda_{X^N}[F]. \tag{2.3.8}$$

Definition 2.3.3. The *projection map* $\pi_X^n : \widehat{\Lambda}_X \to \widehat{\Lambda}_{\{x_1, \ldots, x_n\}}$ is defined as the continuous map sending $x_{n+1}, x_{n+2}, \ldots$ to 0 and x_i to x_i for $i = 1, \ldots, n$.

For a field $F \supset \mathbb{C}$ and a graded algebra $\mathcal{A}$ over $\mathbb{C}$, we can extend the domain of the projection

$$\pi_X^n : \mathcal{A} \,\widehat{\otimes}\, \Lambda_X[F] \to \mathcal{A} \,\widehat{\otimes}\, \Lambda_{\{x_1, \ldots, x_n\}}[F]$$

by identifying with $1_{\mathcal{A}} \otimes \pi_X^n$ then extending by continuity under the graded topology.

Definition 2.3.4. Let $\mathcal{A}$ and $\mathcal{A}'$ be graded algebras over $\mathbb{C}$ and $\{a_{n,j}\}_j$ be a basis for $\mathcal{A}_n$ for each $n \ge 0$. We say that an element $f \in \mathcal{A} \,\widehat{\otimes}\, \mathcal{A}'[F]$ is $\mathcal{A}$-*projective* if

$$f = \sum_{n,j} a_{n,j} \otimes \alpha'_{n,j}, \quad \alpha'_{n,j} \in \mathcal{A}'_n$$

such that $\lim_{n\to\infty} \min_j \operatorname{ldeg}(\alpha'_{n,j}) = \infty$. This property is independent of the choice of basis.

Elements which are $\mathcal{A}$-projective are closed under addition and multiplication and form a subalgebra of $\mathcal{A} \mathbin{\widehat{\otimes}} \mathcal{A}'[F]$. If $\mathcal{A} = \Lambda_X$, denote the algebra of Λ_X-projective elements by $\mathcal{P}_X(\Lambda_X \mathbin{\widehat{\otimes}} \mathcal{A}'[F])$.

Macdonald Pairing and Residue

Recall the Macdonald scalar product determined by

$$\langle P_\lambda, Q_\mu \rangle = \delta_{\lambda\mu}.$$

Definition 2.3.5. Let $\mathcal{A}, \mathcal{A}'$ be graded algebras over $\mathbb{C}$. Fix a field $F \supset \mathbb{C}$, and let the *Macdonald pairing* be the bilinear map $\langle \cdot, \cdot \rangle_Y : (\mathcal{A} \otimes \Lambda_X)[F] \times (\Lambda_X \otimes \mathcal{A}')[F] \to \mathcal{A} \otimes \mathcal{A}'[F]$ defined by

$$\langle a \otimes P_\lambda, Q_\mu \otimes b \rangle_X := \langle P_\lambda, Q_\mu \rangle a \otimes b = \delta_{\lambda\mu} a \otimes b.$$

This pairing does not extend by continuity to the completions of the domain. However, the pairing does extend continuously to

$$\mathcal{P}_X(\mathcal{A} \mathbin{\widehat{\otimes}} \Lambda_X[F]) \times (\Lambda_X \mathbin{\widehat{\otimes}} \mathcal{A}'[F]).$$

Definition 2.3.6. Given an ordered set $Z = (z_1, \ldots, z_k)$ of variables, denote by $\oint dZ : \mathcal{L}(Z) \to \mathbb{C}$ the *residue operator* which takes an element of $\mathcal{L}(Z)$ and returns the coefficient of $(z_1 \cdots z_k)^{-1}$. For $\oint dZ$ applied to $f \in \mathcal{L}(Z)$ we write $\oint f\, dZ$ or $\oint dZ \cdot f$.

As with the projection map, the residue operator can act on larger domains. For example, we can extend

$$\oint dZ : \widehat{\mathcal{A}}[\mathcal{L}(Z, W^1, \ldots, W^k)] \to \widehat{\mathcal{A}}[\mathcal{L}(W^1, \ldots, W^k)] \tag{2.3.9}$$

by the action $1_{\mathcal{A}} \otimes \oint dZ$ then extension by continuity. In this case, $\oint dZ$ preserves the degree of homogeneous elements. In particular, if we replace $\widehat{\mathcal{A}}$ with $\mathcal{A} \mathbin{\widehat{\otimes}} \mathcal{A}'$, we have that $\oint dZ$ preserves $\mathcal{A}$-projectivity.

The residue operator commutes with continuous maps under the graded topology.

Lemma 2.3.7. *Let $\mathcal{A}$, $\mathcal{A}'$ be graded algebras over $\mathbb{C}$, and let $\varphi : \widehat{\mathcal{A}} \to \widehat{\mathcal{A}}'$ be a continuous map which extends naturally to a continuous map $\widehat{\mathcal{A}}[\mathcal{L}(Z, W)] \to \widehat{\mathcal{A}}'[\mathcal{L}(Z, W)]$. Then*

$$\varphi \circ \left(\oint dZ \right) = \left(\oint dZ \right) \circ \varphi.$$

Lemma 2.3.8. *Let $\mathcal{A}$, $\mathcal{A}'$ be graded algebras over $\mathbb{C}$, let $f \in \mathcal{A} \mathbin{\widehat{\otimes}} \Lambda_X[\mathcal{L}(Z)]$ and $g \in \Lambda_X \mathbin{\widehat{\otimes}} \mathcal{A}'[\mathcal{L}(W)]$.*

If f is Λ_X-projective, then

$$\left\langle \oint f\, dZ, g \right\rangle_X = \oint \langle f, g\rangle_X \, dZ, \tag{2.3.10}$$

$$\left\langle f, \oint g\, dW \right\rangle_X = \oint \langle f, g\rangle_X \, dW \tag{2.3.11}$$

Since the residue operator preserves projectivity, the left hand sides of the equalities above are valid expressions.

Proof. For arbitrary $g \in \Lambda_X \widehat{\otimes} \mathcal{A}'[\mathcal{L}(W)]$, the map $\langle \cdot, g\rangle_X$ is continuous on $\mathcal{P}_X(\mathcal{A} \otimes \Lambda_X[\mathcal{L}(Z)])$. By Lemma 2.3.7, (2.3.10) follows. For $f \in \mathcal{P}_X(\mathcal{A} \widehat{\otimes} \Lambda_X(\mathcal{L}(Z)])$, the map $\langle f, \cdot\rangle$ is continuous on $\Lambda_X \widehat{\otimes} \mathcal{A}'[\mathcal{L}(W)]$. By Lemma 2.3.7, (2.3.11) follows. $\square$

Formal Macdonald Processes

Let X, Y be countable sets of variables. Fix $0 < q, t < 1$ throughout this section. Define the following element of $\Lambda_X \widehat{\otimes} \Lambda_Y$

$$\Pi(X, Y) := \prod_{x\in X, y\in Y} \frac{(txy; q)_\infty}{(xy; q)_\infty}$$

From [43, Chapter VI, Sections 2 and 4], we have the following equalities

$$\Pi(X, Y) = \sum_{\lambda\in\mathbb{Y}} P_\lambda(X) Q_\lambda(Y) = \exp\left(\sum_{n=1}^{\infty} \frac{1-t^n}{1-q^n}\frac{1}{n} p_n(X) p_n(Y)\right).$$

Define the following element of $\Lambda_X \widehat{\otimes} \Lambda_Y$ obtained by taking $q = 0$ above

$$H(X, Y; t) = \prod_{x\in X, y\in Y} \frac{1-txy}{1-xy} = \exp\left(\sum_{n=1}^{\infty} \frac{1-t^n}{n} p_n(X) p_n(Y)\right). \tag{2.3.12}$$

Given countable sets of variables X^1, X^2, the following splitting equality holds

$$\Pi((X^1, X^2), Y) = \Pi(X^1, Y)\Pi(X^2, Y) \tag{2.3.13}$$

and likewise for $H(\cdot, \cdot; t)$. There is also an inversion equality

$$H(X, Y; t)^{-1} = H(tX, Y; t^{-1}) \tag{2.3.14}$$

where by tX we mean the variable set $\{tx\}_{x\in X}$.

Definition 2.3.9. Fix a positive integer N and let $\boldsymbol{U} = (U^1, \ldots, U^N)$ and $\boldsymbol{V} = (V^1, \ldots, V^N)$ be ordered N-tuples of countable sets of variables. A *formal Macdonald process* is a formal probability measure on $\mathbb{Y}^N$ valued in $\widehat{\bigotimes}_{i=1}^N(\Lambda_{U^i}\otimes\Lambda_{V^i})$ with the assignment

$$\begin{aligned}&\mathbb{MP}^{\mathbf{f}}_{\boldsymbol{U},\boldsymbol{V}}(\boldsymbol{\lambda})\\&=\mathscr{Z}^{-1}P_{\lambda^1}(U^1)\left(\sum_{\mu\in\mathbb{Y}}Q_{\lambda^1/\mu}(V^1)P_{\lambda^2/\mu}(U^2)\right)\cdots\left(\sum_{\mu\in\mathbb{Y}}Q_{\lambda^{N-1}/\mu}(V^{N-1})P_{\lambda^N/\mu}(U^N)\right)Q_{\lambda^N}(V^N)\end{aligned}$$

where $\boldsymbol{\lambda}=(\lambda^1,\dots,\lambda^N)$ and $\mathscr{Z}\in\widehat{\bigotimes}_{i=1}^N(\Lambda_{U^i}\otimes\Lambda_{V^i})$ is the normalization constant for which the sum over $\boldsymbol{\lambda}\in\mathbb{Y}^N$ gives unity.

From [7, Section 3],

$$\mathscr{Z}=\prod_{1\le i\le j\le N}\Pi(U^i,V^j). \tag{2.3.15}$$

In terms of the pairing, the formal Macdonald process can be expressed as

$$\mathbb{MP}^{\mathbf{f}}_{\boldsymbol{U},\boldsymbol{V}}(\boldsymbol{\lambda})=\mathscr{Z}^{-1}P_{\lambda^1}(U^1)\left(\prod_{i=1}^{N-1}\langle Q_{\lambda^i}(V^i,Y^i),P_{\lambda^{i+1}}(Y^i,U^{i+1})\rangle_{Y^i}\right)Q_{\lambda^N}(V^N). \tag{2.3.16}$$

This is an immediate consequence of the branching rule (2.3.6).

The Π's introduced earlier also relate well with the pairing

$$\langle\Pi(X^1,Y),\Pi(Y,X^2)\rangle_Y=\Pi(X^1,X^2).$$

Since the power symmetric functions from an algebraic basis for $\Lambda_X[\mathcal{L}(Z)]$, this relation can be further extended as follows. Take graded algebras $\mathcal{A}$ and $\mathcal{A}'$ over $\mathcal{L}(Z)$ with $Z=(z_1,\dots,z_k)$, and sequences $\{a_n\},\{a'_n\}$ in $\mathcal{A}$ and $\mathcal{A}'$ respectively such that $\operatorname{ldeg}(a_n),\operatorname{ldeg}(a'_n)\to\infty$ as $n\to\infty$. Then

$$\left\langle\exp\left(\sum_{n=1}^\infty\frac{a_n}{n}p_n(Y)\right),\exp\left(\sum_{n=1}^\infty\frac{a'_n}{n}p_n(Y)\right)\right\rangle=\exp\left(\sum_{n=1}^\infty\frac{1-q^n}{1-t^n}\frac{a_na'_n}{n}\right). \tag{2.3.17}$$

See [7, Proposition 2.3] for further details.

Lemma 2.3.10. *Let X^1,X^2,X^3,X^4,Y be countable sets of variables. Then*

$$\begin{aligned}&\langle H(X^1,Y;t^{-1})\Pi(X^2,Y),H(X^3,Y;t^{-1})\Pi(X^4,Y)\rangle_Y\\&\quad=H(X^1,X^4;t^{-1})H(X^2,X^3;t^{-1})\Pi(X^2,X^4)\prod\frac{(1-x_1x_3)(1-\frac{q}{t}x_1x_3)}{(1-\frac{1}{t}x_1x_3)(1-qx_1x_3)}\end{aligned} \tag{2.3.18}$$

where the product is over $x_i\in X^i$ for $i=1,2,3,4$. The expression $(1-qx_1x_3)^{-1}$ is interpreted as the formal power series $\sum_{n=0}^\infty(qx_1x_3)^n$ and similarly for $(1-t^{-1}x_1x_3)^{-1}$.

Proof. Use (2.3.17) with

$$a_n = (1-t^{-n})p_n(X^1) + \frac{1-t^n}{1-q^n}p_n(X^2)$$
$$a'_n = (1-t^{-n})p_n(X^3) + \frac{1-t^n}{1-q^n}p_n(X^4).$$

□

Negut's Operator

Define the continuous linear operator $D_{-k}^X : \widehat{\Lambda_X} \to \widehat{\Lambda_X}$ by

$$D_{-k}^X P_\lambda(X;q,t) = \wp_k(\lambda;q,t)P_\lambda(X;q,t).$$

This operator was studied in [49] and an integral form for this operator was obtained in [32]. The action of this operator can be given in terms of the residue operator. We first introduce notation to abbreviate the expression. Let $Z = (z_1, \dots, z_k)$ be an ordered set of variables. Define

$$DZ = \frac{1}{(2\pi\mathbf{i})^{|Z|}} \frac{\sum_{i=1}^k \frac{z_1}{z_i}\frac{q^{i-1}}{t^{i-1}}}{(1-\frac{qz_1}{tz_2})\cdots(1-\frac{qz_{k-1}}{tz_k})} \prod_{i<j} \frac{(1-\frac{z_i}{z_j})(1-\frac{qz_i}{tz_j})}{(1-\frac{z_i}{tz_j})(1-\frac{qz_i}{z_j})} \prod_{i=1}^k \frac{dz_k}{z_k}. \tag{2.3.19}$$

For the instances of $(1-v)^{-1}$ in the expression, we mean the power series expansion into $\sum_{n\geq 0} v^n$. We adopt the shorthand notation $Z^{-1} = (z_1^{-1}, \dots, z_k^{-1})$.

Proposition 2.3.11. *Let X and Y be countable sets of variables. Then*

$$D_{-k}^X \Pi(X,Y) = \Pi(X,Y) \oint DZ \cdot H(qZ^{-1}, X; t^{-1})H(Y, q^{-1}Z; t)^{-1}. \tag{2.3.20}$$

Proof. From [32, Proposition 4.10], we have (2.3.20) where instead of a set of variables Y we have some fixed set $\{u_1, \dots, u_n\}$ of complex numbers, and X is still a countable set of variables. Here we have $\oint : \widehat{\Lambda_X}[\mathcal{L}(Z)] \to \widehat{\Lambda_X}$. The goal is to extend this to a formal equality on $\Lambda_X \widehat{\otimes} \Lambda_Y$ for Y an arbitrary countable set of variables.

We can replace (2.3.20) with a finite set of variables $Y^{(n)} = \{y_1, \dots, y_n\}$ instead of fixed complex numbers. In such a setting, we must consider the residue operator as a map $\oint dZ$: $\Lambda_X \widehat{\otimes} \Lambda_{y_1,\dots,y_n}[\mathcal{L}(Z)] \to \Lambda_X \widehat{\otimes} \Lambda_{y_1,\dots,y_n}$.

Note that if $f, g \in \Lambda_X \widehat{\otimes} \Lambda_Y$ such that $\pi_n^Y f = \pi_n^Y g$ for all n, then $f = g$. One then sees that (2.3.20) holds formally for arbitrary countable sets of variables X, Y. In this setting, the residue operator takes $\Lambda_X \widehat{\otimes} \Lambda_Y[\mathcal{L}(Z)]$ to $\Lambda_X \widehat{\otimes} \Lambda_Y$. □

Formal Multicut Expectations

We obtain formulas for multicut expectations of formal Macdonald processes. The idea is to repeated apply the operators D_{-k}^X to Π.

Theorem 2.3.12. *The following formal identity holds for any nonnegative integers $k_1, \ldots, k_N$*

$$\mathbb{E}_{\mathbb{MP}^{\mathbf{f}}_{\boldsymbol{U},\boldsymbol{V}}}[\wp_{k_1}(\lambda^1;q,t)\cdots\wp_{k_N}(\lambda^N;q,t)] = \oint DZ_1\cdots DZ_N$$
$$\times \prod_{1\le i\le j\le N} H(U^i, qZ_j^{-1};t^{-1})H(q^{-1}Z_i, V^j;t)^{-1} \prod_{1\le i<j\le N} C(Z_i,Z_j)$$

where $|Z_i| = k_i$ and

$$C(Z,W) = \prod_{i,j} \frac{(1-t^{-1}qz_i/w_j)(1-z_i/w_j)}{(1-qz_i/w_j)(1-t^{-1}z_i/w_j)} \tag{2.3.21}$$

for sets of variables $Z = (z_1, \ldots, z_k)$ and $W = (w_1, \ldots, w_\ell)$.

This theorem implies a more general result in which the $\wp_{k_i}(\lambda^i)$ may be taken to higher powers than 1 in the expectation.

Corollary 2.3.13. *Let $1 \le x_1 \le \cdots \le x_m \le N$ and $k_1, \cdots, k_m > 0$ be integers. Then*

$$\begin{aligned}\mathbb{E}_{\mathbb{MP}^{\mathbf{f}}_{\boldsymbol{U},\boldsymbol{V}}}[\wp_{k_1}(\lambda^{x_1};q,t)\cdots\wp_{k_m}(\lambda^{x_m};q,t)] &= \oint DZ_1\cdots DZ_m \\ \times \prod_{a=1}^{m} \prod_{\substack{1\le i\le x_a \\ x_a\le j\le N}} H(U^i, qZ_a^{-1};t^{-1})H(q^{-1}Z_a, V^j;t)^{-1} &\prod_{a<b} C(Z_a,Z_b).\end{aligned} \tag{2.3.22}$$

where $|Z_a| = k_a$.

Proof of Theorem 2.3.12. Choose nonnegative integers $k_1, \ldots, k_N$ and let $Z_i = \{z_{ij}\}_{j=1}^{k_i}$ for $i = 1, \ldots, N$ be disjoint sets of variables.

1. Consider the element

$$\sum_{\boldsymbol{\lambda}} \wp_{k_1}(\lambda^1)\cdots\wp_{k_N}(\lambda^N)\mathbb{MP}^{\mathbf{f}}_{\boldsymbol{U},\boldsymbol{V}}(\boldsymbol{\lambda}) \in \widehat{\bigotimes}_{i=1}^{N}(\Lambda_{U^i}\otimes\Lambda_{V^i}).$$

2. Multiply through by the normalizing constant. Reexpress the sums within $\mathbb{MP}$ in terms of Macdonald pairings as in (2.3.16)

$$\sum_{\boldsymbol{\lambda}} \wp_{k_1}(\lambda^1)\cdots\wp_{k_N}(\lambda^N)P_{\lambda^1}(U^1)\left(\prod_{i=1}^{N-1}\left\langle Q_{\lambda^i}(V^i,Y^i), P_{\lambda^{i+1}}(Y^i,U^{i+1})\right\rangle_{Y^i}\right)Q_{\lambda^N}(V^N).$$

Here we note the spaces which the pairings map:

$$\langle\cdot,\cdot\rangle_{Y^i} : \left(\Lambda_{V^i}\,\widehat{\otimes}\,\Lambda_{Y^i}\right)\times\left(\Lambda_{Y^i}\,\widehat{\otimes}\,\Lambda_{U^{i+1}}\right)\to\Lambda_{V^i}\,\widehat{\otimes}\,\Lambda_{U^{i+1}}.$$

By natural inclusions (2.3.7) and consistency (2.3.8), the domain of this pairing may be extended.

3. Bring the summation inside the pairings and the pairings inside the pairings

$$\langle \mathbf{E}_1, \langle \mathbf{E}_2, \langle \cdots \langle \mathbf{E}_{N-1}, \mathbf{E}_N \rangle_{Y^{N-1}} \cdots \rangle_{Y^2} \rangle_{Y^1}$$

where

$$\mathbf{E}_i = \sum_{\lambda^i} \wp_{k_i} P_{\lambda^1}(Y^{i-1}, U^i) Q_{\lambda^i}(V^i, Y^i)$$

and Y^0, Y^N are empty sets of variables. It was important to use the fact that the first argument of the Y^i Macdonald pairing is Λ_{Y^i}-projective which provides the continuity necessary for bringing the summations inside.

4. We can reexpress the summations in terms of Negut's operator in the residue form (2.3.20)

$$\begin{aligned}\mathbf{E}_i &= D_{-k_i}^{Y^{i-1},U^i} \Pi((Y^{i-1}, U^i), (V^i, Y^i)) \\ &= \Pi((Y^{i-1}, U^i), (V^i, Y^i)) \oint H((Y^{i-1}, U^i), qZ_i^{-1}; t^{-1}) H(q^{-1}Z_i, (V^i, Y^i); t)^{-1}\, DZ_i\end{aligned}$$

5. The domain of the residue operator can be appropriately extended and consistency follows from (2.3.7) and (2.3.8). Note that the integrand in $\mathbf{E}_i$ remains Λ_{Y^i}-projective. Therefore, by (2.3.10) and (2.3.11), we may commute the residue operators with the pairings. After pulling out Y^i independent factors outside the residue operators, we obtain

$$\oint DZ_1 \cdots \oint DZ_N \cdot A \cdot \langle \mathbf{F}_1, \langle \mathbf{F}_2, \cdots \langle \mathbf{F}_{N-1}, \mathbf{F}_N \rangle_{Y^{N-1}} \cdots \rangle_{Y^2} \rangle_{Y^1}$$

where

$$\mathbf{F}_i = H(Y^{i-1}, qZ_i^{-1}; t^{-1}) \Pi(Y^{i-1}, V^i) H(q^{-1}Z_i, Y^i; t)^{-1} \Pi((Y^{i-1}, U^i), Y^i) \tag{2.3.23}$$

$$A = \prod_{i=1}^{N} H(U_i, qZ_i^{-1}; t^{-1}) H(q^{-1}Z_i, V_i; t)^{-1} \Pi(U^i, V^i). \tag{2.3.24}$$

Here, (2.3.13) and (2.3.14) were used to split H and Π.

6. Apply the pairings for Y^i in decreasing order of i. At the $(N-i)$th step, we have

$$\oint DZ_1 \cdots \oint DZ_N \cdot A_i \cdot \langle \mathbf{F}_1, \langle \mathbf{F}_2, \cdots \langle \mathbf{F}_i, \mathscr{F}_i \rangle_{Y^i} \cdots \rangle_{Y^2} \rangle_{Y^1} \tag{2.3.25}$$

where A_i collects the $Y^1, \ldots, Y^i$ independent terms. We show by induction that

$$\mathscr{F}_i = H(Y^i, qZ_{[i+1,N]}^{-1}; t^{-1}) \Pi(Y^i, V^{[i+1,N]}) \tag{2.3.26}$$

where we used shorthand notation $Z_{[i,j]} = (Z_i, Z_{i+1}, \ldots, Z_j)$ and similarly for V. If we suppose (2.3.26) is true, then within the Y^i bracket in (2.3.25), $\mathscr{F}_i$ interacts with the third and fourth terms given in (2.3.23). By (2.3.14) and (2.3.18), this interaction produces

$$\begin{aligned} &C(Z_i, Z_{[i+1,N]})H(q^{-1}Z_i, V^{[i+1,N]}; t)^{-1} \\ &\quad \times H((U^i, Y^{i-1}), qZ^{-1}_{[i+1,N]}; t^{-1})\Pi((U^i, Y^{i-1}), V^{[i+1,N]}) \end{aligned} \tag{2.3.27}$$

where $C(Z, W)$ is defined by (2.3.21). As a formal expression, we expand any terms of the form $(1 - v)^{-1}$ as the geometric series. The $\{Y^j\}$ independent term of (2.3.27) is

$$C(Z_i, Z_{[i+1,N]})H(q^{-1}Z_i, V^{[i+1,N]}; t)^{-1}H(U^i, qZ^{-1}_{[i+1,N]}; t^{-1})\Pi(U^i, V^{[i+1,N]}). \tag{2.3.28}$$

After picking up the first two terms in (2.3.23), the remaining term to interact with the Y^{i-1} pairing is

$$H(Y^{i-1}, qZ^{-1}_{[i,N]}; t^{-1})\Pi(Y^{i-1}, V^{[i,N]})$$

which completes the induction as the starting term and ending terms are consistent, the initial term for $i = N - 1$ is exactly $\mathbf{F}_N$, and the final term is unity because Y^0 is empty. After collecting the Y^j-independent terms (2.3.28) from each $i = N - 1, N - 2, \ldots, 1$ and applying (2.3.13), we complete the proof of Theorem 2.3.12. □

We now illustrate the main idea of the proof of Corollary 2.3.13 via a particular example. For further details, we note that the proof is essentially identical to a corresponding extension in [7] (Theorem 3.10 to Corollary 3.11).

Proof Idea of Corollary 2.3.13. We consider the example of $N = 1$ and $m = 2$. Let $k_1, k_2 > 0$ be integers. Consider auxiliary variables $\boldsymbol{S} = (S^1, S^2), \boldsymbol{T} = (T^1, T^2)$, and the formal expectation

$$\begin{aligned} &\mathbb{E}_{\mathbb{MP}^{\mathbf{f}}_{\boldsymbol{S},\boldsymbol{T}}}[\wp_{k_1}(\lambda^1)\wp_{k_2}(\lambda^2)] \\ &\quad = \mathscr{Z}_0^{-1} \sum_{\lambda^1,\lambda^2\in\mathbb{Y}} \wp_{k_1}(\lambda^1)\wp_{k_2}(\lambda^2)P_{\lambda^1}(S^1)\left(\sum_{\mu\in\mathbb{Y}} Q_{\lambda^1/\mu}(T^1)P_{\lambda^2/\mu}(S^2)\right)Q_{\lambda^1}(T^2) \end{aligned} \tag{2.3.29}$$

where $\mathscr{Z}_0$ is the normalizing factor $\mathbb{MP}^{\mathbf{f}}_{\boldsymbol{S},\boldsymbol{T}}$. Consider the map $\phi : \Lambda_{T^1} \otimes \Lambda_{S^2} \to \mathbb{C}$ which sends $f(T^1)g(S^2) \mapsto f(0)g(0)$ to the constant term for any $f, g \in \Lambda_X$. By applying (the continuous extension of) ϕ to (2.3.29) and rewriting $S^1 = U$ and $T^2 = V$, we get

$$\mathscr{Z}^{-1} \sum_{\lambda\in\mathbb{Y}} \wp_{k_1}(\lambda)\wp_{k_2}(\lambda)P_\lambda(U)Q_\lambda(V) = \mathbb{E}_{\mathbb{MP}^{\mathbf{f}}_{U,V}}[\wp_{k_1}(\lambda)\wp_{k_2}(\lambda)]. \tag{2.3.30}$$

where $\mathscr{Z}$ is the normalizing factor for $\mathbb{MP}^{\mathbf{f}}_{U,V}$. On the other hand, by Theorem 2.3.12, we have a formal residue expression for (2.3.29). By applying ϕ to this expression, we obtain (2.3.22) for this choice of N, m.

In the general case, we consider some formal Macdonald process in a greater number of variables, apply Theorem 2.3.12, then apply variable contractions ϕ to obtain the Corollary. □

2.3.2 RPP Observables

In this section, we derive a formula for (2.3.1), stated below in Theorem 2.3.17. It is convenient to do this in two steps: first apply Corollary 2.3.13 to a formal version of the RPP measures, then specialize the formal RPPs to $\mathbb{P}^{B,r,\boldsymbol{s}}_{\alpha,\mathrm{t}}$.

Formal Random Plane Partition

Fix a measure $\mathbb{P}^{B,r,\boldsymbol{s}}_{\alpha,\mathrm{t}}$. For each $e \in I_E$, let

$$F_{\mu,\lambda}(X;b,q,t) = \begin{cases} P_{\lambda/\mu}(X;q,t) & \text{if } b=1, \\ Q_{\mu/\lambda}(X;q,t) & \text{if } b=0. \end{cases} \tag{2.3.31}$$

Definition 2.3.14. If the domain I of the back wall $B : I \to \mathbb{R}$ has finite length, define the *formal RPP with back wall B* to be the formal probability measure $\mathbb{P}^{B,\mathbf{f}}$ supported on $\mathcal{P}_B$ and valued in $\widehat{\bigotimes}_{e\in I_E}\Lambda_{X_e}$ so that

$$\mathbb{P}^{B,\mathbf{f}}(\pi) = \frac{1}{\mathscr{Z}} \prod_{e\in I_E} F_{\pi^{e-\frac{1}{2}},\pi^{e+\frac{1}{2}}}(X_e; B'(e), q, t). \tag{2.3.32}$$

The partition function can be computed:

$$\mathscr{Z} = \prod_{\substack{e_1,e_2\in I_E, e_1<e_2 \\ B'(e_1)>B'(e_2)}} \Pi(X_{e_1}, X_{e_2}). \tag{2.3.33}$$

We comment on how to obtain (2.3.33) after proving Proposition 2.3.15.

Proposition 2.3.15. *Suppose the domain I of $B : I \to \mathbb{R}$ has finite length. Let $x_1 \le \cdots \le x_m$ be points in I_E and $k_1, \ldots, k_m > 0$ be integers. Then*

$$\begin{aligned} \mathbb{E}_{\mathbb{P}^{B,\mathbf{f}}}\left[\prod_{i=1}^m \wp_{k_i}(\pi^{x_i})\right] &= \oint \cdots \oint \prod_{a<b} C(Z_a, Z_b) \\ \times \prod_{a=1}^m \prod_{\substack{e\in I_E, e<x_a \\ B'(e)=1}} & H(X^e, qZ_a^{-1}; t^{-1}) \prod_{\substack{e\in I_E, e>x_a \\ B'(e)=0}} H(q^{-1}Z_a, X^e; t)^{-1}\, DZ_a \end{aligned}$$

where $|Z_a| = k_a$.

Proof. The proof is specializing Corollary 2.3.13 to the formal RPPs. We find a good way of relabeling the formal Macdonald process indices to make this specialization transparent.

Let $N = |I_E| - 1$, $e' = \min I_E$ and $v' = \min I_V$ (then $v' = e' - \frac{1}{2}$). We may reexpress (2.3.32) as

$$\begin{aligned} \mathbb{P}^{B,\mathbf{f}}(\pi) &= \frac{1}{\mathscr{Z}} F_{\pi^{v'},\pi^{v'+1}}(X_{e'}; B'(e'), q, t) \\ &\times F_{\pi^{v'+1},\pi^{v'+2}}(X_{e'+1}; B'(e'+1); q, t) \cdots F_{\pi^{v'+N},\pi^{v'+N+1}}(X_{e'+N}; B'(e'+N); q, t). \end{aligned} \tag{2.3.34}$$

By (2.3.3) and (2.3.31), we have that $\pi^{v'+1} = (0)$ whenever $B'(e') = 0$. Likewise $\pi^{v'+N} = (0)$ whenever $B'(e'+N) = 1$. We may therefore assume that $B'(e') = 1$ and $B'(e'+N) = 0$.

Let $\widetilde{\boldsymbol{U}} = (\widetilde{U}^1, \ldots, \widetilde{U}^N)$ and $\widetilde{\boldsymbol{V}} = (\widetilde{V}^1, \ldots, \widetilde{V}^N)$ where $N = |I_E| - 1$. Consider the formal Macdonald process $\mathbb{MP}^{\mathbf{f}}_{\widetilde{\boldsymbol{U}}, \widetilde{\boldsymbol{V}}}(\widetilde{\lambda}^1, \ldots, \widetilde{\lambda}^N)$. It will be convenient to consider relabelings $\boldsymbol{U} = (U^e)_{e \in I_E}, \boldsymbol{V} = (V^e)_{e \in I_E}$, $(\lambda^e)_{e \in I_E}$ so that

$$\widetilde{U}^1 = U^{e'}, \quad \widetilde{U}^2 = U^{e'+1}, \quad \ldots, \quad \widetilde{U}^N = U^{e'+N-1} \tag{2.3.35}$$

$$\widetilde{V}^1 = V^{e'+1}, \quad \widetilde{V}^2 = V^{e'+2}, \quad \ldots, \quad \widetilde{V}^N = V^{e'+N} \tag{2.3.36}$$

$$\widetilde{\lambda}^1 = \lambda^{v'+1}, \quad \widetilde{\lambda}^2 = \lambda^{v'+1}, \quad \ldots, \quad \widetilde{\lambda}^N = \lambda^{v'+N}. \tag{2.3.37}$$

Thus

$$\begin{aligned} \mathbb{MP}^{\mathbf{f}}_{\widetilde{U}, \widetilde{V}}(\widetilde{\lambda^1}, \ldots, \widetilde{\lambda^N}) = \frac{1}{\widetilde{\mathscr{Z}}} P_{\lambda^{v'+1}}(U^{e'}) \cdot \sum_{\mu \in \mathbb{Y}} Q_{\lambda^{v'+1}/\mu}(V^{e'+1}) P_{\lambda^{v'+2}/\mu}(U^{e'+1}) \\ \cdots \sum_{\mu \in \mathbb{Y}} Q_{\lambda^{v'+N-1}/\mu}(V^{e'+N-1}) P_{\lambda^{v'+N}/\mu}(U^{e'+N-1}) \cdot Q_{\lambda^{v'+N}}(V^{e'+N}) \end{aligned} \tag{2.3.38}$$

where $\widetilde{\mathscr{Z}}$ is the normalization factor.

For $e \in I_E$, define X^e to be U^e if $B'(e) = 1$ and V^e if $B'(e) = 0$, and $\boldsymbol{X} = (X^e)_{e \in I_E}$. Similarly, define Y^e to be V^e if $B'(e) = 1$ and U^e if $B'(e) = 0$, and $\boldsymbol{Y} = (Y^e)_{e \in I_E}$.

Let ρ_0^X denote the 0-specialization on Λ_X, or equivalently constant term map for Λ_X. Define

$$\rho_0^B := \bigotimes_{e \in I_E} \rho_0^{Y^e} : \bigotimes_{e \in I_E} \Lambda_{Y^e} \to \mathbb{C}.$$

By taking tensor products with the identity on Λ_{X^e} for $e \in I_E$, and extending by continuity, we have a map

$$\rho_0^B : \widehat{\bigotimes}_{e \in I_E} (\Lambda_{X^e} \widehat{\otimes} \Lambda_{Y^e}) \to \widehat{\bigotimes}_{e \in I_E} \Lambda_{X^e}.$$

We may further extend this map by extending the scalars from $\mathbb{C}$ to $\mathcal{L}(Z^1, \ldots, Z^n)$.

Applying ρ_0^B to (2.3.38) gives (2.3.34) for $\lambda^v = \pi^v$, $v \in I_v$. This continues to hold true for expectations, so we have

$$\rho_0^B \left(\mathbb{E}_{\mathbb{MP}^{\mathbf{f}}_{\boldsymbol{U}, \boldsymbol{V}}} [\wp_{k_1}(\pi^{x_1}) \cdots \wp_{k_m}(\pi^{x_m})] \right) = \mathbb{E}_{\mathbb{P}^{B, \mathbf{f}}} [\wp_{k_1}(\lambda^{x_1}) \cdots \wp_{k_m}(\lambda^{x_m})].$$

By Corollary 2.3.13 the left hand side is exactly

$$\rho_0^B\left(\oint\cdots\oint\prod_{a<b}C(Z_a,Z_b)\prod_{a=1}^{m}\left(\prod_{e<x_a,e\in I_E}H(U^e,qZ_a^{-1};t^{-1})\prod_{e>x_a,e\in I_E}H(q^{-1}Z_a,V^e;t)^{-1}\,DZ_a\right)\right)$$
$$=\oint\cdots\oint\prod_{a<b}C(Z_a,Z_b)\prod_{a=1}^{m}\prod_{\substack{e\in I_E,e<x_a\\B'(e)=1}}H(X^e,qZ_a^{-1};t^{-1})\prod_{\substack{e\in I_E,e>x_a\\B'(e)=0}}H(q^{-1}Z_a,X^e;t)^{-1}\,DZ_a.$$

where we have used the fact that the residue operator commutes with continuous maps. This proves the proposition. □

Remark 7. The formula (2.3.33) is then a consequence of applying the specializations in the proof of Proposition 2.3.15 to the partition function for the formal Macdonald process (2.3.15).

Specialization to RPP

Consider the distribution in (2.2.3) where we have a sequence of weights $(r_v)_{v\in I_V}$. Fix an arbitrary $\xi>0$, define

$$a_e:=\begin{cases}\xi\prod_{\substack{v\in I_V\\v<e}}r_v,\\ \xi\prod_{v\in I_V\,e<v<v_0}r_v^{-1} & \text{if } e>v_0.\end{cases}\tag{2.3.39}$$

The dependence of our models on a_e is only through their ratios with one another. In particular, the choice of ξ and v_0 is immaterial. By (2.3.4) and (2.3.5), the distribution defined by

$$\mathbb{P}(\pi)=\frac{1}{\mathscr{Z}((a_e)_{I_E})}\prod_{e\in I_E}F_{\lambda^{e-\frac12},\lambda^{e+\frac12}}(a_e^{1-2B'(e)};B'(e),q,t)\tag{2.3.40}$$

coincides with (2.2.3) if and only if

$$\mathscr{Z}((a_e)_{I_E})=\prod_{\substack{e_1,e_2\in I_E,e_1<e_2\\B'(e_1)>B'(e_2)}}\Pi(a_{e_1}^{-1},a_{e_2})<\infty.$$

This implies the following lemma.

Lemma 2.3.16. *If the weights in (2.2.3) are summable, then for any $e_1,e_2\in I_E$ such that $e_1<e_2$ and $B'(e_1)>B'(e_2)$, we have*

$$a_{e_1}^{-1}a_{e_2}<1$$

where $(a_e)_{e\in I_E}$ is defined in (2.3.39). If I is finite, the inequality is also sufficient to determine the weights are summable.

We note that for I of finite length, each diagonal partition π^v of $\pi\in\mathcal{P}_B$ has bounded length $\ell(\pi^v)$ depending only B and v. Thus for finite I_E, the finiteness of $\mathscr{Z}((a_e)_{I_E})$ implies the existence of the multicut expectations of (2.3.40).

For the measure $\mathbb{P}^{B,r,\boldsymbol{s}}_{\alpha,\mathfrak{t}}$, a suitable choice for a_e is given by

$$a_e = r^v s_0 \cdots s_{\lfloor e \rfloor} = r^v s_0 \cdots s_{\lfloor e \rfloor}. \tag{2.3.41}$$

The main formula for the observables $\wp$ can be obtained by specializing the formal RPPs, taking $X_e \mapsto a_e^{1-2B'(e)}$. For $x \in I_V$, define the function

$$\begin{aligned} G^B_{<x}(z;\varepsilon,\mathfrak{t}) &= \prod_{\substack{e<x, e\in I_E\\ B'(e)=1}} \frac{1-(ta_e z)^{-1}}{1-(a_e z)^{-1}} = \prod_{i=0}^{p-1} \prod_{\substack{e<x, e\in I_E\\ e\in p\mathbb{Z}+i+\frac{1}{2}}} \left(\frac{1-(tr^e(s_0\cdots s_i)z)^{-1}}{1-(r^e(s_0\cdots s_i)z)^{-1}} \right)^{B'(e)} \\ G^B_{>x}(z;\varepsilon,\mathfrak{t}) &= \prod_{\substack{e>x, e\in I_E\\ B'(e)=0}} \frac{1-a_e z}{1-ta_e z} = \prod_{i=0}^{p-1} \prod_{\substack{e>x, e\in I_E\\ e\in p\mathbb{Z}+i+\frac{1}{2}}} \left(\frac{1-r^e(s_0\cdots s_i)z}{1-tr^e(s_0\cdots s_i)z} \right)^{1-B'(e)} \end{aligned} \tag{2.3.42}$$

Given some function $g(z)$ in one-variable and $Z = (z_1, \dots, z_k)$ an ordered collection of variables, we write

$$g(Z) := \prod_{i=1}^{k} g(z_i).$$

Theorem 2.3.17. *Consider the measure $\mathbb{P}^{B,r,\boldsymbol{s}}_{\alpha,\mathfrak{t}}$ where $r = e^{-\varepsilon}$ and let $(\pi^x)_{x\in I}$ denote the (random) diagonal partitions. Let $x_1 \le \cdots \le x_m$ be in I_V and $k_1, \dots, k_m > 0$ be integers. Suppose there exist positively oriented contours $\{\mathcal{C}_{i,j}\}_{\substack{1\le j\le k_i\\ 1\le i\le m}}$ such that*

- *the contour $\mathcal{C}_{i,j}$ is contained in the domain bounded by $t\mathcal{C}_{i',j'}$ whenever $(i,j) < (i',j')$ in lexicographical ordering;*
- *each domain bounded by $\mathcal{C}_{i,j}$ contains 0 and the poles of $G^B_{<x}(z;\varepsilon,\mathfrak{t})$ but not the poles of $G^B_{>x}(z;\varepsilon,\mathfrak{t})$.*

Then

$$\mathbb{E}[\wp_{k_1}(\pi^{x_1};q,t)\cdots\wp_{k_m}(\pi^{x_m};q,t)] = \oint\cdots\oint \prod_{a<b} C(Z_a,Z_b) \prod_{a=1}^{m} G^B_{<x_a}(Z_a;\varepsilon,\mathfrak{t}) G^B_{>x_a}(Z_a;\varepsilon,\mathfrak{t})\, DZ_a$$

where $|Z_i| = k_i$, $Z_i = (z_{i,1},\dots,z_{i,k_i})$, the contour of $z_{i,j}$ is given by $\mathcal{C}_{i,j}$.

Remark 8. By Lemma 2.3.16, given any poles $\mathfrak{p}_1, \mathfrak{p}_2$ of $G^B_{<x}(z;\varepsilon,\mathfrak{t}), G^B_{>x}(z;\varepsilon,\mathfrak{t})$ respectively, we have $\mathfrak{p}_1 < \mathfrak{p}_2$. The existence of the contours $\mathcal{C}_{i,j}$ is then dependent on whether there is enough distance between these two sets of poles.

Proof of Theorem 2.3.17. If the domain I has finite length, then the theorem follows from Proposition 2.3.15. To see how to obtain the contour conditions, we recall the formal definition of (2.3.12) and (2.3.19). The formal expansion of (2.3.12) that we desire amounts to taking contours which contain the poles of $G^B_{<x}(z;\varepsilon,\mathfrak{t})$ but not the poles of $G^B_{>x}(z;\varepsilon,\mathfrak{t})$. The expressions $(1-\frac{z_{i,j}}{tz_{i',j'}})^{-1}$

which appear in DZ and $C(Z,W)$ are expanded as $\sum_{n\geq 0}\frac{z_{i,j}^n}{(tz_{i',j'})^n}$ which requires the condition that $\mathcal{C}_{i,j}$ is contained in the domain bounded by $t\mathcal{C}_{i',j'}$ whenever $(i,j)<(i',j')$ in lexicographical order. Note the change of variables rewriting $q^{-1}Z_a$ as Z_a.

If I has infinite length, define $\mathbb{P}_N := \mathbb{P}_{\alpha,\mathfrak{t}}^{B_N,r,\boldsymbol{s}}$ where the back wall $B_N : I^N \to \mathbb{R}$ is defined to be the restriction of B to $I^N := I\cap[-N,N]$, for $N\in\mathbb{N}$. The summability of the weights of $\mathbb{P} := \mathbb{P}_{\alpha,\mathfrak{t}}^{B,r,\boldsymbol{s}}$ implies the summability of the weights of $\mathbb{P}_N$.

Let $(a_e)_{e\in I_E}$ be the sequence of specializations for $\mathbb{P}$ as in (2.3.41). Then $(a_e)_{e\in I_{N,e}}$ where $I_E^N = I_E\cap[N,N]$ is the sequence of specializations for $\mathbb{P}_N$. Let $\mathscr{Z}$ and $\mathscr{Z}_N$ denote the partition functions for $\mathbb{P}$ and $\mathbb{P}_N$ respectively. Choose $x_1\leq\cdots\leq x_m\in I_E$ and integers $k_1,\dots,k_m>0$. Consider N large enough so that $x_1,\dots,x_m\in[-N,N]$. We have

$$\begin{aligned}\mathscr{Z}_N\mathbb{E}_{\mathbb{P}_N}\left[\prod_{i=1}^m \wp_{k_i}(\pi^{x_i})\right] &= \sum_{(\pi^v)\in\mathbb{Y}^{I_V^N}}\left[\prod_{i=1}^m \wp_{k_i}(\pi^{x_i})\right]\prod_{e\in I_E^N} F_{\pi^{e-\frac{1}{2}},\pi^{e+\frac{1}{2}}}(a_e^{1-2B'(e)};B'(e),q,t)\\ \to \sum_{(\pi^v)\in\mathbb{Y}^{I_V}}\left[\prod_{i=1}^m \wp_{k_i}(\pi^{x_i})\right]\prod_{e\in I_E} F_{\pi^{e-\frac{1}{2}},\pi^{e+\frac{1}{2}}}(a_e^{1-2B'(e)};B'(e),q,t) &= \mathscr{Z}\mathbb{E}_{\mathbb{P}}\left[\prod_{i=1}^m \wp_{k_i}(\pi^{x_i})\right]\end{aligned}$$

as $N\to\infty$ since the sequence is monotonically increasing. Since $\mathscr{Z}_N\to\mathscr{Z}$, we have as $N\to\infty$

$$\mathbb{E}_{\mathbb{P}_N}\left[\prod_{i=1}^m \wp_{k_i}(\pi^{x_i})\right]\to\mathbb{E}_{\mathbb{P}}\left[\prod_{i=1}^m \wp_{k_i}(\pi^{x_i})\right].$$

On the other hand, for any $x\in I_V$, we have $G_{<x}^{B_N}(z;\varepsilon,\mathfrak{t})\to G_{<x}^{B}(z;\varepsilon,\mathfrak{t})$ as $N\to\infty$ uniformly away from the poles of $G_{<x}^{B}(z;\varepsilon,\mathfrak{t})$, and likewise for $G_{>x}^{B}(z;\varepsilon,\mathfrak{t})$. By applying the theorem for the known case of B_N and taking $N\to\infty$, the general theorem follows. □

2.4 Limit Conditions and Back Walls

2.4.1 Consequences of the Limit Conditions

In this section, we identify the class of functions which can be realized as limits of back walls. As mentioned in Section 2.2.2, our limit theorems restrict to a dense subset of this class. We motivate this restriction through the concept of singular points and some examples from the literature. Our study of singular points is also used in Section 2.5 for asymptotics.

We recall the Limit Conditions.

Limit Conditions. Fix a p-periodic, bi-infinite sequence $\boldsymbol{s}=(\cdots,s_{-1},s_0,s_1,\cdots)\in\mathbb{R}_{>0}^{\infty}$ such that $s_0\cdots s_{p-1}=1$. Let $\mathbb{P}_{\alpha,\mathfrak{t}}^{B,r,\boldsymbol{s}}$ be a family parametrized by a small parameter $\varepsilon>0$ where $B:I^\varepsilon\to\mathbb{R}$ and $r:=e^{-\varepsilon}$ vary with ε so that

1. there exist elements in $\mathbb{Z}\cup\{\pm\infty\}$

$$\inf I^\varepsilon = v_0(\varepsilon)<\cdots<v_n(\varepsilon)=\sup I^\varepsilon$$

such that for each $1 \leq \ell \leq n$, $B'(x)$ is p-periodic on $(v_{\ell-1}, v_\ell) \cap (\mathbb{Z} + \frac{1}{2})$;

2. there exists an interval $I \subset \mathbb{R}$ and a piecewise linear $\mathcal{B} : I \to \mathbb{R}$ with non-differentiable points

$$\inf I = V_0 < \cdots < V_n = \sup I$$

such that

$$\varepsilon v_\ell(\varepsilon) \to V_\ell \quad (0 \leq \ell \leq n), \quad \varepsilon B^\varepsilon(x/\varepsilon) \to \mathcal{B}(x)$$

as $\varepsilon \to 0$, where the latter convergence is uniform over any compact subset of I.

In the setting of global limits under this limit regime, some of the information encoded by $\boldsymbol{s}$ is washed away. The dependence on $\boldsymbol{s}$ is only through the values and corresponding multiplicities of the sequence $\{s_0 \cdots s_i\}_{i=0}^{p-1}$. This motivates the following definition.

Definition 2.4.1. We associate to $\boldsymbol{s}$ the multiset $\mathcal{S} = \{s_0 \cdots s_i\}_{i=0}^{p-1}$. Given $\sigma \in \mathcal{S}$, let

$$S_\sigma = \{i \in [[0, p-1]] : \sigma = s_0 \cdots s_i\}.$$

In particular, we remember the multiplicity of each member $\sigma \in \mathcal{S}$ in $(s_0 \cdots s_i)_{i=0}^{p-1}$. The multiplicity of σ is given by $|S_\sigma|$. In replacing $\boldsymbol{s}$ with $\mathcal{S}$, we forget about the particular order of the sequence $(s_0 \cdots s_i)_{i=0}^{p-1}$.

For fixed $\mathcal{S}$, it is not the case that *any* piecewise linear $\mathcal{B}$ may be realized as the limiting back wall of some $\mathbb{P}_{\alpha,\mathrm{t}}^{B,r,\boldsymbol{s}}$ satisfying the Limit Conditions. This is due to the fact that $\mathbb{P}_{\alpha,\mathrm{t}}^{B,r,\boldsymbol{s}}$ is not a probability measure for arbitrary B; the conditions required for the summability of the weights, summarized by Lemma 2.3.16, severely restricts the class of B which give rise to probability measures. We now characterize the set of $\mathcal{B}$ which can be achieved by the Limit Conditions.

Given a real-valued function f on an interval I, let $f(x^\pm) = \lim_{u \to x^\pm} f(u)$.

Definition 2.4.2. Let $\tau_1 \geq \cdots \geq \tau_p$ be a labeling of the elements of $\mathcal{S}$ and set $\tau_0 = \infty, \tau_{p+1} = 0$. Let $\mathfrak{B}(\mathcal{S})$ denote the set of continuous piecewise linear functions $\mathcal{B} : I \to \mathbb{R}$ on some interval domain such that

1. the non-differentiable points of $\mathcal{B}$ are given by

$$\inf I = V_0 < \cdots < V_n = \sup I;$$

2. for each $x \in I \setminus \{V_\ell\}_{\ell=0}^n$, $\mathcal{B}'(x) \in \{\frac{i}{p}\}_{i=0}^p$;

3. if $V \leq W$ are non-differentiable points of $\mathcal{B}$ with $\mathcal{B}'(V^-) = \frac{i}{p}$, $\mathcal{B}'(W^+) = \frac{j}{p}$, then

$$\tau_i^{-1} \tau_{j+1} e^{-(W-V)} \leq 1. \tag{2.4.1}$$

Remark 9. In the definition above, we take the convention that $\mathcal{B}'(V_0^-) = 0$ and $\mathcal{B}'(V_n^+) = 1$.

Theorem 2.4.3. *A function $\mathcal{B} : I \to \mathbb{R}$ is a limiting back wall of some $\mathbb{P}_{\alpha,\mathrm{t}}^{B,r,\boldsymbol{s}}$ satisfying the Limit Conditions if and only if $\mathcal{B} \in \mathfrak{B}(\mathcal{S})$.*

Before proving this theorem, we provide an important link between the slopes of $\mathcal{B}'$ and that of the prelimit $B'(x)$ on $p\mathbb{Z}+\frac{1}{2}$.

Lemma 2.4.4. *Suppose $\mathbb{P}^{B,r,\boldsymbol{s}}_{\alpha,\mathrm{t}}$ satisfies the Limit Conditions. Fix $\ell \in [[1,n]]$ and let $\mathcal{B}'(V_{\ell-1},V_\ell)=\frac{i}{p}$. Then for sufficiently small $\varepsilon>0$, there exists a set $A\subset[[0,p-1]]$ (potentially varying in ε) of size i such that*

$$s_0\cdots s_a \le s_0\cdots s_b, \quad \text{for all } a\in A,\ b\in[[0,p-1]]\setminus A$$

and

$$B' = \mathbf{1}\left[A+p\mathbb{Z}+\frac{1}{2}\right]$$

on $I_E^\varepsilon\cap(v_{\ell-1},v_\ell)$.

Proof. By the Limit Conditions, we know that for sufficiently small $\varepsilon>0$ there exists a (potentially varying in ε) subset $A\subset[[0,p-1]]$ of size i such that

$$B' = \mathbf{1}\left[A+p\mathbb{Z}+\frac{1}{2}\right].$$

on $I_E^\varepsilon\cap(v_{\ell-1},v_\ell)$.

Assume for contradiction that for arbitrarily small $\varepsilon>0$, there exist (potentially varying in ε) pairs $\sigma=s_0\cdots s_a$ and $\tau=s_0\cdots s_b$ for some $a\in A$ and $b\in[[0,p-1]]\setminus A$ such that $\sigma>\tau$. For ε small enough so that $(v_{\ell-1},v_\ell)$ has more than p points, we may choose $e_1\in(v_{\ell-1},v_\ell)\cap(a+p\mathbb{Z}+\frac{1}{2})$, $e_2\in(v_{\ell-1},v_\ell)\cap(b+p\mathbb{Z}+\frac{1}{2})$ such that $0<e_2-e_1<p$. By (2.3.41), we have

$$a_{e_1}^{-1}a_{e_2}=\sigma^{-1}\tau r^{e_2-e_1}>\sigma^{-1}\tau e^{-\varepsilon p}$$

where $(a_e)_{e\in I_E^\varepsilon}$ is the specialization sequence for $\mathbb{P}^{B,r,\boldsymbol{s}}_{\alpha,\mathrm{t}}$ as defined in (2.3.39). For ε sufficiently small, the latter is >1 which violates Lemma 2.3.16. $\square$

Proof of Theorem 2.4.3. Suppose $\mathcal{B}:I\to\mathbb{R}$ is a limiting back wall of $\mathbb{P}^{B,r,\boldsymbol{s}}_{\alpha,\mathrm{t}}$ satisfying the Limit Conditions. It is clear that $\mathcal{B}$ satisfies properties (1) and (2) of Definition 2.4.2 as a direct consequence of the p-periodicity and convergence in the Limit Conditions. We check property (3) of Definition 2.4.2. Let $1\le\ell\le m<n$ and suppose $\mathcal{B}'(V_\ell^-)=\frac{i}{p}$ and $\mathcal{B}'(V_m^+)=\frac{j}{p}$; note that checking the cases $\ell=0$ and $m=n$ is trivial. By Lemma 2.4.4, for small enough $\varepsilon>0$ there exist subsets $A_1,A_2\in[[0,p-1]]$ of sizes i,j respectively such that

$$B'\big|_{I_E^\varepsilon\cap(v_{\ell-1},v_\ell)}=\mathbf{1}[A_1], \quad B'\big|_{I_E^\varepsilon\cap(v_m,v_{m+1})}=\mathbf{1}[A_2].$$

Moreover,

$$s_0\cdots s_a\le s_0\cdots s_b, \quad \text{for } a\in A_1,\ b\in[[0,p-1]]\setminus A_1$$

so that the multiset $\{s_0\cdots s_a\}_{a\in A_1}$ coincides with $\{\tau_1,\ldots,\tau_i\}$. Likewise, $\{s_0\cdots s_a\}_{a\in A_2}$ coincides with $\{\tau_1,\ldots,\tau_j\}$. In particular, there exist

$$e_1\in(A_1+p\mathbb{Z}+\frac{1}{2})\cap(v_{\ell-1},v_\ell), \quad e_2\in([[0,p-1]]\setminus A_2+p\mathbb{Z}+\frac{1}{2})\cap(v_m,v_{m+1})$$

where $s_0 \cdots s_a = \tau_i$, $s_0 \cdots s_b = \tau_{j+1}$, such that

$$B'(e_1) = 1, \quad B'(e_2) = 0$$

for small enough $\varepsilon > 0$. By p-periodicity, we may add that

$$0 \le v_\ell - e_1 < p, \qquad 0 \le e_2 - v_m < p.$$

By (2.3.41) and Lemma 2.3.16, we have

$$a_{e_1}^{-1} a_{e_2} = \tau_i^{-1} \tau_{j+1} r^{e_2 - e_1} < 1$$

where $(a_e)_{e \in I_E^\varepsilon}$ is the specialization sequence for $\mathbb{P}_{\alpha,\mathrm{t}}^{B,r,\boldsymbol{s}}$ as defined in (2.3.39). Taking $\varepsilon \to 0$, we obtain

$$\tau_i^{-1} \tau_{j+1} e^{-(V_m - V_\ell)} \le 1.$$

Conversely, suppose $\mathcal{B} \in \mathfrak{B}(\mathcal{S})$. For small $\varepsilon > 0$, set

$$v_\ell = \lfloor V_\ell / \varepsilon \rfloor + \ell, \quad 0 \le \ell \le n;$$

in the case $V_0 = -\infty$ ($V_n = +\infty$) let $v_0 = -\lfloor 1/\varepsilon^2 \rfloor$, $v_n = \lfloor 1/\varepsilon^2 \rfloor$. Let B be a back wall of a skew diagram such that $B'(x)$ is p-periodic in $x \in (v_{\ell-1}, v_\ell)$. For each $1 \le \ell \le n$, if we have $\mathcal{B}'(V_{\ell-1}, V_\ell) = \frac{i}{p}$, choose some subset $A_\ell \subset [[0, p-1]]$ of size i so that

$$s_0 \cdots s_a \le s_0 \cdots s_b, \quad a \in A_\ell, b \in [[0, p-1]] \setminus A_\ell, \quad B'\big|_{I_E^\varepsilon \cap (v_{\ell-1}, v_\ell)} = \mathbf{1}[A_\ell]$$

By fixing $\varepsilon B(\lfloor x/\varepsilon \rfloor) = \mathcal{B}(x)$ at some point $x \in I$, we have the convergence

$$\varepsilon B(\lfloor x/\varepsilon \rfloor) \to \mathcal{B}(x).$$

It remains to check that $\mathbb{P}_{\alpha,\mathrm{t}}^{B,r,\boldsymbol{s}}$ defines a probability measure, at least for ε sufficiently small. By Lemma 2.3.16, it suffices to check that $a_{e_1}^{-1} a_{e_2} < 1$ over all edges $e_1 < e_2$ where $B'(e_1) = 1$ and $B'(e_2) = 0$. We divide this into two cases.

Case 1: $e_1, e_2 \in (v_{\ell-1}, v_\ell)$. By construction of B, we have $B' = \mathbf{1}[A_\ell]$ on $(v_{\ell-1}, v_\ell)$. Thus $e_1 \in a + p\mathbb{Z} + \frac{1}{2}$, $e_2 \in b + p\mathbb{Z} + \frac{1}{2}$ for some $a \in A_\ell, b \in [[0, p-1]] \setminus A_\ell$. Thus

$$a_{e_1}^{-1} a_{e_2} = (s_0 \cdots s_a)^{-1} (s_0 \cdots s_b) r^{e_2 - e_1} < 1.$$

Note that this only relies on p-periodicity and did not require property (3) in Definition 2.4.2.

Case 2: $e_1 \in (v_{\ell-1}, v_\ell)$, $e_2 \in (v_m, v_{m+1})$ *where* $0 < \ell \le m < n$. Again, by construction of B we have $B = \mathbf{1}[A_\ell]$ on $(v_{\ell-1}, v_\ell)$ and $B = \mathbf{1}[A_{m+1}]$ on (v_m, v_{m+1}). If $\mathcal{B}'(V_\ell^-) = \frac{i}{p}$ and $\mathcal{B}'(V_m^+) = \frac{j}{p}$, then we may argue as before to establish that A_ℓ coincides with $\{\tau_1, \ldots, \tau_i\}$ and A_{m+1} coincides with $\{\tau_1, \ldots, \tau_j\}$. Then

$$a_{e_1}^{-1} a_{e_2} = \sigma^{-1} \tau r^{e_2 - e_1}$$

where $\sigma \geq \tau_i$ and $\tau \leq \tau_{j+1}$. Since $e_1 < v_\ell \leq v_m < e_2$, we have

$$a_{e_1}^{-1} a_{e_2} \leq \tau_i^{-1} \tau_{j+1} r^{v_m - v_\ell + 1} = \tau_i^{-1} \tau_{j+1} e^{-\varepsilon(m-\ell+1)} e^{-\varepsilon(\lfloor V_m/\varepsilon \rfloor - \lfloor V_\ell/\varepsilon \rfloor)} < \tau_i^{-1} \tau_{j+1} e^{-(V_m - V_\ell)} \leq 1$$

where the latter inequality follows from property (3) for $\mathfrak{B}(\mathcal{S})$. $\square$

2.4.2 Well-Behaved Back Walls

In this subsection, we introduce a subset $\mathfrak{B}^\Delta(\mathcal{S})$ of $\mathfrak{B}(\mathcal{S})$ which corresponds to well-behaved back walls. This good behavior is characterized by the presence of only *finitely many* singular points which we describe further below.

Before providing the definition of the subset $\mathfrak{B}^\Delta(\mathcal{S})$, we begin by introducing and motivating the notion of a singular point. Fix $\mathcal{B} \in \mathfrak{B}(\mathcal{S})$, and define

$$\begin{aligned} \rho_<(\mathbf{x}) &= \max\{\tau_i^{-1} e^\xi : \xi \leq \mathbf{x}, \mathcal{B}'(\xi^-) = \frac{i}{p}\}, \\ \rho_>(\mathbf{x}) &= \min\{\tau_{j+1}^{-1} e^\xi : \xi \geq \mathbf{x}, \mathcal{B}'(\xi^+) = \frac{j}{p}\}. \end{aligned} \tag{2.4.2}$$

Lemma 2.4.5. *If $\mathcal{B} \in \mathfrak{B}(\mathcal{S})$ and $\mathbf{x} \in I$, then*

$$\rho_<(\mathbf{x}) \leq \rho_>(\mathbf{x}).$$

Proof. Suppose $\xi_1 \leq \xi_2$ and $\mathcal{B}'(\xi_1^-) = \frac{i}{p}, \mathcal{B}'(\xi_2^+) = \frac{j}{p}$. It is enough to show that

$$\tau_i^{-1} e^{\xi_1} \leq \tau_{j+1}^{-1} e^{\xi_2}. \tag{2.4.3}$$

If $\xi_1, \xi_2 \in (V_{\ell-1}, V_\ell)$ for some $\ell \in [[1, n]]$, then $i = j$ so that (2.4.3) holds. Otherwise, there exists a non-differentiable points V, W such that

$$\xi_1 \leq V \leq W \leq \xi_2.$$

Assume that V is the minimal such point and W is the maximal such point. Then

$$\mathcal{B}'(V^-) = \mathcal{B}'(\xi_1^-) = \frac{i}{p}, \quad \mathcal{B}'(W^+) = \mathcal{B}'(\xi_2^+) = \frac{j}{p}.$$

Since

$$\tau_i^{-1} \tau_{j+1} e^{-(W-V)} \leq 1$$

it follows that

$$\tau_i^{-1} e^{\xi_1} \leq \tau_i^{-1} e^V \leq \tau_{j+1}^{-1} e^W \leq \tau_{j+1} e^{\xi_2}.$$

$\square$

We define the singular points to be those points which achieve equality:

Definition 2.4.6. Given $\mathcal{B} \in \mathfrak{B}(\mathcal{S})$, we say that $\mathbf{x} \in I$ is a *singular point of $\mathcal{B}$* if $\rho_<(\mathbf{x}) = \rho_>(\mathbf{x})$.

Definition 2.4.7. Denote by $\mathfrak{B}^{\Delta}(\mathcal{S})$ the subset of $\mathfrak{B}(\mathcal{S})$ consisting of $\mathcal{B}$ with finitely many singular points.

The concept of singular points is significant due to the following connection with the limit shape. Recall the content of Theorem 2.2.7: if $\mathbb{P}_{\alpha,\mathrm{t}}^{\mathcal{B},r,\boldsymbol{s}}$ satisfies the Limit Condition with $\mathcal{B} \in \mathfrak{B}^{\Delta}(\mathcal{S})$, then the random rescaled height function $\varepsilon h(x/\varepsilon, y/\varepsilon)$ converges to a deterministic limit $\mathcal{H}(x,y)$. Then the local proportions $p_{▱}, p_{⏢}, p_{\lozenge}$ converge at each point (x,y). Recall the liquid region is the set of (x,y) where all the proportions are nonzero. We may view the closure of the liquid region as the set of points (x,y) where the local picture is random.

In Section 2.6, we characterize the liquid region as the set of (x,y) for which some equation $\mathcal{G}_x^{\mathcal{B}}(\zeta) = e^{-y}$ determined by $\mathcal{B}$ has a pair of nonreal complex roots, see (2.6.2). The map $\mathcal{B} \mapsto \mathcal{G}_x^{\mathcal{B}}$ is continuous with respect to the topology on $\mathfrak{B}(\mathcal{S})$ induced by the coordinates

$$(V_0, \dots, V_n, \mathcal{B}'(V_0, V_1), \dots, \mathcal{B}'(V_{n-1}, V_n)) \in (\mathbb{R} \cup \{\pm\infty\})^{2n+1}$$

and convergence in compactum on $\mathbb{H}$ in the image. Thus one can *formally* extend the definition of the liquid region associated to some $\mathcal{B} \in \mathfrak{B}(\mathcal{S}) \setminus \mathfrak{B}^{\Delta}(\mathcal{S})$ to be the set of (x,y) such that $\mathcal{G}_x^{\mathcal{B}}(\zeta) = e^{-y}$ has a pair of nonreal roots.

Under this alternative definition of the liquid region, one can determine that the singular points of $\mathcal{B}$ are exactly the points x such that (x,y) is in the liquid region for arbitrarily large y. In other words, the singular points of $\mathcal{B}$ correspond to the horizontal coordinates along which the liquid region is vertically unbounded.

For certain examples of $\mathcal{S}$ and $\mathcal{B} \in \mathfrak{B}(\mathcal{S}) \setminus \mathfrak{B}^{\Delta}(\mathcal{S})$, one can prove the limit shape phenomenon. In these cases, the alternative definition of the liquid region coincides with the original definition of the liquid region in terms of the local proportions of lozenges. Although it is not present in the literature, we believe that one may use the method of correlation kernels to verify the limit shape phenomenon for arbitrary $\mathcal{S}$ and $\mathcal{B} \in \mathfrak{B}(\mathcal{S}) \setminus \mathfrak{B}^{\Delta}(\mathcal{S})$ in the *non-interacting* ($q = t$) case.

We now provide several examples in the literature which illustrate the connection between singular points and unboundedness of the limit shape, then give some references to later sections which give suggestions for generalizing this connection to arbitrary $\mathcal{B}$.

Example 2.4.1.

1. *p arbitrary,* $\mathcal{S} = \{1, \dots, 1\}$. The set $\mathfrak{B}(\mathcal{S})$ consists of $\mathcal{B}$ such that $\mathcal{B}'(V_{\ell-1}, V_\ell) \in [0,1] \cap \frac{1}{|\mathcal{S}|}\mathbb{Z}$ since (2.4.1) trivially holds. The singular points in $I \setminus \{V_\ell\}_{\ell=0}^n$ are precisely $x \in (V_{\ell-1}, V_\ell)$ where $\mathcal{B}'(V_{\ell-1}, V_\ell) \notin \{0,1\}$. It was demonstrated in [14], that the horizontal coordinate $x \in (V_{\ell-1}, V_\ell)$ of the limit shape is vertically unbounded if and only if $\mathcal{B}'(V_{\ell-1}, V_\ell) \notin \{0,1\}$, see also [45, Section 1.2]. The set $\mathfrak{B}^{\Delta}(\mathcal{S})$ consists of $\mathcal{B}$ such that $\mathcal{B}'(V_{\ell-1}, V_\ell) = 1$ or 0 for every $\ell = 1, \dots, n$. In this case, the singular points are precisely those V_ℓ where

$$0 = \mathcal{B}'(V_\ell^+) < \mathcal{B}'(V_\ell^-) = 1,$$

and these are exactly the horizontal coordinates where the limit shape is vertically unbounded.

2. $p = 2$, $\mathcal{S} = \{1, \alpha\}, \alpha > 1$. Consider the case where $n = 3$ and we take

$$V_0 = -b, \quad V_1 = -a, \quad V_2 = a, \quad V_3 = b, \qquad a < b,$$
$$\mathcal{B}'(-b, -a) = 1, \quad \mathcal{B}'(-a, a) = 1/2, \quad \mathcal{B}'(a, b) = 0.$$

   There exists a threshold value a_0 such that if $a < a_0$ then $\mathcal{B} \notin \mathfrak{B}(\mathcal{S})$ (thus this does not correspond to a limit of a plane partition), and if $a \geq a_0$ then $\mathcal{B} \in \mathfrak{B}(\mathcal{S})$. If $a > a_0$, then $\mathcal{B} \in \mathfrak{B}^{\Delta}(\mathcal{S})$ and the singular points occur at $-a, a$. If $a = a_0$, then $\mathcal{B} \in \mathfrak{B}(\mathcal{S}) \setminus \mathfrak{B}^{\Delta}(\mathcal{S})$ and $[-a, a]$ is the set of singular points. In both of these cases, the singular points correspond to the set of horizontal coordinates where the limit shape is vertically unbounded, see [46, Sections 1.1.1 & 4].

3. In general, we show in Section 2.6.3 that the singular points for $\mathcal{B} \in \mathfrak{B}^{\Delta}(\mathcal{S})$ are exactly the horizontal coordinates where the limit shape is vertically unbounded. The method for computing the frozen boundary in Section 2.6.3 can also be used to see that $\mathcal{B} \in \mathfrak{B}(\mathcal{S}) \setminus \mathfrak{B}^{\Delta}(\mathcal{S})$ give rise to limit shapes (as defined above) that are vertically unbounded over an entire nonempty open interval, and these unbounded parts correspond to components of singular points.

Although Theorem 2.4.7 has the advantage of simplicity, it is not as useful for application. We have the following equivalent definition and characterization of singular points for $\mathcal{B} \in \mathfrak{B}^{\Delta}(\mathcal{S})$.

Definition 2.4.8. Let $\sigma_1 > \cdots > \sigma_d$ be the *distinct* elements of $\mathcal{S}$ in decreasing order, and set $\sigma_0 = \infty$, $\sigma_{d+1} = 0$. Let $\varsigma_a = \sum_{j=1}^{a} \frac{|S_{\sigma_j}|}{p}$ for $0 \leq a \leq d$.

Proposition 2.4.9. *We have that $\mathcal{B} \in \mathfrak{B}^{\Delta}(\mathcal{S})$ if and only if $\mathcal{B} \in \mathfrak{B}(\mathcal{S})$ such that*

1. *for each $x \in I \setminus \{V_\ell\}_{\ell=0}^{n}$, we have $\mathcal{B}'(x) \in \{\varsigma_i\}_{i=0}^{d}$;*

2. *if $V < W$ are non-differentiable points of $\mathcal{B}$, then $\rho_<(V) < \rho_>(W)$.*

If $\mathcal{B} \in \mathfrak{B}^{\Delta}(\mathcal{S})$, then the singular points of $\mathcal{B}$ are exactly the non-differentiable points V of $\mathcal{B}$ such that $\mathcal{B}'(V^+) < \mathcal{B}'(V^-)$. In this case, if $\mathcal{B}'(V^-) = \varsigma_i$ then $\mathcal{B}'(V^+) = \varsigma_{i-1}$ and

$$\rho_<(V) = \rho_>(V) = \sigma_i^{-1} e^V.$$

Before providing the proof, we highlight a few features. The first condition in Proposition 2.4.9 restricts the possible values of the slopes of $\mathcal{B}$ whereas the second condition is a refinement of property (3) in Definition 2.4.2. This is transparent when we rewrite the second condition as the following equivalent statement:

If $V < W$ are non-differentiable points of $\mathcal{B}$ with $\mathcal{B}'(V^-) = \varsigma_i$, $\mathcal{B}'(W^+) = \varsigma_j$ for some $0 \leq i, j \leq d$, then

$$\sigma_i^{-1} \sigma_{j+1} e^{-(W-V)} < 1. \tag{2.4.4}$$

Although this refinement requires the inequality to be strict for pairs of non-differentiable points $V < W$, it does not require the same for $V = W$; namely if V is a non-differentiable point of $\mathcal{B}$

such that $\mathcal{B}'(V^-) = \varsigma_i$ and $\mathcal{B}'(V^+) = \varsigma_j$ then we still have the *weak inequality*

$$\sigma_i^{-1}\sigma_{j+1}e^{-(W-V)} \le 1.$$

This way of viewing $\mathfrak{B}^\Delta(\mathcal{S})$ also has the advantage of realizing the subset as dense in $\mathfrak{B}(\mathcal{S})$.

Corollary 2.4.10. *If we endow $\mathfrak{B}(\mathcal{S})$ with the topology induced from the disjoint union $\bigsqcup_{n=1}^{\infty}(\mathbb{R} \cup \{\pm\infty\})^{2n+1}$ by identifying $\mathcal{B} \in \mathfrak{B}(\mathcal{S})$ with the point*

$$(V_0, \dots, V_n, \mathcal{B}'(V_0, V_1), \dots, \mathcal{B}'(V_{n-1}, V_n)).$$

Then $\mathfrak{B}^\Delta(\mathcal{S})$ is a dense subset of $\mathfrak{B}(\mathcal{S})$ by Proposition 2.4.9. Note that the parametrization identifies $\mathcal{B}$ which differ up to translation.

To prove Proposition 2.4.9, we require a lemma which describes $\rho_<(\mathbf{x}), \rho_>(\mathbf{x})$ in terms of maximizing, minimizing over finite sets.

Lemma 2.4.11. *Let $\mathcal{B} \in \mathfrak{B}(\mathcal{S})$. Then*

$$\rho_<(\mathbf{x}) = \max_{\ell \in [[1,n]]}\{\tau_i^{-1}e^{\min(\mathbf{x},V_\ell)} : V_{\ell-1} < \mathbf{x}, \mathcal{B}'((V_{\ell-1}, V_\ell)) = \frac{i}{p}\}$$
$$\rho_>(\mathbf{x}) = \min_{\ell \in [[1,n]]}\{\tau_{i+1}^{-1}e^{\max(V_{\ell-1},\mathbf{x})} : V_\ell > \mathbf{x}, \mathcal{B}'((V_{\ell-1}, V_\ell)) = \frac{i}{p}\}$$

Moreover,

1. *if $\mathbf{x} \in (V_{\ell-1}, V_\ell]$ and $\mathcal{B}'(\mathbf{x}^-) = \frac{i}{p}$, then*

$$\rho_<(\mathbf{x}) = \max(\rho_<(V_{\ell-1}), \tau_i^{-1}e^{\mathbf{x}}),$$

2. *and if $\mathbf{x} \in [V_{\ell-1}, V_\ell)$ and $\mathcal{B}'(\mathbf{x}^+) = \frac{i}{p}$, then*

$$\rho_>(\mathbf{x}) = \min(\rho_>(V_\ell), \tau_{i+1}^{-1}e^{\mathbf{x}}).$$

Proof. Suppose $\xi \le \mathbf{x}$ with $\xi \in (V_{\ell-1}, V_\ell]$ and $\mathcal{B}'(\xi^-) = \frac{i}{p}$. Observe that

$$\mathcal{B}'(\xi^-) = \mathcal{B}'((V_{\ell-1}, V_\ell)) = \mathcal{B}'(\min(\mathbf{x}, V_\ell)^-)$$

which is $\frac{i}{p}$. Then

$$\tau_i^{-1}e^{\xi} \le \tau_i^{-1}e^{\min(\mathbf{x},V_\ell)} \le \rho_<(\mathbf{x}).$$

Maximizing over all $\ell \in [[1,n]]$ proves the statement for $\rho_<(\mathbf{x})$. A similar argument yields the expression for $\rho_>(\mathbf{x})$. The rest of the lemma follows from the obtained form for $\rho_<(\mathbf{x})$ and $\rho_>(\mathbf{x})$. $\square$

Proof of Proposition 2.4.9. Suppose $\mathbf{x} \in (V_{\ell-1}, V_\ell)$ is a singular point of $\mathcal{B}$. By Lemma 2.4.11, one of the following equalities must hold

$$\rho_<(V_{\ell-1}) = \rho_>(V_\ell), \quad \tau_i^{-1}e^{\mathbf{x}} = \tau_{i+1}^{-1}e^{\mathbf{x}}, \quad \rho_<(V_{\ell-1}) = \tau_{i+1}^{-1}e^{\mathbf{x}}, \quad \tau_i^{-1}e^{\mathbf{x}} = \rho_>(V_\ell).$$

Observe that the latter two equalities cannot hold, we have

$$\rho_<(V_{\ell-1}) \le \rho_>(V_{\ell-1}) \le \tau_{i+1}^{-1} e^{V_{\ell-1}} < \tau_{i+1}^{-1} e^{\mathbf{x}}$$

and similarly $\tau_i^{-1} e^{\mathbf{x}} < \rho_>(V_\ell)$. Thus if $\mathbf{x} \in (V_{\ell-1}, V_\ell)$ is singular, then

$$\rho_<(V_{\ell-1}) = \rho_>(V_\ell), \quad \text{or} \quad \tau_i^{-1} = \tau_{i+1}^{-1}.$$

However note that these equalities are independent of $\mathbf{x} \in (V_{\ell-1}, V_\ell)$. In particular, this means that $\mathcal{B}$ has a singular point in $(V_{\ell-1}, V_\ell)$ if and only if every point is singular in $(V_{\ell-1}, V_\ell)$.

Therefore, $\mathcal{B}$ has finitely many singular points if and only if there are no singular points in $I \setminus \{V_\ell\}_{\ell=0}^n$. By our discussion above, this is true if and only if

$$\rho_<(V_{\ell-1}) < \rho_>(V_\ell)$$

for every $\ell \in [[1, n]]$ and

$$\tau_i^{-1} < \tau_{i+1}^{-1}$$

for any $0 \le i \le p$ such that $\mathcal{B}'((V_{\ell-1}, V_\ell)) = \frac{i}{p}$. The latter condition is equivalent to $i = \sum_{j=1}^a |S_{\sigma_j}|$ for some $0 \le a \le d$ in which case $\mathcal{B}'(x) \in \{\varsigma_i\}_{i=0}^d$. This proves the desired equivalent description of $\mathfrak{B}^\Delta(\mathcal{S})$.

We now characterize the singular points. Since the singular points of $\mathcal{B} \in \mathfrak{B}^\Delta(\mathcal{S})$ are necessarily non-differentiable points of $\mathcal{B}$, we may assume our singular point is V_ℓ for some $1 \le \ell < n$ with $\mathcal{B}'(V_\ell^-) = \varsigma_i$ and $\mathcal{B}'(V_\ell^+) = \varsigma_j$. Since

$$\max(\rho_<(V_{\ell-1}), \sigma_i^{-1} e^{V_\ell}) = \rho_<(V_\ell) = \rho_>(V_\ell) = \min(\rho_>(V_{\ell+1}), \sigma_{j+1}^{-1} e^{V_\ell})$$

Since $\rho_<(V_{\ell-1}) < \rho_>(V_\ell)$ and $\rho_<(V_\ell) < \rho_>(V_{\ell+1})$, we must have

$$\sigma_i^{-1} e^{V_\ell} = \rho_<(V_\ell) = \rho_>(V_\ell) = \sigma_{j+1}^{-1} e^{V_\ell}.$$

This is the case when $j = i - 1$. $\square$

2.5 Asymptotics

In this section, we obtain asymptotics of the observables from Section 2.3 under our limit regime. We begin with relevant definitions prior to the statement of the main theorems for this section (Theorems 2.5.2 and 2.5.3).

Define

$$
\begin{aligned}
\mathcal{G}_{<\mathbf{x}}(z) &= \prod_{\sigma\in\mathcal{S}} \prod_{\substack{1\le\ell\le n\\ V_{\ell-1}<\mathbf{x}}} \left(\frac{1-e^{\min(V_\ell,\mathbf{x})}(\sigma z)^{-1}}{1-e^{V_{\ell-1}}(\sigma z)^{-1}}\right)^{\frac{1}{p}\mathbf{1}[\mathcal{B}'(V_\ell^-)\ge\varsigma_\sigma]}, \\
\mathcal{G}_{>\mathbf{x}}(z) &= \prod_{\sigma\in\mathcal{S}} \prod_{\substack{1\le\ell\le n\\ V_\ell>\mathbf{x}}} \left(\frac{1-e^{-\max(V_{\ell-1},\mathbf{x})}\sigma z}{1-e^{-V_\ell}\sigma z}\right)^{\frac{1}{p}\mathbf{1}[\mathcal{B}'(V_\ell^-)<\varsigma_\sigma]}.
\end{aligned}
\tag{2.5.1}
$$

where we set $\varsigma_\sigma = \varsigma_{\sigma_i}$ if $\sigma = \sigma_i$. Here the branches are chosen so that the argument is 0 for $|z|$ large and real. Recall that $\mathcal{S}$ is a multiset of p elements up to multiplicity with d distinct values. In particular, the product over $\sigma \in \mathcal{S}$ is a product over p terms.

Definition 2.5.1. Let $\mathbb{P}^{B,r,\boldsymbol{s}}_{\alpha,\mathrm{t}}$ satisfy the Limit Conditions and let $V_{\ell_1}, \dots, V_{\ell_\nu}$ be the singular points of $\mathcal{B}$. Given $d > 0$, we say that $x(\varepsilon)$ is *d-separated from singular points* if

$$
|x(\varepsilon) - v_{\ell_i}(\varepsilon)| \ge d, \quad 1 \le i \le \nu
$$

for all sufficiently small $\varepsilon > 0$.

We now present the main results for this section. Let $(\pi^x)_{x\in I^\varepsilon}$ denote the diagonal sections of $\mathbb{P}^{B,r,\boldsymbol{s}}_{\alpha,\mathrm{t}}$.

Theorem 2.5.2. *Suppose $\mathbb{P}^{B,r,\boldsymbol{s}}_{\alpha,\mathrm{t}}$ satisfies the Limit Conditions with $\mathcal{B} \in \mathfrak{B}^\Delta(\mathcal{S})$. Fix $\mathbf{x} \in I, k \in \mathbb{Z}_{\ge 0}$. Let $x \in I^\varepsilon_V$ be $k\mathrm{t}$-separated from singular points and satisfy $\varepsilon x \to \mathbf{x}$. Then*

$$
\mathbb{E}\wp_k(\pi^x; r^{\mathrm{t}\alpha}, r^{\mathrm{t}}) \to \frac{1}{2\pi\mathbf{i}} \oint [\mathcal{G}^{\mathcal{B}}_{<\mathbf{x}}(z)\mathcal{G}^{\mathcal{B}}_{>\mathbf{x}}(z)]^{k\mathrm{t}} \frac{dz}{z} \tag{2.5.2}
$$

as $\varepsilon \to 0$, where the contour is positively oriented around $[0, \rho_<(\mathbf{x}))$ and does not contain $(\rho_>(\mathbf{x}), \infty)$; recall $\rho_<(\mathbf{x})$ and $\rho_>(\mathbf{x})$ are defined in (2.4.2).

Theorem 2.5.3. *Suppose $\mathbb{P}^{B,r,\boldsymbol{s}}_{\alpha,\mathrm{t}}$ satisfies the Limit Conditions with $\mathcal{B} \in \mathfrak{B}^\Delta(\mathcal{S})$. Fix $\mathbf{x}_1 \le \cdots \le \mathbf{x}_m$, $k_1, \dots, k_m \in \mathbb{Z}_{\ge 0}$. Let $x_1 \le \cdots \le x_m$ in I^ε_V be such that x_i is $2k_i\mathrm{t}$-separated from singular points and satisfies $\varepsilon x_i \to \mathbf{x}_i$ for each $i \in [[1, m]]$. Then the vector*

$$
\left(\frac{1}{\varepsilon}(\wp_{k_1}(\pi^{x_1}; r^{\mathrm{t}\alpha}, r^{\mathrm{t}}) - \mathbb{E}\wp_{k_1}(\pi^{x_1}; r^{\mathrm{t}\alpha}, r^{\mathrm{t}})), \dots, \frac{1}{\varepsilon}(\wp_{k_m}(\pi^{x_m}; r^{\mathrm{t}\alpha}, r^{\mathrm{t}}) - \mathbb{E}\wp_{k_m}(\pi^{x_m}; r^{\mathrm{t}\alpha}, r^{\mathrm{t}}))\right)
$$

converges in distribution as $\varepsilon \to 0$ to the centered gaussian vector $(\mathfrak{D}_{k_1}(\mathbf{x}_1), \dots \mathfrak{D}_{k_m}(\mathbf{x}_m))$ with covariance defined by

$$
\begin{aligned}
&\mathrm{Cov}\left(\mathfrak{D}_{k_i}(\mathbf{x}_i), \mathfrak{D}_{k_j}(\mathbf{x}_j)\right) \\
&\quad = \alpha\mathrm{t}^2 \frac{k_i k_j}{(2\pi\mathbf{i})^2} \oint\oint \frac{[\mathcal{G}^{\mathcal{B}}_{<\mathbf{x}_i}(z)\mathcal{G}^{\mathcal{B}}_{>\mathbf{x}_i}(z)]^{k_i\mathrm{t}}[\mathcal{G}^{\mathcal{B}}_{<\mathbf{x}_j}(w)\mathcal{G}^{\mathcal{B}}_{>\mathbf{x}_j}(w)]^{k_j\mathrm{t}}}{(z-w)^2} dz\, dw, \quad 1 \le i < j \le m
\end{aligned}
\tag{2.5.3}
$$

where

- *the z-contour is positively oriented around $[0, \rho_<(\mathbf{x}_i))$ but does not contain $(\rho_>(\mathbf{x}_i), \infty)$,*
- *the w-contour is positively oriented around $[0, \rho_<(\mathbf{x}_j))$ but does not contain $(\rho_>(\mathbf{x}_j), \infty)$,*
- *and the z-contour is enclosed by the w-contour.*

If $\rho_<(\mathbf{x}_i) = \rho_>(\mathbf{x}_j) =: \rho$, then the z- and w-contours intersect at ρ.

The remainder of this section is devoted to the proofs of Theorems 2.5.2 and 2.5.3. We begin by collecting some asymptotic preliminaries. We then give an outline of the proofs to illustrate the key ideas, followed by the rigorous proofs.

2.5.1 Preliminary Asymptotics

In preparation for the asymptotics of the formula from Theorem 2.3.17, we study the asymptotics $G^B_{<x}, G^B_{>x}$ and some asymptotic properties of their poles, formulated in two propositions. The latter is important for understanding the placement of contours in the analysis of moments. Before presenting the propositions, we introduce some notation to work with the poles of $G^B_{<x}, G^B_{>x}$.

Given $\mathbb{P}^{B,r,\boldsymbol{s}}_{\alpha,\mathrm{t}}$ satisfying the Limit Conditions such that $\mathcal{B} \in \mathfrak{B}^\Delta(\mathcal{S})$, we define the ($\varepsilon$-dependent) sets

$$\begin{aligned} R^{a,\ell}_{<x} &= \{(s_0 \cdots s_a r^e)^{-1} : e \in (v_{\ell-1}, v_\ell) \cap (a + p\mathbb{Z} + \frac{1}{2}), e < x, \mathcal{B}'(V_{\ell-1}, V_\ell) \geq \varsigma_{s_0 \cdots s_a}\}, \\ R^{a,\ell}_{>x} &= \{(ts_0 \cdots s_a r^e)^{-1} : e \in (v_\ell, v_{\ell+1}) \cap (a + p\mathbb{Z} + \frac{1}{2}), e > x, \mathcal{B}'(V_\ell, V_{\ell+1}) < \varsigma_{s_0 \cdots s_a}\}. \end{aligned} \tag{2.5.4}$$

Recall the q-Pochhammer symbol

$$(a;q)_N = \prod_{i=0}^{N-1} (1 - aq^i), \quad (a;q)_\infty = \prod_{i=0}^{\infty} (1 - aq^i)$$

Lemma 2.5.4. *Suppose $\mathbb{P}^{B,r,\boldsymbol{s}}_{\alpha,\mathrm{t}}$ satisfies the Limit Conditions such that $\mathcal{B} \in \mathfrak{B}^\Delta(\mathcal{S})$. For $\varepsilon > 0$ sufficiently small,*

$$\begin{aligned} G^B_{<x}(z; \varepsilon, \mathrm{t}) &= \prod_{R^{a,\ell}_{<x} \neq \emptyset} \frac{(t^{-1}(\max R^{a,\ell}_{<x}) z^{-1}; r^p)_\infty}{((\max R^{a,\ell}_{<x}) z^{-1}; r^p)_\infty} \cdot \frac{(r^p(\min R^{a,\ell}_{<x}) z^{-1}; r^p)_\infty}{(t^{-1} r^p (\min R^{a,\ell}_{<x}) z^{-1}; r^p)_\infty} \\ G^B_{>x}(z; \varepsilon, \mathrm{t}) &= \prod_{R^{a,\ell}_{>x} \neq \emptyset} \frac{(t^{-1}(\min R^{a,\ell}_{>x})^{-1} z; r^p)_\infty}{((\min R^{a,\ell}_{>x})^{-1} z; r^p)_\infty} \cdot \frac{(r^p(\max R^{a,\ell}_{>x})^{-1} z; r^p)_\infty}{(t^{-1} r^p (\max R^{a,\ell}_{>x})^{-1} z; r^p)_\infty}. \end{aligned} \tag{2.5.5}$$

Proof. If $\mathcal{B}'((V_{\ell-1}, V_\ell)) = \varsigma_i$, then Lemma 2.4.4 implies

$$B' = \mathbf{1}\left[\bigcup_{1 \leq j \leq i} (S_{\sigma_j} + p\mathbb{Z} + \frac{1}{2})\right]$$

on $I_E^\varepsilon \cap (v_{\ell-1}, v_\ell)$. By the definitions of $G^B_{<x}$ and $R^{a,\ell}_{<x}$, we have

$$G^B_{<x}(z;\varepsilon,\mathfrak{t}) = \prod_{R^{a,\ell}_{<x}\neq\emptyset} \prod_{\rho\in R^{a,\ell}_{<x}} \frac{1-t^{-1}\rho z^{-1}}{1-\rho z^{-1}}.$$

Since the elements of $R^{a,\ell}_{<x}$ are

$$\max R^{a,\ell}_{<x}, \quad r^p \max R^{a,\ell}_{<x}, \quad r^{2p}\max R^{a,\ell}_{<x}, \quad \ldots, \quad \min R^{a,\ell}_{<x},$$

in decreasing order, we can rewrite the product over ρ as the ratios of Pochhammer symbols in (2.5.5). The argument is similar for $G^B_{>x}$. $\square$

The first proposition in this section describes the behavior of the largest pole of $G^B_{<x}$ and the smallest pole of $G^B_{>x}$.

Proposition 2.5.5. *Suppose $\mathbb{P}^{B,r,\boldsymbol{s}}_{\alpha,\mathfrak{t}}$ satisfies the Limit Conditions such that $\mathcal{B} \in \mathfrak{B}^\Delta(\mathcal{S})$. For $x \in I^\varepsilon$, let $\rho^\varepsilon_<(x)$ and $\rho^\varepsilon_>(x)$ denote the maximal pole of $G^B_{<x}(z;\varepsilon,\mathfrak{t})$ and minimal pole of $G^B_{>x}(z;\varepsilon,\mathfrak{t})$ respectively. Suppose $\mathbf{x} \in I$ and $x \in I^\varepsilon$ satisfy $\varepsilon x \to \mathbf{x}$ as $\varepsilon \to 0$.*

(a) If $\mathbf{x} \in I \setminus \{V_\ell\}_{\ell=0}^n$, then

$$\lim_{\varepsilon\to 0} \rho^\varepsilon_<(x) = \rho_<(\mathbf{x}), \qquad \lim_{\varepsilon\to 0}\rho^\varepsilon_>(x) = \rho_>(\mathbf{x}).$$

(b) If $\mathbf{x} = V_\ell$ is not a singular point and

(i) $x > v_\ell$ for all $\varepsilon > 0$, then

$$\lim_{\varepsilon\to 0} \rho^\varepsilon_<(x) = \rho_<(V_\ell^+), \qquad \lim_{\varepsilon\to 0}\rho^\varepsilon_>(x) = \rho_>(V_\ell^+) = \rho_>(V_\ell).$$

(ii) $x < v_\ell$ for all $\varepsilon > 0$, then

$$\lim_{\varepsilon\to 0} \rho^\varepsilon_<(x) = \rho_<(V_\ell^-) = \rho_<(V_\ell), \qquad \lim_{\varepsilon\to 0}\rho^\varepsilon_>(x) = \rho_>(V_\ell^-).$$

(iii) $x = v_\ell$ for all $\varepsilon > 0$, then

$$\lim_{\varepsilon\to 0} \rho^\varepsilon_<(x) = \rho_<(V_\ell), \qquad \lim_{\varepsilon\to 0}\rho^\varepsilon_>(x) = \rho_>(V_\ell).$$

(c) If $\mathbf{x} = V_\ell$ is a singular point, then

$$\lim_{\varepsilon\to 0} \rho^\varepsilon_<(x) = \rho_<(\mathbf{x}) = \rho_>(\mathbf{x}) = \lim_{\varepsilon\to 0}\rho^\varepsilon_>(x).$$

Furthermore, if $\mathcal{B}'(V_\ell) = \varsigma_\sigma$, then

$$\lim_{\varepsilon\to 0} \max\left(\bigcup_{a\in S_\sigma} R^{a,\ell}_{<x}\right) = \rho_<(V_\ell) > \limsup_{\varepsilon\to 0} \max\{\textit{poles of } G^B_{<x}(z;\varepsilon,\mathfrak{t})\} \setminus \left(\bigcup_{a\in S_\sigma} R^{a,\ell}_{<x}\right), \tag{2.5.6}$$

$$\lim_{\varepsilon\to 0} \min\left(\bigcup_{a\in S_\sigma} R^{a,\ell}_{>x}\right) = \rho_>(V_\ell) < \liminf_{\varepsilon\to 0} \min\{\textit{poles of } G^B_{>x}(z;\varepsilon,\mathfrak{t})\} \setminus \left(\bigcup_{a\in S_\sigma} R^{a,\ell}_{>x}\right), \tag{2.5.7}$$

$$t^{-1} r^{-|x-v_\ell|-1} \le \frac{\rho^\varepsilon_>(x)}{\rho^\varepsilon_<(x)} \le t^{-1} r^{-|x-v_\ell|-2p+1}, \tag{2.5.8}$$

for $\varepsilon > 0$ sufficiently small.

The next proposition gives asymptotics of $G^B_{<x}, G^B_{>x}$ with special precision given to points near maximal and minimal poles respectively.

Let $\theta \in (0,\pi)$, $\delta > 0$. Define

$$D^{\varepsilon,\theta,\delta} = \big\{w \in \mathbb{C} : \operatorname{dist}\big(w, [1,\infty)\big) \ge \delta\big\} \cup \big\{w \in \mathbb{C} : |\arg\big(w - (1-\varepsilon)\big)| \ge \theta\big\},$$

that is the neighborhood δ-separated from $[1,\infty)$ between the rays of arguments $\pm\theta$ started from $1-\varepsilon$ on the left side of $\mathbb{C}$.

Proposition 2.5.6. *Suppose $\mathbb{P}^{B,r,\boldsymbol{s}}_{\alpha,\mathfrak{t}}$ satisfies the Limit Conditions such that $\mathcal{B} \in \mathfrak{B}^\Delta(\mathcal{S})$. For $x \in I^\varepsilon$, let $\rho^\varepsilon_<(x)$ and $\rho^\varepsilon_>(x)$ denote the maximal pole of $G^B_{<x}(z;\varepsilon,\mathfrak{t})$ and minimal pole of $G^B_{>x}(z;\varepsilon,\mathfrak{t})$ respectively. Suppose $x \in I^\varepsilon_V$ such that $\varepsilon x \to \mathbf{x} \in I$. Further assume that if $\mathbf{x} = V_\ell$ with V_ℓ not a singular point, then either $x > v_\ell$ for all $\varepsilon > 0$, $x = v_\ell$ for all $\varepsilon > 0$, or $x < v_\ell$ for all $\varepsilon > 0$. Then*

$$G^B_{<x}(t^{-1}z^{-1}\cdot\rho^\varepsilon_<(x);\varepsilon,\mathfrak{t}) = \mathcal{G}_{<\mathbf{x}}(z^{-1}\cdot\rho_<(\mathbf{x}))^{\mathfrak{t}} \exp\left(O\left(\varepsilon\frac{|z|\vee|z|^2}{|1-z|}\right)\right)$$
$$G^B_{>x}(tz\cdot\rho^\varepsilon_>(x);\varepsilon,\mathfrak{t}) = \mathcal{G}_{>\mathbf{x}}(z\cdot\rho_>(\mathbf{x}))^{\mathfrak{t}} \exp\left(O\left(\varepsilon\frac{|z|\vee|z|^2}{|1-z|}\right)\right)$$

uniformly for $z \in D^{\varepsilon,\theta,\delta}$ and $\varepsilon > 0$ sufficiently small.

The remainder of this subsection is devoted to the proofs of Propositions 2.5.5 and 2.5.6.

Proof of Proposition 2.5.5. Suppose throughout the proof, ε is small enough so that the conclusion of Lemma 2.5.4 is true and so that $v_\ell - v_{\ell-1} > p$ for each $\ell \in [[1,n]]$ (that is we have at least one period in $(v_{\ell-1}, v_\ell)$). Then

$$\rho^\varepsilon_<(x) = \max\left(\bigcup_{a\in[[0,p-1]],m\in[[1,n]]} R^{a,m}_{<x}\right) \tag{2.5.9}$$

where the above set being maximized over is the set of poles of $G^B_{<x}(z;\varepsilon,\mathfrak{t})$. Observe that if $R^{a,m}_{<x}$ is nonempty, then

$$\max R^{a,m}_{<x} = (s_0\cdots s_a)^{-1} e^{\min(\mathbf{x},V_m)} + o(1)$$

as $\varepsilon \to 0$. The set $R^{a,m}_{<x}$ is nonempty if $v_{m-1} < x$ and $\mathcal{B}'((V_{m-1}, V_m)) \geq \varsigma_{s_0 \cdots s_a}$; here we used the fact that $v_m - v_{m-1} > p$, otherwise it is possible that $\mathbb{Z} + a + \frac{1}{2}$ and (v_{m-1}, v_m) do not intersect.

We first prove the convergence statements in (a), (b), (c) for $\rho^\varepsilon_<(x)$. The argument for $\rho^\varepsilon_>(x)$ is similar.

(a). If $\mathbf{x} \in I \setminus \{V_\ell\}_{\ell=0}^n$, then $R^{a,m}_{<x}$ is nonempty if and only if $V_{m-1} < \mathbf{x}$ and $\mathcal{B}'((V_{m-1}, V_m)) \geq \varsigma_{s_0 \cdots s_a}$. Then for $\varepsilon x \to \mathbf{x}$, we have

$$\begin{aligned} \rho^\varepsilon_<(x) &\to \max\{(s_0 \cdots s_a)^{-1} e^{\min(\mathbf{x}, V_m)} : V_{m-1} < \mathbf{x}, \mathcal{B}'((V_{m-1}, V_m)) \geq \varsigma_{s_0 \cdots s_a}\} \\ &= \max\{(s_0 \cdots s_a)^{-1} e^{\min(\mathbf{x}, V_m)} : V_{m-1} < \mathbf{x}, \mathcal{B}'((V_{m-1}, V_m)) = \varsigma_{s_0 \cdots s_a}\} = \rho_<(\mathbf{x}) \end{aligned} \tag{2.5.10}$$

as $\varepsilon \to 0$, where the first equality follows from the fact that $\varsigma_{s_0 \cdots s_a}$ is an increasing function of $(s_0 \cdots s_a)^{-1}$ and the second equality follows from Lemma 2.4.11.

(bii), (biii) and (c). Suppose $\mathbf{x} = V_\ell$ for some $0 < \ell \leq n$ and $x \leq v_\ell$ for all $\varepsilon > 0$. In this case, we again have that $R^{a,m}_{<x}$ is nonempty if and only if $V_{m-1} < \mathbf{x} = V_\ell$ and $\mathcal{B}'((V_{m-1}, V_m)) \geq \varsigma_{s_0 \cdots s_a}$. Then (2.5.10) holds for this case as well.

(bi) and (c). Suppose $\mathbf{x} = V_\ell$ for some $0 \leq \ell < n$ and $x > v_\ell$ for all $\varepsilon > 0$. Then $R^{a,m}_{<x}$ is nonempty if and only if $V_{m-1} \leq V_\ell = \mathbf{x}$ and $\mathcal{B}'((V_{m-1}, V_m)) \geq \varsigma_{s_0 \cdots s_a}$. Then

$$\begin{aligned} \rho^\varepsilon_<(x) &\to \max\{(s_0 \cdots s_a)^{-1} e^{\min(\mathbf{x}, V_m)} : V_{m-1} \leq \mathbf{x} = V_\ell, \mathcal{B}'((V_{m-1}, V_m)) \geq \varsigma_{s_0 \cdots s_a}\} \\ &= \max\{(s_0 \cdots s_a)^{-1} e^{\min(\mathbf{x}, V_m)} : V_{m-1} \leq \mathbf{x} = V_\ell, \mathcal{B}'((V_{m-1}, V_m)) = \varsigma_{s_0 \cdots s_a}\} = \rho_<(V_\ell^+) \end{aligned}$$

as $\varepsilon \to 0$, by the same reasoning as before.

Note that combining the latter two cases gives us the complete case of the convergence of $\rho^\varepsilon_<(x)$ for (c). It remains to show (2.5.6), (2.5.7), (2.5.8). For the remainder of the proof, assume $\mathbf{x} = V_\ell$ is a singular point.

(c): (2.5.6). By the argument preceding the case analysis above, $R^{a,m}_{<x}$ is empty if $m > \ell + 1$. We rewrite the union

$$\begin{aligned} \left(\bigcup_{a \in [[0,p-1]], m \in [[1,n]]} R^{a,m}_{<x} \right) &= \left(\bigcup_{a \in [[0,p-1]], m < \ell} R^{a,m}_{<x} \right) \cup \left(\bigcup_{a \in [[0,p-1]]} R^{a,\ell+1}_{<x} \right) \\ &\quad \cup \left(\bigcup_{a \in [[0,p-1]] \setminus S_\sigma} R^{a,\ell}_{<x} \right) \cup \left(\bigcup_{a \in S_\sigma} R^{a,\ell}_{<x} \right) \end{aligned} \tag{2.5.11}$$

as four smaller unions. We want to show that the lim sup of the maximum of the first three unions converges to $\rho < \rho_<(V_\ell)$, then by (2.5.9) the maximum of the fourth union must be equal to $\rho^\varepsilon_<(x)$. Since $\rho^\varepsilon_<(x) \to \rho_<(V_\ell)$ as $\varepsilon \to 0$, this establishes (2.5.11). Note that the maximum of the first union is exactly $\rho^\varepsilon_<(v_{\ell-1})$ and therefore converges to $\rho_<(V_{\ell-1})$ as $\varepsilon \to 0$. By Proposition 2.4.9,

$$\rho_<(V_{\ell-1}) < \rho_>(V_\ell) = \rho_<(V_\ell)$$

because V_ℓ is a singular point. We have thus shown

$$\lim_{\varepsilon\to 0}\max\left(\bigcup_{a\in[[0,p-1]],m<\ell} R^{a,m}_{<x}\right) < \rho_<(V_\ell).$$

For the second union in (2.5.11), observe that by Proposition 2.4.9, if $\sigma = \sigma_i$ then

$$\mathcal{B}'((V_\ell, V_{\ell+1})) = \mathcal{B}'(V_\ell^+) = \varsigma_{i-1}.$$

Thus, arguing as in the case analysis above,

$$\begin{aligned}\limsup_{\varepsilon\to 0}\max\left(\bigcup_{a\in[[0,p-1]]} R^{a,\ell+1}_{<x}\right) &\leq \max\{(s_0\cdots s_a)^{-1}e^{V_\ell} : \varsigma_{i-1} = \mathcal{B}'((V_\ell,V_{\ell+1})) \geq \varsigma_{s_0\cdots s_a}\}\\ &\leq \sigma_{i-1}^{-1}e^{V_\ell} < \sigma_i^{-1}e^{V_\ell} = \rho_<(V_\ell)\end{aligned}$$

where we recall $\varsigma_{i-1} = \varsigma_{\sigma_{i-1}}$ and the final equality follows from Proposition 2.4.9. It is necessary to take the $\limsup$ since the set above may be empty depending on whether $x < v_\ell$ or $x > v_\ell$. Also, note that the second inequality is equality if the preceding set is nonempty (this is when $i > 0$). Otherwise it is strict since $\max\emptyset = -\infty < 0$; recall that $\sigma_0 = \infty$ so we take $\sigma_0^{-1} = 0$. The third union is similar, we have $\mathcal{B}'((V_{\ell-1}, V_\ell)) = \varsigma_\sigma = \varsigma_i$ by assumption. Then

$$\begin{aligned}\lim_{\varepsilon\to 0}\max\left(\bigcup_{a\in[[0,p-1]]\setminus S_\sigma} R^{a,\ell}_{<x}\right) &= \max\{(s_0\cdots s_a)^{-1}e^{V_\ell} : \varsigma_i = \mathcal{B}'((V_\ell,V_{\ell+1})) \geq \varsigma_{s_0\cdots s_a}, s_0\cdots s_a \neq \sigma_i\}\\ &\leq \sigma_{i-1}^{-1}e^{V_\ell} < \sigma_i^{-1}e^{V_\ell} = \rho_<(V_\ell).\end{aligned}$$

This proves (2.5.6).

<u>(c): (2.5.7).</u> Uses an analogous argument as for (2.5.6).

<u>(c): (2.5.8).</u> Using (2.5.6) and (2.5.7), notice that

$$\rho^\varepsilon_<(x) = \max\left(\bigcup_{a\in S_\sigma} R^{a,\ell}_{<x}\right) = \sigma^{-1}r^{-e_1}, \quad \rho^\varepsilon_>(x) = \min\left(\bigcup_{a\in S_\sigma} R^{a,\ell}_{>x}\right) = t^{-1}\sigma^{-1}r^{-e_2}$$

where

$$e_1 = \max\{e \in (v_{\ell-1}, v_\ell)\cap(S_\sigma + p\mathbb{Z} + \frac{1}{2}) : e < x\}, \quad e_2 = \min\{e \in (v_\ell, v_{\ell+1})\cap(S_\sigma + p\mathbb{Z} + \frac{1}{2}) : e > x\}.$$

We have the inequalities

$$\begin{aligned}\min(v_\ell, x) - p + \frac{1}{2} \leq e_1 \leq \min(v_\ell, x) - \frac{1}{2}\\ \max(v_\ell, x) + \frac{1}{2} \leq e_2 \leq \max(v_\ell, x) + p - \frac{1}{2}.\end{aligned}$$

Thus we conclude (2.5.8). $\square$

Before going into the proof of Proposition 2.5.6, we first require a lemma on the asymptotics of Pochhammer symbols.

Lemma 2.5.7. *Let $\mathfrak{a} > 0$ and suppose $N(\varepsilon) \in \mathbb{Z}_{>0}$ such that $\liminf_{\varepsilon \to 0} \varepsilon N(\varepsilon) > 0$ as $\varepsilon \to 0$. Then for any fixed $\theta \in (0, \pi), \delta > 0$, we have*

$$\frac{(z; e^{-\varepsilon})_{N(\varepsilon)}}{(e^{-\varepsilon \mathfrak{a}} z; e^{-\varepsilon})_{N(\varepsilon)}} = \left(\frac{1-z}{1-r^N z} \right)^{\mathfrak{a}} \exp\left(O\left(\varepsilon \frac{|z| \vee |z|^2}{|1-z|} \right) \right)$$

uniformly for z in $D^{\varepsilon,\theta,\delta}$ and ε arbitrarily small.

Proof. Throughout the proof, the constant symbol C is independent of $\varepsilon > 0$ (though it may depend on $\mathfrak{a}, \theta$), and may change from line to line.

Set $r = e^{-\varepsilon}$, $N = N(\varepsilon)$, $D^\varepsilon = D^{\varepsilon,\theta,\delta}$. Define

$$\begin{aligned}
E_1^\varepsilon(z) &= \log \frac{(z;r)_N}{(r^{\mathfrak{a}} z; r)_N} - \frac{1-r^{\mathfrak{a}}}{1-r} \log\left(\frac{1-z}{1-r^N z} \right), \\
E_2^\varepsilon(z) &= \frac{1-r^{\mathfrak{a}}}{1-r} \log\left(\frac{1-z}{1-r^N z} \right) - \mathfrak{a} \log\left(\frac{1-z}{1-r^N z} \right) \\
E^\varepsilon(z) &= E_1^\varepsilon(z) + E_2^\varepsilon(z) = \log \frac{(z;r)_N}{(r^{\mathfrak{a}} z; r)_N} - \mathfrak{a} \log\left(\frac{1-z}{1-r^N z} \right).
\end{aligned}$$

Using the fact that

$$1 - r^{\mathfrak{a}} = 1 - (1 - (1-r))^{\mathfrak{a}} = \mathfrak{a}(1-r) - \binom{\mathfrak{a}}{2}(1-r)^2 + O(\varepsilon^3),$$

we have

$$E_2^\varepsilon(z) = \left(\frac{1-r^{\mathfrak{a}}}{1-r} - \mathfrak{a} \right) \log\left(\frac{1-z}{1-r^N z} \right) = \left(-\binom{\mathfrak{a}}{2}\varepsilon + O(\varepsilon^2) \right) \log\left(\frac{1-z}{1-r^N z} \right).$$

The log term is bounded for large $|z|$, behaves as $C|z|$ for small $|z|$, and has logarithmic singularities at $z = 1$ and $z = r^{-N}$. Since D^ε is separated by a constant distance from the r^{-N} singularity by the assumption $\liminf_{\varepsilon \to 0} \varepsilon N(\varepsilon) > 0$, we may disregard the singularity at $z = r^{-N}$ and have

$$|E_2^\varepsilon(z)| \le C\varepsilon \frac{|z|}{|1-z|}$$

for $z \in D^\varepsilon$. Note that we accounted for the log singularity at 1 with the "wasteful" $|1-z|$ term in the denominator and the boundedness for $|z|$ large by balancing the right hand side.

Observe that

$$\log \frac{(z;r)_N}{(r^{\mathfrak{a}}z;r)_N} = -\sum_{j=0}^{N-1} \int_{r^{\mathfrak{a}+j}}^{r^j} \frac{z}{1-uz}\,du = -\frac{1-r^{\mathfrak{a}}}{1-r}\int_{r^N}^{1} \frac{z}{1-u(v)z}\,dv$$

by the change of variables

$$(r^{j+1}, r^j] \ni v \mapsto u(v) = \frac{1-r^{\mathfrak{a}}}{1-r}(v-r^j) + r^j \in (r^{j+\mathfrak{a}}, r^j]. \tag{2.5.12}$$

We may similarly write

$$\frac{1-r^{\mathfrak{a}}}{1-r}\log\left(\frac{1-z}{1-r^N z}\right) = -\frac{1-r^{\mathfrak{a}}}{1-r}\int_{r^N}^{1}\frac{z}{1-vz}\,dv.$$

Thus

$$E_1^{\varepsilon}(z) = \frac{1-r^{\mathfrak{a}}}{1-r}\int_{r^N}^{1}\left(\frac{z}{1-u(v)z} - \frac{z}{1-vz}\right)dv = \frac{1-r^{\mathfrak{a}}}{1-r}\int_{r^N}^{1}\frac{(u(v)-v)z^2}{(1-u(v)z)(1-vz)}\,dv.$$

For $v \in [r^{j+1}, r^j]$,

$$u(v) - v = \frac{r-r^{\mathfrak{a}}}{1-r}(v-r^j),$$

so that

$$|u(v)-v| < C\varepsilon$$

where the constant is also uniform over $v \in [r^N, 1]$. Also for $v \in [r^{j+1}, r^j]$,

$$\left|\frac{1-vz}{1-u(v)z}\right| = \left|1 + \frac{z(u(v)-v)}{1-u(v)z}\right| \le C$$

for $z \in D^{\varepsilon}$. Indeed, $u(v)$ ranges from $r^{N-1+\mathfrak{a}}$ to 1 so that the buffer provided by D^{ε} bounds $|1-u(v)z|$ from below by some constant times ε. Thus

$$|E_1^{\varepsilon}(z)| \le \frac{1-r^{\mathfrak{a}}}{1-r}\left|\int_{r^N}^{1}\frac{(u(v)-v)z^2}{(1-u(v)z)(1-vz)}\,dv\right| \le \int_{r^N}^{1}\frac{C\,\varepsilon\,|z|^2}{|1-vz|^2}\,dv$$

By writing

$$\frac{1}{|1-vz|^2} = \frac{1}{z-\bar{z}}\left(\frac{z}{1-vz} - \frac{\bar{z}}{1-v\bar{z}}\right), \quad v \in [r^N, 1],$$

we obtain

$$|E_1^{\varepsilon}(z)| \le \frac{C\,\varepsilon|z|^2}{z-\bar{z}}\left[\log\left(\frac{1-\bar{z}}{1-r^N\bar{z}}\right) - \log\left(\frac{1-z}{1-r^N z}\right)\right] = C\,\varepsilon\cdot\frac{|z|^2}{\Im z}\arg\left(\frac{1-\bar{z}}{1-r^N\bar{z}}\right)$$

Observe that

$$0 \le \frac{|z|}{\Im z} \arg\left(\frac{1-\bar{z}}{1-r^N\bar{z}}\right) = -\frac{|z|}{|1-z|}\frac{1}{\sin\arg(1-z)} \arg\left(\frac{1-\bar{z}}{1-r^N z}\right) \le C\frac{|z|}{|1-z|}$$

for $z \in D^\varepsilon$. The latter bound follows from the fact that the $\sin\arg(1-z)$ denominator term is balanced by $\arg\left(\frac{1-\bar{z}}{1-r^N z}\right)$ away from $[1,\infty)$; note that as z approaches $\mathbb{R}$ we only need the fact that $\arg\left(\frac{1-\bar{z}}{1-r^N z}\right)$ approaches 0, however near $z=1$ we really need $\arg(1-\bar{z})$ to balance $\sin\arg(1-z)$. Thus

$$|E_1^\varepsilon(z)| \le C\varepsilon\frac{|z|^2}{|1-z|}$$

uniformly for $z \in D^\varepsilon$. Combining our bounds, we obtain

$$|E^\varepsilon(z)| \le C\varepsilon\frac{|z| \vee |z|^2}{|1-z|}$$

uniformly for $z \in D^\varepsilon$. Our lemma now follows from

$$\frac{(z;r)_N}{(r^{\mathfrak{a}}z;r)_N} = \exp(E^\varepsilon(z)).$$

□

Proof of Proposition 2.5.6. By Lemma 2.5.4, for sufficiently small $\varepsilon > 0$, we have

$$G^B_{>x}(tz\cdot\rho^\varepsilon_>(x);\varepsilon,\mathfrak{t}) = \prod_{R^{a,\ell}_{>x}\neq\emptyset} \frac{((\min R^{a,\ell}_{>x})^{-1}z\cdot\rho^\varepsilon_>(x);r^p)_\infty}{(t(\min R^{a,\ell}_{>x})^{-1}z\cdot\rho^\varepsilon_>(x);r^p)_\infty}\cdot\frac{(tr^p(\max R^{a,\ell}_{>x})^{-1}z\cdot\rho^\varepsilon_>(x);r^p)_\infty}{(r^p(\max R^{a,\ell}_{>x})^{-1}z\cdot\rho^\varepsilon_>(x);r^p)_\infty}$$

Fix $R^{a,\ell}_{>x} \neq \emptyset$. As established in the various cases in the proof of Proposition 2.5.5, the nonemptiness of $R^{a,\ell}_{>x}$ for a,ℓ fixed is independent of $\varepsilon > 0$ when ε is sufficiently small, under our assumptions on x. We have

$$\frac{((\min R^{a,\ell}_{>x})^{-1}z\cdot\rho^\varepsilon_>(x);r^p)_\infty}{(t(\min R^{a,\ell}_{>x})^{-1}z\cdot\rho^\varepsilon_>(x);r^p)_\infty}\cdot\frac{(tr^p(\max R^{a,\ell}_{>x})^{-1}z\cdot\rho^\varepsilon_>(x);r^p)_\infty}{(r^p(\max R^{a,\ell}_{>x})^{-1}z\cdot\rho^\varepsilon_>(x);r^p)_\infty} = \frac{((\min R^{a,\ell}_{>x})^{-1}z\cdot\rho^\varepsilon_>(x);r^p)_N}{(t(\min R^{a,\ell}_{>x})^{-1}z\cdot\rho^\varepsilon_>(x);r^p)_N}$$

where $N = N(\varepsilon)$ satisfies

$$r^{p(N-1)} = \frac{\min R^{a,\ell}_{>x}}{\max R^{a,\ell}_{>x}}.$$

By definition of $\rho^\varepsilon_>(x)$,

$$\frac{\rho^\varepsilon_>(x)}{\min R^{a,\ell}_{>x}} = r^\nu$$

for some $\nu = \nu(\varepsilon) \ge 0$. Note that $\varepsilon\nu \to \hat{\nu}$ as $\varepsilon \to 0$ for some $\hat{\nu}$; this follows from the fact that $\min R^{a,\ell}_{>x}$ converges to $(s_0\cdots s_a)^{-1}e^{\max(\mathbf{x},V_\ell)}$ and that $\rho^\varepsilon_>(x)$ converges by Proposition 2.5.5.

There are two main cases to consider: $\hat{\nu} > 0$ and $\hat{\nu} = 0$. First suppose $\hat{\nu} > 0$. Note that if $a_1 = a_1(\varepsilon), a_2 = a_2(\varepsilon)$ converge to some $\hat{a} > 0$ as $\varepsilon \to 0$, then

$$\frac{1 - r^a z}{1 - r^b z} = \exp\left(\log\left(1 + \frac{z(r^b - r^a)}{1 - r^b z}\right)\right) = e^{O(\varepsilon \cdot (|z| \wedge 1))}$$

uniformly over $z \in D^{0,\theta,\delta}$. Thus Lemma 2.5.7 implies

$$\frac{((\min R^{a,\ell}_{>x})^{-1} z \cdot \rho^{\varepsilon}_{>}(x); r^p)_N}{(t(\min R^{a,\ell}_{>x})^{-1} z \cdot \rho^{\varepsilon}_{>}(x); r^p)_N} = \left(\frac{1 - (s_0 \cdots s_a)^{-1} e^{\max(\mathbf{x}, V_\ell)} z \cdot \rho_{>}(\mathbf{x})}{1 - (s_0 \cdots s_a)^{-1} e^{V_{\ell+1}} z \cdot \rho_{>}(\mathbf{x})}\right)^{\frac{\mathfrak{t}}{p}} e^{O(\varepsilon(|z| \wedge 1))}$$

uniformly over $z \in D^{0,\theta,\delta}$.

If $\hat{\nu} = 0$, then write

$$\frac{((\min R^{a,\ell}_{>x})^{-1} z / \rho^{\varepsilon}_{>}(x); r^p)_N}{(t(\min R^{a,\ell}_{>x})^{-1} z / \rho^{\varepsilon}_{>}(x); r^p)_N} = \frac{(r^\nu z; r^p)_N}{(t r^\nu z; r^p)_N}$$

If, in addition, $\lim_{\varepsilon \to 0} \varepsilon N > 0$, then by Lemma 2.5.7, we have

$$\frac{(r^\nu z; r^p)_N}{(t r^\nu z; r^p)_N} = \left(\frac{1 - r^\nu z}{1 - (s_0 \cdots s_a)^{-1} e^{V_\ell + 1} z \cdot \rho_{>}(\mathbf{x})}\right)^{\frac{\mathfrak{t}}{p}} \exp\left(O\left(\varepsilon \frac{|z| \vee |z|^2}{|1 - z|}\right)\right)$$

uniformly over $z \in D^{\varepsilon,\theta,\delta}$. We may further replace the $1 - r^\nu z$ with $1 - z$ without changing the right hand side since

$$\frac{1 - r^\nu z}{1 - z} = \exp\left(\log\left(1 + \frac{z(1 - r^\nu)}{1 - z}\right)\right) = \exp\left(O\left(\varepsilon \frac{|z|}{|1 - z|}\right)\right) \tag{2.5.13}$$

uniformly over $z \in D^{\varepsilon,\theta,\delta}$.

Otherwise, if $\lim_{\varepsilon \to 0} \varepsilon N = 0$, take $M = M(\varepsilon) \in \mathbb{Z}_{>0}$ large so that $\lim_{\varepsilon \to 0} \varepsilon M > 0$. Then by Lemma 2.5.7,

$$\begin{aligned}
\frac{(r^\nu z; r^p)_N}{(t r^\nu z; r^p)_N} &= \frac{(r^\nu z; r^p)_{N+M}}{(t r^\nu z; r^p)_{N+M}} \cdot \frac{(t r^{\nu + pN} z; r^p)_M}{(r^{\nu + pN} z; r^p)_M} \\
&= \left(\frac{1 - r^\nu z}{1 - r^{\nu + pN} z}\right)^{\frac{\mathfrak{t}}{p}} \exp\left(O\left(\varepsilon \frac{|z| \vee |z|^2}{|1 - z|}\right)\right) = \exp\left(O\left(\varepsilon \frac{|z| \vee |z|^2}{|1 - z|}\right)\right)
\end{aligned}$$

uniformly over $z \in D^{\varepsilon,\theta,\delta}$ where the last equality follows from applying (2.5.13) to $\frac{1 - r^\nu z}{1 - z}$ and $\frac{1 - r^{pN+\nu} z}{1 - z}$, then taking their quotient.

Multiplying over all $R^{a,\ell}_{>x} \neq \emptyset$ and comparing with $\mathcal{G}_{>\mathbf{x}}$ as defined in (2.5.1), we obtain the desired asymptotics for $G^B_{>x}$. The argument for $G^B_{>x}$ is similar. $\square$

2.5.2 Outline of proofs

We first outline the proof of Theorem 2.5.2 to indicate the main obstacles. Part of the proof of Theorem 2.5.3 will mirror the outlined proof of Theorem 2.5.2.

By Theorem 2.3.17 and (2.3.19), we can express $\mathbb{E}\wp_k(\pi^x; q, t)$ as

$$\frac{1}{(2\pi\mathbf{i})^k}\oint_{\mathcal{C}_1'}\cdots\oint_{\mathcal{C}_k'}\frac{\sum_{i=1}^k \frac{1}{z_i}\frac{q^{i-1}}{t^{i-1}}}{(z_2-\frac{q}{t}z_1)\cdots(z_k-\frac{q}{t}z_{k-1})}\prod_{i<j}\frac{(z_j-z_i)(z_j-\frac{q}{t}z_i)}{(z_j-\frac{1}{t}z_i)(z_j-qz_i)}\prod_{i=1}^k G_{<x}^{B^\varepsilon}(z_i;\varepsilon,\mathfrak{t})G_{>x}^{B^\varepsilon}(z_i;\varepsilon,\mathfrak{t})\,dz_i \tag{2.5.14}$$

where $\mathcal{C}_i'$ is the z_i-contour and satisfies the conditions in Theorem 2.3.17. If the $\mathcal{C}_i'$ converge to $\mathcal{C}_i$ and are separated from one another so that the contours do not pass through any singularities of the integrand, then the integrand converges on the contour and the integral (2.5.14) converges to

$$\frac{1}{(2\pi\mathbf{i})^k}\oint_{\mathcal{C}_1}\cdots\oint_{\mathcal{C}_k}\frac{\sum_{i=1}^k\frac{1}{z_i}}{(z_2-z_1)\cdots(z_k-z_{k-1})}\prod_{i=1}^k[\mathcal{G}_{<\mathbf{x}}^{\mathcal{B}}(z)\mathcal{G}_{>\mathbf{x}}^{\mathcal{B}}(z)]^{\mathfrak{t}}\,dz_i. \tag{2.5.15}$$

The dimension of this contour is in general higher than what we desire. However, we can obtain the desired form by applying the following Theorem from [32, Corollary A.2].

Theorem 2.5.8 ([32])**.** *Let s be a positive integer. Let $f, g_1, \ldots, g_s$ be meromorphic functions with possible poles at $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_m\}$. Then for $k \geq 2$,*

$$\frac{1}{(2\pi\mathbf{i})^k}\oint\cdots\oint\frac{1}{(v_2-v_1)\cdots(v_k-v_{k-1})}\prod_{j=1}^s\left(\sum_{i=1}^k g_j(v_i)\right)\prod_{i=1}^k f(v_i)\,dv_i = \frac{k^{s-1}}{2\pi\mathbf{i}}\oint f(v)^k\prod_{j=1}^s g_j(v)\,dv.$$

where the contours contain $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_m\}$ and on the left side we require that the v_i-contour is contained in the v_j-contour whenever $i < j$.

We note that this was how the asymptotics of moments were carried out in [32].

In the case that $\mathbf{x} = V_\ell$ is a singular point, there is some additional difficulty due to the order of $\operatorname{dist}(\rho_<^\varepsilon(x), \rho_>^\varepsilon(x))$ being $O(\varepsilon)$ where $\rho_<^\varepsilon(x), \rho_>^\varepsilon(x)$ are defined as in Proposition 2.5.5. By the conditions on the contours in Theorem 2.3.17, our kth moment formula only makes sense if there is some separation between v_ℓ and x. This is where the $k\mathfrak{t}$-separation condition is needed. Even with this separation condition, the contours $\mathcal{C}_i'$ in (2.5.14) are $O(\varepsilon)$ from one another on the positive real axis; this is the main technical complication and is a byproduct of $\operatorname{dist}(\rho_<^\varepsilon(x), \rho_>^\varepsilon(x))$ being $O(\varepsilon)$. For points $(z_1, \ldots, z_n)$ where $|z_i - z_j| \sim O(\varepsilon)$, $i \neq j$, we take care to show that the integrand does not diverge. We still have convergence to (2.5.15) but with the limiting contours sharing a common point; this means that the integration goes over some singularities of the integrand. However, we are still able to obtain (2.5.2) by finding a sequence of integrals which converge to (2.5.15) for which we can apply the dimension reduction formula to complete the proof of Theorem 2.5.2.

The proof of Theorem 2.5.3 requires the asymptotics of higher cumulants, see the Appendix for a review of the definition and some facts about cumulants. Modulo a reduction step, the proof of Theorem 2.5.3 follows a similar line of argument as Theorem 2.5.2. Therefore some of the more repetitive points will be done in less detail.

2.5.3 Proof of Theorem 2.5.2

As outlined above, we want to show that (2.5.14) converges to (2.5.15). We suppress the dependence on B, α and $\mathfrak{t}$ in the indexing. Throughout the proof, constants C are uniform in ε and may vary from line to line. We let $\rho_<^\varepsilon(x), \rho_>^\varepsilon(x)$ be as in Proposition 2.5.5. For the proof, we assume that if $x = V_\ell$, then either $x > v_\ell$ for all $\varepsilon > 0$, $x = v_\ell$ for all $\varepsilon > 0$ or $x < v_\ell$ for all $\varepsilon > 0$. The purpose of this assumption is to ensure $\rho_{<x}^\varepsilon$ and $\rho_{>x}^\varepsilon$ both converge as in Proposition 2.5.5. Note that this condition is not restrictive, if we show (2.5.14) converges to (2.5.15) under each separate regime $x > v_\ell$, $x = v_\ell$, and $x < v_\ell$, then certainly we have that (2.5.14) converges to (2.5.15) under a general limit $\varepsilon x \to \mathbf{x}$ without restriction on the inequality between x and v_ℓ, since the limit is independent of this ordering.

The first step is to find contours such that the conditions in Theorem 2.3.17 are satisfied. As we will see, the $k\mathfrak{t}$-separation condition gives us existence of such contours. In order for the conditions in Theorem 2.3.17 to be met, we require the contours $\mathcal{C}_1', \ldots, \mathcal{C}_k'$ in (2.5.14) to satisfy the following: $\mathcal{C}_i'$ contains $[0, \rho_<^\varepsilon(x)]$ but does not intersect $[\rho_>^\varepsilon(x), \infty)$, and $\mathcal{C}_j'$ encircles $t\mathcal{C}_i'$ for any $i < j$. In particular, we require that $\mathcal{C}_i'$ intersects $(\rho_<^\varepsilon(x), \rho_>^\varepsilon(x))$ at some point a_i, and these points must satisfy

$$\rho_<^\varepsilon(x) < a_1 < ta_2 < \cdots < t^{k-1}a_k < t^{k-1}\rho_>^\varepsilon(x). \tag{2.5.16}$$

Let $0 < \theta_k, \ldots, \theta_1 < \pi$, and set $\mathcal{C}_1'', \ldots, \mathcal{C}_k''$ to be contours in $\mathbb{C}$ such that $\mathcal{C}_i''$ is the contour consisting of line segments and circular arc

$$\{1 + ue^{\pm \mathbf{i}\theta_j} : u \in [0,1]\}, \quad \{z \in \mathbb{C} : |z| = |1 + e^{\mathbf{i}\theta_j}|, \arg z > \arg(1 + e^{\mathbf{i}\theta_j})\}$$

where the arg branches are in $(-\pi, \pi]$, positively oriented around 0. Then $\mathcal{C}_1'', \ldots, \mathcal{C}_k''$ intersect pairwise at 1 and $\mathcal{C}_j''$ encircles $\mathcal{C}_i'' \setminus \{1\}$ whenever $i < j$.

The $k\mathfrak{t}$-separation condition guarantees the existence of points $a_1 := a_1(\varepsilon), \ldots, a_k := a_k(\varepsilon)$ satisfying (2.5.16). Indeed, by Proposition 2.5.5, if

$$t^{-M}r^{-1} := \frac{\rho_>^\varepsilon(x)}{\rho_<^\varepsilon(x)}$$

then $M \geq k+1$. Indeed, if $\mathbf{x}$ is not a singular point then $\lim_{\varepsilon\to 0} \rho_<^\varepsilon(x) < \lim_{\varepsilon \to 0} \rho_>^\varepsilon(x)$ so that there is enough space to guarantee this inequality for ε sufficiently small, and if $\mathbf{x}$ is a singular point then the $k\mathfrak{t}$-separation condition implies the right hand side is $\geq t^{-1}r^{-k\mathfrak{t}-1} = t^{-(k+1)}r^{-1}$. In particular, we may set

$$a_i = (t^{-M}r^{-1})^{\frac{i}{k+1}} \rho_<^\varepsilon(x) = (t^{-M}r^{-1})^{-\frac{k+1-i}{k+1}} \rho_>^\varepsilon(x).$$

Since $a_i/a_{i-1} > t^{-1}$, by setting $\mathcal{C}_i' = a_i\mathcal{C}_i''$, we obtain contours satisfying the conditions in Theorem 2.3.17.

If $\mathbf{x}$ is not a singular point, then $\lim_{\varepsilon\to 0}\rho_<^\varepsilon(x) < \lim_{\varepsilon\to 0}\rho_>^\varepsilon(x)$ so that $\lim_{\varepsilon\to 0} t^{-M} > 1$. Then the integrand in (2.5.14) is bounded, having no singularities. Thus by Proposition 2.5.6, (2.5.14) converges to (2.5.15) with $\mathcal{C}_i = (\lim_{\varepsilon\to 0} a_i)\mathcal{C}_i''$.

In the case that $\mathbf{x}$ is a singular point, then $t^{-M} \to 1$, so there are singularities in (2.5.14) to take care of. Let $F_\varepsilon(z_1, \ldots, z_k)$ denote the integrand in (2.5.14). We may change variables in (2.5.14) to replace the $\mathcal{C}_i'$ contours with $\mathcal{C}_i''$ via $z_i = a_iw_i$. By dominated convergence, to prove that (2.5.14) converges to (2.5.15), we seek a function g so that

$$|F_\varepsilon(a_1w_1, \ldots, a_kw_k)| \leq g(w_1, \ldots, w_k)$$

for $(w_1, \ldots, w_k) \in \mathcal{C}_1'' \times \cdots \times \mathcal{C}_k''$ with g integrable on $\mathcal{C}_1'' \times \cdots \times \mathcal{C}_k''$ with respect to $d|w_1|\cdots d|w_k|$.

We may write

$$\begin{aligned} G_{<x}(a_iw_i;\varepsilon,\mathfrak{t}) &= G_{<x}(t^{-1}\cdot t(t^{-M}r^{-1})^{\frac{i}{k+1}}w_i\cdot\rho_<^\varepsilon(x)) \\ G_{>x}(a_iw_i;\varepsilon\mathfrak{t}) &= G_{>x}(t\cdot t^{-1}(t^{-M}r^{-1})^{\frac{k+1-i}{k+1}}w_i\cdot\rho_>^\varepsilon(x)) \end{aligned}$$

If we denote by $\mathcal{C}_i''^{-1} = \{w : w^{-1} \in \mathcal{C}_i''\}$, then there exists $\theta \in (0,\pi)$ such that $\mathcal{C}_i'', \mathcal{C}_i''^{-1} \subset D^{0,\theta,\delta}$ for $1 \leq i \leq k$ and some fixed $\delta > 0$. This is clear for $\mathcal{C}_i''$. For $\mathcal{C}_i''^{-1}$, this follows from the map $w \mapsto w^{-1}$ being conformal and an involution taking $(-\infty, 0)$ onto intself and $(0,1)$ to $(1,\infty)$. This implies that given $A > 0$, then $e^{-\varepsilon\mathfrak{a}}\mathcal{C}_i''^{-1}, e^{-\varepsilon\mathfrak{a}}\mathcal{C}_i'' \subset D^{c\varepsilon,\theta,\delta}$ for $1 \leq i \leq k$, $\mathfrak{a} \geq A$, $\delta > 0$ fixed, and c depending on A. In particular, since

$$t(t^{-M}r^{-1})^{\frac{i}{k+1}} > r^{-\frac{1}{k+1}},$$

we have

$$\begin{aligned} \left(t\frac{a_i}{\rho_<^\varepsilon(x)}w_i\right)^{-1} &= (t(t^{-M}r^{-1})^{\frac{i}{k+1}}w_i)^{-1} \in D^{c\varepsilon,\theta,\delta} \\ t^{-1}\frac{a_i}{\rho_>^\varepsilon(x)}w_i &= t^{-1}(t^{-M}r^{-1})^{-\frac{k+1-i}{k+1}}w_i \in D^{c\varepsilon,\theta,\delta} \end{aligned}$$

for $1 \leq i \leq k$ and some fixed $\delta, c > 0$, $\theta \in (0,\pi)$. Thus by Proposition 2.5.6, we have

$$\begin{aligned} G_{<x}(a_iw_i;\varepsilon,\mathfrak{t}) &= \mathcal{G}_{<\mathbf{x}}(t(t^{-M}r^{-1})^{\frac{i}{k+1}}w_i\cdot\rho_<(\mathbf{x}))^{\mathfrak{t}}\exp\left(O\left(\frac{\varepsilon}{|1-\frac{ta_iw_i}{\rho_<^\varepsilon(x)}|}\right)\right), \\ G_{>x}(a_iw_i;\varepsilon,\mathfrak{t}) &= \mathcal{G}_{>\mathbf{x}}(t^{-1}(t^{-M}r^{-1})^{\frac{k+1-i}{k+1}}w_i\cdot\rho_>(\mathbf{x}))^{\mathfrak{t}}\exp\left(O\left(\frac{\varepsilon}{|1-\frac{ta_iw_i}{\rho_>^\varepsilon(x)}|}\right)\right) \end{aligned} \tag{2.5.17}$$

for $w_i \in \mathcal{C}_i'', 1 \leq i \leq k$; note that we dropped the $|z| \vee |z|^2$ term since $\mathcal{C}_i''$ is bounded and separated from 0. Observe that away from $[0,1]$,

$$\mathcal{G}_{<\mathbf{x}}(z)^p = R(z)(1-z)$$

where $R(z)$ is a rational functions with poles < 1, we have a similar statement for $\mathcal{G}_{<\mathbf{x}}(z)$ but we will only need the fact that it is bounded (we could alternatively flip the roles of $\mathcal{G}_{<\mathbf{x}}$ and $\mathcal{G}_{>\mathbf{x}}$ here). Using this and the fact that we may replace the exponential term in (2.5.17) with a crude constant order term, we have

$$|G_{<x}(a_i w_i; \varepsilon, \mathfrak{t}) \cdot G_{>x}(a_i w_i; \varepsilon, \mathfrak{t})| \leq C|1 - t^{\delta} w_i|^c \tag{2.5.18}$$

for $w_i \in \mathcal{C}_i''$ and some fixed constant $c > 0$, $\delta > 0$. Next, we need the following lemma.

Lemma 2.5.9. *Fix $\nu > 0$. For $\varepsilon > 0$ arbitrarily small, there exists a constant C independent of ε such that*

$$\frac{|1 - e^{-\nu\varepsilon} z|}{|w - e^{-\varepsilon} z|} \leq C$$

on $(z, w) \in \mathcal{C}_i'' \times \mathcal{C}_j''$ whenever $1 \leq i < j \leq k$.

Proof. It suffices to check the inequality for (z, w) near $(1, 1)$ where $\mathcal{C}_i'' \times \mathcal{C}_j''$ looks like

$$\{1 + ue^{\pm \mathbf{i}\theta_i} : u \in [0, \delta]\} \cup \{1 + ue^{\pm \mathbf{i}\theta_j} : u \in [0, \delta]\}$$

for some small $\delta > 0$. Since $\frac{1 - e^{-\nu\varepsilon} z}{1 - e^{-\varepsilon} z}$ is bounded over this region, independent of ε, it suffices to bound

$$\left| \frac{1 - e^{-\varepsilon} z}{w - e^{-\varepsilon} z} \right| = \left| 1 + \frac{1 - w}{(w - 1) + (1 - e^{-\varepsilon} z)} \right| = 1 + \left| \frac{w - 1}{(w - 1) + (1 - e^{-\varepsilon} z)} \right| = 1 + \left| \frac{1}{1 + \frac{1 - e^{-\varepsilon} z}{w - 1}} \right|.$$

Without loss of generality, we may suppose $w = 1 + ue^{\mathbf{i}\theta_j}$ for $u \in [0, \delta]$. Also note that if $z \in \mathbb{H}$ then $|w - e^{-\varepsilon} z| < |w - e^{-\varepsilon} \bar{z}|$, so we may suppose that $z \in \mathbb{H}$. Then $1 - e^{-\varepsilon} z = u' e^{\mathbf{i}\theta}$ for some $0 < \theta < \theta_i < \theta_j$ and $u' \in [0, \delta]$. Thus

$$\left| \frac{1}{1 + \frac{1 - e^{-\varepsilon} z}{w - 1}} \right| = \left| \frac{1}{1 + ve^{\mathbf{i}(\theta - \theta_j)}} \right|$$

for some $v \in [0, \infty)$. We can maximize the right hand side over $v \in [0, \infty)$ for each fixed $\theta < \theta_j$; the maximizing value is given by $\frac{1}{\sin(\theta - \theta_j)}$. Thus

$$\left| \frac{1 - e^{-\varepsilon} z}{w - e^{-\varepsilon} z} \right| \leq 1 + \frac{1}{\sin(\theta_i - \theta_j)}$$

which proves the lemma. $\square$

Using (2.5.18) and the fact that $\left|\frac{(z_j-z_i)(z_j-\frac{q}{t}z_i)}{(z_j-\frac{1}{t}z_i)(z_j-qz_i)}\right|$ is bounded on $\mathcal{C}_1'\times\cdots\times\mathcal{C}_k'$ uniformly in ε, we can dominate $F_\varepsilon(z_1,\ldots,z_k)$ as follows

$$|F_\varepsilon(z_1,\ldots,z_k)| \le C\cdot\left|\frac{1}{(z_2-\frac{q}{t}z_1)\cdots(z_k-\frac{q}{t}z_{k-1})}\right|\prod_{i=1}^k |1-t^\delta a_i^{-1}z_i|^c \tag{2.5.19}$$

$$\le C\left|\frac{1}{(z_2-\frac{q}{t}z_1)\cdots(z_k-\frac{q}{t}z_{k-1})}\right|^{1-c} \tag{2.5.20}$$

on $\mathcal{C}_1'\times\cdots\times\mathcal{C}_k'$, where the second inequality follows from Lemma 2.5.9. With $z_i=a_iw_i$, (2.5.20) becomes

$$\begin{aligned}|F_\varepsilon(z_1,\ldots,z_k)| &\le C\left|\frac{1}{(a_2w_2-\frac{q}{t}a_1w_1)\cdots(a_kw_k-\frac{q}{t}a_{k-1}w_{k-1})}\right|^{1-c}\\ &\le C\left|\frac{1}{(w_2-w_1)\cdots(w_k-w_{k-1})}\right|^{1-c} =: g(w_1,\ldots,w_k)\end{aligned} \tag{2.5.21}$$

for $(w_1,\ldots,w_k)\in\mathcal{C}_1''\times\cdots\times\mathcal{C}_k''$. The dominating function g is integrable, since all its singularities are integrable; e.g. integrate w_1, then w_2, etc. We may thus apply dominated convergence and see that (2.5.14) converges to (2.5.15) up to a change of variables where we need to take $\mathcal{C}_i=\rho_<(V_\ell)\mathcal{C}_i''$.

Now that we have shown (2.5.14) converges to (2.5.15), we want to show that (2.5.15) is the right hand side of (2.5.2). By a similar argument, we have the integral

$$\frac{1}{(2\pi\mathbf{i})^k}\oint_{\mathcal{C}_1'}\cdots\oint_{\mathcal{C}_k'}\frac{\sum_{i=1}^k\frac{1}{z_i}}{(z_2-z_1)\cdots(z_k-z_{k-1})}\prod_{i=1}^k G_{<x}(z_i;\varepsilon,\mathfrak{t})\cdot G_{>x}(z_i;\varepsilon,\mathfrak{t})\,dz_i, \tag{2.5.22}$$

with $\mathcal{C}_i'$ and G_x depending on ε, converges to (2.5.15) as well. On the other hand, Theorem 2.5.8 says that (2.5.22) may be reexpressed as

$$\frac{1}{2\pi\mathbf{i}}\oint_{\mathcal{C}'}[G_{<x}(z;\varepsilon,\mathfrak{t})\cdot G_{>x}(z;\varepsilon,\mathfrak{t})]^k\frac{dz}{z}$$

where $\mathcal{C}'$ is some contour containing the poles of $G^B_{<x}(z;\varepsilon,\mathfrak{t})$ and 0 and containing no pole of $G^B_{>x}(z;\varepsilon,\mathfrak{t})$; for example it suffices to pick $\mathcal{C}_i'$ for some $i\in[[1,k]]$. This converges to the right hand side of (2.5.2). Therefore, (2.5.15) coincides with the right hand side of (2.5.2), completing the proof of Theorem 2.5.2. □

2.5.4 Proof of Theorem 2.5.3

We begin by defining

$$D_k^\varepsilon(x)=\frac{1}{\varepsilon}(\wp_k(\pi^x;q,t)-\mathbb{E}\wp_k(\pi^x;q,t)). \tag{2.5.23}$$

As alluded to in proof outline, the idea is to compute the cumulants of $(D^\varepsilon_{k_1}(x_1), \dots, D^\varepsilon_{k_m}(x_m))$ and check that they are asymptotically Gaussian as $\varepsilon \to 0$; that is the order ≥ 3 cumulants vanish and the order 2 cumulants (i.e. the covariances) have the structure asserted by (2.5.3). The asymptotics of the cumulants have many elements similar to the asymptotics from the proof of Theorem 2.5.2. However, the higher order cumulants require greater separation in order for the formulas from Theorem 2.3.17 to apply which poses a problem yet again for the singular points. To ameliorate this, we prove Theorem 2.5.3 by first reducing to a seemingly weaker claim.

Claim. *Fix integers $\nu \geq 2$, $k_1, \dots, k_\nu \in \mathbb{Z}_{>0}$ and let $x_1 \leq \cdots \leq x_\nu$ be points in I^ε (depending on ε) such that*

$$\begin{aligned} &\varepsilon x_i \to \mathbf{x}_i \in I \quad \textit{for } i \in [[1,\nu]], \textit{ and} \\ &\quad x_i \quad \textit{is } (k_i + \cdots + k_\nu)\mathfrak{t}\textit{-separated from singular points, for } i \in [[1,\nu]]. \end{aligned} \tag{2.5.24}$$

Then

$$\kappa(D^\varepsilon_{k_1}(x_1), \dots, D^\varepsilon_{k_\nu}(x_\nu)) \to \begin{cases} \kappa(\mathfrak{D}_{k_1}(x_1), \mathfrak{D}_{k_2}(x_2)) & \textit{if } \nu = 2, \\ 0 & \textit{if } \nu > 2. \end{cases} \tag{2.5.25}$$

Lemma 2.5.10. *The Claim above implies Theorem 2.5.3*

Proof. We first show that for any integer $k > 0$ and any $\boldsymbol{x} \in I$, we have

$$\begin{aligned} &\mathbb{E}(D^\varepsilon_k(x) - D^\varepsilon_k(\tilde{x}))^2 \to 0 \\ &\quad \text{for any } x, \tilde{x} \in I^\varepsilon \text{ that are } (2k)\mathfrak{t}\text{-separated from singular points with } \varepsilon x, \varepsilon\tilde{x} \to \boldsymbol{x}. \end{aligned} \tag{2.5.26}$$

To see (2.5.26), rewrite

$$\begin{aligned} \mathbb{E}(D^\varepsilon_k(x) - D^\varepsilon_k(\tilde{x}))^2 &= \mathbb{E}(D^\varepsilon_k(x))^2 - 2\mathbb{E}D^\varepsilon_k(x)D^\varepsilon_k(\tilde{x}) + \mathbb{E}(D^\varepsilon_k(\tilde{x}))^2 \\ &= \kappa(D^\varepsilon_k(x), D^\varepsilon_k(x)) - 2\kappa(D^\varepsilon_k(x), D^\varepsilon_k(\tilde{x})) + \kappa(D^\varepsilon_k(\tilde{x}), D^\varepsilon_k(\tilde{x})). \end{aligned} \tag{2.5.27}$$

By the Claim, each cumulant appearing in (2.5.27) converges as $\varepsilon \to 0$ to the same limit:

$$\kappa(D^\varepsilon_k(x), D^\varepsilon_k(x)),\ \kappa(D^\varepsilon_k(x), D^\varepsilon_k(\tilde{x})),\ \kappa(D^\varepsilon_k(\tilde{x}), D^\varepsilon_k(\tilde{x})) \to \kappa(\mathfrak{D}_{k_1}(\mathbf{x}), \mathfrak{D}_{k_2}(\mathbf{x})).$$

This implies (2.5.26).

Let $x_1, \dots, x_m \in I^\varepsilon$ and integers $k_1, \dots, k_m \in \mathbb{Z}_{>0}$ be as in the statement of Theorem 2.5.3. We can find $\tilde{x}_1, \dots, \tilde{x}_m \in I^\varepsilon$ such that

$$\begin{aligned} &\varepsilon \tilde{x}_i \to \mathbf{x}_i \in I \quad \text{for } i \in [[1,\nu]], \text{ and} \\ &\quad \tilde{x}_i \quad \text{is } \chi\text{-separated from singular points for every } \chi > 0, \text{ for } i \in [[1,\nu]]. \end{aligned} \tag{2.5.28}$$

By the Claim, we can choose χ large enough so that

$$\kappa(D^\varepsilon_{k_{i_1}}(\tilde{x}_{i_1}), \dots, D^\varepsilon_{k_{i_\nu}}(\tilde{x}_{i_\nu})) \to \begin{cases} \kappa(\mathfrak{D}_{k_1}(\mathbf{x}_{i_1}), \mathfrak{D}_{k_2}(\mathbf{x}_{i_2})) & \text{if } \nu = 2, \\ 0 & \text{if } \nu > 2. \end{cases} \tag{2.5.29}$$

for any $i_1, \dots, i_\nu \in [[1, m]]$ and $\nu > 0$. By Lemma 2.8.3, this implies that $(D^\varepsilon_{k_1}(\tilde{x}_1), \dots, D^\varepsilon_{k_m}(\tilde{x}_m))$ converges in distribution to the Gaussian vector $(\mathfrak{D}_{k_1}(\mathbf{x}_1), \dots, \mathfrak{D}_{k_m}(\mathbf{x}_m))$.

By (2.5.26), we have

$$\mathbb{E}(D^\varepsilon_{k_i}(x_i) - D^\varepsilon_{k_i}(\tilde{x}_i))^2 \to 0$$

for each $i \in [[1, m]]$. This implies that $(D^\varepsilon_{k_1}(x_1), \dots, D^\varepsilon_{k_m}(x_m))$ also converges in distribution to the Gaussian vector $(\mathfrak{D}_{k_1}(\mathbf{x}_1), \dots, \mathfrak{D}_{k_m}(\mathbf{x}_m))$. □

We are left to prove the Claim. Adding a constant vector to a random vector adds only a constant order term to the logarithm of the characteristic function. By Definitions 2.8.1 and 2.8.2, we have for $\nu > 1$

$$\begin{aligned}\kappa(D^\varepsilon_{k_1}(x_1), \dots, D^\varepsilon_{k_\nu}(x_\nu)) &= \varepsilon^{-\nu}\kappa\left(\wp_{k_1}(\pi^{x_1}; q, t), \dots, \wp_{k_\nu}(\pi^{x_\nu}; q, t)\right) \\ &= \sum_{\substack{d \in \mathbb{Z}_{>0} \\ \{U_1, \dots, U_d\} \in \Theta_\nu}} \varepsilon^{-\nu}(-1)^{d-1}(d-1)! \prod_{\ell=1}^{d} \mathbb{E}\left[\prod_{i \in U_\ell} \wp_{k_i}(\pi^{x_i}; q, t)\right]\end{aligned} \tag{2.5.30}$$

where we use the notation from the Appendix with Θ_ν the collection of all set partitions of $[[1, \nu]]$.

If $x_1 \le \cdots \le x_\nu$, then given that the conditions of Theorem 2.3.17 are met, (2.5.30) can be expressed as

$$\varepsilon^{-\nu} \oint \cdots \oint \mathfrak{C}(Z_1, \dots, Z_\nu) \prod_{i=1}^{\nu} G_{<x_i}(Z_i) G_{>x_i}(Z_i)\, DZ_i \tag{2.5.31}$$

where $Z_i = (z_{i,1}, \dots, z_{i,k_i})$,

$$\mathfrak{C}(Z_1, \dots, Z_\nu) = \sum_{\substack{d \in \mathbb{Z}_{>0} \\ \{U_1, \dots, U_d\} \in \Theta_\nu}} (-1)^{d-1}(d-1)! \prod_{\ell=1}^{d} \prod_{\substack{(i,j) \in U_\ell \\ i<j}} \mathfrak{C}(Z_i, Z_j). \tag{2.5.32}$$

Let $\mathcal{C}'_{i,j}$ denote the $z_{i,j}$-contour in (2.5.31).

As before, the separation condition (2.5.24) ensures that the conditions of Theorem 2.3.17 are met. We verify that this is the case. We require the existence of contours $\mathcal{C}'_{i,j}$, $j \in [[1, k_i]]$ and $i \in [[1, \nu]]$, satisfying the following: $\mathcal{C}'_{i,j}$ contains $[0, \rho^\varepsilon_<(x_i)]$ but no elements of $[\rho^\varepsilon_>, \infty)$, and $t\mathcal{C}'_{i',j'}$ encircles $\mathcal{C}'_{i,j}$ whenever $(i', j') > (i, j)$ in lexicographical order. In particular, we require $\mathcal{C}'_{i,j}$ intersects $(\rho^\varepsilon_<(x_i), \rho^\varepsilon_>(x_i))$ at some point $a_{i,j}$, and these points must satisfy

$$a_{i,j} < t a_{i',j'}, \qquad (i, j) < (i', j') \tag{2.5.33}$$

in lexicographical order. We can construct such contours as follows.

Let $0 < \theta_{i,j} < \pi$ such that $\theta_{i',j'} < \theta_{i,j}$ whenever $(i, j) < (i', j')$ in lexicographical order and set $\mathcal{C}_{i,j}$ to be the contour in $\mathbb{C}$ consisting of line segments and a circular arc

$$\{1 + u e^{\pm \mathbf{i}\theta_{i,j}} : u \in [0, 1]\}, \quad \{z \in \mathbb{C} : |z| = |1 + e^{\mathbf{i}\theta_{i,j}}|, \arg z > \arg(1 + e^{\mathbf{i}\theta_{i,j}})\}$$

where the arg brsnches are in $(-\pi, \pi]$, positively oriented around 0. Then $\mathcal{C}''_{i,j}$, $1 \le j \le k_i, 1 \le i \le \nu$, intersect pairwise at 1 and $\mathcal{C}''_{i',j'}$ encircles $\mathcal{C}''_{i,j} \setminus \{1\}$ whenver $(i,j) < (i',j')$ in lexicographical order.

From $\rho^{\varepsilon}_{<}(x_i) < \rho^{\varepsilon}_{>}(x_i)$, we have

$$\lim_{\varepsilon \to 0} \rho^{\varepsilon}_{<}(x_i) \le \lim_{\varepsilon \to 0} \rho^{\varepsilon}_{>}(x_i).$$

By monotonicity of $\rho^{\varepsilon}_{<}$ and $\rho^{\varepsilon}_{>}$, we also have that the left hand side and right hand side grow as i increases; recall that $x_1 \le \cdots \le x_\nu$. By definition of singular points and Proposition 2.5.5, we have equality if and only if $\mathbf{x}_i$ is a singular point. Let $\Psi \subset [[1,\nu]]$ be defined so that $i \in \Psi$ if and only if $\mathbf{x}_i$ is singular. Then, we can find $a_{i,j} := a_{i,j}(\varepsilon)$, for each $(i,j), (i',j')$ where $j \in [[1,k_i]], j' \in [[1,k_{i'}]]$ and $i, i' \in [[1,\nu]] \setminus \Psi$, such that

$$\begin{aligned} &\lim_{\varepsilon \to 0} \rho^{\varepsilon}_{<}(x_i) < \lim_{\varepsilon \to 0} a_{i,j} < \lim_{\varepsilon \to 0} \rho^{\varepsilon}_{>}(x_i), \\ &\lim_{\varepsilon \to 0} a_{i,j} < \lim_{\varepsilon \to 0} a_{i',j'}, \qquad (i,j) < (i',j') \end{aligned} \tag{2.5.34}$$

where we order lexicographically. For $i \in \Psi$ we define $a_{i,j}$ using the separation condition. Suppose we have,

$$\mathbf{x}_{\eta-1} < \mathbf{x}_\eta = \cdots = \mathbf{x}_{\eta+\xi-1} < \mathbf{x}_{\eta+\xi}$$

with $\mathbf{x}_i = V_\ell$ a singular point, in particular $\eta, \ldots, \eta+\xi-1 \in \Psi$. By Proposition 2.5.5 the separation condition implies that

$$\frac{\rho^{\varepsilon}_{>}(x_i)}{\rho^{\varepsilon}_{<}(x_i)} \ge t^{-1} r^{-(k_i+\cdots+k_\nu)\mathrm{t}-1} = t^{-(k_i+\cdots+k_\nu+1)} r^{-1}, \quad i = \eta, \ldots, \eta+\xi-1.$$

Thus, we may define $a_{i,j}$ for $i \in \Psi$ such that for any pair $a_{i,j}, a_{i',j'}$ with $j \in [[1,k_i]], j' \in [[1,k_{i'}]]$ and $i, i' \in [[\eta, \eta+\xi-1]]$, we have

$$\frac{a_{i,j}}{\rho^{\varepsilon}_{<}(x_i)} > t^{-1-\delta}, \quad \frac{\rho^{\varepsilon}_{>}(x_i)}{a_{i,j}} > t^{-1-\delta}, \quad \frac{a_{i',j'}}{a_{i,j}} > t^{-1-\delta}, \quad (i,j) < (i',j') \tag{2.5.35}$$

for some fixed, small $\delta > 0$. Indeed, this follows from the separation condition and telescoping over $a_{i,j+1}/a_{i,j}$ with boundary cases

$$a_{i,1}/\rho^{\varepsilon}_{<}(x_i), \quad \rho^{\varepsilon}_{>}(x_i)/a_{i,k_i}.$$

Although the separation condition is not optimal, it is nearly saturated in the case where $x_1 = \cdots = x_\nu$ all converge to the same singular point V_ℓ and are as close to v_ℓ as the separation condition allows.

From this choice of $a_{i,j}$, we construct the $z_{i,j}$-contour $\mathcal{C}'_{i,j} = a_{i,j}\mathcal{C}''_{i,j}$ similar to the proof of Theorem 2.5.2. By (2.5.34) and (2.5.35), the points $a_{i,j}$ satisfy (2.5.33). Therefore the contours $\mathcal{C}'_{i,j}$ meet the conditions of Theorem 2.3.17, and so we may express (2.5.30) as (2.5.31).

Our next objective is to apply dominated convergence to (2.5.31). To this end, we rewrite (2.5.31) in a different form. Let $U \subset [[1,\nu]]$, $\mathcal{T}(U)$ denote the set of undirected simple graphs with vertices labeled by U, and $\mathcal{L}(U) \subset \mathcal{T}(U)$ the subset of connected graphs. Given a graph Ω, we denote by $E(\Omega)$ the edge set of Ω. We show

$$\mathfrak{C}(Z_1, \ldots, Z_m) = \sum_{\Omega \in \mathcal{L}([[1,\nu]])} \prod_{\substack{i<j \\ (i,j)\in E(\Omega)}} (C(Z_i, Z_j) - 1). \tag{2.5.36}$$

Define

$$\mathcal{K}(U) = \sum_{\Omega \in \mathcal{L}([[1,\nu]])} \prod_{\substack{i<j \\ (i,j)\in E(\Omega)}} (C(Z_i, Z_j) - 1), \quad \mathcal{E}(U) = \sum_{\Omega \in \mathcal{T}([[1,\nu]])} \prod_{\substack{i<j \\ (i,j)\in E(\Omega)}} (C(Z_i, Z_j) - 1). \tag{2.5.37}$$

Then

$$\mathcal{E}(U) = \sum_{\substack{d>0 \\ \{U_1,\ldots,U_d\}\in\Theta_U}} \prod_{\ell=1}^{d} \mathcal{K}(U_\ell). \tag{2.5.38}$$

By Lemma 2.8.4, we have

$$\mathcal{K}(U) = \sum_{\substack{d>0 \\ \{U_1,\ldots,U_d\}\in\Theta_U}} (-1)^{d-1}(d-1)! \prod_{\ell=1}^{d} \mathcal{E}(U_\ell) \tag{2.5.39}$$

which agrees with the right hand side of (2.5.32) when $U = [[1,\nu]]$. This proves (2.5.36).

We also record that

$$C(Z_i, Z_j) - 1 = \sum_{\substack{\emptyset \neq S \subset [[1,k_i]] \\ \emptyset \neq T \subset [[1,k_j]]}} \prod_{(a,b)\in S\times T} \frac{(1-q)(t^{-1}-1) z_{i,a} z_{j,b}}{(z_{j,b} - q z_{i,a})(z_{j,b} - \frac{1}{t} z_{i,a})} \tag{2.5.40}$$

For each $i \in [[1,\nu]]$ and $S \subset [[1,k_i]]$, let a_S be the minimal member of S (we note this choice is arbitrary). For each edge $i, j \in [[1,\nu]]$ with $i < j$, and any pair of subsets $S \subset [[1,k_i]], T \subset [[1,k_j]]$, we have

$$\left| \prod_{(a,b)\in S\times T} \frac{(1-q)(t^{-1}-1) z_{i,a} z_{j,b}}{(z_{j,b} - q z_{i,a})(z_{j,b} - \frac{1}{t} z_{i,a})} \right| \leq C \left| \frac{\varepsilon^2}{(z_{j,a_T} - q z_{i,a_S})(z_{j,a_T} - \frac{1}{t} z_{i,a_S})} \right| \tag{2.5.41}$$

for $z_{i,j} \in \mathcal{C}'_{i,j}$. The inequality follows from removing all but one $(a_S, a_T) \in S \times T$ using the fact that $|z_{j,b} - q z_{i,a}|, |z_{j,b} - \frac{1}{t} z_{i,a}| \geq C\varepsilon$ by our choice of contours. For each $\Omega \in \mathcal{L}([[1,\nu]])$, fix a complete subtree Ω' of Ω. Then by (2.5.36), (2.5.40) and (2.5.41), we have

$$\begin{aligned} |\mathfrak{C}(Z_1,\ldots,Z_\nu)| &\le C \sum_{\Omega\in\mathcal{L}([[1,\nu]])} \prod_{\substack{i<j\\(i,j)\in E(\Omega)}} \sum_{\substack{S\subset[[1,k_i]]\\T\subset[[1,k_j]]}} \left| \frac{\varepsilon^2}{(z_{j,a_T}-qz_{i,a_S})(z_{j,a_T}-\frac{1}{t}z_{i,a_S})} \right| \\ &\le C\varepsilon^{2\nu-2} \sum_{\Omega\in\mathcal{L}([[1,\nu]])} \prod_{\substack{i<j\\(i,j)\in E(\Omega')}} \sum_{\substack{S\subset[[1,k_i]]\\T\subset[[1,k_j]]}} \left| \frac{1}{(z_{j,a_T}-qz_{i,a_S})(z_{j,a_T}-\frac{1}{t}z_{i,a_S})} \right| \end{aligned} \tag{2.5.42}$$

for $z_{i,j}\in\mathcal{C}'_{i,j}$. In the last line, we used the fact that the number of edges in Ω' is $\nu-1$ in pulling out the ε factor.

On the other hand, as with (2.5.19) in the proof of Theorem 2.5.2, we have the bound

$$\left| G_{x_i}(Z_i)\frac{DZ_i}{dZ_i} \right| \le C \left| \frac{1}{(z_{i,2}-\frac{q}{t}z_{i,1})\cdots(z_{i,k}-\frac{q}{t}z_{i,k-1})} \right| \prod_{j=1}^{k_i} |1-t^\delta a_{ij}^{-1}z_{ij}|^{c'} \tag{2.5.43}$$

for some $c'>0$ and $\delta>0$ fixed along $z_{i,j}\in\mathcal{C}'_{i,j}$ where we write $\frac{DZ_i}{dZ_i}$ to indicate that we remove the differentials from DZ_i. Then by (2.5.42), (2.5.43), we obtain the following bound for the integrand $F_\varepsilon(Z_1,\ldots,Z_\nu)$ of (2.5.31) as we had done with (2.5.20) in the proof of Theorem 2.5.2:

$$\begin{aligned} |F_\varepsilon(Z_1,\ldots,Z_\nu)| &\le C\varepsilon^{\nu-2} \prod_{i=1}^{\nu} \left| \frac{1}{(z_{i,2}-\frac{q}{t}z_{i,1})\cdots(z_{i,k_i}-\frac{q}{t}z_{i,k_i-1})} \right|^{1-c} \\ &\times \sum_{\Omega\in\mathcal{L}([[1,\nu]])} \prod_{\substack{i<j\\(i,j)\in E(\Omega')}} \sum_{\substack{S\subset[[1,k_i]]\\T\subset[[1,k_j]]}} \left| \frac{1}{(z_{j,a_T}-qz_{i,a_S})(z_{j,a_T}-\frac{1}{t}z_{i,a_S})} \right|^{1-c} \end{aligned} \tag{2.5.44}$$

for some $c>0$ along $z_{i,j}\in\mathcal{C}'_{i,j}$ for each $j\in[[1,k_i]]$ and $i\in[[1,\nu]]$.

If we expand out the right hand side of (2.5.44), we have that $|F_\varepsilon(Z_1,\ldots,Z_\nu)|$ is bounded on along $\mathcal{C}'_{i,j}$ by a sum of finitely many terms of the form

$$\begin{aligned} T_\varepsilon(Z_1,\ldots,Z_\nu) &= C\varepsilon^{\nu-2} \prod_{i=1}^{\nu} \left| \frac{1}{(z_{i,2}-\frac{q}{t}z_{i,1})\cdots(z_{i,k_i}-\frac{q}{t}z_{i,k_i-1})} \right|^{1-c} \\ &\times \prod_{\substack{i<j\\(i,j)\in E(\Omega')}} \left| \frac{1}{(z_{j,b_{(i,j)}}-qz_{i,a_{(i,j)}})(z_{j,b_{(i,j)}}-\frac{1}{t}z_{i,a_{(i,j)}})} \right|^{1-c} \end{aligned} \tag{2.5.45}$$

for some $\Omega\in\mathcal{L}([[1,\nu]])$ and some $a_{(i,j)}\in[[1,k_i]]$, $b_{(i,j)}\in[[1,k_j]]$ for each $(i,j)\in E(\Omega')$. Since we are seeking an integrable function which dominates $F_\varepsilon(Z_1,\ldots,Z_\nu)$, it suffices to dominate $T_\varepsilon(Z_1,\ldots,Z_\nu)$ by an integrable function.

We may assume $c<1$. Choose a distinguished element $(i_\Omega,j_\Omega)\in\Omega'$ where $i_\Omega<j_\Omega$. Then

(2.5.45) may be replaced by

$$
\begin{aligned}
T_\varepsilon(Z_1,\dots,Z_\nu) \le C\varepsilon^{(\nu-2)c} \prod_{i=1}^{\nu} &\left| \frac{1}{(z_{i,2}-\frac{q}{t}z_{i,1})\cdots(z_{i,k_i}-\frac{q}{t}z_{i,k_i-1})} \right|^{1-c} \\
\times &\left| \frac{1}{(z_{j_\Omega,b_{(i_\Omega,j_\Omega)}} - qz_{i_\Omega,a_{(i_\Omega,j_\Omega)}})} \prod_{\substack{i<j \\ (i,j)\in E(\Omega')}} \frac{1}{(z_{j,b_{(i,j)}} - \frac{1}{t}z_{i,a_{(i,j)}})} \right|^{1-c}
\end{aligned}
\tag{2.5.46}
$$

where we used $\varepsilon^{(\nu-2)(1-c)}$ to remove each $|z_{j,b_{(i,j)}} - qz_{i,a_{(i,j)}}|$ term except for the one corresponding to the distinguished edge (i_Ω, j_Ω), recalling again that the number of edges of Ω' is $\nu-1$.

We set $z_{i,j} = \frac{a_{i,j}}{\lim_{\varepsilon to 0} a_{i,j}} w_{i,j}$ for the rest of the proof so that $w_{i,j}$ runs along $\mathcal{C}_{i,j} := (\lim_{\varepsilon to 0} a_{i,j})\mathcal{C}''_{i,j}$. Then, like (2.5.21) in the proof of Theorem 2.5.2, we have

$$
\begin{aligned}
T_\varepsilon(Z_1,\dots,Z_\nu) \le C\varepsilon^{(\nu-2)c} \prod_{i=1}^{\nu} &\left| \frac{1}{(w_{i,2}-w_{i,1})\cdots(w_{i,k_i}-w_{i,k_i-1})} \right|^{1-c} \\
\times &\left| \frac{1}{(w_{j_\Omega,b_{(i_\Omega,j_\Omega)}} - w_{i_\Omega,a_{(i_\Omega,j_\Omega)}})} \prod_{\substack{i<j \\ (i,j)\in E(\Omega')}} \frac{1}{(w_{j,b_{(i,j)}} - w_{i,a_{(i,j)}})} \right|^{1-c} \\
&=: \varepsilon^{(\nu-2)c} g(W_1,\dots,W_\nu)
\end{aligned}
\tag{2.5.47}
$$

We show that $g(W_1,\dots,W_\nu)$ is integrable on $w_{i,j} \in \mathcal{C}_{i,j}$, $j \in [[1,k_i]]$, $i \in [[1,\nu]]$ with respect to the measure $\prod_{\substack{i\in[[1,\nu]] \\ j\in[[1,k_i]]}} d|w_{i,j}|$. We use the following lemma to this end.

Lemma 2.5.11. *Let Γ be a graph with vertices labeled by some subset of $\{(i,j)\}_{\substack{i\in[[1,\nu]] \\ j\in[[1,k_i]]}}$. If Γ is a tree and $c>0$, then for any $(v_1', v_2') \in E(\Gamma)$ we have*

$$
\oint \cdots \oint \frac{1}{|w_{v_1'} - w_{v_2'}|^{1-c}} \prod_{(v_1,v_2)\in E(\Gamma)} \frac{1}{|w_{v_1}-w_{v_2}|^{1-c}} \prod_{\substack{i\in[[1,\nu]] \\ j\in[[1,k_i]]}} |dw_{i,j}| < \infty
$$

where $\mathcal{C}_{i,j}$ is the $w_{i,j}$ contour.

Proof. Since the contours $\mathcal{C}_{i,j}$ have finite length, we can integrate out the variables independent of the integrand, so it suffices to show

$$
\oint \cdots \oint \frac{1}{|w_{v_1'} - w_{v_2'}|^{1-c}} \prod_{(v_1,v_2)\in E(\Gamma)} \frac{1}{|w_{v_1}-w_{v_2}|^{1-c}} \prod_{v\in V(\Gamma)} |dw_v| \tag{2.5.48}
$$

where $V(\Gamma)$ is the vertex set of Γ.

For the case where the number of edges of Γ is 1, we have

$$\oint \oint \frac{1}{|w_{v_1'} - w_{v_2'}|^{2(1-c)}} |dw_{v_1'}||dw_{v_2'}|$$

is integrable even if $\mathcal{C}_{v_1}$ and $\mathcal{C}_{v_2}$ meet at a point.

For general $|E(\Gamma)| \geq 1$, choose a leaf vertex v_1 of Γ so that v_1 is not a vertex of the edge (v_1', v_2'). There is a unique edge $e_1 = (v_1, v_2) \in E(\Gamma)$ containing v_1. If we integrate over w_{v_1}, then the w_{v_1}-dependent part of the integrand is $|w_{v_1} - w_{v_2}|^{c-1}$ which is integrable even if $\mathcal{C}_{v_2}$ and $\mathcal{C}_{v_1}$ intersect. Then repeat this procedure with the graph $\Gamma \setminus \{v_1\}$ which is still a tree. After repeating this procedure, we will eventually return to the case $|E(\Gamma)| = 1$. □

We check that $g(W_1, \ldots, W_\nu)$ meets the conditions of Lemma 2.5.11. Our graph Γ is the union of graph Γ_i for $i \in [[0, \nu]]$ defined as follows. Let Γ_i be the graph with edge set $E(\Gamma_i) = \{((i,1),(i,2)), \ldots, ((i,k_i-1),(i,k_i))\}$ for $i \in [[1,\nu]]$ and $E(\Gamma_0) = \{((i, a_{(i,j)}), (j, b_{(i,j)})) : (i,j) \in \Omega'\}$. We must check that Γ is a tree. Indeed, if we collapse each Γ_i to a point v_i for $i \in [[1,\nu]]$ (alternatively said, we project away the second coordinate for the vertex labels), then Γ becomes the tree Ω'. Since each Γ_i for $i \in [[1,\nu]]$ is a tree, this implies that Γ is a tree. Thus by Lemma 2.5.11,

$$\int \cdots \int g(W_1, \ldots, W_\nu) \prod_{\substack{i \in [[1,\nu]] \\ j \in [[1,k_i]]}} d|w_{i,j}| < +\infty \tag{2.5.49}$$

where the $w_{i,j}$-contour is $\mathcal{C}_{i,j}$.

If $\nu > 2$, then since $\varepsilon^{(\nu-2)c} \to 0$, we have that (2.5.47) converges to 0. Thus for $\nu > 2$, the cumulant (2.5.30) converges to 0. If $\nu = 2$, then (2.5.47) allows us to apply dominated convergence so that (2.5.30) converges to

$$\begin{aligned} &\frac{1}{(2\pi\mathbf{i})^{k_1+k_2}} \oint \cdots \oint \left(\alpha \mathrm{t}^2 \sum_{\substack{a \in [[1,k_1]] \\ b \in [[1,k_2]]}} \frac{w_{1,a} w_{2,b}}{(w_{1,a} - w_{2,b})^2} \right) \\ &\quad \times \prod_{i=1}^{2} \frac{\sum_{j=1}^{k_i} \frac{1}{w_{i,j}}}{(w_{i,2} - w_{i,1}) \cdots (w_{i,k_i} - w_{i,k_i-1})} \prod_{j=1}^{k_i} [\mathcal{G}_{<\mathbf{x}_i}(w_{i,j}) \mathcal{G}_{>\mathbf{x}_i}(w_{i,j})]^{\mathrm{t}} dw_{i,j} \end{aligned} \tag{2.5.50}$$

where $\mathcal{C}_{i,j}$ is the $z_{i,j}$-contour. As in the proof of Theorem 2.5.2, we can consider the family of integrals replacing $\mathcal{G}^{\mathrm{t}}_{<\mathbf{x}_i}$ with $G_{<x_i}$, $\mathcal{G}^{\mathrm{t}}_{>\mathbf{x}_i}$ with $G_{>x_i}$ and $\mathcal{C}_{i,j}$ with $\mathcal{C}'_{i,j}$ in (2.5.50). By applying Theorem 2.5.8 twice to this family and taking the limit as $\varepsilon \to 0$, we have that (2.5.50) is equal to

$$\alpha \mathrm{t}^2 \frac{k_1 k_2}{(2\pi\mathbf{i})^2} \oint_{\mathcal{C}_2} \oint_{\mathcal{C}_1} \frac{1}{(z-w)^2} [\mathcal{G}_{<\mathbf{x}_1}(z) \mathcal{G}_{>\mathbf{x}_1}(z)]^{k_1 \mathrm{t}} [\mathcal{G}_{<\mathbf{x}_2}(w) \mathcal{G}_{>\mathbf{x}_2}(w)]^{k_2 \mathrm{t}} \, dz \, dw$$

where $\mathcal{C}_1 = \mathcal{C}_{1,i}$ for some $i \in [[1,k_1]]$ and $\mathcal{C}_2 = \mathcal{C}_{2,j}$ for some $j \in [[1,k_2]]$. This proves the Claim, and therefore proves Theorem 2.5.3.

2.6 Complex Structure of the Liquid Region

Throughout this section, we fix a family $\mathbb{P}^{B,r,\boldsymbol{s}}_{\alpha,\mathrm{t}}$ satisfying the Limit Conditions, as defined in the beginning of Section 2.5, with limiting back wall $\mathcal{B} : I \to \mathbb{R}$. The *liquid region*, denoted by $\mathscr{L}$, is the subset of $\mathbb{R}^2$ consisting of (x, y) such that the local proportions of $\square$, $\square$, $\lozenge$ lozenges at (x, y) are all positive. The goal of this section is to give a natural complex structure on $\mathscr{L}$ through $\mathcal{G}_x$. This complex structure will be used in Section 2.7 to describe the GFF fluctuations of $\mathbb{P}^{B,r,\boldsymbol{s}}_{\alpha,\mathrm{t}}$ in the limit as $\varepsilon \to 0$.

The key object of study is the function

$$\mathcal{G}^{\mathcal{B}}_x(\zeta) = e^{-\mathcal{B}(x)}\mathcal{G}^{\mathcal{B}}_{<x}(\zeta) \cdot \mathcal{G}^{\mathcal{B}}_{>x}(\zeta) \tag{2.6.1}$$

Throughout this section, we write $\mathcal{G}_x := \mathcal{G}^{\mathcal{B}}_x$.

Definition 2.6.1. For $(x, y) \in \mathbb{R}^2$, we define the (x, y)-*companion equation*

$$\mathcal{G}_x(\zeta) = e^{-y}. \tag{2.6.2}$$

The importance of the companion equation is due to its connection to the liquid region.

Definition 2.6.2. The *liquid region* denoted by $\mathscr{L}$ is the set of $(x, y) \in \mathbb{R}^2$ for which the (x, y)-companion equation has a nonreal pair of roots.

We provided a different definition of the liquid region earlier: the *liquid region* is the set of (x, y) such that all the local proportions $p_{\square}, p_{\square}, p_{\lozenge}$ are positive. The equivalence of these definitions can be seen in Section 2.7. For now, we use the definition of the liquid region in terms of the companion equation. The following theorem is the main result of this section.

Theorem 2.6.3. *For each $(x, y) \in \mathscr{L}$, there exists a unique root $\zeta(x, y)$ of the (x, y)-companion equation in the upper half plane $\mathbb{H}$. The map $\zeta : \mathscr{L} \to \mathbb{H}$ is a diffeomorphism.*

Theorem 2.6.3 endows $\mathscr{L}$ with the complex structure of $\mathbb{H}$ given by the pullback of $\zeta(x, y)$.

We prove Theorem 2.6.3 in two parts. In Section 2.6.1, we show that the map $\zeta(x, y)$ in Theorem 2.6.3 exists. In Section 2.6.2, we prove that the map $\zeta(x, y)$ is a diffeomorphism. In Section 2.6.3, we give a parametrization of the boundary of $\mathscr{L}$ which we call the *frozen boundary*; this involves studying the double roots of (2.6.2).

Before proceeding, we show a convenient expression for $\mathcal{G}_x$ as a product of two functions where one is independent of x and the other depends on x. We also introduce some useful definitions.

Define

$$P_x(z) = \prod_{\sigma\in\mathcal{S}} (1 - e^{-x}\sigma z)^{\frac{1}{p}} \tag{2.6.3}$$

$$Q(z) = e^{\mathcal{B}(V_0)} \prod_{\substack{\ell\in[[0,n]]\\ \sigma\in\mathcal{S}}} (1 - e^{-V_\ell}\sigma z)^{\frac{1}{p}(\mathbf{1}[\mathcal{B}'(V_\ell^+)\geq\varsigma_\sigma]-\mathbf{1}[\mathcal{B}'(V_\ell^-)\geq\varsigma_\sigma])} \tag{2.6.4}$$

$$= e^{\mathcal{B}(V_0)} \prod_{\sigma\in\mathcal{S}} \left[(1 - e^{-V_0}\sigma z)^{\frac{1}{p}} \prod_{\ell=1}^{n} \left(\frac{1 - e^{-V_\ell}\sigma z}{1 - e^{-V_{\ell-1}}\sigma z} \right)^{\frac{1}{p}\mathbf{1}[\mathcal{B}'(V_{\ell-1},V_\ell)<\varsigma_\sigma]} \right] \tag{2.6.5}$$

$$= e^{\mathcal{B}(V_0)} \prod_{\sigma\in\mathcal{S}} \left[(1 - e^{-V_n}\sigma z)^{\frac{1}{p}} \prod_{\ell=1}^{n} \left(\frac{1 - e^{-V_{\ell-1}}\sigma z}{1 - e^{-V_\ell}\sigma z} \right)^{\frac{1}{p}\mathbf{1}[\mathcal{B}'(V_{\ell-1},V_\ell)\geq\varsigma_\sigma]} \right] \tag{2.6.6}$$

where the branches of $(1-vz)^s$ above are taken so that $z < v^{-1}$ give positive real values. If $V_0 = -\infty$, then the expression above is to be interpreted as the limit of the expression as $V_0 \to -\infty$.

Lemma 2.6.4. *For $x \in \mathbb{R}$, we have*

$$\mathcal{G}_x(z) = \frac{P_x(z)}{Q(z)}.$$

Proof. It is sufficient to prove this lemma for $V_0 > -\infty$, since we may then take $V_0 \to -\infty$. Suppose $a \in [[0, p-1]]$ is maximal so that $V_{a-1} < x$, then (2.5.1) implies

$$e^{-\mathcal{B}(x)}\mathcal{G}_{<x}(z) = e^{-\mathcal{B}(x)} \prod_{\sigma\in\mathcal{S}} \prod_{\ell=1}^{a} \left(\frac{1 - e^{\min(V_\ell,x)}(\sigma z)^{-1}}{1 - e^{V_{\ell-1}}(\sigma z)^{-1}} \right)^{\frac{1}{p}\mathbf{1}[\mathcal{B}'(V_\ell^-)\geq\varsigma_\sigma]}. \tag{2.6.7}$$

Observe that if $x_1, x_2 \in [V_{\ell-1}, V_\ell]$, then

$$\mathcal{B}(x_1) - \mathcal{B}(x_2) = (x_1 - x_2)\mathcal{B}'(V_{\ell-1}, V_\ell).$$

This implies

$$e^{-\mathcal{B}(x)} = e^{-\mathcal{B}'(V_a^-)(x-V_{a-1})} e^{-\mathcal{B}(V_{a-1})}. \tag{2.6.8}$$

By Lemma 2.4.4 and the definition of ς_σ, we have $\mathcal{B}'(V_\ell^-) = \sum_{\sigma\in\mathcal{S}} \frac{1}{p}\mathbf{1}[\mathcal{B}'(V_\ell^-) \geq \varsigma_\sigma]$. Thus the first exponential term on the right hand side of (2.6.8) can be used to turn

$$\frac{1 - e^{x}(\sigma z)^{-1}}{1 - e^{V_{a-1}}(\sigma z)^{-1}} \quad \text{into} \quad \frac{1 - e^{-x}\sigma z}{1 - e^{-V_{a-1}}\sigma z}$$

for each $\sigma \in \mathcal{S}$ in (2.6.7). The second factor in (2.6.8) can be used to iterate this procedure where V_{a-1} and V_{a-2} play the role of x and V_{a-1} respectively; then with V_{a-2} and V_{a-3}, etc. At the end of this procedure, we obtain

$$
\begin{aligned}
e^{-\mathcal{B}(x)}\mathcal{G}_{<x}(z) &= e^{-\mathcal{B}(V_0)}\prod_{\sigma\in\mathcal{S}}\prod_{\ell=1}^{a}\left(\frac{1-e^{-\min(V_\ell,x)}\sigma z}{1-e^{-V_{\ell-1}}\sigma z}\right)^{\frac{1}{p}\mathbf{1}[\mathcal{B}'(V_\ell^-)\geq\varsigma_\sigma]} \\
&= e^{-\mathcal{B}(V_0)}\prod_{\sigma\in\mathcal{S}}\left[(1-e^{-x}\sigma z)^{\frac{1}{p}\mathbf{1}[\mathcal{B}'(V_a^-)\geq\varsigma_\sigma]}\prod_{\ell=0}^{a-1}(1-e^{-V_a}\sigma z)^{\frac{1}{p}(\mathbf{1}[\mathcal{B}'(V_a^-)\geq\varsigma_\sigma]-\mathbf{1}[\mathcal{B}'(V_a^+)\geq\varsigma_\sigma])}\right]
\end{aligned}
$$

where the second equality uses the fact that $\mathcal{B}'(V_{\ell+1}^-)=\mathcal{B}'(V_\ell^+)$. From (2.5.1), we may also write

$$
\mathcal{G}_{>x}(z)=\prod_{\sigma\in\mathcal{S}}\left[(1-e^{-x}\sigma z)^{\frac{1}{p}\mathbf{1}[\mathcal{B}'(V_a^-)<\varsigma_\sigma]}\prod_{\ell=a}^{n}(1-e^{-V_a}\sigma z)^{\frac{1}{p}(\mathbf{1}[\mathcal{B}'(V_a^+)<\varsigma_\sigma]-\mathbf{1}[\mathcal{B}'(V_a^-)<\varsigma_\sigma])}\right].
$$

From (2.6.1) and the fact that

$$
\mathbf{1}[\mathcal{B}'(V_a^+)<\varsigma_\sigma]-\mathbf{1}[\mathcal{B}'(V_a^-)<\varsigma_\sigma]=\mathbf{1}[\mathcal{B}'(V_a^-)\geq\varsigma_\sigma]-\mathbf{1}[\mathcal{B}'(V_a^+)\geq\varsigma_\sigma],
$$

we obtain

$$
\mathcal{G}_x(z)=e^{-\mathcal{B}(V_0)}\prod_{\sigma\in\mathcal{S}}\left[(1-e^{-x}\sigma z)^{\frac{1}{p}}\prod_{\ell=0}^{n}(1-e^{-V_a}\sigma z)^{\frac{1}{p}\mathbf{1}[\mathcal{B}'(V_a^-)\geq\varsigma_\sigma]-\mathbf{1}[\mathcal{B}'(V_a^+)\geq\varsigma_\sigma]}\right]=\frac{P_x(z)}{Q(z)}.
$$

□

Definition 2.6.5. Suppose $f:U\to\hat{\mathbb{C}}$ is such that f^p is meromorphic. We say z_0 is a *branch pole* (*branch zero*) of f if z_0 is a pole (zero) of f^p. The *order* of a branch pole (zero) of f at z_0 is defined to be the order of the pole (zero) of f^p at z_0 divided by p.

Definition 2.6.6. Suppose f is a product of terms of the form $a-z$ or $1-a^{-1}z$ for some $a\in\mathbb{R}$. Define

$$
\arg_+ f(u)=\lim_{\substack{z\to u\\ \arg z>0}}\arg f(z)
$$

for $u\in\mathbb{R}$ away from a branch point of f, where we take arg so that $\arg(a-z),\arg(1-a^{-1}z)=0$ for $z<a$.

Definition 2.6.7. Let V be an non-differentiable point of $\mathcal{B}$. Define

$$
J_V=\{\sigma^{-1}e^V:\mathbf{1}[\mathcal{B}'(V^+)\geq\varsigma_\sigma]\neq\mathbf{1}[\mathcal{B}(V^-)\geq\varsigma_\sigma]\}
$$

and for $x\in I$ define $J_x=\{\sigma^{-1}e^x:\sigma\in\mathcal{S}\}$. If $V_0=-\infty$ (or $V_n=+\infty$), take $J_{V_0}=\{0\}$ ($J_{V_n}=\{\infty\}$). Let $\mathcal{J}=\bigcup_{\ell=0}^n J_V$ and $\mathcal{J}_x=J_x\cup\mathcal{J}$. By (2.6.4), (2.6.3), and (2.6.1), $\mathcal{J}_x$ contains the set of branch poles and zeros of $\mathcal{G}_x^{\mathcal{B}}$.

By the definition of J_V and the fact that Theorem 2.4.9 classifies the situation where $\mathcal{B}'(V^+)-\mathcal{B}'(V^-)<0$, we have the following lemma.

Lemma 2.6.8. *Let $\mathbb{P}^{B,r,\boldsymbol{s}}_{\alpha,\mathfrak{t}}$ satisfy the Limit Conditions. Let V be a non-differentiable point of $\mathcal{B}$.*

- *If $\mathcal{B}'(V^+) - \mathcal{B}'(V^-) > 0$, then $J_V = \{\sigma_i^{-1} e^V : \varsigma_i \in (\mathcal{B}'(V^-), \mathcal{B}'(V^+)], 1 \le i \le d\}$.*
- *If $\mathcal{B}'(V^+) - \mathcal{B}'(V^-) < 0$, then $J_V = \{\sigma_i^{-1} e^V\}$ for $1 \le i \le d$ such that $\mathcal{B}'(V^-) = \varsigma_i$.*

As a consequence of Lemma 2.6.8 and (2.4.4), we have $J_V < J_W$ whenever $V < W$.

2.6.1 Existence of $\zeta(x, y)$

It will be convenient to consider the phase and magnitude equations for (2.6.2):

$$e^{-y} = |\mathcal{G}_x(\zeta)| \tag{2.6.9}$$

$$0 = \arg_+ \mathcal{G}_x(\zeta). \tag{2.6.10}$$

From (2.6.9) and (2.6.10), we see that the zeros of (2.6.2) approach branch poles and branch zeros of $\mathcal{G}_x$ as $y \to -\infty$ and $y \to +\infty$ respectively. The idea is to use this condition to restrict the number of nonreal roots to a single pair.

The following definitions will be useful.

Definition 2.6.9. Denote by $\widehat{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ the one-point compactification of $\mathbb{R}$. By an *interval* in $\widehat{\mathbb{R}}$ we mean either an interval in $\mathbb{R}$ or a set of the form $(a, +\infty) \cup \{\infty\} \cup (-\infty, b)$. For $a > b$, we denote the $\widehat{\mathbb{R}}$-interval $(a, +\infty) \cup \{\infty\} \cup (-\infty, b)$ by (a, b). The set $\widehat{\mathbb{R}} \setminus \mathcal{J}$ is a union of $|\mathcal{J}|$ disjoint intervals.

Given a set of points A in $\widehat{\mathbb{R}}$, we say that an *external (internal) component* of $\widehat{\mathbb{R}} \setminus A$ is a connected component of $\widehat{\mathbb{R}} \setminus A$ of the form (a, b) for some $a > b$ ($a < b$).

The following basic facts are immediate consequences of the definition (2.6.1) of $\mathcal{G}_x$.

Recall that the branch poles and branch zeros of $\mathcal{G}_x$ are contained in $\mathcal{J}_x$. When $J_x \cap \mathcal{J}$ is nonempty, there may be cancellations of poles and zeros, thus this containment may be strict for some values of x. Note that if $a \in J_x \setminus \mathcal{J}$, then a is a branch zero of $\mathcal{G}_x$. Also, note that any branch pole a of $\mathcal{G}_x$ must be a point in J_V for some non-differentiable point V of $\mathcal{B}$ such that $a = \sigma^{-1} e^V$ and $\mathcal{B}'(V^+) - \mathcal{B}'(V^-) > 0$.

The following lemma gives a restriction on branch poles.

Lemma 2.6.10. *Fix $x \in \mathbb{R}$. Suppose L is a connected component of $\widehat{\mathbb{R}} \setminus \mathcal{J}_x$ such that $\arg_+ \mathcal{G}_x(L) = 0$. If L is the external component and $x \in (V_0, V_n)$, then both endpoints of L are branch poles of $\mathcal{G}_x$. Otherwise, at most one endpoint of L is a branch pole of $\mathcal{G}_x$.*

Proof. Since $\mathcal{B}'(V_0^-) = 0$ and $\mathcal{B}'(V_n^+) = 1$, we have $\min \mathcal{J} = \min J_{V_0} = \sigma_1^{-1} e^{V_0}$ and $\max \mathcal{J} = \max J_{V_n} = \sigma_d^{-1} e^{V_n}$. Furthermore, $\mathcal{B}'(V_0^+) - \mathcal{B}'(V_0^-) > 0$ and $\mathcal{B}'(V_n^+) - \mathcal{B}'(V_n^-) > 0$.

We prove the lemma by examining the case of the external component and internal components separately.

<u>External.</u> Consider the case where L is the external component of $\widehat{\mathbb{R}} \setminus \mathcal{J}_x$. If $x \in (V_0, V_n)$, then the endpoints of L are $\min \mathcal{J}$ and $\max \mathcal{J}$, both of which are elements of $\mathcal{J} \setminus J_x$. Thus these are branch

poles of $\mathcal{G}_x$. If $x > V_n$ ($< V_0$), then one endpoint of L is a branch pole given by $\min \mathcal{J}$ ($\max \mathcal{J}$) and the other endpoint is a branch zero given by $\max J_x$ ($\min J_x$).

The situation for $x = V_n$ and $x = V_0$ is more complicated due to cancellation of elements in $\mathcal{J}$ and J_x. Suppose $x = V_n$, then one endpoint of L is still a branch pole given by $\min \mathcal{J}$, and if J_{V_n} is a strict subset of J_x then $\max(J_x \setminus J_{V_n})$ is a branch zero and the other endpoint of L. However, we may have $J_{V_n} = J_x$. In this case, $\mathcal{B}'(V_{n-1}^+) = \mathcal{B}'(V_n^-) = 0$ which implies $\mathcal{B}'(V_{n-1}^+) - \mathcal{B}'(V_{n-1}^-) < 0$. Then $\max \mathcal{J}_x \setminus J_x = \max J_{V_{n-1}}$ is a branch zero of $\mathcal{G}_x$ and the other endpoint of L. The argument for $x = V_0$ is similar. This concludes the analysis for the external components.

Internal. Now consider the case where L is some internal component. Suppose for contradiction that both endpoints of L are branch poles of $\mathcal{G}_x$.

We claim that $L = (\sigma_i^{-1} e^V, \sigma_{i+1}^{-1} e^W)$ for non-differentiable points $V \le W$ of $\mathcal{B}$ and for some $i \in [[1, d-1]]$. By Lemma 2.6.8, each endpoint of L must be an element of J_U for some non-differentiable point U of positive change in slope. The endpoints of L must be adjacent elements of $\mathcal{J}$. If $\sigma_i^{-1} e^V = \max J_V$, then the right endpoint of L must be $\sigma_{i+1}^{-1} e^W = \min J_W$ where W is the adjacent non-differentiable point $V < W$ by Lemma 2.6.8. Otherwise, the right endpoint of L must be $\sigma_{i+1}^{-1} e^V$. This proves the claim.

Observe that

$$\begin{aligned}
\arg_+ Q(L) &= -\frac{\pi}{p} \sum_{\ell=0}^{n} \sum_{\substack{\sigma \in \mathcal{S}: \\ \sigma^{-1} e^{V_\ell} \in J_{V_\ell}}} (\mathbf{1}[\mathcal{B}'(V_\ell^+) \ge \varsigma_\sigma] - \mathbf{1}[\mathcal{B}'(V_\ell^-) \ge \varsigma_\sigma]) \mathbf{1}[\sigma^{-1} e^{V_\ell} < L] \\
&= -\frac{\pi}{p} \sum_{\sigma \in \mathcal{S}} \left(\mathbf{1}[\mathcal{B}'(V^+) \ge \varsigma_\sigma] \mathbf{1}[\sigma^{-1} \le \sigma_i^{-1}] + \mathbf{1}[\mathcal{B}'(V^-) \ge \varsigma_\sigma] \mathbf{1}[\sigma^{-1} > \sigma_i^{-1}] \right) \\
&= -\frac{\pi}{p} \sum_{\sigma \in \mathcal{S}} \mathbf{1}[\sigma^{-1} \le \sigma_i^{-1}] = -\pi \varsigma_i
\end{aligned}$$

where the first equality follows from (2.6.4), the second from telescoping in ℓ, the third follows from the fact that $\mathcal{B}'(V^-) < \varsigma_i$ and $\mathcal{B}'(V^+) \ge \varsigma_i$ by Theorem 2.6.8 and monotonicity of ς_σ in σ^{-1}.

We require $\arg_+ P_x(L) - \arg_+ Q(L) = \arg_+ \mathcal{G}_x(L) = 0$, that is $\arg_+ P_x(L) = -\pi \varsigma_i$. This happens exactly when

$$\sigma_i^{-1} e^x < L < \sigma_{i+1}^{-1} e^x. \tag{2.6.11}$$

Since, $L = (\sigma_i^{-1} e^V, \sigma_{i+1}^{-1} e^W)$ where $V \le W$, this is only possible if $x = V = W$. However, this is the case where the branch pole $\sigma_i^{-1} e^V \in J_V$ is cancelled by the branch zero $\sigma_i^{-1} e^x$ from P_x, likewise for $\sigma_{i+1}^{-1} e^W$. Thus neither endpoints of L are branch poles. This completes the proof of the lemma. $\square$

The lemma below allows us to extract data from the phase equation.

Lemma 2.6.11. *Fix $x \in \mathbb{R}$ and suppose a continuous root function $\zeta(y)$ of (2.6.2) approaches*

$a \in \mathcal{J}_x$ as $y \to +\infty$ $(-\infty)$. Writing $\zeta = a + \varepsilon e^{i\theta}$ with $\theta \in [0, \pi]$, (2.6.10) has the form

$$\arg_+ \mathcal{G}_x(a^-) + (\arg_+ \mathcal{G}_x(a^+) - \arg_- \mathcal{G}_x(a^-)) \cdot \frac{\theta}{2\pi} + \varepsilon O(\theta \wedge (\pi - \theta)) = 0. \tag{2.6.12}$$

Proof. Clearly, $\arg(1 - a^{-1}\zeta) = \theta$. For $b \neq a$, we have

$$\arg(1 - b^{-1}\zeta) = \begin{cases} -\varepsilon O(\theta \wedge (\pi - \theta)) & \text{if } a < b, \\ -(\pi - \varepsilon O(\theta \wedge (\pi - \theta))) & \text{if } a > b. \end{cases} \tag{2.6.13}$$

In the case $b = 0$, consider $\arg(-\zeta)$ in which case we still have (2.6.13). The lemma follows from (2.6.1). ☐

The next lemma connects roots of (2.6.2) with branch poles of $\mathcal{G}_x$.

Lemma 2.6.12. *Fix $x \in \mathbb{R}$.*

1. *For each component $L \subset \widehat{\mathbb{R}} \setminus \mathcal{J}_x$ such that $\arg_+ \mathcal{G}_x(L) = 0$ and each endpoint $\mathfrak{a}$ of L such that $\mathfrak{a}$ is a branch pole of $\mathcal{G}_x$, there exists a continuous root function $\zeta_{\mathfrak{a}}(y)$ of (2.6.2) such that*

$$\lim_{y \to -\infty} \zeta_{\mathfrak{a}}(y) = \mathfrak{a}$$

   which is uniquely defined on $(-\infty, y_0)$ for some suitably negative y_0.

2. *Conversely, there exists some suitably negative y_0 such that if ζ is a root of (2.6.2) at (x, y) for some $y < y_0$, then $\zeta = \zeta_{\mathfrak{a}}(y)$ for some component $L \subset \widehat{\mathbb{R}} \setminus \mathcal{J}_x$ such that $\arg_+ \mathcal{G}_x(L) = 0$ with endpoint $\mathfrak{a}$ which is a branch pole of $\mathcal{G}_x$.*

3. *A component $L \subset \widehat{\mathbb{R}} \setminus \mathcal{J}_x$ contains exactly one root of (2.6.2) for y near $-\infty$ if and only if L contains at least one root of (2.6.2) for all $y \in \mathbb{R}$.*

Proof. By (2.6.9), as $y \to -\infty$, any root of (2.6.2) must get arbitrarily close to a branch pole of $\mathcal{G}_x$. By Lemma 2.6.11, the set of such branch poles $\mathfrak{a}$ must be those for which $\arg \mathcal{G}_x(\mathfrak{a}^+) = 0$ or $\arg \mathcal{G}_x(\mathfrak{a}^-) = 0$ (both cannot hold simultaneously); thus $\mathfrak{a}$ is an endpoint of some component $L \subset \widehat{\mathbb{R}} \setminus \mathcal{J}_x$ such that $\arg_+ \mathcal{G}_x(L) = 0$. Theorem 2.6.11 further indicates that if ζ is near $\mathfrak{a}$ for y sufficiently negative, then $\arg_+ \mathcal{G}_x(\zeta) = 0$ which means that $\zeta \in L$.

(1) For each $\mathfrak{a}$ which is a branch pole of $\mathcal{G}_x$ and an endpoint of some L with $\arg_+ \mathcal{G}_x(L) = 0$, we can see from (2.6.2) that there exists some root $\zeta_{\mathfrak{a}}(y) \in L$ near $\mathfrak{a}$ for y sufficiently negative. By (2.6.2) and the constraint that roots near $\mathfrak{a}$ must be real for y sufficiently negative, we have that $\zeta_{\mathfrak{a}}(y)$ is uniquely defined on $(-\infty, y_0)$ for some suitably negative y_0. This root function is necessarily continuous for $y \in (-\infty, y_0)$.

(2) By the discussion in the first paragraph and uniqueness of $\zeta_{\mathfrak{a}}$, we have that any root ζ of (2.6.2) must coincide with some $\zeta_{\mathfrak{a}}(y)$ for y sufficiently small.

(3) If L contains exactly one root of (2.6.2), then one endpoint is a branch pole and the other is a branch zero. Roots are added to L if nonreal pairs coalesce and are removed from L if real roots coalesce. This implies that the parity of the number of roots in L must be preserved. Thus for any value of y, L must contain a root of (2.6.2). ☐

We are now in a position to prove the existence of the map $\zeta(x, y)$.

Proposition 2.6.13. *The (x, y)-companion equation has at most 2 nonreal roots for a given $(x, y) \in \mathbb{R}^2$. If (2.6.2) has a nonreal root, then $x \in (V_0, V_n)$*

Proof. Fix $x \in \mathbb{R}$. By the contour integral formula for roots of functions applied to $\mathcal{G}_x(\zeta) - e^{-y}$ which is analytic in $w = e^{-y}$, we have that the number ν of roots of (2.6.2) is constant as y varies in $\mathbb{R}$.

Let $\mathcal{L}_x$ be the set of components of $\widehat{\mathbb{R}} \setminus \mathcal{J}_x$ containing exactly one root of (2.6.2) for y near $-\infty$. By Lemma 2.6.12, if $x \in (V_0, V_n)$ then the total number of roots of (2.6.2) is

$$\nu = |\mathcal{L}_x| + 2$$

where the 2 is contributed by the two endpoints of the external component. Moreover, since there is at least one root contained in each component $L \in \mathcal{L}_x$ for all $y \in \mathbb{R}$, we must have at least $|\mathcal{L}_x|$ real roots of (2.6.2) at any $y \in \mathbb{R}$. Thus there are at most two nonreal roots of (2.6.2) at any given $(x, y) \in \mathbb{R}^2$.

Similarly by Lemma 2.6.12, if $x \notin (V_0, V_n)$ then the total number of roots of (2.6.2) is $\nu = |\mathcal{L}_x|$. Moreover, since there is at least one root contained in each component $L \in \mathcal{L}_x$ for all $y \in \mathbb{R}$, all the roots are real for all $y \in \mathbb{R}$. $\square$

Remark 10. We note that a continuous extension $\widetilde{\zeta}(x, y)$ of $\zeta(x, y)$ onto $\mathbb{R}^2$ is not uniquely defined. However, the component of $\widehat{\mathbb{R}} \setminus \mathcal{J}_x$ which $\widetilde{\zeta}(x, y)$ belongs to when it is real, i.e. $(x, y) \notin \mathscr{L}$, *is* well-defined. From the proof of Proposition 2.6.13, for any fixed $x \in I$ we see that $\widetilde{\zeta}(x, y)$ is contained in the external component for y sufficiently negative. This is because the external component has 2 roots and all other components have at most one root for y sufficiently negative, and we require a coalescing of real roots to get a pair of nonreal roots. Thus we have the freedom to choose $\widetilde{\zeta}$ so that $\widetilde{\zeta}(x, y) \to \min \mathcal{J}_x$ as $y \to -\infty$ for all $x \in I$, or so that $\widetilde{\zeta}(x, y) \to \max \mathcal{J}_x$ as $y \to -\infty$ for all $x \in I$; note that it is not clear that $\widetilde{\zeta}$ is uniquely determined upon making this choice, and we will not need this fact.

Remark 11. As y increases from $-\infty$, we have that the first coalescing of roots (i.e. the first double root of (2.6.2)) appears when the two roots starting at opposite endpoints of the external component coalesce. If $y = y_0$ is when the first double root appears, then each point in the external component must have been a root of the (x, y)-companion equation for some $y \leq y_0$.

2.6.2 Diffeomorphism between Roots and $\mathbb{H}$

Proposition 2.6.13 gives us the existence of a map $\zeta : \mathscr{L} \to \mathbb{H}$. We now show that this map is a diffeomorphism. Throughout this subsection, it will be convenient to use the symbol ζ as a variable in $\mathbb{C}$. To avoid ambiguity we will write $\zeta(x, y)$ when referring to the root function.

It is useful to consider the spectral curve of the RPP (see [39]), as they can be viewed as the source of the surjectivity of $\zeta(x, y)$. To this end, we start by rewriting the companion equation into an alternative convenient form.

Definition 2.6.14. Define the analytic function

$$\mathcal{Q}(\zeta, \eta) = \eta \cdot Q(\zeta)^{-1} - 1 \tag{2.6.14}$$

and the *spectral curve for periodic weights* $s_0, \ldots, s_{p-1}$

$$P(z, w) = \prod_{\sigma \in \mathcal{S}} (1 - \sigma z)^{\frac{1}{p}} - w \tag{2.6.15}$$

where the branch is chosen so that the argument of $1 - \sigma z$ is 0 when $z < \sigma^{-1}$.

We will write $(z, w) \in P$ to mean $P(z, w) = 0$. Then $\zeta \in \mathbb{H}$ is a solution to the companion equation for some $(x, y) \in \mathbb{R}^2$ if and only if there exists $(z, w) \in P$ such that

$$\mathcal{Q}(\zeta, \eta) = 0, \tag{2.6.16}$$

$$(\zeta, \eta) = (e^x z, e^y w). \tag{2.6.17}$$

Remark 12. This definition of the spectral curve differs from the definition in [39], which gives $\prod_{\sigma \in \mathcal{S}} (1 - \sigma z) - w^p$. This difference arises from a rescaling of the x-axis, namely that [39] contracts the x-direction by p to make the corresponding fundamental domain $\mathbb{Z} \times \mathbb{Z}$ periodic whereas our fundamental domain is $p\mathbb{Z} \times \mathbb{Z}$ periodic.

Lemma 2.6.15. *The set of $(z, w) \in P$ with $z \in \mathbb{H}$ satisfies $0 < -\arg w < \pi - \arg z$. Moreover, for any pair φ, θ with $0 < \varphi < \pi - \theta$, there exists a unique $(z, w) \in P$ with $\arg w = -\varphi$, $\arg z = \theta$.*

Proof. Fix $\theta \in (0, \pi)$. By geometric considerations, the map $\rho \mapsto \arg(1 - \rho e^{i\theta})$ strictly decreases on $(0, \infty)$, approaching 0 as $\rho \to 0$ and approaching $-(\pi - \theta)$ as $\rho \to \infty$. Thus the map

$$\vartheta : \rho \mapsto \frac{1}{p} \sum_{\sigma \in \mathcal{S}} \arg(1 - \sigma \rho e^{i\theta})$$

also strictly decreases on $(0, \infty)$, approaching 0 as $\rho \to 0$ and approaching $-(\pi - \theta)$ as $\rho \to \infty$.

Taking $z = \rho e^{i\theta}$, we have that $\vartheta(\rho) = \arg w$. Since ϑ is a bijection, any point $(\theta, -\varphi)$ such that $0 < \varphi < \pi - \theta$ uniquely determines a point $(z, w) \in P$ where $(\arg z, \arg w) = (\theta, -\varphi)$. ☐

Proof of Theorem 2.6.3. We first establish that $\zeta(x, y)$ is a bijection from $\mathscr{L}$ onto $\mathbb{H}$. Given $\zeta \in \mathbb{H}$, the pair $(\zeta, \eta) = (\zeta, Q(\zeta))$ solves (2.6.16). To show that ζ is a solution to the companion equation for some $(x, y) \in \mathbb{R}^2$ (and therefore in $\mathscr{L}$), we must find $(z, w) \in P$ so that $(\zeta, \eta) = (e^x z, e^y w)$. Alternatively said, we want $(z, w) \in P$ so that $\arg(\zeta, \eta) = \arg(z, w)$, and by Lemma 2.6.15 this $(z, w) \in P$ is unique if it exists. By Lemma 2.6.15 again, it is enough to show that $0 < -\arg \eta < \pi - \arg \zeta$.

By geometric considerations (see proof of Lemma 2.6.15), given $\zeta \in \mathbb{H}$ and $a, b \in [0, \infty]$ such that $a < b$, we have

$$-(\pi - \arg \zeta) \leq \arg(1 - a^{-1}\zeta) < \arg(1 - b^{-1}\zeta) \leq 0$$

where we mean $\arg(-\zeta) = -(\pi - \arg\zeta)$ for $\arg(1-\infty\zeta)$. By (2.6.5)

$$\begin{aligned}\arg\eta = \frac{1}{p}\sum_{\sigma\in\mathcal{S}}\Bigg[&\arg(1-e^{-V_0}\sigma\zeta)\\ &+\sum_{\ell=1}^{n}\mathbf{1}[\mathcal{B}'(V_{\ell-1},V_\ell)<\varsigma_\sigma]\left(\arg(1-e^{-V_\ell}\sigma\zeta)-\arg(1-e^{-V_{\ell-1}}\sigma\zeta)\right)\Bigg]\end{aligned} \tag{2.6.18}$$

and by (2.6.6)

$$\begin{aligned}\arg\eta = \frac{1}{p}\sum_{\sigma\in\mathcal{S}}\Bigg[&\arg(1-e^{-V_n}\sigma\zeta)\\ &+\sum_{\ell=1}^{n}\mathbf{1}[\mathcal{B}'(V_{\ell-1},V_\ell)\geq\varsigma_\sigma]\left(\arg(1-e^{-V_{\ell-1}}\sigma\zeta)-\arg(1-e^{-V_\ell}\sigma\zeta)\right)\Bigg].\end{aligned} \tag{2.6.19}$$

In (2.6.18), each summand in the summation over ℓ is ≥ 0 with some > 0. This implies

$$\arg\eta > \sum_{\sigma\in\mathcal{S}}\frac{1}{p}\arg(1-e^{-V_0}\sigma\zeta) \geq -(\pi-\arg\zeta).$$

In (2.6.19), each summand in the summation over ℓ is ≤ 0 with some < 0. This implies

$$\arg\eta < \sum_{\sigma\in\mathcal{S}}\frac{1}{p}\arg(1-e^{-V_n}\sigma\zeta) \leq 0.$$

So we have $0 < -\arg\eta < \pi - \arg\zeta$. This proves $\zeta(x,y)$ is surjective. Moreover, we see that the procedure from ζ to (x,y) uniquely determines (x,y), thus $\zeta(x,y)$ is also injective.

Since $\zeta(x,y)$ is differentiable on $\mathscr{L}$, it remains to show the differentiability of the inverse map. However, the chain of procedures $\zeta \mapsto (\zeta,\eta) = (\zeta, Q(\zeta)) \mapsto (\arg\zeta, \arg\eta)$ is differentiable. Furthermore, the map $(\theta,-\varphi)\mapsto(z,w)\in P$ where $\arg(z,w) = (\theta,-\varphi)$ is differentiable. Therefore $\zeta \mapsto \left(\frac{\zeta}{z},\frac{\eta}{w}\right) = (e^x, e^y)$ is differentiable. It follows that the inverse of $\zeta(x,y)$ is differentiable. □

2.6.3 Double Roots and the Frozen Boundary

In this section, we describe the *frozen boundary* which we define to be the boundary of the liquid region $\mathscr{L}$. This requires understanding the double roots of the companion equation.

By taking the logarithm and differentiating, a root of the companion equation is a double root if it solves

$$\frac{1}{p}\sum_{\sigma\in\mathcal{S}}\frac{1}{\zeta-e^x\sigma^{-1}} - \Sigma(\zeta) = 0 \tag{2.6.20}$$

where

$$\Sigma(\zeta) = \frac{1}{p} \sum_{\substack{\ell \in [[0,n]] \\ \sigma \in \mathcal{S}}} \left(\mathbf{1}[\mathcal{B}'(V_\ell^+) \geq \varsigma_\sigma] - \mathbf{1}[\mathcal{B}'(V_\ell^-) \geq \varsigma_\sigma] \right) \frac{1}{\zeta - e^{V_\ell} \sigma^{-1}} \tag{2.6.21}$$

and we take $e^{V_0} = 0$ if $V_0 = -\infty$ and $\frac{1}{\zeta - e^{V_n} \alpha^{-1}} = 0$ if $V_n = +\infty$. We have the alternative expressions, corresponding to (2.6.5) and (2.6.6) respectively,

$$\Sigma(\zeta) = \frac{1}{p} \sum_{\sigma \in \mathcal{S}} \left[\frac{1}{\zeta - e^{V_0} \sigma^{-1}} + \sum_{\ell=1}^{n} \mathbf{1}[\mathcal{B}'(V_{\ell-1}, V_\ell) < \varsigma_\sigma] \left(\frac{1}{\zeta - e^{V_\ell} \sigma^{-1}} - \frac{1}{\zeta - e^{V_{\ell-1}} \sigma^{-1}} \right) \right] \tag{2.6.22}$$

$$= \frac{1}{p} \sum_{\sigma \in \mathcal{S}} \left[\frac{1}{\zeta - e^{V_n} \sigma^{-1}} + \sum_{\ell=1}^{n} \mathbf{1}[\mathcal{B}'(V_{\ell-1}, V_\ell) \geq \varsigma_\sigma] \left(\frac{1}{\zeta - e^{V_{\ell-1}} \sigma^{-1}} - \frac{1}{\zeta - e^{V_\ell} \sigma^{-1}} \right) \right] \tag{2.6.23}$$

taking the same modifications as above for $V_0 = -\infty$ and $V_n = +\infty$.

We reexpress the double root equation to obtain a parametrization of the double roots of (2.6.2) in terms of $\zeta \in \mathbb{R}$. Define the function

$$f(u) = \frac{1}{p} \sum_{\sigma \in \mathcal{S}} \frac{1}{1 - \sigma^{-1} u}$$

to rewrite the double root equation as

$$f(e^x \zeta^{-1}) = \zeta \Sigma(\zeta). \tag{2.6.24}$$

We want to invert f, though we require multiple inverses.

Take $f(\infty) = 0$ so that $\lim_{|u| \to \infty} f(u) = f(\infty)$. Then f is defined on $\widehat{\mathbb{R}} \setminus \{\sigma\}_{\sigma \in \mathcal{S}}$ and is invertible on each connected component. Define $E_0 = (\sigma_1, 0)$ (that is $(\sigma_1, +\infty) \cup \{\infty\} \cup (-\infty, 0)$), $E_j = (\sigma_{j+1}, \sigma_j)$ for $1 \leq j < d$, and $E_d = (0, \sigma_d)$. Let φ_j denote the inverse of f restricted to E_j. For $j \in [[1, d-1]]$, we have $\varphi_j : \mathbb{R} \to E_j$ whereas $\varphi_0 : (-\infty, 1) \to E_0$ and $\varphi_d : (1, +\infty) \to E_d$. Note that φ_0 maps $(-\infty, 0)$ to $(\sigma_1, +\infty)$, 0 to ∞, and $(0, 1)$ to $(-\infty, 0)$.

Suppose $e^x \zeta^{-1} \in E_j$ for some $j \in [[0, d]]$. Then $\zeta \in (-\infty, \sigma_1^{-1} e^x)$ if $j = 0$, $\zeta \in (e^x \sigma_j^{-1}, e^x \sigma_{j+1}^{-1})$ if $j \in [[1, d-1]]$, and $\zeta \in (e^x \sigma_d^{-1}, +\infty)$ if $j = d$. In particular, $\arg_+ P_x(\zeta) = \varsigma_j$.

By inverting (2.6.24), the condition for being a double root of (2.6.2) becomes

$$e^x = \zeta \varphi(\zeta \Sigma(\zeta)) \tag{2.6.25}$$

$$e^y = \frac{Q(\zeta)}{P_x(\zeta)} = \frac{Q(\zeta)}{\prod_{\sigma \in \mathcal{S}} (1 - \frac{\sigma}{\varphi(\zeta \Sigma(\zeta))})^{\frac{1}{p}}} \tag{2.6.26}$$

where the choice of inverse φ must make the right hand side of (2.6.26) positive. More specifically, we have $\arg_+ Q(\zeta) = \varsigma_j$ for some $j \in [[0, d]]$. Then choose $\varphi = \varphi_j$ which implies $\arg_+ P_x(\zeta) = \varsigma_j$ by our discussion above. Thus the argument $\arg_+$ of the right hand side of (2.6.26) is 0. Moreover, φ_j is the only inverse of f which allows the right hand side of (2.6.26) to be positive. We record our choice of φ below:

$$\varphi = \varphi_j : \quad \arg_+ Q(\zeta) = \varsigma_j \tag{2.6.27}$$

The following proposition shows that (2.6.25) and (2.6.26) provides an $\mathbb{R} \setminus \mathcal{J}$-parametrization of the frozen boundary.

Proposition 2.6.16. *Each $\zeta \in \mathbb{R} \setminus \mathcal{J}$ is a double root of (2.6.2) for some unique $(x, y) \in \mathbb{R}^2$.*

Proof. Let $\zeta \in \mathbb{R} \setminus \mathcal{J}$. Consider first $\zeta = 0$ (when $0 \notin \mathcal{J}$); note that $V_0 > -\infty$ is equivalent to $0 \notin \mathcal{J}$. Then (2.6.20) becomes

$$\frac{1}{p} \sum_{\sigma \in \mathcal{S}} e^{-x} \sigma = -\Sigma(0).$$

The right hand side is positive by (2.6.23), and so we can solve for x. The companion equation also gives us

$$e^{-y} = \mathcal{G}_x(0) = e^{-\mathcal{B}(V_0)}$$

so that we can solve for y. This proves the $\zeta = 0$ case.

For $\zeta \neq 0$, we show that the right hand sides of (2.6.25) and (2.6.26) are positive. The positivity of the right hand side of (2.6.26) follows from the choice of inverse φ. We note that the uniqueness of (x, y) follows from the fact that the choice of inverse φ of f is the only one that gives positivity (of (2.6.26)) as mentioned above.

It remains to show that the right hand side of (2.6.25) is positive. Suppose $j \in [[0, d]]$ so that $\varphi = \varphi_j$. We proceed by case analysis.

Suppose $j \neq 0, d$. Then $\varphi(\zeta\Sigma(\zeta)) \in E_j \subset \mathbb{R}_{>0}$ and $\arg_+ P_x(\zeta) = \varsigma_j$. These imply $\varphi_j(\zeta\Sigma(\zeta)) > 0$ and $\zeta > 0$. Thus the right hand side of (2.6.25) is positive.

In preparation for the remaining cases, we note that (2.6.22) and (2.6.23) imply

$$\zeta\Sigma(\zeta) = \frac{1}{p} \sum_{\sigma \in \mathcal{S}} \left[\frac{1}{1 - e^{V_0}(\sigma\zeta)^{-1}} + \sum_{\ell=1}^{n} \mathbf{1}[\mathcal{B}'(V_\ell^-) < \varsigma_\sigma] \left(\frac{1}{1 - e^{V_\ell}(\sigma\zeta)^{-1}} - \frac{1}{1 - e^{V_{\ell-1}}(\sigma\zeta)^{-1}} \right) \right] \tag{2.6.28}$$

$$= \frac{1}{p} \sum_{\sigma \in \mathcal{S}} \left[\frac{1}{1 - e^{V_n}(\sigma\zeta)^{-1}} + \sum_{\ell=1}^{n} \mathbf{1}[\mathcal{B}'(V_\ell^-) \geq \varsigma_\sigma] \left(\frac{1}{1 - e^{V_{\ell-1}}(\sigma\zeta)^{-1}} - \frac{1}{1 - e^{V_\ell}(\sigma\zeta)^{-1}} \right) \right] \tag{2.6.29}$$

We also record

$$\begin{aligned} \arg_+ Q(\zeta) &= -\frac{\pi}{p} \sum_{\ell=0}^{n} \sum_{\substack{\sigma \in \mathcal{S} \\ \sigma^{-1} e^{V_\ell} \in J_{V_\ell}}} (\mathbf{1}[\mathcal{B}'(V_\ell^+) \geq \varsigma_\sigma] - \mathbf{1}[\mathcal{B}'(V_\ell^-) \geq \varsigma_\sigma]) \mathbf{1}[\sigma^{-1} e^{V_\ell} < \zeta] \\ &= -\frac{\pi}{p} \sum_{\sigma \in \mathcal{S}} \mathbf{1}[\mathcal{B}'(V_{\ell_\sigma - 1}^+) \geq \varsigma_\sigma] = -\frac{\pi}{p} \sum_{\sigma \in \mathcal{S}} \mathbf{1}[\mathcal{B}'(V_{\ell_\sigma}^-) \geq \varsigma_\sigma] \end{aligned} \tag{2.6.30}$$

where $\ell_\sigma = \max\{\ell : \sigma^{-1} e^{V_{\ell-1}} < \zeta\}$.

If $j = d$, we must check $\zeta\Sigma(\zeta) > 1$, i.e. is in the domain of φ_d. Since $\arg_+ Q(\zeta) = -\pi$, by (2.6.30) we have $\mathcal{B}'(V_{\ell_\sigma}^-) \geq \varsigma_\sigma$ for all $\sigma \in \mathcal{S}$. Plugging this into (2.6.28), along with the fact that $\zeta \in (\sigma^{-1}e^{V_{\ell_\sigma - 1}}, \sigma^{-1}e^{V_{\ell_\sigma}})$ with the convention that $V_{n+1} = +\infty$, we have $\zeta\Sigma(\zeta) > 1$. Thus $\zeta\Sigma(\zeta)$ is in the domain of φ. Since $\varphi(\zeta\Sigma(\zeta)) \in E_0$ and $\zeta > 0$, the right hand side of (2.6.25) is positive.

If $j = 0$, we must check that $\zeta\Sigma(\zeta) < 0$ or $0 < \zeta\Sigma(\zeta) < 1$, i.e. is in the domain of φ_0. If $\zeta < 0$, then (2.6.28) implies $\zeta\Sigma(\zeta) > 0$ and (2.6.29) implies $\zeta\Sigma(\zeta) < 1$. We then have $\varphi(\zeta\Sigma(\zeta)) < 0$. Thus $\zeta\varphi(\zeta\Sigma(\zeta)) > 0$.

Otherwise, $\zeta > 0$. Since $\arg_+ Q(\zeta) = 0$, by (2.6.30) we have $\mathcal{B}'(V_{\ell_\sigma}^-) < \varsigma_\sigma$ for all $\sigma \in \mathcal{S}$. We note that $\zeta < \sigma^{-1}e^{V_n}$ for each $\sigma \in \mathcal{S}$, otherwise it cannot be the case that $\arg_+ Q(\zeta) = 0$. Plugging this into (2.6.29), we have $\zeta\Sigma(\zeta)) < 0$. Then $\varphi(\zeta\Sigma(\zeta)) > 0$ so that the right hand side of (2.6.25) is positive. $\square$

The next proposition explains what happens as ζ approaches an element of $\mathcal{J}$.

Proposition 2.6.17. *Let V be an non-differentiable point of $\mathcal{B}$. As $\zeta \to \sigma^{-1}e^V \in J_V$ in (2.6.25) and (2.6.26), we have $x \to V$ and y converges to a finite limit if $\mathcal{B}'(V^+) - \mathcal{B}'(V^-) > 0$, and $y \to +\infty$ if $\mathcal{B}'(V^+) - \mathcal{B}'(V^-) < 0$.*

Proof. Let $\eta = \sigma_i^{-1}e^V$ for some $i \in [[1, d]]$. By (2.6.30)

$$\arg_+ Q(\eta_+) = \frac{1}{p} \sum_{\tau^{-1} \leq \sigma_i^{-1}} \mathbf{1}[\mathcal{B}'(V^+) \geq \varsigma_\tau] + \frac{1}{p} \sum_{\tau^{-1} > \sigma_i^{-1}} \mathbf{1}[\mathcal{B}'(V^-) \geq \varsigma_\tau]. \tag{2.6.31}$$

If $\mathcal{B}'(V^+) - \mathcal{B}'(V^-) > 0$ (< 0), then by Lemma 2.6.8,

$$\arg_+ Q(\eta^+) = -\pi\varsigma_i \ (= -\pi\varsigma_{i-1}),$$
$$\arg_+ Q(\eta^-) = -\pi\varsigma_{i-1} \ (= -\pi\varsigma_i),$$

and $\Sigma(\eta^\pm) = \pm\infty (= \mp\infty)$. By (2.6.27) and the manner in which φ_j maps into E_j for $j \in [[0, d]]$, we have $\lim_{\zeta \to \eta^\pm} \varphi(\zeta\Sigma(\zeta)) = \sigma_i$. Thus $\lim_{\zeta \to \eta^\pm} \zeta\varphi(\zeta\Sigma(\zeta)) = e^V$. We also have that Q has a branch zero (pole) at $\sigma_i^{-1}e^V$ of order $\frac{|S_{\sigma_i}|}{p}$.

Thus if $\mathcal{B}'(V^+) - \mathcal{B}'(V^-) > 0$, then as $\zeta \to \eta$ we have $e^y = \frac{Q(\zeta)}{P_x(\zeta)}$ has a finite limit since the branch zero of Q is met by the branch zero of P_x of equal order. In the case $\mathcal{B}'(V^+) - \mathcal{B}'(V^-) < 0$, then $e^y = \frac{Q(\zeta)}{P_x(\zeta)}$ approaches $+\infty$ as $\zeta \to \eta$. This proves the proposition. $\square$

Remark 13. Proposition 2.6.17 asserts that the points $\sigma^{-1}e^V \in J_V$ such that $\mathcal{B}'(V^+) - \mathcal{B}'(V^-) < 0$ correspond to parts of the frozen region where the vertical coordinate is unbounded. These are the *tentacles* in the frozen region which arise due to singular points of the back wall, see Figures 2.11 and 2.12. The cusps correspond to other points in J_V.

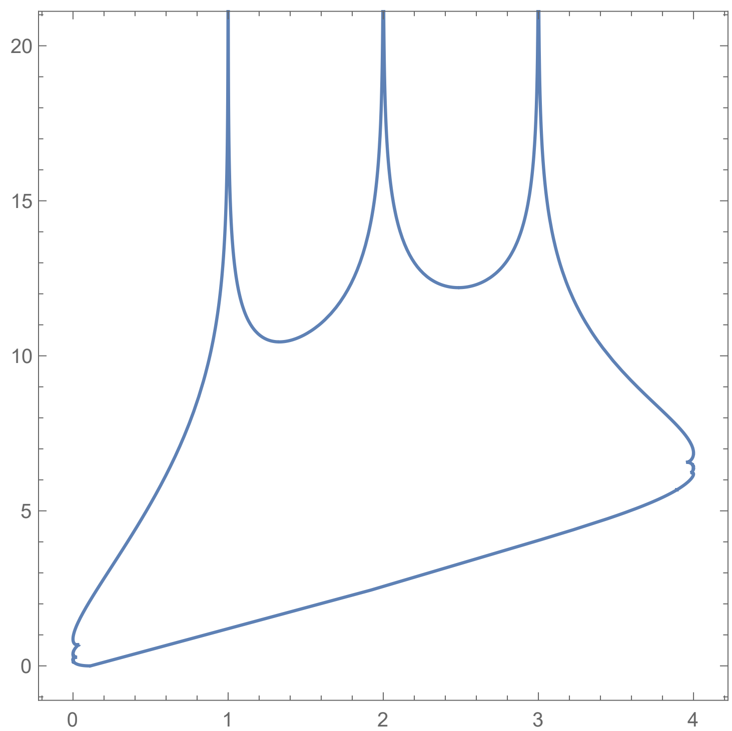


Figure 2.11: Frozen boundary for periodic weights $2, 2, \frac{1}{4}$, 4 linear back wall pieces with slopes $0, \frac{1}{3}, \frac{2}{3}, 1$ from left to right. Each non-differentiable point is equally spaced from one another and corresponds to a singular point.

2.7 Limit Shapes and the Gaussian Free Field

In this section, we combine the results from the previous sections to obtain the main results of this chapter: the limit shape and Gaussian free field fluctuations for the height functions. These correspond to Theorems 2.2.7 and 2.2.8 given in Section 2.2. However, we provide more precise reformulations of these theorems.

Throughout this section, we fix a family $\mathbb{P}^{B,r,\boldsymbol{s}}_{\alpha,\mathfrak{t}}$ satisfying the Limit Conditions, as defined in the beginning of Section 2.4.1, with limiting back wall $\mathcal{B} : I \to \mathbb{R}$. We write $\mathcal{G}_{\mathbf{x}} := \mathcal{G}^{\mathcal{B}}_{\mathbf{x}}$.

Definition 2.7.1. We say $x(\varepsilon) \in I^{\varepsilon}$ is *separated from singular points* if $x(\varepsilon)$ is *d-separated from singular points* (see Definition 2.5.1) for every $d > 0$.

Let $\zeta = \zeta^{\mathcal{B}} : \mathscr{L} \to \mathbb{H}$ denote the diffeomorphism from Theorem 2.6.3. As in Remark 10, we can extend ζ to a function $\widetilde{\zeta} : I \times \mathbb{R} \to \mathbb{H}$ such that

$$\lim_{\mathbf{y} \to -\infty} \widetilde{\zeta}(\mathbf{x}, \mathbf{y}) = \min \mathcal{J}_{\mathbf{x}} = \sigma_1^{-1} e^{V_0}.$$

Theorem 2.7.2. *Suppose* $\mathbb{P}^{B,r,\boldsymbol{s}}_{\alpha,\mathfrak{t}}$ *satisfies the Limit Conditions with* $\mathcal{B} \in \mathfrak{B}^{\Delta}(\mathcal{S})$. *There exists a deterministic Lipschitz* 1 *function* $\mathcal{H} := \mathcal{H}^{\mathcal{B}} : I \times \mathbb{R} \to \mathbb{R}$ independent of $\alpha, \mathfrak{t}$ *such that for any*

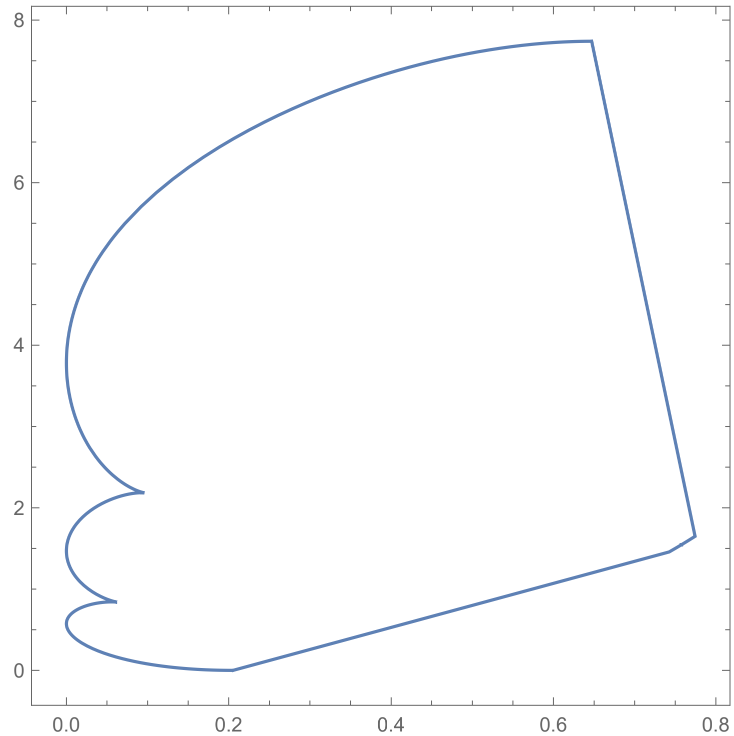


Figure 2.12: Frozen boundary for periodic weights $4, \frac{1}{4}, 2, \frac{1}{2}, \frac{5}{4}, \frac{4}{5}, 1$ linear back wall piece with slope $\frac{1}{2}$. See Figure 2.2 for a large sample.

$(\mathbf{x}, \mathbf{y}) \in I \times \mathbb{R}$ and $x(\varepsilon) \in I^{\varepsilon}$ separated from singular points with $\varepsilon x(\varepsilon) \to \mathbf{x}$, we have the convergence of random measures on $\mathbf{y} \in \mathbb{R}$

$$\varepsilon h\left(x(\varepsilon), \frac{\mathbf{y}}{\varepsilon}\right) \to \frac{1}{\alpha}\mathcal{H}(\mathbf{x}, \mathbf{y}) \tag{2.7.1}$$

weakly in probability as $\varepsilon \to 0$. We have the following explicit description of $\mathcal{H}$. For each $\mathbf{x} \in I$, we have $\mathcal{H}(\mathbf{x}, \mathbf{y}) = 0$ for sufficiently negative $\mathbf{y}$, and

$$\begin{aligned} \nabla\mathcal{H}(\mathbf{x}, \mathbf{y}) &:= (\partial_{\mathbf{x}}\mathcal{H}(\mathbf{x}, \mathbf{y}), \partial_{\mathbf{y}}\mathcal{H}(\mathbf{x}, \mathbf{y})) \\ &= \frac{1}{\pi}\left(\sum_{\sigma \in \mathcal{S}} \frac{1}{p} \arg\left(1 - \sigma^{-1}e^{\mathbf{x}}\widetilde{\zeta}(\mathbf{x}, \mathbf{y})\right), \pi \cdot \mathbf{1}_{e^{-y} < \mathcal{G}_{\mathbf{x}}(0)} - \arg\widetilde{\zeta}(\mathbf{x}, \mathbf{y})\right) \end{aligned} \tag{2.7.2}$$

for $(\mathbf{x}, \mathbf{y}) \in I \times \mathbb{R}$ such that the right hand side is well-defined and where the arg branches are chosen so that $\arg z = 0$ for $z > 0$.

Remark 14. Although the choice of $\widetilde{\zeta}$ was not shown to be well-defined, we note that $\arg\widetilde{\zeta}(\mathbf{x}, \mathbf{y})$ and $\arg(1 - \sigma^{-1}e^{\mathbf{x}}\widetilde{\zeta}(\mathbf{x}, \mathbf{y}))$ are well-defined (upon specification of the branch). This follows from Remark 10.

The condition in the indicator function $e^{-\mathbf{y}} < \mathcal{G}_{\mathbf{x}}(0)$ is always true in the liquid region. To see this, let $\mathscr{L}_{\mathbf{x}_0}$ be the slice $\{(\mathbf{x}_0, \mathbf{y}) : \mathbf{y} \in \mathbb{R}, (\mathbf{x}_0, \mathbf{y}) \in \mathscr{L}\}$. Observe that 0 is a root of the $(\mathbf{x}, -\log \mathcal{G}_{\mathbf{x}}(0))$-companion equation. By Remark 11, if $e^{-\mathbf{y}} \geq \mathcal{G}_{\mathbf{x}}(0)$, then $-\log \mathcal{G}_{\mathbf{x}}(0) \leq \min \mathscr{L}_{\mathbf{x}}$.

By the preceding discussion, we have the following corollary:

Corollary 2.7.3. *The limiting height function $\mathcal{H}$ from Theorem 2.7.2 has the following gradient for $(\mathbf{x}, \mathbf{y}) \in \mathscr{L}$:*

$$\nabla \mathcal{H}(\mathbf{x}, \mathbf{y}) = \frac{1}{\pi} \left(\sum_{\sigma \in \mathcal{S}} \frac{1}{p} \arg \left(1 - \sigma^{-1} e^{\mathbf{x}} \zeta(\mathbf{x}, \mathbf{y})\right), \pi - \arg \zeta(\mathbf{x}, \mathbf{y}) \right). \tag{2.7.3}$$

Theorem 2.7.4. *Suppose $\mathbb{P}^{B,r,\boldsymbol{s}}_{\alpha,\mathfrak{t}}$ satisfies the Limit Conditions with $\mathcal{B} \in \mathfrak{B}^{\Delta}(\mathcal{S})$, $\zeta(x, y) := \zeta^{\mathcal{B}}(x, y)$ denotes the homeomorphism of Theorem 2.6.3, and*

$$\overline{h}(x, y) = h(x, y) - \mathbb{E}h(x, y)$$

denotes the centered height function. Then $\sqrt{\alpha\pi} \cdot \overline{h}$ converges to the ζ-pullback of the GFF in the following sense. For any $\mathbf{x}_1, \ldots, \mathbf{x}_m \in I$, integers $k_1, \ldots, k_m > 0$, and $x_1(\varepsilon), \ldots, x_m(\varepsilon) \in I^{\varepsilon}$ such that $x_i(\varepsilon)$ is $\mathfrak{t}k_i$-separated from singular points and $\varepsilon x_i(\varepsilon) \to \mathbf{x}_i$ for each $i \in [[1, m]]$, we have the convergence of random vectors

$$\left(\sqrt{\alpha\pi} \int \overline{h}\left(x_i(\varepsilon), \frac{\mathbf{y}}{\varepsilon}\right) e^{-k_i \mathfrak{t} \mathbf{y}}\, d\mathbf{y} \right)_{i \in [[1,m]]} \to \left(\int \mathfrak{H}(\zeta(\mathbf{x}_i, \mathbf{y})) e^{-k_i \mathfrak{t} \mathbf{y}}\, d\mathbf{y} \right)_{i \in [[1,m]]}. \tag{2.7.4}$$

in distribution as $\varepsilon \to 0$, where $\mathfrak{H}$ is the Gaussian free field on $\mathbb{H}$.

Let $x_0 \in \mathbb{R}$ and define

$$\mathcal{D}_{x_0} = \{\zeta(x, y) : x \leq x_0, y \in \mathbb{R}, (x, y) \in \mathscr{L}\}, \tag{2.7.5}$$

$$\mathcal{D}^{\mathbb{C}}_{x_0} = \mathrm{cl}(\{z : z \in \mathcal{D}_{x_0} \text{ or } \bar{z} \in \mathcal{D}_{x_0}\}). \tag{2.7.6}$$

The boundary of $\mathcal{D}^{\mathbb{C}}_{x_0}$ is

$$\partial \mathcal{D}^{\mathbb{C}}_{x_0} = \mathrm{cl}\left(\{\zeta(x_0, y) : y \in \mathbb{R}, (x_0, y) \in \mathscr{L}\} \cup \{\overline{\zeta}(x_0, y) : y \in \mathbb{R}, (x_0, y) \in \mathscr{L}\}\right). \tag{2.7.7}$$

The points gained in taking the closure are all in $\mathbb{R}$.

By the parametrization of the frozen boundary in Section 2.6.3 and Proposition 2.6.17, we have:

Lemma 2.7.5. *The set $\mathcal{D}^{\mathbb{C}}_{x_0}$ contains a neighborhood of J_{V_ℓ} if $V_\ell < x_0$, and is separated from J_{V_ℓ} if $V_\ell > x_0$.*

2.7.1 Limit Shape: Proof of Theorem 2.7.2

Let $\mathbf{x} \in I$ and $x(\varepsilon) \in I^{\varepsilon}$ be separated from singular points with $\varepsilon x(\varepsilon) \to \mathbf{x}$. By Theorem 2.5.2 and Proposition 2.3.1, the rescaled height function $\varepsilon h\left(x(\varepsilon), \frac{\mathbf{y}}{\varepsilon}\right)$ converges to some limiting distribution

$\mathcal{H}^{\alpha,\mathfrak{t}}$ determined by

$$\int e^{-\mathfrak{t}k\mathbf{y}}\mathcal{H}^{\alpha,\mathfrak{t}}(\mathbf{x},\mathbf{y})\,d\mathbf{y} = \frac{1}{\alpha k^2\mathfrak{t}^2}\frac{1}{2\pi\mathbf{i}}\oint_{\mathcal{C}}\frac{\mathcal{G}_{\mathbf{x}}(z)^{k\mathfrak{t}}}{z}\,dz \tag{2.7.8}$$

for $k \in \mathbb{Z}_{>0}$ and where the contour $\mathcal{C}$ is described in Theorem 2.5.2.

We first show that $\mathcal{H}^{\alpha,\mathfrak{t}}$ is independent of $\mathfrak{t}$, and dependent on α by a simple scaling factor. The function $y \mapsto \mathbb{E}\varepsilon h\left(x(\varepsilon), \frac{y}{\varepsilon}\right)$ is $\frac{1}{\alpha}$-Lipschitz by (2.3.2), nonnegative and monotonically increasing. By considering a subsequence of $y \mapsto \mathbb{E}\varepsilon h\left(x(\varepsilon), \frac{y}{\varepsilon}\right)$, we deduce that $\mathcal{H}^{\alpha,\mathfrak{t}}(x,y)$ is $\frac{1}{\alpha}$-Lipschitz, nonnegative, and monotonically increasing in y in the sense that its weak derivative is a positive measure.

Definition 2.7.6. Given a measure μ on $\mathbb{R}$, let $\left\{\int e^{-k\mathfrak{t}u}\mu(du)\right\}_{k\in\mathbb{Z}_{>0}}$ be the set of $\mathfrak{t}$-*exponential moments* of μ; for $\mathfrak{t} = 1$ we simply say *exponential moments* of μ. Denote by $E^{\mathfrak{t}}\mu$ the measure defined by

$$(E^{\mathfrak{t}}\mu)(du) = \mu(d(\exp(-\mathfrak{t}u))).$$

Let us write $E\mu = E^1\mu$.

Lemma 2.7.7. *Fix $x \in I$. There exists some M such that $\mathcal{H}^{\alpha,\mathfrak{t}}(x,y) = 0$ on $\{y \le M\}$, and $\mathcal{H}^{\alpha,\mathfrak{t}}$ is determined by its $\mathfrak{t}$-exponential moments.*

Proof. Introduce the auxiliary measure $\mu_x^{\mathfrak{t}}$ with moments defined

$$\int_0^\infty u^{\mathfrak{t}(k-1)}\mu_x^{\mathfrak{t}}(du) = \frac{1}{\alpha(k\mathfrak{t})^2}\frac{1}{2\pi\mathbf{i}}\oint_{\mathcal{C}}\frac{G_x(z)^{k\mathfrak{t}}}{z}\,dz \tag{2.7.9}$$

for $k \in \mathbb{Z}_{>0}$. Then $\mu_x^{\mathfrak{t}} = E^{\mathfrak{t}}\mathcal{H}^{\alpha,\mathfrak{t}}(x,\cdot)$. It follows that $\mu_x^{\mathfrak{t}}$ is a finite positive measure supported in $[0,+\infty)$. By scaling by an appropriate $c > 0$, we may view $c\mu_x^{\mathfrak{t}}$ as a probability measure. From the form of the contour integral, we see that there is a large enough $L > 0$ such that $\mathbb{E}_{c\mu_x^{\mathfrak{t}}}u^{\mathfrak{t}k} \le L^k$. By a standard Markov inequality argument, this implies $\mu_x^{\mathfrak{t}}$ is compactly supported in $[0,+\infty)$. By changing variables, we obtain the lemma. $\square$

Let $\mathcal{H} := \mathcal{H}^{1,1}$. The exponential moments of $\mathcal{H}$ can be analytically extended in the variable k. By the preceding lemma and (2.7.8), for any $\alpha, \mathfrak{t} > 0$ we have

$$\mathcal{H}^{\alpha,\mathfrak{t}}(x,y) = \frac{1}{\alpha}\mathcal{H}^{1,1}(x,y) =: \mathcal{H}(x,y).$$

To prove Theorem 2.7.2, it remains to show (2.7.2). Note that we don't know $\partial_x\mathcal{H}(x,y)$ exists at the moment. However, by Lemma 2.7.7, $\mathcal{H}(x,y)$ is determined by $\partial_y\mathcal{H}(x,y)$, and we will first compute $\partial_y\mathcal{H}(x,y)$ from which it will be apparent that $\partial_x\mathcal{H}(x,y)$ exists.

We reduce to the case $-\infty < V_0$, $V_n < \infty$. Suppose $V_0 = -\infty$ or $V_n = +\infty$. Let us revive the superscripts for the back walls for the sake of this reduction. We have our limiting back wall $\mathcal{B}$, and construct a sequence $\mathcal{B}_n$ of back walls which is simply the restriction of $\mathcal{B}$ to $I \cap [-N,N]$ so $V_0 = -N$ and $V_n = N$. Since $\mathcal{G}_x^{\mathcal{B}_N} \to \mathcal{G}_x^{\mathcal{B}}$ uniformly on compactum in $\mathbb{C}\setminus\mathbb{R}$, we see that the exponential moments of $\mathcal{H}^{\mathcal{B}^N}$ converge to the exponential moments of $\mathcal{H}^{\mathcal{B}}$. Moreover, $\widetilde{\zeta}^{\mathcal{B}_N}$ which solves (2.6.2) for $\mathcal{G}^{\mathcal{B}_N}$ converges to $\widetilde{\zeta}^{\mathcal{B}}$ pointwise. Thus the reduction is valid by (2.7.2). We suppress the $\mathcal{B}$ superscript for the remainder of the proof.

As mentioned above we first compute $\partial_y \mathcal{H}(x,y)$. The computation for $\partial_x \mathcal{H}(x,y)$ will then be similar and use many of facts collected from the computation of $\partial_y \mathcal{H}(x,y)$.

Computation of Density for $\partial_y \mathcal{H}(x,y)$. By (2.7.8), we have

$$\int \partial_y \mathcal{H}(x,y) e^{-ky}\, dy = k \int \mathcal{H}(x,y) e^{-ky}\, dy = \frac{1}{2\pi \mathbf{i}} \oint_{\mathcal{C}} \frac{\mathcal{G}_x(z)^k}{kz}\, dz$$

for $k \in \mathbb{Z}_{>0}$. By Lemma 2.7.7, we know $\nu_x = E(\partial_y \mathcal{H}(x,\cdot))$ is compactly supported in $[0,\infty)$. Since $V_0 > -\infty$, from our explicit formula (2.6.1) we see that the poles of $\mathcal{G}_x$ are strictly positive. The set of poles of the integrand contained in $\mathcal{C}$ is exactly $\{0\} \cup \mathfrak{P}_x$ where

$$\mathfrak{P}_x = \{\text{branch poles } p \text{ of } \mathcal{G}_x^{\mathcal{B}} \text{ in } J_V \text{ such that } V < x\} \subset \bigcup_{\substack{\mathcal{B}'(V^+) - \mathcal{B}'(V^-) < 0 \\ V < x}} J_V. \tag{2.7.10}$$

Let us split $\mathcal{C}$ into a contour γ_0 containing 0 but no poles in $\mathfrak{P}_x$ and a contour γ which does not contain 0 but contains $\mathfrak{P}_x$. For v large enough so that $\operatorname{supp}(\nu_x) < v$ and $\left|\frac{\mathcal{G}_x(z)}{v}\right| < 1$ along $z \in \gamma$, we have the Stieltjes transform

$$\begin{aligned} S_{\nu_x}(v) &= \int_0^\infty \frac{\nu_x(du)}{u - v} = -\sum_{k=0}^\infty \int_0^\infty \frac{u^k \nu_x(du)}{v^{k+1}} = -\frac{1}{2\pi \mathbf{i}} \sum_{k=1}^\infty \oint_{\mathcal{C}} \frac{\mathcal{G}_x(z)^k}{v^k k} \frac{dz}{z} \\ &= -\sum_{k=1}^\infty \frac{\mathcal{G}_x(0)^k}{v^k k} - \frac{1}{2\pi \mathbf{i}} \sum_{k=1}^\infty \oint_\gamma \frac{\mathcal{G}_x(z)^k}{v^k k} \frac{dz}{z} \\ &= \log\left(1 - \frac{\mathcal{G}_x(0)}{v}\right) + \frac{1}{2\pi \mathbf{i}} \oint_\gamma \log\left(1 - \frac{\mathcal{G}_x(z)}{v}\right) \frac{dz}{z}. \end{aligned} \tag{2.7.11}$$

For v large enough so that $\operatorname{supp}(\nu_x) < v$ and $\left|\frac{\mathcal{G}_x(z)}{v}\right| < 1$ along γ, define the auxiliary function

$$\Upsilon_x(v) = \log\left(1 - \frac{\mathcal{G}_x(0)^p}{v^p}\right) + \frac{1}{2\pi \mathbf{i}} \oint_\gamma \log\left(1 - \frac{\mathcal{G}_x(z)^p}{v^p}\right) \frac{dz}{z}. \tag{2.7.12}$$

We can extend $\Upsilon_x(v)$ to the domain $\{|\arg z| < 2\pi/p\} \setminus \operatorname{supp} \nu_x$. Indeed, setting $\omega = e^{2\pi \mathbf{i}/p}$, we have

$$\Upsilon_x(v) = \sum_{j=0}^{p-1} \log\left(1 - \frac{\omega^j \mathcal{G}_x(0)}{v}\right) + \frac{1}{2\pi \mathbf{i}} \oint_\gamma \sum_{j=0}^{p-1} \frac{\log\left(1 - \frac{\omega^j G_x(z)}{v}\right)}{z}\, dz = \sum_{j=0}^{p-1} S_{\nu_x}(v\omega^{-j}) \tag{2.7.13}$$

for v large, but since the right hand side is an analytic function in $\{|\arg z| < 2\pi/p\} \setminus \operatorname{supp} \nu_x$ this identity extends to this larger domain.

The idea is that we want to evaluate the residues for the contour integral formula of the Stieltjes transform, but since the integrand is not meromorphic we consider an the auxiliary function $\Upsilon_x(v)$.

The auxiliary function allows us to consider an alternative meromorphic integrand. The following lemma establishes that $\Upsilon_x(v)$ may be considered instead of the Stieltjes transform.

Lemma 2.7.8. *For $v_0 > 0$,*

$$\frac{1}{\pi}\Im_+\Upsilon_x(v_0) = \frac{1}{\pi}\Im_+ S_{\nu_x}(v_0) = \rho_x(v_0) \tag{2.7.14}$$

where ρ_x is the density of ν_x (with respect to the Lebesgue measure) and $\Im_+$ is the imaginary part taken from the limit from the upper half plane.

Proof. The second equality of (2.7.14) is a property of the Stieltjes transform of a measure. We prove the first equality of (2.7.14).

Take $v_0 > 0$. Since $\overline{S_{\nu_x}(v)} = S_{\nu_x}(\bar{v})$ for $v \in \mathbb{C} \setminus \operatorname{supp} \nu_x = \mathbb{C} \setminus [0, \infty)$,

$$\Im_+(S_{\nu_x}(v_0\omega^{-j}) + S_{\nu_x}(v_0\omega^j)) = 0$$

for $j \neq 0$ and $j \neq p/2$ if p is even. If p is even, then

$$\Im_+ S_{\nu_x}(v_0\omega^{p/2}) = \Im_+ S_{\nu_x}(-v_0) = 0$$

since $\operatorname{supp} \nu_x \subset [0, \infty)$. By (2.7.13), this proves the lemma. $\square$

By Lemma 2.7.8, we are left to find $\Im_+\Upsilon_x(v) = \rho_x(v)$ and change variables $v = e^{-y}$ to compute $\partial_y \mathcal{H}(x, y)$. To this end we consider the following equation

$$\mathcal{G}_x(z)^p = v^p. \tag{2.7.15}$$

The solutions of (2.6.2) are exactly solutions of

$$\mathcal{G}_x(z) = \omega^j v \tag{2.7.16}$$

for some $j \in [[0, p-1]]$. A solution to the companion equation (2.6.2) is a solution to (2.7.16) with $j = 0$ and $v = e^{-y}$.

Definition 2.7.9. We say that $z(x, v)$ is a *root function of* (2.7.15) if $z(x, v)$ is a root of (2.7.15) for each (x, v) and continuous in (x, v).

Let $z(x, v)$ be a root function of (2.7.15). Note that as $v \to +\infty$, $z(x, v)$ converges to a pole of $\mathcal{G}_x(z)^p$, equivalently a branch pole of $\mathcal{G}_x(z)$.

Definition 2.7.10. If $\mathfrak{p}$ is a branch pole of $\mathcal{G}_x(z)$, let us say that $\mathfrak{p}$ is the *source* of a root function $z(x, v)$ of (2.7.15) if $\lim_{v\to+\infty} z(x, v) = \mathfrak{p}$.

Let $\mathfrak{Z}_x$ be the multiset of $\mathfrak{P}_x$ where the multiplicity of an element $\mathfrak{p} \in \mathfrak{P}_x$ in $\mathfrak{Z}_x$ is the multiplicity of the pole in $\mathcal{G}_x(z)^p$. We will abuse notation and treat $\mathfrak{p} \in \mathfrak{Z}_x$ as an element of $\mathfrak{P}_x$ in statements such as "$\lim_{v\to+\infty} z(x, v) = \mathfrak{p}$". Let $\mathcal{M}$ be the set of $v \in \mathbb{C}$ such that $\mathcal{G}_x(z)^p - v^p$ has a root of order > 1.

Lemma 2.7.11. *For $v \in \mathbb{R}_{>0}$, we have that*

$$\Upsilon_x(v) = \log\left(1 - \frac{\mathcal{G}_x(0)}{v}\right) - \sum_{\mathfrak{p} \in \mathfrak{Z}_x} (\log z_{\mathfrak{p}}(x,v) - \log \mathfrak{p}) \tag{2.7.17}$$

where $z_{\mathfrak{p}}(x,v)$ is a root function of (2.7.15). Furthermore the root functions $z_{\mathfrak{p}}(x,v)$ satisfy the following properties:

(i) For $v \in \mathbb{R}_{>0} \setminus \mathcal{M}$, the root functions $z_{\mathfrak{p}}(x,v)$ are distinct from one another.

(ii) If $z_{\mathfrak{p}}(x,v_0)$ is a root of (2.7.16) for $j = 0$, then $z_{\mathfrak{p}}(x,v)$ is a root of (2.7.16) for $j = 0$ for all $v > 0$.

(iii) If $z_{\mathfrak{p}}(x,v_0) \in \mathcal{D}_x^{\mathbb{C}}$ is a root of (2.7.16) for some $j \neq 0$, then $z_{\mathfrak{p}}(x,v) \in \mathcal{D}_x^{\mathbb{C}}$ for all $v > 0$.

(iv) If $v > 0$ and $z_{\mathfrak{p}}(x,v)$ is a root of (2.7.16) for $j \neq 0$ which is either a double root of (2.7.15) or $z_{\mathfrak{p}}(x,v) \in \mathbb{C} \setminus \mathbb{R}$, then there exists $\mathfrak{q} \in \mathfrak{Z}_x$ different from $\mathfrak{p}$ such that $z_{\mathfrak{p}}(x,v) = \overline{z_{\mathfrak{q}}(x,v)}$.

Proof. Since the poles of $1 - \frac{G_x(z)^p}{v^p}$ in γ occur exactly in $\mathfrak{P}_x$, we have

$$\Upsilon_x(v) = \log\left(1 - \frac{\mathcal{G}_x(0)}{v}\right) - \frac{1}{2\pi\mathbf{i}} \oint_\gamma (\log z) \frac{d}{dz} \log\left(1 - \frac{\mathcal{G}_x(z)^p}{v^p}\right) dz \tag{2.7.18}$$

$$= \log\left(1 - \frac{\mathcal{G}_x(0)}{v}\right) - \sum_{\mathfrak{p} \in \mathfrak{Z}_x} (\log z_{\mathfrak{p}}(x,v) - \log \mathfrak{p}) \tag{2.7.19}$$

for $z_{\mathfrak{p}}(x,v)$ a root function with source $\mathfrak{p}$, where this formula is initially valid for v large. For large enough v, there is no ambiguity in the choice of $z_{\mathfrak{p}}(x,v)$. We want to extend this formula to $v \in (0,+\infty)$. To do this, we require care in the choice of analytic extension to avoid ambiguity near double roots.

The set $\mathcal{M}$ is finite. Indeed, the discriminant of $\mathcal{G}_x(z)^p - v^p \in (\mathbb{R}[v])[z]$ is a polynomial in $\mathbb{R}[v]$, and the set of roots of the discriminant is exactly the set $\mathcal{M}$. Thus, from some large $v_0 > 0$, $z_{\mathfrak{p}}(x,v)$ can be extended to any $v \in \mathbb{R}_{>0} \setminus \mathcal{M}$ via a path from v_0 to v which avoids $\mathcal{M}$. This extension is not unique, as windings around $\mathcal{M}$ may change the value. This ambiguity is lost if we enforce the path from v_0 to v to lie in a strip $U = \{0 \leq \Im z < \delta\}$ where δ is chosen so that $\mathcal{M} \cap U \subset \mathbb{R}$. For $v \in \mathbb{R} \setminus \mathcal{M}$, let $z_{\mathfrak{p}}(x,v)$ denote the root function extended via a path from some large v_0 to v within the strip $U \setminus \mathcal{M}$, see Figure 2.13. We extend the definition to all $v \in \mathbb{R}$ by continuity.

We now prove that the root function $z_{\mathfrak{p}}(x,v)$ satisfy properties (i)-(iv). Property (i) follows from the definition of our root function, and since the roots must remain separated on $U \setminus \mathcal{M}$.

For properties (ii) and (iii), we note that a point $z \in \mathbb{C}$ cannot be both a root of (2.7.16) for $j_1 = 0$ at (x,v_1) and $j_2 \in [[1,p-1]]$ at (x,v_2); the special position of $j = 0$ is a consequence of our choice of branch cut for $\mathcal{G}_x$. This immediately implies property (ii). For property (iii), we use the additional fact that $z_{\mathfrak{p}}(x,v_1) \in \mathcal{D}_x^{\mathbb{C}}$ and $z_{\mathfrak{p}}(x,v_2) \notin \mathcal{D}_x^{\mathbb{C}}$ implies that $z_{\mathfrak{p}}(x,v)$ crosses the boundary of $\mathcal{D}_x^{\mathbb{C}}$ at some v_3. The boundary of $\mathcal{D}_x^{\mathbb{C}}$ is the closure of

$$\{\zeta(x,y) : y \in \mathbb{R}, (x,y) \in \mathscr{L}\} \cup \{\bar{\zeta}(x,y) : y \in \mathbb{R}, (x,y) \in \mathscr{L}\}$$

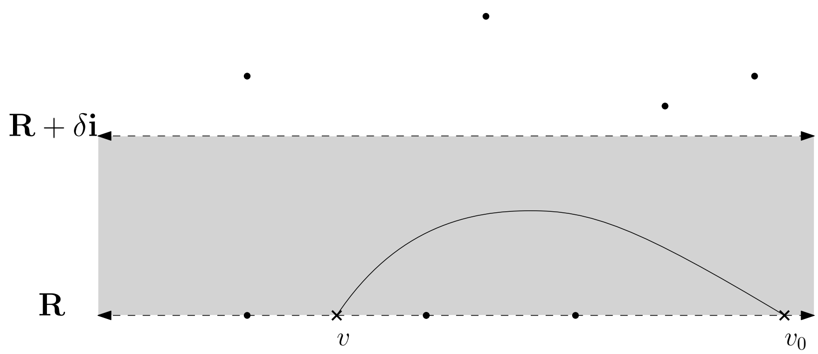


Figure 2.13: Dots represent points of $\mathcal{M}$. We extend from v_0 to v along a path that remains in the grey strip.

which means that $z_{\mathfrak{p}}(x, v_3)$ is a root of (2.7.16) for $j = 0$. By Property (ii), this implies $z_{\mathfrak{p}}(x, v)$ is a root of (2.7.16) for $j = 0$ for all $v > 0$.

For property (iv), observe that the set of root functions $z_{\mathfrak{p}}(x, v)$ of (2.7.16) for $j \neq 0$ is exactly the set of all roots of (2.7.16) for $j \in [[1, p-1]]$ contained in $\mathcal{D}_x^{\mathbb{C}}$. Indeed, this is the case as $v \to +\infty$, and by property (iii) this is therefore the case for all $v > 0$. If $z_{\mathfrak{p}}(x, v) \in \mathcal{D}_x^{\mathbb{C}} \setminus \mathbb{R}$ for some $\mathfrak{p} \in \mathfrak{Z}_{x,1}$, then $\overline{z_{\mathfrak{p}}(x, v)} \in \mathcal{D}_x^{\mathbb{C}}$ is a root of (2.7.16) for some $j \in [[1, p-1]]$. Property (iv) follows from this fact; note that the double root case is a limiting case. $\square$

Given Lemma 2.7.11, we may define the sub(multi)set $\mathfrak{Z}_{x,0}$ of $\mathfrak{Z}_x$ consisting of $\mathfrak{p}$ such that $z_{\mathfrak{p}}(x, v)$ is a root of (2.7.16) for $j = 0$. Similarly, let $\mathfrak{Z}_{x,1}$ be the sub(multi)set of $\mathfrak{Z}_x$ consisting of $\mathfrak{q}$ such that $z_{\mathfrak{q}}(x, v)$ is a root of (2.7.16) for $j \in [[1, p-1]]$.

We rewrite (2.7.17) as

$$\Upsilon_x(v) = \log\left(1 - \frac{\mathcal{G}_x(0)^p}{v^p}\right) - \sum_{\mathfrak{p} \in \mathfrak{Z}_{x,0}} (\log z_{\mathfrak{p}}(x, v) - \log \mathfrak{p}) - \sum_{\mathfrak{p} \in \mathfrak{Z}_{x,1}} (\log z_{\mathfrak{p}}(x, v) - \log \mathfrak{p}).$$

By property (iv) of Lemma 2.7.11, this expression for $\Upsilon_x(v)$ implies

$$\Im_+ \Upsilon_x(v) = \pi \cdot \mathbf{1}_{v < \mathcal{G}_x(0)} - \sum_{\mathfrak{p} \in \mathfrak{Z}_{x,0}} \arg z_{\mathfrak{p}}(x, v). \tag{2.7.20}$$

The argument branch is taken so that $\arg z_{\mathfrak{p}}(x, v) = 0$ for v sufficiently large. This formula is valid except where $v = \mathcal{G}_x(0)$ and $z_{\mathfrak{p}}(x, v) = 0$ for some $\mathfrak{p} \in \mathfrak{Z}_{x,0}$. Since we are evaluating the density of a measure, these finitely many exceptional points are immaterial.

By our choice of $\widetilde{\zeta}$, we have $\mathfrak{p}_0 = \min \mathcal{J}_x \in \mathfrak{Z}_{x,0}$ such that $\widetilde{\zeta}(x, y) = z_{\mathfrak{p}_0}(x, v)$ for some $v > 0$ where $e^{-y} = v$. From Remark 10, we have that $z_{\mathfrak{p}_0}(x, v_0) = \widetilde{\zeta}(x, y_0)$ at some point $v_0 = e^{-y_0}$ where $\widetilde{\zeta}(x, y) \in \mathbb{H}$; recall that nonreal pair of roots first appear (as we increase y) when the two roots in the external component coalesce. Moreover, we can find y_0 sufficiently negative so that the other $z_{\mathfrak{p}}(x, v)$ for $\mathfrak{p} \in \mathfrak{Z}_x \setminus \{\mathfrak{p}_0\}$ are confined in their respective components (which are not the external component). We note that $z_{\mathfrak{p}_0}(x, v_0) = \widetilde{\zeta}(x, y_0)$, rather than its conjugate, because we must have

$0 \leq \frac{1}{\pi}\Im_+\Upsilon_x = \rho_x \leq 1$ by the positivity of $\partial_y \mathcal{H}$ and 1-Lipschitz condition on $\mathcal{H}$. Define the set

$$\mathscr{C} = \{(x,y) \in \mathscr{L} : y \in \mathbb{R}, \widetilde{\zeta}(x,y) = z_{\mathfrak{p}}(x, e^{-y}) \text{ for some } \mathfrak{p} \in \mathfrak{Z}_{x,0}, \\ \text{and } \widetilde{\zeta}(x,y) \neq \overline{z_{\mathfrak{p}}}(x,e^{-y}) \text{ for all } \mathfrak{p} \in \mathfrak{Z}_{x,0}\}.$$

Since $(x, y_0) \in \mathscr{C}$, we have $\mathscr{C}$ is nonempty. We use a connectedness argument to show that $\mathscr{C} = \mathscr{L}$. Since the measures ν_x are compactly supported and bounded, and the moments depend continuously in x, we have that the Stieltjes transforms $S_{\nu_x}(v)$ are continuous in (x,v) for $x \in I$ and $v \in \mathbb{C} \setminus \mathbb{R}$. Then $\Upsilon_x(v)$ is continuous in (x,v) for $x \in I$ and $v \in \{|\arg z| < 2\pi/p\} \setminus \operatorname{supp} \nu_x$. By the form (2.7.18), this continuity can be extended to (x,v) where $v \in \mathbb{R}_{>0}$ as long as $z_{\mathfrak{p}}(x,v) \neq 0$ for all $\mathfrak{p} \in \mathfrak{Z}_x$ and $v \neq \mathcal{G}_x(0)$ (which is generically the case). It follows that $\mathscr{C}$ must be both an open and closed subset of $\mathscr{L}$, where the open condition is immediate and the closed condition follows from continuity. Since we showed that $\mathscr{C}$ is nonempty, we deduce that $\mathscr{C} = \mathscr{L}$ since $\mathscr{L}$ is homeomorphic to the simply connected set $\mathbb{H}$ by Theorem 2.6.3. With this fact and (2.7.20), we have (for $v = e^{-y}$)

$$\frac{1}{\pi}\Im_+\Upsilon_x(v) = \mathbf{1}_{v<\mathcal{G}_x(0)} - \frac{1}{\pi}\arg\widetilde{\zeta}(x, -\log v) \tag{2.7.21}$$

where the argument is chosen so that $\arg z = 0$ when $z > 0$. Indeed, the indicator takes value 1 whenever $\widetilde{\zeta}$ is nonreal (see the discussion following Remark 14) and we require $0 \leq \frac{1}{\pi}\Upsilon_x \leq 1$, so this determines the choice of argument. Changing variables, we have

$$\partial_y\mathcal{H}(x,y) = \mathbf{1}_{e^{-y}<\mathcal{G}_x(0)} - \frac{1}{\pi}\arg\widetilde{\zeta}(x,y).$$

Computation of Density for $\partial_x\mathcal{H}(x,y)$. We follow a similar argument, and have already done most of the work for it. Notice that

$$\frac{\partial_x\mathcal{G}_x(z)}{\mathcal{G}_x(z)} = \frac{1}{p}\sum_{\sigma\in\mathcal{S}}\frac{\sigma e^{-x}z}{1-\sigma e^{-x}z} \tag{2.7.22}$$

so that by (2.7.8), we have

$$\int \partial_x\mathcal{H}(x,y)e^{-ky} = \frac{1}{2\pi\mathbf{i}}\oint_{\mathcal{C}}\frac{\mathcal{G}_x(z)^{k-1}\partial_x\mathcal{G}_x(z)}{kz}\,dz = \frac{1}{p}\sum_{\sigma\in\mathcal{S}}\frac{1}{2\pi\mathbf{i}}\oint_{\mathcal{C}}\frac{\mathcal{G}_x(z)^k}{k}\frac{\sigma e^{-x}}{1-\sigma e^{-x}z}\,dz \tag{2.7.23}$$

Recall that we had split $\mathcal{C}$ into γ and γ_0 before in the computation of $\partial_y\mathcal{H}(x,y)$. This time, we can simply deform $\mathcal{C}$ to γ which encircle $\mathfrak{P}_x$ and not 0 since there is no pole at 0. Let $\xi_x = E(\partial_x\mathcal{H}(x,\cdot))$. As before, we have for large v,

$$S_{\xi_x}(v) = -\sum_{k=1}^{\infty}\frac{1}{p}\sum_{\sigma\in\mathcal{S}}\frac{1}{2\pi\mathbf{i}}\oint_{\gamma}\frac{\mathcal{G}_x(z)^k}{v^k k}\frac{\sigma e^{-x}}{1-\sigma e^{-x}z}\,dz = \frac{1}{p}\sum_{\sigma\in\mathcal{S}}\frac{1}{2\pi\mathbf{i}}\oint_{\gamma}\log\left(1-\frac{\mathcal{G}_x(z)}{v}\right)\frac{\sigma e^{-x}}{1-\sigma e^{-x}z}\,dz$$

We consider, also for large v,

$$\begin{aligned}\Xi_x(v) &= \frac{1}{p}\sum_{\sigma\in\mathcal{S}}\frac{1}{2\pi\mathbf{i}}\oint_\gamma \log\left(1-\frac{\mathcal{G}_x(z)^p}{v^p}\right)\frac{\sigma e^{-x}}{1-\sigma e^{-x}z}\,dz \\ &= \frac{1}{p}\sum_{\sigma\in\mathcal{S}}\frac{1}{2\pi\mathbf{i}}\oint_\gamma \log(1-\sigma e^{-x}z)\frac{d}{dz}\log\left(1-\frac{\mathcal{G}_x(z)^p}{v^p}\right)\,dz.\end{aligned} \tag{2.7.24}$$

As before with $\Upsilon_x(v)$, we can extend $\Xi_x(v)$ to $\{|\arg z|<2\pi/p\}\setminus\operatorname{supp}\xi_x$, and we have an analogue of Lemma 2.7.8

$$\frac{1}{\pi}\Im_+\Xi_x(v) = \frac{1}{\pi}\Im_+ S_{\xi_x}(v) = \varrho(v) \tag{2.7.25}$$

where $\varrho(v)$ is the density of ξ_x with respect to Lebesgue. From (2.7.24), we have the following formula

$$\Xi_x(v) = \frac{1}{p}\sum_{\sigma\in\mathcal{S}}\sum_{\mathfrak{p}\in\mathfrak{Z}_x}(\log(1-\sigma e^{-x}z_{\mathfrak{p}}(x,v)) - \log(1-\sigma e^{-x}\mathfrak{p})) \tag{2.7.26}$$

for $v>0$ where $z_{\mathfrak{p}}(x,v)$ are defined as in Lemma 2.7.11. The log branches here are chosen so that $\log(1-\sigma e^{-x}z_{\mathfrak{p}}(x,v)) - \log(1-\sigma e^{-x}\mathfrak{p}) = 0$ for v large (so that $z_{\mathfrak{p}}(x,v)$ is near $\mathfrak{p}$). Define for $\sigma\in\mathcal{S}$

$$f_\sigma(x,v) = \sum_{\mathfrak{p}\in\mathfrak{Z}_{x,1}}(\arg(1-\sigma e^{-x}z_{\mathfrak{p}}(x,v)) - \arg(1-\sigma e^{-x}\mathfrak{p})) \tag{2.7.27}$$

$$g_\sigma(x,v) = \sum_{\mathfrak{p}\in\mathfrak{Z}_{x,0}}(\arg(1-\sigma e^{-x}z_{\mathfrak{p}}(x,v)) - \arg(1-\sigma e^{-x}\mathfrak{p})) \tag{2.7.28}$$

so that

$$\Im_+\Xi_x(v) = \frac{1}{p}\sum_{\sigma\in\mathcal{S}}(f_\sigma(x,v)+g_\sigma(x,v)).$$

Given the following lemma, we can compute that

$$\frac{1}{\pi}\Im_+\Xi_x(e^{-y}) = \frac{1}{\pi}\sum_{\sigma\in\mathcal{S}}\frac{1}{p}\log\left(1-\sigma^{-1}e^x\widetilde{\zeta}(x,y)\right).$$

Changing variables gives us $\partial_x\mathcal{H}(x,y)$ and completes the proof of the Theorem.

Lemma 2.7.12. *For any $\sigma\in\mathcal{S}$, both sets $\{y\in\mathbb{R} : f_\sigma(x,e^{-y})=0\}$ and $\{y\in\mathbb{R} : g_\sigma(x,e^{-y}) = \arg(1-\sigma e^{-x}\widetilde{\zeta}(x,y))\}$ are equal to $\mathbb{R}$.*

Proof. We use a connectedness argument. Fix $\sigma\in\mathcal{S}$, let $A_1 = \{y\in\mathbb{R} : f_\sigma(x,e^{-y})=0\}$ and $A_2 = \{y\in\mathbb{R} : g_\sigma(x,e^{-y}) = \arg(1-\sigma e^{-x}\widetilde{\zeta}(x,y))\}$. Both A_1 and A_2 are nonempty and closed subsets of $\mathbb{R}$. It suffices to prove that A_1 and A_2 are open subsets of $\mathbb{R}$.

Suppose $y_0\in A_1$. We show that $[y_0,y_0+\delta)\subset A_1$ for some $\delta>0$. The argument for $(y_0-\delta,y_0]$ is identical, and so this shows that A_1 is open. Let $\mathfrak{p}\in\mathfrak{Z}_{x,1}$. If $z_{\mathfrak{p}}(x,e^{-y})$ is real for $y\in[y_0,y_0+\delta)$ for some $\delta > 0$, then $z_{\mathfrak{p}}(x, e^{-y})$ must be in some connected component of $\widehat{\mathbb{R}} \setminus \mathcal{J}_x$. Note that it cannot be in the external component since $\arg \mathcal{G}_x = 0$ on the external component. Then $\arg(1 - \sigma^{-1} z_{\mathfrak{p}}(x, e^{-y}))$ is constant on $[y_0, y_0 + \delta)$.

Otherwise, for sufficiently small $\delta > 0$, we have $z_{\mathfrak{p}}(x, e^{-y}) \in \mathbb{C} \setminus \mathbb{R}$ for all $y \in (y_0, y_0 + \delta)$. This means that $z_{\mathfrak{p}}(x, e^{-y_0})$ is either a nonreal root of (2.7.16) or a double root. In either case, by property (iv) of Lemma 2.7.11, there exists some $\mathfrak{q} \in \mathfrak{Z}_{x,1}$ different from $\mathfrak{p}$ such that $z_{\mathfrak{q}}(x, e^{-y_0}) = \overline{z_{\mathfrak{p}}}(x, e^{-y_0})$. If $z_{\mathfrak{p}}(x, e^{-y}) \in \mathbb{C} \setminus \mathbb{R}$ for $y \in (y_0, y_0 + \delta)$, then $z_{\mathfrak{q}}(x, e^{-y}) = \overline{z_{\mathfrak{p}}}(x, e^{-y})$ for $y \in (y_0, y_0 + \delta)$ by continuity and property (i) of Lemma 2.7.11. Then

$$\begin{aligned}&(\arg(1 - \sigma e^{-x} z_{\mathfrak{p}}(x, y)) - \arg(1 - \sigma e^{-x} z_{\mathfrak{p}}(x, y_0))) \\ &\quad + (\arg(1 - \sigma e^{-x} z_{\mathfrak{q}}(x, y)) - \arg(1 - \sigma e^{-x} z_{\mathfrak{q}}(x, y_0))) = 0\end{aligned}$$

for $y \in (y_0, y_0 + \delta)$. Summing over all $\mathfrak{p} \in \mathfrak{Z}_{x,1}$, we see that $f_\sigma(x, e^{-y}) - f_\sigma(x, e^{-y_0}) = 0$ for $y \in [y_0, y_0 + \delta)$. Thus A_1 is open.

The argument for the openness of A_2 is similar in structure. Before proceeding, note that from the argument that $\mathscr{C} = \mathscr{L}$ in the computation of $\partial_y \mathcal{H}(x, y)$, we have that $(-\infty, y'] \subset A_2$ where y' is the minimal y such that (2.6.2) has a double root. We have that $\widetilde{\zeta}(x, y)$ is contained in the external component of $\widehat{\mathbb{R}} \setminus \mathcal{J}_x$ if and only if $y \in (-\infty, y']$ (see Remark 11).

Take $y_0 \in A_2 \setminus (-\infty, y')$. As before, we show that $[y_0, y_0 + \delta) \subset A_2$ for some $\delta > 0$, and the argument for $(y_0 - \delta, y_0]$ is identical for $y_0 \in A_2 \setminus (-\infty, y']$. If $z_{\mathfrak{p}}(x, e^{-y})$ is real for $y \in [y_0, y_0 + \delta)$, then it must be confined to some component of $\widehat{\mathbb{R}} \setminus \mathcal{J}_x$ which is not the external component. As before, this implies $\arg(1 - \sigma e^{-x} z_{\mathfrak{p}}(x, e^{-y}))$ is constant for $y \in [y_0, y_0 + \delta)$.

Otherwise, there exists $\delta > 0$ small enough so that $z_{\mathfrak{p}}(x, e^{-y}) \in \mathbb{C} \setminus \mathbb{R}$ for $y \in (y_0, y_0 + \delta)$. Since $\mathscr{C} = \mathscr{L}$, this corresponds exactly to the case where $\widetilde{\zeta}(x, y) \in \mathbb{C} \setminus \mathbb{R}$, for $y \in (y_0, y_0 + \delta)$. Moreover, we must have $\widetilde{\zeta}(x, y) = z_{\mathfrak{p}}(x, e^{-y})$ for some $\mathfrak{p} \in \mathfrak{Z}_{x,0}$ and $z_{\mathfrak{q}}(x, e^{-y})$ are real for $\mathfrak{q} \in \mathfrak{Z}_{x,0}$, $\mathfrak{q} \neq \mathfrak{p}$ for $y \in [y_0, y_0 + \delta)$. In particular, $z_{\mathfrak{q}}(x, e^{-y})$ are confined to some junction component of $\widehat{\mathbb{R}} \setminus \mathcal{J}_x$ which is not the external component. Then

$$g_\sigma(x, e^{-y}) - g_\sigma(x, e^{-y_0}) = -\arg(1 - \sigma e^{-x} \widetilde{\zeta}(x, y)) + \arg(1 - \sigma e^{-x} \widetilde{\zeta}(x, y_0)).$$

Thus A_2 is open. $\square$

2.7.2 Fluctuations: Proof of Theorem 2.7.4

Proof of Theorem 2.2.8. For $x \in I$ and $k \in \mathbb{Z}_{>0}$, by Proposition 2.3.1 we have for $x(\varepsilon) \in I^\varepsilon$

$$\begin{aligned}\sqrt{\alpha\pi} \int \left(h\left(x(\varepsilon), \frac{\mathbf{y}}{\varepsilon}\right) - \mathbb{E} h\left(x(\varepsilon), \frac{\mathbf{y}}{\varepsilon}\right) \right) e^{-k\mathfrak{t}\mathbf{y}} \, d\mathbf{y} &= \frac{\varepsilon \sqrt{\pi} t^{kB(x)}}{\sqrt{\alpha} k^2 (\log t)^2} \left(\wp_k(\pi^x; q, t) - \mathbb{E} \wp_k(\pi^x; q, t) \right) \\ &= \frac{\sqrt{\pi} t^{kB(x)}}{\sqrt{\alpha} k^2 \mathfrak{t}^2 \varepsilon} \left(\wp_k(\pi^x; q, t) - \mathbb{E} \wp_k(\pi^x; q, t) \right).\end{aligned}$$

Let $\mathbf{x}_1 \leq \cdots \leq \mathbf{x}_m$ in I and $k_1, \ldots, k_m \in \mathbb{Z}_{>0}$. Let $x_1(\varepsilon) \leq \cdots \leq x_m(\varepsilon)$ be points in I^ε such that $x_i(\varepsilon)$ is $k_i\mathfrak{t}$-separated from singular points. By Theorem 2.5.3, the random vector indexed by

$i \in [[1,m]]$ whose components are given by

$$\sqrt{\alpha\pi}\int \left(h\left(x_i(\varepsilon),\frac{\mathbf{y}}{\varepsilon}\right)-\mathbb{E}h\left(x_i(\varepsilon),\frac{\mathbf{y}}{\varepsilon}\right)\right)e^{-k_i\mathbf{t}\mathbf{y}}\,d\mathbf{y}$$

converges to the Gaussian vector whose ith and jth component for $i<j$ has the covariance

$$\frac{\pi}{(2\pi\mathbf{i})^2k_ik_j\mathbf{t}^2}\oint_{\mathcal{C}_j}\oint_{\mathcal{C}_i}\frac{\mathcal{G}_{\mathbf{x}_i}(z)^{k_i\mathbf{t}}\mathcal{G}_{\mathbf{x}_j}(w)^{k_j\mathbf{t}}}{(z-w)^2}dz\,dw \tag{2.7.29}$$

where $\mathcal{C}_1,\ldots,\mathcal{C}_m$ are contours meeting the criteria in Theorem 2.5.3, in particular $\mathcal{C}_i$ is enclosed by $\mathcal{C}_j$ for $i<j$ and the set of branch poles of $\mathcal{G}_{x_i}$ enclosed by $\mathcal{C}_i$ is exactly $\mathfrak{P}_{x_i}$ as defined in (2.7.10).

Suppose $\mathbf{x}_i<\mathbf{x}_j$ so that the contours $\mathcal{C}_i$ and $\mathcal{C}_j$ are separated. By Lemma 2.7.5, we can deform $\mathcal{C}_i$ to the boundary of $\mathcal{D}^{\mathbb{C}}_{\mathbf{x}_i}$ in (2.7.29). From (2.7.7), we can express the boundary as the union of

$$\{\zeta(\mathbf{x}_i,\mathbf{y}):\mathbf{y}\in\mathbb{R},(\mathbf{x}_i,\mathbf{y})\in\mathscr{L}\}\cup\{\overline{\zeta(\mathbf{x}_i,\mathbf{y})}:\mathbf{y}\in\mathbb{R},(\mathbf{x}_i,\mathbf{y})\in\mathscr{L}\}$$

up to finitely many points in the set difference. Let $Y_{\mathbf{x}_i}=\{\mathbf{y}\in\mathbb{R}:(\mathbf{x}_i,\mathbf{y})\in\mathscr{L}\}$ which is a union of finitely many open intervals. Parametrizing the boundary of $\mathcal{D}^{\mathbb{C}}_{\mathbf{x}_i}$ by $\zeta(\mathbf{x}_i,\mathbf{y})$ and its conjugate, we may rewrite (2.7.29) as the sum

$$\begin{aligned}
&-\frac{1}{4\pi k_ik_j\mathbf{t}^2}\int_{Y_{\mathbf{x}_j}}\int_{Y_{\mathbf{x}_i}}\frac{e^{-k_i\mathbf{t}\mathbf{y}_1}e^{-k_j\mathbf{t}\mathbf{y}_2}}{(\zeta(\mathbf{x}_i,\mathbf{y}_1)-\zeta(\mathbf{x}_j,\mathbf{y}_2))^2}\frac{\partial\zeta}{\partial\mathbf{y}_1}(\mathbf{x}_i,\mathbf{y}_1)\frac{\partial\zeta}{\partial\mathbf{y}_2}(\mathbf{x}_j,\mathbf{y}_2)\,d\mathbf{y}_1\,d\mathbf{y}_2\\
&+\frac{1}{4\pi k_ik_j\mathbf{t}^2}\int_{Y_{\mathbf{x}_j}}\int_{Y_{\mathbf{x}_i}}\frac{e^{-k_i\mathbf{t}\mathbf{y}_1}e^{-k_j\mathbf{t}\mathbf{y}_2}}{(\zeta(\mathbf{x}_i,\mathbf{y}_1)-\overline{\zeta}(\mathbf{x}_j,\mathbf{y}_2))^2}\frac{\partial\zeta}{\partial\mathbf{y}_1}(\mathbf{x}_i,\mathbf{y}_1)\frac{\partial\overline{\zeta}}{\partial\mathbf{y}_2}(\mathbf{x}_j,\mathbf{y}_2)\,d\mathbf{y}_1\,d\mathbf{y}_2\\
&+\frac{1}{4\pi k_ik_j\mathbf{t}^2}\int_{Y_{\mathbf{x}_j}}\int_{Y_{\mathbf{x}_i}}\frac{e^{-k_i\mathbf{t}\mathbf{y}_1}e^{-k_j\mathbf{t}\mathbf{y}_2}}{(\overline{\zeta}(\mathbf{x}_i,\mathbf{y}_1)-\zeta(\mathbf{x}_j,\mathbf{y}_2))^2}\frac{\partial\overline{\zeta}}{\partial\mathbf{y}_1}(\mathbf{x}_i,\mathbf{y}_1)\frac{\partial\zeta}{\partial\mathbf{y}_2}(\mathbf{x}_j,\mathbf{y}_2)\,d\mathbf{y}_1\,d\mathbf{y}_2\\
&-\frac{1}{4\pi k_ik_j\mathbf{t}^2}\int_{Y_{\mathbf{x}_j}}\int_{Y_{\mathbf{x}_i}}\frac{e^{-k_i\mathbf{t}\mathbf{y}_1}e^{-k_j\mathbf{t}\mathbf{y}_2}}{(\overline{\zeta}(\mathbf{x}_i,\mathbf{y}_1)-\overline{\zeta}(\mathbf{x}_j,\mathbf{y}_2))^2}\frac{\partial\overline{\zeta}}{\partial\mathbf{y}_1}(\mathbf{x}_i,\mathbf{y}_1)\frac{\partial\overline{\zeta}}{\partial\mathbf{y}_2}(\mathbf{x}_j,\mathbf{y}_2)\,d\mathbf{y}_1\,d\mathbf{y}_2.
\end{aligned} \tag{2.7.30}$$

Integrate by parts on $\mathbf{y}_1$ and $\mathbf{y}_2$ for each summand in (2.7.30), observing that the boundary terms cancel since the value of $\zeta(\mathbf{x},\cdot)$ at the end points of any connected component of $Y_{\mathbf{x}}$ is real, to obtain

$$\begin{aligned}
&-\frac{1}{4\pi}\int_{Y_{\mathbf{x}_j}}\int_{Y_{\mathbf{x}_i}}e^{-k_i\mathbf{t}\mathbf{y}_1}e^{-k_j\mathbf{t}\mathbf{y}_1}\big(\log(\zeta(\mathbf{x}_i,\mathbf{y}_1)-\zeta(\mathbf{x}_j,\mathbf{y}_1))-\log(\zeta(\mathbf{x}_i,\mathbf{y}_1)-\overline{\zeta}(\mathbf{x}_j,\mathbf{y}_1))\\
&\qquad-\log(\overline{\zeta}(\mathbf{x}_i,\mathbf{y}_1)-\zeta(\mathbf{x}_j,\mathbf{y}_1))+\log(\overline{\zeta}(\mathbf{x}_i,\mathbf{y}_1)-\overline{\zeta}(\mathbf{x}_j,\mathbf{y}_1))\big)\,d\mathbf{y}_1\,d\mathbf{y}_1\\
&\qquad\qquad=-\frac{1}{2\pi}\int_{Y_{\mathbf{x}_j}}\int_{Y_{\mathbf{x}_i}}e^{-k_i\mathbf{t}\mathbf{y}_1}e^{-k_j\mathbf{t}\mathbf{y}_1}\log\left|\frac{\zeta(\mathbf{x}_i,\mathbf{y}_1)-\zeta(\mathbf{x}_j,\mathbf{y}_1)}{\zeta(\mathbf{x}_i,\mathbf{y}_1)-\overline{\zeta}(\mathbf{x}_j,\mathbf{y}_1)}\right|
\end{aligned}$$

where the final equality follows from the fact that this covariance is real so there is no imaginary part arising from branch considerations of the logarithms. The latter term is exactly the covariance

$$\mathrm{Cov}\left(\int \mathfrak{H}(\zeta(\mathbf{x}_i,\mathbf{y}))e^{-k_i \mathbf{t}\mathbf{y}}\,d\mathbf{y}, \int \mathfrak{H}(\zeta(\mathbf{x}_j,\mathbf{y}))e^{-k_j \mathbf{t}\mathbf{y}}\,d\mathbf{y}\right).$$

The case $\mathbf{x}_i = \mathbf{x}_j$ follows from taking the limit. $\square$

2.8 Appendix

We recall the notion of cumulants and some basic properties.

Definition 2.8.1. For any positive integer ν, let Θ_ν be the collection of all set partitions of $[[1,\nu]]$, namely

$$\Theta_\nu = \left\{ \{U_1,\dots,U_d\} : d>0,\ \bigcup_{i=1}^{d} U_i = [[1,\nu]],\ U_i \cap U_j = \emptyset\ \forall i \neq j,\ U_i \neq \emptyset\ \forall i \in [[1,d]] \right\}.$$

For a random vector $\boldsymbol{u} = (u_1,\dots,u_m)$ and any $v_1,\dots,v_\nu \in \{u_1,\dots,u_m\}$, define the *(order ν) cumulant* $\kappa(v_1,\dots,v_\nu)$ as

$$\kappa(v_1,\dots,v_\nu) = \sum_{\substack{d>0 \\ \{U_1,\dots,U_d\}\in\Theta_\nu}} (-1)^{d-1}(d-1)! \prod_{\ell=1}^{d} \mathbb{E}\left[\prod_{i\in U_\ell} v_i\right]. \tag{2.8.1}$$

From the definition we see that for any random vector $\boldsymbol{u}$, the existence of all cumulants of order up to ν is equivalent to the existence of all moments of order up to ν. Note that the cumulants of order 2 are exactly the covariances:

$$\kappa(v_1,v_2) = \mathrm{Cov}(v_1,v_2).$$

We have the following alternative definition for cumulants.

Definition 2.8.2. Let $\boldsymbol{u} = (u_1,\dots,u_m)$ be a random vector. For any $v_1,\dots,v_\nu \in \{u_1,\dots,u_m\}$, define the *(order ν) cumulant* $\kappa(v_1,\dots,v_\nu)$ as

$$\kappa(v_1,\dots,v_\nu) = (-i)^\nu \frac{\partial^\nu}{\partial t_1 \cdots \partial t_m} \log \mathbb{E}\left[\exp\left(\mathbf{i}\sum_{j=1}^{\nu} t_j v_j\right)\right]\Bigg|_{t_1=\cdots=t_\nu=0}. \tag{2.8.2}$$

For further details see [56, Section 3.1, Section 3.2] wherein the agreement between Definitions 2.8.1 and 2.8.2 is shown by taking the second definition and proving (2.8.1).

As a consequence of the second definition we have the following lemma.

Lemma 2.8.3. *A random vector is Gaussian if and only if all cumulants of order ≥ 3 vanish.*

We have the following formal versions of (2.8.1) and (2.8.2). Let $E_{n_1,\ldots,n_\nu} \in \mathbb{C}$ with $E_{0,\ldots,0} = 1$. Define the following formal power series

$$\begin{aligned}
E(t_1,\ldots,t_\nu) &= \sum_{n_1,\ldots,n_\nu \geq 0} \frac{E_{n_1,\ldots,n_\nu}}{n_1!\cdots n_\nu!} t_1^{n_1}\cdots t_\nu^{n_\nu} \\
K(t_1,\ldots,t_\nu) &= \log E(t_1,\ldots,t_\nu) =: \sum_{n_1,\ldots,n_\nu \geq 0} \frac{K_{n_1,\ldots,n_\nu}}{n_1!\cdots n_\nu!} t_1^{n_1}\cdots t_\nu^{n_\nu}.
\end{aligned}$$

Let $E(U) = E_{n_1,\ldots,n_\nu}$ where $n_j = 1$ if $j \in U$ and 0 otherwise, and likewise define $K(U)$. Letting Θ_U be the collection of all set partitions of U, we have

$$K(U) = \sum_{\substack{d>0 \\ \{U_1,\ldots,U_d\}\in\Theta_U}} (-1)^{d-1}(d-1)! \prod_{\ell=1}^{d} E(U_\ell) \tag{2.8.3}$$

By exponentiating $K(t_1,\ldots,t_\nu)$ we also obtain

$$E(U) = \sum_{\substack{d>0 \\ \{U_1,\ldots,U_d\}\in\Theta_U}} \prod_{\ell=1}^{d} K(U_\ell) \tag{2.8.4}$$

where the sum is over all set partitions of U. This gives us the following lemma.

Lemma 2.8.4. *Suppose that K and E are functions which take values on nonempty subsets of $[[1,\nu]]$. Then*

$$E(U) = \sum_{\substack{d>0 \\ \{U_1,\ldots,U_d\}\in\Theta_U}} \prod_{i=1}^{d} K(U_i) \tag{2.8.5}$$

if and only if

$$K(U) = \sum_{\substack{d>0 \\ \{U_1,\ldots,U_d\}\in\Theta_U}} (-1)^{d-1}(d-1)! \prod_{i=1}^{d} E(U_i). \tag{2.8.6}$$

Chapter 3

β-Jacobi Product Process

3.1 Introduction

Let $\mathbb{F}_\beta$ denote the real, complex, quaternion skew field and $\mathbb{U}_\beta(L)$ denote the L-dimensional orthogonal, unitary, symplectic group for $\beta = 1, 2, 4$ respectively. In this chapter, we study Markov chains $(\mathbf{y}^{(T)})_{T \in \mathbb{Z}_{>0}}$ formed by the squared singular values $\mathbf{y}^{(T)} := (y_1^{(T)} \geq \cdots \geq y_N^{(T)})$ of matrices

$$Y_T := X_1 \cdots X_T \tag{3.1.1}$$

where $X_1, X_2, \ldots$ are independent random $N \times N$ matrices and time T corresponds to the number of factors. The distributions of $X_1, X_2, \ldots$ are taken to be invariant under the right action of $\mathbb{U}_\beta(N)$, thus β indicates the symmetry class of our model. We focus on the case where the squared singular values of X_T are distributed as the β-Jacobi ensemble and refer to the resulting Markov chain $(\mathbf{y}^{(T)})_{T>0}$ as the *β-Jacobi product process*. The β-Jacobi ensemble depends on two parameters $\alpha > 0$, $M \in \mathbb{Z}_{>0}$, $M \geq N$. If we write $M = L - N' - N$, $\alpha = N' - N + 1$, then this is the squared singular value distribution of an $N' \times N$ submatrix A of a Haar distributed $\mathbb{U}_\beta(L)$ matrix. The matrix A is sometimes referred to as a *truncated orthogonal/unitary/symplectic matrix* in the literature for $\beta = 1, 2, 4$.

Through an extrapolation procedure which combines ideas from [29] and [11], we extend the notion of a β-Jacobi product process to arbitrary $\beta > 0$. The general β-Jacobi ensemble is then a distribution parametrized by $\alpha > 0$, $M \in \mathbb{Z}_{>0}$ such that $M \geq N$ on N-particles $(x_1, \ldots, x_N)$ supported in $[0,1]^N$ with density is proportional to

$$\prod_{1 \leq i < j \leq N} |x_i - x_j|^\beta \prod_{i=1}^N x_i^{(\beta/2)\alpha - 1} (1 - x_i)^{(\beta/2)(M-N+1)-1} \, dx_i \tag{3.1.2}$$

If $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \ldots$ are independent random N-vectors where $\mathbf{x}^{(T)}$ is distributed as the β-Jacobi ensemble with parameters α_T, M_T, then the β-Jacobi product process with parameters $(\alpha_T, M_T)_{T \in \mathbb{Z}_{>0}}$ is a Markov chain $(\mathbf{y}^{(T)})_{T>0}$ where the distribution of $\mathbf{y}^{(T)}$ conditioned on $\mathbf{y}^{(T-1)}$ depends only on $\mathbf{x}^{(T)}$. This dependence is a generalization to arbitrary $\beta > 0$ of the effect that matrix products within a symmetry class have on singular values.

In general, a β-ensemble (see e.g. [3, C20]) is a particle system $(x_1, \ldots, x_N)$ with density proportional to

$$\prod_{1\le i<j\le N} |x_i - x_j|^\beta \prod_{i=1}^{N} w(x_i)$$

for some weight (or potential) $w(x)$. We note that the Hermite ($w(x) = e^{-x^2/2}$), Laguerre ($w(x) = x^p e^{-x}$) and Jacobi ($w(x) = x^p(1-x)^q$) ensembles correspond to eigenvalues of classical $\mathbb{U}_\beta$-invariant matrix models (see e.g. [5, C4]). Since the former two ensembles can be realized as degenerations of the Jacobi ensemble, the Jacobi ensemble may be viewed as the most general among the classical β-ensembles.

The main objective of this chapter is to study the fluctuations of $(\mathbf{y}^{(T)})_{T>0}$ in the limit as $N \to \infty$. In particular, we consider the following two settings: (i) global fluctuations where we do not rescale time as N grows and (ii) local fluctuations at the right edge (that is, of the rightmost particles) where time grows linearly with N. Along the way, we prove a limit shape result for arbitrary $\beta > 0$ which extends known results [19, 67] for $\beta = 1, 2, 4$.

We show that the global fluctuations are described by a Gaussian process whose covariance function exhibits logarithmic correlation on short-scales. This is reminiscent of the Gaussian free field.

For local fluctuations, our results are twofold. We consider the regime where time T grows to infinity such that $\lim_{N\to\infty} T/N = \widehat{T}$ for some $\widehat{T} > 0$. First, we demonstrate that for $\beta = 2$, the fluctuations of the right edge are described by a process in time $\widehat{T}$ whose fixed-time marginals interpolate between the Airy point process as $\widehat{T} \to 0$ and a deterministic, "picket fence" configuration as $\widehat{T} \to \infty$. Second, for arbitrary $\beta > 0$, we show tightness of the point process at the right edge where we expect to see a β-generalization of the interpolating process. Moreover, we provide exponential moment formulas for this conjectural limit process.

The method's in this chapter combine ideas about Macdonald processes, β-Jacobi ensembles, and products of matrices into a unified picture [6–8, 11, 29]. Our approach involves Macdonald symmetric, Jack symmetric, Heckman-Opdam hypergeometric functions, and their correspondence [29] to product processes in order to obtain moment formulas. By the asymptotic analysis of these moment formulas, we access the fluctuations of β-Jacobi product processes.

For local fluctuations at $\beta = 2$, previous works [4, 42] established *fixed-time convergence results* to a family of point processes indexed by $\widehat{T}$ for the case of Ginibre matrices. These point processes interpolate between the Airy point process and picket fence statistics as $\widehat{T}$ ranges from 0 to ∞. Our main theorems on local fluctuations extend these results to *joint convergence across time* $\widehat{T}$ for Ginibre *and* Jacobi matrices. As a consequence, the interpolating process that appears in our work links together this $\widehat{T}$-parametrized family of point processes across time; we provide more details below. The appearance of this process is independent of the choice of Jacobi parameters. This suggests that there may be a wider universality class of products of matrices with generic distributions where this interpolating process appears. For $\beta > 0$ arbitrary, we find universality of limiting moment formulas which do not depend on the parameters of the β-Jacobi product process. We conjecture that there exists a β-generalization of the interpolating process. Under this conjecture, these moment formulas are expressions for the Laplace transforms of the generalized interpolating processes evaluated at positive integer values.

We now proceed to a more detailed discussion of the main results and methods. The main results on global fluctuations are provided in Section 3.1.1 along with some additional background. Similarly, Section 3.1.2 contains background and the main results on local fluctuations at the right edge. We conclude the introduction with a description of our methods in Section 3.1.3.

3.1.1 Global Fluctuations

Our first asymptotic regime preserves the time scaling as the number of particles N grows to ∞. We begin with some background on known limit shape and fluctuation results under this regime.

In a series of breakthrough articles (see e.g. [66, 67]), Voiculescu established that the squared singular values of products of certain large unitarily invariant random matrices concentrate around a limit shape. These results have since been generalized to include orthogonally and symplectically invariant matrix ensembles including the β-Hermite, β-Laguerre and β-Jacobi ensembles for $\beta = 1, 4$, see e.g. [19, Theorem 5.1].

Under general assumptions on unitarily invariant random matrices $X_1, X_2, \ldots$, the global fluctuations of the squared singular values of Y_T are known to be Gaussian due to Collins-Mingo-Sniady-Speicher [18, Theorems 7.9 and 8.3] via second-order freeness and due to Guionnet-Novak [33] via Schwinger-Dyson equations. An explicit form for the covariance was recently discovered by Gorin-Sun [31] who used a difference operators approach on multivariate Bessel functions. In particular, the authors found that the covariance can be identified by a Gaussian field related to the Gaussian free field in β-ensembles, discussed further below.

On a related note, a variety of authors established the Gaussianity of global fluctuations for eigenvalues of other types of product matrix ensembles. By asymptotic analysis of the Stieltjes transform, Vasilchuk [64] proved a central limit theorem for linear statistics of eigenvalues of a certain product of unitary matrices. A recent, general result of Coston-O'Rourke [20] showed Gaussian fluctuations for the eigenvalues of products of Wigner matrices.

While the fluctuations for $\beta = 2$ are well-studied, the same cannot be said for $\beta = 1, 4$. It appears that the exact form of global fluctuations for $\beta = 1, 4$ were not accessed prior to this work; see however [44, 58] for generalizations of the notion of second order freeness for orthogonal and symplectic ensembles. This gap in the literature is one of the motivations for this work.

From the perspective of β-ensembles, our work seeks to extend results on fluctuations of β-ensembles to β-product processes. Several prior works established Gaussianity of fluctuations for ordinary β-ensembles. The first work of this kind was due to Johansson [37] where Gaussian fluctuations for β-ensembles with analytic potentials are shown using loop equations. This approach was further extended in the articles of Kriecherbauer-Shcherbina [41], Borot-Guionnet [12, 13], Shcherbina [59], Borodin-Gorin-Guionnet [10], and Dimitrov-Knizel [23]. Dumitriu-Paquette [26] established global fluctuations for the general β-Jacobi ensemble through the tridiagonalization representation of this ensemble [25]. Borodin-Gorin [8] generalized this result by considering a 2-dimensional extension called the β-Jacobi corners process, named so as it extended the matrix corners process for $\beta = 1, 2, 4$. Using an approach related to ours involving Macdonald processes (discussed further in Section 3.1.3), they showed that the global fluctuations can be described by the Gaussian free field.

We now introduce some notation for our main results on global asymptotics.

Definition 3.1.1. For $\beta \in \{1, 2, 4\}$, let X_T be a right $\mathbb{U}_\beta(N)$-invariant $N \times N$ matrix with squared singular values distributed as the β-Jacobi ensemble with parameters $\alpha_T > 0, M_T \in \mathbb{Z}_{>0}$ (recall $M_T \geq N$). We say that a Markov chain $(\mathbf{y}^{(T)})_{T \in \mathbb{Z}_{>0}}$ is distributed as the *N-particle β-Jacobi product process with parameters* $(\alpha_T, M_T)_{T>0}$ if $\mathbf{y}^{(T)}$ is distributed as the squared singular value of Y_T defined by (3.1.1). As commonly done in the literature, this measure can be extended for arbitrary $\beta > 0$; the specifics of this extension are described in Section 3.3. Let

$$\mathrm{m}_{\mathbf{y}^{(T)}} := \frac{1}{N} \sum_{i=1}^{N} \delta_{y_i^{(T)}}. \tag{3.1.3}$$

We first provide the limit shape theorem for arbitrary $\beta > 0$, extending known results for $\beta = 1, 2, 4$.

Theorem 3.1.2 (Limit Shape)**.** *Suppose $(\mathbf{y}^{(T)})_{T \in \mathbb{Z}_{>0}}$ is distributed as the N-particle β-Jacobi product process with parameters $(\alpha_T := \alpha_T(N), M_T := M_T(N))_{T \in \mathbb{Z}_{>0}}$ and $\widehat{\alpha}_T \geq 0$, $\widehat{M}_T \geq 1$ such that*

$$\lim_{N \to \infty} (\alpha_T/N, M_T/N) = (\widehat{\alpha}_T, \widehat{M}_T)$$

for each $T \in \mathbb{Z}_{>0}$. Then for any positive integers k, T there exists a probability measure $\mathbf{m}_{(\widehat{\alpha}_\tau, \widehat{M}_\tau)_{\tau=1}^T}$ (independent of β) such that

$$\lim_{N \to \infty} \mathrm{m}_{\mathbf{y}^{(T)}} = \mathbf{m}_{(\widehat{\alpha}_\tau, \widehat{M}_\tau)_{\tau=1}^T}$$

weakly in probability. Moreover, we have

$$\int x^k \, d\mathbf{m}_{(\widehat{\alpha}_\tau, \widehat{M}_\tau)_{\tau=1}^T} = -\frac{1}{k} \cdot \frac{1}{2\pi \mathbf{i}} \oint \left(\frac{v}{v+1} \prod_{\tau=1}^{T} \frac{v - \widehat{\alpha}_\tau}{v - \widehat{\alpha}_\tau - \widehat{M}_\tau} \right)^k dv$$

where the contour is positively oriented around the pole at -1 but does not enclose $\widehat{\alpha}_\tau + \widehat{M}_\tau$ for $1 \leq \tau \leq T$.

Remark 15. Note that the limit of the β-Jacobi ensemble with parameters α_τ, M_τ converges to a limit measure $\mathbf{m}_{\widehat{\alpha}_\tau, \widehat{M}_\tau}$. We can express the limit of products of β-ensembles as

$$\mathbf{m}_{(\widehat{\alpha}_\tau, \widehat{M}_\tau)_{\tau=1}^T} = \mathbf{m}_{\widehat{\alpha}_1, \widehat{M}_1} \boxtimes \cdots \boxtimes \mathbf{m}_{\widehat{\alpha}_T, \widehat{M}_T}$$

where $\boxtimes$ denotes the *free multiplicative convolution* (see e.g. [66]).

Our main result for global asymptotics states that the fluctuations are Gaussian with an explicit covariance structure.

Theorem 3.1.3 (Global Fluctuations)**.** *Suppose $(\mathbf{y}^{(T)})_{T \in \mathbb{Z}_{>0}}$ is distributed as the N-particle β-Jacobi product process with parameters $(\alpha_T := \alpha_T(N), M_T := M_T(N))_{T \in \mathbb{Z}_{>0}}$ and $\widehat{\alpha}_T \geq 0$, $\widehat{M}_T \geq 1$ such that*

$$\lim_{N \to \infty} (\alpha_T/N, M_T/N) = (\widehat{\alpha}_T, \widehat{M}_T)$$

for each $T \in \mathbb{Z}_{>0}$. Then for any positive integers m, $k_1, \dots k_m$ and $T_1 \geq \ldots \geq T_m$, we have that the random vector

$$\left(\int x^{k_1}\, \mathrm{dm}_{\mathbf{y}^{(T_1)}}(x) - \mathbb{E}\int x^{k_1}\, \mathrm{dm}_{\mathbf{y}^{(T_1)}}(x), \dots, \int x^{k_m}\, \mathrm{dm}_{\mathbf{y}^{(T_m)}}(x) - \mathbb{E}\int x^{k_m}\, \mathrm{dm}_{\mathbf{y}^{(T_m)}}(x)\right) \tag{3.1.4}$$

converges in distribution to a Gaussian vector as $N \to \infty$ where the covariance between the ith and jth component is given by

$$\frac{(\beta/2)^{-1}}{(2\pi\mathbf{i})^2} \oint\oint \frac{1}{(v_2 - v_1)^2} \left(\frac{v_1}{v_1+1}\prod_{\tau=1}^{T_i} \frac{v_1 - \widehat{\alpha}_\tau}{v_1 - \widehat{\alpha}_\tau - \widehat{M}_\tau}\right)^{k_i} \left(\frac{v_2}{v_2+1}\prod_{\tau=1}^{T_j} \frac{v_2 - \widehat{\alpha}_\tau}{v_2 - \widehat{\alpha}_\tau - \widehat{M}_\tau}\right)^{k_j} dv_1\, dv_2 \tag{3.1.5}$$

where the v_2 contour encloses the v_1 contour, both the v_1, v_2 contours are positively oriented around -1, but the v_1 contour does not contain $\widehat{\alpha}_\tau + \widehat{M}_\tau$ for $1 \leq \tau \leq T_i$ and the v_2 contour does not contain $\widehat{\alpha}_\tau + \widehat{M}_\tau$ for $1 \leq \tau \leq T_j$.

We observe that the covariance depends on the symmetry class β only through a factor of β^{-1}. This is a common feature feature among β-ensembles in the literature; compare with e.g. [8]. In the case $\beta = 2$, our result intersects that of [31]. In particular, this means that the covariance can be described in terms of a Gaussian process whose covariance function is logarithmic on short-scales, as in [8, Proof of Theorem 4.13; 31, Proof of Corollary 4.10]. This is related to the appearance of the Gaussian free field in β-ensembles which has a distinguished logarithmic covariance structure given by the Green's kernel on the upper half plane.

3.1.2 Local Fluctuations for Growing Products

Our second asymptotic regime takes the number of products T and particles N to grow linearly with respect to one another. Let $\widehat{T} = \lim_{N\to\infty} T/N$. Under this regime, we study the fluctuations of the right edge (the rightmost particles).

A recent result due to Akemann-Burda-Kieburg [4] and Liu-Wang-Wang [42, Theorem 1.2] considers this asymptotic regime for squared singular values of Y_T as defined by (3.1.1) with $X_1, X_2, \dots$ taken to be $N \times N$ complex Ginibre matrices. We recall that an $N \times N$ complex Ginibre matrix is a matrix of i.i.d. complex standard Gaussians. Define $(\xi_1 := \xi_1(N) \geq \cdots \geq \xi_N := \xi_N(N))$ from the squared singular values $\mathbf{y}^{(T)} = (y_1^{(T)}, \dots, y_N^{(T)})$ of Y_T via

$$y_i^{(T)} = N^{T+1} e^{\xi_i}. \tag{3.1.6}$$

These authors showed that in the limit $N \to \infty$, the point process $\xi_1, \xi_2, \dots$ converges to a $\widehat{T}$-parametrized limiting determinantal point process [5, C4.2] $\mathfrak{x}_i^{(\widehat{T})}, \mathfrak{x}_2^{(\widehat{T})}, \dots$ whose correlation kernel is given by

$$K_{\widehat{T}}(\mathfrak{x},\mathfrak{y}) = \int_{1-\mathbf{i}\infty}^{1+\mathbf{i}\infty} \frac{ds}{2\pi\mathbf{i}} \oint \frac{dt}{2\pi\mathbf{i}} \frac{1}{s-t} \frac{\Gamma(t)}{\Gamma(s)} \frac{e^{\frac{\widehat{T}(s^2-s)}{2}-\mathfrak{y}s}}{e^{\frac{\widehat{T}(t^2-t)}{2}-\mathfrak{x}s}} \tag{3.1.7}$$

where the t-contour is to the left of 1, starting from $-\infty-\mathbf{i}\varepsilon$, positively looping around $0,-1,-2,\ldots$, and then going to $-\infty+\mathbf{i}\varepsilon$. We note that our description of this point process differs from that of Liu-Wang-Wang [42, Equation 1.8] by a translation factor of $\widehat{T}/2$. Akemann-Burda-Kieburg have a different form [4, Equation 19] for the correlation kernel of $\mathfrak{x}_1^{(\widehat{T})},\mathfrak{x}_2^{(\widehat{T})},\ldots$ where their point process differs from that of ours by a multiplicative factor of $\widehat{T}^{2/3}$ and a translation factor of $1+\log\widehat{T}$.

The following convergence statements are given in [42, Theorem 3.2] with a proof sketch. As $\widehat{T}\to 0$, the process $\zeta_1^{(\widehat{T})},\zeta_2^{(\widehat{T})},\ldots$ defined by

$$\mathfrak{x}_i^{(\widehat{T})} = 2^{-1/3}\widehat{T}^{2/3}\zeta_i^{(\widehat{T})} + 1 + \log\widehat{T}$$

converges to the Airy point process [5, p232-234] defined by the correlation kernel

$$K_{\mathrm{Airy}}(\zeta,\eta) = \frac{\mathrm{Ai}(\zeta)\mathrm{Ai}'(\eta)-\mathrm{Ai}(\eta)\mathrm{Ai}'(\zeta)}{\zeta-\eta}.$$

As $\widehat{T}\to+\infty$, we have the convergence

$$\left\{\widehat{T}^{-1}\mathfrak{x}_i^{(\widehat{T})}\right\}_{i=1}^{\infty} \to \left\{-i+\frac{1}{2}\right\}_{i=1}^{\infty}$$

to deterministic particles, referred to as *picket fence statistics* due to the associated measure being a semi-infinite set of equally spaced unit delta masses. Furthermore, as $\widehat{T}\to+\infty$, the fluctuations of the largest particle can be described by a Gaussian random variable. More precisely,

$$\widehat{T}^{-1/2}\left(\mathfrak{x}_1^{(\widehat{T})}+\frac{\widehat{T}}{2}\right) \to \text{(real) standard Gaussian random variable.}$$

While the convergence of the second largest, third largest, etc eigenvalues are not explicitly shown in [4, 42], there is good evidence (e.g. [42, Theorem 1.1]) that each should be Gaussian and a rigorous proof should be accessible through the correlation kernels. We note that Liu-Wang-Wang also consider Ginibre matrices of different rectangular sizes and show that the point process $(\mathfrak{x}_i^{(\widehat{T})})_{i=1}^{\infty}$ appears universally in this setting [42, Section 3.4].

Although the correlation kernel (3.1.7) determines $\mathfrak{x}^{(\widehat{T})} := (\mathfrak{x}_i^{(\widehat{T})})_{i=1}^{\infty}$, we use an alternative characterization via the Laplace transform. By combining our result (in particular Theorem 3.6.1) for the Ginibre case with that of [42], we obtain a formula for the Laplace transform of $\mathfrak{x}^{(\widehat{T})}$.

Theorem 3.1.4 (Formula for Laplace Transform)**.** *For $c_1, \ldots, c_m > 0$, we have*

$$\mathbb{E}\left[\prod_{i=1}^{m}\sum_{j=1}^{\infty} e^{c_i \mathfrak{x}_j^{(\widehat{T})}}\right] = \frac{(-1)^m}{(2\pi \mathbf{i})^m} \oint \cdots \oint \prod_{1\le i<j\le m} \frac{(u_j - u_i)(u_j + c_j - u_i - c_i)}{(u_j - u_i - c_i)(u_j + c_j - u_i)}$$
$$\times \prod_{i=1}^{m} \frac{\Gamma(-u_i - c_i)}{\Gamma(-u_i)} \exp\left(-\widehat{T}\frac{c_i(c_i+1)}{2} - \widehat{T}c_i u_i\right)\frac{du_i}{c_i}$$

where the u_i contour $\mathfrak{U}_i$ is a positively oriented contour around $-c_i, -c_i+1, -c_i+2, \ldots$ which starts and ends at $+\infty$, and $\mathfrak{U}_i$ is enclosed by $\mathfrak{U}_j - c_i$ for $i < j$.

Taking $\widehat{T} \to 0$ and c_i to grow linearly with $\widehat{T}^{-2/3}$, we can recover a formula for the Laplace transform of the Airy point process (see [9, Equation 14]). Taking $\widehat{T} \to +\infty$ and c_i to decay linearly with $\widehat{T}^{-1/2}$, we obtain the Laplace transform of a real Gaussian; if instead we let c_i decay as $\widehat{T}^{-1}$, we obtain the Laplace transform for picket fence statistics. In the former two cases, these limits are directly accessible from the contour integral formula. In the latter case, one must take the residue expansion.

Our main results for $\beta = 2$ consider a natural extension of $\mathfrak{x}^{(\widehat{T})} = (\mathfrak{x}_1^{(\widehat{T})}, \mathfrak{x}_2^{(\widehat{T})}, \ldots)$ across time $\widehat{T}$.

Definition 3.1.5. By an *interpolating process* (that is, interpolating between the Airy point process and picket fence statistics), we mean a process $(\mathfrak{x}^{(\widehat{T})})_{\widehat{T}>0} := (\mathfrak{x}_i^{(\widehat{T})})_{\widehat{T}>0, i\in\mathbb{Z}_{>0}}$ defined by the joint Laplace transform

$$\mathbb{E}\left[\prod_{i=1}^{m}\sum_{j=1}^{\infty} e^{c_i \mathfrak{x}_j^{(\widehat{T}_j)}}\right] = \frac{(-1)^m}{(2\pi \mathbf{i})^m} \oint \cdots \oint \prod_{1\le i<j\le m} \frac{(u_j - u_i)(u_j + c_j - u_i - c_i)}{(u_j - u_i - c_i)(u_j + c_j - u_i)}$$
$$\times \prod_{i=1}^{m} \frac{\Gamma(-u_i - c_i)}{\Gamma(-u_i)} \exp\left(-\widehat{T}_i\frac{c_i(c_i+1)}{2} - \widehat{T}_i c_i u_i\right)\frac{du_i}{c_i}$$

where $\widehat{T}_1 \ge \cdots \ge \widehat{T}_m > 0$, $c_1, \ldots, c_m > 0$, the u_i contour $\mathfrak{U}_i$ is a positively oriented contour around $-c_i, -c_i+1, -c_i+2, \ldots$ which starts and ends at $+\infty$, and $\mathfrak{U}_i$ is enclosed by $\mathfrak{U}_j - c_i$ for $i < j$.

Indeed, Theorem 3.1.4 implies that the distribution of $(\mathfrak{x}^{(\widehat{T})})_{\widehat{T}>0}$ for a fixed time slice $\widehat{T}$ is exactly the point process with correlation kernel (3.1.7). Given a vector $U = (u_1, \ldots, u_k)$, we write $e^U := (e^{u_1}, \ldots, e^{u_k})$. We now state the main results for local fluctuations for $\beta = 2$.

Theorem 3.1.6 ($\beta = 2$ Ginibre, Edge Fluctuations)**.** *Suppose $(\mathbf{y}^{(T)})_{T\in\mathbb{Z}_{>0}}$ are the squared singular values of Y_T defined as in* (3.1.1) *where $X_1, X_2, \ldots$ are independent $N\times N$ complex Ginibre matrices. Then*

$$\lim_{N\to\infty} \log\left(\frac{1}{N^{\lfloor N\widehat{T}\rfloor+1}}\mathbf{y}^{(\lfloor N\widehat{T}\rfloor)}\right) = \mathfrak{x}^{(\widehat{T})}$$

in finite dimensional distributions across time $\widehat{T} > 0$.

Theorem 3.1.7 ($\beta = 2$ Jacobi, Edge Fluctuations)**.** *Suppose $(\mathbf{y}^{(T)})_{T\in\mathbb{Z}_{>0}}$ are the squared singular values of Y_T defined as in (3.1.1) where $X_1, X_2, \ldots$ are independent $N \times N$ matrices with squared singular values distributed as the ($\beta = 2$) Jacobi ensemble with parameters $\alpha := \alpha(N), M := M(N)$. If $\widehat{\alpha} > 0$, $\widehat{M} \geq 1$ such that*

$$\lim_{N\to\infty} (\alpha/N, M/N) = (\widehat{\alpha}, \widehat{M}),$$

then

$$\lim_{N\to\infty} \log\left(\frac{1}{N}\left(\frac{M+N+\alpha-1}{N+\alpha-1}\right)^{\lfloor CN\widehat{T}\rfloor} \mathbf{y}^{(\lfloor CN\widehat{T}\rfloor)}\right) = \mathfrak{r}^{(\widehat{T})}$$

in finite dimensional distributions across time $\widehat{T} > 0$, where

$$C = \left(\frac{1}{1+\widehat{\alpha}} - \frac{1}{\widehat{M}+1+\widehat{\alpha}}\right)^{-1}.$$

Theorems 3.1.6 and 3.1.7 generalize [42, Theorem 1.2] in two directions. In one direction, we show convergence of the right edge jointly across time instead of a single fixed time. To the best of the author's knowledge, this is the first appearance of the limit process $(\mathfrak{r}^{(\widehat{T})})_{\widehat{T}>0}$. In another direction, we establish that this convergence is universal among products of Jacobi ensembles. Our methods also extend to Laguerre ensembles, although we do not state any results in this direction beyond the Ginibre case (note that Ginibre singular values are a special case of the Laguerre ensemble).

We expect that $(\mathfrak{r}^{(\widehat{T})})_{\widehat{T}>0}$ is a determinantal point process. We believe that the correlation kernel for $(\mathfrak{r}^{(\widehat{T})})_{\widehat{T}>0}$ can be computed via the correlation kernels arising in [11] for the truncated unitary product process. However, working with Laplace transforms has the advantage of generalizing to arbitrary $\beta > 0$.

We now move on to the main result for arbitrary $\beta > 0$.

Theorem 3.1.8 ($\beta > 0$, Edge Fluctuations)**.** *Let $k_1, \ldots, k_m > 0$ be integers. Suppose $(\mathbf{y}^{(T)})_{T\in\mathbb{Z}_{>0}}$ is distributed as the N-particle β-Jacobi product process with parameters $\alpha := \alpha(N), M := M(N)$. If $\widehat{\alpha} > 0, \widehat{M} \geq 1$ such that*

$$\lim_{N\to\infty} (\alpha/N, M/N) = (\widehat{\alpha}, \widehat{M}),$$

then

$$\begin{aligned}
&\lim_{N\to\infty} \mathbb{E}\left[\prod_{i=1}^m \frac{1}{N^{k_i}} \left(\frac{M+N+\alpha-1}{N+\alpha-1}\right)^{k_i\lfloor CN\widehat{T}_i\rfloor} \int x^{k_i} d\mathrm{m}_{\mathbf{y}^{(\lfloor CN\widehat{T}_i\rfloor)}}(x)\right] \\
&= \frac{(-\beta/2)^{-m}}{(2\pi\mathbf{i})^{\sum_{i=1}^m k_i}} \oint\cdots\oint \prod_{\substack{1\leq i,i'\leq m, 1\leq j\leq k_i, 1\leq j'\leq k_{i'} \\ (i,j)<(i,j')}} \frac{(u_{i',j'}-u_{i,j})(u_{i',j'}-u_{i,j}+1-\beta/2)}{(u_{i',j'}-u_{i,j}+1)(u_{i',j'}-u_{i,j}-\beta/2)} \\
&\times \prod_{i=1}^m \left(\prod_{a=1}^{k_i-1} \frac{1}{u_{i,a+1}-u_{i,a}+1-\beta/2}\right)\left(\prod_{j=1}^{k_i} \frac{e^{-\widehat{T}_i(\beta/2)^{-1}u_{i,j}}}{-(\beta/2)^{-1}u_{i,j}}\, du_{i,j}\right)
\end{aligned}$$

where the $u_{i,j}$ contour is positively oriented around 0, *the $u_{i',j'}$ contour encloses a* $\max(\theta, 1)$ *neighborhood of the $u_{i,j}$ contour for $(i,j) < (i',j')$ in lexicographical order, and*

$$C = \left(\frac{1}{1+\widehat{\alpha}} - \frac{1}{\widehat{M}+1+\widehat{\alpha}} \right)^{-1}.$$

As in the $\beta = 2$ case, we expect the right hand side of Theorem 3.1.8 to describe the Laplace transform of a β-generalization of the interpolation point processes. However, it is not clear that the right hand side determines a limiting point process since the moment problem is indeterminate. Despite the indeterminacy, our result implies tightness.

Corollary 3.1.9. *For $(\mathbf{y}^{(T)})_{T\in\mathbb{Z}_{>0}}$ distributed as in Theorem 3.1.8, the finite dimensional distributions of*

$$\log\left(\frac{1}{N} \left(\frac{M+N+\alpha-1}{N+\alpha-1} \right)^{\lfloor CN\widehat{T} \rfloor} \mathbf{y}^{(\lfloor CN\widehat{T} \rfloor)} \right) \tag{3.1.8}$$

are tight in N.

We conjecture that (3.1.8) converges to a universal limit process $(\mathfrak{x}^{(\widehat{T},\beta)})_{\widehat{T}>0}$ with the property that $\widehat{T} \to 0$ yields the β-Airy point process and $\widehat{T} \to +\infty$ yields a Gaussian limit under proper rescaling. Assuming this conjecture, the right hand side of Theorem 3.1.8 gives formulas for Laplace transform of the conjectural point process

$$\mathbb{E}\left[\prod_{i=1}^{m} \sum_{j=1}^{\infty} e^{k_i \mathfrak{x}_i^{(\widehat{T},\beta)}} \right]$$

evaluated at positive integer values $(k_1, \ldots, k_m) \in \mathbb{Z}_{>0}^m$. In addition, this conjecture implies that if $k = k_1 = \cdots = k_m$ are taken to grow on the order $\widehat{T}^{-2/3}$ as $\widehat{T} \to 0$, then we should obtain the Laplace transform of the β-Airy point process (see [30]) in the limit.

3.1.3 Method

Our method relies on deep connections between the Macdonald symmetric functions and random matrices. More specifically, we draw upon connections between (i) Macdonald processes and β-ensembles, and (ii) product processes and Macdonald symmetric functions. The key fact that we leverage is that certain observables of Macdonald processes are accessible through special operators which diagonalize the Macdonald symmetric functions. This idea was pioneered by [6] and further extended in [2, 8, 32].

The connection between Macdonald processes and β-ensembles was first observed by Borodin-Gorin [8]. They showed that the Heckman-Opdam limit (defined below) of certain Macdonald processes produce the β-Jacobi corners process. Using formulas for observables of Macdonald processes, they accessed the global fluctuations of the β-Jacobi corners process. On the other hand, the connection between product ensembles and Schur symmetric functions (a special case of the Macdonald symmetric functions) was established recently by Borodin-Gorin-Strahov [11] for $\beta = 2$. They showed that limits of certain Schur processes yield singular value processes of products of truncated unitary matrices. Our method combines and extends these two approaches to obtain observables for products of β-Jacobi ensembles. We note that the observables we use are derived from a different set of operators than that of [8]. The operators we use were introduced by Negut [49] in an algebraic setting, reexpressed as contour integral formulas by Gorin-Zhang [32] for the asymptotic analysis of the β-Jacobi corners process and further applied to a broad class of Macdonald processes by the author [2]. These operators have the advantage of giving exact moment formulas for the β-Jacobi product process, thus a streamlined asymptotic analysis.

Through these connections, we establish that the β-Jacobi product processes are limits of certain Macdonald processes. We prove this limit statement by describing the Markov transition kernels of the β-Jacobi product process in terms of expectations of Jack symmetric functions and demonstrating convergence of analogous expectations on the side of Macdonald processes. Our computations are generalizable to arbitrary $\beta > 0$ by the extension introduced by Gorin-Marcus [29].

With this limit relation established, we obtain moment formulas for the β-Jacobi product process through formulas for moments of Macdonald processes. Global and local fluctuations are obtained through the moment's method. The global asymptotic analysis uses techniques related to the approaches of [2, 8, 32]. Our method of accessing fluctuations of the right edge (as $T, N \to \infty$) is inspired by the method of analyzing large moments to access the edge of particle systems, pioneered by Sinai-Soshnikov [61] and further refined by Soshnikov [63] who established the universality of the Airy point process at the spectral edge of Wigner matrices. Soshnikov's approach used moments which grow as the size of the matrices grow to obtain the Laplace transform of the Airy point process in the limit. Our approach is based on the observation that the growing order of moments can be replaced by the growing order of products in our setting. For further background on the moment's method, see the survey [62] and references therein.

The remainder of this chapter is organized as follows. In Section 3.2, we provide the necessary background on Macdonald symmetric, Jack symmetric and Heckman-Opdam hypergeometric functions along with references for additional study. We detail the connections between these functions and the β-Jacobi product process in Section 3.3, and obtain moment formulas in Section 3.4 by passing through the Macdonald process formalism. In the remaining Sections 3.5 and 3.6, we prove our main theorems on global and respectively local fluctuations via asymptotic analysis of the moment formulas.

3.2 Preliminaries on Special Functions

3.2.1 Symmetric Functions

Let $\mathbb{Y}$ denote the set of partitions. We represent $\lambda \in \mathbb{Y}$ as the nondecreasing sequence $(\lambda_1, \lambda_2, \ldots)$ of its parts. Denote by $\ell(\lambda)$ the number of indices i such that $\lambda_i \neq 0$ and by $|\lambda|$ the size the partition $\sum_{i \geq 1} \lambda_i$. Let Λ denote the algebra (over $\mathbb{C}$) of symmetric functions in countably many variables

$x_1, x_2, \ldots$. Define $p_0 = 1$ and

$$\begin{aligned} p_k &= \sum_{i \geq 1} x_i^k, \quad k \in \mathbb{Z}_{>0} \\ p_\lambda &= \prod_{i=1}^{\ell(\lambda)} p_{\lambda_i}, \quad \lambda \in \mathbb{Y}. \end{aligned} \tag{3.2.1}$$

Then $\{p_\lambda\}_{\lambda \in \mathbb{Y}}$ forms a linear basis of Λ. Fixing $0 < q, t < 1$, we have the scalar product

$$\langle p_\lambda, p_\mu \rangle_{(q,t)} = \delta_{\lambda\mu} \prod_{i=1}^{\ell(\lambda)} \frac{1 - q^{\lambda_i}}{1 - t^{\lambda_i}} \prod_{i=1}^{\infty} i^{m_i(\lambda)} m_i(\lambda)! \tag{3.2.2}$$

where $m_i(\lambda)$ is the multiplicity of i in λ. We remove the subscript (q, t) when the dependence on (q, t) is clear.

The *Macdonald symmetric functions* $\{P_\lambda(X; q, t)\}_{\lambda \in \mathbb{Y}}$ are the unique (homogeneous) symmetric functions satisfying

$$\langle P_\lambda(X; q, t), P_\mu(X; q, t) \rangle = 0 \tag{3.2.3}$$

for $\lambda \neq \mu$ and with leading monomial $x_1^{\lambda_1} x_2^{\lambda_2} \cdots$ with respect to lexicographical ordering of the powers $(\lambda_1, \lambda_2, \ldots)$. This implies that $\{P_\lambda(X; q, t)\}_{\lambda \in \mathbb{Y}}$ forms a linear basis for Λ. Let $Q_\lambda(X; q, t)$ represent the multiple of $P_\lambda(X; q, t)$ satisfying

$$\langle P_\lambda(X; q, t), Q_\lambda(X; q, t) \rangle = 1. \tag{3.2.4}$$

Given $\mu, \nu \in \mathbb{Y}$, we may expand $P_\mu(X; q, t) P_\nu(X; q, t)$ in the basis of Macdonald symmetric functions. By orthogonality (3.2.3) and duality (3.2.4), the coefficient of $P_\lambda(X; q, t)$ is given by

$$\langle P_\mu(X; q, t) P_\nu(X; q, t), Q_\lambda(X; q, t) \rangle \tag{3.2.5}$$

and is nonzero only if

$$|\mu| + |\nu| = |\lambda|, \qquad \lambda \supset \mu, \qquad \lambda \supset \nu \tag{3.2.6}$$

where the first condition is a consequence of the homogeneity and the justification for the latter two conditions can be found in [43, Chapter VI, (7.4)].

The *skew Macdonald symmetric functions* $P_{\lambda/\mu}(X; q, t), Q_{\lambda/\mu}(X; q, t)$ are defined by

$$\begin{aligned} \langle P_{\lambda/\mu}(X; q, t), Q_\nu(X; q, t) \rangle &= \langle P_\lambda(X; q, t), Q_\mu(X; q, t) Q_\nu(X; q, t) \rangle \\ \langle Q_{\lambda/\mu}(X; q, t), P_\nu(X; q, t) \rangle &= \langle Q_\lambda(X; q, t), P_\mu(X; q, t) P_\nu(X; q, t) \rangle. \end{aligned} \tag{3.2.7}$$

As a consequence of (3.2.5), we have that $P_{\lambda/\mu}(X; q, t) \neq 0$ only if $\lambda \supset \mu$ and likewise for $Q_{\lambda/\mu}(X; q, t)$.

Let $\mathbb{Y}_N$ denote the set of partitions of length $\leq N$, Λ_N the complex algebra of symmetric polynomials in N-variables $x_1, \ldots, x_N$ and $\pi_N : \Lambda \to \Lambda_N$ the restriction homomorphism which effectively takes $x_{N+1} = x_{N+2} = \cdots = 0$. Then

$$P_\lambda(x_1, \ldots, x_N; q, t) := \pi_N P_\lambda(X; q, t) \tag{3.2.8}$$

for $\lambda \in \mathbb{Y}_N$ form a basis for Λ_N. We have

$$a_1, \ldots, a_N \geq 0 \quad \Longrightarrow \quad P_{\lambda/\mu}(a_1, \ldots, a_N; q, t) \geq 0; \tag{3.2.9}$$

we refer to [43, Chapter VI,(7.9') and (7.14')] for the ingredients to prove this nonnegativity. For $A = (a_1, a_2, \ldots)$ and $B = (b_1, b_2, \ldots)$ such that $\sup_{1 \leq i,j \leq \infty} |a_i b_j| < 1$, we also have the Cauchy identity

$$\sum_\lambda P_\lambda(A; q, t) Q_\lambda(B; q, t) = \prod_{i=1}^{\infty} \prod_{j=1}^{\infty} \frac{(t a_i b_j; q)_\infty}{(a_i b_j; q)_\infty} =: \Pi(A, B; q, t) \tag{3.2.10}$$

(see [43, Chapter VI,(4.13)]). The following key properties for the skew Macdonald symmetric functions can be found in [43, Chapter VI, Section 7.4]:

$$P_{\lambda/\nu}(X; q, t) = \sum_{\mu \in \mathbb{Y}} P_{\lambda/\mu}(X; q, t) P_{\mu/\nu}(X; q, t), \tag{3.2.11}$$

$$\sum_{\lambda \in \mathbb{Y}} P_\lambda(X) Q_{\lambda/\mu}(Y) = \Pi(X, Y; q, t) P_\mu(X). \tag{3.2.12}$$

the first equality is known as the *branching rule.*

We define a normalized version of the Macdonald symmetric polynomial

$$\widehat{P}_\lambda(x_1, \ldots, x_N; q, t) = \frac{P_\lambda(x_1, \ldots, x_N; q, t)}{P_\lambda(1, t, \ldots, t^{N-1}; q, t)}, \quad \lambda \in \mathbb{Y}_N \tag{3.2.13}$$

which satisfies the label-variable symmetry for $\lambda, \mu \in \mathbb{Y}_N$

$$\widehat{P}_\lambda(q^{\mu_1} t^{N-1}, q^{\mu_2} t^{N-2}, \ldots, q^{\mu_N}; q, t) = \widehat{P}_\mu(q^{\lambda_1} t^{N-1}, q^{\lambda_2} t^{N-2}, \ldots, q^{\lambda_N}; q, t). \tag{3.2.14}$$

Since the normalized Macdonald symmetric functions form a linear basis for the algebra of symmetric functions, we have the existence of coefficients $c^\lambda_{\mu\nu}(\widehat{P}; q, t)$ which satisfy

$$\widehat{P}_\mu(x_1, \ldots, x_N; q, t) \widehat{P}_\nu(x_1, \ldots, x_N; q, t) = \sum_\lambda c^\lambda_{\mu\nu}(\widehat{P}; q, t) \widehat{P}_\lambda(x_1, \ldots, x_N; q, t). \tag{3.2.15}$$

3.2.2 Jack Symmetric and Heckman-Opdam Hypergeometric Functions

We now describe two degenerations of the Macdonald symmetric polynomials. For $\theta > 0$, the Jack symmetric polynomials are

$$J_\lambda(x_1, \dots, x_N; \theta) := \lim_{q \to 1} P_\lambda(x_1, \dots, x_N; q, q^\theta), \quad \lambda \in \mathbb{Y}_N, \tag{3.2.16}$$

(see [43, Chapter VI, Section 10]). Let $\widehat{J}$ denote the limit obtained by replacing P_λ with $\widehat{P}_\lambda$.

For $\theta > 0$ and $\mathbf{r} = (r_1 > \cdots > r_N > 0 = r_{N+1} = \cdots = r_M)$, define the *Heckman-Opdam hypergeometric function* (for Type A root systems [34–36, 54, 55]) by

$$\begin{aligned} t = q^\theta, \quad q = \exp(-\varepsilon), \quad \lambda = \lfloor \varepsilon^{-1}(r_1, \dots, r_N) \rfloor, \quad x_i = \exp(\varepsilon z_i), \\ \mathcal{F}_{\mathbf{r}}(z_1, \dots, z_M; \theta) := \lim_{\varepsilon \to 0} \varepsilon^{\theta N(N-1)/2 + N(M-N)} P_\lambda(x_1, \dots, x_M; q, t) \\ \widetilde{\mathcal{F}}_{\mathbf{r}}(z_1, \dots, z_M; \theta) := \lim_{\varepsilon \to 0} \varepsilon^{N(\theta-1) + \theta N(N-1)/2 + N(M-N)} Q_\lambda(x_1, \dots, x_M; q, t) \\ \widehat{\mathcal{F}}_{\mathbf{r}}(z_1, \dots, z_M; \theta) := \lim_{\varepsilon \to 0} \widehat{P}_\lambda(x_1, \dots, x_M; q, t) \end{aligned}$$

see [8, Propositions 6.4 and 6.5] for more details. In particular, we may write the normalized form as

$$\widehat{\mathcal{F}}_{\mathbf{r}}(z_1, \dots, z_N) = \frac{\mathcal{F}_{\mathbf{r}}(z_1, \dots, z_N)}{\mathcal{F}_{\mathbf{r}}(0, -\theta, -2\theta, \dots, -(N-1)\theta)} \tag{3.2.17}$$

and extend the limit to $\mathbf{r} = (r_1 \geq \cdots \geq r_N)$.

Let

$$\mathcal{R}^N := \{\mathbf{r} \in \mathbb{R}^N_{\geq 0} : r_1 \geq \cdots \geq r_N\}, \quad \mathcal{U}^N := \mathcal{R}^N \cap [0,1]^N.$$

We may view $\boldsymbol{r} \in \mathcal{R}^N$ as a member of $\mathcal{R}^M$ for $M \geq N$ via identification with $(r_1, \dots, r_N, 0, \dots, 0) \in \mathcal{R}^M$.

The Cauchy identity (3.2.10) degenerates to

$$\int_{\mathcal{R}^{\min(N,M)}} \widetilde{\mathcal{F}}_{\mathbf{r}}(a_1, \dots, a_N; \theta) \mathcal{F}_{\mathbf{r}}(b_1, \dots, b_M) \prod_{i=1}^{\min(N,M)} dr_i = \prod_{i=1}^{N} \prod_{j=1}^{M} \frac{\Gamma(-a_i - b_j)}{\Gamma(\theta - a_i - b_j)} \tag{3.2.18}$$

for $a_1, \dots, a_N, b_1, \dots, b_M$ such that $\Re(a_i + b_j) < 0$ for all $i \in [[1, N]], j \in [[1, M]]$ (see [8, Proposition 6.6]). The variable-index symmetry (3.2.14) degenerates to

$$\widehat{\mathcal{F}}_{\mathbf{r}}(-\lambda_1 - (N-1)\theta, -\lambda_2 - (N-2)\theta, \dots, -\lambda_N; \theta) = \widehat{J}_\lambda(\exp(-r_1), \exp(-r_2), \dots, \exp(-r_N); \theta) \tag{3.2.19}$$

for $\mathbf{r} \in \mathcal{R}^N$ and $\lambda \in \mathbb{Y}_N$ (see [29, Section 2]).

We also have the existence of coefficients $c^{\lambda}_{\mu\nu}(\widehat{J};\theta), c^{\boldsymbol{s}}_{\boldsymbol{\ell},\boldsymbol{r}}(\widehat{\mathcal{F}};\theta)$ which satisfy

$$\widehat{J}_\mu(x_1,\dots,x_N;\theta)\widehat{J}_\nu(x_1,\dots,x_N;\theta) = \sum_\lambda c^{\lambda}_{\mu\nu}(\widehat{J};\theta)\widehat{J}_\lambda(x_1,\dots,x_N;\theta)$$
$$\widehat{\mathcal{F}}_{\boldsymbol{\ell}}(z_1,\dots,z_N;\theta)\widehat{\mathcal{F}}_{\boldsymbol{r}}(z_1,\dots,z_N;\theta) = \int_{\boldsymbol{s}} c^{\boldsymbol{s}}_{\boldsymbol{\ell},\boldsymbol{r}}(\widehat{\mathcal{F}};\theta)\widehat{\mathcal{F}}(z_1,\dots,z_N;\theta)$$

where

$$c^{\lambda}_{\mu\nu}(\widehat{P};q,t) \to c^{\lambda}_{\mu\nu}(\widehat{J};\theta), \quad t = q^\theta \to 1. \tag{3.2.20}$$

and $c^{\boldsymbol{s}}_{\boldsymbol{\ell},\boldsymbol{r}}(\widehat{\mathcal{F}};\theta)$ is a (possibly signed) measure in $\boldsymbol{s}$ with total mass 1 supported in the set of $\boldsymbol{s} \in \mathcal{R}^N$ satisfying

$$s_1 + \cdots + s_N = (r_1+\ell_1) + \cdots (r_N + \ell_N), \qquad r_N + \ell_N \le s_i \le r_1 + \ell_1, 1 \le i \le N,$$

see [29, Section 2]. For $\theta = 1/2, 1, 2$, the measures $c^{\boldsymbol{s}}_{\boldsymbol{\ell},\boldsymbol{r}}$ are nonnegative and therefore probability measures.

3.3 β-Jacobi Product Process

In this section, we describe the connection between the special functions discussed in the previous section with the β-Jacobi product process. The main result of this section (Theorem 3.3.12) realizes the β-Jacobi product processes as limits of discrete *Macdonald processes*, a Markov chain whose distribution has a special structure in terms of Macdonald symmetric functions. We begin by finding a description for the transition kernels of the β-Jacobi product process for $\beta = 1, 2, 4$. We then provide the interpolation to $\beta > 0$ via a limit transition from the Macdonald processes detailed below, after which point Theorem 3.3.12 is a straightforward consequence.

Definition 3.3.1. Suppose X, Y denote two random, independent, $N \times N$ self adjoint random matrices with right $\mathbb{U}_\beta$-invariant distributions. Let $\mathbf{x}$ and $\mathbf{y}$ denote the random squared singular values of X, Y respectively. Define

$$\mathbf{x} \boxtimes_\beta \mathbf{y} \tag{3.3.1}$$

to be the squared singular values of XY.

Definition 3.3.2. Let $\theta, \alpha > 0$, and $M, N \in \mathbb{Z}_{>0}$ such that $M \ge N$. Denote by $\mathbb{P}^{\alpha,M,N}_\theta$ the measure on N-particles $\mathbf{x} \in \mathcal{U}^N$ with density proportional to

$$\prod_{1\le i<j\le N} |x_i - x_j|^{2\theta} \prod_{i=1}^N x_i^{\theta\alpha-1}(1-x_i)^{\theta(M-N)+\theta-1}\,dx_i. \tag{3.3.2}$$

Remark 16. Although we work with $M \in \mathbb{Z}_{>0}$, one can extend the results in this section for $M > 0$ by analytic continuation.

This is the β-Jacobi ensemble with $\beta = 2\theta$. For $\theta = 1/2, 1, 2$, this is the eigenvalue distribution of a certain matrix ensemble.

Proposition 3.3.3 ([27, Proposition 3.8.2]). *Let $\beta \in \{1, 2, 4\}$, $L, N', N \in \mathbb{Z}_{>0}$ such that $N' \geq N$, $L \geq N' + N$ and $U \in \mathbb{U}_\beta(L)$. If A is the $N' \times N$ top left rectangular submatrix of U, then the eigenvalues of A^*A are distributed as (3.3.2) with*

$$\alpha = N' - N + 1, \quad M = L - N'. \tag{3.3.3}$$

The density of the β-Jacobi ensemble can also be given in terms of Heckman-Opdam hypergeometric functions. The following Proposition is a consequence of a general convergence statement [8, Theorem 2.8] for the β-Jacobi *corners* process; a multilevel extension of the β-Jacobi ensemble.

We adopt the following notation for brevity, for any $\mu \in \mathbb{Y}_N$ set

$$\begin{aligned} \rho_\mu^N(q,t) &= (q^{\mu_1} t^{N-1}, q^{\mu_2} t^{N-2}, \ldots, q^{\mu_N}), \\ \varrho_\mu^N(\theta) &= -(\mu_1 + \theta(N-1), \mu_2 + \theta(N-2), \ldots, \mu_N), \\ (a_1, \ldots, a_N) + c &= (a_1 + c, \ldots, a_N + c) \quad \text{for} \quad a_1, \ldots, a_N, c \in \mathbb{R}. \end{aligned} \tag{3.3.4}$$

The dependence on (q,t) and θ will be clear from the context so we write $\rho_\mu^N = \rho_\mu^N(q,t)$ and $\varrho_\lambda^N = \varrho_\lambda^N(\theta)$.

Proposition 3.3.4. *Let $\theta, \alpha > 0$, and $M, N \in \mathbb{Z}_{>0}$ such that $M \geq N$. Let $\varepsilon > 0$ be a small parameter with $t = q^\theta$, $\quad q = \exp(-\varepsilon)$, and $\lambda \in \mathbb{Y}_N$ distributed as*

$$\operatorname{Prob}(\lambda = \mu) = \frac{1}{\Pi(\rho_0^N, t^\alpha \rho_0^M; q, t)} P_\mu(\rho_0^N; q, t) Q_\mu(t^\alpha \rho_0^M; q, t). \tag{3.3.5}$$

If $\mathbf{y} \sim \mathbb{P}_\theta^{\alpha,M,N}$, then $\exp(-\varepsilon\lambda) \to \mathbf{y}$ in distribution as $\varepsilon \to 0$. In particular, the density $\mathbb{P}_\theta^{\alpha,M,N}$ has the form

$$\frac{1}{H(\varrho_0^N, \varrho_0^M - \alpha\theta; \theta)} \mathcal{F}_{\mathbf{r}}(\varrho_0^N; \theta) \widetilde{\mathcal{F}}_{\mathbf{r}}(\varrho_0^M - \alpha\theta; \theta) \, d\mathbf{r} \tag{3.3.6}$$

where $\mathbf{y} = \exp(-\mathbf{r})$.

Definition 3.3.5. Let $\beta \in \{1, 2, 4\}$ and $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \ldots$ be independent, random elements of $\mathcal{R}^N$. Define the *β-product process* on $(\mathbf{x}^{(T)})_{T \in \mathbb{Z}_{>0}}$ to be the random sequence $(\mathbf{y}^{(T)})_{T \in \mathbb{Z}_{>0}}$ where $\mathbf{y}^{(1)} := \mathbf{x}^{(1)}$ and

$$\mathbf{y}^{(T+1)} = \mathbf{y}^{(T)} \boxtimes_\beta \mathbf{x}^{(T+1)}, \quad T \in \mathbb{Z}_{>0}. \tag{3.3.7}$$

By the independence of $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \ldots$, the β-product process is a Markov process in discrete time T. We compute the Markov transition probabilities of this process with $\mathbf{x}^{(T)}$ distributed as the β-Jacobi ensemble for some set of parameters for each $T \in \mathbb{Z}_{>0}$.

Proposition 3.3.6. *Let $\theta \in \{1/2, 1, 2\}$, $\alpha > 0$, and $M, N \in \mathbb{Z}_{>0}$ such that $M \geq N$. If $\mathbf{x} \sim \mathbb{P}_\theta^{\alpha,M,N}$, $\mathbf{y} \in \mathcal{U}^N$ and $\mathbf{z} = \mathbf{y} \boxtimes_{2\theta} \mathbf{x}$, then*

$$\mathbb{E}\widehat{J}_\kappa(\mathbf{z};\theta) = \widehat{J}_\kappa(\mathbf{y};\theta)\frac{H(\varrho_\kappa^N, \varrho_0^M - \alpha\theta)}{H(\varrho_0^N, \varrho_0^M - \alpha\theta)}, \quad \kappa \in \mathbb{Y}_N. \tag{3.3.8}$$

This distribution is determined by (3.3.8).

Proof. If $\boldsymbol{\ell}, \mathbf{r} \in \mathcal{R}^N$, then by [29, Proposition 2.2] the probability measure of the random vector $\mathbf{s} \in \mathcal{R}^N$ defined by

$$\exp(-\mathbf{s}) = \exp(-\boldsymbol{\ell}) \boxtimes_{2\theta} \exp(-\mathbf{r}) \tag{3.3.9}$$

is given by $c_{\boldsymbol{\ell},\mathbf{r}}^{\mathbf{s}}(\widehat{\mathcal{F}};\theta)$. If we let $\mathbf{r} \in \mathcal{R}^N$ be random such that $\exp(-\mathbf{r}) \sim \mathbb{P}_\theta^{\alpha,M,N}$, then for any $\kappa \in \mathbb{Y}_N$

$$\begin{aligned}
\mathbb{E}\widehat{J}_\kappa(\exp(-\mathbf{s});\theta) &= \int_{\mathbf{r}} \frac{1}{H(\varrho_0^N, \varrho_0^M - \alpha\theta)} \mathcal{F}_{\mathbf{r}}(\varrho_0^N;\theta)\widetilde{\mathcal{F}}_{\mathbf{r}}(\varrho_0^M - \alpha\theta;\theta) \int_{\mathbf{s}} c_{\boldsymbol{\ell},\mathbf{r}}^{\mathbf{s}}(\widehat{\mathcal{F}};\theta)\widehat{J}_\kappa(\exp(-\mathbf{s});\theta) \\
&= \frac{1}{H(\varrho_0^N, \varrho_0^M - \alpha\theta)} \int_{\mathbf{r}} \mathcal{F}_{\mathbf{r}}(\varrho_0^N;\theta)\widetilde{\mathcal{F}}_{\mathbf{r}}(\varrho_0^M - \alpha\theta;\theta)\widehat{\mathcal{F}}_{\boldsymbol{\ell}}(\varrho_\kappa^N;\theta)\widehat{\mathcal{F}}_{\mathbf{r}}(\varrho_\kappa^N;\theta) \\
&= \frac{\widehat{J}_\kappa(\exp(-\boldsymbol{\ell});\theta)}{H(\varrho_0^N, \varrho_0^M - \alpha\theta)} \int_{\mathbf{r}} \mathcal{F}_{\mathbf{r}}(\varrho_\kappa^N;\theta)\widetilde{\mathcal{F}}_{\mathbf{r}}(\varrho_0^M - \alpha\theta;\theta) \\
&= \widehat{J}_\kappa(\exp(-\boldsymbol{\ell});\theta)\frac{H(\varrho_\kappa^N, \varrho_0^M - \alpha\theta)}{H(\varrho_0^N, \varrho_0^M - \alpha\theta)}
\end{aligned} \tag{3.3.10}$$

where the first equality uses Theorem 3.3.4, the second and third equalities use (3.2.19) and the fourth equality uses (3.2.18). Taking $\mathbf{y} = \exp(-\boldsymbol{\ell})$ and $\mathbf{z} = \exp(-\mathbf{s})$ proves (3.3.8). Since the Jack symmetric functions in N-variables form a basis for the symmetric polynomials in N-variables and since $\mathbf{z}$ is supported in $\mathcal{U}^N$, (3.3.8) determines the distribution of $\mathbf{z}$. □

The proof of Theorem 3.3.6 suggests that the probability density of $\boldsymbol{s}$ conditioned on $\boldsymbol{r}$, where $\exp(-\boldsymbol{s}) = \exp(-\boldsymbol{\ell}) \boxtimes_\beta \exp(-\boldsymbol{r})$, can be expressed as

$$\frac{1}{H(\varrho_0^N, \varrho_0^M - \alpha\theta;\theta)} \int_{\mathbf{r}} c_{\boldsymbol{\ell},\mathbf{r}}^{\mathbf{s}}(\widehat{\mathcal{F}};\theta)\mathcal{F}_{\mathbf{r}}(\varrho_0^N;\theta)\widetilde{\mathcal{F}}_{\mathbf{r}}(\rho_0^M - \alpha\theta;\theta). \tag{3.3.11}$$

However, $c_{\boldsymbol{\ell},\mathbf{r}}^{\mathbf{s}}$ is a measure on $\mathbf{s} \in \mathcal{R}^N$ so that (3.3.11) is integrating over measures. Thus without reference to a density, the integral (3.3.11) is ill-defined. We therefore take the approach of identifying the Markov kernel by moments.

The transition probabilities of Proposition 3.3.6 can be extended to arbitrary $\beta > 0$. We detail this extrapolation below. The idea is seeing the Markov kernels of Proposition 3.3.6 as limits of a family of kernels derived from Macdonald symmetric functions.

Definition 3.3.7. Let $0 < q, t < 1$, $\alpha > 0$ and $M, N \in \mathbb{Z}_{>0}$ such that $M \geq N$. Define the kernel

$$K_{q,t}^{\alpha,M,N}(\mu,\lambda) = \frac{1}{\Pi(\rho_0^N, t^\alpha \rho_0^M; q, t)} \sum_{\nu \in \mathbb{Y}} c_{\mu\nu}^{\lambda}(\widehat{P}; q, t) P_\nu(\rho_0^N; q, t) Q_\nu(t^\alpha \rho_0^M; q, t), \quad \mu, \lambda \in \mathbb{Y}_N \tag{3.3.12}$$

Expressing (3.2.15) as

$$\frac{P_\mu(x_1, \ldots, x_N; q, t)}{P_\mu(\rho_0^N; q, t)} \frac{P_\nu(x_1, \ldots, x_N; q, t)}{P_\nu(\rho_0^N; q, t)} = \sum_\lambda c_{\mu\nu}^\lambda(\widehat{P}; q, t) \frac{P_\lambda(x_1, \ldots, x_N; q, t)}{P_\lambda(\rho_0^N; q, t)},$$

we see that (3.2.4) and (3.2.7) imply

$$c_{\mu\nu}^\lambda(\widehat{P}; q, t) = \frac{P_\lambda(\rho_0^N; q, t)}{P_\mu(\rho_0^N; q, t) P_\nu(\rho_0^N; q, t)} \langle Q_{\lambda/\mu}(\cdot; q, t), P_\nu(\cdot, q, t) \rangle$$

so that

$$\begin{aligned} K_{q,t}^{\alpha,M,N}(\mu,\lambda) &= \frac{P_\lambda(\rho_0^N; q, t)}{P_\mu(\rho_0^N; q, t) \Pi(\rho_0^N, t^\alpha \rho_0^M; q, t)} \sum_{\nu \in \mathbb{Y}} \langle Q_{\lambda/\mu}(\cdot; q, t), P_\nu(\cdot; q, t) \rangle Q_\nu(t^\alpha \rho_0^M; q, t) \\ &= \frac{1}{\Pi(\rho_0^N, t^\alpha \rho_0^M; q, t)} \frac{P_\lambda(\rho_0^N; q, t)}{P_\mu(\rho_0^N; q, t)} Q_{\lambda/\mu}(t^\alpha \rho_0^M; q, t). \end{aligned} \tag{3.3.13}$$

Lemma 3.3.8. *For each $\kappa, \mu \in \mathbb{Y}_N$, we have*

$$\sum_{\lambda \in \mathbb{Y}} \widehat{P}_\kappa(\rho_\lambda^N; q, t) K_{q,t}^{\alpha,M,N}(\mu, \lambda) = \widehat{P}_\kappa(\rho_\mu^N; q, t) \frac{\Pi(\rho_\kappa^N, t^\alpha \rho_0^M; q, t)}{\Pi(\rho_0^N, t^\alpha \rho_0^M; q, t)}. \tag{3.3.14}$$

In particular, $K_{q,t}^{\alpha,M,N}(\mu, \cdot)$ defines a probability distribution.

Proof. We contract the notation by dropping the (q,t) from $\Pi, P, \widehat{P}, Q$. We have

$$\begin{aligned} \sum_{\lambda \in \mathbb{Y}} \widehat{P}_\kappa(\rho_\lambda^N) K_{q,t}^{\alpha,M,N}(\mu,\lambda) &= \frac{1}{P_\mu(\rho_0^N) \Pi(\rho_0^N, t^\alpha \rho_0^M)} \sum_{\lambda \in \mathbb{Y}} \widehat{P}_\lambda(\rho_\kappa^N) P_\lambda(\rho_0^N) Q_{\lambda/\mu}(t^\alpha \rho_0^M) \\ &= \frac{1}{P_\mu(\rho_0^N) \Pi(\rho_0^N, t^\alpha \rho_0^M)} \sum_{\lambda \in \mathbb{Y}} P_\lambda(\rho_\kappa^N) Q_{\lambda/\mu}(t^\alpha \rho_0^M) \\ &= \widehat{P}_\mu(\rho_\kappa^N) \frac{\Pi(\rho_\kappa^N, t^\alpha \rho_0^M)}{\Pi(\rho_0^N, t^\alpha \rho_0^M)} \end{aligned} \tag{3.3.15}$$

where we use (3.2.14) in the first equality and (3.2.12) in the final equality. By taking $\kappa = (0)$, we see that $K_{q,t}^{\alpha,M,N}(\mu, \cdot)$ defines a probability distribution. By (3.2.14), the lemma follows. $\square$

Proposition 3.3.9. *Suppose $\mathbf{v} \in \mathcal{U}^N$, $q = e^{-\varepsilon}$, $t = q^\theta$, and $\mu = \mu(\varepsilon) \in \mathbb{Y}_N$ such that $q^\mu \to \mathbf{v}$ as $\varepsilon \to 0$. If $\lambda \sim K_{q,t}^{\alpha,M,N}(\mu, \cdot)$, then as $\varepsilon \to 0$, q^λ converges in distribution to a random variable*

$\mathbf{u} \in \mathcal{U}^N$ *with probability measure* $\mathcal{K}_\theta^{\alpha,M,N}(\mathbf{v},\mathbf{u})$ *determined by*

$$\int_{\mathbf{u}} \widehat{J}_\kappa(\mathbf{u};\theta)\mathcal{K}_\theta^{\alpha,M,N}(\mathbf{v},\mathbf{u}) = \widehat{J}_\kappa(\mathbf{v};\theta)\frac{H(\varrho_\kappa^N,\varrho_0^M-\alpha\theta;\theta)}{H(\varrho_0^N,\varrho_0^M-\alpha\theta;\theta)}, \quad \kappa \in \mathbb{Y}_N. \tag{3.3.16}$$

Proof. By Lemma 3.3.8, for any $\kappa \in \mathbb{Y}_N$ we have the convergence

$$\sum_{\upsilon \in \exp(-\varepsilon \mathbb{Y}_N)} \widehat{P}_\kappa\left(\upsilon_1 t^{N-1}, \upsilon_2 t^{N-2}, \ldots, \upsilon_N; q, t\right) K_{q,t}^{\alpha,M,N}\left(\mu, -\varepsilon^{-1}\log\upsilon\right) \to \widehat{J}_\kappa(\mathbf{v};\theta)\frac{H(\varrho_\kappa^N,\varrho_0^M-\alpha\theta;\theta)}{H(\varrho_0^N,\varrho_0^M-\alpha\theta;\theta)} \tag{3.3.17}$$

as $\varepsilon \to 0$. For any given $\kappa \in \mathbb{Y}_N$, we also have the uniform convergence

$$\widehat{P}_\kappa\left(u_1 t^{N-1}, u_2 t^{N-2}, \ldots, u_N; q, t\right) \to \widehat{J}_\kappa(u_1,u_2,\ldots,u_N;\theta) \tag{3.3.18}$$

as $\varepsilon \to 0$ over $(u_1,\ldots,u_N) \in \mathcal{U}^N$. Therefore

$$\sum_{\upsilon \in \exp(-\varepsilon \mathbb{Y}_N)} \widehat{J}_\kappa(\upsilon_1,\ldots,\upsilon_N;\theta) K_{q,t}^{\alpha,M,N}\left(\mu, -\varepsilon^{-1}\log\upsilon\right) \to \widehat{J}_\kappa(\mathbf{v};\theta)\frac{H(\varrho_\kappa^N,\varrho_0^M-\alpha\theta;\theta)}{H(\varrho_0^N,\varrho_0^M-\alpha\theta;\theta)}. \tag{3.3.19}$$

Since $\mathcal{U}^N$ is compact and the Jack symmetric functions are dense in $C(\mathcal{U}^N,\mathbb{R})$, (3.3.19) implies the convergence in distribution of $K_{q,t}^{\alpha,M,N}(\mu,-\varepsilon^{-1}\log\upsilon)$ as probability measures on $\upsilon \in \exp(-\varepsilon\mathbb{Y}_N)$ to a probability measure $\mathcal{K}_\theta^{\alpha,M,N}(\mathbf{v},\mathbf{u})$ on $\mathbf{u} \in \mathcal{U}^N$. The limiting probability measure satisfies (3.3.16). $\square$

Definition 3.3.10. Let $\theta > 0$, $N \in \mathbb{Z}_{>0}$, $\alpha_T \geq 0$, and $M_T \in \mathbb{Z}_{>0}$ such that $M_T \geq N$. The β-Jacobi product process with parameters $(\alpha_T, M_T)_{T\in\mathbb{Z}_{>0}}$, N is a Markov chain $(\mathbf{y}^{(T)})_{T\in\mathbb{Z}_{>0}}$ with state space $\mathcal{U}^N$ such that (i) $\mathbf{y}^{(1)} \sim \mathbb{P}_\theta^{\alpha_1,M_1,N}$ and (ii) $\mathcal{K}_\theta^{\alpha_T,M_T,N}(\mathbf{v},\mathbf{u})$ is the conditional probability of $\mathbf{y}^{(T)} = \mathbf{u}$ given $\mathbf{y}^{(T-1)} = \mathbf{v}$. We denote this probability measure by $\mathbb{P}_\theta^{(\alpha_T,M_T)_{T>0},N}$.

We refer to Markov chains of the form above as *β-Jacobi product processes* where $\beta = 2\theta$. Informally, we may view $\mathbf{y}^{(T)}$ as satisfying

$$\mathbf{y}^{(T+1)} = \mathbf{y}^{(T)} \boxtimes_\beta \mathbf{x}^{(T)}$$

where $\mathbf{x}^{(T)} \sim \mathbb{P}_\theta^{\alpha_T,M_T,N}$, $\mathbf{y}^{(1)} = \mathbf{x}^{(1)}$ and $\boxtimes_\beta$ is some analytic extension of the operation for $\beta = 1, 2, 4$.

Definition 3.3.11. Let $\mathbb{P}_{q,t}^{(\alpha_T,M_T)_{T>0},N}$ be the measure on random Young diagrams $(\lambda^T)_{T\in\mathbb{Z}_{>0}}$ such that $\ell(\lambda^T) \leq N$,

$$\mathrm{Prob}(\lambda^1 = \mu) = \frac{1}{\Pi(\rho_0^N, t^{\alpha_1}\rho_0^{M_1})} P_\mu(\rho_0^N;q,t) Q_\mu(t^{\alpha_1}\rho_0^{M_1};q,t),$$

$$\mathrm{Prob}(\lambda^T = \lambda | \lambda^{T-1} = \mu) = K_{q,t}^{\alpha_T,M_T,N}(\mu,\lambda), \quad T > 1.$$

The measure $\mathbb{P}_{q,t}^{(\alpha_T, M_T)_{T>0},N}$ is a special example of a so-called *(infinite) ascending Macdonald process* [8, Section 2.1].

Theorem 3.3.12. *Let $\theta > 0$, $N \in \mathbb{Z}_{>0}$, $\alpha_T \geq 0$, and $M_T \in \mathbb{Z}_{>0}$ such that $M_T \geq N$ for $T \in \mathbb{Z}_{>0}$. Let $\varepsilon > 0$ be a small parameter with $t = q^\theta, q = e^{-\varepsilon}$, $(\lambda^T := \lambda^T(\varepsilon))_{T \in \mathbb{Z}_{>0}} \sim \mathbb{P}_{q,t}^{(\alpha_T, M_T)_{T>0},N}$. Then the finite dimensional distributions of $(\exp(-\varepsilon\lambda^T))_{T\in\mathbb{Z}_{>0}}$ converge to those of $\mathbb{P}_{\theta}^{(\alpha_T, M_T)_{T>0},N}$ as $\varepsilon \to 0$.*

Proof. We suppress the dependence on (q,t) and θ, though all limits we take are $\varepsilon \to 0$ with $q = \exp(-\varepsilon)$, $t = q^\theta$. Let $(\mathbf{y}^{(T)})_{T>0} \sim \mathbb{P}_{\theta}^{(\alpha_T, M_T)_{T>0},N}$. It suffices to show that

$$\mathbb{E}\widehat{P}_{\kappa^1}(\rho_{\lambda^1}^N) \cdots \widehat{P}_{\kappa^T}(\rho_{\lambda^T}^N) \to \mathbb{E}\widehat{J}_{\kappa^1}(\mathbf{y}^{(1)}) \cdots \widehat{J}_{\kappa^T}(\mathbf{y}^{(T)})$$

for any $T \geq 1$. We induct on T, observing that the statement for $T = 1$ follows from Theorem 3.3.9, and if the statement is known for $T-1$ for some $T > 1$ then

$$\begin{aligned}
\mathbb{E}\widehat{P}_{\kappa^1}(\rho_{\lambda^1}^N) \cdots \widehat{P}_{\kappa^T}(\rho_{\lambda^T}^T) &= \frac{\Pi(\rho_{\kappa^T}^N, t^\alpha \rho_0^M)}{\Pi(\rho_0^N, t^\alpha \rho_0^M)} \mathbb{E}\widehat{P}_{\kappa^1}(\rho_{\lambda^1}^N) \cdots \widehat{P}_{\kappa^{T-1}}(\rho_{\lambda^{T-1}}) \widehat{P}_{\kappa^T}(\rho_{\lambda^{T-1}}) \\
&= \frac{\Pi(\rho_{\kappa^T}^N, t^\alpha \rho_0^M)}{\Pi(\rho_0^N, t^\alpha \rho_0^M)} \sum_\mu c_{\kappa^{T-1}\kappa^T}^\mu(\widehat{P}) \mathbb{E}\widehat{P}_{\kappa^1}(\rho_{\lambda^1}^N) \cdots \widehat{P}_{\kappa^{T-2}}(\rho_{\lambda^{T-2}}) \widehat{P}_\mu(\rho_{\lambda^{T-1}})
\end{aligned}$$

where the first equality follows from Lemma 3.3.8. The latter converges, by our induction hypothesis and (3.2.20), to

$$\frac{H(\varrho_{\kappa^T}^N, \varrho_0^M - \alpha\theta)}{H(\varrho_0^N, \varrho_0^M - \alpha\theta)} \sum_\mu c_{\kappa^{T-1}\kappa^T}^\mu(\widehat{J}) \mathbb{E}\widehat{J}_{\kappa^1}(\mathbf{y}^{(1)}) \cdots \widehat{J}_{\kappa^{T-2}}(\mathbf{y}^{(T-2)}) \widehat{J}_\mu(\mathbf{y}^{(T-1)})$$

which can be written as

$$\frac{H(\varrho_{\kappa^T}^N, \varrho_0^M - \alpha\theta)}{H(\varrho_0^N, \varrho_0^M - \alpha\theta)} \mathbb{E}\widehat{J}_{\kappa^1}(\mathbf{y}^{(1)}) \cdots \widehat{J}_{\kappa^{T-2}}(\mathbf{y}^{(T-2)}) \widehat{J}_{\kappa^{T-1}}(\mathbf{y}^{(T-1)}) \widehat{J}_{\kappa^T}(\mathbf{y}^{(T-1)}) = \mathbb{E}\widehat{J}_{\kappa^1}(\mathbf{y}^{(1)}) \cdots \widehat{J}_{\kappa^T}(\mathbf{y}^{(T)})$$

by Proposition 3.3.9. $\square$

3.4 Observables

The main results of this section are formulas for the joint moments of β-Jacobi product process. We provide formulas for general $\beta > 0$ and more convenient formulas for $\beta = 2$.

Given $U = (u_1, \dots, u_k)$, $V = (v_1, \dots, v_\ell)$, define

$$
\begin{aligned}
\mathcal{A}_T(U) := \mathcal{A}_T(U;\theta) &:= \prod_{i=1}^{k} \frac{u_i}{u_i + \theta N} \prod_{\tau=1}^{T} \frac{u_i - \theta(\alpha_\tau - 1)}{u_i - \theta(\alpha_\tau + M_\tau - 1)}, \\
\mathcal{B}(U) := \mathcal{B}(U;\theta) &:= \frac{1}{(u_2 - u_1 + 1 - \theta) \cdots (u_k - u_{k-1} + 1 - \theta)} \\
&\qquad \times \prod_{1 \le i < j \le k} \frac{(u_j - u_i)(u_j - u_i + 1 - \theta)}{(u_j - u_i + 1)(u_j - u_i - \theta)}, \\
\mathcal{C}(U,V) := \mathcal{C}(U,V;\theta) &:= \prod_{i=1}^{k} \prod_{j=1}^{\ell} \frac{(v_j - u_i)(v_j - u_i + 1 - \theta)}{(v_j - u_i + 1)(v_j - u_i - \theta)}.
\end{aligned}
\tag{3.4.1}
$$

Definition 3.4.1. Given $(x_1, \dots, x_N) \in \mathbb{R}^N$, define for any $k > 0$

$$
\mathfrak{P}_k(x_1, \dots, x_N) = \sum_{i=1}^{N} x_i^k.
$$

Theorem 3.4.2. *Let $\theta, \alpha_T > 0$, $M_T, N \in \mathbb{Z}_{>0}$ such that $M_T \ge N$ for $T \in \mathbb{Z}_{>0}$, and $(\mathbf{y}^{(T)})_{T\in\mathbb{Z}_{>0}} \sim \mathbb{P}_\theta^{(\alpha_T, M_T)_{T>0}, N}$. If $T_1 \ge \cdots \ge T_m > 0$ and $k_1, \dots, k_m > 0$ are integers, then*

$$
\mathbb{E}\left[\prod_{i=1}^{m} \mathfrak{P}_{k_i}(\mathbf{y}^{(T_i)})\right] = \frac{(-\theta)^{-m}}{(2\pi\mathbf{i})^{k_1 + \cdots + k_m}} \oint \cdots \oint \prod_{1 \le i < i' \le m} \mathcal{C}(U_i, U_{i'}) \prod_{i=1}^{m} \mathcal{B}(U_i) \mathcal{A}_{T_i}(U_i) dU_i \tag{3.4.2}
$$

where $U_i = (u_{i,1}, \dots, u_{i,k_i})$,

1. *the $u_{i,j}$ contour $\mathfrak{U}_{i,j}$ is positively oriented around the pole at $-\theta N$ and does not enclose $\theta(\alpha_\tau + M_\tau - 1)$ for $1 \le \tau \le T_i$;*
2. *whenever $(i,j) < (i',j')$ in lexicographical order, $\mathfrak{U}_{i,j}$ is enclosed by the contour $\mathfrak{U}_{i',j'} + c$ for $c \in [-\theta, 1]$;*

given that such contours exist.

Remark 17. The existence of the contours is guaranteed for N large. Since our applications are for N large, the question of existence is not a hindrance.

Theorem 3.4.3. *Let $\alpha_T > 0$, $M_T, N \in \mathbb{Z}_{>0}$ such that $M_T \ge N$ for $T \in \mathbb{Z}_{>0}$, and $(\mathbf{y}^{(T)})_{T\in\mathbb{Z}_{>0}} \sim \mathbb{P}_1^{(\alpha_T, M_T)_{T>0}, N}$. If $T_1 \ge \cdots \ge T_m > 0$ are integers and $c_1, \dots, c_m \in \mathbb{R}_{>0}$, then*

$$
\begin{aligned}
\mathbb{E}\left[\prod_{i=1}^{m} \mathfrak{P}_{c_i}(\mathbf{y}^{(T_i)})\right] = \frac{\prod_{i=1}^{m}(-c_i)^{-1}}{(2\pi\mathbf{i})^m} \oint \cdots \oint \prod_{1 \le i < j \le m} \frac{(u_j - u_i)(u_j + c_j - u_i - c_i)}{(u_j - u_i - c_i)(u_j + c_j - u_i)} \\
\times \prod_{i=1}^{m} \left(\prod_{\ell=1}^{N} \frac{u_i + \ell - 1}{u_i + c_i + \ell - 1} \cdot \prod_{\tau=1}^{T_i} \prod_{\ell=1}^{M_\tau} \frac{u_i + c_i - \alpha_\tau - \ell + 1}{u_i - \alpha_\tau - \ell + 1} \right) du_i
\end{aligned}
$$

where the u_i contour $\mathfrak{U}_i$ is positively oriented around $\{-c_i-\ell+1\}_{\ell=1}^N$ but does not enclose $\alpha_\tau+\ell-1$ for $1 \le \ell \le M_\tau$, $1 \le \tau \le T_i$, and $\mathfrak{U}_i$ is contained in $\mathfrak{U}_j - c_i$ for $i < j$; given that such contours exist.

Remark 18. We note that Theorem 3.4.2 holds for $\theta > 0$ arbitrary, but is restricted to taking $\mathfrak{P}_{k_i}$ where k_i are positive *integers*. Moreover the contours have dimension $k_1 + \cdots + k_m$. In contrast, the contours in Theorem 3.4.3 for $\theta = 1$ have dimension m and $k_i = c_i$ can be an arbitrary positive real. In this chapter, we come to these two formulas from seemingly different approaches. However, it was pointed out by E. Dimitrov through private communication that the $\theta > 0$ case is a true generalization of the $\theta = 1$ case. Via residue expansion and combinatorics, the higher dimensional contour integral formulas reduce to m-dimensional contour integral formulas when $\theta = 1$. The reason we can take $c_1, \ldots, c_m > 0$ arbitrary follows then from analytic continuation.

The remainder of this section is devoted to the proofs of Theorems 2.3.17 and 3.4.3. The starting point is a set of contour integral formulas for exponential moments of Macdonald processes obtained in [32] and further generalized in [2] in a form well-suited for our purposes. These formulas give us expressions for the β-Jacobi product process via the connection established by Theorem 3.3.12.

3.4.1 Formulas for Macdonald Processes

We state and prove discrete analogues of Theorems 2.3.17 and 3.4.3 in the setting of Macdonald processes.

Definition 3.4.4. For $\lambda \in \mathbb{Y}_n$ define

$$\mathfrak{p}_k(\lambda; q, t) = (1 - t^{-k}) \sum_{i=1}^n q^{k\lambda_i} t^{k(-i+1)} + t^{-kn} \tag{3.4.3}$$

for integers $k > 0$. In the special case $q = t$ with $0 < t < 1$, define

$$\mathfrak{p}_t(\lambda) := \mathfrak{p}_1(\lambda; t, t) = (1 - t^{-1}) \sum_{i=1}^n t^{\lambda_i - i + 1} + t^{-n}. \tag{3.4.4}$$

Observe that the definitions of $\mathfrak{p}_t$ and $\mathfrak{p}_k$ are independent of n as long as $\ell(\lambda) \le n$.

If $Z := (z_1, \ldots, z_k)$, then let $Z^{-1} := (z_1^{-1}, \ldots, z_k^{-1})$, $qZ := (qz_1, \ldots, qz_k)$, $dZ := dz_1 \cdots dz_k$. Given $Z = (z_1, \ldots, z_k)$ and $W = (w_1, \ldots, w_\ell)$, let

$$\mathfrak{A}_T(Z) := \mathfrak{A}_T(Z; q, t) := \prod_{i=1}^k \frac{z_i - 1}{z_i - t^N} \prod_{\tau=1}^T \frac{1 - t^{\alpha_\tau - 1} z_i}{1 - t^{\alpha_\tau + M_\tau - 1} z_i}, \tag{3.4.5}$$

$$\mathfrak{B}(Z) := \mathfrak{B}(Z; q, t) := \frac{\sum_{i=1}^k \frac{1}{z_i} \frac{q^{i-1}}{t^{i-1}}}{(z_2 - \frac{q}{t} z_1) \cdots (z_k - \frac{q}{t} z_{k-1})} \prod_{1 \le i < j \le k} \frac{(z_j - z_i)(z_j - \frac{q}{t} z_i)}{(z_j - q z_i)(z_j - \frac{1}{t} z_i)}, \tag{3.4.6}$$

$$\mathfrak{C}(Z, W) := \mathfrak{C}(Z, W; q, t) := \prod_{i=1}^k \prod_{j=1}^\ell \frac{(w_j - z_i)(w_j - \frac{q}{t} z_i)}{(w_j - q z_i)(w_j - \frac{1}{t} z_i)}. \tag{3.4.7}$$

Proposition 3.4.5. *Let $(\lambda^{(T)})_{T>0} \sim \mathbb{P}_{q,t}^{(\alpha_T,M_T)_{T>0},N}$. If $T_1 \geq \cdots \geq T_m > 0$ and $k_1,\ldots,k_m > 0$ are integers, then*

$$\mathbb{E}\left[\mathfrak{p}_{k_1}(\lambda^{(T_1)})\cdots\mathfrak{p}_{k_m}(\lambda^{(T_m)})\right] = \frac{1}{(2\pi\mathbf{i})^{k_1+\cdots+k_m}}\oint\cdots\oint \prod_{1\leq i<i'\leq m}\mathfrak{C}(Z_i,Z_{i'})\prod_{i=1}^{m}\mathfrak{B}(Z_i)\mathfrak{A}_{T_i}(Z_i)\,dZ_i$$

where $Z_i = (z_{i,1},\ldots,z_{i,k_i})$,

1. *the $z_{i,j}$ contour is positively oriented around the poles at $0, t^N$ but does not enclose $t^{-\alpha_\tau-M_\tau+1}$ for $1 \leq \tau \leq T_i$;*
2. *the contours satisfy $|z_{i,j}| < t|z_{i',j'}|$ for any $(i,j) < (i',j')$ in lexicographical order;*

given that such contours exist.

Remark 19. The contours above exist for N large enough as with Theorem 3.4.2.

Proof. Choose $T \geq \max(T_1,\ldots,T_m)$. Consider the measure $\mathfrak{M}$ on $\mathbb{Y}^T$ defined by

$$\mathfrak{M}(\mu^1,\ldots,\mu^T) \\ = \frac{1}{\Pi(\rho^+;\rho_1^-,\ldots,\rho_T^-)}P_{\mu^T}(\rho^+)Q_{\mu^T/\mu^{T-1}}(\rho_T^-)Q_{\mu^{T-1}/\mu^{T-2}}(\rho_{T-1}^-)\cdots Q_{\mu^2/\mu^1}(\rho_2^-)Q_{\mu^1}(\rho_1^-)$$

where $\rho^+ = (a_1,\ldots,a_N), \rho_\tau^- = (b_{\tau,1},\ldots,b_{\tau,M_\tau})$ and $a_i,\ b_{\tau,j} > 0$ are chosen so that the quantity above is summable over $(\mu^1,\ldots,\mu^T) \in \mathbb{Y}^T$. Define

$$G_\tau(z) = \prod_{i=1}^{N}\frac{1-t^{-1}a_iz^{-1}}{1-a_iz^{-1}}\prod_{\tau'=1}^{\tau}\prod_{i=1}^{M_{\tau'}}\frac{1-b_{\tau',j}z}{1-tb_{\tau',j}z}.$$

For such a distribution, [2, Theorem 3.8] shows that

$$\mathbb{E}_{\mathfrak{M}}[\mathfrak{p}_{k_1}(\mu^{T_1})\cdots\mathfrak{p}_{k_m}(\mu^{T_m})] = \frac{1}{(2\pi\mathbf{i})^{k_1+\cdots+k_m}}\oint\cdots\oint\prod_{1\leq i<i'\leq m}\mathfrak{C}(Z_i,Z_{i'})\prod_{i=1}^{m}\mathfrak{B}(Z_i)G_{T_i}(Z_i)\,dZ_i$$

where $|Z_i| = k_i$, the z_{ij} contour is positively oriented around all the poles of G_{T_i} among $0, a_1,\ldots,a_N$, but does not contain any poles of G_{T_i} among $b_{\tau,1},\ldots,b_{\tau,M_\tau}$ for $1 \leq \tau \leq T_i$, and $|z_{i,j}| < t|z_{i',j'}|$ for $(i,j) < (i',j')$ in lexicographical order. Setting $\rho^+ = \rho_0^N$ and $\rho_\tau^- = t^{\alpha_\tau}\rho_0^{M_\tau}$ gives the proposition. □

Proposition 3.4.6. *Let $(\lambda^{(T)})_{T>0} \sim \mathbb{P}_{q,q}^{(\alpha_T,M_T)_{T>0},N}$. If $T_1 \geq \cdots \geq T_m > 0$ and $0 < t_1,\ldots,t_m$, then*

$$\mathbb{E}\left[\prod_{i=1}^{m}\mathfrak{p}_{t_i}(\lambda^{(T_i)})\right] = \frac{1}{(2\pi\mathbf{i})^m}\oint\cdots\oint\prod_{1\leq i<j\leq m}\frac{(z_j-z_i)(t_iz_j-t_jz_i)}{(t_iz_j-z_i)(z_j-t_jz_i)} \\ \times\prod_{i=1}^{m}\left(\prod_{\ell=1}^{N}\frac{z_i-q^{\ell-1}}{z_i-t_iq^{\ell-1}}\cdot\prod_{\tau=1}^{T_i}\prod_{\ell=1}^{M_\tau}\frac{1-t_i^{-1}q^{\alpha_\tau+\ell-1}z_i}{1-q^{\alpha_\tau+\ell-1}z_i}\right)\frac{dz_i}{z_i}$$

where the z_i contour is positively oriented around $0, \{t_i q^{\ell-1}\}_{\ell=1}^N$ but does not enclose $q^{-\alpha_\tau-\ell+1}$ for $1 \le \ell \le M_\tau$, $1 \le \tau \le T_i$, and the contours satisfy $|z_i| < |t_i z_j|$ for $i < j$; given that such contours exist.

Proof. Let $(\lambda^{(T)})_{T\in\mathbb{Z}_{>0}} \sim \mathbb{P}_{q,q}^{(\alpha_T,M_T)_{T>0},N}$. By (3.3.13), for any integers $T_1 > \cdots > T_m > 0$, we have

$$\mathbb{P}(\lambda^{(T)} = \mu^1, \lambda^{(T-1)} = \mu^2, \ldots, \lambda^{(1)} = \mu^T) \\ = s_{\mu^1}(\rho_0^N) s_{\mu^1/\mu^2}(q^{\alpha_T}\rho_0^{M_T}) s_{\mu^2/\mu^3}(q^{\alpha_{T-1}}\rho_0^{M_{T-1}}) \cdots s_{\mu^T}(q^{\alpha_1}\rho_0^{M_1})$$

where $\rho_0^N := \rho_0^N(q,q) = (1, q, \ldots, q^{N-1})$. The distribution of $(\lambda^{(T)}, \ldots, \lambda^{(1)})$ is given by a Schur process (see Theorem 3.8.1). If $\rho = (q^{\alpha_T}\rho_0^{M_T}, \ldots, q^{\alpha_1}\rho_0^{M_1})$ and $(\nu^1, \ldots, \nu^{1+\sum_{i=1}^T M_i}) \sim \mathbb{SP}_{\rho_0^N,\rho}$, then

$$(\lambda^{(T)}, \lambda^{(T-1)}, \ldots, \lambda^{(1)}) \overset{distr}{=} (\nu^1, \nu^{1+M_T}, \ldots \nu^{1+\sum_{i=1}^T M_i})$$

by the branching rule on (3.8.3). Now Theorem 3.4.6 can be seen as a special case of Theorem 3.8.2. □

3.4.2 Excision of Poles at Zero

The proofs of Theorems 2.3.17 and 3.4.3 are essentially taking the appropriate limit transitions of Theorems 3.4.5 and 3.4.6. However, from a technical standpoint there is one preprocessing step. Namely, we want to remove the t^{-kN} term from $\mathfrak{p}_k$ (likewise t_i^{-N} from $\mathfrak{p}_{t_i}$); under the Jacobi limit the summand t^{-kN} grows faster than the term $\sum_{i=1}^N q^{\lambda_i} t^{-i+1}$ which contains the important information. It turns out that the removal of t^{-kN} corresponds to the excision of the pole at 0 in each of the contours.

Proposition 3.4.7. *Let $(\lambda^{(T)})_{T>0} \sim P_{q,t}^{(\alpha_T,M_T)_{T>0},N}$. If $T_1 \ge \cdots \ge T_m > 0$ and $k_1, \ldots, k_m > 0$ are integers, then*

$$\mathbb{E}\left[\prod_{i=1}^m (\mathfrak{p}_{k_i}(\lambda^{(T_i)}) - t^{-k_i N})\right] = \frac{1}{(2\pi\mathbf{i})^{k_1+\cdots+k_m}} \oint \cdots \oint \prod_{1\le i<i'\le m} \mathfrak{C}(Z_i, Z_{i'}) \prod_{i=1}^m \mathfrak{B}(Z_i)\mathfrak{A}_{T_i}(Z_i)\, dZ_i$$

where $Z_i = (z_{i,1}, \ldots, z_{i,k_i})$,

1. *the $z_{i,j}$ contour $\mathfrak{Y}_{i,j}$ is positively oriented around the pole at t^N but does not enclose $t^{-\alpha_\tau - M_\tau + 1}$ for $1 \le \tau \le T_i$;*
2. *$\mathfrak{Y}_{i,j}$ is contained in the contour $c\mathfrak{Y}_{i',j'}$ for any $(i,j) < (i',j')$ in lexicographical order and any $c \in [t, \frac{1}{q}]$;*

given that such contours exist.

Remark 20. There is an asymptotic version of Theorem 3.4.7 given by [32, Lemmas 4.14-4.17]; instead of asserting equality it asserts equality after the Jacobi limit. There are overlapping ideas in the proofs of Theorem 3.4.7 and the asymptotic version in [32].

Proposition 3.4.8. *Let $(\lambda^{(T)})_{T>0} \sim \mathbb{P}_{q,q}^{(\alpha_T,M_T)_{T>0},N}$. If $T_1 \geq \cdots \geq T_m > 0$ and $0 < t_1, \ldots, t_m < 1$, then*

$$\mathbb{E}\left[\prod_{i=1}^{m}(\mathfrak{p}_{t_i}(\lambda^{(T_i)}) - t_i^{-N})\right] = \frac{1}{(2\pi\mathbf{i})^m} \oint \cdots \oint \prod_{1\leq i<j\leq m} \frac{(z_j - z_i)(t_i z_j - t_j z_i)}{(t_i z_j - z_i)(z_j - t_j z_i)}$$
$$\times \prod_{i=1}^{m}\left(\prod_{\ell=1}^{N} \frac{z_i - q^{\ell-1}}{z_i - t_i q^{\ell-1}} \cdot \prod_{\tau=1}^{T_i}\prod_{\ell=1}^{M_\tau} \frac{1 - t_i^{-1} q^{\alpha_\tau+\ell-1} z_i}{1 - q^{\alpha_\tau+\ell-1} z_i}\right) \frac{dz_i}{z_i}$$

where the z_i contour $\mathfrak{Y}_i$ is positively oriented around $\{t_i q^{\ell-1}\}_{\ell=1}^N$ but does not enclose $q^{-\alpha_\tau-\ell+1}$ for $1 \leq \ell \leq M_\tau$, $1 \leq \tau \leq T_i$, and $\mathfrak{Y}_i$ is contained in $t_i\mathfrak{Y}_j$ for $i < j$; given that such contours exist.

Example 3.4.1. Consider the simple case where $m = 1$ with $k_1 = k$ and $T_1 = T$. We may change the contours in Theorem 3.4.5 to obtain the formula

$$\mathbb{E}\mathfrak{p}_k(\lambda^{(T)}) = \frac{1}{(2\pi\mathbf{i})^k} \oint_{\mathfrak{Y}_1\cup\mathfrak{Z}_1} \cdots \oint_{\mathfrak{Y}_k\cup\mathfrak{Z}_k} \mathfrak{B}(Z)\mathfrak{A}_T(Z)dZ \tag{3.4.8}$$

where $Z = (z_1, \ldots, z_k)$, $\mathfrak{Y}_i \cup \mathfrak{Z}_i$ is the z_i contour such that: (i) $\mathfrak{Y}_i$ is contained in $c\mathfrak{Y}_j$ and $\mathfrak{Z}_i$ is contained in $c\mathfrak{Z}_j$ for $1 \leq i < j \leq k$ and $c \in [t, \frac{1}{q}]$, (ii) $\mathfrak{Y}_i$ is positively oriented around the pole t^N but does not enclose $\{t^{-\alpha_\tau-M_\tau+1}\}_{\tau=1}^T$ and $\mathfrak{Z}_i$ is positively oriented around the pole at 0 but does not enclose $\{t^{-\alpha_\tau-M_\tau+1}\}_{\tau=1}^T$, (iii) $\mathfrak{Y}_k$ and $\mathfrak{Z}_k$ are disjoint from one another.

On the other hand, Proposition 3.4.7 asserts that

$$\mathbb{E}\left[\mathfrak{p}_k(\lambda^{(T)}) - t^{-kN}\right] = \frac{1}{(2\pi\mathbf{i})^k} \oint_{\mathfrak{Y}_1} \cdots \oint_{\mathfrak{Y}_k} \mathfrak{B}(Z)\mathfrak{A}_T(Z)\, dZ. \tag{3.4.9}$$

We show how to go from (3.4.8) to (3.4.9).

Proof of (3.4.9). Let $\mathcal{P}(Z)$ be the power set of $\{z_1, \ldots, z_k\}$. For each element $\Upsilon \in \mathcal{P}(Z)$, let $\Upsilon_i = \mathfrak{Z}_i$ if $z_i \in \Upsilon$ and $\Upsilon_i = \mathfrak{Y}_i$ if $z_i \notin \Upsilon$; Υ is exactly the set of z_i such that Υ_i encircles 0. Let

$$\mathfrak{I}_\Upsilon = \frac{1}{(2\pi\mathbf{i})^k} \oint_{\Upsilon_1} \cdots \oint_{\Upsilon_k} \mathfrak{B}(Z)\mathfrak{A}_T(Z)\, dZ.$$

Expand (3.4.8) so the z_i-contour is either $\mathfrak{Y}_i$ or $\mathfrak{Z}_i$. Thus

$$\mathbb{E}\mathfrak{p}_k(\lambda^{(T)}) = \sum_{\Upsilon\in\mathcal{P}(Z)} \mathfrak{I}_\Upsilon.$$

Observe that $\mathfrak{I}_{\{z_1,\ldots,z_k\}} = \mathfrak{A}_T(0)^k = t^{-kN}$ by evaluating residues. Also observe that due to the $\mathfrak{B}(Z)$ term, $\mathfrak{I}_\Upsilon = 0$ unless Υ is of the form $\{z_{r+1}, \ldots, z_{r+d}\}$. Identify $\Upsilon = \{z_{r+1}, \ldots, z_{r+d}\}$ with

(r, d) so that

$$\mathbb{E}\mathfrak{p}_k(\lambda^{(T)}) = t^{-kN} + \mathfrak{I}_\emptyset + \sum_{d=1}^{k-1}\sum_{r=0}^{k-d} \mathfrak{I}_{(r,d)}. \tag{3.4.10}$$

Consider the following three cases for $(r, d) = \Upsilon = \{z_{r+1}, \ldots, z_{r+d}\}$:

(I) $r = 0$. Evaluating the residues at 0 for $z_1, \ldots, z_d$, we obtain that $\mathfrak{I}_{(r,d)}$ is

$$\frac{1}{(2\pi\mathbf{i})^{k-d}} \oint \cdots \oint \frac{\mathfrak{A}_T(0)^d}{z_{d+1}} \frac{1}{(z_{d+2} - \frac{q}{t}z_{d+1}) \cdots (z_k - \frac{q}{t}z_{k-1})} \times \prod_{1 \le i < j \le r} \frac{(z_j - z_i)(z_j - \frac{q}{t}z_i)}{(z_j - qz_i)(z_j - \frac{1}{t}z_i)} \prod_{d < i \le k} \mathfrak{A}_T(z_i) dz_i.$$

We may change the contours so that

$$\mathfrak{I}_{(r,d)} = \frac{1}{(2\pi\mathbf{i})^{k-d}} \oint \cdots \oint \frac{\mathfrak{A}_T(0)^d}{w_1} \mathfrak{B}(w_1, \ldots, w_{k-d}) \prod_{i=1}^{k-d} \mathfrak{A}_T(w_i) dw_i$$

where the w_i-contour is $\mathfrak{Y}_i$.

(II) $0 < r < k - d$. Evaluating the residues at 0 for $z_{r+1}, \ldots, z_{r+d}$, we obtain that $\mathfrak{I}_{(r,d)}$ is

$$\frac{-1}{(2\pi\mathbf{i})^{k-d}} \oint \cdots \oint \frac{\mathfrak{A}_T(0)^d \frac{q^{r-1}}{t^{r-1}}}{z_r z_{r+d+1}} \prod_{\substack{1 \le i < k \\ i \notin [r, r+d]}} (z_{i+1} - \frac{q}{t}z_i)^{-1} \times \prod_{\substack{1 \le i < j \le k \\ i,j \notin (r, r+d]}} \frac{(z_j - z_i)(z_j - \frac{q}{t}z_i)}{(z_j - qz_i)(z_j - \frac{1}{t}z_i)} \prod_{\substack{1 \le i \le k \\ i \notin (r, r+d]}} \mathfrak{A}_T(z_i) dz_i.$$

Reindexing the variables via

$$(z_1, \ldots, z_r) \mapsto (w_1, \ldots, w_r),$$
$$(z_{r+d+1}, \ldots, z_k) \mapsto (w_{r+1}, \ldots, w_{k-d}),$$

we obtain

$$\mathfrak{I}_{(r,d)} = -\frac{1}{(2\pi\mathbf{i})^{k-d}} \oint \cdots \oint \frac{\mathfrak{A}_T(0)^d \frac{q^{r-1}}{t^{r-1}}}{w_r w_{r+1}} (w_{r+1} - \frac{q}{t}w_r) \mathfrak{B}(w_1, \ldots, w_{k-d}) \prod_{i=1}^{k-d} \mathfrak{A}_T(w_i) dw_i$$

where we may take the w_i-contour to be $\mathfrak{Y}_i$.

(III) $r = k - d$. Evaluating the residues at 0 for $z_{r+1}, \dots, z_k$, we obtain that $\mathfrak{I}_{(r,d)}$ is

$$\frac{-1}{(2\pi\mathbf{i})^{k-d}} \oint \cdots \oint \frac{\mathfrak{A}_T(0)^d \frac{q^{r-1}}{t^{r-1}}}{z_r} \frac{1}{(z_2 - \frac{q}{t}z_1)\cdots(z_r - \frac{q}{t}z_{r-1})} \times \prod_{1 \le i < j \le r} \frac{(z_j - z_i)(z_j - \frac{q}{t}z_i)}{(z_j - qz_i)(z_j - \frac{1}{t}z_i)} \prod_{1 \le i \le r} \mathfrak{A}_T(z_i) dz_i.$$

This is exactly

$$\mathfrak{I}_{(r,d)} = -\frac{1}{(2\pi\mathbf{i})^{k-d}} \oint \cdots \oint \frac{\mathfrak{A}_T(0)^d \frac{q^{k-d-1}}{t^{k-d-1}}}{w_{k-d}} \mathfrak{B}(w_1, \dots, w_{k-d}) \prod_{i=1}^{k-d} \mathfrak{A}_T(w_i) dw_i$$

where the w_i-contour is $\mathfrak{Y}_i$.

Fix $0 < d < k$. Combining the three cases and letting $W = (w_1, \dots, w_{k-d})$, we may write

$$\sum_{r=0}^{k-d} \mathfrak{I}_{(r,d)} = \frac{\mathfrak{A}_T(0)^d}{(2\pi\mathbf{i})^{k-d}} \oint \cdots \oint dW \cdot \mathfrak{B}(W)\mathfrak{A}_T(W) \left(\frac{1}{w_1} - \sum_{r=1}^{k-d-1} \frac{\frac{q^{r-1}}{t^{r-1}}}{w_r w_{r+1}} (w_{r+1} - \frac{q}{t} w_r) - \frac{\frac{q^{k-d-1}}{t^{k-d-1}}}{w_{k-d}} \right)$$

because the w_i-contours from each of the different cases were set as $\mathfrak{Y}_i$ for $1 \le i \le k - d$. The integral vanishes because the parenthesized term may be rewritten as

$$\frac{1}{w_1} + \sum_{r=1}^{k-d-1} \left(-\frac{1}{w_r} \frac{q^{r-1}}{t^{r-1}} + \frac{1}{w_{r+1}} \frac{q^r}{t^r} \right) - \frac{1}{w_{k-d}} \frac{q^{k-d-1}}{t^{k-d-1}} = 0.$$

Therefore (3.4.10) simplifies to

$$\mathbb{E}\mathfrak{p}_k(\lambda^{(T)}) = t^{-kN} + \mathfrak{I}_\emptyset$$

which is exactly (3.4.9). □

The general proof of Proposition 3.4.7 is just a factorization into cases of the example above and an inclusion-exclusion argument.

Proof of Theorem 3.4.7. Define contours $\mathfrak{Y}_{i,j}, \mathfrak{Z}_{i,j}$ such that (i) $\mathfrak{Y}_{i,j}$ is contained in $c\mathfrak{Y}_{i',j'}$ and $\mathfrak{Z}_{i,j}$ is contained in $c\mathfrak{Z}_{i',j'}$ whenever $(i,j) < (i',j')$ in lexicographical order and $c \in [t, \frac{1}{q}]$, (ii) $\mathfrak{Y}_{i,j}$ is positively oriented around t^N but does not enclose $t^{-\alpha_\tau - M_\tau + 1}$ for $1 \le \tau \le T_i$ and $\mathfrak{Z}_{i,j}$ is positively oriented around 0 but does not enclose $t^{-\alpha_\tau - M_\tau + 1}$ for $1 \le \tau \le T_i$, (iii) $\mathfrak{Y}_{m,k_m}$ and $\mathfrak{Z}_{m,k_m}$ are disjoint from one another. Note that if contours $\mathfrak{Y}_{i,j}$ $(1 \le j \le k_i, 1 \le i \le m)$ satisfying (i) and (ii) exist, then the existence of contours $\mathfrak{Z}_{i,j}$ $(1 \le j \le k_i, 1 \le i \le m)$ satisfying (i), (ii), (iii) is guaranteed by choosing these contours close enough to 0.

Let $\mathcal{P}(Z_1, \dots, Z_m)$ be the power set of $\{z_{i,j}\}_{1 \le j \le k_i, 1 \le i \le m}$. For each element $\Upsilon \in \mathcal{P}(Z_1, \dots, Z_m)$, let $\Upsilon_{i,j} = \mathfrak{Z}_{i,j}$ if $(i,j) \in \Upsilon$ and $\Upsilon_{i,j} = \mathfrak{Y}_{i,j}$ if $(i,j) \notin \Upsilon$. Let

$$\mathfrak{I}_\Upsilon = \frac{1}{(2\pi\mathbf{i})^{k_1+\cdots+k_m}} \oint_{\Upsilon_{1,1}} \cdots \oint_{\Upsilon_{m,k_m}} \prod_{1\le i<i'\le m} \mathfrak{C}(Z_i, Z_{i'}) \prod_{i=1}^m \mathfrak{B}(Z_i)\mathfrak{A}_{T_i}(Z_i)dZ_i.$$

As in Example 3.4.1, we can use Proposition 3.4.5 and expand the contours as either $\mathfrak{Y}_{i,j}$ or $\mathfrak{Z}_{i,j}$ so that

$$\mathbb{E}\left[\mathfrak{p}_{k_1}(\lambda^{(T_1)})\cdots\mathfrak{p}_{k_m}(\lambda^{(T_m)})\right] = \sum_{\Upsilon\in\mathcal{P}(Z_1,\ldots,Z_m)} \mathfrak{I}_\Upsilon.$$

For an analogous reason as in Example 3.4.1, $\mathfrak{I}_\Upsilon = 0$ unless $\Upsilon \cap Z_i$ has the form $\{z_{i,r_i+1}, \ldots, z_{i,r_i+d_i}\}$ for each $1 \le i \le m$ in which case we identify Υ with the ordered pair $(\boldsymbol{r}, \boldsymbol{d})$ where $\boldsymbol{r} = (r_1, \ldots, r_m)$ and $\boldsymbol{d} = (d_1, \ldots, d_m)$; if $\Upsilon \cap Z_i = \emptyset$ then $d_i = 0$ and take $r_i = 0$ as convention. Let $\mathcal{R}_{\boldsymbol{d}}$ denote the set of $\boldsymbol{r}$ where $0 \le r_i \le k_i - d_i$ if $d_i > 0$ and $r_i = 0$ if $d_i = 0$. Then

$$\mathbb{E}\left[\mathfrak{p}_{k_1}(\lambda^{(T_1)})\cdots\mathfrak{p}_{k_m}(\lambda^{(T_m)})\right] = \sum_{\substack{0\le d_i\le k_i\\ 1\le i\le m}} \sum_{\boldsymbol{r}\in\mathcal{R}_{\boldsymbol{d}}} \mathfrak{I}_{(\boldsymbol{r},\boldsymbol{d})}. \tag{3.4.11}$$

Fix $1 \le i \le k$. Let $W = (w_1, \ldots, w_k)$, define $A^{(i)}_{0,0}(W) = 1$ and $A^{(i)}_{0,k}(W) = \mathfrak{A}_{T_i}(0)^k = t^{-kN}$. For $0 < d < k$ and $0 \le r \le k - d$ define

$$A^{(i)}_{r,d}(W) = \begin{cases} \mathfrak{A}_{T_i}(0)^d \dfrac{1}{w_1} & r = 0, \\ -\mathfrak{A}_{T_i}(0)^d \dfrac{\frac{q^{r-1}}{t^{r-1}}}{w_r w_{r+1}}(w_{r+1} - \dfrac{q}{t}w_r) & 0 < r < k-d, \\ -\mathfrak{A}_{T_i}(0)^d \dfrac{\frac{q^{k-d-1}}{t^{k-d-1}}}{w_{k-d}} & r = k-d. \end{cases}$$

These were the differing parts of the integrand in Example 3.4.1 from the three cases where we showed

$$\sum_{r=0}^{k-d} A^{(i)}_{r,d}(W) = 0.$$

Consider $\mathfrak{I}_{(\boldsymbol{r},\boldsymbol{d})}$ and evaluate the residues of the $\mathfrak{Z}_{i,j}$ contours (which are necessarily at 0) in lexicographical order of (i,j). Then by similar reasoning as in Example 3.4.1, we have

$$\mathfrak{I}_{(\boldsymbol{r},\boldsymbol{d})} = \frac{1}{(2\pi\mathbf{i})^{\sum_{i=1}^m (k_i-d_i)}} \oint \cdots \oint \prod_{1\le i<i'\le k} \mathfrak{C}(W_i, W_{i'}) \prod_{i=1}^m \mathfrak{B}(W_i)\mathfrak{A}_{T_i}(W_i)A^{(i)}_{r_i,d_i}(W_i)dW_i$$

where $W_i = (w_{i,1}, \ldots, w_{i,k_i-d_i})$ and the $w_{i,j}$-contour can be taken to be $\mathfrak{Y}_{i,j}$. Notice that for any fixed $\boldsymbol{d}$, we have

$$\sum_{\boldsymbol{r}\in\mathcal{R}_{\boldsymbol{d}}}\prod_{i=1}^{m}A^{(i)}_{r_i,d_i}(W_i)=\prod_{i:d_i=0}A^{(i)}_{0,0}(W_i)\prod_{i:d_i>0}\sum_{r_i=0}^{k_i-d_i}A^{(i)}_{r_i,d_i}(W_i)=\prod_{i:d_i>0}\sum_{r_i=0}^{k_i-d_i}A^{(i)}_{r_i,d_i}(W_i)$$

where the latter is zero unless we have that $d_i=0$ or $d_i=k_i$ for each $1\leq i\leq m$. Thus (3.4.11) becomes

$$\mathbb{E}\left[\mathfrak{p}_{k_1}(\lambda^{(T_1)})\cdots\mathfrak{p}_{k_m}(\lambda^{(T_m)})\right]=\sum_{\substack{\Upsilon\cap Z_i=\emptyset\text{ or }Z_i\\1\leq i\leq m}}\mathfrak{I}_{\Upsilon}. \tag{3.4.12}$$

Given a subset $S\subset[[1,m]]$, let

$$\mathcal{I}_S=\frac{1}{(2\pi\mathbf{i})^{\sum_{i\in S}k_i}}\oint\cdots\oint\prod_{\substack{1\leq i<i'\leq k\\i,i'\in S}}\mathfrak{C}(W_i,W_{i'})\prod_{i\in S}\mathfrak{B}(W_i)\mathfrak{A}_{T_i}(W_i)dW_i$$

where the $w_{i,j}$ contour is $\mathfrak{Y}_{i,j}$. Once again, by evaluating the residues of the $\mathfrak{Z}_{i,j}$ contours in lexicographical order of (i,j), we have

$$\mathfrak{I}_{\bigcup_{i\notin S}Z_i}=\mathcal{I}_S\prod_{i\notin S}\mathfrak{A}_{T_i}(0)^{k_i}=t^{-N\sum_{i\in[[1,m]]\setminus S}k_i}\cdot\mathcal{I}_S.$$

Since (3.4.12) is over those Υ which are unions of Z_i, we may write

$$\mathbb{E}\left[\mathfrak{p}_{k_1}(\lambda^{(T_1)})\cdots\mathfrak{p}_{k_m}(\lambda^{(T_m)})\right]=\sum_{S\subset[[1,m]]}t^{-N\sum_{i\in[[1,m]]\setminus S}k_i}\cdot\mathcal{I}_S.$$

In particular, observe that for any $S\subset[[1,m]]$,

$$\mathbb{E}\left[\prod_{i\in S}\mathfrak{p}_{k_i}(\lambda^{(T_i)})\right]=\sum_{S'\subset S}t^{-N\sum_{i\in S\setminus S'}k_i}\cdot\mathcal{I}_{S'}.$$

Then by inclusion-exclusion

$$\mathcal{I}_{[[1,m]]}=\sum_{S\subset[[1,m]]}(-1)^{m-|S|}t^{-N\sum_{i\in[[1,m]]\setminus S}k_i}\mathbb{E}\left[\prod_{i\in S}\mathfrak{p}_{k_i}(\lambda^{(T_i)})\right]=\mathbb{E}\left[\prod_{i=1}^{m}\left(\mathfrak{p}_{k_i}(\lambda^{(T_i)})-t^{-k_iN}\right)\right]$$

which completes the proof. □

The proof of Theorem 3.4.8 follows in the same manner as that of Theorem 3.4.7; we omit the proof to avoid repetition.

3.4.3 Jacobi Limit

Using the observables from Proposition 3.4.7 and the Jacobi limit in Theorem 3.3.12, we obtain observables for the β-Jacobi product process.

Proof of Theorem 3.4.2. For $\varepsilon > 0$, let $q = e^{-\varepsilon}, t = q^\theta$ and $(\lambda^{(T)} := \lambda^{(T)}(\varepsilon))_{T\in\mathbb{Z}_{>0}} \sim \mathbb{P}_{q,t}^{(\alpha_T, M_T)_{T>0}, N}$. Observe that for $k \geq 0$ we have

$$t^{-kN} - \mathfrak{p}_k(\lambda^{(T)}; q, t) = (e^{\theta k\varepsilon} - 1)\sum_{i=1}^N e^{-k\varepsilon\lambda_i^{(T)}} e^{-\theta k\varepsilon(-i+1)}.$$

By the convergences

$$\frac{e^{\theta k\varepsilon} - 1}{k\theta\varepsilon} \to 1, \qquad e^{\theta k\varepsilon(-i+1)} \to 1 \qquad (1 \leq i \leq N)$$

as $\varepsilon \to 0$ and Theorem 3.3.12, we have the convergence

$$\lim_{\varepsilon\to 0}\mathbb{E}\left[\frac{t^{-k_1 N} - \mathfrak{p}_{k_1}(\lambda^{(T_1)}; q, t)}{k_1\theta\varepsilon}\cdots\frac{t^{-k_m N} - \mathfrak{p}_{k_m}(\lambda^{(T_m)}; q, t)}{k_m\theta\varepsilon}\right] \to \mathbb{E}\left[\mathfrak{P}_{k_1}(\mathbf{y}^{(T_1)})\cdots\mathfrak{P}_{k_m}(\mathbf{y}^{(T_m)})\right] \tag{3.4.13}$$

as $\varepsilon \to 0$, where $(\mathbf{y}^{(T)})_{T\in\mathbb{Z}_{>0}} \sim \mathbb{P}_\theta^{(\alpha_T, M_T)_{T>0}}$.

We compute the limit on the left hand side of (3.4.13) to obtain an expression for the right hand side. Proposition 3.4.7 yields the contour integral formula

$$\mathbb{E}\left[\prod_{i=1}^m \frac{t^{-k_i N} - \mathfrak{p}_{k_i}(\lambda^{(T_i)}; q, t)}{k_i\theta\varepsilon}\right] = \frac{\prod_{i=1}^m(-k_i\theta\varepsilon)^{-1}}{(2\pi\mathbf{i})^{\sum_{i=1}^m k_i}}\oint\cdots\oint\prod_{1\leq i<i'\leq m}\mathfrak{C}(Z_i, Z_{i'})\prod_{i=1}^m\mathfrak{B}(Z_i)\mathfrak{A}_{T_i}(Z_i)\,dZ_i$$

where $Z_i = (z_{i,1}, \ldots, z_{i,k_i})$,

- the $z_{i,j}$ contour $\mathfrak{Y}_{i,j}$ is positively oriented around t^N but does not enclose $t^{-\alpha_\tau - M_\tau + 1}$ for $1 \leq \tau \leq T_i$;
- the $\mathfrak{Y}_{i,j}$ contour is contained in the contour $c\mathfrak{Y}_{i',j'}$ for any $(i,j) < (i',j')$ in lexicographical order and any $c \in [t, \frac{1}{q}]$.

Note that we may take the $z_{i,j}$ contours close to t^N. Changing variables $z_{i,j} = e^{\varepsilon u_{i,j}}$, we have

$$\begin{aligned}&\mathbb{E}\left[\prod_{i=1}^m \frac{t^{-k_i N} - \mathfrak{p}_{k_i}(\lambda^{(T_i)}; q, t)}{k_i\theta\varepsilon}\right]\\ &\quad= \frac{\prod_{i=1}^m(-k_i\theta)^{-1}}{(2\pi\mathbf{i})^{\sum_{i=1}^m k_i}}\oint\cdots\oint\prod_{1\leq i<i'\leq m}\mathfrak{C}(e^{\varepsilon U_i}, e^{\varepsilon U_{i'}})\prod_{i=1}^m \varepsilon^{k_i - 1}\mathfrak{B}(e^{\varepsilon U_i})\mathfrak{A}_{T_i}(e^{\varepsilon U_i})\,e^{\varepsilon\sum_{j=1}^{k_i}u_{i,j}}\,dU_i\end{aligned}$$

where $U_i = (u_{i,1}, \ldots, u_{i,k_i})$, we denote $e^{\varepsilon U_i} := (e^{\varepsilon u_{i,1}}, \ldots, e^{\varepsilon u_{i,k_i}})$,

- the $u_{i,j}$ contour $\mathfrak{U}_{i,j}$ is positively oriented around the pole at $-\theta N$ but does not enclose $\theta(\alpha_\tau + M_\tau - 1)$ for $1 \le \tau \le T_i$;
- whenever $(i,j) < (i',j')$ in lexicographical order, $\mathfrak{U}_{i,j}$ is enclosed by the contour $\mathfrak{U}_{i',j'} + c$ for $c \in [-\theta, 1]$.

These contours are independent of $\varepsilon > 0$. We have the convergences

$$\mathfrak{A}_T(e^{\varepsilon U_i}) \to \mathcal{A}_T(U_i), \qquad \varepsilon^{k_i - 1}\mathfrak{B}(e^{\varepsilon U_i}) \to k_i \mathcal{B}(U_i), \qquad \mathfrak{C}(e^{\varepsilon U_i, \varepsilon U_{i'}}) \to \mathcal{C}(U_i, U_{i'})$$

as $\varepsilon \to 0$ uniformly over $u_{i,j} \in \mathfrak{U}_{i,j}$ where $1 \le j \le k_i, 1 \le i \le m$. Therefore

$$\lim_{\varepsilon \to 0} \mathbb{E}\left[\prod_{i=1}^m \frac{t^{-k_i N} - \mathfrak{p}_{k_i}(\lambda^{(T_i)}; q, t)}{k_i \theta \varepsilon}\right] = \frac{(-\theta)^{-m}}{(2\pi\mathbf{i})^{\sum_{i=1}^m k_i}} \oint \cdots \oint \prod_{1 \le i < i' \le m} \mathcal{C}(U_i, U_{i'}) \prod_{i=1}^m \mathcal{B}(U_i) \mathcal{A}_{T_i}(U_i)\, dU_i$$

where the $u_{i,j}$ is $\mathfrak{U}_{i,j}$ as defined above. The theorem now follows from (3.4.13). □

The proof of Theorem 3.4.3. We highlight some of the modifications, but omit more details to avoid repetition.

Proof of Theorem 3.4.3. For $\varepsilon > 0$, let $q = e^{-\varepsilon}$, $t_i = q^{c_i}$ for $1 \le i \le m$ and $(\lambda^{(T)} := \lambda^{(T)}(\varepsilon))_{T \in \mathbb{Z}_{>0}} \sim \mathbb{P}_{q,q}^{(\alpha_T, M_T)_{T>0}, N}$. Theorem 3.4.3 now follows by

$$\lim_{\varepsilon \to 0} \mathbb{E}\left[\prod_{i=1}^m \frac{t_i^{-k_i N} - \mathfrak{p}_{t_i}(\lambda^{(T_i)})}{c_i \varepsilon}\right] \to \mathbb{E}\left[\prod_{i=1}^m \mathfrak{P}_{c_i}(\mathbf{y}^{(T_i)})\right]$$

and the change of variables $z_i = e^{\varepsilon u_i}$ on the formula given by Theorem 3.4.8. □

3.5 Global Asymptotics

In this section we prove Theorems 3.1.2 and 3.1.3. We begin by reformulating the theorems in terms of convergence of moments.

Assumption 1. *Fix $\theta > 0$. Let $\alpha_T := \alpha_T(N) > 0, M_T := M_T(N) \in \mathbb{Z}_{>0}$ such that $M_T \ge N$ satisfy*

$$\frac{\alpha_T}{N} \to \widehat{\alpha}_T \quad \text{and} \quad \frac{M_T}{N} \to \widehat{M}_T \tag{3.5.1}$$

as $N \to \infty$ for some $\widehat{\alpha}_T \ge 0$, $\widehat{M}_T \ge 1$. Let $(\mathbf{y}^{(T)} := \mathbf{y}^{(T)}(N))_{T \in \mathbb{Z}_{>0}} \sim \mathbb{P}_\theta^{(\alpha_T, M_T)_{T>0}, N}$.

Theorem 3.5.1. *Under Assumption 1, for any positive integers k, T we have*

$$\lim_{N \to \infty} \frac{1}{N} \mathbb{E}[\mathfrak{P}_k(\mathbf{y}^{(T)})] = -\frac{1}{k} \cdot \frac{1}{2\pi\mathbf{i}} \oint \left(\frac{v}{v+1} \prod_{\tau=1}^T \frac{v - \widehat{\alpha}_\tau}{v - \widehat{\alpha}_\tau - \widehat{M}_\tau}\right)^k dv$$

where the contour is positively oriented around the pole at -1 but does not enclose $\widehat{\alpha}_\tau + \widehat{M}_\tau$ for $1 \leq \tau \leq T$.

Theorem 3.5.2. *Under Assumption 1, for any positive integers $k_1, k_2, T_1 \geq T_2$ we have*

$$\lim_{N\to\infty} \operatorname{Cov}\left(\mathfrak{P}_{k_1}(\mathbf{y}^{(T_1)}), \mathfrak{P}_{k_2}(\mathbf{y}^{(T_2)})\right) = \frac{\theta^{-1}}{(2\pi\mathbf{i})^2} \oint\oint \frac{1}{(v_2-v_1)^2} \prod_{i=1}^{2} \left(\frac{v_i}{v_i+1} \prod_{\tau=1}^{T_i} \frac{v_i - \widehat{\alpha}_\tau}{v_i - \widehat{\alpha}_\tau - \widehat{M}_\tau}\right)^{k_i} dv_i$$

where the v_2 contour encloses the v_1 contour, both v_1, v_2 contours are positively oriented around -1, but the v_1 contour does not contain $\widehat{\alpha}_\tau + \widehat{M}_\tau$ for $1 \leq \tau \leq T_1$ and the v_2 contour does not contain $\widehat{\alpha}_\tau + \widehat{M}_\tau$ for $1 \leq \tau \leq T_2$.

Theorem 3.5.3. *Under Assumption 1, for any positive integers $m \geq 3, k_1, \ldots, k_m, T_1, \ldots, T_m$ we have*

$$\kappa\left(\mathfrak{P}_{k_1}(\mathbf{y}^{(T_1)}), \cdots, \mathfrak{P}_{k_m}(\mathbf{y}^{(T_m)})\right) \to 0$$

as $N \to \infty$.

We show how these theorems imply Theorems 3.1.2 and 3.1.3.

Proof of Theorem 3.1.2. This follows from the convergence Theorem 3.5.1 and the fact that

$$\lim_{N\to\infty} \operatorname{Var}\left(\frac{1}{N}\mathfrak{P}_k(\mathbf{y}^{(T)})\right) = 0$$

by Theorem 3.5.2. □

Proof of Theorem 3.1.3. Theorem 3.5.2 implies the covariance formula (3.1.5). To prove asymptotic Gaussianity, first observe that for $\nu \geq 3$ and $1 \leq i_1, \ldots, i_\nu \leq k$, we have

$$\begin{aligned}\kappa\Big(\mathfrak{P}_{k_{i_1}}(\mathbf{y}^{(T_{i_1})}) - \mathbb{E}[\mathfrak{P}_{k_{i_1}}(\mathbf{y}^{(T_{i_1})})], \ldots, \mathfrak{P}_{k_{i_\nu}}(\mathbf{y}^{(T_{i_\nu})}) - \mathbb{E}[\mathfrak{P}_{k_{i_\nu}}(\mathbf{y}^{(T_{i_\nu})})]\Big) \\ = \kappa\Big(\mathfrak{P}_{k_{i_1}}(\mathbf{y}^{(T_{i_1})}), \ldots, \mathfrak{P}_{k_{i_\nu}}(\mathbf{y}^{(T_{i_\nu})})\Big)\end{aligned}$$

by (3.7.2). By Theorem 3.5.3, the latter converges to 0. Theorem 3.7.3 then implies asymptotic Gaussianity of (3.1.4). □

The remainder of this section is devoted to the proofs of Theorems 3.5.1 to 3.5.3. We first state a useful lemma.

Lemma 3.5.4 ([32, Corollary A.2])**.** *Let s be a positive integer. Let $f, g_1 \ldots, g_s$ be meromorphic* *functions with possible poles at $\{P_1, \ldots, P_m\}$. Then for $n \geq 2$,*

$$\frac{1}{(2\pi\mathbf{i})^n} \oint \cdots \oint \frac{1}{(v_2 - v_1) \cdots (v_n - v_{n-1})} \prod_{i=1}^{n} f(v_i) dv_i \prod_{i=1}^{s} \left(\sum_{j=1}^{n} g_i(v_j) \right)$$
$$= \frac{n^{s-1}}{2\pi\mathbf{i}} \oint f(v)^n \prod_{i=1}^{s} g_i(v) dv,$$

where the contours in both sides are positively oriented around all of $\{P_1, \ldots, P_m\}$, and for the left hand side we require the v_j contour to contain the v_i contour for $1 \leq i < j \leq n$.

3.5.1 Proof of Theorem 3.5.1

By Theorem 3.4.2, we have

$$\mathbb{E}\left[\mathfrak{P}_k(\mathbf{y}^{(T)})\right] = \frac{-\theta^{-1}}{(2\pi\mathbf{i})^k} \oint \cdots \oint \frac{1}{(u_2 - u_1 + 1 - \theta) \cdots (u_k - u_{k-1} + 1 - \theta)}$$
$$\times \prod_{1 \leq i < j \leq k} \frac{(u_j - u_i)(u_j - u_i + 1 - \theta)}{(u_j - u_i + 1)(u_j - u_i - \theta)} \cdot \prod_{i=1}^{k} \frac{u_i}{u_i + \theta N} \prod_{\tau=1}^{T} \frac{u_i - \theta(\alpha_\tau - 1)}{u_i - \theta(\alpha_\tau + M_\tau - 1)} du_i.$$

where

1. the u_i contour $\mathfrak{U}_i$ is positively oriented around $-\theta N$ but does not enclose $\theta(\alpha_\tau + M_\tau - 1)$ for $1 \leq \tau \leq T$;
2. whenever $i < j$, $\mathfrak{U}_i$ is enclosed by the contour $\mathfrak{U}_j + c$ for $c \in [-\theta, 1]$.

Changing variables $u_i = v_i \theta N$, we obtain

$$\frac{1}{N}\mathbb{E}\left[\mathfrak{P}_k(\mathbf{y}^{(T)})\right] = \frac{-1}{(2\pi\mathbf{i})^k} \oint \cdots \oint \frac{1}{(v_2 - v_1 + \frac{1-\theta}{\theta N}) \cdots (v_k - v_{k-1} + \frac{1-\theta}{\theta N})}$$
$$\times \prod_{1 \leq i < j \leq k} \frac{(v_j - v_i)(v_j - v_i + \frac{1-\theta}{\theta N})}{(v_j - v_i + \frac{1}{\theta N})(v_j - v_i - \frac{1}{N})} \cdot \prod_{i=1}^{k} \frac{v_i}{v_i + 1} \prod_{\tau=1}^{T} \frac{v_i - \frac{\alpha_\tau - 1}{N}}{v_i - \frac{\alpha_\tau + M_\tau - 1}{N}} dv_i.$$

Then

$$\lim_{N \to \infty} \frac{1}{N}\mathbb{E}\left[\mathfrak{P}_k(\mathbf{y}^{(T)})\right] = \frac{-1}{(2\pi\mathbf{i})^k} \oint \cdots \oint \frac{1}{(v_2 - v_1) \cdots (v_k - v_{k-1})} \prod_{i=1}^{k} \frac{v_i}{v_i + 1} \prod_{\tau=1}^{T} \frac{v_i - \widehat{\alpha}_\tau}{v_i - \widehat{\alpha}_\tau - \widehat{M}_\tau} dv_i.$$

where

1. the v_i contour $\mathfrak{V}_i$ is positively oriented around -1 but does not enclose $\widehat{\alpha}_\tau + \widehat{M}_\tau$ for $1 \leq \tau \leq T$;
2. whenever $i < j$, $\mathfrak{V}_i$ is enclosed by the contour $\mathfrak{V}_j$.

The theorem now follows by applying Theorem 3.5.4 to the multidimensional integral above. □

3.5.2 Proof of Theorem 3.5.2

By Theorem 3.4.2, we have

$$\operatorname{Cov}\left(\mathfrak{P}_{k_1}(\mathbf{y}^{(T_1)}),\mathfrak{P}_{k_2}(\mathbf{y}^{(T_2)})\right)=\frac{\theta^{-2}}{(2\pi\mathbf{i})^{k_1+k_2}}\oint\cdots\oint(\mathcal{C}(U_1,U_2)-1)\prod_{i=1}^{2}\mathcal{B}(U_i)\mathcal{A}_{T_i}(U_i)\,dU_i$$

where $U_i=(u_{i,1},\dots,u_{i,k_i})$ for $i=1,2$,

1. the $u_{i,j}$ contour $\mathfrak{U}_{i,j}$ is positively oriented around $-\theta N$ but does not enclose $\theta(\alpha_\tau+M_\tau-1)$ for $1\le\tau\le T_i$;
2. whenever $(i,j)<(i',j')$ in lexicographical order, $\mathfrak{U}_{i,j}$ is enclosed by the contour $\mathfrak{U}_{i',j'}+c$ for $c\in[-\theta,1]$.

Changing variables $u_{i,j}=v_{i,j}\theta N$, we obtain

$$\begin{aligned}&\operatorname{Cov}\left(\mathfrak{P}_{k_1}(\mathbf{y}^{(T_1)}),\mathfrak{P}_{k_2}(\mathbf{y}^{(T_2)})\right)\\&\quad=\frac{1}{(2\pi\mathbf{i})^{k_1+k_2}}\oint\cdots\oint N^2\left(\prod_{i=1}^{k_1}\prod_{j=1}^{k_2}\frac{(v_{2,j}-v_{1,i})(v_{2,j}-v_{1,i}+\frac{1-\theta}{\theta N})}{(v_{2,j}-v_{1,i}+\frac{1}{\theta N})(v_{2,j}-v_{1,i}-\frac{1}{N})}-1\right)\\&\times\prod_{i=1}^{2}\frac{1}{(v_{i,2}-v_{i,1}+\frac{1-\theta}{\theta N})\cdots(v_{i,k_i}-v_{i,k_i-1}+\frac{1-\theta}{\theta N})}\left(\prod_{1\le j<j'\le k_i}\frac{(v_{i,j'}-v_{i,j})(v_{i,j'}-v_{i,j}+\frac{1-\theta}{\theta N})}{(v_{i,j'}-v_{i,j}+\frac{1}{\theta N})(v_{i,j'}-v_{i,j}-\frac{1}{N})}\right)\\&\qquad\times\prod_{j=1}^{k_i}\frac{v_{i,j}}{v_{i,j}+1}\prod_{\tau=1}^{T_i}\frac{v_{i,j}-\frac{\alpha_\tau-1}{N}}{v_{i,j}-\frac{\alpha_\tau+M_\tau-1}{N}}\,dv_{i,j}.\end{aligned}$$

We may take the $v_{i,j}$ contours to be independent of N for N sufficiently large. Observe that

$$\begin{aligned}\prod_{i=1}^{k_1}\prod_{j=1}^{k_2}\frac{(v_{2,j}-v_{1,i})(v_{2,j}-v_{1,i}+\frac{1-\theta}{\theta N})}{(v_{2,j}-v_{1,i}+\frac{1}{\theta N})(v_{2,j}-v_{1,i}-\frac{1}{N})}&=\prod_{i=1}^{k_1}\prod_{j=1}^{k_2}\left(1+\frac{\theta^{-1}}{N^2}\cdot\frac{1}{(v_{2,j}-v_{1,i}+\frac{1}{\theta N})(v_{2,j}-v_{1,i}-\frac{1}{N})}\right)\\&=1+\frac{\theta^{-1}}{N^2}\sum_{i=1}^{k_1}\sum_{j=1}^{k_2}\frac{1}{(v_{2,j}-v_{1,i})^2}+O\left(\frac{1}{N^3}\right)\end{aligned}\tag{3.5.2}$$

where the $O(1/N^3)$ term is uniform over the $v_{i,j}$ contours. Then

$$\begin{aligned}\lim_{N\to\infty}\operatorname{Cov}\left(\mathfrak{P}_{k_1}(\mathbf{y}^{(T_1)}),\mathfrak{P}_{k_2}(\mathbf{y}^{(T_2)})\right)&=\frac{\theta^{-1}}{(2\pi\mathbf{i})^{k_1+k_2}}\oint\cdots\oint\sum_{i=1}^{k_1}\sum_{j=1}^{k_2}\frac{1}{(v_{2,j}-v_{1,i})^2}\\&\times\prod_{i=1}^{2}\frac{1}{(v_{i,2}-v_{i,1})\cdots(v_{i,k_i}-v_{i,k_i-1})}\prod_{j=1}^{k_i}\frac{v_{i,j}}{v_{i,j}+1}\prod_{\tau=1}^{T_i}\frac{v_{i,j}-\widehat{\alpha}_\tau}{v_{i,j}-\widehat{\alpha}_\tau-\widehat{M}_\tau}\,dv_{i,j}.\end{aligned}$$

where

1. the $v_{i,j}$ contour $\mathfrak{V}_{i,j}$ is positively oriented around -1 and does not enclose $\widehat{\alpha}_\tau + \widehat{M_\tau}$ for $1 \leq \tau \leq T_i$;

2. whenever $(i,j) < (i',j')$, $\mathfrak{V}_{i,j}$ is enclosed by the contour $\mathfrak{V}_{i',j'}$.

The theorem now follows by applying Theorem 3.5.4 twice to the multidimensional integral above. $\square$

3.5.3 Proof of Theorem 3.5.3

By Definition 3.7.1, we have

$$\kappa\left(\mathfrak{P}_{k_1}(\mathbf{y}^{(T_1)}), \cdots, \mathfrak{P}_{k_m}(\mathbf{y}^{(T_m)})\right) = \sum_{\substack{d>0\\ \{S_1,\ldots,S_d\}\in\Theta_m}} (-1)^{d-1}(d-1)! \prod_{\ell=1}^{d} \mathbb{E}\left[\prod_{i\in S_\ell} \mathfrak{P}_{k_i}(\mathbf{y}^{(T_i)})\right]$$

where we use the notation from the Appendix with Θ_m denoting the collection of all set partitions of $[[1,m]]$. Assume $T_1 \geq \cdots \geq T_m$ so that the conditions of Theorem 3.4.2 are met. Then

$$\kappa\left(\mathfrak{P}_{k_1}(\mathbf{y}^{(T_1)}), \cdots, \mathfrak{P}_{k_m}(\mathbf{y}^{(T_m)})\right) = \frac{(-\theta)^{-m}}{(2\pi\mathbf{i})^{\sum_i k_i}} \oint \cdots \oint \widetilde{\mathcal{C}}(U_1,\ldots,U_m) \prod_{i=1}^{m} \mathcal{B}(U_i)\mathcal{A}_{T_i}(U_i)\, dU_i \tag{3.5.3}$$

where $U_i = (u_{i,1},\ldots,u_{i,k_i})$ and

$$\widetilde{\mathcal{C}}(U_1,\ldots,U_m) = \sum_{\substack{d>0\\ \{S_1,\ldots,S_d\}\in\Theta_m}} (-1)^{d-1}(d-1)! \prod_{\ell=1}^{d} \prod_{\substack{(i,j)\in S_\ell\\ i<j}} \mathcal{C}(U_i,U_j). \tag{3.5.4}$$

We note that the contours are such that

1. the $u_{i,j}$ contour $\mathfrak{U}_{i,j}$ is positively oriented around $-\theta N$ but does not enclose $\theta(\alpha_\tau + M_\tau - 1)$ for $1 \leq \tau \leq T_i$;

2. whenever $(i,j) < (i',j')$, $\mathfrak{U}_{i,j}$ is enclosed by the contour $\mathfrak{U}_{i',j'} + c$ for $c \in [-\theta, 1]$.

Let $S \subset [[1,m]]$, $\mathcal{T}(S)$ denote the set of undirected simple graphs with vertices labeled by S and $\mathcal{L}(S) \subset \mathcal{T}(S)$ denote the subset of connected graphs. Given a graph Ω, we denote by $E(\Omega)$ the edge set of Ω. We claim that

$$\widetilde{\mathcal{C}}(U_1,\ldots,U_m) = \sum_{\Omega\in\mathcal{L}([[1,m]])} \prod_{\substack{(i,j)\in E(\Omega)\\ i<j}} (\mathcal{C}(U_i,U_j) - 1). \tag{3.5.5}$$

Define

$$\mathcal{K}(S) := \sum_{\Omega \in \mathcal{L}(S)} \prod_{\substack{(i,j)\in E(\Omega)\\ i<j}} (\mathcal{C}(U_i,U_j)-1), \quad \mathcal{E}(S) := \sum_{\Omega \in \mathcal{T}(S)} \prod_{\substack{(i,j)\in E(\Omega)\\ i<j}} (\mathcal{C}(U_i,U_j)-1).$$

Then

$$\mathcal{E}(S) = \sum_{\substack{d>0\\ \{S_1,\ldots,S_d\}\in\Theta_S}} \prod_{\ell=1}^{d} \mathcal{K}(S_\ell)$$

where Θ_S is the collection of partitions of S. By Theorem 3.7.4, we have

$$\mathcal{K}(S) = \sum_{\substack{d>0\\ \{S_1,\ldots,S_d\}\in\Theta_S}} (-1)^{d-1}(d-1)! \prod_{\ell=1}^{d} \mathcal{E}(S_\ell)$$

which agrees with the right hand side of (3.5.4) when $S = [[1,m]]$. Thus (3.5.5) follows.

Applying (3.5.5), (3.5.3) becomes

$$\begin{aligned}
&\kappa\left(\mathfrak{P}_{k_1}(\mathbf{y}^{(T_1)})\cdots\mathfrak{P}_{k_m}(\mathbf{y}^{(T_m)})\right)\\
&\qquad = \sum_{\Omega\in\mathcal{L}([[1,m]])} \frac{(-\theta)^{-m}}{(2\pi\mathbf{i})^{\sum_i k_i}} \oint\cdots\oint \prod_{\substack{(i,j)\in E(\Omega)\\ i<j}} (\mathcal{C}(U_i,U_j)-1) \prod_{i=1}^{m} \mathcal{B}(U_i)\mathcal{A}_{T_i}(U_i)\,dU_i\\
&\qquad =: \sum_{\Omega\in\mathcal{L}([[1,m]])} \mathcal{I}_\Omega.
\end{aligned}$$

We want to show that $\mathcal{I}_\Omega \sim o(1)$ for each $\Omega \in \mathcal{L}([[1,m]])$ as $N \to \infty$. Fix $\Omega \in \mathcal{L}([[1,m]])$. Changing variables $u_{i,j} = \theta N v_{i,j}$, we obtain

$$\begin{aligned}
\mathcal{I}_\Omega = \frac{(-N)^m}{(2\pi\mathbf{i})^{\sum_i k_i}} \oint\cdots\oint \prod_{\substack{(i,j)\in E(\Omega)\\ i<j}} &\left(\prod_{a=1}^{k_i}\prod_{b=1}^{k_j} \frac{(v_{j,b}-v_{i,a})(v_{j,b}-v_{i,a}+\frac{1-\theta}{N})}{(v_{j,b}-v_{i,a}+\frac{1}{\theta N})(v_{j,b}-v_{i,a}-\frac{1}{N})} - 1\right)\\
\times \prod_{i=1}^{m} \frac{1}{(v_{i,2}-v_{i,1}+\frac{1-\theta}{\theta N})\cdots(v_{i,k_i}-v_{i,k_i-1}+\frac{1-\theta}{\theta N})} &\left(\prod_{1\le a<b\le k_i} \frac{(v_{i,b}-v_{i,a})(v_{i,b}-v_{i,a}+\frac{1-\theta}{\theta N})}{(v_{i,b}-v_{i,a}+\frac{1}{\theta N})(v_{i,b}-v_{i,a}-\frac{1}{N})}\right)\\
&\times \prod_{j=1}^{k_i} \frac{v_{i,j}}{v_{i,j}+1} \prod_{\tau=1}^{T_i} \frac{v_{i,j}-\frac{\alpha_\tau-1}{N}}{v_{i,j}-\frac{\alpha_\tau+M_\tau-1}{N}} dv_{i,j}.
\end{aligned} \tag{3.5.6}$$

The contours may be chosen to be fixed for sufficiently large N. In the same manner that we have

(3.5.2) in the proof of Theorem 3.5.2, we have

$$\prod_{a=1}^{k_i}\prod_{b=1}^{k_j}\frac{(v_{j,b}-v_{i,a})(v_{j,b}-v_{i,a}+\frac{1-\theta}{\theta N})}{(v_{j,b}-v_{i,a}+\frac{1}{\theta N})(v_{j,b}-v_{i,a}-\frac{1}{N})}=1+\frac{\theta^{-1}}{N^2}\sum_{a=1}^{k_i}\sum_{b=1}^{k_j}\frac{1}{(v_{j,b}-v_{i,a})^2}+O\left(\frac{1}{N^3}\right).$$

This implies that $\mathcal{I}_\Omega \sim O(N^{m-2|E(\Omega)|})$. For any $\Omega \in \mathcal{L}([[1,m]])$, we have that $|E(\Omega)| \geq m-1$. Therefore $\mathcal{I}_\Omega \sim o(1)$ whenever $m \geq 3$. This completes the proof. □

3.6 Local Fluctuations at the Edge

In this section, we prove the main results Theorems 3.1.6 to 3.1.8 for local fluctuations. We begin by reformulating the main theorems.

Assumption 2. *Fix $\theta > 0$. Let $T_1 := T_1(N), \ldots, T_m := T_m(N)$ be integers, $\alpha := \alpha(N) \geq 1$ and $M := M(N) \in \mathbb{Z}_{>0}$ such that $M \geq N$ satisfy*

$$\frac{T_i}{N} \to \widehat{T}_i, \quad \frac{\alpha}{N} \to \widehat{\alpha}, \quad \frac{M}{N} \to \widehat{M}$$

as $N \to \infty$ for some $\widehat{\alpha} \geq 0, \widehat{M} \geq 1$ and $\widehat{T}_i > 0$, $1 \leq i \leq m$. Let $(\mathbf{y}^{(T)} := \mathbf{y}^{(T)}(N))_{T\in\mathbb{Z}_{>0}} \sim \mathbb{P}_\theta^{(\alpha,M)^{T>0},N}$.

Theorem 3.6.1. *Let $c_1, \ldots, c_m > 0$ such that $c_2+\cdots+c_m < 1$. Let $T_1 := T_1(N), \ldots, T_m := T_m(N)$ be integers satisfying*

$$\frac{T_i}{N} \to \widehat{T}_i$$

as $N \to \infty$ for some $\widehat{T}_i > 0$, $1 \leq i \leq m$. If $(\mathbf{y}^{(T)} := \mathbf{y}^{(T)}(N))_{T\in\mathbb{Z}_{>0}}$ is distributed as the squared singular values of Y_T as in (3.1.1) where $X_1, X_2, \ldots$ are independent $N \times N$ complex Ginibre matrices, then

$$\lim_{N\to\infty}\mathbb{E}\left[\prod_{i=1}^m \frac{1}{N^{c_i(T_i+1)}}\mathfrak{P}_{c_i}(\mathbf{y}^{(T_i)})\right]=\frac{(-1)^m}{(2\pi\mathbf{i})^m}\oint\cdots\oint\prod_{1\leq i<j\leq m}\frac{(u_j-u_i)(u_j+c_j-u_i-c_i)}{(u_j-u_i-c_i)(u_j+c_j-u_i)}$$
$$\times\prod_{i=1}^m\frac{\Gamma(-u_i-c_i)}{\Gamma(-u_i)}\exp\left(-\widehat{T}_i\frac{c_i(c_i+1)}{2}-\widehat{T}_ic_iu_i\right)\frac{du_i}{c_i}$$

where the u_i contour $\mathfrak{U}_i$ is a positively oriented contour around $-c_i, -c_i+1, -c_i+2, \ldots$ which starts and ends at $+\infty$, and $\mathfrak{U}_i$ is enclosed by $\mathfrak{U}_j - c_i$ for $i < j$.

This gives us Theorem 3.1.6. By combining [42, Theorem 3.2] with Theorem 3.6.1, we obtain Theorem 3.1.4. The next theorem implies Theorem 3.1.7.

Theorem 3.6.2. *Let $c_1, \ldots, c_m > 0$ such that $c_2+\cdots+c_m < 1$. Under Assumption 2 with $\theta = 1$,* *we have*

$$\lim_{N\to\infty} \mathbb{E}\left[\prod_{i=1}^{m} \frac{(M+N+\alpha-1)^{c_iT_i}}{N^{c_i}(N+\alpha-1)^{c_iT_i}} \mathfrak{P}_{c_i}(\mathbf{y}^{(T_i)})\right]$$
$$= \frac{(-1)^m}{(2\pi\mathbf{i})^m} \oint \cdots \oint \prod_{1\le i<j\le m} \frac{(u_j-u_i)(u_j+c_j-u_i-c_i)}{(u_j-u_i-c_i)(u_j+c_j-u_i)}$$
$$\times \prod_{i=1}^{m} \frac{\Gamma(-u_i-c_i)}{\Gamma(-u_i)} \exp\left[-\widehat{T}_i\left(\frac{1}{1+\widehat{\alpha}} - \frac{1}{\widehat{M}+1+\widehat{\alpha}}\right)\left(\frac{c_i(c_i+1)}{2} + c_iu_i\right)\right]\frac{du_i}{c_i}$$

where the u_i contour $\mathfrak{U}_i$ is a positively oriented contour around $-c_i, -c_i+1, -c_i+2, \ldots$ which starts and ends at $+\infty$, and $\mathfrak{U}_i$ is enclosed by $\mathfrak{U}_j - c_i$ for $i<j$.

Our final theorem for arbitrary $\beta > 0$ implies Theorem 3.1.8.

Theorem 3.6.3. *Let $k_1, \ldots, k_m$ be positive integers. Under Assumption 2, we have*

$$\lim_{N\to\infty} \mathbb{E}\left[\prod_{i=1}^{m} \frac{1}{N^{k_i}} \left(\frac{M+N+\alpha-1}{N+\alpha-1}\right)^{k_iT_i} \mathfrak{P}_{k_i}(\mathbf{y}^{(T_i)})\right]$$
$$= \frac{(-\beta/2)^{-m}}{(2\pi\mathbf{i})^{\sum_{i=1}^m k_i}} \oint \cdots \oint \prod_{\substack{1\le i,i'\le m, 1\le j\le k_i, 1\le j'\le k_{i'} \\ (i,j)<(i,j')}} \frac{(u_{i',j'}-u_{i,j})(u_{i',j'}-u_{i,j}+1-\beta/2)}{(u_{i',j'}-u_{i,j}+1)(u_{i',j'}-u_{i,j}-\beta/2)}$$
$$\times \prod_{i=1}^{m} \left(\prod_{a=1}^{k_i-1} \frac{1}{u_{i,a+1}-u_{i,a}+1-\beta/2}\right)\left(\prod_{j=1}^{k_i} \frac{e^{-((1+\widehat{\alpha})^{-1}-(\widehat{M}+1+\widehat{\alpha})^{-1})\widehat{T}_i(\beta/2)^{-1}u_{i,j}}}{-(\beta/2)^{-1}u_{i,j}}\, du_{i,j}\right)$$

where the $u_{i,j}$ contour is positively oriented around 0, the $u_{i',j'}$ contour encloses a $\max(\theta,1)$ neighborhood of the $u_{i,j}$ contour for $(i,j) < (i',j')$ in lexicographical order.

Although Theorem 3.6.3 is not a perfect analogue of Theorem 3.6.2, for example the dimension of the contour integral formulas and the range of values for the Laplace transform, our approach for both the $\theta > 0$ and $\theta = 1$ can be viewed as stemming from the same general moment formulas (see Remark 18). The differentiation between the two cases then comes from additional structure available in the $\theta = 1$ case.

The remainder of this section is organized as follows. In Section 3.6.1, we obtain formulas for the Ginibre case via Theorem 3.4.2 under the appropriate limit. We then gather some lemmas in Section 3.6.2 to ease the presentation of the proofs of Theorems 3.6.1 to 3.6.3 in Section 3.6.3.

3.6.1 Formulas for the Ginibre Case

We obtain formulas for the joint moments of the squared singular values of complex Ginibre products by taking the appropriate limits of Theorems 2.3.17 and 3.4.3. These limit transitions rely on the following lemma:

Lemma 3.6.4. *Suppose* $(\mathbf{u}^{(T)} := \mathbf{u}^{(T)}(M))_{T\in\mathbb{Z}_{>0}} \sim \mathbb{P}_1^{(\alpha,M)^{T>0},N}$. *Then* $(M^T\mathbf{u}^{(T)})_{T\in\mathbb{Z}_{>0}}$ *converges as* $M \to \infty$ *in finite dimensional distributions to the squared singular values* $(\mathbf{y}^{(T)})_{T\in\mathbb{Z}_{>0}}$ *of* Y_T *as in* (3.1.1) *where* $X_1, X_2, \ldots$ *are independent,* $N \times N$ *complex Ginibre matrices.*

The lemma follows from the observation that $M \to \infty$ corresponds to growing the size of the ambient unitary matrix. Recall that the matrices X_T in the square case can be obtained as submatrices of a Haar unitary matrix of size $M+N$. After renormalizing by $1/\sqrt{M}$ (renormalizing the singular value by $1/M$) the entries of the unitary matrix behave like independent complex standard Gaussians in the limit.

Given $U = (u_1, \ldots, u_k)$, $V = (v_1, \ldots, v_\ell)$, let $\mathcal{B}(U)$, $\mathcal{C}(U, V)$ be defined as in (3.4.1) and define

$$\widetilde{\mathcal{A}}_T(U) = \theta^{-kT} \prod_{i=1}^{k} \frac{(-u_i)^{T+1}}{-u_i - \theta N}. \tag{3.6.1}$$

Proposition 3.6.5. *Suppose* $(\mathbf{y}^{(T)})_{T\in\mathbb{Z}_{>0}}$ *are the squared singular values of* Y_T *as in* (3.1.1) *where* $X_1, X_2, \ldots$ *are independent,* $N \times N$ *complex Ginibre matrices. If* $T_1 \geq \cdots \geq T_m > 0$ *are integers and* $c_1, \ldots, c_m \in \mathbb{R}_{>0}$*, then*

$$\mathbb{E}\left[\prod_{i=1}^{m} \mathfrak{P}_{c_i}(\mathbf{y}^{(T_i)})\right] = \frac{\prod_{i=1}^{m}(-c_i)^{-1}}{(2\pi\mathbf{i})^m} \oint \cdots \oint \prod_{1\leq i<j\leq m} \frac{(u_j - u_i)(u_j + c_j - u_i - c_i)}{(u_j - u_i - c_i)(u_j + c_j - u_i)}$$
$$\times \prod_{i=1}^{m} \left(\prod_{\ell=1}^{N} \frac{u_i + \ell - 1}{u_i + c_i + \ell - 1} \cdot \left(\frac{\Gamma(1 - u_i)}{\Gamma(1 - u_i - c_i)} \right)^{T_i} \right) du_i$$

where the u_i *contour* $\mathfrak{U}_i$ *is positively oriented around* $\{-c_i - \ell + 1\}_{\ell=1}^{N}$ *but does not enclose* $1, 2, 3, \ldots$*, and* $\mathfrak{U}_i$ *is contained in* $\mathfrak{U}_j - c_i$ *for* $i < j$*; given that such contours exist.*

Proof. Suppose $(\mathbf{u}^{(T)})_{T\in\mathbb{Z}_{>0}} \sim \mathbb{P}_1^{(\alpha_T,M_T)_{T>0},N}$ with $\alpha_T = 1$ and $M_T = 1$ for all $T > 0$. By Theorem 3.4.3, we have

$$\mathbb{E}\left[\prod_{i=1}^{m} \mathfrak{P}_{c_i}(\mathbf{u}^{(T_i)})\right] = \frac{\prod_{i=1}^{m}(-c_i)^{-1}}{(2\pi\mathbf{i})^m} \oint \cdots \oint \prod_{1\leq i<j\leq m} \frac{(u_j - u_i)(u_j + c_j - u_i - c_i)}{(u_j - u_i - c_i)(u_j + c_j - u_i)}$$
$$\times \prod_{i=1}^{m} \left(\prod_{\ell=1}^{N} \frac{u_i + \ell - 1}{u_i + c_i + \ell - 1} \cdot \left(\prod_{\ell=1}^{M} \frac{\ell - u_i - c_i}{\ell - u_i} \right)^{T_i} \right) du_i$$

where the u_i contour $\mathfrak{U}_i$ is positively oriented around $\{-c_i - \ell + 1\}_{i=1}^{N}$ but does not enclose $1, \ldots, M$, and $\mathfrak{U}_i$ is contained in $\mathfrak{U}_j - c_i$. Taking the Ginibre limit gives

$$\lim_{M\to\infty} \mathbb{E}\left[\prod_{i=1}^{m} M^{c_i} \mathfrak{P}_{c_i}(\mathbf{u}^{(T_i)})\right] = \mathbb{E}\left[\prod_{i=1}^{m} \mathfrak{P}_{c_i}(\mathbf{y}^{(T_i)})\right].$$

This convergence, along with the following asymptotics, then implies this proposition:

$$\begin{aligned}\lim_{M\to\infty} M^c \prod_{\ell=1}^{M} \frac{\ell-u-c}{\ell-u} &= \frac{\Gamma(1-u)}{\Gamma(1-u-c)} \lim_{M\to\infty} M^c \frac{\Gamma(M+1-u-c)}{\Gamma(M+1-u)}\\ &= \frac{\Gamma(1-u)}{\Gamma(1-u-c)}\end{aligned}$$

uniformly in u on compact subsets of $\mathbb{C}$ by Stirling's approximation for the Gamma function. □

3.6.2 Preliminary Asymptotics

Lemma 3.6.6. *For any $c > 0$, we have*

$$\log \frac{\Gamma(z+c)}{\Gamma(z)} = c\log(z+c) - \frac{1}{2}\frac{c(c+1)}{z} + R(z)$$

where the remainder satisfies

$$|R(z)| \le \frac{C_\delta}{|z|^2}$$

over $\{z \in \mathbb{C} : \Re z \ge 0, |z| \ge c(1+\delta)\}$ for any fixed $\delta > 0$ with C_δ some constant depending on δ.

Proof. By Stirling's approximation, we have

$$\log\Gamma(z) = \left(z - \frac{1}{2}\right)\log z - z + \frac{1}{2}\log(2\pi) + \frac{1}{12z} + \widetilde{R}(z), \qquad |\widetilde{R}(z)| \le \frac{C}{|z|^2}$$

for some uniform constant C over $\{z \in \mathbb{C} : \Re z \ge 0\}$, see [53, p.141]. We have

$$\begin{aligned}\log\frac{\Gamma(z+c)}{\Gamma(z)} &= \left(z+c-\frac{1}{2}\right)\log(z+c) - (z+c) + \frac{1}{12}\frac{1}{z+c} - \left(z-\frac{1}{2}\right)\log z + z - \frac{1}{12}\frac{1}{z} + R_1(z)\\ &= \left(z-\frac{1}{2}\right)\log\left(1+\frac{c}{z}\right) - c + c\log(z+c) + R_2(z)\end{aligned}$$

where

$$\max(|R_1(z)|, |R_2(z)|) \le \frac{C}{|z|^2}$$

over $\{\Re z \ge 0\}$ for some constant C. By expanding the first logarithm, we get

$$\begin{aligned}\log\frac{\Gamma(z+c)}{\Gamma(z)} &= \left(z-\frac{1}{2}\right)\left(\frac{c}{z} - \frac{1}{2}\frac{c^2}{z^2}\right) - c + c\log(z+c) + R_3(z)\\ &= -\frac{1}{2}\frac{c(c+1)}{z} + c\log(z+c) + R_4(z)\end{aligned}$$

where for any given $\delta > 0$, we have

$$\max(|R_3(z)|, |R_4(z)|) \leq \frac{C_\delta}{|z|^2}$$

over $\{\Re z \geq 0, |z| \geq c(1+\delta)\}$ for some constant C_δ depending on δ. $\square$

Define

$$G_L(u) := \frac{L^{-c}\Gamma(L-u)}{\Gamma(L-u-c)}. \tag{3.6.2}$$

Lemma 3.6.7. *Fix $\delta, c, \eta > 0$ and for each $N \in \mathbb{Z}_{>0}$, define the region*

$$\mathfrak{R}_\eta^{(N)} := \{u \in \mathbb{C} : |\Im u| \leq \eta, \quad -\eta \leq \Re u \leq N - \delta\}.$$

Suppose $T := T(N)$ and $L := L(N) \geq N$ satisfy

$$\lim_{N\to\infty} T/N \to \widehat{T}, \quad \lim_{N\to\infty} L/N \to \widehat{L}$$

for some $\widehat{T} > 0$ and $\widehat{L} \geq 1$. Then the following statements are true.

1. *For each fixed $u \in \bigcup_{N=1}^\infty \mathfrak{R}_\eta^{(N)}$, we have*

$$\lim_{L\to\infty} G_L(u)^T = \exp\left[-\frac{\widehat{T}}{\widehat{L}}\left(\frac{c(c+1)}{2} + cu\right)\right]. \tag{3.6.3}$$

2. *If $\widehat{L} > 1$, then*

$$|G_L(u)| \geq \exp\left(-\frac{C_{\delta,c}}{L-N}\right), \quad u \in \mathfrak{R}_\eta^{(N)} \tag{3.6.4}$$

   for sufficiently large N and L.

3. *In general, we have*

$$|G_L(u)| \leq \exp\left(-c\frac{\Re u}{L} + c\frac{\eta}{L}\right), \quad u \in \mathfrak{R}_\eta^{(N)} \cap \{u \in \mathbb{C} : |L-u-c| \geq \delta\} \tag{3.6.5}$$

   for sufficiently large N.

Proof. By Theorem 3.6.6, we have

$$G_L(u) = \exp\left[c\log\left(1-\frac{u}{L}\right) - \frac{1}{2}\frac{c(c+1)}{L-u-c} + R(L-u-c)\right] \tag{3.6.6}$$

where $R(z)$ is independent of L and satisfies

$$|R(z)| \leq \frac{C_{\delta'}}{|z|^2}$$

over $\{\Re z \geq 0, |z| \geq c(1+\delta')\}$ for any fixed $\delta' > 0$. By choosing δ' sufficiently small given our choice of c, δ, we have

$$|R(L-u-c)| \leq \frac{C_{\delta,c}}{|L-u-c|^2}, \quad u \in \mathfrak{R}_\eta^{(N)}, \quad L \geq N. \tag{3.6.7}$$

If we fix $u \in \mathfrak{R}_\eta^{(N)}$, we see that (3.6.6) implies (3.6.3).

Observe that

$$|L-u-c| \geq L - \Re u - c \geq L - N - c, \quad u \in \mathfrak{R}_\eta^{(N)}$$

for $L \gg N$. Then (3.6.6) implies (3.6.4).

Thus we are left to prove (3.6.5). For this, we bound $G_L(u)$ separately on the regions $\mathfrak{R}_\eta^{(N)} \cap \{\Re u \leq L - N_0\}$ and $\mathfrak{R}_\eta^{(N)} \cap \{L - N_0 \leq \Re u \leq L\}$ where N_0 is some large number independent of N and L determined below. We have

$$\left|\frac{L - \Re u - c}{L - u - c}\right| \leq C_{\delta,c,\eta}$$

for $u \in \mathfrak{R}_\eta^{(N)} \cap \{|L-u-c| \geq \delta\}$ and in particular on $u \in \mathfrak{R}_\eta^{(N)} \cap \{\Re u \leq L - N_0\}$ where N_0 is some fixed number. Thus

$$\begin{aligned}
&\Re\left(-\frac{1}{2}\frac{c(c+1)}{L-u-c} + R(L-u-c)\right) \\
&\qquad \leq \Re\left(-\frac{1}{2}\frac{c(c+1)}{L-\Re u-c} + \frac{c(c+1)}{L-\Re u-c} - \frac{c(c+1)}{L-u-c} + \frac{C_{\delta,c}}{|L-u-c|^2}\right) \\
&\qquad = -\frac{1}{2}\frac{c(c+1)}{L-\Re u-c} + \Re\left(\frac{-c(c+1)\mathbf{i}\Im u}{(L-\Re u-c)(L-u-c)} + \frac{C_{\delta,c}}{|L-u-c|^2}\right) \\
&\qquad \leq -\frac{1}{2}\frac{c(c+1)}{L-\Re u-c} + \frac{C_{\delta,c,\eta}}{|L-\Re u-c|^2}.
\end{aligned}$$

for $u \in \mathfrak{R}_\eta^{(N)} \cap \{\Re u \leq L - N_0\}$ where we use (3.6.7) in the first inequality. By choosing N_0 sufficiently large, the first term on the right hand side dominates. Then (3.6.6) implies

$$|G_L(u)| \leq \left|1 - \frac{u}{L}\right|^c \exp\left(\frac{-C_{\delta,N_0}}{L-\Re u-c}\right) \leq \left|1 - \frac{u}{L}\right|^c, \quad u \in \mathfrak{R}_\eta^{(N)} \cap \{\Re u \leq L - N_0\}$$

where $L \gg N_0$. Since

$$\Re \log\left(1 - \frac{u}{L}\right) \leq \log\left(1 - \frac{\Re u}{L} + \frac{\eta}{L}\right) \leq -\frac{\Re u}{L} + \frac{\eta}{L},$$

we obtain

$$|G_L(u)| \leq \exp\left(-c\frac{\Re u}{L} + c\frac{\eta}{L}\right), \quad u \in \mathfrak{R}_\eta^{(N)} \cap \{\Re u \leq L - N_0\} \tag{3.6.8}$$

for sufficiently large N.

We also have that (3.6.6) implies the bound

$$|G_L(u)| \leq \left(\frac{N_0}{N} e^{C_{\delta,\eta,N_0}}\right)^c, \quad u \in \mathfrak{R}_\eta^{(N)} \cap \{|L-u-c| \geq \delta\} \cap \{L - N_0 \leq \Re u \leq L\}.$$

Since

$$\log \frac{N_0}{L} + C_{\delta,\eta,N_0} \leq -\frac{\Re u}{L}$$

for large N (recall $L \geq N$), we obtain

$$|G_L(u)| \leq \exp\left(-c\frac{\Re u}{L}\right), \quad u \in \mathfrak{R}_\eta^{(N)} \cap \{|L-u-c| \geq \delta\} \cap \{L - N_0 \leq \Re u \leq L\} \tag{3.6.9}$$

for sufficiently large N. Combining (3.6.8) and (3.6.9), we obtain (3.6.5). □

3.6.3 Proofs of Local Theorems

We prove Theorem 3.6.3, then prove Theorems 3.6.1 and 3.6.2 simultaneously to avoid redundancy.

Proof of Theorem 3.6.3. By Theorem 3.4.2, and replacing u_i with $u_i - \theta N$, we get (given the existence of the appropriate contours described below)

$$\begin{aligned}\mathbb{E}&\left[\prod_{i=1}^m \frac{(M+N+\alpha-1)^{k_i T_i}}{N^{k_i}(N+\alpha-1)^{k_i T_i}} \mathfrak{P}_{k_i}(\mathbf{y}^{(T_i)})\right] \\ &= \frac{(-\theta)^{-m}}{(2\pi\mathbf{i})^{\sum_{i=1}^m k_i}} \oint \cdot \oint \prod_{1\leq i<i'\leq m} \mathcal{C}(U_i, U_{i'}) \prod_{i=1}^m \mathcal{B}(U_i) I_{T_i}(U_i)\, dU_i.\end{aligned} \tag{3.6.10}$$

where $I_T := I_T^{(N)}$ is defined

$$I_T(U) := \prod_{i=1}^k \frac{1 - \frac{u_i}{\theta N}}{-u_i/\theta} \left(\frac{1 - \frac{u_i}{\theta(N+\alpha-1)}}{1 - \frac{u_i}{\theta(N+\alpha+M-1)}}\right)^T$$

for $U = (u_1, \ldots, u_k)$. The contours in (3.6.10) are given as follows. The $u_{i,j}$ contour $\mathfrak{U}_{i,j}^{(N)}$ is positively oriented around 0, avoid the pole at $\theta(\alpha + M - 1)$ and $\mathfrak{U}_{i,j}^{(N)}$ is enclosed by the contour $\mathfrak{U}_{i',j'}^{(N)} + c$ for $c \in [-\theta, 1]$ whenever $(i,j) < (i',j')$ in lexicographical order. Since $\theta(\alpha + M - 1) \to \infty$, we may take the contour $\mathfrak{U}_{i,j}^{(N)}$ to be fixed contours $\mathfrak{U}_{i,j}$ for sufficiently large N.

If $U = (u_1, \ldots, u_k) \in \mathbb{C}^k \setminus \{\mathbf{0}\}$, $T := T(N)$ and $\widehat{T} > 0$ such that $T/N \to \widehat{T} > 0$, then

$$\lim_{N\to\infty} I_T(U) = \prod_{i=1}^{k} \exp\left[-\widehat{T}\left(\frac{1}{1+\widehat{\alpha}} - \frac{1}{1+\widehat{\alpha}+\widehat{M}}\right) u_i/\theta\right] \cdot \frac{1}{-u_i/\theta}$$

uniformly for U in compact subsets of $\mathbb{C}^k \setminus \{\mathbf{0}\}$. This completes the proof of the theorems. □

Proofs of Theorems 3.6.1 and 3.6.2. Throughout this proof, C is a positive constant which may vary from line to line. If C is dependent on additional parameters, we explicitly denote this dependence including the related parameter as a subscript.

Both Theorems 3.6.1 and 3.6.2 are in the setting $\theta = 1$, the difference being that the former is under the assumption that $\mathbf{y}^{(T)}$ are obtained from Ginibre matrices (we refer to this as the *Ginibre case*) and the latter uses Assumption 2 (which we refer to as the *Jacobi case*). We begin by writing our expressions in a common way. Let

$$E^{(N)} := \begin{cases} \mathbb{E}\left[\prod_{i=1}^{m} \frac{1}{N^{c_i(T_i+1)}} \mathfrak{P}_{c_i}(\mathbf{y}^{(T_i)})\right] & \text{for the Ginibre case,} \\ \mathbb{E}\left[\prod_{i=1}^{m} \frac{(M+N+\alpha-1)^{c_i T_i}}{N^{c_i}(N+\alpha-1)^{c_i T_i}} \mathfrak{P}_{c_i}(\mathbf{y}^{(T_i)})\right] & \text{for the Jacobi case.} \end{cases}$$

We are interested in the limit $E^{(N)}$ as $N \to \infty$. By Theorem 3.4.3, Theorem 3.6.5 and replacing u_i with $u_i - N + 1$, we get (given the existence of the appropriate contours described below)

$$E^{(N)} = \frac{(-1)^m}{(2\pi\mathbf{i})^m} \oint \cdots \oint \prod_{1\le i<j\le m} \frac{(u_j-u_i)(u_j+c_j-u_i-c_i)}{(u_j-u_i-c_i)(u_j+c_j-u_i)} \prod_{i=1}^{m} I_{c_i,T_i}(u_i)\, \frac{du_i}{c_i} \tag{3.6.11}$$

where

$$I_{c,T}(u) := I_{c,T}^{(N)}(u) := \begin{cases} \frac{\Gamma(-u-c)}{\Gamma(-u)} G_N(u)^{T+1} & \text{for the Ginibre case,} \\ \frac{\Gamma(-u-c)}{\Gamma(-u)} G_N(u) \left(\frac{G_{N+\alpha-1}(u)}{G_{M+N+\alpha-1}(u)}\right)^T & \text{for the Jacobi case} \end{cases}$$

and we recall $G_L(u)$ is defined by (3.6.2). The contours in (3.6.11) are given as follows. In the Ginibre case, the u_i contour $\mathfrak{U}_i^{(N)}$ is positively oriented around $\{-c_i, \ldots, -c_i + N - 1\}$ but does not enclose $N, N+1, N+2, \ldots$, and $\mathfrak{U}_i^{(N)}$ is contained in $\mathfrak{U}_j^{(N)} - c_i$ for $i < j$; given that such contours exist. In the Jacobi case, the u_i contour $\mathfrak{U}_i^{(N)}$ is positively oriented around $\{-c_i, \ldots, -c_i + N - 1\}$ but does not enclose $N + \alpha - 1, N + \alpha, \ldots, M + N + \alpha - 2$.

The idea of the proof is to use dominated convergence where most of the work in finding a suitable dominating function for $I_{c,T}$ is done by Theorem 3.6.7. We divide the proof into three steps. In the first step, we choose contours $\mathfrak{U}_1^{(N)}, \ldots, \mathfrak{U}_m^{(N)}$ satisfying the conditions above. In the second step, we examine the $N \to \infty$ asymptotics of the function $I_{c,T}(u)$ where $c > 0$ and $T := T(N)$ satisfies $\lim_{N\to\infty} T/N \to \widehat{T}$ for some $\widehat{T} > 0$. In the third step, we apply the asymptotics to (3.6.11) to prove the theorem.

Step 1. We choose $\mathfrak{U}_1^{(N)}, \dots, \mathfrak{U}_m^{(N)}$ which satisfy the conditions for (3.6.11). Fix $\delta > 0$ such that

$$\delta(m+1) < 1 - \sum_{j=2}^{m} c_j.$$

Let $\mathfrak{U}_i^{(N)}$ be a counterclockwise contour around the boundary of the rectangle

$$\left\{ u \in \mathbb{C} : |\Im u| \le i, -\sum_{j=1}^{i-1}(c_j + \delta) \le \Re u \le N - \sum_{j=i}^{m}(c_j + \delta) \right\}.$$

Then $\mathfrak{U}_1^{(N)}, \dots, \mathfrak{U}_m^{(N)}$ satisfy the containment conditions, thus (3.6.11) is valid with this choice. The choice of bound $|\Im u| \le i$ is somewhat arbitrary, we just need a bound which is increasing in i.

We claim that for any set of numbers $a_1, \dots, a_m \in \mathbb{R}$, there exists a sufficiently small $\delta > 0$ such that

$$\operatorname{dist}(\{a_1 + \delta, \dots, a_m + m\delta\}, \mathbb{Z}) \ge \delta.$$

Indeed, if a_i is an integer, then this clearly holds for $\delta < 1/(2m)$. If $a_i \notin \mathbb{Z}$, then this holds for $\delta < \operatorname{dist}(a_i, \mathbb{Z})/(2m)$. By taking δ small enough to satisfy these inequalities, the claim follows. Thus we may choose $\delta > 0$ small enough so that

$$\operatorname{dist}\left(\mathfrak{U}_i^{(N)}, \mathbb{Z} \cup (\mathbb{Z} - c_i)\right) \ge \delta, \quad 1 \le i \le m. \tag{3.6.12}$$

For $1 \le i \le m$, let $\mathfrak{U}_i$ be the counterclockwise contour around the boundary of the semi-infinite region

$$\left\{ u \in \mathbb{C} : |\Im u| \le i, -\sum_{j=1}^{i-1}(c_j + \delta) \le \Re u \right\}.$$

Step 2. Let

$$\mathfrak{R}_\eta^{(N)} := \{u \in \mathbb{C} : |\Im u| \le \eta, \quad -\eta \le \Re u \le N - \delta\}$$

for some fixed parameter $\eta > 0$ and suppose $T := T(N)$ satisfies $\lim_{N\to\infty} T/N \to \widehat{T}$ for some $\widehat{T} > 0$. The goal is to control the term $I_{c,T}(u)$ as $N \to \infty$ on the region

$$\mathfrak{R}_\eta^{(N)} \cap \{u : \operatorname{dist}(u, \mathbb{Z} \cup (\mathbb{Z} - c)) \ge \delta\}$$

where $\eta > 0$ is some fixed parameter. In particular, we prove there are constants $C_{\delta,c,\eta}, C'_{\delta,c,\eta} > 0$ such that

$$|I_{c,T}(u)| \le C_{\delta,c,\eta} \exp\left(-C'_{\delta,c,\eta} \Re u\right) =: f_{\eta,c,\widehat{T}}(u), \quad u \in \mathfrak{R}_\eta^{(N)} \cap \{\operatorname{dist}(u, \mathbb{Z} \cup (\mathbb{Z} - c)) \ge \delta\} \tag{3.6.13}$$

for sufficiently large N, and for fixed $u \in \bigcup_{N=1}^{\infty} \mathfrak{R}_\eta^{(N)} \setminus (\mathbb{Z} \cup (\mathbb{Z} - c))$, we have

$$\lim_{N\to\infty} I_{c,T}(u) = g_{c,\widehat{T}}(u) \tag{3.6.14}$$

where

$$g_{c,\widehat{T}}(u) := \begin{cases} \dfrac{\Gamma(-u-c)}{\Gamma(-u)} \exp\left[-\widehat{T}\left(\dfrac{c(c+1)}{2} + \widehat{T}cu\right)\right] & \text{in the Ginibre case,} \\ \dfrac{\Gamma(-u-c)}{\Gamma(-u)} \exp\left[-\widehat{T}\left(\dfrac{1}{1+\widehat{\alpha}} - \dfrac{1}{\widehat{M}+1+\widehat{\alpha}}\right)\left(\dfrac{c(c+1)}{2} + cu\right)\right] & \text{in the Jacobi case.} \end{cases} \tag{3.6.15}$$

By Euler's reflection formula, we have

$$\frac{\Gamma(-u-c)}{\Gamma(-u)} = \frac{\sin \pi u}{\sin \pi(u+c)} \frac{\Gamma(1+u)}{\Gamma(1+u+c)}.$$

Then

$$I_{c,T}(u) = J_1(u) \cdot J_2(u) \cdot J_3(u) \tag{3.6.16}$$

where

$$\begin{aligned} J_1(u) &:= \frac{\sin \pi u}{\sin \pi(u+c)}, \\ J_2(u) &:= \frac{\Gamma(1+u)}{\Gamma(1+u+c)}, \\ J_3(u) &:= \begin{cases} G_N(u)^{T+1} & \text{for the Ginibre case,} \\ G_N(u)\left(\dfrac{G_{N+\alpha-1}(u)}{G_{M+N+\alpha-1}(u)}\right)^T & \text{for the Jacobi case.} \end{cases} \end{aligned} \tag{3.6.17}$$

We bound each of the terms J_1, J_2, J_3.

By the periodicity of sin in the real direction for any fixed imaginary part, we have

$$|J_1(u)| \le C_{\delta,\eta}, \quad u \in \{\operatorname{dist}(u, \mathbb{Z} \cap (\mathbb{Z} - c) \ge \delta, |\Im u| \le \eta\}. \tag{3.6.18}$$

By Theorem 3.6.6, we have

$$|J_2(u)| \le C_{\delta,c,\eta}|1+u+c|^{-c}, \quad u \in \mathfrak{R}_\eta^{(N)} \tag{3.6.19}$$

The bounds (3.6.4), (3.6.5) combined with (3.6.17) imply

$$|J_3(u)| \le C_{\delta,c,\eta} \exp\left(-C'_{\delta,c,\eta}\Re u\right), \quad u \in \mathfrak{R}_\eta^{(N)} \tag{3.6.20}$$

for sufficiently large N.

Combining the bounds (3.6.18), (3.6.19) and (3.6.20) implies (3.6.13). Moreover, from (3.6.16) we have

$$I_{c,T}(u) = \frac{\Gamma(-u-c)}{\Gamma(-u)} J_3(u).$$

Thus, (3.6.17) and (3.6.3) imply the convergence (3.6.14).

Step 3. Let $\mathfrak{U}_{i,1}^{(N)}$ be the vertical contour along

$$\left\{ u \in \mathbb{C} : |\Im u| \le i, \Re u = N - \sum_{j=i}^{m} (c_j + \delta) \right\},$$

alternatively the rightmost part of $\mathfrak{U}_i^{(N)}$. Let

$$\mathfrak{U}_{i,0}^{(N)} := \mathfrak{U}_i^{(N)} \setminus \mathfrak{U}_{i,1}^{(N)}$$

denote the complementary contour. We may rewrite (3.6.11) as

$$E^{(N)} = \sum_{\boldsymbol{s} \in \{0,1\}^m} \mathcal{I}_{\boldsymbol{s}} \tag{3.6.21}$$

where

$$\mathcal{I}_{\boldsymbol{s}} := \frac{(-1)^m}{(2\pi \mathbf{i})^m} \oint \cdots \oint \prod_{1 \le i < j \le m} \frac{(u_j - u_i)(u_j + c_j - u_i - c_i)}{(u_j - u_i - c_i)(u_j + c_j - u_i)} \prod_{i=1}^{m} I_{c_i, T_i}(u_i) \frac{du_i}{c_i}$$

and the u_i contour is $\mathfrak{U}_{i,s_i}^{(N)}$ for $1 \le i \le m$ if $\boldsymbol{s} = (s_1, \ldots, s_m)$. We prove that

$$\mathcal{I}_{\mathbf{0}} \to \frac{(-1)^m}{(2\pi \mathbf{i})^m} \oint \cdots \oint \prod_{1 \le i < j \le m} \frac{(u_j - u_i)(u_j + c_j - u_i - c_i)}{(u_j - u_i - c_i)(u_j + c_j - u_i)} \prod_{i=1}^{m} g_{c_i, \widehat{T}_i}(u_i) \frac{du_i}{c_i}, \tag{3.6.22}$$

$$\mathcal{I}_{\boldsymbol{s}} \to 0, \qquad \boldsymbol{s} \ne \mathbf{0} \tag{3.6.23}$$

as $N \to \infty$, where the u_i contour in the right hand side of (3.6.22) is given by $\mathfrak{U}_i$ for $1 \le i \le m$. By (3.6.21), this completes the proof of Theorems 3.6.1 and 3.6.2.

Observe that

$$\prod_{1 \le i < j \le m} \frac{(u_j - u_i)(u_j + c_j - u_i - c_i)}{(u_j - u_i - c_i)(u_j + c_j - u_i)}$$

is bounded on $(u_1, \ldots, u_m) \in \bigcup_{N=1}^{\infty} (\mathfrak{U}_1^{(N)} \times \cdots \times \mathfrak{U}_m^{(N)})$. Moreover, for a suitably chosen η, we have $\mathfrak{U}_i^{(N)} \subset \mathfrak{R}_\eta^{(N)} \cap \{\mathrm{dist}(u, \mathbb{Z} \cup (\mathbb{Z} - c_i) \ge \delta\}$. We may use the bound (3.6.13) so that by dominated convergence and the convergence (3.6.14), we obtain (3.6.22).

To prove (3.6.23), we use the boundedness of the cross-term to write

$$|\mathcal{I}_{\boldsymbol{s}}| \le C \int_{\mathfrak{U}_{1,s_1}^{(N)}} d|u_1| \cdots \int_{\mathfrak{U}_{m,s_m}^{(N)}} d|u_m| \cdot \prod_{i=1}^{m} |I_{c_i,T_i}(u_i)| = C \prod_{i=1}^{m} \int_{\mathfrak{U}_{i,s_i}^{(N)}} |I_{c_i,T_i}(u)| d|u|.$$

Using the bound (3.6.13), we have that

$$\int_{\mathfrak{U}_{i,s_i}^{(N)}} |I_{c_i,T_i}(u)|\, d|u|$$

is of constant order when $s_i = 0$ and is $o(1)$ when $s_i = 1$. Thus (3.6.23) follows. $\square$

3.7 Appendix: Cumulants

We recall the notion of cumulants and some basic properties.

Definition 3.7.1. For any positive integer ν, let Θ_ν be the collection of all set partitions of $[[1,\nu]]$, that is

$$\Theta_\nu = \left\{ \{S_1, \dots, S_d\} : d > 0,\ \bigcup_{i=1}^{d} S_i = [[1,\nu]],\ S_i \cap S_j = \emptyset\ \forall i \neq j,\ S_i \neq \emptyset\ \forall i \in [[1,d]] \right\}.$$

For a random vector $\boldsymbol{u} = (u_1, \dots, u_m)$ and any $v_1, \dots, v_\nu \in \{u_1, \dots, u_m\}$, define the *(order ν) cumulant* $\kappa(v_1, \dots, v_\nu)$ by

$$\kappa(v_1, \dots, v_\nu) = \sum_{\substack{d>0 \\ \{S_1,\dots,S_d\}\in\Theta_\nu}} (-1)^{d-1}(d-1)! \prod_{\ell=1}^{d} \mathbb{E}\left[\prod_{i\in S_\ell} v_i\right]. \tag{3.7.1}$$

From the definition we see that for any random vector $\boldsymbol{u}$, the existence of all cumulants of order up to ν is equivalent to the existence of all moments of order up to ν. Note that the cumulants of order 2 are exactly the covariances:

$$\kappa(v_1, v_2) = \mathrm{Cov}(v_1, v_2).$$

We have the following alternative definition for cumulants.

Definition 3.7.2. Let $\boldsymbol{u} = (u_1, \dots, u_m)$ be a random vector. For any $v_1, \dots, v_\nu \in \{u_1, \dots, u_m\}$, define the *(order ν) cumulant* $\kappa(v_1, \dots, v_\nu)$ as

$$\kappa(v_1, \dots, v_\nu) = (-\mathbf{i})^\nu \frac{\partial^\nu}{\partial t_1 \cdots \partial t_m} \log \mathbb{E}\left[\exp\left(\mathbf{i}\sum_{j=1}^{\nu} t_j v_j\right)\right]\Bigg|_{t_1=\cdots=t_\nu=0}. \tag{3.7.2}$$

For further details see [56, Section 3.1, Section 3.2] wherein the agreement between Definitions 3.7.1 and 3.7.2 is shown by taking the second definition and proving (3.7.1).

As a consequence of the second definition we have the following lemma.

Lemma 3.7.3. *A random vector is Gaussian if and only if all cumulants of order ≥ 3 vanish.*

We have the following formal versions of (3.7.1) and (3.7.2). Let $E_{n_1,\dots,n_\nu} \in \mathbb{C}$ with $E_{0,\dots,0} = 1$. Define the following formal power series

$$E(t_1, \dots, t_\nu) = \sum_{n_1,\dots,n_\nu \geq 0} \frac{E_{n_1,\dots,n_\nu}}{n_1! \cdots n_\nu!} t_1^{n_1} \cdots t_\nu^{n_\nu}$$

$$K(t_1, \dots, t_\nu) = \log E(t_1, \dots, t_\nu) =: \sum_{n_1,\dots,n_\nu \geq 0} \frac{K_{n_1,\dots,n_\nu}}{n_1! \cdots n_\nu!} t_1^{n_1} \cdots t_\nu^{n_\nu}.$$

Let $S \subset [[1,\nu]]$ and $E(S) := E_{n_1,\ldots,n_\nu}$ where $n_j = 1$ if $j \in S$ and 0 otherwise, and likewise define $K(S)$. Letting Θ_S be the collection of all set partitions of S, we have

$$K(S) = \sum_{\substack{d>0 \\ \{S_1,\ldots,S_d\}\in\Theta_S}} (-1)^{d-1}(d-1)! \prod_{\ell=1}^{d} E(S_\ell) \tag{3.7.3}$$

and

$$E(S) = \sum_{\substack{d>0 \\ \{S_1,\ldots,S_d\}\in\Theta_S}} \prod_{\ell=1}^{d} K(S_\ell) \tag{3.7.4}$$

are equivalent. This follows from the formal identity $e^{K(t_1,\ldots,t_\nu)} = E(t_1,\ldots,t_\nu)$. This gives us the following lemma.

Lemma 3.7.4. *Suppose that* $\widetilde{K}$ *and* $\widetilde{E}$ *are functions which take values on nonempty subsets of* $[[1,\nu]]$*. Further suppose that*

$$\widetilde{E}(S) = \sum_{\substack{d>0 \\ \{S_1,\ldots,S_d\}\in\Theta_S}} \prod_{i=1}^{d} \widetilde{K}(S_i) \tag{3.7.5}$$

Then

$$\widetilde{K}(S) = \sum_{\substack{d>0 \\ \{S_1,\ldots,S_d\}\in\Theta_S}} (-1)^{d-1}(d-1)! \prod_{i=1}^{d} \widetilde{E}(S_i) \tag{3.7.6}$$

3.8 Appendix: Observables of Schur Processes

The goal of this appendix is to prove a contour integral formula for joint moments of *Schur processes*; Macdonald processes in the case $q = t$. We review several facts about Schur functions. Additional details can be found in [43, Chapters I].

For $q = t$, (3.2.2) is defined by

$$\langle p_\lambda, p_\mu \rangle := \langle p_\lambda, p_\mu \rangle_{(t,t)} = \delta_{\lambda\mu} \prod_{i=1}^{\infty} i^{m_i(\lambda)} m_i(\lambda)!$$

which is independent of $0 < t < 1$. In this case, the Macdonald symmetric functions become the *Schur functions*

$$s_\lambda(X) := P_\lambda(X;t,t) = Q_\lambda(X;t,t), \quad s_{\lambda/\mu}(X) := P_{\lambda/\mu}(X;t,t) = Q_{\lambda/\mu}(X;t,t).$$

Thus the properties for Macdonald symmetric functions are inherited by the Schur functions. To list a few, we have that (3.2.3), (3.2.4), (3.2.11) imply

$$\langle s_\lambda(X), s_\mu(X)\rangle = \delta_{\lambda\mu}, \tag{3.8.1}$$

$$s_{\lambda/\nu}(X,Y) = \sum_{\mu\in\mathbb{Y}} s_{\lambda/\mu}(X) s_{\mu/\nu}(Y). \tag{3.8.2}$$

Given a countably infinite set $X = (x_1, x_2, \dots)$ of variables, let Λ_X denote the algebra of symmetric functions on X over $\mathbb{C}$. For sets $X^{(1)}, \dots, X^{(n)}$ of variables, let $(X^{(1)}, \dots, X^{(n)})$ denote the disjoint union of these sets. We have that $\{s_\lambda(X)\}_{\lambda\in\mathbb{Y}}$ forms a linear basis for Λ_X.

Definition 3.8.1. Suppose $\boldsymbol{a} := (a_1, \dots, a_N) \in \mathbb{R}^N_{>0}$, $\boldsymbol{b} \in (b_1, \dots, b_M) \in \mathbb{R}^M_{>0}$ such that $a_i b_j < 1$ for $1 \le i \le N$, $1 \le j \le M$. Let $\mathbb{SP}_{\boldsymbol{a},\boldsymbol{b}}$ denote the measure on $\boldsymbol{\lambda} = (\lambda^1, \dots, \lambda^M) \in \mathbb{Y}^N = \mathbb{Y}\times\cdots\times\mathbb{Y}$ (this is different from the subscripted $\mathbb{Y}_N$ notation which is the subset of $\mathbb{Y}$ containing partitions λ with $\ell(\lambda) \le N$) where

$$\mathbb{SP}_{\boldsymbol{a},\boldsymbol{b}}(\boldsymbol{\lambda}) = \prod_{\substack{1\le i\le N\\ 1\le j\le M}} (1-a_ib_j)\cdot s_{\lambda^1}(a_1,\dots,a_N) s_{\lambda^1/\lambda^2}(b_1)\cdots s_{\lambda^{M-1}/\lambda^M}(b_{M-1}) s_{\lambda^M}(b_M). \tag{3.8.3}$$

Remark 21. The measure $\mathbb{SP}_{\boldsymbol{a},\boldsymbol{b}}$ is the Schur ($q=t$) case of the so-called *ascending Macdonald process*. We note that

$$\sum_{\lambda^1,\dots,\lambda^M\in\mathbb{Y}} s_{\lambda^1}(a_1,\dots,a_N) s_{\lambda^1/\lambda^2}(b_1)\cdots s_{\lambda^{M-1}/\lambda^M}(b_{M-1}) s_{\lambda^M}(b_M) = \prod_{\substack{1\le i\le N\\ 1\le j\le M}} \frac{1}{1-a_ib_j}$$

is a consequence of the branching rule (3.8.2) and the $q=t$ case of the Cauchy identity (3.2.10).

We prove the following contour integral formula for joint expectations of $\mathfrak{p}_t$ (recall (3.4.4)).

Theorem 3.8.2. *Suppose $\boldsymbol{a} := (a_1, \dots, a_N) \in \mathbb{R}^N_{>0}$, $\boldsymbol{b} \in (b_1, \dots, b_M) \in \mathbb{R}^M_{>0}$ such that $a_i b_j < 1$ for $1 \le i \le M$, $1 \le j \le N$. For $0 < t_1, \dots, t_m < 1$ and $1 \le n_1 \le \cdots n_m \le M$, we have*

$$\mathbb{E}\left[\prod_{i=1}^m \mathfrak{p}_{t_i}(\lambda^{n_i})\right] = \frac{1}{(2\pi\mathbf{i})^m}\oint\cdots\oint \prod_{1\le i<j\le m}\frac{(z_j-z_i)(t_iz_j-t_jz_i)}{(t_iz_j-z_i)(z_j-t_jz_i)}\prod_{i=1}^m\left(\prod_{\ell=1}^N\frac{z_i-a_\ell}{z_i-t_ia_\ell}\cdot\prod_{\ell=n_i}^M\frac{1-t_i^{-1}b_\ell z_i}{1-b_\ell z_i}\right)\frac{dz_i}{z_i}$$

where the z_i contour is positively oriented around $0, \{t_ia_\ell\}_{\ell=1}^N$ but does not encircle $\{b_\ell^{-1}\}_{\ell=n_i}^M$, and the contours satisfy $|z_i| < |t_iz_j|$ for $i<j$; given that such contours exist.

The ideas involved in the proof of Theorem 3.8.2 are based off of the approaches of [6], [7] and [2]. In particular, the organization below follows that of [2, Section 3]. The general approach is to first prove a formal version of Theorem 3.8.2, then specialize this formal version to obtain Theorem 3.8.2.

We organize the subappendices below as follows. In Subappendix 3.8.1, we provide some notions and basic facts about formal symmetric functions. This provides the setting to state and prove a formal version of Theorem 3.8.2 in Subappendix 3.8.2. In Subappendix 3.8.3, we prove Theorem 3.8.2 by specializing the formal analogue.

3.8.1 Preliminaries on Formal Symmetric Functions

Graded Topology

Let F be a field and $\mathcal{A}$ be a ($\mathbb{Z}_{\geq 0}$-)graded algebra over F. Let $\mathcal{A}_n$ denote the nth homogeneous component of $\mathcal{A}$. Throughout this section, let us assume that all of our graded algebras have $\dim \mathcal{A}_n < \infty$ for every $n \geq 0$.

Definition 3.8.3. Given $a \in \mathcal{A}$, define $\operatorname{ldeg}(a)$ to be the minimum degree among the homogeneous components of a. The *graded topology* is the topology on $\mathcal{A}$ where a sequence $a_n \in \mathcal{A}$ converges to $a \in \mathcal{A}$ if and only if

$$\operatorname{ldeg}(a_n - a) \to \infty$$

as $n \to \infty$. Denote the completion of $\mathcal{A}$ under this topology by $\widehat{\mathcal{A}}$.

The completion $\widehat{\mathcal{A}}$ consists of formal sums $\sum_{n=1}^{\infty} a_n$ where $a_n \in \mathcal{A}_n$. Given two graded algebras $\mathcal{A}$ and $\mathcal{A}'$ over F, we give the following grading to $\mathcal{A} \otimes_F \mathcal{A}'$. If $a \in \mathcal{A}_m$ and $a' \in \mathcal{A}'_n$, then $a \otimes a' \in (\mathcal{A} \otimes_F \mathcal{A}')_{m+n}$.

For a field $F \supset \mathbb{C}$ and a graded algebra $\mathcal{A}$ over $\mathbb{C}$, denote by $\mathcal{A}[F]$ the graded algebra $\mathcal{A} \otimes_{\mathbb{C}} F$ over F. Given graded algebras $\mathcal{A}^{(1)}, \ldots, \mathcal{A}^{(k)}$ over $\mathbb{C}$, we denote the completion of $(\mathcal{A}^{(1)} \otimes \cdots \otimes \mathcal{A}^{(n)})[F]$ under the graded topology by

$$\mathcal{A}^{(1)} \,\widehat{\otimes}\, \cdots \,\widehat{\otimes}\, \mathcal{A}^{(k)}[F] \quad \text{or} \quad \widehat{\bigotimes}_{i=1}^{k} \mathcal{A}^{(i)}[F].$$

Let $\Lambda_X[F]$ denote the F-algebra of symmetric functions in $X = \{x_1, x_2, \ldots\}$, a set of variables, with coefficients in F. Take the natural grading on $\Lambda_X[F]$ in which $(\Lambda_X[F])_n$ is spanned by monomials of total degree n. Given disjoint ordered sets of variables $Z_1, \ldots, Z_n$ with $Z_i = (z_{i,1}, \ldots, z_{i,k_i})$, let $\mathcal{L}(Z_1, \ldots, Z_n)$ denote the field of formal Laurent series in the variables

$$\bigcup_{i=1}^{n} \left\{ \frac{z_{i,1}}{z_{i,2}}, \ldots, \frac{z_{i,k_i-1}}{z_{i,k_i}}, z_{i,k_i} \right\}.$$

The space $\widehat{\bigotimes}_{i=1}^{k} \Lambda_{X^i}[F]$ consists of formal sums

$$\sum_{\lambda^1, \ldots, \lambda^N \in \mathbb{Y}} c_{\lambda^1, \ldots, \lambda^N} s_{\lambda^1}(X^1) \cdots s_{\lambda^N}(X^N)$$

where $c_{\lambda^1, \ldots, \lambda^N} \in F$.

For fields $\mathbb{C} \subset F_1 \subset F_2$, $1 \leq k \leq N$, there is the natural inclusion map

$$\widehat{\bigotimes}^k_{i=1} \Lambda_{X^i}[F_1] \hookrightarrow \widehat{\bigotimes}^k_{i=1} \Lambda_{X^i}[F_2]. \tag{3.8.4}$$

We also have consistency

$$\Lambda_{X^1}[F] \otimes_F \cdots \otimes_F \Lambda_{X^N}[F] \cong \Lambda_{X^1} \otimes \cdots \otimes \Lambda_{X^N}[F]. \tag{3.8.5}$$

Definition 3.8.4. The *projection map* $\pi^n_X : \widehat{\Lambda}_X \to \widehat{\Lambda}_{\{x_1,\ldots,x_n\}}$ is defined as the continuous map sending $x_{n+1}, x_{n+2}, \ldots$ to 0 and x_i to x_i for $i = 1, \ldots, n$.

For a field $F \supset \mathbb{C}$ and a graded algebra $\mathcal{A}$ over $\mathbb{C}$, we can extend the domain of the projection

$$\pi^n_X : \mathcal{A} \mathbin{\widehat{\otimes}} \Lambda_X[F] \to \mathcal{A} \mathbin{\widehat{\otimes}} \Lambda_{\{x_1,\ldots,x_n\}}[F]$$

by identifying with $1_{\mathcal{A}} \otimes \pi^n_X$ then extending by continuity under the graded topology.

Definition 3.8.5. Let $\mathcal{A}$ and $\mathcal{A}'$ be graded algebras over $\mathbb{C}$ and $\{a_{n,j}\}_j$ be a basis for $\mathcal{A}_n$ for each $n \geq 0$. We say that an element $f \in \mathcal{A} \mathbin{\widehat{\otimes}} \mathcal{A}'[F]$ is $\mathcal{A}$-*projective* if

$$f = \sum_{n,j} a_{n,j} \otimes \alpha'_{n,j}, \quad \alpha'_{n,j} \in \mathcal{A}'_n$$

such that $\lim_{n\to\infty} \min_j \operatorname{ldeg}(\alpha'_{n,j}) = \infty$. This property is independent of the choice of basis.

Elements which are $\mathcal{A}$-projective are closed under addition and multiplication and form a subalgebra of $\mathcal{A} \mathbin{\widehat{\otimes}} \mathcal{A}'[F]$. If $\mathcal{A} = \Lambda_X$, denote the algebra of Λ_X-projective elements by $\mathcal{P}_X(\Lambda_X \mathbin{\widehat{\otimes}} \mathcal{A}'[F])$.

Schur Pairing and Residue

Definition 3.8.6. Let $\mathcal{A}, \mathcal{A}'$ be graded algebras over $\mathbb{C}$. Fix a field $F \supset \mathbb{C}$, and let the *Schur pairing* be the bilinear map $\langle \cdot, \cdot \rangle_Y : (\mathcal{A} \otimes \Lambda_X)[F] \times (\Lambda_X \otimes \mathcal{A}')[F] \to \mathcal{A} \otimes \mathcal{A}'[F]$ defined by

$$\langle a \otimes s_\lambda, s_\mu \otimes b \rangle_X := \langle s_\lambda, s_\mu \rangle a \otimes b = \delta_{\lambda\mu} a \otimes b.$$

This pairing does not extend by continuity to product of the completions of $(\mathcal{A} \otimes \Lambda_X)[F]$ and $(\Lambda_X \otimes \mathcal{A}')[F]$. However, the pairing does extend continuously to

$$\mathcal{P}_X(\mathcal{A} \mathbin{\widehat{\otimes}} \Lambda_X[F]) \times (\Lambda_X \mathbin{\widehat{\otimes}} \mathcal{A}'[F]).$$

Definition 3.8.7. Given an ordered set $Z = (z_1, \ldots, z_k)$ of variables, denote by $\oint dZ : \mathcal{L}(Z) \to \mathbb{C}$ the *residue operator* which takes an element of $\mathcal{L}(Z)$ and returns the coefficient of $(z_1 \cdots z_k)^{-1}$. For $\oint dZ$ applied to $f \in \mathcal{L}(Z)$ we write $\oint f\, dZ$ or $\oint dZ \cdot f$.

As with the projection map, the residue operator can act on larger domains. For example, we can extend

$$\oint dZ : \widehat{\mathcal{A}}[\mathcal{L}(Z, W^1, \ldots, W^k)] \to \widehat{\mathcal{A}}[\mathcal{L}(W^1, \ldots, W^k)] \tag{3.8.6}$$

by the action $1_{\mathcal{A}} \otimes \oint dZ$ then extension by continuity. In this case, $\oint dZ$ preserves the degree of homogeneous elements. In particular, if we replace $\widehat{\mathcal{A}}$ with $\mathcal{A} \mathbin{\widehat{\otimes}} \mathcal{A}'$, we have that $\oint dZ$ preserves $\mathcal{A}$-projectivity.

The residue operator commutes with continuous maps under the graded topology.

Lemma 3.8.8. *Let $\mathcal{A}$, $\mathcal{A}'$ be graded algebras over $\mathbb{C}$, and let $\varphi : \widehat{\mathcal{A}} \to \widehat{\mathcal{A}}'$ be a continuous map which extends to a continuous map $\widehat{\mathcal{A}}[\mathcal{L}(Z, W)] \to \widehat{\mathcal{A}}'[\mathcal{L}(Z, W)]$. Then*

$$\varphi \circ \left(\oint dZ \right) = \left(\oint dZ \right) \circ \varphi.$$

Lemma 3.8.9. *Let $\mathcal{A}$, $\mathcal{A}'$ be graded algebras over $\mathbb{C}$, let $f \in \mathcal{A} \mathbin{\widehat{\otimes}} \Lambda_X[\mathcal{L}(Z)]$ and $g \in \Lambda_X \mathbin{\widehat{\otimes}} \mathcal{A}'[\mathcal{L}(W)]$. If f is Λ_X-projective, then*

$$\left\langle \oint f\, dZ, g \right\rangle_X = \oint \langle f, g \rangle_X \, dZ, \tag{3.8.7}$$

$$\left\langle f, \oint g\, dW \right\rangle_X = \oint \langle f, g \rangle_X \, dW \tag{3.8.8}$$

Since the residue operator preserves projectivity, the left hand sides of the equalities above are valid expressions.

Proof. For arbitrary $g \in \Lambda_X \mathbin{\widehat{\otimes}} \mathcal{A}'[\mathcal{L}(W)]$, the map $\langle \cdot, g \rangle_X$ is continuous on $\mathcal{P}_X(\mathcal{A} \otimes \Lambda_X[\mathcal{L}(Z)])$. By Theorem 3.8.8, (3.8.7) follows. For $f \in \mathcal{P}_X(\mathcal{A} \mathbin{\widehat{\otimes}} \Lambda_X(\mathcal{L}(Z)])$, the map $\langle f, \cdot \rangle$ is continuous on $\Lambda_X \mathbin{\widehat{\otimes}} \mathcal{A}'[\mathcal{L}(W)]$. By Theorem 3.8.8, (3.8.8) follows. ∎

3.8.2 Formal Schur Processes

Let X, Y be countable sets of variables. Define the following elements of $\Lambda_X \mathbin{\widehat{\otimes}} \Lambda_Y$

$$\Pi(X, Y) := \Pi(X, Y; t, t) = \prod_{x \in X, y \in Y} \frac{1}{1 - xy},$$

$$H(X, Y; t) := \Pi(X, Y; 0, t) = \prod_{x \in X, y \in Y} \frac{1 - txy}{1 - xy}.$$

Definition 3.8.10. Fix a positive integer N and let $\boldsymbol{U} = (U^1, \ldots, U^N)$ and $\boldsymbol{V} = (V^1, \ldots, V^N)$ be ordered N-tuples of countable sets of variables. A *formal Schur process* is a formal probability measure on $\mathbb{Y}^N$, valued in $\widehat{\bigotimes}_{i=1}^{N}(\Lambda_{U^i} \otimes \Lambda_{V^i})$ with the assignment

$$\mathbb{SP}^{\mathbf{f}}_{\boldsymbol{U},\boldsymbol{V}}(\boldsymbol{\lambda}) \\ = \mathscr{Z}^{-1} s_{\lambda^1}(U^1) \left(\sum_{\mu \in \mathbb{Y}} s_{\lambda^1/\mu}(V^1) s_{\lambda^2/\mu}(U^2) \right) \cdots \left(\sum_{\mu \in \mathbb{Y}} s_{\lambda^{N-1}/\mu}(V^{N-1}) s_{\lambda^N/\mu}(U^N) \right) s_{\lambda^N}(V^N)$$

where $\boldsymbol{\lambda} = (\lambda^1, \ldots, \lambda^N)$ and $\mathscr{Z} \in \widehat{\bigotimes}_{i=1}^{N}(\Lambda_{U^i} \otimes \Lambda_{V^i})$ is the normalization constant for which the sum over $\boldsymbol{\lambda} \in \mathbb{Y}^N$ gives unity.

From [7, Section 3],

$$\mathscr{Z} = \prod_{1 \le i \le j \le N} \Pi(U^i, V^j). \tag{3.8.9}$$

Theorem 3.8.11. *The following formal identity holds for any nonzero $t_1, \ldots, t_N$*

$$\mathbb{E}_{\mathbb{SP}^{\mathbf{f}}_{\boldsymbol{U},\boldsymbol{V}}}[\mathfrak{p}_{t_1}(\lambda^1) \cdots \mathfrak{p}_{t_N}(\lambda^N)] = \frac{1}{(2\pi \mathbf{i})^N} \oint \frac{dz_1}{z_1} \cdots \frac{dz_N}{z_N} \\ \times \prod_{1 \le i \le j \le N} H(U^i, z_j^{-1}; t_j)^{-1} H(z_i, V^j; t_i^{-1}) \prod_{1 \le i < j \le N} \frac{(1 - \frac{z_i}{z_j})(1 - \frac{t_j z_i}{t_i z_j})}{(1 - \frac{z_i}{t_i z_j})(1 - \frac{t_j z_i}{z_j})}.$$

This theorem implies a more general corollary.

Corollary 3.8.12. *Let $1 \le n_1 \le \cdots \le n_m \le N$ and $t_1, \cdots, t_m \neq 0$. Then*

$$\mathbb{E}_{\mathbb{SP}^{\mathbf{f}}_{\boldsymbol{U},\boldsymbol{V}}}[\mathfrak{p}_{t_1}(\lambda^{n_1}) \cdots \mathfrak{p}_{t_m}(\lambda^{n_m})] = \frac{1}{(2\pi \mathbf{i})^m} \oint \frac{dz_1}{z_1} \cdots \oint \frac{dz_m}{z_m} \\ \times \prod_{a=1}^{m} \prod_{\substack{1 \le i \le n_a \\ n_a \le j \le N}} H(U^i, z_a^{-1}; t_a)^{-1} H(z_a, V^j; t_a^{-1}) \prod_{1 \le a < b \le m} \frac{(1 - \frac{z_a}{z_b})(1 - \frac{t_b z_a}{t_a z_b})}{(1 - \frac{z_a}{t_a z_b})(1 - \frac{t_b z_a}{z_b})}. \tag{3.8.10}$$

We prove Theorem 3.8.11 and Theorem 3.8.12 below.

Basic Properties

From [43, Chapter VI, Sections 2 and 4], we have the equalities

$$\Pi(X, Y) = \sum_{\lambda \in \mathbb{Y}} s_\lambda(X) s_\lambda(Y) = \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} p_n(X) p_n(Y) \right), \\ H(X, Y; t) = \exp\left(\sum_{n=1}^{\infty} \frac{1 - t^n}{n} p_n(X) p_n(Y) \right). \tag{3.8.11}$$

Given countable sets of variables X^1, X^2, the following splitting equality holds

$$\Pi((X^1, X^2), Y) = \Pi(X^1, Y)\Pi(X^2, Y) \tag{3.8.12}$$

and likewise for $H(\cdot, \cdot; t)$. There is also an inversion equality

$$H(X, Y; t)^{-1} = H(tX, Y; t^{-1}) \tag{3.8.13}$$

where by tX we mean the variable set $\{tx\}_{x\in X}$.

In terms of the pairing, the formal Schur process can be expressed as

$$\mathbb{SP}^{\mathbf{f}}_{\boldsymbol{U},\boldsymbol{V}}(\boldsymbol{\lambda}) = \mathscr{Z}^{-1} s_{\lambda^1}(U^1) \left(\prod_{i=1}^{N-1} \langle s_{\lambda^i}(V^i, Y^i), s_{\lambda^{i+1}}(Y^i, U^{i+1}) \rangle_{Y^i} \right) s_{\lambda^N}(V^N). \tag{3.8.14}$$

This is an immediate consequence of the branching rule (3.8.2).

The Π's introduced earlier also relate well with the pairing

$$\langle \Pi(X^1, Y), \Pi(Y, X^2) \rangle_Y = \Pi(X^1, X^2).$$

Since the power symmetric functions form an algebraic basis for $\Lambda_X[\mathcal{L}(Z)]$, this relation can be further extended as follows. Take graded algebras $\mathcal{A}$ and $\mathcal{A}'$ over $\mathcal{L}(Z)$ with $Z = (z_1, \ldots, z_k)$, and sequences $\{a_n\}, \{a'_n\}$ in $\mathcal{A}$ and $\mathcal{A}'$ respectively such that $\operatorname{ldeg}(a_n), \operatorname{ldeg}(a'_n) \to \infty$ as $n \to \infty$. Then

$$\left\langle \exp\left(\sum_{n=1}^{\infty} \frac{a_n}{n} p_n(Y) \right), \exp\left(\sum_{n=1}^{\infty} \frac{a'_n}{n} p_n(Y) \right) \right\rangle_Y = \exp\left(\sum_{n=1}^{\infty} \frac{a_n a'_n}{n} \right). \tag{3.8.15}$$

See [7, Proposition 2.3] for further details.

Macdonald/Negut Operator

Define the continuous linear operator $D_t^X : \widehat{\Lambda}_X \to \widehat{\Lambda}_X$ by

$$D_t^X s_\lambda(X) = \mathfrak{p}_t(\lambda) s_\lambda(X).$$

Proposition 3.8.13. *Let X and Y be countable sets of variables. Then*

$$D_t^X \Pi(X, Y) = \Pi(X, Y) \frac{1}{2\pi \mathbf{i}} \oint H(z^{-1}, X; t)^{-1} H(Y, z; t^{-1}) \frac{dz}{z}. \tag{3.8.16}$$

Proof. From [32, Proposition 4.10], we have (3.8.16) where instead of a set of variables Y we have some fixed set $\{u_1, \ldots, u_n\}$ of complex numbers, and X is still a countable set of variables. Here we have $\oint : \widehat{\Lambda}_X[\mathcal{L}(z)] \to \widehat{\Lambda}_X$. The goal is to extend this to a formal equality on $\Lambda_X \mathbin{\widehat{\otimes}} \Lambda_Y$ for Y an arbitrary countable set of variables.

We can replace (3.8.16) with a finite set of variables $Y^{(n)} = \{y_1, \ldots, y_n\}$ instead of fixed complex numbers. In such a setting, we must consider the residue operator as a map $\oint dZ$:

$\Lambda_X \,\widehat{\otimes}\, \Lambda_{\{y_1,\ldots,y_n\}}[\mathcal{L}(z)] \to \Lambda_X \,\widehat{\otimes}\, \Lambda_{\{y_1,\ldots,y_n\}}$.

Note that if $f, g \in \Lambda_X \,\widehat{\otimes}\, \Lambda_Y$ such that $\pi_n^Y f = \pi_n^Y g$ for all n, then $f = g$. One then sees that (3.8.16) holds formally for arbitrary countable sets of variables X, Y. In this setting, the residue operator takes $\Lambda_X \,\widehat{\otimes}\, \Lambda_Y[\mathcal{L}(z)]$ to $\Lambda_X \,\widehat{\otimes}\, \Lambda_Y$. □

Proofs of Formal Theorems

Proof of Theorem 3.8.11. Choose nonzero $t_1, \ldots, t_N$ and let $z_1, \ldots, z_N$ be variables.

1. Consider the element

$$\sum_{\boldsymbol{\lambda}} \mathfrak{p}_{t_1}(\lambda^1) \cdots \mathfrak{p}_{t_N}(\lambda^N) \mathbb{SP}^{\mathbf{f}}_{\boldsymbol{U},\boldsymbol{V}}(\boldsymbol{\lambda}) \in \widehat{\bigotimes}_{i=1}^{N} (\Lambda_{U^i} \otimes \Lambda_{V^i}).$$

2. Multiply through by the normalizing constant. Reexpress the sums within $\mathbb{SP}$ in terms of Schur pairings as in (3.8.14)

$$\sum_{\boldsymbol{\lambda}} \mathfrak{p}_{t_1}(\lambda^1) \cdots \mathfrak{p}_{t_N}(\lambda^N) s_{\lambda^1}(U^1) \left(\prod_{i=1}^{N-1} \left\langle s_{\lambda^i}(V^i, Y^i), s_{\lambda^{i+1}}(Y^i, U^{i+1}) \right\rangle_{Y^i} \right) s_{\lambda^N}(V^N).$$

Here we note the spaces which the pairings map:

$$\langle \cdot, \cdot \rangle_{Y^i} : \left(\Lambda_{V^i} \,\widehat{\otimes}\, \Lambda_{Y^i}\right) \times \left(\Lambda_{Y^i} \,\widehat{\otimes}\, \Lambda_{U^{i+1}}\right) \to \Lambda_{V^i} \,\widehat{\otimes}\, \Lambda_{U^{i+1}}.$$

By natural inclusions (3.8.4) and consistency (3.8.5), the domain of this pairing may be extended.

3. Bring the summation inside the pairings and the pairings inside the pairings

$$\langle \mathbf{E}_1, \langle \mathbf{E}_2, \langle \cdots \langle \mathbf{E}_{N-1}, \mathbf{E}_N \rangle_{Y^{N-1}} \cdots \rangle_{Y^2} \rangle_{Y^1}$$

where

$$\mathbf{E}_i = \sum_{\lambda^i} \mathfrak{p}_{t_i} s_{\lambda^1}(Y^{i-1}, U^i) s_{\lambda^i}(V^i, Y^i)$$

and Y^0, Y^N are empty sets of variables. It was important to use the fact that the first argument of the Y^i Macdonald pairing is Λ_{Y^i}-projective which provides the continuity necessary for bringing the summations inside.

4. We can reexpress the summations in terms of the operators D_t in the residue form (3.8.16)

$$\begin{aligned}\mathbf{E}_i &= D_{t_i}^{Y^{i-1},U^i} \Pi((Y^{i-1}, U^i), (V^i, Y^i)) \\ &= \Pi((Y^{i-1}, U^i), (V^i, Y^i)) \frac{1}{2\pi \mathbf{i}} \oint H((Y^{i-1}, U^i), z_i^{-1}; t_i)^{-1} H(z_i, (V^i, Y^i); t_i^{-1}) \, \frac{dz_i}{z_i}.\end{aligned}$$

5. The domain of the residue operator can be appropriately extended and consistency follows from

(3.8.4) and (3.8.5). Note that the integrand in $\mathbf{E}_i$ remains Λ_{Y^i}-projective. Therefore, by (3.8.7) and (3.8.8), we may commute the residue operators with the pairings. After pulling out Y^i independent factors outside the residue operators, we obtain

$$\frac{1}{(2\pi\mathbf{i})^N}\oint\frac{dz_1}{z_1}\cdots\oint\frac{dz_N}{z_N}\cdot A\cdot\langle\mathbf{F}_1,\langle\mathbf{F}_2,\cdots\langle\mathbf{F}_{N-1},\mathbf{F}_N\rangle_{Y^{N-1}}\cdots\rangle_{Y^2}\rangle_{Y^1}$$

where

$$\mathbf{F}_i = H(Y^{i-1},z_i^{-1};t_i)^{-1}\Pi(Y^{i-1},V^i)H(z_i,Y^i;t_i^{-1})\Pi((Y^{i-1},U^i),Y^i) \tag{3.8.17}$$

$$A = \prod_{i=1}^{N} H(U_i,z_i^{-1};t_i)^{-1}H(z_i,V_i;t_i^{-1})\Pi(U^i,V^i). \tag{3.8.18}$$

Here, (3.8.12) and (3.8.13) were used to split H and Π.

6. Apply the pairings for Y^i in decreasing order of i. We claim by induction that at the $(N-i)$th step, we have

$$\frac{1}{(2\pi\mathbf{i})^N}\oint\frac{dz_1}{z_1}\cdots\oint\frac{dz_N}{z_N}\cdot A\prod_{i<\ell\leq N}A_\ell\cdot\langle\mathbf{F}_1,\langle\mathbf{F}_2,\cdots\langle\mathbf{F}_i,\mathscr{F}_i\rangle_{Y^i}\cdots\rangle_{Y^2}\rangle_{Y^1} \tag{3.8.19}$$

where

$$\begin{aligned} A_i &= \prod_{j:i<j\leq N}\frac{(1-\frac{z_i}{z_j})(1-\frac{t_iz_i}{t_jz_j})}{(1-\frac{z_i}{t_jz_j})(1-\frac{t_iz_i}{z_j})}H(z_i,V^j;t_i^{-1})H(U^i,z_j^{-1};t_j)^{-1}\Pi(U^i,V^j),\\ \mathscr{F}_i &= \prod_{j:i<j\leq N}H(Y^i,z_j^{-1};t_j)^{-1}\Pi(Y^i,V^j)\end{aligned} \tag{3.8.20}$$

and $V^{[i+1,N]} := (V^{i+1},\ldots,V^N)$. The base case is true since $A_N = 1$ and $\mathscr{F}_{N-1} = \mathbf{F}_N$. If we suppose (3.8.20) is true for $1\leq i\leq N$, then

$$\begin{aligned}\langle\mathbf{F}_i,\mathscr{F}_i\rangle_{Y_i} &= H(Y^{i-1},z_i^{-1};t_i)^{-1}\Pi(Y^{i-1},V^i)\\ &\quad\times\left\langle H(z_i,Y^i;t_i^{-1})\Pi((Y^{i-1},U^i),Y^i)\,,\prod_{j:i<j\leq N}H(Y^i,z_j^{-1};t_j)^{-1}\Pi(Y^i,V^j)\right\rangle_{Y_i}\end{aligned}$$

By (3.8.11) and (3.8.15), the Y_i-bracket on the right hand side yields

$$\prod_{j:i<j\leq N}\frac{(1-\frac{z_i}{z_j})(1-\frac{t_jz_i}{t_iz_j})}{(1-\frac{z_i}{t_iz_j})(1-\frac{t_jz_i}{z_j})}H(z_i,V^j;t_i^{-1})H((U^i,Y^{i-1}),z_j^{-1};t_j)^{-1}\Pi((U^i,Y^{i-1}),V^j).$$

This implies

$$\langle\mathbf{F}_i,\mathscr{F}_i\rangle_{Y_i} = A_i\cdot\mathscr{F}_{i-1}$$

where we note that A_i is independent of $\{Y^j\}$. This completes the induction. Since Y^0 is empty, in the final step we get $\mathscr{F}_0 = 1$. Thus, we obtain the formula

$$\frac{1}{(2\pi\mathbf{i})^N} \oint \frac{dz_1}{z_1} \cdots \oint \frac{dz_N}{z_N} \cdot A \prod_{i=1}^{N} A_i.$$

Substituting (3.8.18) and (3.8.20), we get

$$\frac{1}{(2\pi\mathbf{i})^N} \oint \frac{dz_1}{z_1} \cdots \oint \frac{dz_N}{z_N} \times \prod_{1\le i\le j\le N} \Pi(U_i, V_j) H(U^i, z_j^{-1}; t_j)^{-1} H(z_i, V^j; t_i^{-1}) \prod_{1\le i<j\le N} \frac{(1-\frac{z_i}{z_j})(1-\frac{t_j z_i}{t_i z_j})}{(1-\frac{z_i}{t_i z_j})(1-\frac{t_j z_i}{z_j})}.$$

Recall that we multiplied by the normalizing constant $\mathscr{Z}$. Dividing back by $\mathscr{Z}$ and using (3.8.9), we complete the proof. □

We now illustrate the main idea of the proof of Corollary 3.8.12 via a particular example. For further details, we note that the proof is essentially identical to a corresponding extension in [7] (Theorem 3.10 to Corollary 3.11).

Proof Idea of Theorem 3.8.12. We consider the example of $N = 1$ and $m = 2$. Let $t_1, t_2 \neq 0$. Consider auxiliary variables $\boldsymbol{S} = (S^1, S^2), \boldsymbol{T} = (T^1, T^2)$, and the formal expectation

$$\mathbb{E}_{\mathbb{SP}^{\mathbf{f}}_{\boldsymbol{S},\boldsymbol{T}}}[\mathfrak{p}_{t_1}(\lambda^1), \mathfrak{p}_{t_2}(\lambda^2)] = \mathscr{Z}_0^{-1} \sum_{\lambda^1,\lambda^2\in\mathbb{Y}} \mathfrak{p}_{t_1}(\lambda^1)\mathfrak{p}_{t_2}(\lambda^2) s_{\lambda^1}(S^1) \left(\sum_{\mu\in\mathbb{Y}} s_{\lambda^1/\mu}(T^1) s_{\lambda^2/\mu}(S^2) \right) s_{\lambda^1}(T^2) \tag{3.8.21}$$

where $\mathscr{Z}_0$ is the normalizing factor for $\mathbb{SP}^{\mathbf{f}}_{\boldsymbol{S},\boldsymbol{T}}$. Consider the map $\phi : \Lambda_{T^1} \otimes \Lambda_{S^2}$ defined by $f(T^1)g(S^2) \mapsto f(0)g(0)$ for any $f, g \in \Lambda_X$. By applying (the continuous extension of) ϕ to (3.8.21) and rewriting $S^1 = U$ and $T^2 = V$, we get

$$\mathscr{Z}^{-1} \sum_{\lambda\in\mathbb{Y}} \mathfrak{p}_{t_1}(\lambda)\mathfrak{p}_{t_2}(\lambda) s_\lambda(U) s_\lambda(V) = \mathbb{E}_{\mathbb{SP}^{\mathbf{f}}_{U,V}}[\mathfrak{p}_{t_1}(\lambda)\mathfrak{p}_{t_2}(\lambda)]. \tag{3.8.22}$$

where $\mathscr{Z}$ is the normalizing factor for $\mathbb{SP}^{\mathbf{f}}_{U,V}$. On the other hand, by Theorem 3.8.11, we have a formal residue expression for (3.8.21). By applying ϕ to this expression, we obtain (3.8.10) for this choice of N and m.

In the general case, we consider some formal Macdonald process in a greater number of variables, apply Theorem 3.8.11, then apply variable contractions ϕ to obtain the Corollary. □

3.8.3 Proof of Theorem B.3

Let $\boldsymbol{U} = (U^1, \dots, U^{M+N})$, $\boldsymbol{V} = (V^1, \dots, V^{M+N})$ be tuples of countable sets of variables. Given $a \geq 0$ and a countable set of variables U, let $\rho_a^U : \Lambda_U \to \mathbb{R}$ be the unital algebra homomorphism defined by

$$\rho_a^U : p_n(U) \mapsto a^n.$$

This uniquely determines the specialization ρ because the power symmetric functions generate the algebra of symmetric functions. We may think of ρ as evaluating $f \in \Lambda_U$ at $(a, 0, 0, \dots)$. We can certainly extend ρ_a^U to $\widehat{\Lambda}_U$. Let

$$\rho = \bigotimes_{i=1}^{N} (\rho_{a_i}^{U^i} \otimes \rho_0^{V^i}) \otimes \bigotimes_{i=1}^{M} (\rho_0^{U^{N+i}} \otimes \rho_{b_i}^{V^{N+i}}).$$

Then

$$\rho\left(\mathbb{SP}^{\mathbf{f}}_{\boldsymbol{U},\boldsymbol{V}}(\boldsymbol{\lambda})\right) = \mathbb{SP}_{\boldsymbol{a},\boldsymbol{b}}(\boldsymbol{\lambda}).$$

This implies that for any $1 \leq n_1 \leq \cdots \leq n_m \leq M$ and $t_1, \dots, t_m \neq 0$, we have

$$\rho\left(\mathbb{E}_{\mathbb{SP}^{\mathbf{f}}_{\boldsymbol{U},\boldsymbol{V}}}\left[\prod_{i=1}^{m} \mathfrak{p}_{t_i}(\lambda^{n_i+N})\right]\right) = \mathbb{E}_{\mathbb{SP}_{\boldsymbol{a},\boldsymbol{b}}}\left[\prod_{i=1}^{m} \mathfrak{p}_{t_i}(\lambda^{n_i})\right].$$

By Theorem 3.8.12, this is exactly

$$\rho\left(\frac{1}{(2\pi\mathbf{i})^m} \oint \cdots \oint \prod_{1 \leq a < b \leq m} \frac{(1 - \frac{z_a}{z_b})(1 - \frac{t_b z_a}{t_a z_b})}{(1 - \frac{z_a}{t_a z_b})(1 - \frac{t_b z_a}{z_b})} \prod_{a=1}^{m} \prod_{\substack{1 \leq i \leq n_a \\ n_a \leq j \leq M+N}} H(U^i, z_a^{-1}; t_a)^{-1} H(z_a, V^j; t_a^{-1}) \frac{dz_a}{z_a}\right)$$

which gives precisely the contour integral formula stated by Theorem 3.8.2. Note we used the fact that the residue operator commutes with continuous maps. $\square$

Chapter 4

Moments and Supersymmetric Lifts

4.1 Introduction and Main Results

In this chapter, we introduce a new approach to study particle systems arising from certain convolution operations on random matrices and representations of the unitary group. The main results of this chapter are moment formulas for each of these models. We do not pursue any asymptotic questions. However, the formulas we present are amenable for asymptotic analysis in the global and local regimes — this will be explored in a future article. We begin by introducing our two models.

The first model is obtained by projections and sums of independent random matrices whose distributions are invariant under conjugation by unitary matrices. Let $\mathcal{R}_N := \{\boldsymbol{x} \in \mathbb{R}^N : x_1 \geq \cdots \geq x_N\}$ and $\mathcal{R}_N^\circ := \{\boldsymbol{x} \in \mathcal{R}_N : x_1 > \cdots > x_N\}$. Given $\boldsymbol{a}, \boldsymbol{b} \in \mathcal{R}_N$, let $\boldsymbol{a} \boxplus \boldsymbol{b}$ denote the random element of $\mathcal{R}_N$ whose distribution is given by the eigenvalue distribution of

$$U\mathrm{diag}(\boldsymbol{a})U^\dagger + V\mathrm{diag}(\boldsymbol{b})V^\dagger$$

where U, V are independent Haar distributed $N \times N$ unitary matrices. Denote by $\pi_{N\to k}\boldsymbol{a}$ the random element of $\mathcal{R}^k$ whose distribution is given by the eigenvalues of the $k \times k$ top left corner submatrix of $U\mathrm{diag}(\boldsymbol{a})U^\dagger$, for $1 \leq k \leq N$.

The second, closely related model is obtained from restrictions and tensor products of representations of the unitary group. A *signature of length* N is an N-tuple $(\lambda_1 \geq \cdots \geq \lambda_N)$ of non-increasing integers. Let $\mathbb{GT}_N$ denote the set of all signatures of length N and $\mathcal{U}(N)$ denote the N-dimensional unitary group. There is a bijection between $\mathbb{GT}_N$ and the set of irreducible representations $\mathcal{U}(N)$ where $\lambda \mapsto \mathrm{V}_\lambda$ if λ is the highest weight of ρ_λ. Let $\lambda^{(1)}, \ldots, \lambda^{(m)} \in \mathbb{GT}_N$ and for $1 \leq h \leq N$, let

$$\rho = \rho^{\lambda^{(1)}} \otimes \cdots \otimes \rho^{\lambda^{(m)}}\Big|_{\mathcal{U}(h)} = \bigoplus_{\lambda \in \mathbb{GT}_h} c^\lambda \rho_\lambda \tag{4.1.1}$$

be its decomposition into irreducible components of $\mathcal{U}(h)$. We define the measure $\mathbb{P}^\rho$ on $\mathbb{GT}_N$ by

$$\mathbb{P}^\rho(\lambda) = \frac{c^\lambda \dim \rho_\lambda}{\dim \rho}, \tag{4.1.2}$$

in other words the probability of λ is proportional to the dimension of the isotypic component of V_λ in V.

The main results of this chapter are formulas for the joint moments

$$\mathbb{E}\left[\prod_{i=1}^{k}\sum_{j=1}^{h} e^{c_i s_j}\right] \qquad \text{and} \qquad \mathbb{E}\left[\prod_{i=1}^{k}\sum_{j=1}^{h} e^{c_i \lambda_j}\right]$$

where $c_1, \ldots, c_k > 0$, $\boldsymbol{s} = \pi_{N\to h}(\boldsymbol{r}^{(1)} \boxplus \cdots \boxplus \boldsymbol{r}^{(m)})$ for some deterministic $\boldsymbol{r}^{(1)}, \ldots, \boldsymbol{r}^{(m)} \in \mathcal{R}_N$, and $\lambda = (\lambda_1, \ldots, \lambda_N) \sim \rho^{\lambda^{(1)}} \otimes \cdots \otimes \rho^{\lambda^{(m)}}\Big|_{\mathcal{U}(h)}$ for some $\lambda^{(1)}, \ldots, \lambda^{(m)} \in \mathbb{GT}_N$. The formal statements are given by Theorems 4.5.1 and 4.5.2. The formulas for the joint moments are in terms of certain lifts of the multivariate Bessel functions, continuous analogues of the Schur functions, for which we obtain contour integral representations for, summarized by Theorem 4.4.2.

The remainder of this chapter is organized as follows. In Section 4.2, we give some necessary background on the symmetric functions that we will encounter. In Section 4.3, we obtain expressions for the joint moments in terms of certain lifts of Schur and multivariate Bessel functions. In Section 4.4, we find contour integral representations for a class of these lifts. Finally, Section 4.5 gives the formal statements of our main results, along with some directions for future research.

Notation. Given a function $f(w_1, \ldots, w_N)$ we write

$$f(w_1, \ldots, w_k, a^{N-k}) := f(w_1, \ldots, w_k, a, \ldots, a),$$

rather than the usual indication where a^{N-k} is exponentiation.

4.2 Preliminaries

The rational Schur functions are defined for $\lambda \in \mathbb{GT}_N$ by

$$s_\lambda(x_1, \ldots, x_N) := \frac{\det\left(x_i^{\lambda_j+N-j}\right)_{i,j=1}^{N}}{\Delta(x_1, \ldots, x_N)}, \qquad \Delta(x_1, \ldots, x_N) := \det\left(x_i^{N-j}\right)_{i,j=1}^{N} = \prod_{1\le i<j\le N}(x_i - x_j) \tag{4.2.1}$$

If $\lambda_N \ge 0$, then s_λ is a homogeneous symmetric polynomial in $x_1, \ldots, x_N$ of degree $\sum_{i=1}^N \lambda_i$. Indeed, the numerator is antisymmetric in $(x_1, \ldots, x_N)$ and therefore divisible by the Vandermonde determinant in the denominator. In general, we also have the relation

$$s_{\lambda+(1^N)}(x_1, \ldots, x_N) = x_1 \cdots x_N s_\lambda(x_1, \ldots, x_N), \quad \lambda \in \mathbb{GT}_N$$

where (1^N) is the N-vector of all ones. Thus, $s_\lambda(x_1,\dots,x_N)$ is a homogeneous rational function in $x_1,\dots,x_N$ of degree $\sum_{i=1}^N \lambda_i$ with possible poles at $x_i=0$. For $h\le N$, we define the *normalized Schur function*

$$\widehat{s}_\lambda(x_1,\dots,x_h;N,q) := \frac{s_\lambda(x_1,\dots,x_h,1,q,\dots,q^{N-h-1})}{s_\lambda(1,q,\dots,q^{N-1})}. \tag{4.2.2}$$

The rational Schur functions are identified with the irreducible characters of $\mathcal{U}(N)$. Observe that any class function on $\mathcal{U}(N)$ is a symmetric function of the eigenvalues of $U\in\mathcal{U}(N)$. More precisely, let $u_1,\dots,u_N$ be the eigenvalues of U, then a class function on $\mathcal{U}(N)$ is of the form

$$U\mapsto f(u_1,\dots,u_N)$$

for some symmetric f.

Proposition 4.2.1. *The irreducible character of $\mathcal{U}(N)$ associated to $\lambda\in\mathbb{GT}_N$ is the symmetric function $s_\lambda(u_1,\dots,u_N)$ on the eigenvalues of $U\in\mathcal{U}(N)$.*

We consider a semiclassical limit of the rational Schur functions. The *multivariate Bessel functions* are defined for $\boldsymbol{\ell}\in\mathcal{R}_N^\circ$ by

$$\mathcal{B}_{\boldsymbol{\ell}}(z_1,\dots,z_N) := \frac{\det\left(e^{z_i\ell_j}\right)_{i,j=1}^N}{\Delta(\boldsymbol{z})} = \lim_{\varepsilon\to0}\varepsilon^{N(N-1)/2}s_{\lfloor\boldsymbol{\ell}/\varepsilon\rfloor}(e^{\varepsilon z_1},\dots,e^{\varepsilon z_N}) \tag{4.2.3}$$

for $z_1,\dots,z_N$ distinct and extended to $z_1,\dots,z_N\in\mathbb{C}$ arbitrary by continuity. We now define *normalized multivariate Bessel functions* for $\boldsymbol{\ell}\in\mathcal{R}_N$ by

$$\widehat{\mathcal{B}}_{\boldsymbol{\ell}}(z_1,\dots,z_h;N,\varepsilon) := \lim_{\varepsilon\to0}\widehat{s}_{\lfloor\boldsymbol{\ell}/\varepsilon\rfloor}(e^{\varepsilon z_1},\dots,e^{\varepsilon z_h};N,e^{\varepsilon}).$$

For $\boldsymbol{\ell}\in\mathcal{R}_N^\circ$, we have the exact formula

$$\widehat{\mathcal{B}}_{\boldsymbol{\ell}}(z_1,\dots,z_N;N,\alpha) := \frac{\mathcal{B}_{\boldsymbol{\ell}}(z_1,\dots,z_N)}{\mathcal{B}_{\boldsymbol{\ell}}(0,\alpha,\dots,(N-1)\alpha)} = \frac{\Delta((N-1)\alpha,(N-2)\alpha,\dots,0)}{\Delta(e^{\alpha\ell_1},\dots,e^{\alpha\ell_N})}\cdot\frac{\det[e^{z_i\ell_j}]_{i,j=1}^N}{\Delta(\boldsymbol{z})} \tag{4.2.4}$$

which extends to a continuous function in $(\boldsymbol{\ell},\boldsymbol{z})\in\mathbb{C}^N\times\mathbb{C}^N$. We note that

$$s_\lambda(e^{u_1},\dots,e^{u_N}) = \frac{\Delta(u_1,\dots,u_N)}{\Delta(e^{u_1},\dots,e^{u_N})}\mathcal{B}_{\lambda+\delta_N}(u_1,\dots,u_N) \tag{4.2.5}$$

where $\delta_N := (N-1,N-2,\dots,0)$.

There exists an analogue of the Littlewood-Richardson coefficients for normalized multivariate Bessel functions

$$\widehat{\mathcal{B}}_{\boldsymbol{\ell}}(z_1,\dots,z_N;N,0)\widehat{\mathcal{B}}_{\boldsymbol{r}}(z_1,\dots,z_N;N,0) = \int_{\boldsymbol{s}} c_{\boldsymbol{\ell},\boldsymbol{r}}^{\boldsymbol{s}}\,\widehat{\mathcal{B}}_{\boldsymbol{s}}(z_1,\dots,z_N;N,0)\,d\boldsymbol{s} \tag{4.2.6}$$

where $c^{\boldsymbol{s}}_{\boldsymbol{\ell},\boldsymbol{r}}$ is a probability measure supported in the set of $\boldsymbol{s}$ which satisfy

$$s_1 + \cdots + s_N = (r_1 + \ell_1) + \cdots (r_N + \ell_N), \qquad r_N + \ell_N \le s_i \le r_1 + \ell_1, \quad 1 \le i \le N$$

which is in particular compact, see [29, Section 2].

Definition 4.2.2. Given $\boldsymbol{r} \in \mathcal{R}_N$, define the *corners process with top row* $\boldsymbol{r}$ to be the random point process $\{x_i^k\}_{1 \le i \le k \le N}$ where $(x_1^k \ge \cdots \ge x_k^k) = \pi_{N,k}\boldsymbol{r}$.

The normalized multivariate Bessel function indexed by $\boldsymbol{r}$ is the (partial) Laplace transform of the corners process $\{x_i^k\}_{1 \le i \le k \le N}$ with top row $\boldsymbol{r}$:

$$\widehat{\mathcal{B}}_{\boldsymbol{r}}(z_1, \ldots, z_N; N, 0) = \mathbb{E}_{\{x_i^k\}} \left[\exp\left(\sum_{k=1}^N z_k \cdot \left(\sum_{i=1}^k x_i^k - \sum_{j=1}^{k-1} x_j^{k-1} \right) \right) \right], \tag{4.2.7}$$

see [29, Equation (20)]. Moreover, the measures $c^{\boldsymbol{s}}_{\boldsymbol{\ell},\boldsymbol{r}}$ are intimately linked to the operation $\boxplus$.

Proposition 4.2.3 ([29, Proposition 2.3])**.** *Let $\boldsymbol{\ell}, \boldsymbol{r} \in \mathcal{R}_N$. Then $\boldsymbol{s} = \boldsymbol{\ell} \boxplus \boldsymbol{r}$ is $c^{\boldsymbol{s}}_{\boldsymbol{\ell},\boldsymbol{r}}$-distributed.*

Corollary 4.2.4. *Let $\boldsymbol{r}^1, \ldots, \boldsymbol{r}^m \in \mathcal{R}_N$. For any $1 \le h \le N$, if $\boldsymbol{s} = \pi_{N \to h}(\boldsymbol{r}^1 \boxplus \cdots \boxplus \boldsymbol{r}^m)$, then*

$$\mathbb{E}\widehat{\mathcal{B}}_{\boldsymbol{s}}(z_1, \ldots, z_h; h, 0) = \prod_{i=1}^n \widehat{\mathcal{B}}_{\boldsymbol{r}^i}(z_1, \ldots, z_h; N, 0).$$

Proof. By repeated applications of Theorem 4.2.3, we have for $\boldsymbol{r} = \boldsymbol{r}^1 \boxplus \cdots \boxplus \boldsymbol{r}^m$ that

$$\mathbb{E}\widehat{\mathcal{B}}_{\boldsymbol{r}}(z_1, \ldots, z_N; N, 0) = \prod_{i=1}^m \widehat{\mathcal{B}}_{\boldsymbol{r}^i}(z_1, \ldots, z_N; N, 0).$$

For $\boldsymbol{s} = \pi_{N \to h}\boldsymbol{r}$, we have by (4.2.7) that

$$\mathbb{E}\widehat{\mathcal{B}}_{\boldsymbol{s}}(z_1, \ldots, z_h; h, 0) = \mathbb{E}\widehat{\mathcal{B}}_{\boldsymbol{r}}(z_1, \ldots, z_h; N, 0) = \prod_{i=1}^m \widehat{\mathcal{B}}_{\boldsymbol{r}^i}(z_1, \ldots, z_h; N, 0).$$

□

4.3 Moments and Operators

For $t \in \mathbb{C}$, define the operator D_t^N on functions in $(x_1, \ldots, x_N)$ by

$$D_t^N := \sum_{i=1}^N \prod_{j \ne i} \frac{t x_i - x_j}{x_i - x_j} T_{t, x_i}$$

where T_{t,x_i} maps $g(x_1, \dots, x_i, \dots, x_N)$ to $g(x_1, \dots, tx_i, \dots, x_N)$. By direct computation, we have the eigenrelations

$$D_t^N s_\lambda(x_1, \dots, x_N) = \left(\sum_{i=1}^{N} t^{\lambda_i + N - i} \right) s_\lambda(x_1, \dots, x_N), \quad \lambda \in \mathbb{Y}_N \tag{4.3.1}$$

Definition 4.3.1. Let $\Omega \subset \mathbb{C}$ be a neighborhood and F be a continuous symmetric function on Ω^N. We say that a family

$$\left\{ \widetilde{F}_k(x_1, \dots, x_{N+k}/y_1, \dots, y_k) \right\}_{k \geq 0}$$

is a *supersymmetric lift of F (on the domain Ω)* if $\widetilde{F}_k$ is continuous and doubly symmetric in $(x_1, \dots, x_{N+k}) \in \Omega^{N+k}$ and $(y_1, \dots, y_k) \in \Omega^k$, and satisfies

$$\widetilde{F}_k(x_1, \dots, x_{N+k}/y_1, \dots, y_k)\Big|_{x_{N+k} = y_k} = \widetilde{F}_{k-1}(x_1, \dots, x_{N+k-1}/y_1, \dots, y_{k-1}), \tag{4.3.2}$$

$$\widetilde{F}_0(x_1, \dots, x_N) = F(x_1, \dots, x_N). \tag{4.3.3}$$

In this case, we drop the subscript k and say that $\widetilde{F}$ is a supersymmetric lift of F. We refer to property (4.3.2) as the *cancellation property.*

The choice of supersymmetric lift is not unique. This freedom of choice will be relevant for us later.

Example 4.3.1. For multiplicative functions, there is a natural multiplicative supersymmetric lift. If

$$F(x_1, \dots, x_N) = \prod_{i=1}^{N} f(x_i)$$

for some continuous function f on Ω, then

$$\widetilde{F}(x_1, \dots, x_{N+k}/y_1, \dots, y_k) = \prod_{i=1}^{N+k} f(x_i) \prod_{j=1}^{k} \frac{1}{f(y_j)}$$

is a supersymmetric lift of F on $\Omega \setminus \{z : f(z) = 0\}$.

Example 4.3.2. If F and G are symmetric functions with supersymmetric lifts $\widetilde{F}$ and $\widetilde{G}$ on Ω respectively, then $\widetilde{F} \cdot \widetilde{G}$ is a supersymmmetric lift of $F \cdot G$ on Ω.

Example 4.3.3. Suppose $\{\widetilde{F}_k\}_{k \geq 0}$ is a supersymmetric lift of a function F. Then $\{\widetilde{F}_k\}_{k \geq 1}$ is a supersymmetric lift of $\widetilde{F}_1$.

Example 4.3.4. We briefly describe the connection between supersymmetric lifts and algebras of supersymmetric polynomials; further details can be found in [43, I.3, Examples 23 and 24], [48, Chapter 2]. Let Λ_n denote the algebra of symmetric polynomials in n-variables $x_1, \dots, x_n$ and

$$\Lambda = \varprojlim_n \Lambda_n$$

be the algebra of symmetric functions where the projection $\Lambda_{n+1} \to \Lambda_n$ is the map setting $x_{n+1} = 0$. Denote by $\mathrm{pr}_n : \Lambda \to \Lambda_n$ the natural projections. Since the Newton power sums p_k form an algebraic basis for Λ, an algebra homomorphism on Λ is determined by its action on the Newton power sums. The *algebra of supersymmetric polynomials in ordinary variables* $(x_1, \dots, x_m)$ *and dual variables* $(y_1, \dots, y_n)$, denoted $\Lambda_{m,n}$, is the image of the map $\Lambda \to \mathbb{C}[x_1, \dots, x_m, y_1, \dots, y_n]$ determined by

$$1 \mapsto 1, \quad p_k \mapsto \sum_{i=1}^{m} x_i^k - \sum_{j=1}^{n} y_j^k, \quad quadk > 0$$

From this definition, it is apparent that setting $x_m = y_n$ "cancels" these two variables. More precisely, for $f \in \Lambda_{m,n}$, $f\big|_{x_m = y_n = u}$ is independent of u. Given $F \in \Lambda_n$, fix $f \in \mathrm{pr}_n^{-1}(F)$. The images $\widetilde{F} := \widetilde{F}_k$ of f in $\Lambda_{n+k,n}$ for $k \geq 0$ provide a supersymmetric lift of F. This choice is not unique because the choice of lift f is not unique.

The action of D_t^N on a function can be described as a contour integral in terms of a supersymmetric lift of that function.

Proposition 4.3.2. *Let $a_1, \dots, a_N, t_1, \dots, t_k \in \mathbb{C}$ and $\Omega \subset \mathbb{C}$ be a neighborhood of $\{a_1, \dots, a_N\}$. If $F : \Omega^N \to \mathbb{C}$ is a continuous symmetric function, $\widetilde{F}$ is a supersymmetric lift of F on Ω, then*

$$\begin{aligned} D_{t_1}^N \cdots D_{t_k}^N F(a_1, \dots, a_N) = \frac{1}{(2\pi\mathbf{i})^k} \oint \cdots \oint \widetilde{F}(a_1, \dots, a_N, t_1 u_1, \dots, t_k u_k / u_1, \dots, u_k) \\ \times \prod_{1 \leq i < j \leq k} \frac{(t_j u_j - t_i u_i)(u_j - u_i)}{(u_j - t_i u_i)(t_j u_j - u_i)} \prod_{i=1}^{k} \left(\prod_{j=1}^{N} \frac{t_i u_i - a_j}{u_i - a_j} \right) \frac{du_i}{(t_i - 1) u_i} \end{aligned} \tag{4.3.4}$$

where the u_i-contour is contained in Ω and positively oriented around the poles $a_1, \dots, a_N$ for $1 \leq i \leq k$, and the u_j-contour contains $t_i u_i$ whenever $i < j$, assuming such a set of contours in Ω exists.

Proof. By continuity, we may assume $a_1, \dots, a_N$ are distinct. For $k = 1$, we have

$$D_t^N F(a_1, \dots, a_N) = \frac{1}{2\pi\mathbf{i}} \oint \widetilde{F}(a_1, \dots, a_N, tu/u) \prod_{j=1}^{N} \frac{tu - a_j}{u - a_j} \frac{du}{(t-1)u}$$

by the residue theorem and the cancellation property (4.3.2). The general case follows by induction and Examples 4.3.1 to 4.3.3. $\square$

Proposition 4.3.3. *Let N, m, k, h be integers such that $1 \leq h \leq N$. Let $t_1, \dots, t_k > 0$, $\lambda^1, \dots, \lambda^m \in \mathbb{GT}_N$, and*

$$\rho = \rho^{\lambda^1} \otimes \cdots \otimes \rho^{\lambda^m} \Big|_{\mathcal{U}(h)}.$$

Suppose that $S_1, \dots, S_m$ are supersymmetric lifts of $\widehat{s}_{\lambda^1}(z_1, \dots, z_N; N, 1), \dots, \widehat{s}_{\lambda^m}(z_1, \dots, z_N; N, 1)$

respectively in some neighborhood $\Omega \subset \mathbb{C}$ of 1 and $\lambda \sim \mathbb{P}^\rho$. Then

$$\begin{aligned}\mathbb{E}\left[\prod_{i=1}^k \sum_{j=1}^h t_i^{\lambda_j+N-j}\right] &= \frac{1}{(2\pi\mathbf{i})^k}\oint\cdots\oint \prod_{i=1}^m S_i(1^N, t_1u_1,\dots,t_ku_k/u_1,\dots,u_k)\\ &\times \prod_{1\le i<j\le k}\frac{(t_ju_j-t_iu_i)(u_j-u_i)}{(u_j-t_iu_i)(t_ju_j-u_i)}\prod_{i=1}^k\left(\frac{t_iu_i-1}{u_i-1}\right)^h\frac{du_i}{(t_i-1)u_i}\end{aligned} \tag{4.3.5}$$

where the u_i-contour is contained in Ω and positively oriented around the pole at 0, and the u_j-contour contains t_iu_i whenever $i<j$, assuming such a set of contours in Ω exists.

Proof. By (4.1.2), we have

$$\mathbb{E}\left(\prod_{i=1}^k \sum_{j=1}^h t_i^{\lambda_j+N-j}\right)\frac{s_\lambda(x_1,\dots,x_h)}{s_\lambda(1^h)} = \sum_{\lambda\in\mathbb{GT}_N}\left(\prod_{i=1}^k \sum_{j=1}^h t_i^{\lambda_j+N-j}\right)\frac{c^\lambda \dim\rho_\lambda}{\dim\rho}\frac{s_\lambda(x_1,\dots,x_h)}{s_\lambda(1^h)}$$

where c^λ are the coefficients from (4.1.1). By (4.3.1) and the equality

$$\dim\rho_\lambda = s_\lambda(1^h)$$

which follows from Theorem 4.2.1, this becomes

$$\sum_{\lambda\in\mathbb{GT}_N}\left(\prod_{i=1}^k \sum_{j=1}^h t_i^{\lambda_j+N-j}\right)c^\lambda\frac{s_\lambda(x_1,\dots,x_h)}{\dim\rho} = D_{t_1}^h\cdots D_{t_k}^h\sum_{\lambda\in\mathbb{GT}_N}c^\lambda\frac{s_\lambda(x_1,\dots,x_h)}{\dim\rho}.$$

Using the equalities

$$\dim\rho = \prod_{i=1}^m s_{\lambda^i}(1^N), \quad \prod_{i=1}^m s_{\lambda^i}(x_1,\dots,x_h,1^{N-h}) = \sum_{\lambda\in\mathbb{GT}_N}c^\lambda s_\lambda(x_1,\dots,x_h)$$

which follow from (4.1.1) and Theorem 4.2.1, we obtain

$$D_{t_1}^h\cdots D_{t_k}^h\prod_{i=1}^m\frac{s_{\lambda^i}(x_1,\dots,x_h,1^{N-h})}{s_{\lambda^i}(1^N)} = D_{t_1}^h\cdots D_{t_k}^h\prod_{i=1}^m\widehat{s}_{\lambda^i}(x_1,\dots,x_h;N,1).$$

By setting $x_1=\cdots=x_h=1$, we have shown

$$\mathbb{E}\left(\prod_{i=1}^k \sum_{j=1}^h t_i^{\lambda_j+N-j}\right) = D_{t_1}^h\cdots D_{t_k}^h\prod_{i=1}^m\widehat{s}_{\lambda^i}(x_1,\dots,x_h;N,h)\Big|_{x_1=\cdots=x_h=1}.$$

Applying (4.3.4) and Example 4.3.1 to the right hand side implies (4.3.5). ☐

There is an analogous connection between the normalized multivariate Bessel functions and random matrices. For $c \in \mathbb{C}$, define the operator $\mathcal{D}_c^N$ on functions in $(z_1, \ldots, z_N)$ by

$$\mathcal{D}_c^N := \sum_{i=1}^{N} \prod_{j \neq i} \frac{c + z_i - z_j}{z_i - z_j} \mathcal{T}_{c,z_i}$$

where $\mathcal{T}_{c,z_i}$ maps $g(z_1, \ldots, z_i, \ldots, z_N)$ to $g(z_1, \ldots, z_i + c, \ldots, z_N)$. We have the eigenrelations

$$\mathcal{D}_c^N \widehat{\mathcal{B}}_{\boldsymbol{\ell}}(z_1, \ldots, z_N; N, 0) = \left(\sum_{i=1}^{N} e^{c\ell_i} \right) \widehat{\mathcal{B}}_{\boldsymbol{\ell}}(z_1, \ldots, z_N; N, 0), \quad \boldsymbol{\ell} \in \mathcal{R}_N \tag{4.3.6}$$

which can be checked directly for $\boldsymbol{\ell} \in \mathcal{R}_N^\circ$ and extended to $\boldsymbol{\ell} \in \mathcal{R}_N$ by continuity. We have an analogue of Theorem 4.3.2 for $\mathcal{D}_c$.

Proposition 4.3.4. *Let $a_1, \ldots, a_N, c_1, \ldots, c_k \in \mathbb{C}$ and $\Omega \subset \mathbb{C}$ be a neighborhood of $\{a_1, \ldots, a_N\}$. If $F : \Omega^N \to \mathbb{C}$ is a continuous symmetric function, $\widetilde{F}$ is a supersymmetric lift of F, and Ω is suitably large, then*

$$\begin{aligned} \mathcal{D}_{c_1}^N \cdots \mathcal{D}_{c_k}^N F(a_1, \ldots, a_N) = \frac{1}{(2\pi \mathbf{i})^k} \oint \cdots \oint \widetilde{F}(a_1, \ldots, a_N, u_1 + c_1, \ldots, u_k + c_k / u_1, \ldots, u_k) \\ \times \prod_{1 \le i < j \le k} \frac{(u_j + c_j - u_i - c_i)(u_j - u_i)}{(u_j - u_i - c_i)(u_j + c_j - u_i)} \prod_{i=1}^{k} \left(\prod_{j=1}^{N} \frac{u_i + c_i - a_j}{u_i - a_j} \right) \frac{du_i}{c_i} \end{aligned} \tag{4.3.7}$$

where the u_i-contour is contained in Ω and positively oriented around the poles $a_1, \ldots, a_N$ for $1 \le i \le k$. Moreover, the u_j-contour contains $u_i + c_i$ whenever $i < j$.

Proposition 4.3.5. *Let N, m, k, h be integers such that $1 \le h \le N$. For any $c_1, \ldots, c_k > 0$ and $\boldsymbol{r}^1, \ldots, \boldsymbol{r}^m \in \mathcal{R}_N$, let $\boldsymbol{s} = \pi_{N \to h}(\boldsymbol{r}^1 \boxplus \cdots \boxplus \boldsymbol{r}^m)$. Suppose that $\mathfrak{B}_1, \ldots, \mathfrak{B}_m$ are supersymmetric lifts of $\widehat{\mathcal{B}}_{\boldsymbol{r}^1}(z_1, \ldots, z_N; N, 0), \ldots, \widehat{\mathcal{B}}_{\boldsymbol{r}^m}(z_1, \ldots, z_N; N, 0)$ in some neighborhood $\Omega \subset \mathbb{C}$ of 0. If Ω is suitably large, then*

$$\begin{aligned} \mathbb{E}\left[\prod_{i=1}^{k} \sum_{j=1}^{h} e^{c_i s_j} \right] = \frac{1}{(2\pi \mathbf{i})^k} \oint \cdots \oint \prod_{i=1}^{m} \mathfrak{B}_i(0^N, u_1 + c_1, \ldots, u_k + c_k / u_1, \ldots, u_k) \\ \times \prod_{1 \le i < j \le k} \frac{(u_j + c_j - u_i - c_i)(u_j - u_i)}{(u_j - u_i - c_i)(u_j + c_j - u_i)} \prod_{i=1}^{k} \left(\prod_{j=1}^{N} \frac{u_i + c_i}{u_i} \right) \frac{du_i}{c_i} \end{aligned} \tag{4.3.8}$$

where the u_i-contour is contained in Ω and positively oriented around the pole at 0. Moreover, the u_j-contour contains $u_i + c_i$ whenever $i < j$.

Proof. By Theorem 4.2.4 and (4.3.6), we have

$$\mathbb{E}\left(\prod_{i=1}^{k}\sum_{j=1}^{h} e^{c_i s_j}\right)\widehat{\mathcal{B}}_{\boldsymbol{s}}(z_1,\ldots,z_h)\Bigg|_{z_i=0} = \mathcal{D}_{c_1}^{h}\cdots\mathcal{D}_{c_k}^{h}\prod_{i=1}^{m}\widehat{\mathcal{B}}_{\boldsymbol{r}^i}(z_1,\ldots,z_h,0^{N-h})\Bigg|_{z_i=0}.$$

The left hand side of the above matches the left hand side of (4.3.8) because $\widehat{\mathcal{B}}_{\boldsymbol{s}}(0^h)=1$, and the right hand side of the above matches the right hand side of (4.3.8) by Theorem 4.3.2. □

4.4 Formulas for Supersymmetric Lifts

Let $\boldsymbol{x}=(x_1,\ldots,x_m)$, $\boldsymbol{y}=(y_1,\ldots,y_n)$, $p\in\mathbb{C}$,

$$D(\boldsymbol{x};\boldsymbol{y}) := \prod_{i=1}^{m}\prod_{j=1}^{n}(x_i+y_j), \quad E_p(\boldsymbol{x};\boldsymbol{y}) := \left(\frac{e^{p(x_i-y_j)}}{x_i-y_j}\right)_{\substack{1\le i\le m\\ 1\le j\le n}}.$$

Let $\delta_m=(m-1,m-2,\ldots,0)$, and for any $\boldsymbol{\ell}=(\ell_1,\ldots,\ell_m)\in\mathbb{C}^m$, let

$$A_{\boldsymbol{\ell}}(\boldsymbol{x}) := \left(e^{x_i\ell_j}\right)_{1\le i,j\le m}.$$

Given integers $N,k\ge 0$, $p\in\mathbb{C}$, and $\boldsymbol{\ell}\in\mathcal{R}_N^{\circ}$, set $\boldsymbol{u}=(u_1,\ldots,u_{N+k})$, $\boldsymbol{v}=(v_1,\ldots,v_k)$, and define

$$\mathcal{B}_{\boldsymbol{\ell},p}(\boldsymbol{u}/\boldsymbol{v}) := (-1)^{Nk}\frac{D(\boldsymbol{u};-\boldsymbol{v})}{\Delta(\boldsymbol{u})\Delta(-\boldsymbol{v})}\det\begin{pmatrix}E_p(\boldsymbol{u};\boldsymbol{v}) & A_{\boldsymbol{\ell}}(\boldsymbol{u})\end{pmatrix} \tag{4.4.1}$$

which is valid for $u_1,\ldots,u_{N+k},v_1,\ldots,v_k$ distinct but extends by continuity. This is a doubly symmetric function in $\boldsymbol{u}\in\mathbb{C}^{N+k}$ and $\boldsymbol{v}\in\mathbb{C}^k$. Given $\alpha\in\mathbb{C}$, the function

$$\frac{\mathcal{B}_{\boldsymbol{\ell},p}(u_1,\ldots,u_{N+k}/v_1,\ldots,v_k)}{\mathcal{B}_{\boldsymbol{\ell}}(0,\alpha,2\alpha,\ldots,(N-1)\alpha)}. \tag{4.4.2}$$

extends to an analytic function in $(\alpha,\boldsymbol{\ell},\boldsymbol{u},\boldsymbol{v})\in\mathbb{C}^{1+N+(N+k)+k}$ by (4.2.3) and (4.4.1), so that this function is defined for $\boldsymbol{\ell}\in\mathcal{R}_N$. Thus, we can let

$$\widehat{\mathcal{B}}_{\boldsymbol{\ell},p}(u_1,\ldots,u_k/v_1,\ldots,v_k;N,\alpha) := \frac{\mathcal{B}_{\boldsymbol{\ell},p}(u_1,\ldots,u_k,0,\alpha,2\alpha,\ldots,(N-1)\alpha/v_1,\ldots,v_k)}{\mathcal{B}_{\boldsymbol{\ell}}(0,\alpha,2\alpha,\ldots,(N-1)\alpha)} \tag{4.4.3}$$

for $\boldsymbol{\ell}\in\mathcal{R}_N$.

Given $\lambda\in\mathbb{GT}_N$, define

$$s_{\lambda,p}(e^{\boldsymbol{u}}/e^{\boldsymbol{v}}) := (-1)^{Nk}\prod_{i=1}^{k}e^{-v_i}\cdot\frac{D(e^{\boldsymbol{u}};-e^{\boldsymbol{v}})}{\Delta(e^{\boldsymbol{u}})\Delta(-e^{\boldsymbol{v}})}\det\begin{pmatrix}E_p(\boldsymbol{u};\boldsymbol{v}) & A_{\lambda+\delta_N}(\boldsymbol{u})\end{pmatrix} \tag{4.4.4}$$

which is a doubly symmetric function in $\boldsymbol{u}\in\mathbb{C}^{N+k}$ and $\boldsymbol{v}\in\mathbb{C}^k$.

Remark 22. Note that $s_{\lambda,p}(\boldsymbol{x}/\boldsymbol{y})$ is not well-defined, for $\boldsymbol{x} = (x_1, \ldots, x_{N+k})$ and $\boldsymbol{y} = (y_1, \ldots, y_k)$, without specifying branches of $\log x_i$ and $\log y_j$. Throughout this article, we write $s_{\lambda,p}(e^{\boldsymbol{u}}/e^{\boldsymbol{v}})$ to indicate the branch $u_i = \log x_i$ and $v_j = \log y_j$, or otherwise explicitly state the choice of branch.

Define

$$\widehat{s}_{\lambda,p}(e^{u_1}, \ldots, e^{u_k}/e^{v_1}, \ldots, e^{v_k}; N, e^{\alpha}) := \frac{s_{\lambda,p}(e^{u_1}, \ldots, e^{u_k}, 1, e^{\alpha}, \ldots, e^{(N-1)\alpha}/e^{v_1}, \ldots, e^{v_k})}{s_{\lambda}(1, e^{\alpha}, \ldots, e^{(N-1)\alpha})}. \tag{4.4.5}$$

We have the relation

$$s_{\lambda,p}(e^{\boldsymbol{u}}/e^{\boldsymbol{v}}; N, e^{\alpha}) = \prod_{i=1}^{k} e^{-v_i} \cdot \frac{D(e^{\boldsymbol{u}}; -e^{\boldsymbol{v}})}{D(\boldsymbol{u}; -\boldsymbol{v})} \frac{\Delta(\boldsymbol{u})\Delta(\boldsymbol{v})}{\Delta(e^{\boldsymbol{u}})\Delta(e^{\boldsymbol{v}})} \mathcal{B}_{\lambda+\delta_N,p}(\boldsymbol{u}/\boldsymbol{v}; N, \alpha). \tag{4.4.6}$$

Combining (4.2.5) and (4.4.6), we have

$$\begin{aligned}
&\widehat{s}_{\lambda,p}(e^{u_1}, \ldots, e^{u_k}/e^{v_1}, \ldots, e^{v_k}; N, e^{\alpha}) \\
&\quad = \prod_{i=1}^{k} \frac{1}{e^{v_i}} \left(\prod_{j=1}^{N} \frac{u_i - (j-1)\alpha}{e^{u_i} - e^{(j-1)\alpha}} \cdot \frac{e^{v_i} - e^{(j-1)\alpha}}{v_i - (j-1)\alpha} \right) \frac{D(e^{u_1}, \ldots, e^{u_k}; -e^{v_1}, \ldots, -e^{v_k})}{D(u_1, \ldots, u_k; -v_1, \ldots, -v_k)} \\
&\qquad \times \frac{\Delta(u_1, \ldots, u_k)\Delta(v_1, \ldots, v_k)}{\Delta(e^{u_1}, \ldots, e^{u_k})\Delta(e^{v_1}, \ldots, e^{v_k})} \widehat{\mathcal{B}}_{\lambda+\delta_N,p}(u_1, \ldots, u_k/v_1, \ldots, v_k; N, \alpha)
\end{aligned} \tag{4.4.7}$$

We note that (4.4.7) allows us to treat the quantum and classical problems by one unified approach using the multivariate Bessel functions.

Lemma 4.4.1. *For fixed $p, \alpha \in \mathbb{C}$, (4.4.2) and (4.4.4) are respective supersymmetric lifts of $\widehat{\mathcal{B}}_{\boldsymbol{\ell}}(u_1, \ldots, u_N; N, \alpha)$ and $s_\lambda(e^{u_1}, \ldots, e^{u_N})$ on $\mathbb{C}$ for $\boldsymbol{\ell} \in \mathcal{R}_N$ and $\lambda \in \mathbb{GT}_N$.*

Proof. We prove that $\mathcal{B}_{\boldsymbol{\ell},p}$ is a supersymmetric lift of $\mathcal{B}_{\boldsymbol{\ell}}$ with $\boldsymbol{\ell} \in \mathcal{R}_N^\circ$. Dividing by $\mathcal{B}_{\boldsymbol{\ell}}(0, \alpha, \ldots, (N-1)\alpha)$ and continuity in $\boldsymbol{\ell}$ implies the statement for (4.4.2). The statement for (4.4.4) then follows from (4.4.6). For $k = 0$,

$$\mathcal{B}_{\boldsymbol{\ell},p}(u_1, \ldots, u_N) = \mathcal{B}_{\boldsymbol{\ell}}(u_1, \ldots, u_N).$$

It remains to check the cancellation property

$$\mathcal{B}_{\boldsymbol{\ell},p}(u_1, \ldots, u_{N+k}/v_1, \ldots, v_k)\Big|_{u_1=v_1} = \mathcal{B}_{\boldsymbol{\ell},p}(u_2, \ldots, u_{N+k}/v_2, \ldots, v_k).$$

In (4.4.1), bring the term $u_1 - v_1$ from $D(\boldsymbol{u}; -\boldsymbol{v})$ into the first column of the matrix in the determinant. Sending $u_1 \to v_1$ and observing

$$\lim_{u_1 \to v_1} (u_1 - v_1) \frac{e^{p(u_1 - v_1)}}{u_1 - v_1} = 1$$

proves the cancellation property. $\square$

Theorem 4.4.2. *Let $p, \alpha \in \mathbb{C}$, $\boldsymbol{\ell} \in \mathcal{R}_N$, $\boldsymbol{u} = (u_1, \ldots, u_k)$, $\boldsymbol{v} = (v_1, \ldots, v_k) \in (\mathbb{C} \setminus \{0, \alpha, \ldots, (N-1)\alpha\})^k$ where $u_1, \ldots, u_k, v_1, \ldots, v_k$ are distinct. Then*

$$\widehat{\mathcal{B}}_{\boldsymbol{\ell},p}(\boldsymbol{u}/\boldsymbol{v}; N, \alpha) = \frac{D(\boldsymbol{u}; -\boldsymbol{v})}{\Delta(\boldsymbol{u})\Delta(-\boldsymbol{v})} \det\left(\frac{1}{u_i - v_j}\widehat{\mathcal{B}}_{\boldsymbol{\ell},p}(u_i/v_j; N, \alpha)\right)_{i,j=1}^{k}. \tag{4.4.8}$$

If $\alpha \neq 0$, $p \notin \{\ell_1, \ldots, \ell_N\}$ and $u, v \in \mathbb{C} \setminus \{0, \alpha, \ldots, (N-1)\alpha\}$ such that $u \neq v$, then

$$\begin{aligned}&\widehat{\mathcal{B}}_{\boldsymbol{\ell},p}(u/v; N, \alpha)\\ &= (u-v)\prod_{i=1}^{N}\frac{v-(i-1)\alpha}{u-(i-1)\alpha}\left(\frac{e^{p(u-v)}}{u-v} + \int_p^\infty dw \oint \frac{dz}{2\pi\mathbf{i}} \cdot \frac{\alpha}{e^{(w-z)\alpha}-1}\frac{e^{zu}}{e^{wv}}\prod_{i=1}^{N}\frac{e^{w\alpha}-e^{\ell_i\alpha}}{e^{z\alpha}-e^{\ell_i\alpha}}\right)\end{aligned} \tag{4.4.9}$$

where the z-contour is positively oriented around $\ell_1, \ldots, \ell_N$, and the w-contour is a ray from p to ∞ which is disjoint from the z-contour such that

$$\min_{0 \le i < N} \Re\Big(w\big(v - (N-b)\alpha\big)\Big) > 0$$

for $|w|$ large.

By taking $\alpha \to 0$ in (4.4.9), we have

Corollary 4.4.3. *If $p \notin \{\ell_1, \ldots, \ell_N\}$ and $u, v \in \mathbb{C}^*$ such that $u \neq v$, then*

$$\widehat{\mathcal{B}}_{\boldsymbol{\ell},p}(u/v; N, 0) = (u-v)\left(\frac{v}{u}\right)^N\left(\frac{e^{p(u-v)}}{u-v} + \int_p^\infty dw \oint \frac{dz}{2\pi\mathbf{i}} \cdot \frac{1}{w-z}\frac{e^{zu}}{e^{wv}}\prod_{i=1}^{N}\frac{w-\ell_i}{z-\ell_i}\right) \tag{4.4.10}$$

where the z-contour is positively oriented around $\ell_1, \ldots, \ell_N$, and the w-contour is a ray from p to ∞ which is disjoint from the z-contour such that

$$\Re(wv) > 0$$

for $|w|$ large.

4.4.1 Proofs of Formulas

We record a formula for more general normalizations of $\mathcal{B}_{\boldsymbol{\ell},p}$ as one of the main steps in the proof of Theorem 4.4.2.

Proposition 4.4.4. *Let $\boldsymbol{\ell} \in \mathcal{R}_N$, $\boldsymbol{\xi} := (\xi_1, \ldots, \xi_N) \in \mathbb{C}^N$ with distinct components, and $\boldsymbol{u} := (u_1, \ldots, u_k)$, $\boldsymbol{v} := (v_1, \ldots, v_k) \in (\mathbb{C} \setminus \{\xi_1, \ldots, \xi_N\})^k$ such that $u_1, \ldots, u_k$, $v_1, \ldots, v_k$ are distinct. Define the vectors*

$$\mathrm{a}(u) = (e^{\ell_1 u}, \ldots, e^{\ell_N u}), \quad \mathrm{b}(v) = \left(\frac{e^{p(\xi_1 - v)}}{v-\xi_1}, \frac{e^{p(\xi_2 - v)}}{v-\xi_2}, \ldots, \frac{e^{p(\xi_N - v)}}{v-\xi_N}\right)$$

and denote the transpose of a vector u *by* u$'$*. Then*

$$\frac{\mathcal{B}_{\boldsymbol{\ell},p}(\boldsymbol{u},\boldsymbol{\xi}/\boldsymbol{v})}{\mathcal{B}_{\boldsymbol{\ell}}(\boldsymbol{\xi})} = \frac{D(\boldsymbol{u};-\boldsymbol{v})}{\Delta(\boldsymbol{u})\Delta(-\boldsymbol{v})}\det\left(\frac{1}{u_i-v_j}\frac{\mathcal{B}_{\boldsymbol{\ell},p}(u_i,\boldsymbol{\xi}/v_j)}{\mathcal{B}_{\boldsymbol{\ell}}(\boldsymbol{\xi})}\right)_{i,j=1}^{k}. \tag{4.4.11}$$

If $\xi_1,\dots,\xi_N$ *are distinct and* $u,v\in\mathbb{C}\setminus\{\xi_1,\dots,\xi_N\}$ *such that* $u\neq v$*, then*

$$\frac{\mathcal{B}_{\boldsymbol{\ell},p}(u,\boldsymbol{\xi}/v)}{\mathcal{B}_{\boldsymbol{\ell}}(\boldsymbol{\xi})} = (u-v)\prod_{i=1}^{N}\frac{v-\xi_i}{u-\xi_i}\left(\frac{e^{p(u-v)}}{u-v}+\mathrm{a}(u)'A_{\boldsymbol{\ell}}(\boldsymbol{\xi})^{-1}\mathrm{b}(v)\right). \tag{4.4.12}$$

Proof. We may assume $\boldsymbol{\ell}\in\mathcal{R}_N^{\circ}$ by continuity. Suppose $u_1,\dots,u_k,v_1,\dots,v_k,\xi_1,\dots,\xi_N$ are distinct. By (4.4.4), we have

$$\mathcal{B}_{\boldsymbol{\ell},p}(\boldsymbol{u},\boldsymbol{\xi}/\boldsymbol{v}) = \frac{D(\boldsymbol{v};-\boldsymbol{\xi})}{D(\boldsymbol{u};-\boldsymbol{\xi})}\frac{D(\boldsymbol{u};-\boldsymbol{v})}{\Delta(\boldsymbol{u})\Delta(-\boldsymbol{v})}\frac{1}{\Delta(\boldsymbol{\xi})}\det\begin{pmatrix}E_p(\boldsymbol{u};\boldsymbol{v}) & A_{\boldsymbol{\ell}}(\boldsymbol{u})\\ E_p(\boldsymbol{\xi};\boldsymbol{v}) & A_{\boldsymbol{\ell}}(\boldsymbol{\xi})\end{pmatrix}.$$

Using the block matrix determinant formula

$$\det\begin{pmatrix}M_{11} & M_{12}\\ M_{21} & M_{22}\end{pmatrix} = \det\left(M_{22}\right)\det\left(M_{11}-M_{12}M_{22}^{-1}M_{21}\right)$$

where M_{11},M_{22} are square matrices and M_{22} is invertible, we obtain

$$\mathcal{B}_{\boldsymbol{\ell},p}(\boldsymbol{u},\boldsymbol{\xi}/\boldsymbol{v}) = \frac{D(\boldsymbol{v};-\boldsymbol{\xi})}{D(\boldsymbol{u};-\boldsymbol{\xi})}\frac{D(\boldsymbol{u};-\boldsymbol{v})}{\Delta(\boldsymbol{u})\Delta(-\boldsymbol{v})}\mathcal{B}_{\boldsymbol{\ell}}(\boldsymbol{\xi})\det\left(E_p(\boldsymbol{u};\boldsymbol{v})-A_{\boldsymbol{\ell}}(\boldsymbol{u})A_{\boldsymbol{\ell}}(\boldsymbol{\xi})^{-1}E_p(\boldsymbol{\xi};\boldsymbol{v})\right)$$

where we used the alternant formula for multivariate Bessel functions (4.2.3). Then

$$\frac{\mathcal{B}_{\boldsymbol{\ell},p}(\boldsymbol{u},\boldsymbol{\xi}/\boldsymbol{v})}{\mathcal{B}_{\boldsymbol{\ell}}(\boldsymbol{\xi})} = \frac{D(\boldsymbol{u};-\boldsymbol{v})}{\Delta(\boldsymbol{u})\Delta(-\boldsymbol{v})}\det\left(\prod_{l=1}^{N}\frac{v_j-\xi_l}{u_i-\xi_l}\left(\frac{e^{p(u_i-v_j)}}{u_i-v_j}+\mathrm{a}(u_i)'A_{\boldsymbol{\ell}}(\boldsymbol{\xi})^{-1}\mathrm{b}(v_j)\right)\right)_{i,j=1}^{k}. \tag{4.4.13}$$

When $k=1$, this gives us (4.4.12). For general k, observe that the (i,j) entry of the matrix in (4.4.13) is

$$\frac{1}{u_i-v_j}\frac{\mathcal{B}_{\boldsymbol{\ell},p}(u_i,\boldsymbol{\xi}/v_j)}{\mathcal{B}_{\boldsymbol{\ell}}(\boldsymbol{\xi})}$$

by the $k=1$ case (4.4.12). This proves (4.4.11). □

Proof of Theorem 4.4.2. We have (4.4.8) as a direct consequence of (4.4.11) with $\Xi=((N-$

$1)\alpha, (N-2)\alpha, \ldots, 0)$. We now prove (4.4.9). Using (4.4.12), we have

$$\begin{aligned}
&\widehat{\mathcal{B}}_{\boldsymbol{\ell},p}(u/v; N, \alpha) \\
&\quad = (u-v)\prod_{i=1}^{N}\frac{v-(i-1)\alpha}{u-(i-1)\alpha}\left(\frac{e^{p(u-v)}}{u-v} + \sum_{a,b=1}^{N} e^{\ell_a u}\left[A_{\boldsymbol{\ell}}((N-1)\alpha,\ldots,0)^{-1}\right]_{a,b}\frac{e^{p(N-b)\alpha-pv}}{v-(N-b)\alpha}\right)
\end{aligned} \tag{4.4.14}$$

and note that

$$A_{\boldsymbol{\ell}}((N-1)\alpha, (N-2)\alpha, \ldots, 0) = A_{\delta_N}(\ell_1\alpha, \ell_2\alpha, \ldots, \ell_N\alpha)'.$$

This is a Vandermonde matrix in $e^{\ell_1\alpha}, \ldots, e^{\ell_N\alpha}$ whose inverse is given by

$$\left[A_{\boldsymbol{\ell}}((N-1)\alpha,\ldots,0)^{-1}\right]_{a,b} = (-1)^{b-1}e_{b-1}(e^{\ell_1\alpha},\ldots,\widehat{e^{\ell_a\alpha}},\ldots,e^{\ell_N\alpha})\prod_{i\neq a}\frac{1}{e^{\ell_a\alpha}-e^{\ell_i\alpha}}$$

where the e_b's denote the elementary symmetric polynomials. The summation over a, b in (4.4.14) becomes

$$\sum_{a=1}^{N} e^{\ell_a u}\prod_{i\neq a}\frac{1}{e^{\ell_a\alpha}-e^{\ell_i\alpha}}\sum_{b=1}^{N}(-1)^{b-1}e_{b-1}(e^{\ell_1\alpha},\ldots,\widehat{e^{\ell_a\alpha}},\ldots,e^{\ell_N\alpha})\int_p^\infty \frac{e^{w(N-b)\alpha}}{e^{wv}}\,dw$$

where the line integral is a ray from p to ∞ such that

$$\Re\Big(w\big((N-b)\alpha - v\big)\Big) < 0$$

for every $b = 1, \ldots, N$ and $|w|$ large. By the generating series for the elementary symmetric polynomials, this becomes

$$\sum_{a=1}^{N}\int_p^\infty \frac{e^{\ell_a u}}{e^{wv}}\prod_{i\neq a}\frac{e^{w\alpha}-e^{\ell_i\alpha}}{e^{\ell_a\alpha}-e^{\ell_i\alpha}}\,dw = \int_p^\infty dw\oint\frac{dz}{2\pi\mathbf{i}}\cdot\frac{\alpha}{e^{(w-z)\alpha}-1}\frac{e^{zu}}{e^{wv}}\prod_{i=1}^{N}\frac{e^{w\alpha}-e^{\ell_i\alpha}}{e^{z\alpha}-e^{\ell_i\alpha}}$$

where the z-contour is positively oriented around $\ell_1, \ldots, \ell_N$ and does not intersect the w-contour. Inputting the double contour integral formula into (4.4.14) yields (4.4.9). $\square$

4.4.2 Alternative Choices of Lifts

Given some function $g : \Omega \subset \mathbb{C} \to \mathbb{C}$, let

$$g(\boldsymbol{x}, \boldsymbol{y}) = (g(x_i, y_j))_{\substack{1\le i\le N+k \\ 1\le j\le k}}.$$

The same idea as in the proof of Theorem 4.4.1 shows that

$$(-1)^{Nk}\frac{D(\boldsymbol{x};-\boldsymbol{y})}{\Delta(\boldsymbol{x})\Delta(-\boldsymbol{y})}\det\big(g(\boldsymbol{x},\boldsymbol{y}) \quad A_{\lambda+\delta_N}(\boldsymbol{x})\big) \tag{4.4.15}$$

defines a supersymmetric lift of s_λ as long as

$$\lim_{x\to y}(x-y)g(x,y)=1.$$

For $g(x,y)=1/(x-y)$, (4.4.15) is a determinantal formula for the so-called *supersymmetric Schur function indexed by* λ due to Moens and Van der Jeugt [47]. The supersymmetric Schur functions are the $\Lambda_{m,n}$ image of the Schur functions $s_\lambda\in\Lambda$ (see Example 4.3.4). In Moens and Van der Jeugt's more general setting, they do not require the condition $m-n=N$ for $\lambda\in\mathbb{GT}_N$. Their notation differs from ours by $y_i\mapsto -y_i$. Analogues of Theorem 4.4.2, Theorem 4.4.3, and Theorem 4.4.4 exist for the supersymmetric Schur function, though the integral becomes more complicated.

Another choice of lift can be obtained by considering a certain analytic extension of

$$\frac{s_\lambda(1,q,\ldots,q^{b-1},q^a,q^{b+1},\ldots,q^{N-1})}{s_\lambda(1,q,\ldots,q^{N-1})}$$

as a function in q^a,q^b for which double contour integrals exist, see [31, Equation 3.10], [21, Theorem 3.4]. Our choice of lift (4.4.4) appears the most suitable for our asymptotics among these presented choices of lifts. However, the ideal choice of lift may depend on the application.

4.5 Main Results and Further Discussions

Combining Theorems 4.3.5 and 4.4.1, we get our first main result:

Theorem 4.5.1. *Let N,m,k,h be integers such that $1\le h\le N$. For any $c_1,\ldots,c_k>0$ and $\boldsymbol{r}^1,\ldots,\boldsymbol{r}^m\in\mathcal{R}_N$, let $\boldsymbol{s}=\pi_{N\to h}(\boldsymbol{r}^1\boxplus\cdots\boxplus\boldsymbol{r}^m)$. Then for any $p_1,\ldots,p_m\in\mathbb{C}$, we have*

$$\begin{aligned}\mathbb{E}\left[\prod_{i=1}^k\sum_{j=1}^h e^{c_is_j}\right]&=\frac{1}{(2\pi\mathbf{i})^k}\oint\cdots\oint\prod_{i=1}^m\widehat{\mathcal{B}}_{\boldsymbol{r}^i,\mathfrak{p}_i}(0^N,u_1+c_1,\ldots,u_k+c_k/u_1,\ldots,u_k)\\&\times\prod_{1\le i<j\le k}\frac{(u_j+c_j-u_i-c_i)(u_j-u_i)}{(u_j-u_i-c_i)(u_j+c_j-u_i)}\prod_{i=1}^k\left(\prod_{j=1}^N\frac{u_i+c_i}{u_i}\right)\frac{du_i}{c_i}\end{aligned} \tag{4.5.1}$$

where the u_i-contour is positively oriented around the pole at 0, and the u_j-contour contains u_i+c_i whenever $i<j$.

The utility of this formula is the multiple contour integral expression for the supersymmetric lift $\widehat{\mathcal{B}}_{\boldsymbol{\ell},p}$ of the normalized multivariate Bessel function $\widehat{\mathcal{B}}_{\boldsymbol{\ell}}$ given by Theorem 4.4.3.

Similarly, we can combine Theorems 3.8.2 and 4.4.1 to obtain our second main result:

Theorem 4.5.2. *Let N, m, k, h be integers such that $1 \leq h \leq N$. Let any $c_1, \dots, c_k > 0$, $\lambda^1, \dots, \lambda^m \in \mathbb{GT}_N$, and*

$$\rho = \rho^{\lambda^1} \otimes \cdots \otimes \rho^{\lambda^m}\Big|_{\mathcal{U}(h)}.$$

Then for any $p_1, \dots, p_m \in \mathbb{C}$, we have

$$\begin{aligned}\mathbb{E}\left[\prod_{i=1}^{k}\sum_{j=1}^{h} e^{c_i(\lambda_j+N-j)}\right] = \frac{1}{(2\pi\mathbf{i})^k}\oint\cdots\oint\prod_{i=1}^{m}\widehat{s}_{\lambda^i,p_i}(1^N, e^{u_1+c_1}, \dots, e^{u_k+c_k}/e^{u_1}, \dots, e^{u_k})& \\ \times\prod_{1\leq i<j\leq k}\frac{(e^{u_j+c_j}-e^{u_i+c_i})(e^{u_j}-e^{u_i})}{(e^{u_j}-e^{u_i+c_i})(e^{u_j+c_j}-e^{u_i})}\prod_{i=1}^{k}\left(\prod_{j=1}^{N}\frac{e^{u_i+c_i}-1}{e^{u_i}-1}\right)\frac{du_i}{e^{c_i}-1}&\end{aligned} \tag{4.5.2}$$

where the u_i-contour is positively oriented around the pole at 0, and the u_j-contour contains $u_i + c_i$ whenever $i < j$.

As with the random matrix case, we have an expression for $\widehat{s}_{\lambda,p}$ through our formula for $\mathcal{B}_{\lambda+\delta_N,p}$ and the relation (4.2.5).

The contour integral representation for $\mathcal{B}_{\boldsymbol{\ell},p}$ is in a form that is amenable to steepest descent analysis. This suggests that the asymptotics of the moments of our models may be analyzable via (4.5.1) and (4.5.2). While global limits of the additive random matrix model have been studied under general hypotheses (see [31]), local limits are less understood. In particular, few results are known about the local fluctuations of sums of random matrices at the edge except for a few cases where the distributions of one of the summands is special, such as the semicircle distribution. For representations of the unitary group, almost no edge results exist for tensor products of representations. By considering large moments of our models, we may access the local fluctuations for the largest particles. We investigate this further in a future article. We note that similar moment formulas are also obtainable for products of unitarily invariant random matrices and the analogous questions are pursuable in this setting.

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