On Durable and Nondurable Consumption with Transactions Costs

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Abstract

The first chapter considers the optimal purchase of durable and nondurable goods when adjusting one's durable stock is costly and when durables and nondurables interact in utility. The solution to the optimization problem is characterized by a two-band control of the ratio of the consumer's durable stock to nondurable consumption. The complexity of the solution of the model is reduced from solving a differential equation to the solution of six simultaneous non-linear equations, utilizing techniques found elsewhere in the optimal consumption/investment literature. The paper also analyzes the effects of transactions costs of durable goods on nondurable consumption. When the durable stock is updated, nondurable consumption also jumps. These results are discussed in the context of previous work on Euler equation and asset pricing models.

The second chapter uses a simplified version of the model of the optimal purchase of durable and nondurable goods under transactions costs presented

in the first chapter. In this model durables and nondurables enter additively separable. The chapter estimates the important parameters of the model using a data set of individual families' holdings and purchases. The model is then aggregated and applied to macro data. The dynamics of the steady state distribution is developed in order to predict the movements in the ratio of aggregate durables to aggregate nondurables. The empirical model explains a great deal of movements in the aggregate ratio of durables to nondurables over the post-war period, and the two empirical procedures yield consistent results.

The third chapter investigates an important non-price contract terms in consumer loan markets: the length of the loan contract. It is supposed that these non-price terms matter to consumers because they are unable to borrow to fund nondurable consumption. A simple model is presented where consumers borrow to fund a durable purchase and choose to repay the loan in the next period or over two periods. Attitudes towards default influence the choice of loan contract length, and the equilibrium term structure of loan rates reflects the partial signaling that occurs in the market. Microeconomic and macroeconomic evidence is presented that suggests that the observed risk premium and the contract length are positively correlated.

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Introduction

This thesis investigates the theoretical and empirical dynamics of durable and nondurable consumption in the U.S. economy. The thesis considers durable expenditures because the growth rate of durable expenditures is much more volatile than of nondurables over the post-war period. Durable purchases are an important component of the business cycle. The thesis considers nondurables because important interactions may exist between durables and nondurables (Chapters 1 and 3), or nondurables can serve as a useful tool to normalize durables across individuals (Chapter 2).

Though the goal of the thesis is to understand aggregate fluctuations, the theoretical sections explicitly model individual optimization, and it uses the individual rules to derive aggregate implications. The empirical sections study both microeconomic and macroeconomic data. In Chapter 2 the microeconomic results are used in the macroeconomic estimation, and the consistency of the results between the two data sets is tested. The purpose is to establish firm microeconomic foundations for the study of aggregate durable fluctuations.

Two important elements of the consumer's decision problem that are important for understanding aggregate fluctuations are emphasized in the thesis. Early research on the individual's problem used convex costs of adjustment to justify observed dynamics, especially when aggregated, but later developments found such short-cuts wanting. After all, individual consumers adjust their stocks in lumps, waiting quite a bit of time before making relatively large purchases. The discrete nature of individual decision rules has important implications for aggregate dynamics, and this is an explicit element of the models used in this thesis. In chapters 1 and 2 consumers do not adjust their durable stocks continuously because of transactions costs. Consumers who sell their durables receive only a fraction of their value. Chapter 3 takes up an additional element of the consumer's problem that is important for very large durable purchases such as automobiles, the consumer loan market. Because durable goods can serve as collateral for loans, individuals can more easily buy durables on loan than nondurables. If in fact they are prohibited from borrowing against future income to fund nondurable purchases, consumers will use their loan contracts on automobiles as substitutes. These deviations from more classical theory are not only important in practice but have important implications for aggregate dynamics.

The first chapter considers the optimal purchase of durable and nondurable goods when adjusting one's durable stock is costly and when durables and nondurables interact in utility. The solution to the optimization problem is characterized by a two-band control of the ratio of the consumer's durable stock to nondurable consumption. When this ratio hits a fixed band, it is returned to a certain point in between. Usually through depreciation the durable stock becomes too low relative to nondurable consumption, and the consumer buys a new durable, instantaneously raising the ratio. Through bad luck, however, the consumer may find that the durable stock is too high. In this case the consumer sells the durable and buys a less expensive one, where the proceeds are devoted to future nondurable consumption. The complexity of the solution of the model is reduced from solving a differential equation to the solution of six simultaneous non-linear equations, utilizing techniques found elsewhere in the optimal consumption/investment literature.

The paper also analyzes the effects of transactions costs of durable goods on nondurable consumption. When the durable stock is updated, nondurable consumption also jumps. These results are discussed in the context of previous work on Euler equation and asset pricing models. In particular the solution technique can be interpreted given the martingale theory in finance. The consumer chooses controls such that the marginal value of wealth grows in expectation equal to the difference between the discount rate and the risk-free rate. The marginal value of wealth is continuous, and hence, the marginal utility of nondurable consumption is continuous. Since the durable stock is an element of marginal utility of nondurable consumption and it jumps at the time of adjustment, nondurable consumption must also jump to preserve the continuity.

The second chapter uses a simplified version of the model of the optimal purchase of durable and nondurable goods under transactions costs presented in the first chapter; in this model durables and nondurables enter additively separable. Even though the dynamics of durables and nondurables are unaffected by one another, the inclusion of nondurables in the analysis of durables is important for two reasons. First, nondurables can be used to normalize the stock of durables instead of total wealth, which is used by others. Nondurable consumption is more easily measured in some circumstances than permanent wealth, and it is more readily available in some data sets. Second, the inclusion of nondurables allows one to consider the effects of changes in relative prices.

The chapter uses both microeconomic and macroeconomic data sets. It first estimates the important parameters of the model using a data set of individual families' holdings and purchases. The steady state distribution of consumers' ratios is used as a maximum likelihood function. The maximum likelihood estimates are consistent with one's intuitive expectations and are consistent with previous research. The model is then aggregated and applied to macro data. The dynamics of the steady state distribution is developed in order to predict the movements in the ratio of aggregate durables to aggregate nondurables. The empirical model explains a great deal of movements in the aggregate ratio of durables to nondurables over the post-war period, and the two empirical procedures yield consistent results.

The third chapter investigates an important non-price contract terms in consumer loan markets: the length of the loan contract. It is supposed that the contract length matters to consumers because they are unable to borrow to fund nondurable consumption. A simple model is presented where consumers borrow to fund a durable purchase and choose to repay the loan in the next period or over two periods. Attitudes towards default influence the choice of loan contract length, and the equilibrium term structure of loan rates reflects the partial signaling that occurs in the market. Microeconomic and macroeconomic evidence is presented that suggests that the observed risk premium and the contract length are positively correlated. Data from the Consumer Expenditure Survey of 1985 suggests that consumers differ in their financing of newly purchased automobiles, and these differences support the model above. Aggregate data since 1972 also indicates that the contract length helps explain movements in the risk premium. Both empirical exercises consider the simultaneous determination of these variables.

Chapter 1:

OPTIMAL DURABLE AND NONDURABLE CONSUMPTION WITH TRANSACTIONS COSTS

1. Introduction

Recent models of durable goods have stressed the lumpy nature of expenditures. Informal observations suggest that individuals update their durable stocks infrequently, and when they do make purchases, their purchases are large. This behavior is opposite from that predicted by convex cost of adjustment models where an individual smoothly through time adjusts the durable stock to its optimal level. As Bertola and Caballero (1990) stress, aggregate dynamics can also differ significantly from models with smoother adjustment costs. Since durable expenditures are such an important element of aggregate fluctuations, it is important to understand the true nature of durable consumption.

Consumers, however, do not derive utility only from the services of their durable stock. They also consume nondurable goods which they can keep at an optimal level. If the optimal amount of nondurable consumption is independent of the level of the durable stock, then the addition of complicated durable dynamics will not affect results of models which only include nondurable consumption. For instance, the Permanent-Income Life-Cycle model of consumption still holds for nondurables separately. Euler equation tests on nondurable consumption such as in Hall (1978) are still valid, and the Consumption CAPM (Breeden, 1979) also obtains. On the other hand, if durables interact with nondurables, complicated durable dynamics will feed into the dynamics of nondurables. Tests which do not allow for these effects may be severely misspecified.

The present model allows for these effects through a general CES utility function over nondurable and durable consumption. The model generalizes Grossman-Laroque (1990) in that the consumer can always purchase nondurable goods. It retains the assumption of transactions costs on the adjustment of the durable good, and thus, a two band policy for the durable stock is still optimal. Since the model allows for interactions between durable and nondurable consumption it is specially suited to study not only the effects of transactions costs on durable goods, but also its effects on nondurables.

The model generates several interesting results which generalize the findings of previous works. Marginal utility of nondurable consumption is a continuous random variable whose instantaneous drift is just the difference between the individual's discount rate and the risk-free interest rate. In this environment Hall's martingale hypothesis, correctly interpreted, still obtains. With durables and nondurables nonseparable in utility, however, nondurables suffer from the same aggregation problems as durables. Even if the correct functional form of individual marginal utility is specified and the correct aggregate level of the durable stock is available, aggregate Euler tests similar to Hall are misspecified. In addition C-CAPM holds at the individual level but not at the aggregate level. That is, the rate of return on any asset is such that a consumer is indifferent between holding the asset and consuming one unit of nondurables. But because aggregation of nondurables is a problem, no simple representative agent exists. Tests of C-CAPM which relate aggregate consumption of nondurables to individual security returns are invalid.

The rest of the paper proceeds as follows. The next section discusses previous models and tests of durable and nondurable consumption relevant to the issues addressed here. In the third section, the model in Grossman and Laroque (1990) is adapted to the present issue. As before, by a homogeneity property the number of state variables can be reduced from two, total wealth and the stock of durables, to one, the ratio of wealth to the durable minus the transactions cost. Techniques developed by Karatzas *et al.* (1986) are applied to reduce the computational burden of the solution. Instead of numerically solving a differential equation with complicated boundary conditions, one need only solve a system of non-linear equations. The fourth section discusses both more general results pertaining to the issues discussed above and numerical solutions to the problem for interesting combinations of parameters. The continuity of marginal utility is proved and the relationship to Hall and C-CAPM is discussed. Aggregation issues play a prominent role here. The final section concludes with directions for further research.

2. Previous Work

2.1 Durable Consumption in Isolation

The benchmark by which all recent models and tests of durable consumption is compared is Mankiw's (1982) extension of Hall (1978) to durable consumption. Assuming no costs of adjusting the individual's durable stock and ignoring the impact of nondurable consumption with separability, Mankiw shows that changes in the expenditures on real durables should follow an MA(1) process

$$\Delta E_t = \epsilon_t - (1 - \alpha)\epsilon_{t-1}, \tag{1}$$

where α is the depreciation rate of the durable stock. Note that if the stock fully depreciates in one period, as is true with nondurables, then one obtains Hall's result. Of course, (1) is derived under the assumption of quadratic utility. Estimates of this MA coefficient using aggregate quarterly data, however, imply a depreciation rate nearly one.

A second paper which considers the Mankiw-PIH as applied to durable goods is Bernanke (1984). Unlike Mankiw, Bernanke allows for sluggish adjustment with a convex cost of adjustment term. He uses the panel data set of Hendricks, Youmans and Keller (1973), which consists of interviews of 1434 families for each year from 1967 to 1970, to test the model at a micro level. Counter to the macro literature, he finds the data fit the model fairly well. Transitory income, apart from its annuity value, has no effect on the adjustment of the durable stock to its desired level as the estimated parameter is small, insignificant, and of the opposite sign from a liquidity constraint model. The annual rate of stock adjustment is estimated to be 70%, larger than the 55% figure Caballero (1990b) finds in aggregate annual data. At a micro level the PIH fits better than at the aggregate level, a finding similar to that found in tests of the PIH using nondurable goods (Hall and Mishkin, 1982).

In fact, Caballero (1990b) stresses the difference in results between micro and macro studies and between studies using quarterly data and those using annual data. Estimating a non-parsimonious MA process for quarterly durable expenditures uncovers important components at further lags. Caballero interprets these additional moving average components as suggesting that consumers slowly adjust their stocks. Using annual data, he shows that in three years the Mankiw model seems to hold. The sum of the three MA coefficients is -.714.

The inconsistency of results and the implausible microeconomic behavior implied by these models have led researchers to consider models of lumpy purchases. After all, agents do not continuously add to their stocks; they do not slowly add a headlight, then a new door, a different grill so that their old 1985 Plymouth Horizon is now the equivalent of a 1991 Ford Thunderbird. Noncombinability (Lancaster, 1979) is a basic feature of durable goods, and any model of durable fluctuations should be rooted in a more sensible microeconomic model. Coupled with the observation that people change their stocks only infrequently and in large doses has led researchers to rationalize this behavior with S-s models of durable. In such models consumers keep their durable stocks normalized by total wealth between two bands. When the normalized stock through depreciation or changes in wealth hit one of these bands, the stock is moved to a point in between. The consumer does not continuously re-optimize as in Mankiw and Bernanke because there exist kinked costs of adjustment that make periods of inaction optimal (Bertola and Caballero, 1990).

One of the first papers to apply the techniques of continuous-time, stochastic optimal control to the problem is Grossman and Laroque (1987, 1990). [1]

[1] The 1987 predecessor is important in its own right since it contains the techniques by

The consumer receives utility from a durable good and income from the portfolio of risky and riskless assets. A basic result of their paper is that while each consumer still holds a mean-variance efficient portfolio and thus CAPM obtains, C-CAPM does not. There is no simple relationship between durable expenditures in their model and the returns to individual assets.[2] The full implications of this model for the aggregate behavior of durable expenditures, however, are unknown. Aggregation becomes important because the number of consumers updating their stocks in an interval of time and the total amount they purchase will depend on the distribution of agents between the bands. This distribution is not constant if shocks hit all individuals similarly.

The problem of aggregation encourages economists to move in two directions. One research agenda investigates microeconomic data instead of aggregate macro data. Similar to what Hall and Mishkin (1982) do with nondurable data, the strategy is to develop Euler equation tests which should hold in the presence of S-s behavior. Eberly (1990) takes this approach with the Survey of Consumer Finances; her results are supportive of the S-s model. The placement of the bands between which the durable is held and the point to which the stock is adjusted are in line with the theoretical predictions of Grossman and Laroque (1990). In addition, the width of the bands and the time between which the

---

which Grossman and Laroque overcome a technical problem briefly discussed below.

[2] The application of C-CAPM to their environment may seem strange since traditional tests of C-CAPM use nondurable consumption data. Grossman and Laroque, however, argue that their model is applicable to much nondurable consumption because components of the nondurable consumption series such as apparel are in fact somewhat durable. The durability of nondurables and its application to PIH tests is explored in Hayashi (1985b) and Heaton (1990). Eichenbaum and Hansen (1990) also find important durability in nondurable data. This research strategy is not considered here.

stock is adjusted are affected only by variables that are theoretically important. She does find, however, that 58% of her sample is constrained in their ability to follow the optimal program, a number far higher than the 20% figure found by Hall and Mishkin (1982) and Zeldes (1989), and the 15% who report being credit constrained in her sample.[3]

The second approach is to confront the issue of aggregation directly.[4] Bar-Ilan and Blinder (1988a, 1988b) show that a sensible approximation to the model in the context of automobiles suggests that the dynamics of the number of cars bought in a quarter and the average value of that which is bought should differ substantially. The latter matches the traditional predictions of the PIH while the former does not. The number of cars bought should be substantially more variable than the average expenditure on cars. They find that the realized differences in U.S. data are generally consistent with the theory.

Several papers consider whether aggregating S-s models can mimic actual expenditure series on U.S. durables. Eberly (1989) discusses approximating the cross-sectional distribution of agents in the two band environment with a one-sided model. Bertola and Caballero (1990) develop a discrete-time approximation and apply it to aggregate durable expenditures. Their model explains a substantial portion of the variation in aggregate durable expenditures left unexplained by the frictionless model, and it produces the serial correlation

[3] Mishkin (1976) investigates the effects of liquidity constraints on aggregate durable purchases.

[4] See Caballero and Engel 1989, 1990 for a fuller discussion of the issue of aggregation in continuous time with idiosyncratic and aggregate uncertainty in general optimal control models.

properties found in the durable data. Eberly (1990) considers the theoretical cross-sectional distribution of a two-sided model with no aggregate shocks. The theoretical distribution matches the distribution of automobiles in her sample using a goodness-of-fit statistic.

Caballero (1990a) develops the process for the aggregate cross-sectional distribution of agents produced from a simplified version of Grossman and Laroque (1990). He applies his model to data on the expenditures of new cars and furniture and furnishings. Given the different levels of development of secondary markets for these two commodity groups, the aggregate dynamics of these two series can be predicted theoretically and tested empirically. Caballero concludes that the evidence is in line with the idea that lumpiness at the microeconomic level can explain the slow response of durable purchases in reacting to aggregate shocks. An aggregate shock moves everyone closer to the adjustment band. Those individuals who do not hit the band immediately, hit the band sooner in the future than they would without an aggregate shock. It is this feature of the model that produces the lagged effects of an aggregate shock at the macro level. In other words, S-s models can explain the slow adjustment in aggregate data found in tests of Mankiw's original model.

2.2 *Interaction between Durables and Nondurables*

The interaction between durables and nondurables in the context of convex cost of adjustment models have been explicitly explored in two papers. [5] Bernanke (1985) develops a model where the consumer faces convex costs of adjustment to the durable stock and where utility from nondurables and durables interact. Thus, the slowness of durable adjustment spills over to the optimal purchase of nondurables. He finds no role, however, for these spillover effects in aggregate data.

A second exploration of the interaction between durables and nondurables is in Eichenbaum and Hansen (1990). They model aggregate consumption as the outcome of maximizing behavior of a representative agent with a utility function that allows them to consider interaction effects. Different auxiliary assumptions on the nature of utility, on the existence of adjustment costs of the durable stock and on the nature of growth (trend or difference stationary) produce different answers. Generally, they find evidence that durables and nondurables are not perfect substitutes. Moreover, for the purposes considered here, they find some evidence that the durable and nondurable preferences are not separable. They also find, counter to the treatment of the nature of nondurables modeled here, that there exists substantial evidence that nondurables are somewhat durable.

These results of tests of the interaction of durables and nondurables with

[5] Caballero (1990b) jointly estimates the processes for durable and nondurable expenditures. The effect of durables on nondurables, however, is only through the budget constraint as he assumes additive separability in the utility function. Dunn and Singleton (1986) model the term structure of interest rates with utility interacting between durables and nondurables. While their results are supportive of such interaction effects, their over-identifying restrictions are strongly rejected.

smooth or no costs of adjustment must be interpreted with caution. The plausibility of S-s microeconomic behavior and the aggregation problems associated with these models suggests these previous tests may be biased. Furthermore, previous aggregate tests of the PIH ignoring durables have failed (Hayashi, 1985a). Thus, the joint behavior of durables and nondurables in an S-s environment is both an open question and one that is of *a priori* importance.

3. Modeling Durable and Nondurable Consumption

3.1 Set-up

Suppose an infinitely lived consumer has a utility function of the form

$$U(K_t, C_t) = \frac{1}{\nu}\left(K_t^{-\rho} + bC_t^{-\rho}\right)^{-\frac{\nu}{\rho}},$$

where K_t is a durable good which depreciates at rate α and C_t is nondurable consumption. Of course, $b > 0$. I also assume $\rho > -1$ and $\nu < 1$. Furthermore, for reasons to be discussed later, if $\rho < 0$ then I assume $\nu + \rho < 0$.[6] Since prices are absent in this model, an implicit assumption is that the relative price of the nondurable in terms of the durable is constant, as is the real interest rate. The parameter b accounts for the relative price in addition to utility.

When the consumer buys a new durable, she must first sell the old durable. There is a transactions cost equal to λK_{t-}, and thus, the consumer can only obtain $(1-\lambda)K_t$ units of a new durable from the sale of the old. The assumption

[6] The case of log utility (set $\rho = -\nu$, subtract $(1+b)/\nu$ and take $\lim_{\nu\to 0}$) can be solved in a similar manner. The solution to the model with utility separable and especially log separable is given below. If $\rho \to \infty$ then the utility function converges to the Leontief utility function.

of a transactions cost is non-trivial and gives the problem the flavor of a durable goods problem. If no transactions costs exist, the consumer continuously buys and sells the durable good in order to keep durable goods services at an optimal level. Such a response is similar to a nondurable goods problem.[7]

If the durable good is the only asset held, the consumer obviously never buys and sells new durables. The consumer also holds a portfolio of two assets, B_t units of a riskless bond with an instantaneous return r_f and X_t units of a risky assets whose value fluctuates randomly. If $\hat{L}_t$ is the value of one unit of the risky asset, then $d\hat{L}_t = \hat{L}_t(\hat{\mu}dt + \sigma dw(t))$. [8] Define total wealth as $Q_t = K_t + B_t + X_t$, and let $\mu \equiv \hat{\mu} - r_f$ be the excess return the risky asset earns on average. One can eliminate B_t from the Q_t process to obtain

$$dQ_t = -\alpha K_t dt + r_f(Q_t - K_t)dt + X_t(\mu dt + \sigma dw(t)) - C_t dt.$$

Note that marketable wealth is equal to $(1-\lambda)K_t + B_t + X_t = Q_t - \lambda K_t$. There is a bankruptcy constraint in that marketable wealth must always be non-negative.

The consumer's problem is to maximize $E\int_0^\infty \frac{1}{\nu}e^{-\delta t}\left(K_t^{-\rho} + bC_t^{-\rho}\right)^{-\frac{\nu}{\rho}}dt$. Given that the durable good is not continuously controlled, the problem is equivalent to maximizing expected utility over X_t, C_t, optional stopping times, $\{\tau_1, \tau_2 \ldots\}$, and new durables bought at those times $\{K_{\tau_1}, K_{\tau_2} \ldots\}$. Using dy-

[7] See the proof of Theorem 2.1 in the Appendix for the solution to the problem when $\lambda = 0$.

[8] Grossman and Laroque allow for a portfolio of n risky assets, and then, they show that a mutual fund theorem holds. The consumer acts as if she first figures how much of the mutual fund she will hold and then allocates the fund in a mean-variance efficient manner among the n risky assets. I side-step this added complexity and just assume there is only one risky asset. One can literally repeat their analysis verbatim to prove two fund separation and the CAPM.

namic programming, the problem is formally:

$$
\begin{aligned}
V(Q_0, K_0) = \sup_{\substack{k,\tau,\\ \{X(t),C(t)\}}} & E\Big[\int_0^\tau e^{-\delta t}\frac{1}{\nu}(K_t^{-\rho} + bC_t^{-\rho})^{-\frac{\nu}{\rho}}dt \\
& + e^{-\delta\tau}V(Q_{\tau_-} - \lambda K_{\tau_-}, k)\Big] \\
\text{s.t.} \quad dQ_t &= -\alpha K_t dt + r_f(Q_t - K_t)dt + X_t(\mu dt + \sigma dw(t)) \\
& \quad - C_t dt \qquad t \in [0, t) \\
dK_t &= -\alpha K_t dt \qquad t \in [0, t) \\
Q_t - \lambda K_t &\geq 0.
\end{aligned}
\tag{2}
$$

It is easily proved, as Grossman and Laroque (1990) and Karatzas *et al.* (1986) do for their problems, that the value function is bounded. In the Appendix the following theorem is proved.

Theorem 2.1

Assume

$$\beta \equiv \delta - \nu r_f - \frac{\mu^2}{2\sigma^2}\frac{\nu}{1-\nu} > 0.$$

There exist two constants, η_1 and η_2, that provide the following bounds on the value function:

$$\eta_1 Q^\nu \geq V(Q, K) \geq \eta_2(Q - \lambda K)^\nu.$$

Remark: This is the same condition Grossman-Laroque and Karatzas *et al.* require for their problems.

The upper bound is found by calculating the return to the problem with no transactions costs; the consumer cannot do better. The lower bound is found by calculating the return to the problem following an arbitrary strategy under conditions that make the consumer worse off.

Solving an optimal control problem with two state variables can be a daunting task because the resulting Hamilton-Jacobi-Bellman equation is a partial differential equation. Grossman and Laroque (1990) prove their problem is homogeneous in the durable good, and thus, the problem can be reduced to a single state variable. The following construction shows this problem also can be rewritten in terms of a single state variable.

3.2 A Single State Variable

From equation (2) when $t \neq \tau$ the value function evolves according to a Hamilton-Jacobi-Bellman equation. Suppressing the maximization operator for convenience

$$\delta V(Q_t, K_t)dt = \frac{1}{\nu}(K_t^{-\rho} + bC_t^{-\rho})^{-\frac{\nu}{\rho}}dt + V_Q EdQ + \tfrac{1}{2}V_{QQ}dQ^2 + V_K dK.$$

Plugging in the values for dQ and dK and dividing by dt yields

$$\delta V(Q_t, K_t) = \frac{1}{\nu}(K_t^{-\rho} + bC_t^{-\rho})^{-\frac{\nu}{\rho}} + V_Q(-\alpha K_t + r_f(Q_t - K_t) + \mu X - C_t)$$
$$+ \tfrac{1}{2}\sigma^2 V_{QQ}X^2 - \alpha K_t V_K.$$

The first-order condition for X_t gives

$$X_t = -\frac{\mu}{\sigma^2}\frac{V_Q}{V_{QQ}}.$$

Let $c = \frac{C}{K}$ and $u(c_t) = \frac{1}{\nu}(1 + bc_t^{-\rho})^{-\frac{\nu}{\rho}}$. Plugging in and rearranging yields

$$\delta V(Q_t, K_t) = K_t^{\nu}\frac{1}{\nu}(1+bc_t^{-\rho})^{-\frac{\nu}{\rho}} + K_t V_Q(-\alpha + r_f(\frac{Q_t}{K_t} - 1) - c_t) - \gamma\frac{V_Q^2}{V_{QQ}} - \alpha K_t V_K, \tag{3}$$

where $\gamma \equiv \frac{1}{2}\frac{\mu^2}{\sigma^2}$.

Now, let

$$h(y_t) \equiv K_t^{-\nu} V(Q_t, K_t); \qquad y_t \equiv \frac{Q_t}{K_t} - \lambda. \tag{4}$$

Differentiating (4) by Q_t and K_t gives expressions for V_Q, V_{QQ} and V_K in terms of $h(\cdot)$. In terms of y, (3) becomes

$$\bar{\delta} h(y) = h'(y)\big(r(y + \lambda - 1) - c\big) - \gamma \frac{h'(y)^2}{h''(y)} + \frac{1}{\nu}(1 + bc_t^{-\rho})^{-\frac{\nu}{\rho}}, \tag{5}$$

where $\bar{\delta} = \delta + \nu\alpha$ and $r = r_f + \alpha$. Also, by the Itô calculus, since $y_t = \frac{Q_t}{K_t} - \lambda$,

$$dy_t = r(y_t + \lambda - 1)dt - 2\gamma \frac{h'(y_t)}{h''(y_t)} dt - c\, dt - \frac{\mu}{\sigma} \frac{h'(y_t)}{h''(y_t)} dw(t). \tag{6}$$

Given the construction of the HJB equation, next to consider is the terminal value of the problem at $t = \tau$. Rewrite the value of the program at τ as:

$$V(Q_\tau - \lambda K_\tau, k) = k^\nu h(\frac{Q_\tau - \lambda K_\tau}{k} - \lambda). \tag{7}$$

Define a new variable, M, as:

$$\begin{aligned} M &= \sup_k (Q_\tau - \lambda K_{\tau_-})^{-\nu} V(Q_{\tau_-} - \lambda K_{\tau_-} - \lambda K_{\tau_-}) \\ &= \sup_k (Q_\tau - \lambda K_\tau)^{-\nu} k^\nu h(\frac{Q_\tau - \lambda K_\tau}{k} - \lambda) \\ &= \sup_y (y + \lambda)^{-\nu} h(y). \end{aligned} \tag{8}$$

Given (7) and (8) and noting that $y_{\tau_-} = Q_{\tau_-}/K_{\tau_-} - \lambda$:

$$V(Q_{\tau_-} - \lambda K_{\tau_-}, k) = k^\nu h(\frac{Q_{\tau_-} - \lambda K_{\tau_-}}{k} - \lambda) = M K_{\tau_-}^\nu y_{\tau_-}^\nu.$$

Putting the technique together yields:

$$
\begin{aligned}
V(Q_0; K_0) &= \sup_{\tau,\{X_t,C_t\}} E\Big[\int_0^\tau e^{-\delta t}\frac{1}{\nu}(K_t^{-\rho}+bC_t^{-\rho})^{-\frac{\nu}{\rho}}dt \\
&\qquad + \sup_k e^{-\delta\tau}V(Q_{\tau_-}-\lambda K_{\tau_-},k)\Big] \\
K_0^{\nu}h(y_0) &= \sup_{\tau,\{x(y_t),C_t\}} E\Big[\int_0^\tau e^{-\delta t}\frac{1}{\nu}(K_t^{-\rho}+bC_t^{-\rho})^{-\frac{\nu}{\rho}}dt + e^{-\delta t}K_0^{\nu}y_{\tau-}^{\nu}M\Big] \\
h(y_0) &= \sup_{\tau,\{x(y_t),c_t\}} E\Big[\int_0^\tau e^{-\delta t}\frac{1}{\nu}(1+bc_t^{-\rho})^{-\frac{\nu}{\rho}}dt + e^{-\delta t}y_{\tau-}^{\nu}M\Big].
\end{aligned}
\tag{9}
$$

Of course the bankruptcy constraint is $Q_t - \lambda K_t \geq 0 \iff y_t \geq 0$. [9] One can verify that the implied HJB equation from (9) is the same as in (5).

To solve the problem, G-L conjecture that optimally $y(t)$ remains between two bands y_1 and y_2. If y_t is too high the consumer can enjoy higher durable services immediately. On the other hand if y_t is too low, financial wealth will not increase fast enough, and future durable purchases and nondurable consumption will suffer. When $y(t)$ hits one of the bands at time τ, $K(\tau_-)$ is sold and a new $K(\tau_+)$ is bought such that $y(\tau_+) = y^*$. y^* is the same for both y_1 and y_2 because in $y \times t$ space the total transactions cost λ is independent of y^*. Since the investor pays the same cost no matter what, she may as well move to the optimal y. The optimal y is independent of the barrier from which the consumer moves due to the Markov nature of the y_t process, and thus, there is only one y^*.

[9] If $U(K_t, C_t) = log(K_t) + b\log(C_t)$, the problem reduces to:

$$h(y) = \sup_{\tau,\{x(y_t)\}} E\int_0^\tau e^{-\delta t}[b\log(bh'(y_t)^{-1}) - \frac{(1+b)\alpha}{\delta}]dt + e^{-\delta t}(M + \frac{(1+b)}{\delta}\log y_t),$$

and M is defined as

$$M \equiv \sup_y[h(y) - \frac{(1+b)}{\delta}\log(y+\lambda)].$$

When $y(t) \in [y_1, y_2]$ the value of y is given by $h(y)$ which solves the non-linear differential equation (5).

The complete mathematical description of the solution is given by the following equations,

$$
\begin{aligned}
\delta h(y) &= h'(y)(r(y+\lambda-1)-c) - \gamma \frac{h'(y)^2}{h''(y)} + \frac{1}{\nu}(1+bc_t^{-\rho})^{-\frac{\nu}{\rho}}; \\
h(y) &\geq y^\nu M \qquad \forall y; \\
h(y_i) &= y_i^\nu M \qquad i=1,2; \\
h'(y_i) &= \nu y_i^{\nu-1} M \qquad i=1,2; \\
M &= \sup_y (y+\lambda)^{-\nu} h(y); \\
y^* &= \arg\sup_y (y+\lambda)^{-\nu} h(y).
\end{aligned} \tag{10}
$$

The inequality is the natural requirement that τ, the optimal time to buy a new durable, be non-negative. If it does not hold then the consumer should immediately sell her durable and purchase another. In fact she should have already done this. The first set of equalities for y_i are the value matching conditions. At y_1 and y_2 the value of jumping minus the cost should be zero. Such a condition is a property of a value function; it is not rooted in optimality (Dumas, 1988). The second set of equalities for y_i are the smooth pasting conditions. The consumer should be indifferent between jumping immediately and waiting a fraction of time dt. The last two equations are simply the definitions of M and y^*.

3.3 The Solution to the Bellman Equation

Following Karatzas *et al.* (1986), define an implicit mapping between y

and marginal utility $u'(c)$. It should not cause confusion when I write $y(c)$ or alternately $y(u'(c))$. Later it will be shown that $y(c)$ is a monotonic function of $u'(c)$, and thus, of c. Since c is continuously controlled, the first-order conditions for c imply that $h'(y) = u'(c)$. Differentiate $h'(y)$ by c to get

$$h''(y) = \frac{u''(c)}{y'(c)}. \tag{11}$$

Differentiate (5); use equation (11) and rearrange to obtain

$$-\gamma\frac{u'(c)^2}{u''(c)}y''(c) + u'(c)(r-\bar{\delta})y'(c) + ru''(c)y(c) - \gamma(2u'(c) - \frac{u'(c)^2u'''(c)}{u''(c)^2})$$
$$+ (r(\lambda-1) - c)u''(c) = 0. \tag{12}$$

The general solution to differential equation (12) is

$$y(c) = B_1u'(c)^{\theta_+} + B_2u'(c)^{\theta_-} + 1 - \lambda + \frac{c}{r}$$
$$-\frac{1}{\gamma(\theta_+ - \theta_-)}\left(\frac{u'(c)^{\theta_+}}{\theta_+}\int_0^c u'(v)^{-\theta_+}dv + \frac{u'(c)^{\theta_-}}{\theta_-}\int_c^\infty u'(v)^{-\theta_-}dv\right) \tag{13}$$

where θ_+ and θ_- are the solutions to the quadratic equation

$$\theta^2 - \frac{r-\gamma-\bar{\delta}}{\gamma}\theta - \frac{r}{\gamma} = 0. \tag{14}$$

The proof is through verification.

The constants B_1 and B_2 are determined by the boundary conditions given implicitly in the value matching and smooth pasting conditions. Of course, it must be verified for particular parameter values that the solutions for B_1 and B_2 yield a monotonic function relating c to y. In order for (13) to be well-defined, $\lim_{s\to\infty}\int_0^s u'(v)^{-\theta_-}dv < \infty$. A necessary condition for this convergence is that $\lim_{v\to\infty} u'(v) = 0$, hence the sufficient condition that $\nu + \rho < 0$ if $\rho < 0$.

The next step in the solution is to solve for the $u'(c)$ process. The next theorem proved in the Appendix gives the process.

Theorem 3.1

Let z_t denote the marginal utility of c_t, $z_t \equiv u'(c_t)$. For $t \in [0, \tau)$, *z_t solves the following stochastic differential equation*

$$dz_t = (\bar{\delta} - r) z_t dt - \frac{\mu}{\sigma} z_t dw(t).$$

Two remarks about the z process are in order. First, as long as $\mu \neq 0$, problems associated with degenerate stochastic processes (zero variance at a point in time) are not a concern. If $\mu = 0$, any risk averse investor will keep $X_t = 0$, investing all her wealth in B_t, and thus, the problem is completely deterministic. Second, this result is similar to that found in the literature on Euler equation tests on nondurable consumption starting with Hall (1978). This similarity and its implications are pursued below.

The proof of Theorem 3.1 in the Appendix follows the technique in Karatzas *et al.* (1986). This result can also be understood given the results in Duffie and Skiadas (1991). Let F_t denote financial assets, that is, $F_t = Q_t - K_t$. From (9), define the problem for $t \in [0, \tau)$ as

$$\max_{\{x(t), c(t)\}} \int_0^{\tau} e^{-\delta t} \frac{1}{\nu} \left(1 + b c_t^{-\rho}\right)^{-\frac{\nu}{\rho}} dt$$

$$\text{s.t.} \qquad df_t = r dt + x_t\big(\mu dt + \sigma dw(t)\big) - c_t dt,$$

where lower case letters denote their upper case counterparts divided by K_t.

Define $\pi_t = e^{-\bar{\delta} t} u'(c_t) = e^{-\bar{\delta} t} z_t$. Using the Itô calculus,

$$\frac{d\pi_t}{\pi_t} = \mu_{\pi_t} dt + \sigma_{\pi_t} dw(t) = -\bar{\delta} dt + \frac{dz_t}{z_t}. \tag{15}$$

From Duffie and Skiadas (1991) any security in a complete market multiplied by π_t is a martingale. Defining $b_t = \frac{B_t}{K_t}$ and $\hat{l}_t = \frac{L_t}{K_t}$, implies that

$$db_t = rdt$$

$$d\hat{l}_t = (\hat{\mu} + \alpha)dt + \sigma dw(t).$$

Given the two securities in this market, $b(t)$ and $\hat{l}_t$, this martingale property implies

$$0 = E[\frac{db_t}{b_t} + \frac{d\pi_t}{\pi_t}]$$

$$0 = E[\frac{d\hat{l}_t}{\hat{l}_t} + \frac{d\pi_t}{\pi_t} + \frac{d\hat{l}_t}{\hat{l}_t}\frac{d\pi_t}{\pi_t}].$$

Thus, $\mu_\pi = -r$ and $\sigma_\pi = -\frac{\mu}{\sigma}$. Combined with (15), this gives Theorem 3.1.

The above has followed §§6–7 of Karatzas *et al.* (1986). The next step is to redefine the value function in terms of c which follows §§8–9. Define $H\big(u'(c)\big) \equiv h\big(y(c)\big)$. Replacing $h(y)$ with $H\big(u'(c)\big)$ leaves

$$H\big(u'(c)\big) = \sup_\tau E\left[\int_0^\tau e^{-\delta t}\frac{1}{\nu}(1 + bc_t^{-\rho})^{-\frac{\nu}{\rho}}dt + e^{-\delta\tau}My_{\tau-}^{a}\right].$$

Applying the Feynman-Kac Formula (Duffie, 1988), for $t \neq \tau$, $H(z_t)$ solves the differential equation

$$\bar{\delta}H(z) = H'(z)E[dz] + \tfrac{1}{2}H''(z)Var[dz] + \frac{1}{\nu}(1 + bc_t^{-\rho})^{-\frac{\nu}{\rho}},$$

where $u'(c) = z$. Plugging in the mean and variance of dz gives the differential equation:

$$\bar{\delta}H(c) = \frac{u'(c)}{u''(c)}(\bar{\delta} - r - \gamma\frac{u'''(c)u'(c)}{u''(c)^2})H'(c) + \gamma\frac{u'(c)^2}{u''(c)^2}H''(c) + \frac{1}{\nu}(1 + bc_t^{-\rho})^{-\frac{\nu}{\rho}}. \tag{16}$$

The general solution to (16) is:

$$H(c) = A_1 u'(c)^{\phi_+} + A_2 u'(c)^{\phi_-} + \frac{u(c)}{\bar{\delta}}$$
$$- \frac{1}{\gamma(\phi_+ - \phi_-)} \left(\frac{u'(c)^{\phi_+}}{\phi_+} \int_0^c u'(v)^{-\theta_+} dv + \frac{u'(c)^{\phi_-}}{\phi_-} \int_c^\infty u'(v)^{-\theta_-} dv \right) \tag{17}$$

where ϕ_+ and ϕ_- are the solutions to the quadratic equation:

$$\phi^2 - \frac{r + \gamma - \bar{\delta}}{\gamma} \phi - \frac{\bar{\delta}}{\gamma} = 0. \tag{18}$$

A relation used later is $\phi_\pm = 1 + \theta_\pm$.

Before showing that $H(c)$ as defined in (17) solves (10), the following Lemma, also proved in the Appendix, is needed.

Lemma 3.2

The constants A_1 and A_2 are related to B_1 and *B_2 by*

$$A_1 = \frac{\theta_+}{1 + \theta_+} B_1 \qquad A_2 = \frac{\theta_-}{1 + \theta_-} B_2$$

Theorem 3.3, the main result, now follows.

Theorem 3.3

Furthermore, $H(c)$ is the value function, and the formulas for $X(t)$ and *$c(t)$ are the optimal controls as are the bounds y_1, y^* and y_2.*

Proof:

One uses Lemma 3.2 and the relationship between h and H to verify that (17) solves (5). The boundaries y_1, y^* and y_2 are optimal because they satisfy the first-order conditions for optimality. $H(c)$ is the value function and $X(t)$ and $C(t)$ are optimal controls by Theorem 4.2, the Verification Theorem for Autonomous Problems, in Flemming and Rishel (1975). Theorem 2.1 shows that $H(c)$ is bounded, and $H(c)$ is obviously in C^2. Given Theorem 3.1, as long as $\mu \neq 0$, the processes do not become degenerately non-stochastic. •

If $\rho = -\nu$, the utility function is additively separable and the integrals in the definitions of y and H can be solved. Let $z \equiv u'(c) = bc^{\nu-1}$, then

$$y(z) = B_1 z^{\theta_+} + B_2 z^{\theta_-} + 1 - \lambda - \psi z^j,$$

$$H(z) = \frac{\theta_+}{\phi_+} B_1 z^{\phi_+} + \frac{\theta_-}{\phi_-} B_2 z^{\phi_-} + \frac{1}{\nu\bar{\delta}} - \frac{1}{\nu}\psi z^{j+1},$$

where

$$j = \frac{1}{\nu - 1}, \qquad \psi = \left(b^j \gamma (j - \theta_+)(j - \theta_-)\right)^{-1}.$$

The final solution to the general problem are six unknown variables, c^*, c_1, c_2, M, B_1, B_2 related by the following equations:

$$\begin{aligned}
&\left(y(c^*, B_1, B_2) + \lambda\right)^{-\nu} H(c^*, B_1, B_2) - M = 0; \\
&\nu H(c^*, B_1, B_2) - \left(y(c^*, B_1, B_2) + \lambda\right) u'(c^*) = 0; \\
&H(c_i, B_1, B_2) - M\left(y(c_i, B_1, B_2)\right)^{\nu} = 0, \qquad i = 1, 2; \\
&M\nu\left(y(z_i, B_1, B_2)\right)^{\nu-1} - z_i = 0, \qquad i = 1, 2;
\end{aligned} \tag{19}$$

where the functions $y(\cdot)$ and $H(\cdot)$ are given by (13) and (17) along with *Lemma 3.2.*

4. Discussion

Several results follow from this solution, some are analytical, others are numerical given the system of equations in (19).

4.1 Microeconomic Orthogonality Tests

The first immediate implication of the above analysis is that durable expenditures are dependent only on the current durable stock and current wealth.

This is immediate from the formula for the jump in the durable stock at the time of adjustment.

Lemma 4.1

At $t = \tau$ the percentage change in the durable stock is given by

$$\frac{K_+ - K_-}{K} = \frac{y_i - y^* - \lambda}{y^* + \lambda}. \tag{20}$$

Moreover, $K_+ - K_- > 0$ if $y = y_2$ and negative if $y = y_1$.

The proof is in the Appendix.

Given K_0, $K_- = e^{-\alpha\tau}K_0$. The expectation of K_+ at time $t = 0$ then involves the joint probability of τ and whether $z = z_1$ or z_2. This distribution function is over z which is itself a Markov process. Therefore, the expectation of K_+ is dependent only on K_0 and $z(0)$. Other expectations, such as K_t for some positive t not conditioned on where z is located also involve joint probabilities. Still, these expectations are dependent only on K_0 and on expectations of the path of z. Since z is Markov the same result obtains for $E[K_t]$.

The importance of this fact is that changes in the stock of durables are independent of any other variables known at time s. Traditional orthogonality tests used in the literature on nondurable consumption can be applied to durable expenditures. For instance Eberly (1990) conditions on the time at which the durable stock is updated. Let τ_1 and τ_0 be two times at which the durable stock is updated. Denote financial assets at τ as F_τ. Then taking logs

$$ln(K_{\tau_1}) - ln(K_{\tau_0}) = ln(F_{\tau_1}) - ln(F_{\tau_0}) = 1 + \xi_{\tau_0,\tau_1}.$$

where ξ_{τ_0,τ_1} is the realized growth rate in financial assets between τ_0 and τ_1. ξ is a random, Markov process. Thus, the change in the durable stock between updates is a Markov process viewed from the last update. It is this fact she exploits in her Euler tests.

4.2 Hall's Martingale Hypothesis

A second implication of the above analysis is the relationship between nondurable consumption, the durable stock and the marginal value of wealth. The effect of δ on the slope of marginal utility is reminiscent of the work on nondurable consumption smoothing. Hall (1978) shows in a discrete time framework with utility only over nondurable consumption that

$$E_t U'(C_{t+1}) = \frac{1+\delta}{1+r_f} U'(C_t), \tag{21}$$

where $E_t U'(C_{t+1})$ is the expectation at time t of the marginal utility of consumption at time $t+1$; δ is the rate of time preference and r_f is the riskless interest rate. Karatzas *et al.* (1986) with a general utility function over nondurable consumption obtain·

$$d\zeta_t = (\delta - r_f)\zeta_t dt - \frac{\mu}{\sigma}\zeta_t dw(t), \tag{22}$$

where $\zeta_t = U'(C_t)$ and the other parameters are as above. [10] In both cases the slopes of expected marginal utility depend upon the difference between the rate of time preference and the riskless rate of return. Heaton (1990) also finds this

[10] Equation 7.4, p. 272 with the parameters interpreted in terms of those used above.

result in a model where the utility of consumption of "nondurable" goods lasts over time. One particular example of his general model is durable goods.

The same result obtains here. Recall from (4) that $V(Q_t, K_t) = K_t^{\nu} h(y_t)$. Define

$$Z_t = V_Q = K_t^{-(1-\nu)} h'(y_t) = K_0^{-(1-\nu)} e^{\alpha(1-\nu)t} z_t.$$

When, $z_t \in (z_2, z_1)$, $dz_t = (\bar{\delta} - r) z_t dt - \frac{\mu}{\sigma} z_t dw(t)$; from (11) $h'(y_t) = u'(c_t) \equiv z_t$. Using the Itô calculus, Z_t solves the stochastic differential equation

$$\begin{aligned} dZ_t &= K_0^{-(1-\nu)} e^{\alpha(1-\nu)t} dz_t + \alpha(1-\nu) K_0^{-(1-\nu)} e^{\alpha(1-\nu)t} z_t dt \\ &= K_0^{-(1-\nu)} e^{\alpha(1-\nu)} z_t \left[(\delta + \nu\alpha - (\alpha + r_f)) dt + \alpha(1-\nu) dt - \frac{\mu}{\sigma} dw(t) \right] \\ &= (\delta - r_f) Z_t dt - \frac{\mu}{\sigma} Z_t dw(t). \end{aligned}$$

Moreover, the V_Q is continuous at the point at which z jumps. Since the process for V_Q is a geometric brownian motion in the open interval and continuous at the boundaries, V_Q is continuous everywhere.[11] The following lemma is proved in the appendix.

Lemma 4.2

The marginal value of total wealth Q_t is continuous at the boundaries. That is

$$V_Q(Q_{\tau-}, K_{\tau-}) = V_Q(Q_{\tau-} - \lambda K_{\tau-}, K_{+}).$$

Since $U_C(K_t, C_t) = V_Q$, the marginal utility of nondurable consumption is continuous at the point at which K_t jumps.

[11] The continuity of V_Q may seem surprising given that $h'(y)$ jumps at the sale of a durable. This continuity is implied by the smooth pasting conditions which is hidden in the homogeneity construction.

Given the discussion earlier on the process of z_t, the results in Duffie and Skiadas (1991) can be applied to this continuity.

The importance of the lemma is illustrated by writing the equality of U_C at the boundaries

$$bC_-^{-(1+\rho)}(K_-^{-\rho} + bC_-^{-\rho})^{-\frac{\nu+\rho}{\rho}} = bC_+^{-(1+\rho)}(K_+^{-\rho} + bC_+^{-\rho})^{-\frac{\nu+\rho}{\rho}}. \qquad (23)$$

Since K_+ differs discretely from K_-, as given in Lemma 4.1, in order to preserve the equality in (23) nondurable consumption must jump. That is C_+ also differs discretely from C_-. The sign of the jump depends on the sign of the cross partial derivative of utility.

$$U_{CK}(K, C) = (\nu + \rho)bK^{-(1+\rho)}C^{-(1+\rho)}(K^{-\rho} + b.C.^{-\rho})^{-\frac{\nu}{\rho}-2}.$$

If $\nu + \rho > 0$, durable and nondurable consumption are complements in utility.[12] Increases in K lead to increases in the marginal utility of nondurable consumption. To lower U_C back to its former level, nondurable consumption has to increase. On the other hand, if K and C are substitutes in utility, when K jumps upward, C jumps downward. Of course if K and C are additively separable, marginal utility depends only on C. In this case continuity of marginal utility implies continuity of nondurable consumption.

Important interaction effects between durables and nondurables imply that former tests of the PIH using only nondurable consumption may be invalid.

[12] I define two goods as complements and substitutes in utility if the cross-derivative of utility is positive or negative. This concepts is different from the modern definition of complements and substitutes which depends on the partial derivative of the Hicksian demand function with respect to prices.

While careful treatment of these interaction effects can remove these problems when using micro data, tests using macro data are more problematical. Both because nondurable consumption for each individual jumps at the individual's boundary, and because the present stock of durables helps determine optimal nondurable consumption, aggregate nondurable consumption depends not only on the aggregate stock of durables, but also on the cross-sectional distribution of agents within their bands. To see this, note that $C_t^A = \int_I C_t(i)di$. The important variable for analysis, however, is

$$Z_t(i) = bC_t(i)^{-(1+\rho)}(K_t(i)^{-\rho} + bC_t(i)^{-\rho})^{-\frac{\nu}{\rho}},$$

but there is no way to write a comparable variable in terms of K_t^A and C_t^A.[13]

Nondurable consumption, then, suffers from the same aggregation problem as do durable goods. This aggregation problem means that a representative agent with stable preferences, *i.e.* independent of the distribution of stocks of durable goods does not exist. In particular, the C-CAPM, derived under the assumption of a representative agent, is invalid. Moreover, specifying the marginal rate of substitution for a representative consumer is also impossible. Aggregate orthogonality tests such as Hansen and Singleton (1982) are misspecified.

4.3 Numerical Calculations

For particular parameter values, the system of equations can be solved numerically. A problem, however, is that initial guesses must be fairly close to the

[13] An exception is if $-\rho = \nu$; that is additive separability; see Beaulieu (1991).

correct value; otherwise the system diverges. For example, under a standard set of parameter values, $B_1 = -.169$. If the remaining five initial guesses are accurate to two decimal places, in order for the system to converge, the initial guess for B_1 must be in the interval $[-.19, -.13]$. To overcome the difficulty of initial guesses, one can first solve the differential equation numerically for a one particular parameter set. Then, one simply iterates slowly over a parameter that is changed, solving the system for each small increase in the parameter and using the resulting solution as initial values for the next iteration. One simple procedure is to start with $\rho = -\nu$ and $b = 0$. This is the original Grossman-Laroque model, and it is easily solved. Then one can iterate first on b and then on ρ and ν.

Besides the computational ease of the solution, there are other advantages to solving the problem in this manner. Two statistics Grossman and Laroque present are $E[\tau]$, which is the expected time it takes to reach y_1 or y_2 starting at y^*, and *Prob.* y_1, which is the probability that y_1 is reached before y_2. They do not explicitly solve for either of these statistics because they do not have a closed form for the wealth process. As equation (6) shows dy is a function of the unknown function, $h'(y)$ and $h''(y)$. Note, however, that these exit time problems can be written in terms of c. y hits y_1 when c hits c_1 and likewise for c_2. Given that $u'(c) = z$ is invertible, one can also write these exit times in terms of z. Since a simple closed form solution for dz exists, one can solve for these statistics, which is done in the Appendix.

Table 1 reports simulation results for a few specific sets of parameter values.

In particular the values of ρ and ν are chosen so that the cases of substitutes, complements and additive separability can all be explored. More simulations are obviously needed to draw any particular conclusions, but a few are suggested. First, compared to results in Grossman-Laroque, the barrier policies shift radically to the right. That is when nondurable consumption is included, the ratio of marketable wealth to the durable stock is much larger at the adjustment points, which makes sense since some wealth is saved for nondurable consumption. Second, the barriers in y widen as ν increases while the expected time to hit one of the barriers decreases. Third, changes in b do not shift the bands in z-space if durables and nondurables enter utility additively separable. Otherwise, they do. This finding is important because b includes the relative price, and accounting for relative price effects is easier if the bands are stable in some space (Beaulieu, 1991). Because the bands in z-space are constant under additive separability, the two statistics $E[\tau]$ and *Prob.* y_1 also do not change with changes in b.

Table 2 considers an issue discussed earlier but not illustrated in Table 1. Lemma 4.1 gives the formula for the percentage jump in the durable stock at the point of adjustment. As for nondurables, that $C_+ = c^* K_+$ and $C_- = c_i K_-$ along with Lemma 1 gives the formula for the percentage jump in nondurable consumption

$$\frac{C_+ - C_-}{C_-} = \frac{c^*}{c_i}\frac{y_i}{y^* + \lambda} - 1, \tag{24}$$

$i = 1, 2$ depending upon whether the durable is adjusted downward ($i = 1$) or whether the durable is adjusted upward ($i = 2$). The percentage jumps in both the durable stock and nondurable consumption are reported for the same parameter values that are found in Table 1.

As was explained above, whether nondurable consumption jumps upwards or downwards when the durable stock is adjusted upwards depends on the sign of the cross-partial derivative of utility. When $\rho + \nu$ is negative, durables and nondurables are substitutes in utility, and as is seen in Table 2, nondurable consumption jumps in the opposite direction from the durable stock. When $\rho + \nu$ is positive, the two are complements in utility and they jump in the same direction. When utility is additively separable nondurable consumption is continuous and therefore there is no jump.

The simulation results suggest that jumps in nondurable consumption are much smaller than for durables. The durable stock jumps anywhere from 60% to 150% while nondurable consumption jumps only 1% to 4%. Of course nondurable consumption varies in between the barriers whereas durables do not, and thus, aggregate variation in durables and nondurables will depend upon the cross-sectional distribution of the population within the bands.

5. Conclusions

The model allows for both nondurable and durable consumption in a model where a consumer's durable stock is adjusted infrequently. When the amount of the durable stock affects the optimal purchase of nondurables, the S-s policy of durable purchases strongly affects the dynamics of nondurable consumption. In fact nondurable consumption in such a case is not continuous as it jumps when the durable stock is adjusted. Such effects make aggregation impossible and call into question rejections based on an assumed representative consumer.

Two responses to the problem of aggregation are possible. First, one can assume additive separability between durables and nondurables, a strategy pursued in Beaulieu (1991) and Caballero (1990a). Once one assumes additive separability, besides the possibility of aggregation, one can use nondurable consumption instead of wealth to normalize the durable stock. Modeling permanent income is not needed as a precise measure of wealth shocks is available with nondurable consumption. Moreover, including nondurables in the analysis of durables allows one to consider relative price effects. Beaulieu discusses these points thoroughly.

A second strategy is to eschew traditional statistical modeling and instead use simulations to draw conclusions. For instance Brainard *et al.* (1991) suggest that one reason why previous C-CAPM tests have performed poorly is that durables and nondurables interact in utility. They find that the C-CAPM model performs better at longer horizons, a finding they reason is consistent with this paper's model of durables and nondurables. One way to test this stipulation is to simulate the model for a large set of people with different levels of durable stocks and financial wealth. The solution technique for an individual's problem outlined in this paper makes such a proposal possible for a wide range of parameter values.

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Table 1: Numerical Solutions

ρ	ν	λ	b	$\bar{y}_1$	$\bar{y}^*$	$\bar{y}_2$	c_1	c^*	c_2	z_1	z^*	z_2	M	B_1	B_2	$E[\tau]$	*Prob* y_1
Substitutes																	
1.0	-1.5	.05	1.0	2.756	6.953	12.902	.045	.117	.220	2397.426	226.543	48.767	-19252	-.113	38.225	34.511	.076
1.0	-1.5	.05	2.0	3.664	9.445	17.627	.064	.165	.310	2813.041	265.582	56.903	-49840	-.109	44.212	34.628	.076
1.0	-1.5	.10	1.0	2.360	7.357	18.926	.037	.123	.254	3805.548	198.959	34.514	-19473	-.099	35.449	43.010	.044
1.0	-1.5	.10	2.0	2.997	9.892	20.277	.053	.174	.357	4465.790	232.986	40.217	-50254	-.096	40.861	43.145	.044
Complements																	
1.0	-.5	.05	1.0	2.217	7.305	16.186	.036	.118	.250	145.068	23.431	7.138	-925	-.254	27.941	23.163	.118
1.0	-.5	.05	2.0	2.928	9.887	22.037	.050	.166	.357	123.412	20.097	6.110	-1269	-.264	25.514	23.142	.119
1.0	-.5	.10	1.0	1.820	7.809	19.228	.028	.125	.292	206.365	21.330	5.564	-931	-.228	27.436	29.479	.079
1.0	-.5	.10	2.0	2.260	10.464	26.012	.040	.176	.417	175.704	18.296	4.775	-1274	-.236	25.095	29.423	.079
Additively Separable																	
1.0	-1.0	.05	1.0	2.446	6.883	13.659	.041	.117	.232	597.849	72.456	18.534	-3432	-.169	27.080	28.922	.093
1.0	-1.0	.05	2.0	3.338	9.445	18.725	.058	.166	.329	597.849	72.456	18.534	-6464	-.169	27.080	28.922	.093
1.0	-1.0	.10	1.0	2.064	7.308	15.980	.033	.124	.270	898.481	64.876	13.741	-3465	-.150	26.004	36.433	.058
1.0	-1.0	.10	2.0	2.791	10.016	21.864	.047	.176	.382	898.483	64.876	13.740	-6509	-.150	26.004	36.443	.058
-.1	.1	.05	1.0	.674	4.153	14.117	.001	.009	.030	387.191	70.341	23.459	2533	-.214	391.241	20.487	.129
-.1	.1	.05	2.0	.952	6.002	20.381	.003	.019	.065	387.191	70.341	23.459	3529	-.214	391.241	20.487	.129
-.1	.1	.10	1.0	.555	4.738	18.955	.001	.010	.041	510.127	61.974	17.880	2513	-.195	379.153	26.219	.090
-.1	.1	.10	2.0	.760	6.866	27.425	.002	.022	.088	510.127	61.974	17.880	3510	-.195	379.153	26.219	.090

Table 2: Percentage Jumps in Durables and Nondurables

ρ	ν	λ	b	*Durable* From y_1	*Durable* From y_2	*Non-Durable* From y_1	*Non-Durable* From y_2
Substitutes							
1.0	-1.5	.05	1.0	-.611	.848	.012	-.017
1.0	-1.5	.05	2.0	-.611	.852	.010	-.013
1.0	-1.5	.10	1.0	-.693	1.015	.021	-.024
1.0	-1.5	.10	2.0	-.693	1.019	.012	-.016
Complements							
1.0	-.5	.05	1.0	-.703	1.209	-.028	.043
1.0	-.5	.05	2.0	-.702	1.212	-.018	.029
1.0	-.5	.10	1.0	-.780	1.449	-.017	.049
1.0	-.5	.10	2.0	-.779	1.449	-.021	.036
Additively Separable							
1.0	-1.0	.05	1.0	-.652	.977	.000	.000
1.0	-1.0	.05	2.0	-.652	.977	.000	.000
1.0	-1.0	.10	1.0	-.731	1.173	.000	.000
1.0	-1.0	.10	2.0	-.731	1.173	.000	.000
-.1	.1	.05	1.0	-.850	2.388	.000	.000
-.1	.1	.05	2.0	-.850	2.388	.000	.000
-.1	.1	.10	1.0	-.904	2.979	.000	.000
-.1	.1	.10	2.0	-.904	2.979	.000	.000

Notes:

1. Other parameter values for Tables 1–2 are: $\alpha = 0.0$, $r_f = .01$, $\sigma = .22$. $\bar{y} = y + \lambda$.

APPENDIX: PROOFS AND STATISTICS

The Appendix contains the proofs missing in the text. It also develops the formulas for the statistics calculated in the simulations.

A.1. Proofs

Proof of Theorem 2.1

It is obvious the consumer can do no better than what is optimal when $\lambda = 0$. Suppose $\lambda = 0$. For any $\lambda > 0$, the consumer can follow the optimal program for that particular value of λ, but every time a new durable is bought, instead of paying the transactions cost, the consumer puts $\lambda K_{\tau-}$ in the riskless asset. Thus, the consumer earns the same utility over an entire period but has a bigger bank account which can be used to increase consumption at any time.

To find the upper bound on the value function, let $\lambda = 0$. The process for Q_t is the same as it is in the text, and Q_t is the only state variable. The Bellman equation becomes

$$\delta V(Q) = \max_{\{X(t),C(t),K(t)\}} \left[V'(Q)E[dQ] + \tfrac{1}{2}V''(Q)dQ^2 + \frac{1}{\nu}\left(K_t^{-\rho} + bC_t^{-\rho}\right)^{-\frac{\nu}{\rho}}\right].$$

Given the form of the utility function $V(Q) = \nu_1 Q^\nu$ is a good guess for the form of the utility function. Rewriting the Bellman equation,

$$\delta\eta_1 Q^\nu = \max_{\{X(t),C(t),K(t)\}} \left[\nu\eta_1 Q^{\nu-1}(r_f Q - (\alpha + r_f)K + \mu X - C) + \tfrac{1}{2}\nu(\nu-1)\eta_1 Q^{\nu-2}\sigma^2 X^2 + \frac{1}{\nu}\left(K_t^{-\rho} + bC_t^{-\rho}\right)^{-\frac{\nu}{\rho}}\right].$$

The first-order conditions give

$$X = \frac{\mu}{\sigma^2}\frac{Q}{1-\nu};$$

$$C = \left(\frac{\nu\eta_1}{bp_1}\right)^{\frac{1}{1-\nu}} Q$$

$$K = (b(\alpha + r_f))^{-\frac{1}{1+\rho}} \left(\frac{\nu\eta_1}{bp_1}\right)^{-\frac{1}{1-\nu}} Q,$$

where

$$p_0 \equiv b + \left[b(\alpha + r_f])\right]^{\frac{\rho}{1+\rho}}$$

$$p_1 \equiv p_0^{-\frac{\nu+\rho}{\rho}}.$$

Plugging in the values for X, C and K and rearranging leaves

$$\beta\eta_1 = (\nu\eta_1)^{-\frac{\nu}{1-\nu}} \left(\frac{1}{\nu}(bp_1)^{\frac{\nu}{1-\nu}} p_0^{-\frac{\nu}{\rho}} - (bp_1)^{\frac{1}{1-\nu}}(b^{-\frac{1}{1+\rho}}(\alpha + r_f)^{\frac{\rho}{1+\rho}} + 1)\right),$$

where

$$\beta \equiv \delta - \nu r_f + \tfrac{1}{2}\frac{\mu^2}{\sigma^2}\frac{\nu}{1-\nu}$$

Given the definitions of p_0 and p_1, rewrite the above as

$$\beta = (\nu\eta_1)^{-\frac{1}{1-\nu}} b^{\frac{\nu}{1-\nu}} p_0^{-\frac{\nu(1+\rho)}{\rho(1-\nu)}}(1-\nu).$$

Note that $\nu\eta_1 > 0$ as is p_0. Thus, the right-hand side is greater than zero. For there to be a well defined solution, β must be greater than zero; hence the assumption in the theorem that $\beta > 0$. If it holds then by the Verification Theorem (Flemming and Rishel, 1975), $\eta_1 Q^a$ is the value function for the problem where

$$\eta_1 = \frac{1}{\nu}\left(\frac{\beta}{1-\nu}\right)^{\nu-1} p_0^{-\frac{\nu(1+\rho)}{\rho}} \tag{25}$$

To find the lower bound of the value function, first assume $\rho < 0$. Two strategies are possible. The consumer sells her durable and either buys a new durable that she holds forever, or she adds the proceeds from the sale to her financial portfolio and only consumes nondurable goods. From Grossman and Laroque (1990), the former strategy yields

$$\frac{(Q - \lambda K)^{\nu}}{\nu(\delta + \nu\alpha)}.$$

From an adaptation of results in Karatzas *et al.* (1986), the latter strategy yields

$$\frac{b^{-\frac{\nu}{\rho}}}{\nu}\left(\frac{1-\nu}{\beta}\right)^{1-\nu}(Q - \lambda K)^{\nu}.$$

If $\rho < 0$ then

$$\nu_2 = \left[\frac{1}{\nu(\delta + \nu\alpha)}, \frac{b^{-\frac{\nu}{\rho}}}{\nu}\left(\frac{1-\nu}{\beta}\right)^{1-\nu}\right],$$

whichever provides a more effective bound.

Now, let $\rho > 0$, and assume $r = 0$. Also, assume after an initial, immediate sale of the durable good, $\lambda = 1$. It is obvious that she can only do better if $r > 0$ and $\lambda < 1$ by the same reasoning given above. Beginning immediately, the consumer implements the following strategy.[14] She sells her old durable and splits the proceeds in two. With half the proceeds she buys a new durable good. Every year she sells and buys a new one. Because $\lambda = 1$, revenue from the sale of the old durable is zero. With the other half of the proceeds she buys a certain stream of the nondurable good. At $t = \infty$ her assets from the purchase of the durable and nondurable good are exhausted.

Define $\tilde{Q} \equiv \frac{1}{2}(Q_0 - \lambda K_0)$ which represents the amount devoted to each good separately over the infinite horizon. Also define

$$g_n \equiv \begin{cases} e^{\frac{\delta n}{2\nu}}(1 - e^{\frac{\delta}{2\nu}}), & \text{if } \nu < 0; \\ e^{-\frac{\delta n}{2\nu}}(1 - e^{-\frac{\delta}{2\nu}}), & \text{if } \nu > 0; \end{cases} \qquad g_n \in (0,1),$$

[14] This strategy is similar to that found in Grossman-Laroque.

where g_n is the fraction of $\tilde{Q}$ invested in year n in the durable. Note that $\sum_{n=0}^{\infty} g_n = 1$. Over the year the durable depreciates leaving

$$K_t = g_n e^{-\alpha(t-n)} \tilde{Q}.$$

As for nondurables, she consumes an amount proportional to her current capital stock. In particular,

$$C_t = \begin{cases} \frac{\alpha}{1-e^{-\alpha}} K_t, & \text{if } \alpha > 0; \\ K_t, & \text{if } \alpha = 0. \end{cases}$$

In either case $\int_0^{\infty} C_t dt = \tilde{Q}$.

The utility from this consumption strategy is

$$\begin{aligned} V &= \frac{1}{\nu} \int_0^{\infty} e^{-\delta t} \frac{1}{\nu} (K_t^{-\rho} + b C_t^{-\rho})^{-\frac{\nu}{\rho}} dt \\ &= \frac{1}{\nu} (1 - e^{\pm \frac{\delta}{2\nu}} \tilde{Q}^{\nu} (1 + b(\frac{\alpha}{1-e^{-\alpha}})^{-\rho})^{-\frac{\nu}{\rho}} \sum_{n=0}^{\infty} \int_n^{n+1} e^{-\delta t \pm \frac{\delta n}{2} - \alpha\nu(t-n)} dt \\ &= \eta_2 (Q_0 - \lambda K_0), \end{aligned}$$

where η_2 depends on the sign of ν.

$$\eta_2 = \begin{cases} \frac{1}{\nu(\delta+\nu\alpha)} (1 - e^{\frac{\delta}{2\nu}}) (1 + b(\frac{\alpha}{1-e^{-\alpha}})^{-\rho})^{-\frac{\nu}{\rho}} \frac{1-e^{-(\delta+\nu\alpha)}}{1-e^{\frac{\delta}{2}}}, & \text{if } \nu < 0; \\ \frac{1}{\nu(\delta+\nu\alpha)} (1 - e^{-\frac{\delta}{2\nu}}) (1 + b(\frac{\alpha}{1-e^{-\alpha}})^{-\rho})^{-\frac{\nu}{\rho}} \frac{1-e^{-(\delta+\nu\alpha)}}{1-e^{\frac{3\delta}{2}}}; & \text{if } \nu > 0. \end{cases}$$

The value function in this case is V. •

Proof of Theorem 3.1

Replacing (11) in (6) yields

$$dy = \left(r(y + \lambda - 1) - 2\gamma \frac{u'(c)}{u''(c)} y'(c) - c \right) dt - \frac{\mu}{\sigma} \frac{u'(c)}{u''(c)} y'(c) dw(t). \qquad (26)$$

Using the Itô calculus,

$$dc = c'(y) dy + \tfrac{1}{2} c''(y) dy^2.$$

Replacing (6) in dc gives

$$dc = r(y + \lambda - 1)dt - 2\gamma\frac{u'(c)}{u''(c)}y'(c) - c)dt - \frac{\mu}{\sigma}\frac{u'(c)}{u''(c)}y'(c)dw(t). \qquad (27)$$

These two facts are needed:

$$c''(y)y'(c)^2 = -c'(y)y''(c)$$

$$\gamma y''(c) = ((r - 2\gamma - \bar{\delta})\frac{u''(c)}{u'(c)} + \gamma\frac{u'''(c)}{u''(c)})y'(c)$$
$$+ r\frac{u''(c)^2}{u'(c)^2}y + (r(\lambda - 1) - c)\frac{u''(c)^2}{u'(c)^2}$$

Using these in dc gives

$$dc = ((\bar{\delta} - r)\frac{u'(c)}{u''(c)} - \gamma\frac{u'(c)^2 u'''(c)}{u''(c)^3})dt - \frac{\mu}{\sigma}\frac{u'(c)}{u''(c)}dw(t). \qquad (28)$$

Using the Itô calculus again gives

$$du'(c) = u''(c)dc + \tfrac{1}{2}u'''(c)dc^2.$$

Replacing dc using (28) and after some manipulations yields

$$dz = (\bar{\delta} - r)z - \frac{\mu}{\sigma}zdw(t),$$

where $z \equiv u'(c)$. •

Proof of 3.3:

Because $H(c) = h(y(c))$ for the compact interval $[c_1, c_2]$, their derivatives with respect to c are equal. Differentiating (13) by c, multiplying by $u'(c)$, and dividing by $u''(c)$ leaves

$$\frac{u'(c)y'(c)}{u''(c)} = \theta_+ B_1 u'(c)^{\theta_+} + \theta_- B_2 u'(c)^{\theta_-}$$
$$- \frac{1}{\gamma(\theta_+ - \theta_-)}\left(u'(c)^{\theta_+}\int_0^c u'(v)^{-\theta_+}dv + u'(c)^{\theta_-}\int_c^\infty u'(v)^{-\theta_-}dv + \frac{1}{\theta_+} - \frac{1}{\theta_-}\right).$$

Differentiating (17) by c and dividing by $u''(c)$ gives

$$\frac{H'(c)}{u''(c)} = \phi_+ A_1 u'(c)^{\theta_+} + \phi_- A_2 u'(c)^{\theta_-}$$
$$- \frac{1}{\gamma(\phi_+ - \phi_-)}\left(u'(c)^{\theta_+} \int_0^c u'(v)^{-\theta_+} dv + u'(c)^{\theta_-} \int_c^\infty u'(v)^{-\theta_-} dv \right).$$

Because $\phi_\pm = 1 + \theta_\pm$, $\theta_+\theta_- = -r/\gamma$, and $\phi_+\phi_- = -\bar{\delta}/\gamma$ these are equal for all c if and only if

$$A_1 = \frac{\theta_+}{1+\theta_+} B_1 \qquad A_2 = \frac{\theta_-}{1+\theta_-} B_2. \bullet$$

Proof of 4.1:

Because $y = \frac{Q}{K} - \lambda$, $K_- = \frac{Q}{y_i + \lambda}$ and $K_+ = \frac{Q - \lambda K_-}{y^* + \lambda}$. Plugging in K_- into the expression for K_+ and $(y_i + \lambda)K_-$ for Q_- and dividing by K_- gives (20). To sign the expression for $y = y_1, y_2$, the following inequality is needed,

$$H'(c) = h'(y)y'(c) > 0.$$

$h'(y) > 0$ because $h'(y) = K^{1-a}V_Q > 0$. $y'(c) > 0$ because $y'(c) = u''(c)/h''(y)$ $= u''(c)K^{a-2}/V_{QQ} > 0$. This gives

$$H(c_2) > H(c^*) > H(c_1), \tag{29}$$

which is true because $c_1 < c^* < c_2$. To prove $\Delta K < 0$ if $y = y_2$ and $\Delta K < 0$ if $y = y_1$ note from the definition of y^* and the smooth pasting conditions that

$$(y^* + \lambda)^{-\nu} = \frac{M}{H(z^*)} \qquad y_1^{-\nu} = \frac{M}{H(z_1)} \qquad y_2^{-\nu} = \frac{M}{H(z_2)}. \tag{30}$$

Now, first assume $\nu < 0$. Thus, $H(z) < 0$ and $M < 0$. This implies

$$\frac{M}{H(z_2)} > \frac{M}{H(z^*)} > \frac{M}{H(z_1)}.$$

Using (30) and noting that $-\nu > 0$ finishes the proof for $\nu < 0$. For $\nu > 0$ (29) is reversed, but because $-\nu < 0$ the inequality is proved. •

Proof of 4.2:

Equation (8) gives

$$M = (Q_- - \lambda K_-)^{-\nu} V(Q_- - \lambda K_-, k).$$

Redefine the smooth pasting conditions at y_1 and y_2 in terms of Q and K replace M with the above to get

$$V_Q(Q_-, K_-) = \nu(Q_- - \lambda K_-)^{-1} V(Q_- - \lambda K_-, K_+).$$

From the definition of y^*, recall it is the argmax in the definition of M, rewrite y^* in terms of Q and K to get

$$\nu\left(\frac{Q_- - \lambda K_-}{K_+}\right)^{-\nu-1} K_+^{-\nu} V(Q_- - \lambda K_-, K_+)$$
$$= \left(\frac{Q_- - \lambda K_-}{K_+}\right)^{-\nu} K_+^{1-\nu} V_Q(Q_- - \lambda K_-, K_+)$$

or $$V_Q(Q_- - \lambda K_-, K_+) = \nu(Q_- - \lambda K_-)^{-1} V(Q_- - \lambda K_-, K_+). \bullet$$

A.2. Statistics

Two statistics, $E[\tau]$, the expected time it takes to hit either y_1 or y_2 starting at y^*, and $Prob.y_1$, the probability y hits y_1 before hitting y_2 starting at y^* provide a useful method by which the patterns of durable purchases among different sets of parameter values can be compared. Because z is a monotonic transformation of y, these statistics can be presented in terms of z. y hits y_i

when z hits z_i. The following two lemmas provide the solutions to these hitting time problems.

Lemma A.1

$E[\tau]$ is the value given by $V_a(z^*)$, where $V_a(z)$ is given by:

$$V_a(z) = \frac{\gamma}{\phi_+ + \phi_-} ln(z/z_1) + \frac{\gamma}{\phi_+ + \phi_-} ln(z_1/z_2) \frac{z_1^{\phi_+ + \phi_-} - z^{\phi_+ + \phi_-}}{z_1^{\phi_+ + \phi_-} - z_2^{\phi_+ + \phi_-}} . \quad (31)$$

Proof:

Karlin and Taylor (1981) show the solution to this hitting time problem is the function $V_a(z)$ which solves the following differential equation:

$$\tfrac{1}{2}V''(z)dz^2 + V'(z)E[dz] = -1, \quad (32)$$

where the two boundary conditions are:

$$V(z_1) = V(z_2) = 0.$$

It is readily verifiable that $V_a(z)$ given in equation (31) solves (32). Note from equation (18) that $\phi_+ + \phi_- = \frac{r+\gamma-\delta}{\gamma}$. •

Lemma A.2

Prob. y_1 is the value given by $V_b(z^*)$, where $V_b(z)$ is given by:

$$V_b(z) = \frac{z^{\phi_+ + \phi_-} - z_2^{\phi_+ + \phi_-}}{z_1^{\phi_+ + \phi_-} - z_2^{\phi_+ + \phi_-}} . \quad (33)$$

Proof:

Karlin and Taylor (1981) show the solution to this hitting time problem is the function $V_b(z)$ which solves the following differential equation:

$$\tfrac{1}{2}V''(z)dz^2 + V'(z)E[dz] = 0, \quad (34)$$

where the two boundary conditions are:

$$V(z_1) = 1 \qquad V(z_2) = 0.$$

It is readily verifiable that $V_b(z)$ given in equation (33) solves (34). •

Remark: The limits of $V_a(z)$ and $V_b(z)$ as $\bar{\delta} - r \rightarrow 0$ are the solutions when $\bar{\delta} - r = 0$.

Chapter 2:

AGGREGATE MOVEMENTS IN DURABLE GOODS WITH FIXED COSTS OF ADJUSTMENT

1. Introduction

Recent models of expenditures stress the lumpy nature of durable purchases. These models have broad appeal because their features are consistent with informal observations of individual behavior. Afterall, consumers do not change their stock of automobiles, appliances and furniture as easily as they change their expenditures on food, apparel and entertainment, and when they do make durable purchases, their purchases are large.

Authors motivate infrequent but large purchases in their models with fixed costs of adjustment, a natural assumption with durable goods. Brokerage costs and sales costs are often an important element of the price of large durable assets such as houses and cars. Some people expend a large amount of resources in gathering information and searching before they commit to a purchase. Due to the nature of the goods, non-combinability is a problem, that is, two $1,500 cars are not the same as one $3,000 car. Thus, people sell their old automobiles when buying a new one even though the trade-in price is oftentimes disappointingly low. Akerlof (1970) justifies such conclusions with an asymmetric information model while Genesove (1990) finds empirical support.

A second reason to consider such lumpy adjustment models is that they have empirical support beyond earlier smooth adjustment models. Eberly (1990) applies the Grossman–Laroque (1990) model to a panel data set and finds results consistent with the model. Bar-Ilan–Blinder (1988a,1988b) suggest that one robust prediction of these models is that the aggregate average amount of durables purchased should be smooth and roughly track wealth while the number of purchases made should be rather volatile and history dependent. They successfully test this prediction using U.S. automobile data. Bertola–Caballero (1990b) and Caballero (1990) present two empirical methodologies that apply a lumpy adjustment model at the microeconomic level to U.S. macroeconomic data. They find their model explains a great deal of the movements in the residuals from a cointegrating relationship between durable purchases and wealth.

In spite of the fact that the main virtue of these models is their solid microeconomic foundations, with the exception of Eberly (1990), little work has been done on the link between the microeconomic and macroeconomic evidence. This paper explicitly considers this link. The theoretical tie that allows one to consider both microeconomic data and macroeconomic data is the cross-sectional distribution of the ratio of individuals' holdings of durable goods to nondurable purchases. On the microeconomic side, parameterizing the cross-sectional distribution allows one to estimate important parameters. On the macroeconomic side, modeling changes in the cross-sectional distribution allows one to describe movements in aggregate purchases of durable goods. Considering both minimizes the difficulty in estimating the macroeconomic model by incorporating some of the microeconomic estimates in the macroeconomic estimation procedure. This is in contrast to Caballero (1990) who assumes symmetry in a subset of the parameters to ease the burden of estimation. Considering both also allows one to test whether the macroeconomic model is consistent with the microeconomic data. Since the appeal of these models is their microeconomic foundations, consistency across data sets is a prerequisite for further investigation.

The rest of the paper is organized as follows. It first presents a model of an individual's optimal purchase of durables and nondurables under fixed costs of adjustment that is thoroughly discussed in Beaulieu (1991). The model predicts that the important variable for analysis is the ratio of the durable stock to nondurable consumption. Individual ratios are distributed between two fixed bands; changes in this distribution drive aggregate dynamics. After discussing the importance of the cross-sectional distribution, the paper estimates this distribution with data on individuals' durable holdings and nondurable purchases. The results are consistent with our intuition and with previous findings. The paper then turns to macroeconomics. The microeconomic model is aggregated, and an empirical model similar to Caballero (1990) is estimated. Several of the parameters from the micro study are used in the estimation, linking the two empirical sections. The macroeconomic model explains 56.7% of the aggregate ratio of durables to nondurables. More importantly, the two estimation procedures yield the same estimates for the amount of idiosyncratic risk, implying consistency between the microeconomic model and macroeconomic model. The final section concludes.

2. The Microeconomic Model

Similar to the model in Grossman–Laroque (1990), assume a consumer solves

$$\max_{\substack{K_{\tau_1}, K_{\tau_2}, \ldots, \{X(t)\} \\ \tau_1, \tau_2, \ldots, \{C(t)\}}} E_t \int_0^\infty e^{-\delta t} \frac{1}{1-A} (K_t^{1-A} + bC_t^{1-A}) dt$$

$$\text{s.t.} \quad dQ_t = -\alpha K_t + r_f(Q_t - K_t) + X_t(\mu dt + \sigma dw^i(t)) - C_t \qquad t \in [0, \tau)$$

$$dK_t = -\alpha K_t \qquad t \in [0, \tau)$$

$$Q_t - \lambda K_t \geq 0, \tag{1}$$

where K_t is the durable stock; C_t is nondurable consumption which is continuously controllable; Q_t is total wealth, and X_t is holdings of the risky asset. The assumption of additive separability between durables and nondurables is important for aggregation; it means, however, that except for a level effect, nondurable consumption behaves as in Merton (1971).[1] The consumer discounts the future at rate δ, and the durable stock depreciates at rate α. The risky asset earns on average an excess return μ with an instantaneous standard deviation equal to σ. The risk-free rate is r_f and $w^i(t)$ is a standard brownian motion whose correlation properties across i are discussed below. When a consumer changes the durable stock at time τ, the consumer sells the old stock, receiving only $(1-\lambda)K_{\tau_-}$. Because the consumer loses λK_{τ_-} at the time of the update, total wealth just after the purchase, Q_{τ_+}, equals total wealth just prior to the purchase less a fraction of the prior durable stock, $Q_{\tau_-} - \lambda K_{\tau_-}$. This jump in total wealth implies that the consumer does not continuously adjust K_t.

[1] Beaulieu (1991) contains an extended discussion of the effects on C_t when the assumption of additive separability is relaxed.

The solution of a more general problem which allows for interactions between durables and nondurables is described in Beaulieu (1991). Through a change in variables similar to Karatzas *et al.* (1986) one can reduce the problem to a system of six nonlinear equations. This technique allows faster and more exact numerical simulations that makes comparative static exercises easier. The method also highlights alternative interpretations of the first-order conditions.

The random variable, Z_t^i, used to solve the resulting Bellman equation, is equal to the ratio of the durable stock to nondurable consumption times $b^{\frac{1}{A}}$ and evolves according to

$$\begin{aligned}\frac{dZ_t^i}{Z_t^i} &= -\alpha dt + \frac{1}{A}\left(\delta - r_f + \tfrac{1}{2}\frac{\mu^2}{\sigma^2}\frac{1-A}{A}\right)dt - \frac{\mu}{A\sigma}dw^i(t) \\ &\equiv \kappa dt + \varsigma dw^i(t),\end{aligned} \tag{2}$$

when the durable stock is not controlled. There are three bands L, C and U. When Z_t^i hits either L or U, the old durable good is sold; a new durable good is bought, and Z_t^i is moved to C where C is between L and U. The initial value Z_0^i is given as a function of the initial durable stock and total wealth.

Optimal nondurable consumption is given by

$$\frac{dC_t^i}{C_t^i} = -\frac{1}{A}\left(\delta - r_f - \tfrac{1}{2}\frac{\mu^2}{\sigma^2}\frac{1+A}{A}\right)dt + \frac{1}{A}\frac{\mu}{\sigma}dw^i(t), \tag{3}$$

which is continuous at the time of adjustment. These results can be understood via the results in Duffie–Skiadas (1991). Given complete markets, marginal utility of nondurable consumption grows in expectation at a rate equal to the difference between the discount rate and the riskless interest rate. Otherwise, marginal utility is a martingale, a result similar to Hall (1978) in a different context. Inverting marginal utility using Itô's lemma gives (3). Marginal utility is continuous at the adjustment point because durables and nondurables enter utility additively separable. The bands and the optimal amount of adjustment come from the envelope condition that the marginal utility of nondurable consumption equals the marginal value of wealth. The timing and amount of adjustment satisfy the conditions of optimality and continuity of the indirect marginal utility of wealth. Solutions such as Karatzas *et al.* (1986) and Beaulieu (1991) can be understood in this light.

Even though it is not the subject of analysis, adding nondurable consumption to the problem is important because other models of durable goods with fixed costs normalize the durable stock by wealth (Bertola–Caballero, 1990b; Caballero, 1990; Eberly, 1990; and Grossman–Laroque, 1990). Measures of wealth, however, must include estimates of permanent income, requiring an additional model whose specification is difficult at best when long time series are available. When there are few time series observations, the model must be simple. Moreover, measuring total wealth requires detailed information on asset holdings. Using nondurable consumption as the measure of permanent income, however, bypasses these problems by relying on individuals to measure their permanent income, limiting the required modeling and widening the number of useful data sets.[2] The addition of nondurable consumption also allows one to consider rel-

[2] Cochrane (1990) finds that aggregate nondurable consumption can be nearly thought of as the stochastic trend component of output, thus passing a weak form of the permanent income hypothesis. The complete behavior of nondurable consumption, however, seems

ative price effects.

It is assumed that each individual in the economy shares the same parameter values and chooses durables and nondurables optimally. Issues such as borrowing constraints found elsewhere are not considered here. Shocks to wealth, however, are assumed to have an idiosyncratic component discussed later. The immediate goal is to estimate important parameters such as L, C, U, the drift rate of Z_t and its variance within the bands and to test the explanatory powers of the model for both microeconomic and macroeconomic data. The final goal is to test whether the results from the two data sets are consistent. To accomplish this, one has to understand how individuals are distributed within the band interval $[L, U]$.

3. Cross-Sectional Distribution

Each individual solving their individual problems will at any point in time hold some K_t^i and consume some C_t^i so that the ratio $Z_t^i = K_t^i/C_t^i$ is located between L and U. At every point in time there will be some density $f(y,t)$ that summarizes the location of each individual's Z_t^i. Modeling changes in this cross-sectional distribution is important because it holds the key to modeling changes in the aggregate ratio K_t^{Ag}/C_t^{Ag}. The amount of durables bought in the economy in a unit of time is related to the mass near the adjustment bands L and U. Through time this distribution changes for two reasons. First, aggregate shocks move everyone in the same direction. For instance a positive aggregate wealth shock shifts the distribution to the left as a positive wealth shock increases C^i

excessively smooth compared to movements in income. See Deaton (1987) for a useful summary.

for everyone but only changes K^i for those individuals close to L. Second, in the absence of aggregate shocks, $f(y,t)$ converges to a steady state $f(y)$.

Examination of the cross-sectional distribution also allows for a richer set of dynamics. Similar aggregate shocks yield different impulse responses for durable goods depending on the density of the distribution near the adjustment bands. The cross-sectional distribution also implies that shocks will have long lasting effects, and the order of the shocks matter. For instance suppose that the initial distribution has a large mass over a small interval near L and no mass near U. Now consider a positive aggregate shock and a negative shock. If the positive shock occurs first, then the mass near L hits L and is moved to C, meaning a large amount of durables is bought. The negative shock then moves this mass now around C to the right. On the other hand, if the negative shock hits first, the distribution moves to the right first, then back to the left with the positive shock. The sum effect is that only a small amount of durables is bought.

Two previous papers call attention to the importance of the cross-sectional distribution in explaining aggregate movements in durable good purchases. Eberly (1990) considers a panel data set from the Survey of Consumer Finances in 1983 and 1986 and analyzes the decisions of individuals in updating their holdings of automobiles. Using her data set, she measures the ratio of total wealth to the value of automobiles in both 1983 and 1986. Given the work of Grossman–Laroque (1990) this ratio also lies between two points and is adjusted to the middle when it hits one of the bands. She calculates a smoothed estimate of the cross-sectional density in 1983 and compares the estimate to the theoretical steady-state density based on an assumed set of parameters. A goodness-of-fit test does not reject the similarity of the two. She then finds that the aggregate shocks in the middle 1980s can explain the differences in the densities she measures in 1983 and 1986.

Caballero (1990) differs from Eberly (1990) in that Caballero does not consider microeconomic data. Instead he develops a mathematical model of changes in the cross-sectional distribution given movements in aggregate permanent income. With this model he estimates the band width and speed of movement within the bands that best explains movements in the aggregate holdings of automobiles and furniture and furnishings. His model explains 81% of the deviations from the permanent income path for automobile expenditures and 92% of the deviations for furniture purchases.

Table 1 reproduces the bands, drift and standard deviations from Caballero (1990) and Eberly (1990). Caballero reports l, c, u, m, and s directly, assuming $c = 0$ and $l = -u$. Eberly on the other hand does not estimate m and s. Instead she reports the fraction of her sample that hits l, that is, sells a more valuable automobile and buys a cheaper one, and she reports the expected time between purchases (see Table 3). From these statistics and the bands, one can infer m and s.[3] The table shows that Caballero estimates a larger band width than Eberly. When an individual hits u, the consumer purchases a new

[3] As is discussed below, this probability and expected time are functions of the bands, drift and standard deviations.. The expected time is evaluated at c. The probability of hitting l is evaluated over all of (l, u) using the steady state density as a measure. This procedure may yield misleading results if her sample in 1983 is far from the steady state. She reports, however, that they are close.

automobile whose value is 2.78 times the value of the old while Eberly estimates this number to be 1.96. For furniture the number is much higher, 6.16. That Eberly estimates a lower band l much closer to c than u is to c calls into question the symmetry Caballero imposes on the model for computational ease. Caballero and Eberly estimate nearly the same drift, though Caballero calculates a much higher standard deviation.

Table 1: Bands from Other Studies – Annual Values

	l	c	u	$\frac{K^l_{\tau_+}}{K_{\tau_-}}$	$\frac{K^u_{\tau_+}}{K_{\tau_-}}$	m	s
Caballero – Cars	-1.021	.000	1.021	.360	2.776	.178	.400
Caballero – Furnit.	-1.818	.000	1.818	.162	6.160	.177	.400
Eberly	-.051	.000	.673	.950	1.960	.173	.127

The next two sections consider estimating parameters of the model of the previous section using microeconomic data and macroeconomic data respectively. The first section differs from Eberly (1990) in that I estimate the bands and speed of adjustment based on the steady-state density. She estimates the bands by observing directly her variable for people just before and after their purchases. The second section differs from Caballero (1990) in that it uses these microeconomic estimates for some of the parameters, making the estimation based on aggregate data less taxing. Otherwise it is similar in spirit though different in the detailed application. Both sections differ from both previous studies in that the following is based on the microeconomic model illustrated in the preceding section. This model uses nondurable consumption to measure permanent income instead of auxiliary estimates of total wealth. These differences and others will be clearer once the applications are described.

4. Microeconomic Testing

As a first pass to understanding the cross-sectional distribution, I consider the durable holdings and nondurable consumption of a set of families at one point in time. The 1985 Consumer Expenditure Survey contains data on appliance holdings and purchases, motor vehicle holdings and purchases, and expenditures on various nondurable goods for 1,965 families in the Fourth Quarter of 1985. Appliance holdings are valued at either the price paid for them if bought recently or at an average price of the purchase of similar used appliances. Motor vehicles are either valued at the purchase price inflated to 1985 dollars and depreciated at 15% per year or at an average price paid for used cars of similar makes and years. The sum of the value of appliances and motor vehicles is the value of the durable stock for each individual family. Two different sets of durable stocks are used according to whether the value of cars and light trucks uses the adjusted purchase price or the used market price when both are available. This definition of durables covers on average 58% of durable purchases.

Nondurable expenditures include purchases of food, alcohol beverages, apparel, housing operations, personal care, reading and tobacco products. A second value of nondurable expenditures adds purchases of gasoline, motor oil and utilities. Including these three covers on average 52% of nondurable consumption plus services other than housing. Four different values of the ratio are available;

the Data Appendix describes the creation of this data set more fully.

To estimate a subset of the parameters, I assume that the distribution of the ratio is in steady state in 1985:4. The steady state distribution is the distribution such that in the absence of correlated shocks the distribution of individuals remains the same, even if a large amount of idiosyncratic shocks move individuals in the interval $[L, U]$. Outside of steady state, the actual distribution converges to the steady state distribution unless correlated shocks move it away. I use maximum likelihood treating the cross-sectional density of Z_t^i as a likelihood function. For what follows, it is easier to work with the log of the ratio; the steady state of the log is given in Proposition 1 .

Proposition 1

Let z_t^i be the log of the ratio of the durable stock to nondurable consumption where the dynamics of Z_t^i are given in (2). Let l, c and u denote the log of L, C and U respectively. The steady state density of z_t^i, $g(z)$, is given by:

$$g(z) = \begin{cases} \alpha_0 \left(e^{\nu z} - e^{\nu l}\right) & \text{if } l \leq z \leq c \\ \alpha_0 \beta_0 \left(e^{\nu z} - e^{\nu u}\right) & \text{if } c \leq z \leq u \\ 0 & \text{otherwise} \end{cases}$$

$$\beta_0 = \frac{e^{\nu l} - e^{\nu c}}{e^{\nu u} - e^{\nu c}}$$

$$\alpha_0 = -\frac{1}{(c - l)e^{\nu l} + \beta_0 (u - c) e^{\nu u}}$$

$$\nu = \frac{2\kappa}{\varsigma^2} - 1.$$

Proof:

The log of the ratio is a brownian motion controlled between l and u. When it hits either l or u it is moved to c. In between, z_t^i solves the stochastic differential equation

$$dz_t^i = (\kappa - \tfrac{1}{2}\varsigma^2)dt + \varsigma dw_t^i, \qquad z_t^i \in (l, u). \tag{4}$$

The steady-state density solves the appropriate forward equation which is done in the fifth section of the Technical Appendix. This density is also found in Caballero (1990), and Eberly (1989) obtains a similar density by different means. •

Suppose, however, that in 1985 for every individual i, z_t^i is measured with error. Specifically suppose

$$\bar{z}_t^i = z_t^i + \epsilon_t^i, \tag{5}$$

where ϵ_t^i is independent across i and is normally distributed with mean μ and variance σ_ϵ^2. The cross-sectional density of $\bar{z}_t^i$ is given in Proposition 2 .

Proposition 2

Suppose $\bar{z}_t^i$ is given as in (5) and z_t^i evolves as in Proposition 1 . Then the steady state density of $\bar{z}_t^i$ is

$$\begin{aligned}
\bar{g}(z) = \alpha_0\beta_0 e^{\nu(z-\mu)+\frac{\nu^2\sigma_\epsilon^2}{2}} &\left(\Phi(\frac{z-c-\mu}{\sigma_\epsilon}+\nu\sigma_\epsilon) - \Phi(\frac{z-u-\mu}{\sigma_\epsilon}+\nu\sigma_\epsilon)\right) \\
&- \alpha_0\beta_0 e^{\nu u}\left(\Phi(\frac{z-c-\mu}{\sigma_\epsilon}) - \Phi(\frac{z-u-\mu}{\sigma_\epsilon})\right) \\
&+ \alpha_0 e^{\nu(z-\mu)+\frac{\nu^2\sigma_\epsilon^2}{2}}\left(\Phi(\frac{z-l-\mu}{\sigma_\epsilon}+\nu\sigma_\epsilon) - \Phi(\frac{z-c-\mu}{\sigma_\epsilon}+\nu\sigma_\epsilon)\right) \\
&- \alpha_0 e^{\nu l}\left(\Phi(\frac{z-l-\mu}{\sigma_\epsilon}) - \Phi(\frac{z-c-\mu}{\sigma_\epsilon})\right).
\end{aligned}$$

The constants α_0, β_0 and ν are as in Proposition 1 , and $\Phi(\cdot)$ is the standard normal cumulative distribution function. Furthermore, there is some continuity in that as ϵ converges to a point mass at zero, $\bar{g}(z) \to g(z)$ pointwise, and as z_t^i converges to a point mass, $\bar{g}(z)$ converges to a normal density.

Proof:

The probability density function of the sum of two independent variables is the convolution of the densities of the two variables. This convolution can be written as above by noting that $g(z) \neq 0$ only over the range (l, u) and by completing the square. The rest of the proof is in the sixth section of the Technical Appendix. •

The density $\tilde{g}(z)$ is used for maximum likelihood estimates because it has several advantages over $g(z)$. First, and most importantly, $g(z)$ violates one assumption that ensures consistency of maximum likelihood. The parameters l and u define the range of observation for $g(z)$ whereas the range for $\tilde{g}(z)$ is $(-\infty, \infty)$. Second, because $g(z) = 0$ for all z outside of (l, u) means that $\hat{l}$ necessarily must be less than the minimum z in the sample and $\hat{u}$ must be greater than the maximum. Any error greatly effects these estimates whereas $\tilde{g}(z)$ is designed to account for measurement error. Third, given the specification in logs, ϵ_t^i can represent cross-sectional variation in the taste parameter b. Consumers may place different relative weights on durables versus nondurables. Fourth, that $g(z)$ has a nondifferentiable point at c makes numerical methods difficult in practice. Fifth, given the approximations needed to construct the durable stock and the incomplete measures of both durables and nondurables, the assumption of some error in K_t^i/C_t^i is natural.[4] And sixth, in principle, $\tilde{g}(z)$ can approximate $g(z)$ arbitrarily well when $\mu = 0$ and $\sigma_\epsilon \rightarrow 0$.

The maximum likelihood estimates for l, c, u, κ/ς^2 and σ_ϵ are reported

[4] The assumption that the errors are independent, however, is problematical if one believes that the error also represents deviations from steady state. In such a case the errors are correlated.

in the first half of Table 2 for the four ratios for the full sample. The mean μ is set to zero as it and the level of l, c and u are jointly unidentified. To see this, add some constant k to μ and subtract k from l, c and u; the density is unchanged. This underidentification also implies that one should not conduct hypothesis tests based on the location of (l, c, u); *i.e.* only $c - l$ and $u - c$ are identified. In addition, the table reports the value of the log likelihood assuming the ratio is distributed according to $\tilde{g}(z)$ and assuming that the ratio is simply normally distributed. A simple normal distribution assumes in effect that the ratio is all error or that the model does not explain the ratio. The mean and variance of Z_t is reported in the columns headed *Normal Distribution.*

From the table, one can see that the model is marginally successful in explaining the data. The estimated parameters L, C and U are well-ordered in that C is between L and U. The values of the log likelihood are somewhat larger than for the normal model, and the standard errors suggest that the three barriers are distinct. [5] In addition, for both series that use primarily net purchase prices to evaluate the automobile stock, C is surprisingly close to U, and the drift rate κ is estimated to be positive, where a negative number is expected.

To test the robustness of the results, I consider one amendment. As discussed earlier, one likely source of error is missing data from both nondurable and durable goods. Even though the model is homogeneous and thus should

[5] Comparisons of the log likelihoods can only be informal as twice the difference is not a valid likelihood ratio test. Assuming a null of normality means that $|\nu| = \infty$ and at least either l or u equal c, as is discussed in the proof of Proposition 2 . The log likelihood of a model whose parameters lie on the boundary of the parameter space is not normally distributed (Godfrey, 1988). The same applies to a Wald test. A typical LM test is unavailable because the hessian is not invertible under the null.

work equally well no matter the level of wealth, error is likely to be especially acute for those families in the tails of the distribution of wealth. A small amount of absolute error in either the numerator or the denominator greatly influences the ratio of durables to nondurables for poor families. Rich families are likely to have large amounts of durables not accounted for or valued incorrectly and a greater amount of nondurable consumption going to other goods not accounted for. To see if such an effect is present, I exclude from the sample the top 20% income earners and the bottom 20%.[6]

The results from this middle income sample are reported in the second half of Table 2, and they are encouraging. The log likelihoods are further from the normal log likelihood. The standard errors of the parameter estimates are smaller than their counterparts in the first half of the table, and the standard deviation of the error, σ_ϵ, is smaller. For used values $\hat{C}$ is further from $\hat{U}$, and for purchases with utilities, the drift rate is negative. Note that even these middle income sample estimates imply that the new durable stock is between 6 and 11 times larger than the old stock when people upgrade their stocks. These jumps are much larger than is found in Caballero (1990) for automobiles or in Eberly (1990), but is near the estimated jump for furniture in Caballero.

Figure 1 plots the kernel density estimate of the observed ratio using the ratio which employs used car values, includes utilities and uses only the middle 60% of income earners. The figure also plots three estimated densities. The

[6] Theoretically, total wealth is the measure that should be used, but it is unavailable. Income should be a good enough proxy for this purpose.

dashed line is the estimated $\bar{g}(z)$ which is the theoretical density that includes an error term. The solid rectangular shaped figure is the steady state density without error. The dotted figure is the normal density assuming that $\tilde{z}_t$ is normally distributed, that is, the model explains none of the observed ratio. The figure explains why the two likelihoods are very close, as the two calculated densities are similar. The function $\bar{g}(z)$, however, has fatter tails and a smaller peak, characteristics that are closer to the observed ratio.

To further evaluate these results, consider Table 3 which contains a couple statistics that facilitate comparisons across studies. The first statistic $E[\tau]$ is the expected time to hit one of the barriers, either l or u. The second p_d is the probability that an individual's wealth decreases sufficiently so that the durable stock must be sold to raise cash. Of course, a new durable good is bought, but this is much smaller than the old one. These statistics can be evaluated at a point in the interval of (l, u) such as c or over the whole interval using some probability distribution over (l, u). A natural candidate to use as a measure over (l, u) is the steady state density developed in Proposition 1 . The formulae for these statistics evaluated at any point are functions that are the solutions to a couple of simple differential equations involving the drift and variance of the stochastic process and boundary conditions at l and u (Karlin–Taylor, 1981).

A few observations emerge from the examination of Table 3. First, Caballero and Eberly agree that the unconditional probability of having to sell one's durable to raise cash is small. Implicitly, he estimates it to be .115 for automobiles and .030 for furniture while she finds it directly to be .04.[7] Their estimates for the expected time between purchases, however, differ somewhat as Caballero estimates values (4.663, 9.910) larger than the 2.5 years Eberly finds in her sample.

The last two thirds of the table report the values for the eight series considered in this paper. Since only m/s^2 is estimated, m and s are separately identified by setting $m = -.299 = 4 \times -.057$, where this value for m is used in the next section on aggregation. Only the ratio m/s^2 is needed to calculate the probability of hitting u, conditionally or unconditionally, while both are needed for $E[\tau]$. The statistics show more graphically why an estimate of c close to u in the CES is problematical.[8] The probability of hitting u is counterfactually too large, especially so for the series that use primarily reported purchase prices.

The last third of the table reports the statistics for the series when the tails of the income distribution are excluded. Even for this sample, the series that use purchase prices is ill-behaved. The series that uses primarily used values for automobiles and light trucks and includes utilities, however, is more in line with previous studies. The probability of having to adjust the durable stock downwards is .087 from C and is .031 unconditionally. The expected times are also in line with Caballero and Eberly. One reason why the results that employ used car values are better than those that use purchase prices is that the

[7] As is discussed earlier, the probability Eberly estimates is not really an unconditional probability but is conditional on the distribution of individuals within the bands in 1983. The true unconditional probability will differ to the extent this distribution differs from the steady state distribution.

[8] Recall u here is having to sell to raise cash, resulting in a stock of durables smaller than the stock immediately preceding the adjustment, the opposite from Caballero and Eberly.

indices used to inflate to 1985:4 prices may poorly reflect the real value while used car prices do not suffer from this problem. The assumption of exponential depreciation, though still maintained, may also be problematical. For these reasons the results that employ used values for cars, nondurables with utilities and include only middle income families is emphasized in the rest of this paper.

5. Macroeconomic Testing

The link between data on individual holdings and aggregate data is the cross-sectional distribution of Z_t^i in $[L, U]$. Assuming that idiosyncratic shocks are uncorrelated across income groups, the ratio of the aggregate durable stock to aggregate nondurable consumption equals $b^{-\frac{1}{\lambda}}$ times the mean of individuals' Z_t^i. The rest of this section is dedicated to developing and testing this implication. The first subsection proves this relationship between the observed aggregate ratio and the individual Z_t^i's by aggregating the microeconomic model. The required assumptions needed for aggregation are highlighted. The second subsection calculates the ratio of the aggregate durable stock to aggregate nondurables less price effects. This removes the term $b^{-\frac{1}{\lambda}}$ and thus should equal the mean of Z_t^i across i. The subsequent subsection models movements in this mean by tracking the behavior of the cross-sectional density. The last subsection then compares the model of this mean to the observed aggregate ratio.

5.1 Aggregating the Optimal Ratio

Assume people can be represented on a continuum in two dimensions, Z and $x \in [0, 1]$. Denote this total space as I and the subspace where all x are the same as I_x. Optimality requires

$$\frac{K_t^i}{C_t^i} = b^{-\frac{1}{A}} Z_t^i \tag{6}$$

for each individual i. The problem of aggregation, however, is that one only observes

$$K_t^{Ag} = \int_I K_t^i di \qquad\qquad C_t^{Ag} = \int_I C_t^i di.$$

Assume there is some cross-sectional density of nondurable consumption $g(x,t)$ where $x \in [0,1]$, and write aggregate nondurable consumption

$$C_t^{Ag} = \int_0^1 g(x) C_t(x) dx,$$

where $C_t(x)$ represents the common consumption of people who are located at percentile x.

Given equation (6) and the cross-sectional density $g(\cdot)$

$$\begin{aligned} K_t(x) &= b^{-\frac{1}{A}} \int_{I_x} Z_t^{i_x} C_t(x) di \\ &= b^{-\frac{1}{A}} C_t(x) \int_{I_x} Z_t^{i_x} di. \end{aligned} \tag{7}$$

Let $f(y,t;x)$ denote the cross-sectional density of Z_t^i for those individuals in percentile x. Rewrite (7) as

$$K_t(x) = b^{-\frac{1}{A}} C_t(x) \int_L^U y f(y,t;x) dy.$$

Integrating over x gives the aggregate durable stock,

$$K_t^{Ag} = b^{-\frac{1}{A}} \int_0^1 g(x) C_t(x) \int_L^U y f(y,t;x) dy dx. \tag{8}$$

In order to employ this technology, assume that $f(y,t;x) = f(y,t)$ for all x. That is the cross-sectional density of Z_t^i within its bands is independent of the percentile consumption of i. This will be true if the initial density of individuals is the same across x and that idiosyncratic shocks to $dw^i(t)$ sum to zero for all x, which is assumed below.

The value of this assumption is given by plugging the assumption into (8).

$$\begin{aligned} K_t^{Ag} &= b^{-\frac{1}{A}} \int_0^1 g(x)C_t(x) \int_L^U yf(y,t)dydx \\ &= b^{-\frac{1}{A}} \int_0^1 g(x)C_t(x)dx \int_L^U yf(y,t)dy \\ \frac{K_t^{Ag}}{C_t^{Ag}} &= b^{-\frac{1}{A}} \int_L^U yf(y,t)dy. \end{aligned} \tag{9}$$

Equation (9) gives a precise prediction of the ratio of the aggregate stock of durables and nondurable consumption.

The parameter b in (9) represents preferences between durable and non-durable consumption. It also represents the relative price P_{ct} between durables and nondurables which has had important movements over the post-war period. The price of durable goods relative to nondurables on average has decreased. Furthermore, the trend in p_{ct} does not fully account for the trend in K_t^{Ag}/C_t^{Ag}.

The model in Beaulieu (1991) assumes that b is constant; he also shows that permanent changes in b do not change the optimal bands (L, C, U) in the Z space. These changes also do not affect Z_t itself. While the unimportance of permanent changes is comforting, two problems arise from the actual price process. First, b is stochastic and has stationary movements. The bands in Z space for a more general model which allows stationary movements in b will depend on b. If future prices are expected to be high, the consumer will update the durable stock early to take advantage of the low price relative to the future. Second, that b trends means the differential equation is no longer autonomous and explicit account of the dependence on time is required.

I use a simple model to accommodate these taste and price movements. I assume that the bands in Z space are fixed, and thus the bands for K_t^{Ag}/C_t^{Ag} move with b_t. This amounts to assuming the stationary movements in b_t are unimportant. I remove the effects of the long run trend and price movements through a cointegration relationship. That which remains is $\overline{Z}_t$ which is controlled between two fixed bands L and U.

5.2 Calculating the Aggregate Ratio Less Price Effects

To estimate $\overline{Z}_t$, assume that b can be represented as

$$b_t = b_0 e^{\vartheta_1 t} p_{ct}^{\vartheta_2},$$

where p_{ct} is the relative price of nondurable goods to durable goods and ϑ_i are a series of nuisance parameters. Taking logs of (9) leaves

$$\ln\left(\frac{K_t}{C_t}\right) = -\frac{1}{A}(\ln(b_0) + \vartheta_1 t + \vartheta_2 \ln(p_{ct})) + \ln(\overline{Z}_t). \tag{10}$$

As discussed above stationary movements in p_{ct} may exist. Such additional movements, however, should not asymptotically bias the coefficients in (10) if one takes (L, C, U) as fixed from the point of view of movements in $\ln(P_{ct})$ and if $\ln(P_{ct})$ and $\ln(K_t^{Ag}/C_t^{Ag})$ are integrated. Absent further aggregate shocks and changes in P_{ct}, $\ln(\overline{Z}_t)$ will converge to a constant as the cross-sectional density converges to its steady state (see the Technical Appendix), and thus, the effects of stationary price changes on $\ln(\overline{Z}_t)$ can be thought of as stationary.

Table 4 reports the results of integration and cointegration tests of

$$\ln(K_t^{Ag}/C_t^{Ag}) \quad \text{and} \quad \ln(P_{ct}).$$

The aggregate durable stock is calculated by assuming an initial stock and adding durable purchases. A depreciation rate of 15% per year is used. [9] The tests are estimated both statically and dynamically by including lags of the changes in the dependent variable. An examination of the table indicates that one cannot reject the hypothesis of integration for $\ln(P_{ct})$. On the other hand $\ln(K_t^{Ag}/C_t^{Ag})$ may not be integrated as the test that includes the three important lags yields a statistic significant at 2.5%. Clearly, 16 lags uses too much degrees of freedom and must be discounted, while no lags is not enough. The test for cointegration suggests that if $\ln(K_t^{Ag}/C_t^{Ag})$ is integrated, K_t^{Ag}/C_t^{Ag} and P_{ct} are cointegrated as both of the preferred statistics are significant at least at the .05% level.

Table 5 describes the results from estimating (10) assuming the two are cointegrated. Both a static and dynamic specification using 2 leads and 8 lags (Stock–Watson, 1989) is estimated. After estimating the nuisance parameters in (10), $\overline{Z}_t$ is found by plugging in the estimated parameters and subtracting from the dependent variable. The table indicates that the effect of P_{ct} on K_t^{Ag}/C_t^{Ag} is small at best.

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[9] The Data Appendix describes the data and the construction of the durable stock in more detail.

Figure 2 illustrates the movements in $\overline{Z}_t$ with the NBER recession dates plotted on the time scale. One important effect consistent with the model is the decline in $\overline{Z}_t$ after recessions. Negative aggregate shocks to wealth lead to an immediate increase in $\overline{Z}_t$ as the cross-sectional density of individuals shifts to the right. Eventually, K_t decreases through depreciation and lack of purchases, moving Z_t back to the left.

5.3 Estimating Movements in the Cross-Sectional Distribution

Given a set of bands L, C and U, values for the drift rate and standard deviation, and an initial density $f(y,0)$, a second estimate of $\int_L^U yf(y,t)dy$ can be obtained; call it $\widehat{\overline{Z}}_t$. First, the process for $f(y,t)$ without aggregate shocks must be developed; the strategy is the same as in Caballero (1990) and Bertola–Caballero (1990a). Assume first, there are no aggregate shocks to Z_t^i, and write

$$dZ_t^i = mZ_t^i dt + sZ_t^i dw^i(t). \tag{11}$$

From Beaulieu (1991), the individual Z_t^i's follow (11) when $Z_t^i \in (L,U)$. When Z_t^i hits L or U, Z_t^i immediately moves to C. Let $f(y,t)$ denote the cross-sectional density of Z_t^i at time t, and assume for the moment that there are no aggregate shocks to the individual Z_t^i's. Theorem 1 gives $f(y,t)$.

Theorem 1 :

Let the Z_t^i process be described as in (11). *Then $f(y,t)$, the cross-sectional density of Z_t, solves*

$$f_t(y,t) = \tfrac{1}{2}s^2y^2f_{yy}(y,t) + (2s^2 - m)yf_y(y,t) + (s^2 - m)f(y,t), \tag{12}$$

in the intervals (L, C) *and* (C, U) *separately. The boundary conditions are*

$$f(L,t) = f(U,t) = 0$$

$$f(C^+, y) = f(C^-, t)$$

$$C^2(f_y(C^+,t) - f_y(C^-,t)) = U^2 f_y(U,t) - L^2 f_y(L,t).$$

Furthermore the solution to (12) *is*

$$f(y,t) = \sum_{k=0}^{\infty} A_k e^{\psi_k t} H(y, \psi_k),$$

where ψ_k, A_k *and* $H(z, \psi_k)$ *are given in the Technical Appendix.*

Proof:

See the Technical Appendix.

Caballero (1990) develops the process for the density of $\ln(Z_t)$ using a discrete time approximation and passing to the limit. The technique in the Appendix uses the forward equation for an instantaneous return process with a random return to the interior and takes the limit as the return is to C with probability one. Both techniques yield the same process, but an advantage of the second approach is that it can accommodate more easily diversity in the return point.

The process for Z_t^i, however, contains aggregate shocks. To accommodate these shocks, write

$$dZ_t^i = (\overline{m} + m_t^{Ag}) Z_t^i dt + s Z_t^i dw^i(t) \qquad (13)$$

where m_t^{Ag} is the aggregate shock in period t and $\overline{m}$ is any drift term additional to the average of m_t^{Ag}. Given an initial density $f(y, t-1)$, the procedure is to approximate $f(y,t)$ by assuming that m_t^{Ag} is deterministically realized through the period. That is, the drift of the Z_t^i process in period t, m_t, equals $\overline{m} + m_t^{Ag}$. Of course next period the drift will likely equal something else as $m_t^{Ag} \neq m_{t+1}^{Ag}$. This assumption, which is also used in Bertola–Caballero (1990a) and Caballero (1990) amounts to ignoring the fact that changes in m_t^{Ag} are not predictable within any period $[t, t+dt]$. Bertola–Caballero discuss this simplifying assumption more fully.

To identify these aggregate shocks, one can use aggregate nondurable consumption. Recall the process for optimal nondurable consumption given by equation (3). Let

$$dw^i(t) = \sigma^{i'} dw^{i'}(t) + \sigma^{Ag} dw^{Ag}(t),$$

where $dw^{i'}(t)$ is idiosyncratic and therefore uncorrelated across i, and $dw^{Ag}(t)$ is aggregate and therefore common across i. The two brownian motions are assumed independent, and $\sigma^{i'\,2} + \sigma^{Ag\,2} = \sigma^2$. Replacing in (3) gives

$$\frac{dC_t^i}{C_t^i} = -\frac{1}{A}\left(\delta - r_f - \frac{1+A}{A}\frac{\mu^2}{2\sigma^2}\right)dt + \frac{1}{A}\frac{\mu}{\sigma}\left(\sigma^{i'} dw^{i'}(t) + \sigma^{Ag} dw^{Ag}(t)\right).$$

Integrate both sides over an index of i, I_x, where each i has the same consumption level, and assume that

$$\int_{I_x} dw^{i'}(t) di = 0.$$

That is, the idiosyncratic shocks wash out for all I_x. This assumption assumes that such shocks as changes in the tax code which differentially affect different wealth percentiles are unimportant.

To use this assumption, multiply both sides by $g(x) C^{i_x}(t)$ and integrate over x giving

$$\int_0^1 g(x)dC_t^{i_x}dx = \left(-\frac{1}{A}\left(\delta - r_f - \frac{1+A}{A}\frac{\mu^2}{2\sigma^2}\right)dt + \frac{\mu\sigma^{Ag}}{A\sigma}dw^{Ag}(t)\right) \cdot \int_0^1 g(x)C_t^{i_x}dx$$

$$\text{or} \quad \frac{\mu\sigma^{Ag}}{A\sigma}dw^{Ag}(t) = \frac{dC_t^{Ag}}{C_t^{Ag}} + \frac{1}{A}\left(\delta - r_f - \frac{1+A}{A}\frac{\mu^2}{2\sigma^2}\right)dt.$$

The process for Z_t^i is

$$\begin{aligned}\frac{dZ_t^i}{Z_t^i} &= -\alpha dt + \frac{1}{A}\left(\delta - r_f + \tfrac{1}{2}\frac{\mu^2}{\sigma^2}\frac{1-A}{A}\right)dt - \frac{\mu}{A\sigma}\left(\sigma^{i'}dw^{i'}(t) + \sigma^{Ag}dw^{Ag}(t)\right) \\ &= -\alpha dt + \tfrac{1}{2}\frac{\mu^2}{\sigma^2 A^2}dt - \frac{dC_t^{Ag}}{C_t^{Ag}} - \frac{\mu\sigma^{i'}}{A\sigma}dw^{i'}(t).\end{aligned} \tag{14}$$

One additional complication is that changes in relative prices lead to movements in nondurable consumption beyond increases in wealth. Since movements in b do not change Z_t or K_t immediately, equation (6) implies that the growth rate in nondurable consumption equals that which is due to the growth rate in wealth plus $\frac{1}{A}db_t$.[10] This effect on C_t is opposite from the effect of b_t on K_t/C_t. Using the notation from section *5.2*, the wealth effect on Z_t is $\frac{dC_t}{C_t} + \nu_1 dt + \nu_2\frac{dP_{ct}}{P_{ct}}$. Equation (14) is modified to

$$\begin{aligned}\frac{dZ_t^i}{Z_t^i} &= \overline{m}dt - \frac{dC_t^{Ag}}{C_t^{Ag}} - \nu_1 dt - \nu_2\frac{dP_{ct}}{P_{ct}} - s dw^{i'}(t) \\ &\equiv m_t - s dw^{i'}(t).\end{aligned} \tag{15}$$

To find the value of $\overline{m}$, let α equal 3.75% or 15% per year, the value used in the creation of $\overline{Z}_t$, and note that

$$\tfrac{1}{2}\frac{\mu^2}{\sigma^2 A^2} = \frac{\overline{\frac{dC}{C}} + \nu_1 + \nu_2\frac{dP_{ct}}{P_{ct}} - (r_f - \delta)/A}{1+A}. \tag{16}$$

[10] Consistent with the assumptions in previous sections, a potential quadratic term is dropped. In effect I assume the shock to b_t is continuously and evenly realized through the unit of time.

The mean quarterly growth rate in C_t from 1947:2–1990:2 is 0.0078 while the mean quarterly growth rate of the relative price from 1950:2 1990:2 is 0.0041. The values of ν_1 and ν_2 are from Table 4. Using annual values for r_f and δ of .03 and .01 and a value of A of 2 yields a value for $\frac{1}{2}\frac{\mu^2}{\sigma^2 A^2} = .0030$ and a value for $\overline{m} = -.0345$. Note that the affect of the assumed values for A, δ and r_f will be small for reasonable parameters, as most of $\overline{m}$ is $-\alpha$.

For each set of parameters $(L, C, U, \overline{m}, s)$ the estimation of $\widehat{\overline{Z}}_t$ is as follows. First, one needs an initial cross-sectional density, $f(y, 0)$. This initial density is assumed to be the steady-state density for the exponential of z_t^i estimated in the previous section. The mean drift of Z_1^i, m_1, is calculated as

$$m_1 = \overline{m} - \frac{dC_1'^{Ag}}{C_1'^{Ag}} - \nu_1 - \nu_2 \frac{dP_{c1}}{P_{c1}}. \tag{17}$$

Given $f(y, 0)$ and m_1, $f(y, 1)$ is calculated as in Theorem 1 . Then,

$$\widehat{\overline{Z}}_1 = \int_L^U y f(y, 1) dy. \tag{18}$$

The process is iterated forward through t as the newly calculated cross-sectional density is used as the initial density for the next period. For a set $(L, C, U, \overline{m}, s)$ this process gives a vector of generated values $\widehat{\overline{Z}}_t$.

An important assumption in the development of the process for the cross-sectional density is the existence of idiosyncratic shocks. Given that the microeconomic model is based on a portfolio policy, idiosyncratic shocks make little sense. Grossman–Laroque (1990) show that CAPM obtains in this model, and thus, all individuals should hold the same market portfolio. If one interprets idiosyncratic shocks as undiversifiable labor income, then the assumption that all income is generated from tradable assets is not true. In either case, the microeconomic model does not accommodate idiosyncratic shocks, and thus, the empirical implementation is only loosely based on the model.

Three considerations, however, argue that the empirical model should allow for the existence of idiosyncratic shocks even if these shocks are not fully developed in the model. First, in principle the empirical tests can estimate s, the standard deviation of the idiosyncratic component of wealth, to be zero. Second, idiosyncratic shocks to wealth must exist since individual consumption is not perfectly correlated with aggregate shocks. How important these shocks are is open to debate (Cochrane, 1991; Heaton–Lucas, 1991; Lucas, 1991; Mace, 1991) and certainly warrants further exploration. Their inclusion, however, seems reasonable. Third, idiosyncratic taste shocks in b_t may also be adequately described in this model as similar to idiosyncratic wealth shocks.

A second interpretation of the idiosyncratic shocks is that they represent stochastic depreciation rates as in Reider (1991). Suppose

$$dK_t = -\alpha_t + \sigma_k dw_k^i(t), \tag{19}$$

where $dw_k^i(t)$ is an idiosyncratic brownian motion increment. As Appendix C shows, the qualitative solution to the problem with stochastic depreciation rates as in (19) is unchanged, though substantially complicated. Therefore, as the problem is written, the idiosyncratic shocks can also come from (19).

5.4 Macroeconomic Results

To generate a theoretical series, $\widehat{\overline{Z}}_t$, based on the technique of the previous subsection, I use the band parameters L, C and U estimated from the CES data set and the value of $\overline{m}$ as calculated in the previous subsection. I estimate s by minimizing the sum of squares between the observed aggregate ratio less price effects and the generated series. One problem, however, is that these bands are truly identified only up to a constant multiple. Recall that one can interpret any multiple W of the three bands in the CES data set as part of the taste/relative price parameter b. This multiple can also be thought of as the exponential of the mean of the log error in the data set. Moreover, the multiple b is removed from the observed $\overline{Z}_t$ series. No estimated parameter should be based on an arbitrary multiple of the bands

A simplifying fact is that the cross-sectional density is homogeneous of degree one in L, C and U. One can calculate $\widehat{\overline{Z}}_t$ as in (18) by dividing the bands L, C, and U by some constant W and by multiplying the distribution by W. The estimation procedure takes advantage of this homogeneity by normalizing the estimated band values by $U - L$ and optimizing on the width also. Formally, the estimation problem is

$$\min_{s,W} \sum_{t=1}^{T} \left([\overline{Z}_t - \overline{Z}] - W [\widehat{\overline{Z}}_t(s) - \widehat{\overline{Z}}(s)] \right)^2, \tag{20}$$

where $\overline{Z}_t$ is calculated as in (10). The values for ν_i to construct $\overline{Z}_t$ are reported in the *Dynamic* column of Table 4. The series $\widehat{\overline{Z}}_t$ is calculated as in (18). The bands for Z_t are $L/(U - L)$, $C/(U - L)$, and $U/(U - L)$ where the values for L,

C and U are the Used/Util/Middle values in Table 2. To minimize the effects of the assumed value for the initial density, the first 16 values of $\widehat{\overline{Z}_t}$ are dropped.

The results from the estimation equation (20) are reported in the first third of Table 6, underneath the heading *Excludes Real Interest Rate Effects.* The first row (Aggregate) reports the estimates; the second (CES) reports analogous results from the CES data set. Since only m/s^2 is identified in the CES data set, s is identified by assuming a value for m. This value is calculated as in (17) where dC_t/C_t and dP_t/P_t are set to their values in 1985:4; these are respectively .0225 and .0121. The value for m is -.0573 giving a value of s equal to .1788. The third row (Difference) reports the difference between the aggregate estimates and those implied in the CES data set. Standard errors of each are in parentheses. The first column reports the estimated width; the second reports the estimated value for idiosyncratic risk s. The third column reports a t-statistic testing whether the estimated s equals .20, the implied quarterly value of s that Caballero (1990) finds. Caballero's standard errors are not included in the t-statistic. The p-value for the two sided test is reported in braces. The fourth column reports a t-statistic testing whether the difference equals zero, while the last column reports the estimated R^2.

Three findings from the table are important. First, the model explains a great deal of the observed $\overline{Z}_t$ as the R^2 is 56.7%. Second, the estimated amount of idiosyncratic risk is close to what Caballero (1990) estimates, though a traditional t-statistic rejects s equal to .20. Third, and most importantly, the estimated s in the macro data set is nearly the same as the estimated s in the micro data set. The third row (Difference) shows that the difference between the two is nearly zero. Moreover, the t-statistic, testing whether the difference is zero, is insignificant at any conceivable level of significance.

Figure 3 plots both the observed series and the generated $\widehat{\overline{Z}}_t$. The figure suggests that the model performs very well in the 50's and 60's. The ratio in the 70's, however, tends to be higher than the model predicts, and the ratio in the mid 80's tends to be lower.

One consideration the model ignores is the important movements in the real interest rate over the post WW II period. Comparative static exercises with the model in Beaulieu (1991) suggests that movements in r_{ft} have important effects on the bands. A complete model of optimizing behavior with stochastic real interest rates is unavailable, instead I use the previous model with movements in U, C and L. To keep the model simple, these bands move so that U/C and C/L remain constant. Their levels move contemporaneously with movements in the real interest rate.

To model these movements I include the real interest rate in the cointegrating relationship between K_t/C_t and p_{ct}, removing the effect of r_{ft} on $\overline{Z}_t$. The results from this new cointegrating equation are reported in Table 7. I then subtract the change in the real interest rate times the coefficient on r_{ft} from the drift rate in the simulation. The bands moving up in this space are like Z_t moving down with the bands constant. Because of the shortened sample, I exclude only the first 8 observations, leaving as the sample period 1964:4–1989:2.

One important consideration in the use of real interest rates is that they are extremely volatile relative to the implications of economic models. These additional movements yield temporary values of ex ante real interest rates anywhere between 9.69% and -3.14%. These large momentary swings probably do not affect the movements in K_t/C_t as much as the fact that the level of r_{ft} in the 1980s was much higher than in the 1970s. To smooth r_{ft} without imposing too much structure, I imagine that the economic relationship is better characterized as depending on the expectation of r_{ft} as of $t-1$. The Data Appendix describes the construction of $E_{t-1}r_{ft}$ in greater detail; the following is a brief sketch.

The nominal interest rate used is the three month Treasury bill rate. Subtracted from the nominal rate is the expected inflation rate in durable prices over the next three months. This expectation is calculating by describing the process of inflation as an ARMA(4,4) with money supply growth and the spread between 6 and 3 month nominal rate included. The expectation of r_{ft} is calculated as is implied in the model of Barro–Sala-i-Martin (1990). It is given as the fitted values from the regression of r_{ft} on one lag each of r_{ft}, real government surplus, the ratio of gross investment to GNP, real crude oil prices, the real rate of return on equities, and a trend. Durable prices are used as a deflator. As is evidenced from Figure A2 in the Appendix, $E_{t-1}r_{ft}$ follows r_{ft} rather closely but with dampened oscillations.

The estimation procedure is the same as before. It uses the same band parameters, but calculates $\overline{m}$ slightly different. It modifies equation (16) by including the average real rate times its coefficient in Table 7. The resulting value for $\overline{m}$ is $-.0343$. The results from this estimation procedure are reported in Table 6 underneath the heading *Includes Real Interest Rate Effects.* The R^2 is larger than before, 63.1%. The estimated standard deviation of idiosyncratic risk is consistent with its previous estimate, but this s is slightly larger, .182. As before, one cannot reject the difference between the estimated s and that implied in the CES data set. Figure 4 plots the observed and generated series. The figure implies that including real interest rate effects helps explain the important drop in the ratio in the mid 80's, but the rise in the ratio in the late 70's is still puzzling. One potential explanation for this 70's effect may be the large increase in real oil prices, requiring a shift to more fuel efficient automobiles.

To test whether the larger R^2 in the procedure that includes real interest rates is from the different sample, I recalculate the generated series without real interest rates, using the same parameters that the first macroeconomic calculation used. The sample series is the same as the sample period that includes real rate effects, that is 1964:4–1989:2.

The results, reported in Table 6, are generally in line with the previous results. The estimated standard deviation of idiosyncratic risk is .1628, which is slightly larger than that previously estimated that excludes real interest rate effects but is smaller than that estimated that includes such effects. The R^2, .4557, is smaller, implying that the higher R^2 estimated when real interest rate effects are included is not a result of the smaller sample. This result is expected given that Figure 3 suggests that the best fit is in the early years and important deviations come in the late 70's and early 80's. Figure 5 plots the observed and generated series. It confirms the results suggested in Table 6; the change in sample does not affect the estimated results.

6. Conclusion

Linking microeconomic and macroeconomic data is important for several reasons. Many economists, searching for "micro-foundations" only believe those aggregate models that are based on individual optimization. Models based on microeconomic behavior can give sharp predictions for the effects of policy interventions and may perform better at medium-term forecasting. They can also explain breaks in ARIMA/trend time series models that are often found to be important in aggregate data.

The present paper uses a description of individual optimizing behavior to model the ratio of durables to nondurable consumption for both cross-sectional microeconomic data and aggregate macroeconomic data. The theoretical link between the two data sets is the cross-sectional distribution. The microeconomic results are reasonable and are in line with previous research and prior expectations. The aggregate results are encouraging in that a large fraction of the residual K_t^{Ag}/C_t^{Ag} series is explained by movements in the cross-sectional density of individuals within their bands.

The most important empirical finding, however, is that the microeconomic and macroeconomic models yield the same estimates for the degree of idiosyncratic risk, s. Not only is the difference between the two estimates statistically insignificant, but the difference is economically not meaningful. So often in the past, aggregate models based on optimizing behavior have been rejected because they have failed such a test. That this model yields consistent results encourages the search for micro foundations along the lines outlined in this paper.

Three assumptions which make the modeling and empirical procedures easier, however, seem unsatisfying. On the microeconomic side, one identifying assumption is that the distribution of individuals within their bands in 1985:4 is in steady state. This may be far from the case considering the sharp movements in nondurable consumption in the early 1980s. One solution is to reverse the procedure, that is, beginning with some initial distribution in the past, model changes in the distribution as is done in the macroeconomic section, picking parameters that produce a distribution closest to the observed CES distribution in 1985. This procedure uses macroeconomic data to explain microeconomic observations.

On the macroeconomic side, the present is a partial equilibrium model. The real interest rates and relative price processes are taken as given, and even then, their effects are modeled simply. To completely understand movements in durable goods, a general equilibrium model linking prices and quantities is needed. For example, a simple extension would be to close the real interest rate process by setting the net supply of risk-free assets equal to zero. With individuals distributed within their bands, such a model would produce complex dynamics in the real interest rate and describe a realistic economy where some people borrow and some lend.

On the modeling side, the optimal band policy is based on a model of portfolio diversification. This leads to stable bands in the space of Z_t. Nondiversifiable income, however, may mean that these bands move. Aggregate shocks then come in two guises. Besides the common movements in wealth, measured by the growth rate of nondurable aggregate consumption, aggregate shocks may also mean common movements in the bands. One example is that the bands may widen in recessions as individual and aggregate uncertainty increases. Such a model may be what scribes have in mind when they explain recessions as people putting off the purchase of big-ticket items. Investigating these considerations is the object of future research.

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Table 2: Steady State Estimates Using Microeconomic Data

Full Sample	*Full Model Distribution*					*Normal Distribution*		*Likelihood*	
	L	C	U	κ/ς^2	σ_ϵ	μ	σ^2	$\mathcal{L}_F$	$\mathcal{L}_N$
Used Market,	2.670	10.382	16.965	-.424	.767	10.062	114.193	-2510.49	-2511.57
No Utilities	(1.641)	(6.693)	(21.195)	(1.749)	(.174)	(.058)	(3.643)		
Used Market,	1.550	8.436	12.649	-1.269	.628	6.026	29.164	-2341.22	-2343.19
Utilties	(2.251)	(2.356)	(7.984)	(.983)	(.064)	(.015)	(.930)		
Purchases,	1.586	9.783	10.418	1.554	.838	10.183	124.918	-2593.30	-2595.07
No Utilities	(1.289)	(10.402)	(10.814)	(1.033)	(.043)	(.064)	(3.985)		
Purchases,	1.329	9.454	9.298	-.203	.686	6.131	33.675	-2451.93	-2454.12
Utilities	(.295)	(4.220)	(3.989)	(1.098)	(.071)	(.017)	(1.074)		
Middle Income Sample	*Full Model Distribution*					*Normal Distribution*		*Likelihood*	
	L	C	U	κ/ς^2	σ_ϵ	μ	σ^2	$\mathcal{L}_F$	$\mathcal{L}_N$
Used Market,	—	—	—	—	—	10.404	94.353	—	-1304.10
No Utilities						(.090)	(4.126)		
Used Market,	1.448	9.793	19.340	-1.791	.521	6.153	27.264	-1214.18	-1217.96
Utilties	(.185)	(1.469)	(13.711)	(1.266)	(.058)	(.026)	(1.192)		
Purchases,	1.784	17.533	18.052	-.128	.672	10.430	103.471	-1356.52	-1359.07
No Utilities	(.435)	(17.437)	(18.264)	(.813)	(.072)	(.099)	(4.524)		
Purchases,	1.268	10.612	10.817	-.427	.628	6.213	31.932	-1289.37	-1291.77
Utilities	(.296)	(5.430)	(5.694)	(1.124)	(.068)	(.031)	(1.396)		

Notes:

1. L, C, and U are exponentiated. μ and σ^2 are means and variances of the ratios not logged. 2. $\mathcal{L}_F$ is the log likelihood assuming the log of the ratio is distributed $\tilde{g}(z)$, while $\mathcal{L}_N$ is the log likelihood assuming it is normally distributed. The full sample contains 1,965 observations. The middle income sample contains 1,046. 3. Missing values indicate nonconvergence.

Table 3 : Summary Statistics for Controlled Processes

	From c		*Unconditional*	
	$E[\tau]$	p_d	$E[\tau]$	p_d
Caballero – Cars	4.663	.093	3.837	.115
Caballero – Furniture	9.910	.018	7.896	.030
Eberly	2.500	.334	1.976	.040
Used/Not/All	1.331	.570	1.233	.362
	(6.600)	(1.016)	(5.340)	(.723)
Used/With/All	4.146	.354	3.360	.131
	(5.612)	(.798)	(1.874)	(.231)
Prch/Not/All	—	.999	—	.978
	—	(.020)	—	(.081)
Prch/With/All	—	—	—	—
	—	—	—	—
Used/Not/Middle	—	—	—	—
	—	—	—	—
Used/With/Middle	7.355	.087	4.683	.031
	(2.717)	(.271)	(.872)	(.084)
Prch/Not/Middle	.041	.983	.534	.547
	(2.731)	(1.114)	(3.425)	(.483)
Prch/With/Middle	.097	.981	1.484	.427
	(4.952)	(.983)	(3.691)	(.479)

Notes:

1. $E[\tau]$ is the expected time to hit one of the two barriers in years. p_d is the probability one hits the barrier that results in a *reduction* in the durable stock. *From c* evaluates these two statistics conditional on starting from the return point c. *Unconditional* evaluates them over the entire range (l, u) using the steady state probability density as the measure. Standard errors are in parentheses.
2. For Caballero the statistics are calculated given the reported bands, the drift and standard deviation. Eberly directly reports the bands, the conditional expected time and unconditional probability (see text for further discussion). The drift and standard deviation are calculated to match these statistics. The other statistics are then calculated from these derived values. The last eight are based on the results from Table 2 . There the bands and m/s^2 are reported. The drift m and standard deviation s are identified by matching assuming $m = -.229$ which is the annualized drift used in the aggregation section ($4 \times -.057$). The Prch/Not/All expected times are unavailable since m/s^2 is estimated to be positive. Prch/With/All is unavailable because C is estimated to be larger than U.

Table 4: Integration and Cointegration Tests

p_{ct}				K_t/C_t				Cointg.			
Period	*Lags*	*t-stat*	*P-Val*	*Period*	*Lags*	*t-stat*	*P-Val*	*Period*	*Lags*	*t-stat*	*P-Val*
50:2–90:2	0	-1.02	—	49:2–90:2	0	-5.36	.01	50:2–90:2	0	-3.48	.125
54:2–90:2	16	-1.74	—	53:2–90:2	16	-2.51	—	54:3–90:2	16	-3.61	.10
54:2–90:2	3	-1.93	—	52:2–90:2	3	-3.82	.025	51:4–90:2	5	-4.34	.025
54:2–90:2	4	-1.91	—	—	—	—	—	54:3–90:2	6	-3.90	.05

Notes:
1. A constant and trend are included in the regression equation.
2. Lags refer to the total number of differences of the dependent variable (for integration tests) or right-hand side variable (for cointegration tests) included in the regression. All lower lags are not necessarily included in the regression. The third row includes those lags with t-stats whose absolute value is greater than 1.6 in the regression of the second row which includes the first 16 lags. The fourth includes those with t-stats greater than 1.4. *P-Val* indicates the p-value for the one-sided regression test. For integration tests, the values are from Dickey-Fuller (1979), and for the cointegration tests, the values are from Phillips-Ouliaris (1990).

Table 5: Removing Price Effects in K_t^A/C_t^A

	Static Regressions			Dynamic Regressions		
	Const.	*Trend*	p_{ct}	*Const.*	*Trend*	p_{ct}
Coeff.	1.249	$.578 \cdot 10^{-3}$	-.165	1.290	$.342 \cdot 10^{-3}$	$-.980 \cdot 10^{-1}$
Std. Err	.017	$.121 \cdot 10^{-3}$	$.308 \cdot 10^{-1}$	.020	$.140 \cdot 10^{-3}$	$.351 \cdot 10^{-1}$
T-Stat	72.569	4.773	-5.351	64.022	2.437	-2.795

Notes:
1. The test regresses the log of K_t/C_t on a constant, linear trend, and the log of p_{ct}. Dynamic regressions also include two leads and eight lags of the change in the log of p_{ct}. The sample period is 1952:2–1989:2. Static regressions include no leads and lags, and the sample period is 1950:1-1990:2.

Table 6: Aggregate Results

	$U-L$	s	t-stat $s=.20$	t-stat $Dff\ s=0$	R^2
	Excludes Real Interest Rate Effects, 1956-1989				
Aggregate	3.7639	.1551	-73.8569		.5673
	(.2883)	(.0006)	{.0000}		
CES	17.8925	.1788	-.3348		
	(13.7673)	(.0632)	{.7379}		
Difference	-14.1286	-.0238		-.3764	
	(12.7704)	(.0632)		{.7067}	
	Includes Real Interest Rate Effects, 1964-1989				
Aggregate	3.8746	.1816	-30.4059		.6308
	(.3025)	(.0006)	{.0000}		
CES	17.8925	.1802	-.3101		
	(13.7673)	(.0637)	{.7565}		
Difference	-14.0179	.0014		.0214	
	(13.7707)	(.0637)		{.9829}	
	Excludes Real Interest Rate Effects, 1964-1989				
Aggregate	3.7882	.1628	-49.5674		.4557
	(.4226)	(.0007)	{.0000}		
CES	17.8925	.1788	-.3348		
	(13.7673)	(.0632)	{.7379}		
Difference	-14.1043	-.0160		-.2532	
	(13.7738)	(.0632)		{.8002}	

Notes:

1. The table is divided in three parts. The top third reports results that exclude the expected real interest rate from the cointegrating relation and does not include changes in the rate as aggregate shocks. It uses data from from 1956:2–1989:2. The second third does include real interest rate effects in both the cointegrating relation and in tracking the cross-sectional density. It uses data from 1964:4–1989:2. For comparison, the last third excludes real interest rate effects but uses the same sample as the middle third table does.
2. Standard errors of estimates are in parentheses; p-values for two-sided tests for whether the t-statistic equals zero are in braces.

Table 7 : Calculating $\tilde{e}_t$ with r_{ft}				
	Static Regressions			
	Const.	*Trend*	p_{ct}	r_{ft}
Coeff.	-.14	$.12 \cdot 10^{-2}$	-.21	$-.90 \cdot 10^{-2}$
Std. Err	.13	$.95 \cdot 10^{-3}$	.19	$.17 \cdot 10^{-2}$
T-Stat	-1.04	1.30	-1.12	-5.15
	Dynamic Regressions			
	Const.	*Trend*	p_{ct}	r_{ft}
Coeff.	-.49	$.35 \cdot 10^{-2}$	-.73	$-.14 \cdot 10^{-1}$
Std. Err	.13	$.94 \cdot 10^{-3}$	.19	$.16 \cdot 10^{-2}$
T-Stat	-3.75	3.78	-3.84	-8.84

Notes:

1. The test regresses the log of K_t/C_t on a constant, linear trend, the log of p_{ct}, and of r_{ft}. Dynamic regressions also include two leads and eight lags of the change in the log of p_{ct}. The sample period is 1962:3–1988:4. Static regressions include no leads and lags, and the sample period is 1962:3-1989:4.

Fig. 1: Estimated Densities

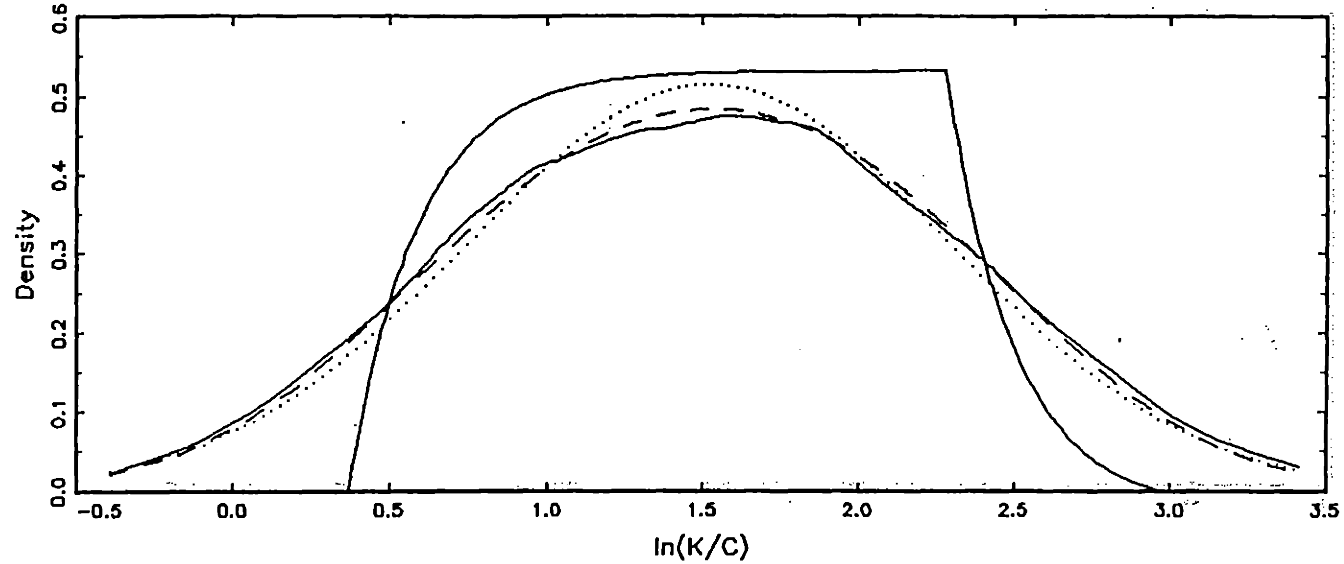


Fig. 2: Observed Z

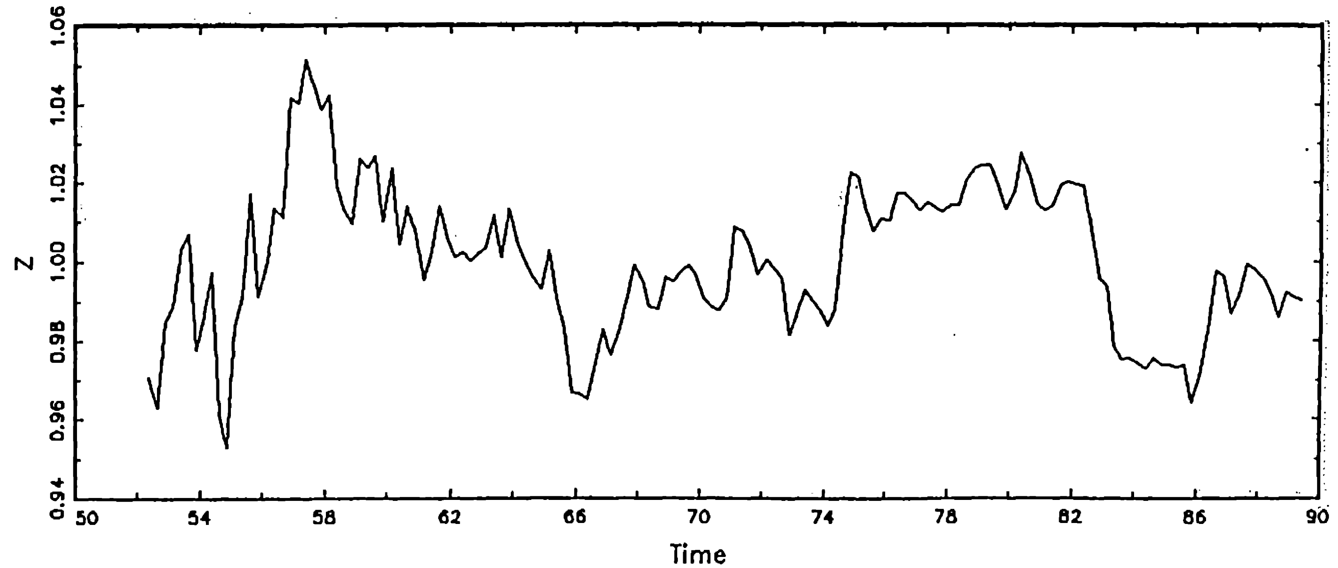


Fig. 3: Estimated Z

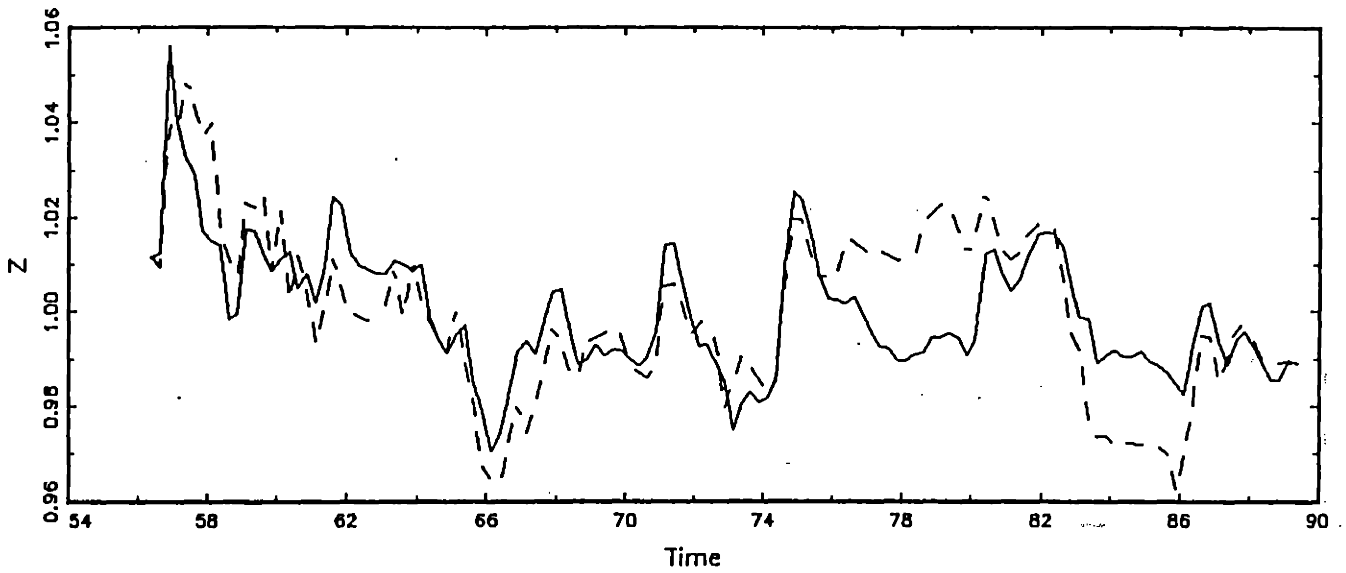


Fig. 4: Estimated Z w/ Rf

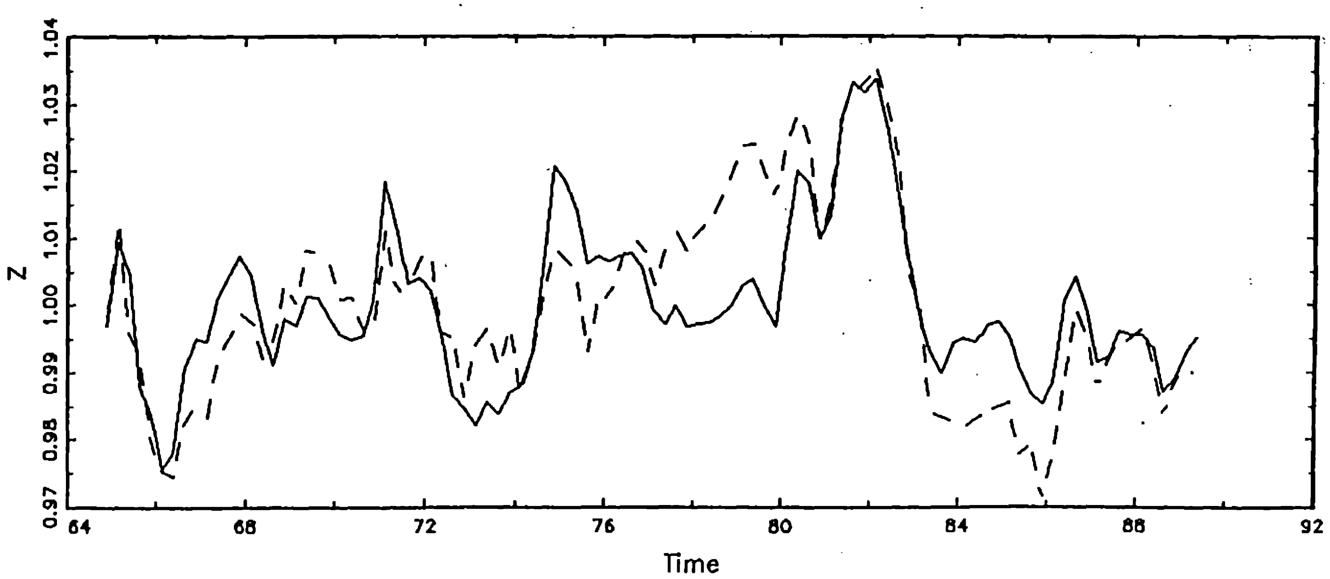
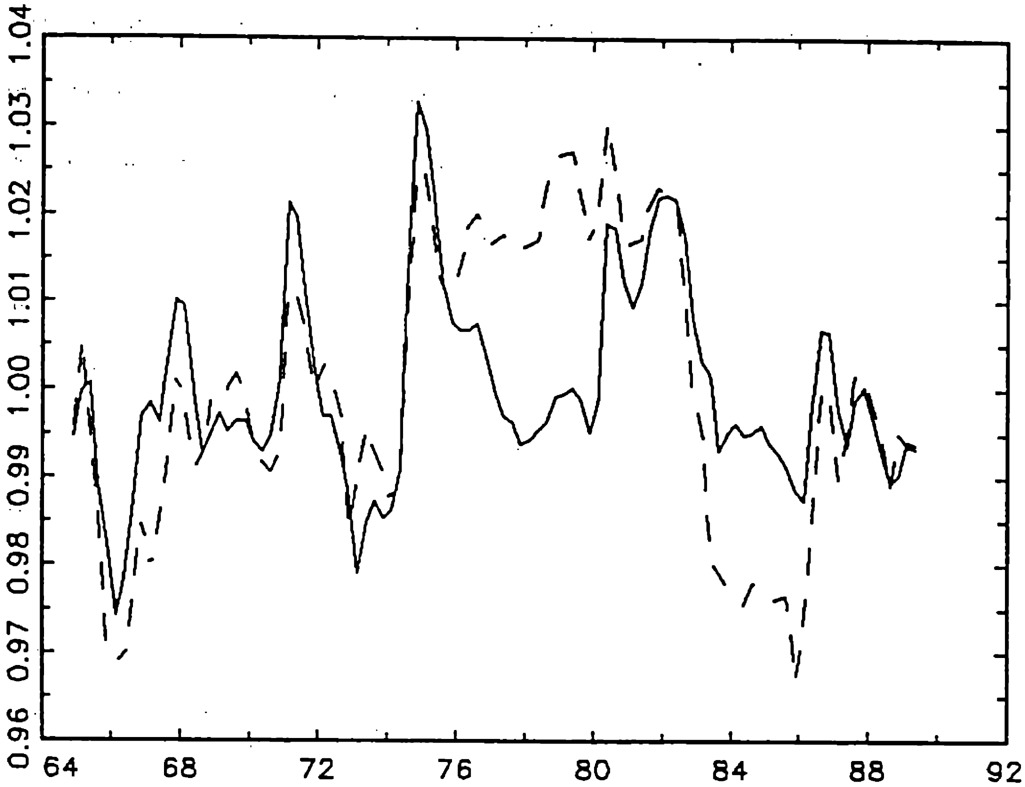


Fig. 5: Estimated Z w/o Rf



APPENDIX A: TECHNICAL APPENDIX

DERIVATION OF THE CROSS-SECTIONAL DISTRIBUTIONS

This appendix develops the cross-sectional densities discussed in the paper. The first section proves the first half of Theorem 1, that is, that the cross-sectional density solves that particular differential equation. The second section then solves the differential equation, while the third section provides a lemma useful for the interpretation of some of the developments in the solution of the forward equation. The fourth section develops the steady state distribution of Z_t showing the connection between the general solution to the cross-sectional density of section two and the steady-state distribution. The fifth section proves Proposition 1, the steady state distribution of z_t, while the sixth section finishes the proof of Proposition 2.

A.1. Derivation of Forward Equation

Define the instantaneous return process, Z_t on the interval $[L, U]$ such that when Z_t hits L or U, it moves to a random point in the interior immediately according to the cdf $p(\bar{y})$. In other words

$$p(\bar{y}) = Prob\,[Z_{t+} \leq \bar{y}], \quad \text{where} \quad Z_{t-} = L, U; \qquad p(L) = 0;\, p(U) = 1.$$

Denote by $a(y)$ one-half of the instantaneous variance and by $b(y)$ the instantaneous drift. For the Z_t process used here $a(y) = \frac{1}{2}s^2y^2$, $b(y) = my$. Feller (1954) (see also Bharucha-Reid, 1960) shows that for any open interval Ω in (L, U), $f(y, t)$ solves the forward equation for the instantaneous return process.

This equation is

$$\frac{\partial}{\partial t}\int_{\Omega} f(y,t)dy = \int_{\Omega} d_y\Big(\frac{\partial[a(y)f(y,t)]}{\partial y} - b(y)f(y,t)$$
$$+ p(y)\lim_{y\to L}[\frac{\partial[a(y)f(y,t)]}{\partial y} - b(y)f(y,t)] \qquad (21)$$
$$- p(y)\lim_{y\to U}[\frac{\partial[a(y)f(y,t)]}{\partial y} - b(y)f(y,t)]\Big).$$

The term on the left equals the change in mass that lies in Ω. It equals the first two terms, the net flux through the two boundaries $\Omega = (\omega_1, \omega_2)$, plus the second two terms, that which is moved from L and U into Ω. The total that is instantaneously moved anywhere is the net flux at the boundaries, the limit terms in the brackets. A proportion equal to $p(\omega_2) - p(\omega_1)$ ends up in Ω. If $p(y)$ is differentiable in Ω then one can neglect the integrals and the equation becomes a partial differential equation.

Feller (1954) also shows that for an accessible boundary, r_j, the boundary condition for $f(\cdot)$ is such that

$$\lim_{y\to r_j} a(y)e^{\int^y b(x)/a(x)dx} f(y,t) = 0.$$

It is easily verified that the terms other than $f(\cdot)$ are bounded and nonzero giving

$$f(L,t) = f(U,t) = 0, \qquad (22)$$

the first two boundary conditions.

For the problem considered above, $p(y) = 0$ for $y < C$ and $p(y) = 1$ for $y > C$. That is, $p(y)$ collapses to a degenerate cdf. Applying this to any Ω to the left of C, the equation becomes the standard forward equation. That is

$$f_t(y,t) = \frac{\partial}{\partial y}[\frac{\partial[a(y)f(y,t)]}{\partial y} - b(y)f(y,t)]. \qquad (23)$$

For any Ω to the right of C, $p(y) = 1$. Extra terms are involved on the right-hand side, but these are constant with respect to y. Therefore, once again the forward equation (21) reduces to (23). Thus, the basic forward equation then holds over the interval (L, C) and (C, U), with the added proviso that two boundary conditions at C are needed.

Given the continuity of the y process, the first boundary condition at C is

$$f(C^+, t) = f(C^-, t). \tag{24}$$

To find the second, let Ω collapse to (C^-, C^+). Equation (21) becomes

$$\begin{aligned} &\frac{\partial}{\partial t}\int_\Omega f(y,t)dy \\ &= \lim_{y \to C+}\left[\frac{\partial[a(y)f(y,t)]}{\partial y} - b(y)f(y,t)\right] - \lim_{y \to C-}\left[\frac{\partial[a(y)f(y,t)]}{\partial y} - b(y)f(y,t)\right] \\ &+ \lim_{y \to L}\left[\frac{\partial[a(y)f(y,t)]}{\partial y} - b(y)f(y,t)\right] - \lim_{y \to U}\left[\frac{\partial[a(y)f(y,t)]}{\partial y} - b(y)f(y,t)\right]. \end{aligned} \tag{25}$$

Plugging in the values for $a(y)$ and $b(y)$, taking the appropriate limits and substituting (22) and (24) in (25), equation (25) becomes

$$0 = C^2(f_y(C^+, t) - f_y(C^-, t)) - (U^2 f_y(U, t) - L^2 f_y(L, t)). \tag{26}$$

Plugging the values for $a(y)$ and $b(y)$ in (23) and collecting equations (22), (24) and (26) finishes the proof. •

A.2. Solution to Forward Equation

Equation (13) of the paper can be solved using the method of the separation of variables. Write $f(y,t) = M(t)H(y)$ where

$$M'(t) = \psi M(t),$$

$$\psi H(y) = \tfrac{1}{2}s^2y^2H''(y) + (2s^2 - m)yH'(y) + (s^2 - m)H(y).$$

It is obvious that $M(t) = e^{\psi t}$. Plugging in $M'(t)H(y)$ for $f_t(y,t)$, it is easily seen that $H(y) = y^{\xi}$ where

$$\xi = -\tfrac{1}{2}(3 - 2m/s^2) \pm \tfrac{1}{2}\sqrt{(3 - 2m/s^2)^2 - 8(1 - m/s^2 - \psi/s^2)}. \tag{27}$$

Denote the two roots of (27) as ξ_+, ξ_-.

A. 2.1 Real Roots

Suppose first the roots are real. Write

$$H(y) = \begin{cases} A_1 y^{\xi_+} + A_2 y^{\xi_-}, & L \le y \le C; \\ A_3 y^{\xi_+} + A_4 y^{\xi_-}, & C \le y \le U. \end{cases}$$

Plugging in (22) implies

$$H(y) = \begin{cases} A_1 (y^{\xi_+} - L^{\xi_+ - \xi_-} y^{\xi_-}), & L \le y \le C; \\ A_3 (y^{\xi_+} - U^{\xi_+ - \xi_-} y^{\xi_-}), & C \le y \le U. \end{cases} \tag{28}$$

Plugging in (24) means

$$A_3 = \frac{C^{\xi_+} - L^{\xi_+ - \xi_-} C^{\xi_-}}{C^{\xi_+} - U^{\xi_+ - \xi_-} C^{\xi_-}} A_1.$$

On the other hand, (26) means

$$A_3 (\xi_+ C^{\xi_+ + 1} - \xi_- U^{\xi_+ - \xi_-} C^{\xi_- + 1} - (\xi_+ - \xi_-) U^{\xi_+ + 1})$$

$$= A_1 (\xi_+ C^{\xi_+ + 1} - \xi_- L^{\xi_+ - \xi_-} C^{\xi_- + 1} - (\xi_+ - \xi_-) L^{\xi_+ + 1}).$$

These last two conditions are compatible if and only if

$$(\xi_+ - \xi_-)(U^{\xi_+ - \xi_-} - L^{\xi_+ - \xi_-}) C^{\xi_+ + \xi_- + 1} - (\xi_+ - \xi_-)(U^{\xi_+ + 1} - L^{\xi_+ + 1}) C^{\xi_+}$$

$$- (\xi_+ - \xi_-)(U^{-1-\xi_-} - L^{-1-\xi_-}) U^{\xi_+ + 1} L^{\xi_+ + 1} C^{\xi_-} = 0.$$

Certainly, the last equation holds if $\xi_+ = \xi_-$, but equation (28) shows that this implies $H(y) = 0 \,\forall y$. It is easily verified, however, that the above equation also holds if ξ_+ or $\xi_- = -1$, which is true if and only if $\psi = 0$. In summary then, if the roots are real,

$$A_0 H(y, \psi_0 = 0) = \begin{cases} A_0 (y^{-1} - L^{-1-\xi} y^{\xi}), & L \le y \le C; \\ B_0 A_0 (y^{-1} - U^{-1-\xi} y^{\xi}), & C \le y \le U, \end{cases} \tag{29}$$

where

$$\xi = -2 + 2m/s^2$$

$$B_0 = \frac{1 - (C/L)^{1+\xi}}{1 - (C/U)^{1+\xi}}$$

$$\psi_0 = 0.$$

With $\psi_0 = 0$, A_0 is found by

$$\int_L^U A_0 H(y, \psi_0) dy = 1.$$

Doing the required integration

$$A_0 = \frac{1}{\ln(C/L) + B_0 \ln(U/C)}. \tag{30}$$

The calculation of A_0 also shows why $\psi_0 = 0$. When ψ is complex, as is shown in Section 3 below, Lemma A1, $\int_L^U H(y; \psi_k) dy = 0$. Thus, for all ψ_n real, the sum over n of $e^{\psi_n t} A_n H(y, \psi_n)$ must integrate to one for all t. But for $\psi_n \neq 0$ the constant A_n must depend on time for this to integrate to one for all t, contradicting the assumption that $A_n H(y, \psi_n)$ is independent of time.

A. 2.2 Complex Roots

Suppose now the roots are complex. Write $\xi_\pm = \phi \pm i\gamma$ where

$$\phi = -\tfrac{1}{2}(3 - 2m/s^2)$$

$$\gamma = \sqrt{2(1 - m/s^2 - \psi/s^2) - \phi^2}$$

$$\theta_y = \gamma \ln y.$$

Note that ξ complex means γ is real which requires that

$$\psi < -\frac{s^2}{8}\left(1 - \frac{2m}{s^2}\right)^2 \leq 0. \tag{31}$$

The implications of (31) are pursued below.

Plugging the definition for ξ into the solution of $H(\cdot)$ gives

$$A_n H(y) = \begin{cases} y^{\phi}(A_1 \cos\theta_y + A_2 \sin\theta_y), & L \le y \le C; \\ y^{\phi}(A_3 \cos\theta_y + A_4 \sin\theta_y), & C \le y \le U. \end{cases}$$

Plugging in (22) implies

$$H(y) = \begin{cases} A_1 y^{\phi}(\cos\theta_y - \cot\theta_L \sin\theta_y), & L \le y \le C; \\ A_3 y^{\phi}(\cos\theta_y - \cot\theta_U \sin\theta_y), & C \le y \le U. \end{cases}$$

Plugging in (24) means

$$A_3 = \frac{\cos\theta_C - \cot\theta_L \sin\theta_C}{\cos\theta_C - \cot\theta_U \sin\theta_C} A_1.$$

On the other hand, (26) means

$$\begin{aligned} &A_3\Big(\phi(\cos\theta_C - \cot\theta_U \sin\theta_C) - \gamma(\sin\theta_C + \cot\theta_U \cos\theta_C) \\ &\qquad + (U/C)^{1+\phi}\gamma\csc\theta_U\Big) \\ &= A_1\Big(\phi(\cos\theta_C - \cot\theta_L \sin\theta_C) - \gamma(\sin\theta_C + \cot\theta_L \cos\theta_C) \\ &\qquad + (L/C)^{1+\phi}\gamma\csc\theta_L\Big). \end{aligned} \tag{32}$$

These last two conditions are compatible if and only if

$$\sin(\gamma\ln(U/L)) - (U/C)^{1+\phi}\sin(\gamma\ln(C/L)) - (L/C)^{1+\phi}\sin(\gamma\ln(U/C)) = 0. \tag{33}$$

There are an infinite number of ψ_n that satisfy this equation.

In summary, then, for $k \ge 1$,

$$A_k H(y, \psi_k) = \begin{cases} A_k y^{\phi}(\cos(\gamma_k \ln y) - \cot(\gamma_k \ln L)\sin(\gamma_k \ln y)), & L \le y \le C; \\ B_k A_k y^{\phi}(\cos(\gamma_k \ln y) - \cot(\gamma_k \ln U)\sin(\gamma_k \ln y)), & C \le y \le U, \end{cases} \tag{34}$$

where

$$B_k = \frac{\cos(\gamma_k \ln C) - \cot(\gamma_k \ln L)\sin(\gamma_k \ln C)}{\cos(\gamma_k \ln C) - \cot(\gamma_k \ln U)\sin(\gamma_k \ln C)}. \tag{35}$$

The coefficients A_k are found through the initial condition that $f(y, 0)$ is given. Let $p(y) = f(y, 0) - A_0 H(y; \psi_0)$. Then

$$\sum_{n=1}^{\infty} A_n H(y; \psi_n) = p(y).$$

Under certain conditions on L, C and U, for example $U/C = C/L$, the problem satisfies the conditions of the Sturm-Liouville problem (Churchill, 1941), and one can appeal to the orthogonality conditions to solve for the coefficients A_k. Generally, however, these conditions are not met. The technique employed here is a simple shortcut. The coefficients A_k are found by approximating H_k as a vector and fitting $\sum_{k=0}^{K} A_k H(y, \psi_k)$ to $f(y, 0)$ by least squares. The upper limit of the sum, K, is chosen such that all the ψ_k that satisfy equation (33) are found where $\psi_k > -200$. Because of non-orthogonality this is only an approximation.

The final solution is just the sum of these particular solutions,

$$f(y, t) = \sum_{k=0}^{\infty} A_k e^{\psi_k t} H(y; \psi_k). \bullet \tag{36}$$

A.3. The Integral of $H(y, \psi_k)$

In finding A_0 in (30), the claim was made that $\int_L^U H(y, \psi_k) dy = 0$ for $k \geq 1$. Lemma A1 proves this claim.

Lemma A1

For all $k \geq 1$, $\int_L^U H(y, \psi_k) dy = 0$.

Proof:

Standard integral tables show that

$$\int e^{au} \sin(nu) du = \frac{e^{au}}{a^2 + n^2}(a \sin(nu) - n \cos(nu)) + C;$$
$$\int e^{au} \cos(nu) du = \frac{e^{au}}{a^2 + n^2}(a \cos(nu) + n \sin(nu)) + C.$$

Making a standard change of variables

$$\int y^{\phi} \sin(\gamma \ln(y))\, dy = \frac{y^{1+\phi}}{(1+\phi)^2+\gamma^2}\left[(1+\phi)\sin(\gamma \ln(y)) - \gamma \cos(\gamma \ln(y))\right];$$

$$\int y^{\phi} \cos(\gamma \ln(y))\, dy = \frac{y^{1+\phi}}{(1+\phi)^2+\gamma^2}\left[(1+\phi)\cos(\gamma \ln(y)) + \gamma \sin(\gamma \ln(y))\right].$$

Denote by θ_y, $\gamma \ln(y)$. Using (34) and the above

$$\begin{aligned}\frac{(1+\phi)^2+\gamma^2}{C^{1+\phi}} \int_L^U H(k,\psi_k) = {} & -(L/C)^{1+\phi}\gamma \csc\theta_L \\ & + (1+\phi)(\cos\theta_C - \cot\theta_L \sin\theta_C) + \gamma(\sin\theta_C + \cot\theta_L \cos\theta_C) \\ & - B_k(1+\phi)(\cos\theta_C - \cot\theta_U \sin\theta_C) + B_k\gamma(\sin\theta_C + \cot\theta_U \cos\theta_C) \\ & + B_k(U/C)^{1+\phi}\gamma\csc\theta_U . \end{aligned} \tag{37}$$

Equation (32) means

$$\begin{aligned} -B_k\gamma[\sin\theta_C + \cot\theta_U \cos\theta_C] + B_k(U/C)^{1+\phi}\gamma\csc\theta_U & \\ = {} & - B_k\phi[\cos\theta_C - \cot\theta_U \sin\theta_C] \\ & + \phi[\cos\theta_C - \cot\theta_L \sin\theta_C] \\ & - \gamma[\sin\theta_C + \cot\theta_L \cos\theta_C] + (L/C)^{1+\phi}\gamma\csc\theta_L . \end{aligned} \tag{38}$$

Replacing (38) in (37) gives

$$\begin{aligned}\frac{(1+\phi)^2+\gamma^2}{C^{1+\phi}} \int_L^U H(k,\psi_k) = {} & (1+\phi)(\cos\theta_C - \cot\theta_L \sin\theta_C) \\ & + \gamma(\sin\theta_C + \cot\theta_L \cos\theta_C) - (L/C)^{1+\phi}\gamma\csc\theta_L \\ & - B_k(1+\phi)(\cos\theta_C - \cot\theta_U \sin\theta_C) - B_k\phi(\cos\theta_C - \cot\theta_U \sin\theta_C) \\ & + \phi(\cos\theta_C - \cot\theta_L \sin\theta_C) - \gamma(\sin\theta_C + \cot\theta_L \cos\theta_C) \\ & + (L/C)^{1+\phi}\gamma\csc\theta_L \\ = {} & (1+2\phi)\left[\cos\theta_C - \cot\theta_L \sin\theta_C - B_k(\cos\theta_C - \cot\theta_U \sin\theta_C)\right]. \end{aligned} \tag{39}$$

But the definition of B_k in equation (35) implies the last line in (39) = 0. •

A.4. Steady State Distribution of $f(y,t)$

The steady state distribution is simply $A_0 H(y,\psi_0)$ as defined in equations (29) and (30). Moreover, as should be expected.

$$\lim_{t \to \infty} f(y,t) = A_0 H(y,\psi_0).$$

If no aggregate shocks hit the system. the distribution converges to the steady state.

To prove the first statement simply let $f(y,0) = A_0 H(y,\psi_0)$, that is, assume the distribution is initially in steady state. In the development of A_k, the function $p(y)$ is defined as $f(y,0) - A_0 H(y,\psi_0)$. In this case $p(y) = 0$, and this implies $A_k = 0 \, \forall k \geq 1$. If, $f(y,0) = A_0 H(y,\psi_0)$ then $f(y,t) = A_0 H(y,\psi_0) \, \forall t \geq 0$ as $A_0 H(\cdot)$ does not change through time. Thus, if the initial distribution is $A_0 H(y,\psi_0)$ and there are no aggregate shocks, the distribution will always be $A_0 H(y,\psi_0)$. This is the definition of a steady state. To prove the second statement rewrite (36) as

$$f(y,t) = A_0 H(y,\psi_0) + \sum_{k=1}^{\infty} A_k e^{\psi_k t} H(y;\psi_k).$$

From equation (31), $\psi_k < 0$, and thus, for all $k \geq 1$

$$\lim_{t \to \infty} A_k e^{\psi_k t} H(y,\psi_k) = 0. \bullet$$

Note that $A_0 H(y,\psi_0)$ solves (23) if one sets the left-hand side of (23), equal to $f_t(y,t)$, to zero.

Figure A1 plots the steady state distribution for a variety of values for s holding L, C, U, and m fixed. The specific values are $(2.363, 19.201, 31.160)$ for the bands and $m = -.525$ for three; $m = .525$ for the other. The values of s

chosen are 0, .654 and .327. For $s = \infty$ one can just set $m = 0$ and s can equal anything. The distribution is just as calculated as in equations (29) and (30).

For $s = 0$, however, one must take limits because ξ explodes. Let m be less than zero as it is in the figure. The analysis for $m > 0$ is just the mirror around C. For $m < 0$, $\xi \to -\infty$. Consider first $y > C$. Then from (29)

$$H(y) = B_0 A_0 (y^{-1} - U^{-1}(y/U)^{\xi}).$$

First off, $B_0 \to 0$, because $(C/L) > 1$ meaning $(C/L)^{\xi} \to 0$ while $(C/U) < 1$ giving $(C/U)^{\xi} \to \infty$. This also gives $A_0 = \frac{1}{\ln(C/L)}$ which is bounded. In the limit then $B_0 A_0 y^{-1} \to 0$. Multiplying the value for B_0 by the second term in parentheses leaves

$$A_0 \frac{1 - (C/L)^{1+\xi}}{1 - (C/U)^{1+\xi}} U^{-1-\xi} y^{\xi} \quad = \quad A_0 \frac{1 - \frac{C}{LU}\left(\frac{Cy}{LU}\right)^{\xi}}{1 - \frac{C}{U}\left(\frac{C}{U}\right)^{\xi}}.$$

Suppose first that $\frac{Cy}{LU} > 1$ then the expression converges to zero, and $H(y) = 0$ for $y > C$. Now suppose that $\frac{Cy}{LU} < 1$. The above limit problem satisfies the conditions of L'Hopital, and the limit of this expression as $\xi \to -\infty$ equals

$$\lim_{\xi \to -\infty} \frac{\frac{C}{LU}\left(\frac{Cy}{LU}\right)^{\xi} \ln\left(\frac{Cy}{LU}\right)}{\frac{C}{U}\left(\frac{C}{U}\right)^{\xi} \ln\left(\frac{C}{U}\right)} = \frac{\frac{1}{L}\left(\frac{y}{L}\right)^{\xi} \ln\left(\frac{Cy}{LU}\right)}{\ln\left(\frac{C}{U}\right)} = 0$$

because $\frac{y}{L} > 1$. In either case $H(y) = 0$ for $y > C$.

The case for $y < C$ is easier. From (29)

$$H(y) = A_0 (y^{-1} - L^{-1}(y/L)^{\xi}).$$

A_0 is given as above and $(y/L)^{\xi} \to 0$. This leaves

$$\lim_{s \to 0} H(y, \psi_0 = 0) = \begin{cases} y^{-1}/\ln(C/L), & L \le y \le C; \\ 0, & C \le y \le U, \end{cases} \quad (40)$$

Equation (40) implies that $\ln(Z)$ is uniformally distributed in (L, C), which is the steady state distribution for a variable controlled by a one-sided $S - s$ rule. As indicated in the figure, from the left. $f(y)$ converges to .025 as $y \to C$, while from the right, $f(y) = 0$.

A.5. Steady State Distribution of $g(y)$

The previous section proved that $H(y, \psi_0 = 0)$ is the steady state distribution of $f(y, t)$. It was noted that $H(y, \psi_0)$ solves the forward equation (23) when one replaces $f_t(y, t)$ with 0. The following does this calculation for $g(y)$.

Recall that z_t solves

$$dz_t = (\kappa - \tfrac{1}{2}\varsigma^2)dt + \varsigma dw(t), \qquad z_t \in (l, u).$$

When $z_t = l$ or u, z_t is immediately moved to c. Define $a(y) = \frac{1}{2}\varsigma^2$ and $b(y) = \kappa - \frac{1}{2}\varsigma^2$. Replacing these values in (23), (22), (24) and (25) and setting $g_t(y) = 0$ gives these systems of equations

$$0 = \tfrac{1}{2}\varsigma^2 g_{yy}(y) - (\kappa - \tfrac{1}{2}\varsigma^2)g_y(y)$$

$$0 = g(l) = g(u)$$

$$0 = g(c^+) - g(c^-)$$

$$0 = (g_y(c^+) - g_y(c^-)) - (g_y(u) - g_y(l)).$$

The solution to the differential equation is

$$g(y) = \begin{cases} \alpha_0(e^{\nu y} - \alpha_1) & \text{if } l \le y \le c \\ \alpha_0\beta_0(e^{\nu y} - \alpha_2) & \text{if } c \le y \le u, \end{cases}$$

where $\nu = \frac{2\kappa}{\varsigma^2} - 1$. The boundary conditions at l and u imply that $\alpha_1 = e^{\nu l}$ and $\alpha_2 = e^{\nu u}$, while the continuity at c means that

$$\beta_0 = \frac{e^{\nu l} - e^{\nu c}}{e^{\nu u} - e^{\nu c}}.$$

It is easily verified that the solution to the differential equation and these values for α_1, α_2 and β_0 also satisfy the last boundary condition. α_0 is calculated to ensure that $g(y_0)$ integrates to one. Thus,

$$g(z) = \begin{cases} \alpha_0(\epsilon^{\nu z} - \epsilon^{\nu l}) & \text{if } l \leq z \leq c \\ \alpha_0\beta_0(\epsilon^{\nu z} - \epsilon^{\nu u}) & \text{if } c \leq z \leq u \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned} \beta_0 &= \frac{\epsilon^{\nu l} - \epsilon^{\nu c}}{\epsilon^{\nu u} - \epsilon^{\nu c}} \\ \alpha_0 &= -\frac{1}{(c-l)\epsilon^{\nu l} + \beta_0(u-c)\epsilon^{\nu u}} \\ \nu &= \frac{2\kappa}{\varsigma^2} - 1, \end{aligned}$$

proving Proposition 1. •

A.6. Continuity of $\tilde{g}(y)$

This section proves the claim that as ϵ_{it} converges to a point mass, $\tilde{g}(z)$ converges pointwise to $g(z)$, and as z_t^i converges to a point mass, $\tilde{g}(z)$ converges to a normal distribution.

The first part of the claim is easy to see. The random variable ϵ_{it} converges to a point mass at its mean when σ_ϵ converges to zero. Set $\mu = 0$ in $\tilde{g}(z)$ for convenience, and let $\sigma_\epsilon \rightarrow 0$. Immediately one can write

$$e^{\nu z + \frac{\nu^2\sigma_\epsilon^2}{2}} = e^{\nu z} \qquad \text{and} \qquad \Phi(\frac{z-c}{\sigma_\epsilon} + \nu\sigma_\epsilon) \approx \Phi(\frac{z-c}{\sigma_\epsilon}).$$

α_0 and β_0 are unaffected. As $\sigma_\epsilon \rightarrow 0$,

$$\Phi(\frac{z-u}{\sigma_\epsilon}) \quad \rightarrow \quad \begin{cases} 1 & \text{if } z > c \\ \frac{1}{2} & \text{if } z = c \\ 0 & \text{if } z < c \end{cases} .$$

Repeating this limit for the other terms and plugging in $\tilde{g}(z)$ gives $g(z)$. Setting $z = c$, l, and u gives these special cases, giving pointwise convergence.

The second part of the claim is more involved. Write

$$\alpha_0 = \frac{\epsilon^{\nu c} - \epsilon^{\nu u}}{(c-l)(\epsilon^{\nu u} - \epsilon^{\nu c})\epsilon^{\nu l} + (u-c)(\epsilon^{\nu l} - \epsilon^{\nu c})\epsilon^{\nu u}}$$
$$\alpha_0\beta_0 = \frac{\epsilon^{\nu c} - \epsilon^{\nu l}}{(c-l)(\epsilon^{\nu u} - \epsilon^{\nu c})\epsilon^{\nu l} + (u-c)(\epsilon^{\nu l} - \epsilon^{\nu c})\epsilon^{\nu u}}.$$

The following claims are easily verified

$$\begin{aligned}
\lim_{\nu\to\infty} \alpha_0\beta_0\epsilon^{\nu u} &= -\frac{1}{u-c} \\
\lim_{\nu\to\infty} \alpha_0\beta_0\epsilon^{\nu c} &= 0 \\
\lim_{\nu\to\infty} \alpha_0\epsilon^{\nu c} &= \frac{1}{u-c} \\
\lim_{\nu\to\infty} \alpha_0\epsilon^{\nu l} &= 0.
\end{aligned} \tag{41}$$

First, let $\nu \to \infty$; the analysis for $\nu \to -\infty$ is the mirror image in (l,c). $g(z) \to U_{[c,u]}$ where $U_{[c,u]}$ is the uniform distribution in $[c,u]$.[1] Then let $c \to u$ which gives a point mass at u. To see the convergence of $g(z)$ rewrite $g(z)$ as

$$g(z) = \begin{cases} \alpha_0 e^{\nu c}(\epsilon^{\nu(z-c)} - e^{\nu(l-c)}) & \text{if } l \le z \le c \\ \alpha_0\beta_0\epsilon^{\nu u}(\epsilon^{\nu(z-u)} - 1) & \text{if } c \le z \le u \\ 0 & \text{otherwise.} \end{cases}$$

Using the above limits in (41)

$$\lim_{\nu\to\infty} g(z) = \begin{cases} \frac{1}{u-c} & \text{if } c \le z < u. \\ 0 & \text{otherwise.} \end{cases}$$

Then as $u \to c$, z converges to a point mass on c.

As for $\tilde{g}(z)$, abusing notation a bit, first note that

$$\begin{aligned}
&\lim_{\nu\to\infty} e^{\nu z + \frac{\nu^2\sigma_\epsilon^2}{2}}(\Phi(\frac{z-c}{\sigma_\epsilon} + \nu\sigma_\epsilon) - \Phi(\frac{z-u}{\sigma_\epsilon} + \nu\sigma_\epsilon)) \\
&= -\frac{1}{\nu\sigma}\frac{1}{\sqrt{2\pi}}(e^{-\frac{1}{2}(\frac{z-c}{\sigma})^2 + \nu c} - e^{-\frac{1}{2}(\frac{z-u}{\sigma})^2 + \nu u}).
\end{aligned} \tag{42}$$

The analogous result is also true for $(\Phi(\frac{z-l}{\sigma_\epsilon} + \nu\sigma_\epsilon) - \Phi(\frac{z-c}{\sigma_\epsilon} + \nu\sigma_\epsilon))$. Also note the same is true if $\nu \to -\infty$. Plugging (42) in the expression for $\tilde{g}(z)$ and using (41) gives

$$\lim_{\nu\to\infty} \tilde{g}(z) = \frac{1}{u-c}(\Phi(\frac{z-c}{\sigma_\epsilon}) - \Phi(\frac{z-u}{\sigma_\epsilon})).$$

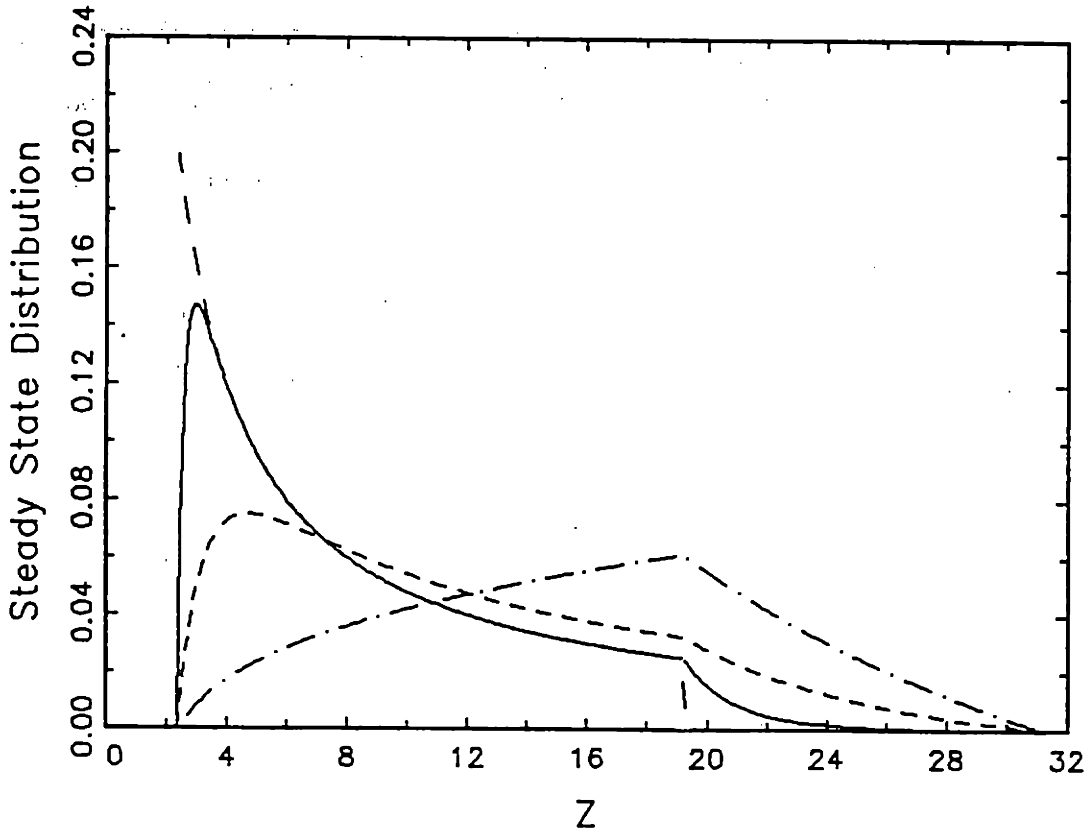
[1] The uniform distribution is the steady state distribution of a one-sided rule for z as is discussed in Section 4.

Now, letting $u \to c$ gives

$$\lim_{u \to c} \lim_{\nu \to \infty} \tilde{g}(z) = \frac{1}{\sigma_\epsilon \sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{z-u}{\sigma_\epsilon}\right)^2}.$$

which is the density of a normally distributed random variable with mean u and variance σ_ϵ^2. •

Fig. A1: Example Steady States



APPENDIX B: DATA SOURCES

The first section describes the creation of the data set that uses the Consumer Expenditure Survey of 1985. One subsection concerns the creation of the appliance data set, a second concerns the development of the motor vehicles data set, while a third the nondurable expenditure data set. The second section describes the aggregate data and the construction of the real interest rate.

B.1. Microeconomic Data Set

B. 1.1 Appliances

There are two data sets on appliances in the CES. One is called APA, and it contains expenditures on major appliances over the last twelve months. The second is called APL, and it contains minimal data on the stock of appliances. The APL has data on eleven appliances, electric stoves, gas stoves, microwaves, other cooking stoves, refrigerators, home freezers, built-in dishwashers, portable dishwashers, garbage disposals, clothes washers, and clothes dryers. The APA has these eleven as well as range hoods and combined major appliances.

The two data sets were merged and matching appliances were deleted. Newly bought appliances were valued at cost while the existing stock was valued at the average price for used purchases in the APA. This is adequate for all but garbage disposals for which no used purchases were made. To calculate the price for garbage disposals, the average markdown (average used/average new) for the other nine was calculated.[1] A weighted average of the markdown was then multiplied by the average new price for garbage disposals to get the used price. This is procedure is summarized in Table A1 . Purchase prices are not available for range hoods and combined appliances. These were set to the weighted average net purchase price, also found in Table A1 .

The values of all appliances for a particular family was summed to obtain a total value of appliances. The summary statistics are reported below. The first line summarizes appliances for each family for which data exists, the second line summarizes appliances for those families in the final sample.

Summary Statistics for Appliances

Obs	*Mean*	*Std Dev*	*Min*	*Max*
9,518	579.73	346.66	20.00	4,202.26
1,965	744.81	367.09	31.00	4202.26

B. 1.2 Automobiles

The relevant data set is OVB which contains data on personal holdings of motor vehicles. Besides automobiles and light trucks, the data set also contains listings on other vehicles. The method by which I value automobiles and light trucks differs from the other vehicles.

One method to evaluate the value of automobiles and light trucks is to use the reported purchase price. This price is deflated for inflation by using the proper deflator according to whether the automobile was a new domestic car,

[1] Other stoves is excluded because it has only one observation.

new foreign car, used car, or truck. These deflators are available from Citibase by dividing current dollar output by constant dollar output. The deflator is divided by the 1985:12 deflator so that the deflator in 1985:12 equals one, and the year chosen is based on the reported time of purchase. To account for wear and obsolescence, a second type of depreciation is needed. An annual depreciation rate of 15% is used where the base time is 1985:12.

A second method that allows one to evaluate the value for those automobiles that lack data on purchase prices is to use the value of used cars for that particular make. Data comes primarily from *The Red Book* (1979-1985) and *Older Car Red Book* (1969-1978), though a few used car valuations are from *Road & Track* or *The Standard Catalog of American Light Duty Trucks. The Red Books* contain the value in the Fourth Quarter of 1985 of used automobiles and light trucks of various makes and years down to 1969. For the most part, automobiles older than 1969 were priced at 1969 values. The OVB for the most part reports the make of the car, for instance a Dodge Diplomat. A problem, however, is that several values for a Diplomat are available according to body style, engine size and special versions. When useful, the transmission type and number of cylinders were used to identify the particular make of car. If the purchase price also was reported, that information was also used. When no other data can be used to pinpoint the exact make, a middle level or slightly lower level make was chosen to represent that particular model.

A second approximation is needed when the exact year is unknown. This happens because the purchase year was unreported, or it happens when the car was bought used. In these cases, I used the approximate year the car was made. It is approximate because the data set only reports groups of years for the year the car was built (1985, 1984, 1980-1983, 1975-1979, 1970-1974 and ≤ 1969). To calculate the value of the car, I use a weighted average of the prices for all possible years. These weights are multiplied by a factor that declines geometrically and is based on the observed number of automobiles of each year known to be owned by members of the sample. For instance, an individual buys a used Ford Mustang in 1980 and reports that the car's year is between 1975 and 1979. Suppose the probability of the car being a 1979 model is y. The probability of it being a 1978 is $x \cdot y$, '77 is $x^2 \cdot y$, '76 is $x^3 \cdot y$, and so on. Since the car is between a 1975 and 1979 model, $y = \frac{1}{1+x+x^2+x^3+x^4}$. Letting n_t denote the number of observations of automobiles of that year and n the total number of observations between '75 and '79, x is chosen to maximize the log likelihood:

$$\mathcal{L} = \max_x - \ln(1 + x + x^2 + x^3 + x^4) + \ln(x) \sum_{j=1}^{4} j n_{79-j}.$$

The resulting x in this case is .747. As an example, the probability weights and values for a Ford Mustang in each year are

Calculating Value of a '75-'79 Ford Mustang

'75	'76	'77	'78	'79
.103	.137	.184	.246	.330
1,575	1,675	1,975	2,500	3,200

giving a total value of $2,425.98. This procedure is repeated for every combination, of years, giving of course different values of x. A particularly difficult calculation was for the Ford Capri which was produced before and after 1978 but not in 1978. Different values for x were used because the exit rate from the data set probably is not constant over the years.

If the automobile was purchased used, it was assumed that the automobile was not made in the same year as the automobile was purchased, unless that is the only year the automobile could be. One such case is if the automobile is reported as being in 75-79 and was bought used in 1975.

A third approximation occurs when an individual reported the make of the car only as a particular division. For instance several families report owning Cheverolets. In such cases a middle level car is chosen as representative of the value.

Several observations were marked as missing since crucial information was unavailable or is inconsistent with other reports. All automobiles for which the make was coded as missing or was topcoded were marked as missing. For 17 observations the make was not produced in years any where close to that reported or were of makes for which no data are available. These were also coded missing. In addition, a few observations were coded as missing when the purchase price implied something far different than the other information provided. In some cases, however, the purchase prices implied a simple correction that made the data consistent. In these cases the corrections were made.

The value of the used car were augmented to reflect the value of reported additional options and atypical transmissions. These values, also from the two *Red Books*, also depend on which option classes the particular make falls into.

One final change to the used value is made. Lemons models such as Akerlof (1970) or Genesove (1990) suggest that the market price of used cars is less or that the quality of the product on the market is less than the average. Another way of putting this is by assuming that the average retail price includes the transactions cost implicit in the model. To adjust for this transactions cost, I took the deflated and depreciated value of the autos and trucks as reported and divide by the used car price when both were available. The average of this ratio is 1.107. I multiplied the used car prices by 1.107, so that both types of data are in the same units.

Two basic data sets emerge. One uses as the final value of the vehicles the used value from the *Red Books* unless the data are missing and thus uses the purchase cost as properly deflated and depreciated in its stead. The second does the opposite. It uses the purchase costs unless that value is missing and thus uses the used car values.

Summary Statistics for Automobiles and Light Trucks

	Obs	*Mean*	*Std Dev*	*Min*	*Max*
Used Values	3,767	3,575.32	2,949.05	379.88	36,223.53
Purchases	1,433	5,789.10	4,674.42	80.82	27,822.49

Of the two vehicles 1,304 have data for both; 2,463 have data from used values but not from purchases; and 129 have data from purchases but not used values.

Other vehicles such as campers and boats are also part of the OVB. Most of these observations contain very little data, mostly just what type the vehicle is. To price these I took those vehicles that have reported purchase prices and purchase times. I deflated and depreciated these prices by the trucks and recreational vehicles deflator and an annual depreciation rate of 15%. I then priced those vehicles without reported prices according to this average. For those observations that have data on purchase times, but not prices, I adjusted the average to reflect this additional information on depreciation. New and used vehicles are treated alike. Table A2 summarizes this information.

The value of all vehicles for a particular family were summed, and this data was merged to the nondurable appliance data. Implicitly, this assumes that those individuals in the Fourth Quarter of 1985 have the vehicles as reported. The summary statistics for these two measures of the stocks of durables, both of which include the values for vehicles other than cars and trucks are reported below.

Summary Statistics for Motor Vehicles

	Obs	*Mean*	*Std Dev*	*Min*	*Max*
Used Values	3,280	4,980.99	4,358.16	80.82	45,232.37
Purchases	3,280	5,152.39	4,829.69	80.82	45,232.37
Used Values	1,965	5,141.04	4,570.33	279.15	49,232.37
Purchases	1,965	5,280.84	4,957.53	89.80	45,232.37

B. 1.3 Nondurable Expenditures

The relevant data set is the Family Characteristics and Income File (FMLY). It contains data on the expenditures of various groups of nondurables and durables. Only a subset of these expenditures were used for nondurables because many groups contain a significant portion of durable expenditures. Those expenditures that are used are alcohol beverages, apparel, food, housing operations, personal care, reading and tobacco. The sum of these is the first measure of nondurables ("Excludes Utilities"); thus I assume that all expenditures in this quarter represent utility gained from nondurable consumption. A second measure of nondurable consumption ("Includes Utilities") adds to the first gasoline, motor oil and utilities. These measures of nondurable expenditures are summarized below.

Summary Statistics for Nondurable Consumption

	Obs	*Mean*	*Std Dev*	*Min*	*Max*
Excl. utilities	5,319	479.44	623.09	0.00	7,042.48
Incl. utilities	5,319	691.86	818.40	0.00	7,299.93
Excl. utilities	1,965	848.39	670.83	34.67	7,042.48
Incl. utilities	1,965	1,246.34	821.69	88.00	7,299.93

The ratio of durables to nondurables then is simply the value of vehicles and appliances divided by current nondurable consumption. Four different combinations are possible, summarized below.

Summary Statistics for the Ratios

	Obs	*Mean*	*Std Dev*	*Min*	*Max*
Ex. utilities/Used	1,965	10.062	10.689	.310	184.208
In. utilities/Used	1,965	6.026	5.402	.275	52.128
Ex. utilities/Prch.	1,965	10.183	11.180	.169	184.208
In. utilities/Prch.	1,965	6.131	5.804	.150	65.354

The number of observations for the different measures of the components of the durable stock and of nondurable expenditures differ for numerous reasons.

Appliances has a large number because it contains data for families who are interviewed in any of the five quarters covered by the CES. The same is true for motor vehicles. Still, the smallest number of observations for any group is 3,280 is much greater than 1,965. A large bit of this discrepancy arises from the fact that families must have data from each component, appliances, motor vehicles and nondurables in order to be in the final sample. Missing data in any group means the family is excluded. In addition, because a ratio is taken, any family that reports zero nondurable consumption is excluded. For these reasons 1,965 families are left.

B.2. Macroeconomic Data

All aggregate data were collected from Citibase and are available quarterly from 1947:1 - 1990:2. Real durable purchases is $GCD82$, and it is measured in billions of 1982 dollars. Nominal durables is GCD, and durable prices is $GCD/GCD82$. Nominal nondurable consumption is nominal nondurables plus nominal services except for housing services. The components that define nondurables are GCN ($GCN82$) – Nondurables, $GCSHO$ ($GCSHO8$) – Housing Operations Services, $GCST$ ($GCST82$) – Transportation Services, $GCSM$ ($GCSM82$) – Medical Services, $GCSOS$ ($GCSOS8$) – Other Services, where the first variable is nominal, the second in parentheses is real. Nominal nondurables is the sum of the nondurable components, while real nondurables is the sum of the real components. The price of nondurables is the ratio of nominal nondurables to real nondurables. The relative price p_{ct} is simply nondurable prices divided by durable prices.

The durable stock is assumed to solve the difference equation

$$K_t = (1-\alpha)K_{t-1} + GCD82,$$

for $t = 47:2 \ldots 90:2$ where the annual depreciation rate is 15% or $\alpha = .0325$. The initial value $K_{1948:1}$ is approximated by

$$K_{1947:1} = e^{\beta_0 + \frac{1}{2}\sigma^2} \sum_{j=0}^{\infty} (1-\alpha)^j e^{-\beta_1 j},$$

where β_0 and β_1 are calculated from a regression of the log of real durable purchases on a constant and trend. The additional $\frac{1}{2}\sigma^2$ term, where σ^2 is the variance of the residuals, is motivated by a Taylor expansion. The specific values for a regression over 1947:1–1990:2 are

$$\beta_0 = 4.0412 \qquad \beta_1 = .0116 \qquad \sigma^2 = .0070.$$

To attenuate the effects of the estimation of $K_{1947:1}$ only values for 1949:1 and greater are used.

The growth rate of non-durable consumption is the log difference of non-durable consumption divided by the average over the quarter of the resident population,

$$\frac{dC_t}{C_t} = \log(C_t/P_t) - \log(C_{t-1}/P_{t-1}),$$

where $P_t = \frac{1}{3}\sum_{m=0}^{2} POPRES_{t+m}$. $POPRES$ is available monthly; $t+m$, $m = 0 \ldots 2$ represents the three months in quarter t.

The price deflators used for motor vehicles divide nominal expenditures by real. Monthly expenditures in Citibase for domestic new cars, foreign new cars, net used cars, and new & used trucks and recreational vehicles are denoted as GMCAND (GMCND8), GMCANF (GMCAF8), GMCAU (GMCAU8), AND GMCAT (GMCAT8). Nominal values are given first, real values in parentheses.

The real interest rate is defined as the three month rate on U.S Treasury Bills minus the expected rate of inflation over the next three months. The inflation rate at time t, π_t, is defined as 400 times either the natural log of the seasonally unadjusted cpi divided by the three month lag of the cpi or the natural log of current durable prices divided by the three month lag of durable prices. The real interest rate series used in the paper is the rate that deflates by durable prices. Durable prices, $PDRB$ are measured as the deflator. Thus r_t, the real interest rate, annualized is defined as

$$r_t = i_t - \mathrm{E}[\pi_{t+1}|\mathcal{F}_t], \tag{43}$$

where i_t is the nominal three month t-bill rate. All data are quarterly; where monthly data is available the first month of the quarter is used. Each variable is described in more detail in Table A3 .

The reduced form inflation rate process is modeled as

$$A(L)\pi_t = \alpha_0 + \alpha_k q_t + C(L)m_t + 2\alpha_5(I_t - i_t) + \alpha_6 t + B(L)\epsilon_t, \tag{44}$$

where π_t is the inflation rate, m_t is the log growth rate of M1, I_t is the six month nominal interest rate while i_t is the three month nominal interest rate, q_t is a quarterly dummy, ϵ_t is white noise, and $A(L)$, $B(L)$, and $C(L)$ are fourth order polynomials in the lag operator, L. The results of the estimation of (44) are reported in Table A4 .

Following Barro and Sala-i-Martin (1990) the real rate of interest is modeled in reduced form as

$$r_t = \beta_0 + \beta_1 m_{t-1} + \beta_2 DEF_{t-1} + \beta_3 I_{t-1}/Y_{t-1} + \beta_4 r_{t-1} + \beta_5 OIL_{t-1} + \beta_6 STOCK_{t-1} + \beta_7 t + \eta_t, \tag{45}$$

where r is the real rate of return as defined in (43), DEF is the federal deficit deflated by the GNP deflator, I/Y is investment over GNP, OIL is the price of oil deflated by the CPI, and $STOCK$ is the rate of return on a value weighted portfolio of stocks including dividends. The results of equation (45) are reported in Table A5 .

The realized rate of return is defined as the nominal rate minus realized inflation. The actual real rate of return is defined as in (43), and the expected real rate is the projected values from (45). These three series are plotted in Figure A2 for the series that uses durable prices for the period 1962:3–1989:3.

Table B1: Summary Statistics for Appliances

	Used		*New*		*Ratio*	
Appliance	*Mean*	*Std. Dev.*	*Mean*	*Std. Dev.*	*Mean*	*Std. Dev.*
Electric Stove	254.50	206.61	506.05	249.23	.50	.44
Gas Stove	117.67	187.38	489.42	217.70	.24	.41
Microwave	161.43	77.89	274.23	214.52	.59	.32
Other Stove	52.00	—	214.50	324.20	—	—
Refrigerator	177.00	160.26	604.68	332.15	.29	.34
Home-Freezer	131.79	89.61	507.37	379.47	.26	.30
Built-in Dishwasher	257.25	196.52	421.74	139.01	.61	.47
Portable Dishwasher	61.33	49.90	433.80	51.65	.14	.12
Garbage Disposal	—	—	101.21	59.51	—	—
Clothes Washer	152.62	107.99	426.51	110.84	.36	.26
Clothes Dryer	96.08	67.81	329.51	91.14	.29	.21
Simple Average	151.77	140.36	398.38	252.39	.37	
Weighted Average	140.82	114.86	385.09	266.27	.25	

Notes:

1. Other Stoves and Garbage Disposals were excluded from the calculation of averages and of the ratio. Weighted averages use the inverse of the standard deviations of each divided by the average of the inverses as the weights and then averages over the CES sample. This effectively gives more weight to those appliances with more observations. The ratio averages only over the nine categories, and therefore does not weight by observation.

Table B2: Non-Car and Truck Vehicles						
Vehicle	*Obs*	*Mean*	*Std. Dev.*	*Min.*	*Max.*	*Total Obs*
Motorized Camper	10	12,405.94	5,264.00	4,709.77	21,693.67	31
Trailer Camper	18	5,038.98	3,908.45	249.73	12,184.38	70
Other Camper	9	1,200.44	1,579.32	141.38	4,418.03	39
Motorcycles	66	1,428.08	1,333.85	100.11	6,065.55	247
Boat w/ Motor	38	5,504.75*	8,381.51	99.91	49,095.20	177
Boat w/o Motor	11	348.80	329.18	58.44	1,201.28	83
Trailer, Not Camper	6	473.31	298.60	95.34	946.09	125
Other Vehicle	15	1,079.38	606.40	139.48	2,422.18	53

Notes:

1. * indicates one observation with reported value equal to zero was valued at the mean.

Table B3: Description of Data	
CPI_t	Consumer Price Index for urban consumers, all items, not seasonally adjusted
DEF_t	Government surplus or deficit (surplus +) deflated by the GNP deflator
i_t	Nominal three month Treasury Bill rate on the secondary market
I_t	Gross Investment in billions of $
$M1$	M1, available monthly
OIL_t	Producer price index of crude petroleum in the U.S., deflated by the CPI or Durable Prices
$PDRB_t$	Nominal durable goods, GCD_t, divided by real durable goods, $GCD82_t$
$STOCK_t$	The real rate of return on a value weighted portfolio of publically traded stocks including dividends. The nominal rate is from CRISP. The CPI inflation rate is subtracted from it
Y_t	Gross National Product in billions of $

Notes:

1. All data except for the rate of return on the stock market are from Citibase.

Table B4: Expected π_t

	0	1	2	3	4
			Durable Prices		
AR	—	.763	.015	-.039	.264
		(.267)	(.323)	(.310)	(.236)
MA	—	-.201	.081	-.038	-.111
		(.283)	(.267)	(.275)	(.173)
m_t	20.014	2.218	-11.541	11.232	-29.274
	(23.543)	(23.684)	(23.174)	(22.811)	(23.304)
$2(R_t - r_t)$	-.854	—	—	—	—
	(.638)				
Trend	.220	—	—	—	—
	(4.597)				
SEAS	51.587	—	—	—	—
	(138.867)				
			CPI		
AR	—	.581	-.141	.828	-.563
		(.137)	(.082)	(.084)	(.133)
MA	—	-.275	.664	-.701	.909
		(.117)	(.114)	(.124)	(.108)
m_t	24.902	79.953	31.727	-29.366	22.452
	(24.764)	(24.116)	(22.808)	(22.220)	(19.814)
$2(R_t - r_t)$	.183	—	—	—	—
	(.562)				
Trend	.038	—	—	—	—
	(.030)				
SEAS	-2.450	.780	1.442	.618	—
	(3.590)	(.464)	(.474)	(.418)	

Notes to Table B4:

1. $A(L)$ is $1 - \sum_{k=1}^{4} a_k L^k$ whereas $B(L)$, and $C(L)$ are of the form $1 + \sum_{k=1}^{4} b_k L^k$. $SEAS_0$ is the constant, while $SEAS_k$ for $k = 1 \ldots 3$ are dummies equal to 1 for quarter k, 0 otherwise.
2. Standard errors are in parentheses. The analysis with durable prices is over 1961:2 – 1990:1 with $\bar{R}^2 = .415$ and $Q = 36.448$ with a p-value of .194. Q is the Ljung-Box statistic taken over 32 autocorrelations. The analysis with the cpi is over 1961:2–1989:2 with $\bar{R}^2 = .675$ and $Q = 25.036$ with a p-value of .723. The Q statistic uses 32 autocorrelations.

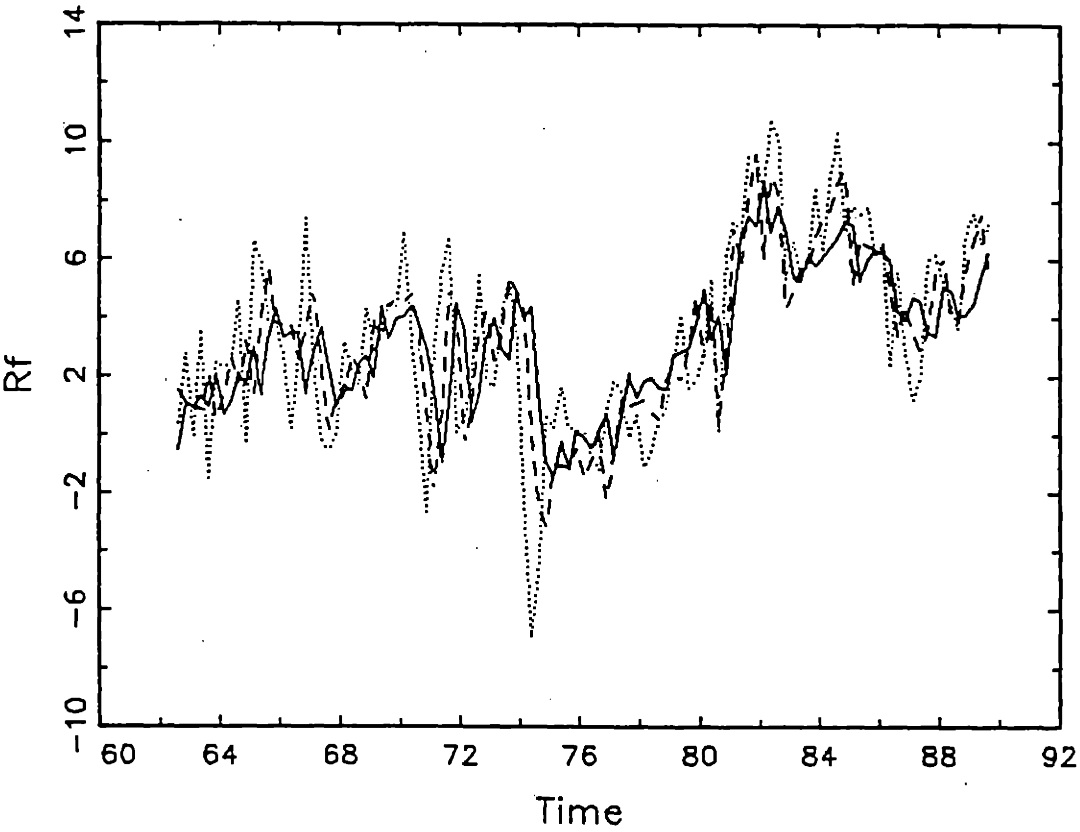
Table B5: Expected r_t

	Durable Prices			*CPI*		
	Coeff.	*Std. Er.*	*T-Stat*	*Coeff.*	*Std. Er.*	*T-Stat*
Const.	1.299	2.825	.460	5.705	3.067	1.860
m_{t-1}	40.841	16.141	2.530	-.399	18.673	-.021
DEF_{t-1}	.011	.007	1.615	.325	.687	.473
I_{t-1}/Y_{t-1}	-15.464	17.160	-.901	-37.097	19.020	-1.950
r_{t-1}	.746	.070	10.599	.514	.085	6.064
OIL_{t-1}	.018	.010	1.714	3.654	1.215	3.008
$STOCK_{t-1}$	.423	2.989	.142	-.591	3.434	-.172
Trend	.010	.010	1.025	-.009	.010	-.875

Notes:

1. The Durable Prices regression is over 1962:3 – 1989:3; $\bar{R}^2 = .651$ and $Q = 36.811$ with a p-value of .183. Q is the Ljung-Box statistic taken over 31 autocorrelations. The CPI regression is over 1961:3 – 1989:2; $\bar{R}^2 = .473$ and $Q = 96.863$ with a p-value of .000. Q is taken over 32 autocorrelations.

Fig. B1: Real Interest Rates



Appendix C: Stochastic Depreciation of the Durable Stock

This appendix describes the solution to the consumer's optimization problem when the durable stock depreciates stochastically. It is based on Reider (1991), though it goes further, using the same techniques as in Chapter 1. It shows that stochastic depreciation of the durable stock can account for the idiosyncratic shocks in the ratio. For simplicity durables and nondurables enter additively separable as in Chapter 2. The development below uses the same terminology as in Chapters 1 and 2, except where noted. The definitions of these variables are in those chapters.

The consumer solves

$$\max_{\tau,k,X_t,C_t} \int_0^{\tau} \frac{1}{a} e^{-\delta t}(K_t^a + bC_t^a)dt + e^{-\delta\tau}V(Q_{\tau_-} - \lambda K_{\tau_-}, k)$$

$$\begin{aligned} \text{s.t.}\quad dQ_t &= -\alpha K_t dt + \sigma_k dw_k(t) + r_f(Q_t - K_t)dt + X_t(\mu dt + \sigma dw(t)) \\ &\quad - C_t dt \qquad t \in [0,t) \\ dK_t &= -\alpha K_t dt + \sigma_k dw_k(t) \qquad t \in [0,t) \\ Q_t &- \lambda K_t \geq 0, \end{aligned} \tag{46}$$

where $dw_k(t)dw(t) = 0$. The solutions to X_t and C_t are the same as before. This gives a similar Bellman equation after plugging in the values for dQ and dK. Suppressing the maximization operator

$$\delta V(Q_t, K_t)dt = Kr_f(\frac{Q}{K} - 1)V_Q - \gamma\frac{V_Q^2}{V_{QQ}} - \alpha K(V_Q + V_K)$$
$$+ \tfrac{1}{2}\sigma_k^2 K^2(V_{QQ} + 2V_{QK} + V_{KK}) + \frac{1}{a} + \frac{1-a}{a} b^{-\frac{1}{a-1}} V_Q^{\frac{a}{a-1}}.$$

Define

$$y \equiv \frac{Q}{K} - 1 \qquad \text{and} \qquad h(y) = K^{-a}V(Q,K).$$

This definition of y is slightly different than that used in the main text, but it should not cause any confusion. It simplifies notation slightly. The reader may believe that because K is stochastic, the problem cannot be reduced to one state variable, but as shown below, that which is of interest, can be found using this same construction. Differentiating by Q and K gives the partial derivatives of V in terms of h:

$$\begin{aligned}\tilde{\delta}h(y) = \tilde{r}yh'(y) + \tfrac{1}{2}\sigma_k^2 y^2 h''(y) - \gamma\frac{h'(y)^2}{h''(y)} \\ + \frac{1}{a} + \frac{1-a}{a}b^{-\frac{1}{a-1}}h'(y)^{\frac{a}{a-1}},\end{aligned} \tag{47}$$

where $\tilde{\delta} \equiv \bar{\delta} + \frac{1}{2}a(1-a)\sigma_k^2$ and $\tilde{r} \equiv r + (1-a)\sigma_k^2$.

As before let $z \equiv h'(y)$. Differentiating (47) by z yields

$$\begin{aligned}-\gamma z^2 y''(z) + (\tilde{r} - \tilde{\delta} - 2\gamma)zy'(z) + (\tilde{r} + \sigma_k^2)y(z) \\ - \tfrac{1}{2}\sigma_k^2 y(z)^2 \frac{y''(z)}{y'(z)^2} - b^{-\frac{1}{a-1}}z^{\frac{1}{a-1}}.\end{aligned} \tag{48}$$

The general solution to (48) is

$$B_1 z_1^{\theta} + B_2 z_2^{\theta} + B_3 z_3^{\theta} + D z^{\frac{1}{a-1}}, \tag{49}$$

where θ_i solves

$$-\gamma\theta^3 + (\tilde{r} - \tilde{\delta} - 2\gamma)\theta^2 + (\tilde{r} + \tfrac{1}{2}\sigma_k^2)\theta + \tfrac{1}{2}\sigma_k^2 = 0,$$

and D is given by

$$D \equiv -b^{\frac{1}{1-a}}\left(\gamma(2-a)(1-a)^{-2} + (\tilde{r} - \tilde{\delta} - 2\gamma)(1-a)^{-1} - (\tilde{r} + \sigma_k^2) - \tfrac{1}{2}\sigma_k^2(2-a)\right)^{-1}.$$

The stochastic differential equation for y is

$$\begin{aligned}dy &= \frac{dQ}{K} - \frac{Q}{K}\frac{dK}{K} + \frac{Q}{K}\frac{dK^2}{K^2} - \frac{dQdK}{K} \\ &= (\tilde{r} - a\sigma_k^2)ydt - 2\gamma zy'(z)dt - b^{-\frac{1}{a-1}}z^{\frac{1}{a-1}}dt \\ &\quad - \frac{\mu}{\sigma}zy'(z)dw(t) - \sigma_k y dw_k(t).\end{aligned} \tag{50}$$

Suppose z solves

$$dz = \mu_z(z)zdt + \sigma_{1z}zdw(t) + \sigma_{2z}(z)zdw_k(t).$$

Then dy is also given by

$$\begin{aligned} dy = y'(z)\mu_z(z)zdt + \tfrac{1}{2}y''(z)z^2(\sigma_{1z}^2 + \sigma_{2z}^2)dt \\ + y'(z)z\big(\sigma_{1z}dw(t) + \sigma_{2z}(z)dw_k(t)\big). \end{aligned} \tag{51}$$

Equations (50) and (51) imply

$$\begin{aligned} \sigma_{1z} &= -\frac{\mu}{\sigma} \\ \sigma_{2z} &= -\sigma_k \frac{y(z)}{zy'(z)} \\ \mu_z &= \tilde{\delta} - \tilde{r} - (1+a)\sigma_k^2 \frac{y(z)}{zy'(z)}. \end{aligned} \tag{52}$$

Equation (52) is the basis of the claim that the introduction of a stochastic durable stock does not qualitatively change the solution of Chapter 1. The pseudo marginal utility variable z_t is still approximately a geometric brownian motion because the last terms in σ_{2z} and μ_z is bounded. Suppose $\theta_1 < \theta_2 < \theta_3$ without loss of generality. As $z \to -\infty$, $\frac{y(z)}{zy'(z)} \to 1/\theta_1$, while as $z \to \infty$, $\frac{y(z)}{zy'(z)} \to 1/\theta_3$.

As for the value function, note that, as before, the problem can be rewritten as

$$\begin{aligned} \tilde{H}(z) = h(y) = \max_{\tau} \int_0^{\tau} e^{-\frac{1}{2}a\sigma_k^2 t + a\sigma_k dw_k(t)} e^{-(\delta + a\alpha)t} \frac{1}{a}\big(1 + b^{-\frac{1}{a-1}}\big)dt \\ + e^{-\frac{1}{2}a\sigma_k^2 \tau + adw_k(\tau)} My_{\tau_-}^a. \end{aligned} \tag{53}$$

The terminal value $My_{\tau_-}^a$ is found as before. Let $\mathcal{M}_t \equiv e^{-\frac{1}{2}a\sigma_k^2 t + a\sigma_k dw_k(t)}$, and $d\mathcal{M}_t = \mathcal{M}_t dw_k(t)$. The true value function $\tilde{H}(z)$ actually depends on two state variables, z_t and $\mathcal{M}_t$. But as (53) makes clear, $\tilde{H}$ can be separated, write $\tilde{H}(z) = \mathcal{M}_t H(z)$. The Bellman equation for $\tilde{H}$ solves is

$$\delta\tilde{H}(z_t, \mathcal{M}_t) = \tilde{H}_z E dz + \tfrac{1}{2}\tilde{H}_{zz}dz^2 + \frac{1}{a}\mathcal{M}_t\big(1 + b^{-\frac{1}{a-1}} z^{\frac{a}{a-1}}\big) + \tilde{H}_{\mathcal{M}} d\mathcal{M}_t$$
$$+ \tfrac{1}{2}\tilde{H}_{\mathcal{M}\mathcal{M}} d\mathcal{M}_t^2 + \tilde{H}_{z\mathcal{M}} dz d\mathcal{M}_t$$
$$\delta\mathcal{M}_t H(z_t) = \mathcal{M}_t H_z E dz + \tfrac{1}{2}\mathcal{M}_t H_{zz}dz^2 + \frac{1}{a}\mathcal{M}_t\big(1 + b^{-\frac{1}{a-1}} z^{\frac{a}{a-1}}\big)$$
$$+ H(z) d\mathcal{M}_t + H_z dz d\mathcal{M}_t \tag{54}$$
$$\delta H(z_t) = H_z E dz + \tfrac{1}{2}H_{zz}dz^2 + \frac{1}{a}\big(1 + b^{-\frac{1}{a-1}} z^{\frac{a}{a-1}}\big) + H(z)\frac{d\mathcal{M}_t}{\mathcal{M}_t}$$
$$+ H_z dz \frac{d\mathcal{M}_t}{\mathcal{M}_t}$$
$$\delta H(z_t) = H_z\big(E dz - a\sigma_k^2 \frac{y(z)}{y'(z)}\big) + \tfrac{1}{2}H_{zz}dz^2 + \frac{1}{a}\big(1 + b^{-\frac{1}{a-1}} z^{\frac{a}{a-1}}\big).$$

Therefore, one can proceed as before in solving the Bellman equation with the proviso that there are "near constant" terms involving $\frac{y(z)}{zy'(z)}$ and its square. The solution of the problem as before is characterized by the two-sided, fixed-band S-s policy, except now, the solution to (54) is not known. The important part, however, is that the qualitative nature of the value function is unchanged, and the process of z_t is similar to a geometric brownian motion, as was claimed.

Chapter 3:

THE RELATIONSHIP AMONG THE MATURITY OF AUTOMOBILE LOANS, NONDURABLE CONSUMPTION AND LIQUIDITY CONSTRAINTS: THEORY AND EVIDENCE

1. Introduction

Several authors have noted the special nature of credit markets, suggesting that the simple price–quantity analysis of introductory microeconomics is deficient. Consumer loan contracts consist of more than an interest rate and loan quantity. They also define the length of the contract, the payment period, collateral requirements, the downpayment, and conditions in case of default. Changes in the economic environment need not result in changes in the stated interest rate if other terms of the contract move.

While there are a myriad of models of credit markets that incorporate non-price terms and/or credit rationing, few papers have provided direct empirical evidence for the effects they describe.[1] Of course several empirical works provide indirect evidence of credit market imperfections. Zeldes (1989), Hansen and Singleton (1982) and others find substantial violations of the intertemporal euler euqtion for nondurable consumption in the direction implied by credit constraints.[2] Chah, Ramey and Starr (1991) find that aggregate durable purchases forecast future nondurable consumption, an implication consistent with a model where consumers can borrow for durables but not nondurables. None of these models, however, take advantage of existing credit markets data except for possibly Treasury Bill interest rate data. They suffer from multiple alternatives such as rule-of-thumb behavior or myopia, which can be difficult to distinguish (Flavin, 1985).

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[1] Exceptions are Jappelli (1990) and Ostas and Zahn (1975).

[2] See Deaton (1987) and Hayashi (1987) for useful reviews.

This paper investigates the relationship among non-price terms and the defined risk premium in consumer credit markets. More specifically, the market considered is the consumer auto loan market. 38.7% of consumer installment credit outstanding at the end of 1990 is devoted to the purchase of motor vehicles, so it is an important piece of consumer credit markets. In addition, specific data on this market are available, including loan rates and non-price terms.

To emphasize non-price terms not discussed previously in the literature, a simple model is developed to explore the importance of loan contract length. In the model consumers borrow to fund a durable purchase and choose to repay the loan in the next period or over two periods. Attitudes towards stochastic future income as well as the slope and variance of future income all influence the maturity decision. An important additional element, however, is future attitudes towards default. Individuals with a propensity towards default in some states of the world prefer longer maturities than those who repay their loans always, and the equilibrium term structure of loan rates reflects the partial signaling that occurs in the market.

The paper then considers empirical evidence both microeconomic and macroeconomic. The microeconomic data set contains 1,561 observations of consumers who bought automobiles in 1985. It has a variety of personal characteristics, financial data and terms of the loan contract for those who financed their purchases. The paper analyzes both the finance/cash outcome and the terms of the loan contracts. The macroeconomic data set contains average monthly observations on the loan rate, the length of the finance contract, the downpayment, and the amount financed over the period 1972–1991. The estimation procedure analyzes the determinants of the risk premium and the contract length.

The theoretical analysis and the empirical evidence suggest two important conclusions. First, non-price terms of consumer auto loan contracts matter for market equilibrium. Both the downpayment and the length of the contract affect the risk premia individual borrowers receive and the average risk premium in the economy over time. Such observations cannot resolve the debate over whether non-price terms move to clear the market and prevent credit constraints. It does suggest, however, that some of the features emphasized in previous models of credit markets may be at work; some of these same attributes are present in models of credit rationing.

Second, that the contract length is one term that matters suggests that a Modigliani-Miller theorem does not hold for individuals in at least one important dimension. That the contract length matters means consumers adjust their planned spending, probably to smooth consumption and reduce variance. That some hold outstanding liabilities while also holding interest bearing assets implies consumers do so at some cost. This paper offers that consumers take these measures because they cannot insure their labor income or even borrow against it to fund nondurable consumption. Given the importance of a downpayment and presumably of collateral requirements such credit constraints on nondurables are not that far-fetched. Such conclusions have implications not only for models of credit markets but for models of intertemporal smoothing, the effect of monetary policy on aggregate activity, and the time series of aggregate durable and nondurable purchases.

The rest of the paper is organized as follows. Section 2 presents a simple model in which the decision to borrow for two periods versus one period to fund durable purchases depends upon the expected future utility of nondurables and attitudes towards default. The model delivers the implication that longer loan maturities are associated with higher default probabilities as consumers sort themselves somewhat on this dimension. Section 3 investigates the relationship between contract length and risk premia in a set of observations on individuals, while Section 4 uses aggregate data through time. Section 5 concludes.

2. A Simple Microeconomic Model

In order for such non-price terms as the length of the contract, the payment period, collateral requirements, the downpayment, and conditions in case of default to influence the stated interest rate, borrowers and/or lenders must have some preferences over these terms. For some terms, the preferences are obvious. Most borrowers probably do not appreciate high downpayments for the same reasons they are borrowing in the first place. They want resources now to invest or to consume and will repay later with interest. Lenders, on the other hand, like downpayments since this raises the ratio of the loan principal to the collateral value, thereby reducing the loss from default. Other terms such as the length of the loan contract are not as obvious and are not strictly modeled in the literature.

If consumers and/or lenders have preferences over non-price loan terms, the market equilibrium echoes these preferences. Lenders may offer combinations of interest rates, loan quantities and non-price terms, reflecting their preferences, and consumers may ask for different sets of combinations, tracing out their indifference curves. Equilibrium may then be a single contract where these two hyper-planes intersect, or it may be a set of contracts reflecting heterogeneity in borrowers or lenders. For example, such terms as a lower downpayment or a longer loan contract may be bought with a higher stated interest rate.

With heterogeneity among consumers, however, an additional consideration arises. If consumers have different default probabilities and these probabilities are correlated with preferences over aspects of the loan contract, the menu of contracts offered by lenders will reflect these probabilities. The information consumers reveal about their default probabilities in their choice of terms may be the single biggest explanation of lenders preferences. As an extreme example, suppose there are two consumers, one with a default probability strictly inside (0,1) and the other with a default probability of exactly zero. Furthermore, suppose the consumer who may default has a strong preference for long loan contracts while the second does not. Lenders offer two contracts, one short-term the other long-term. A separating equilibrium may arise where the defaulter chooses a long-term contract with a large interest rate, while the non-defaulter chooses a short-term contract with a small interest rate. A pooling equilibrium can also arise if the preferences over contract length are not as strong. In the pooling equilibrium, the non-defaulter subsidizes the defaulter because the former cannot signal her difference vis-a-vis the latter.

A final aspect of preferences and equilibrium is that attitudes towards default may also shape the preferences of borrowers over non-price terms. For example, *ceteris paribus* those borrowers who have no qualms towards defaulting at any time weakly prefer longer loan contracts than those who do not default on their loans. The option of defaulting has some value to the former while its value is zero to the latter. Paying a loan off early more quickly dissipates the option.

Informal evidence exists that consumers and lenders care about the length of the loan contract, in spite of the fact that in a Modigliani–Miller world, the term length of any one loan can easily be undone by another financial instrument. The New York *Times* (Sloane, 1990) reports that banks and finance companies are limiting the number of five year loans they are willing to give to consumers. They supposedly charge much higher interest rates on longer contracts, and they have tightened eligibility requirements. In 1988 70% of GMAC loans were for five years; in 1989 it was 58%. Other lenders report the same decrease in five year loans. In the late 1960's and early 1970's GMAC's stated policy in their corporate annual reports was to offer loan contracts no longer than 36 months. Later they admitted that competitive pressures have led to longer contracts. Besides this contemporary evidence, as discussed below, the average contract length has increased steadily from 1972–1990, though one can observe a recent flattening. This trend and the "competitive pressures" suggests consumers prefer long maturities.

This informal analysis suggests that a model of non-price terms should consider several aspects. First, default risk is important and some motivation for default or non-default should be explicit in the model. A simple justification used below is that default results only in the loss of the utility stream from the durable collateral. Second, preferences over non-price terms influence the equilibrium that arises and must be justified. They may be primitive to the utility function; they may arise from attitudes towards default, or they may reflect some other aspect of the economy. One reason consumers in the model presented below have preferences over the length of the loan contract is that the longer the contract, the smaller the monthly payments. Smaller monthly payments may be preferred because of smoothing considerations over time for which the consumer is incapable of doing otherwise, while quick payments may be preferred because of uncertainty over future income. Third, an equilibrium model must incorporate the signaling that occurs in the choice of contracts. The result may be pooling, separating or a combination of the two if consumers differ in many relevant dimensions. The model below has each of these outcomes, depending upon the explicit parameterization.

Several papers have considered the effects of non-price terms on equilibrium interest rates. A particularly popular non-price term is collateral. Several authors suggest that collateral requirements can mitigate the effects of imperfections that lead to credit constraints (Azzi and Cox, 1976; Bestor, 1985; Chan and Kanatas, 1985). Others argue against these claims (Jaffee and Modigliani, 1976; Stiglitz and Weiss, 1981; Wette, 1983). Manage (1990) investigates empirically the effects of collateral requirements on interest rates in states which impose different interest rate ceilings on consumer loans. Luckett (1970) notes informally that banks which are less "sound" hold loans with more stringent terms such as higher collateral requirements or higher assets-to-liabilities ratios.

A second term that has been investigated is downpayments. Koskela (1983) develops a model where the downpayment is an important element of the loan contract, emphasizing one non-price term while noting this is one example. In the model restrictive monetary policy increases the required downpayment while its effects on the stated loan rate and the amount of credit rationing is ambiguous.

Only a few papers, however, have considered the contract length as one of the non-price terms. Harris (1973) and Plaut (1985) present two simple models which include the maturity of the loan as one of the terms that can vary in a loan contract. Consumers pick from a menu of choices lenders offer, trading off interest rates, downpayments, maturity length, *etc.* optimally. Neither, however, explicitly explore the reasons why consumers have preferences over maturity length and what the equilibrium implications of these preferences are. Melenik and Plaut (1986) do the same with loan commitments to businesses. Extending further their analysis, Melenik and Plaut examine a data set of 101 U.S. firms in 1980–81, finding corroborative evidence.

Dhillon, Shilling and Sirmans (1990) discuss factors influencing the decision of mortgage maturities. In their sample some home buyers hold 30-year mortgages while others have 15-year ones. They find that such factors as the region the home is located in, the age of the buyer, the price of the home and tax considerations are important in the outcome of the contract. Other correlated factors, however, are the interest rate and the downpayment. The higher the interest rate, the more likely the consumer has chosen a 30-year mortgage; the higher the downpayment, the more likely the consumer has chosen a 15-year mortgage.

A few papers have dealt with the issue of debt maturity for business investment. Hart and Moore (1991) present a model of business loans for investment purposes where the optimal repayment path depends on the path of project returns and the durability of the projects assets. Their model is devoted to explaining why the maturity of business loans match the payment stream of returns to the loan, that is short-term projects such as working capital are financed with short-term loans while long-term projects such as capital structure loans are financed with long-term loans. Diamond (1991) models the maturity choices of businesses for investment purposes. Some companies may choose long-term bonds in order to insure that they have the requisite capital in the future, whereas others choose short-term debt to obtain the smaller interest rates.

Some of these same issues arise in models of credit constraints. Several models, such as Jaffee and Russell (1976), Stiglitz and Weiss (1981) and Jaffee and Stiglitz (1990), predict that some consumers will be denied credit. Moreover, this constraint has more bite as the supply of funds to lenders is cut by monetary authorities, delivering an important role for monetary policy. A mini industry has grown up as some authors question whether some people are really constrained or just want a cheaper interest rate than their default risk deserves

(Vandell, 1984), whether credit constraints are necessary in equilibrium (Vandell), and whether collateral can screen and separate individuals, eliminating credit constraints. Of course heterogeneous individuals faced with a menu of choices may select different contracts, revealing a set characteristics (Milde and Riley, 1988). As Jaffee and Stiglitz indicate, however, as long as the dimension of risk characteristics is larger than the dimension of observable relevant signals, the equilibrium may be characterized by partial pooling. Pooling equilibria can lead to credit constraints.

As discussed above some non-price terms as downpayments and collateral requirements are thoroughly discussed in the literature. The following model explores the contract length as an additional important non-price term. The model does not introduce elements of credit constraints, not because they are unimportant, but simply to highlight the role of the contract length. Introducing additional elements to allow for constraints would simply complicate the analysis without qualitatively altering any of the conclusions on other non-price terms.

2.1 Model Set-up for the Consumer's Problem

Most previous models of credit markets have been two period models where the consumer borrows in the first period and repays in the second. Obviously, two periods is not enough to study contract length. The simplest extension is to have three periods. Therefore, assume that a consumer maximizes utility over three periods over both nondurable consumption and a durable good. Formally, the consumer solves

$$\max_{C_t, T_i, I[P_1], I[P_2]} E_0 \sum_{t=0}^{2} \beta^t (U^k(K_t) + U(C_t))$$

$$\text{s.t.} \quad K_t = \alpha K_{t-1}$$

$$A_{t+1} = R_f A_t + y_t - C_t - I[P_t] P_t \tag{1}$$

$$A_s \geq 0$$

$$A_{-1} = 0,$$

where $U^k(\cdot)$ is the VNM utility function over durables and $U^c(\cdot)$ is the VNM utility function over nondurables. To enable the consumer to choose to default $U^k(0) > -\infty$. C_t is nondurable consumption; K_t is the durable good; $1/\alpha - 1$ is the depreciation rate; A_t is any savings the consumer makes, earning the gross risk-free rate R_f; y_t is income; and P_t represents per-period payments on the loan contract. When the consumer buys the durable good at time 0, the consumer cannot choose the level of the durable good; it is fixed at $\overline{K}$. Allowing the consumer to optimize on $\overline{K}$ just complicates the analysis. T_i represents the terms of the loan contract signed in period 0, and it is assumed that lenders offer two contracts. In one contract, the consumer pays the entire loan in period 1 with an interest rate R_{B_1}. $P_1 = R_{B_1}\overline{K}$; $P_2 = 0$. In the other the consumers pays half the contract in period 1 and half in period 2. The implicit interest rate is R_{B_2}. $P_i = \frac{R_{B_2}}{1+R_f}\overline{K}$, for $i = 1, 2$. The constraint $A_s \geq 0$ means that the consumer can only borrow to fund the durable good and not nondurable consumption. $I[P_t]$ is an indicator function equal to one if the consumer is current in payments; it equals zero in the case of default.

The revelation of income is also special in order to illustrate the points. In period 0 the consumer earns y_0 for certain. It is assumed that y_0 is sufficiently small so that the consumer does not make any downpayments and does not save in this period. In period 1 income is also known for certain, equal to y_1. In period 2 income can take on one of three values. With probability p_h income is high, $y_2 = y_h$; with probability p_m income is middling, $y_2 = y_m$; and with probability $p_l = 1 - p_h - p_m$ income is low, $y_2 = y_l$, where $y_h > y_m > y_l$. Period 2 income is not known when any decision at time 1 is made. Three outcomes for income occur in period 2 so that one can analyze the effects of variance in states where default does not occur. If there were only two states in period 2 and one of the states was a default state, conditioning on not defaulting in period 2, period 2 income would be known for certain.

The timing of decisions is as follows. In period 0 the consumer borrows $\overline{K}$, earns y_0 and consumes C_0. Because $A_{-1} = 0$, $C_0 = y_0$. At this time the consumer agrees to either a one-period contract or a two-period contract. In period 1 the consumer earns y_1, consumes C_1, thereby deciding what additional assets the consumer will deliver into period 2, and pays what the loan contract calls for or decides to default. If the consumer defaults in period 1, she does not enjoy the utility from the durable in that period or the subsequent period. The only thing stopping consumers from defaulting is that they lose the utility services of the durable good; it's also why the consumer cannot borrow to fund nondurable consumption since there would be no reason to repay the loan. With durable loans, in case of default, the lender repossesses the durable and sells it, receiving only $(1-\lambda)\overline{K}_t$. Period 2 is like period 1, except income is risky and the consumer consumes whatever liquid resources are left since the horizon is over. Decisions in period 2 are taken knowing what income in period 2 is. Figure 1 diagrams the arrival of information and the timing of decisions.

In order to explore the effects of default consumers are of two types. One type values nondurable consumption more so than the other. Consumers who value nondurable consumption highly default if $y_2 = y_l$, while consumers of the second type do not default. The first type of consumers default because they would rather back out of their loans and devote their resources to nondurable consumption. Call these two groups Defaulters (D) and Non-Defaulters (N). $\delta\%$ of consumers fall in group (D) while $(1-\delta)\%$ fall in group (N). Consumers who default in period 2, state l make their plans in earlier periods knowing they will default if $2l$ occurs. There is no myopia in this model.

As an example let $U^k(K) = \psi_i e^{-\mu K}$ and $U(C) = e^{-\mu C}$. The parameter ψ_i controls whether consumers default in the last period, state l, where $\psi_d < \psi_n$, sufficiently so that for defaulters the gain to nondurable utility from defaulting outweighs the loss to durable utility. Assume parameters are such that consumers choose not to default in any other state; this needs to be verified (see Appendix A). Let A denote any additional savings the consumer makes in the second period, and suppose y_0 is small enough that the consumer makes no downpayment and does not save in the first period. Fix the two-period interest rate at some level higher than the one-period interest rate. The optimal decisions of the two atomistic consumers, one who defaults, the other who does not, are given in Table 1.

Table 1: Specific Consumption Patterns

	μ	C_1	C_{2h}	C_{2m}	C_{2l}	A	*Matr.*
Defaulter	1.0	1.223	5.000	1.500	.500	.000	2
Non-Defaulter	1.0	1.223	5.000	1.500	.500	.000	2
Defaulter	3.0	1.321	4.898	1.398	.903	.399	2
Non-Defaulter	3.0	1.146	5.075	1.575	.575	.075	1

Notes:

1. Other parameters values are: $\alpha = .8$, $\beta = .99$, $R_f = 1/\beta$, $R_{B_1} = R_f$, $R_{B_2} = 1.023$, $\psi_d = .095$, $\psi_n = 1.194$, $\overline{K} = .992$, $\delta = .5$, $\lambda = .25$, $p_h = .25$, $p_m = .60$, $y_0 = 0.0$, $y_1 = 2.225$, $y_{2h} = 5.00$, $y_{2m} = 1.50$, and $y_{2l} = .50$. *Matr.* is the number of periods in which the loan is paid.
2. Consumption patterns from Table 2.

The solution points to an important effect that is likely robust in a variety of more complicated models. Consumers who default in period 2 weakly prefer longer maturities to those who do not default. In the particular example of Table 1, for $\mu = 3.0$, the defaulter chooses a two-period contract while the non-defaulter chooses a one-period contract, while for $\mu = 1.0$ they both choose two-period contracts. The proof of this weak preference is straightforward. Ignore the utility from durables since one can take the default decisions as optimal given the definitions for defaulters and non-defaulters. Let V_1 be the value to a consumer of paying the entire loan in period 1:

$$V_1 \equiv U(y_1 - R_{B_1}\overline{K} - A_1) + \beta EU(y_2 + R_f A_1),$$

which is the same for defaulters and non-defaulters. A_1 represents the savings of consumers who choose one-period constracts. Now suppose non-defaulters are indifferent between one- and two-period contracts:

$$V_{2n} \equiv U^c(y_1 - P - A_n) + \beta\big[p_h U^c(y_h - P + R_f A_n)$$
$$+ p_m U^c(y_m - P + R_f A_n) + p_l U^c(y_l - P + R_f A_n)\big] = V_1,$$

where A_n denotes the savings on non-defaulters who choose two-period contracts, and P is the per-period payment. Then defaulters strictly prefer two-period contracts:

$$\begin{aligned} V_1 = V_{2n} &< U(y_1 - P - A_n) + \beta\big[p_h U(y_h - P + R_f A_n) \\ &\quad + p_m U(y_m - P + R_f A_n) + p_l U(y_l + R_f A_n)\big] \\ &\leq U(y_1 - P - A_d) + \beta\big[p_h U(y_h - P + R_f A_d) \\ &\quad + p_m U(y_m - P + R_f A_d) + p_l U(y_l + R_f A_d)\big] \\ &\equiv V_{2d}. \end{aligned}$$

The strict inequality is from the default state $2l$, that is,

$$\beta p_l U(y_l - P + R_f A_n) < \beta p_l U(y_l + R_f A_n).$$

The weak inequality comes from optimizing on A; A_d is the argmax to the defaulter's problem, whereas A_n is not necessarily optimal. Therefore, under different parameterizations for the problem, it is possible that non-defaulters choose one-period contracts, while defaulters choose two-period contracts. The reverse is impossible.

The intuition for this weak preference is straightforward. The choice of maturities arises from two effects. The first effect is a smoothing effect which encourages consumers to choose longer maturities, thereby reducing per-period payments. The second effect is a prudence effect. As explained in Kimball (1990) any consumer with a VNM utility function with a positive third derivative will save for precautionary reasons, which Kimball refers to as prudence. Even if expected future income is equal to present income, the consumer will save in case the future turns out badly. This effect works somewhat opposite of the smoothing effect. Consumers with stronger prudence motives want to deliver more resources into the uncertain future. They can accomplish this by paying the loan off early; it's as if they were saving, earning an interest rate equal to the stated loan rate.[3]

People who default in period 2 prefer longer contracts because they have more of a smoothing motive and less of a prudence motive. These motives differ because for defaulters the relevant comparison is state 1 versus only states $2h$ and $2m$. State $2l$ is irrelevant for this choice because the consumer does not make a payment in it regardless of whether she chooses a one period contract or a two period contract. It does matter, however, in so far as it affects savings decisions in period 1. The smoothing motive is stronger for group (D) because period 2 has a higher expected income than it does for members of group (N). Members of group (D) are more encouraged to transfer some of the burden from the middle period to the last period than are members of group (N). They have less of a prudence motive because the worst outcome in period 2 is irrelevant. The relevant income stream for (D) dominates that for (N) in a first-order sense.

To see this difference in valuations across periods, suppose that defaulters

[3] This model is a bit deceptive in this respect because it only has three periods. Another way to deliver some resources in period 2 is to pay the loan in periods 1, 2, 3 and more, if such additional periods exist.

and non-defaulters have savings fixed and equal to A and that each have signed two-period contracts. Consider the marginal increment in utility from transferring \$1 from period 1 to 2. This marginal value for each consumer is given by

$$MV_d = -U'(y_2 - P - A) + \beta R_f\big[p_h U'(y_h - P + R_f A) + p_m U'(y_m - P + R_f A) + p_l U'(y_l + R_f A)\big]$$

$$MV_n = -U'(y_2 - P - A) + \beta R_f\big[p_h U'(y_h - P + R_f A) + p_m U'(y_m - P + R_f A) + p_l U'(y_l - P + R_f A)\big],$$

while the difference is

$$MV_d - MV_n = p_l \beta R_f\big[U'(y_l + R_f A) - U'(y_l - P + R_f A)\big] < 0.$$

This difference, $MV_d - MV_n$, is readily seen in Table 1 where (D) saves less than (N) when they both have two-period contracts and $\mu = 1.0$. Future dollars relative to current dollars are more valuable to non-defaulters than to defaulters.

One may be tempted to simply assert that a member of group (D) prefers a two-period contract more so than a members of group (N) simply because by choosing a two-period contract, the consumer preserves the option value of defaulting in period 2.[4] This option value is worth something to (D) while it is worthless to (N). While this statement is correct, it provides little in understanding what gives the option its value, especially in comparison to alternatives.

[4] The option value is related to the discounted increment in utility times its probability of exercise. In this particular model, the relevant difference is given by

$$\beta p_l[U_i^k(0) + U(y_l + R_f A_i) - U_i^k(\alpha^2 \overline{K}) - U(y_l - P + R_f A_i)].$$

This difference is positive for defaulters and negative for non-defaulters by definition. See Appendix A for more details.

The option's value is worth more, the higher the value of liquid marginal wealth (marginal utility of nondurable consumption) is in the states the option is exercised. Its value, however, is lower, the less probable its exercise is. Moreover, its value must be compared to its cost, which is the value of marginal wealth in other periods times the relevant interest rate. The analysis in this section explains this more fully.

It is also interesting to note in Table 1 the simultaneous borrowing and lending that occurs, for both defaulters and non-defaulters when $\mu = 3.0$. Rotemberg (1984) discusses this issue with a model similar to the one above. With high values of μ, consumers have relatively high degrees of risk aversion and relatively low elasticities of intertemporal substitution. Under these conditions each type of consumer is concerned with moving resources into the future. Non-defaulters do so both by choosing one-period contracts and by saving. Table 2 shows that as μ increases, non-defaulters first begin to save. A point is reached when instead they choose to pay the loan off earlier and save nothing. Finally, with $\mu = 3.0$ they do both. This makes the point that the contract length of loans on durable goods is a rough instrument consumers can use to smooth nondurable consumption.

To confirm the ideas above and that the results hold in an economy where lenders understand the effects described, the next subsection considers a full equilibrium.

2.2 Equilibrium

The equilibrium concept is simple. For a given set of interest rates R_{B_1} and R_{B_2} and the other parameters of the model, consumers choose either one-period or two-period contracts, whatever is optimal. Assume consumers differ in another dimension so that some (D) consumers and some (N) consumers choose one-period contracts while others choose two-period contracts. It implies on average a certain percentage of consumers who choose two-period contracts will default in the last period. In equilibrium the term structure of interest rates reflects this difference in default probabilities.

As for lenders, it is assumed that lenders are risk-neutral and operate in a competitive environment. Thus the expected return on each type of loan is equal to the return on a riskless bond. Since no borrowers default in period 1, $R_{B_1} = R_f$.[5] As for the two-period rate φ_d is the percentage of group (D) choosing two-period contracts while φ_n is the percentage of group (N) choosing two-period contracts. The proportion of people choosing two-period contracts and who default is

$$\gamma \doteq \frac{\delta \varphi_d p_l}{(1-\delta)\varphi_n + \delta \varphi_d}. \tag{2}$$

A lender who signs a two-period contract receives in expectation at the end of the second period

$$R_f P + (1-\gamma)P + \gamma \alpha^2 (1-\lambda)\overline{K}. \tag{3}$$

All consumers pay P in period 1, and therefore after investing in the riskless

[5] To break this exact tie, one can assume that a certain percentage of consumers, independent of period choice default in period 1. No subsequent analysis is changed.

asset, the lender receives $R_f P$. $(1-\gamma)$ of the consumers pay P in the second period; γ consumers default, leaving only $\alpha^2(1-\lambda)\overline{K}$. The zero profit condition implies (3) equals $R_f^2\overline{K}$ since the lender can also guarantee receipts from a riskless roll-over strategy.[6] Defining $R_{B_2} = (1+R_f)P/\overline{K}$ yields after rearranging

$$R_{B_2} = \frac{R_f + 1}{R_f + 1 - \gamma}\left(R_f R_{B_1} - \gamma\alpha^2(1-\lambda)\right) \geq R_f R_{B_1}. \tag{4}$$

The equilibrium is Nash; it is simply the set of interest rates that deliver the optimizing behavior and the set of contracts, optimal for all consumers, that generates the default rates that justify the loan rates. These loan rates imply zero profit in expectation.

As is indicated above, besides the relative weight they give durables, consumers differ in another dimension. To explore various effects, consider three different economies in which consumers can differ. In each particular economy consumers differ only in one of these added dimensions. First, to explore the effects of attitudes towards risk and intertemporal substitution, suppose $\mu \in [\underline{\mu}, \overline{\mu}]$, where μ is the coefficient of absolute risk aversion. Second, to isolate the effects of the motive to smooth payments let

$$y_1 = p_h\overline{y}_h + p_m\overline{y}_m + p_l\overline{y}_l - v/R_f$$

$$y_2 = y_2 + v,$$

where $v \in [\underline{v}, \overline{v}]$. The lower y_1 is relative to Ey_2, the more need the consumer has to smooth payments over both periods. Third, to explore the effects of risk

[6] One can also introduce a two-period riskless bond whose gross return differs from R_f^2 for some real or nominal term structure issues not modeled. The subsequent analysis remains unchanged.

let

$$y_h = \overline{y_h} + sp_m$$

$$y_m = \overline{y_m} - sp_h,$$

where $s \in [\underline{s}, \overline{s}]$. This last static exercise keeps Ey_2 and y_l constant while changing the variance of y_2. As s increases, y_m decreases, and both defaulters and non-defaulters face more risk. As risk increases the consumer wants to deliver more resources into period 3. They can do this by saving, or they can pay the loan off entirely in the second period. Each of these three variations are such that the parameters are uniformally distributed over the intervals for both (D) and (N).

2.3 Equilibrium Analysis

Table 3 contains the resulting statistics from solving for the equilibrium in the three different economies, while Table 2 contains savings and consumption decisions for several specific consumers in these economies.

The second economy, where y_1 is varied, investigates the effects of smoothing. As the level of y_1 decreases relative to Ey_2 more consumers are inclined to choose two-period contracts. In the parameterizations considered it is non-defaulters who switch from one-period contracts to two period contracts. Individuals with high smoothing motives (low y_1) choose two-period contracts, while those with low smoothing motives (high y_1) choose one-period contracts.

The third economy, where $|y_h - y_m|$ is varied, investigates the effects of prudence. As the variance of y_2 increases more consumers are inclined to choose one-period contracts. As in the second economy this effect hits non-defaulters more strongly than defaulters. Individuals with high prudence motives (high $|y_h - y_m|$) choose one-period contracts, while those with low prudence motives (low $|y_h - y_m|$) choose two-period contracts. As discussed previously, this effect may be ambiguous. With more periods consumers may choose even longer contracts to ensure that payments in any single month are low. This analysis depends on the stochastic process of income. If negative shocks to income are temporary then longer contracts are preferred compared to more permanent shocks. If shocks are permanent, the consumer *ceteris paribus* would rather have the payments already finished.

In equilibrium, a further consideration arises as the price of one- and two-period contracts diverge to reflect the different probabilities of default. (D) consumers may choose a one-period contract because the option value is not worth as much compared to the extra burden of paying a higher rate in non-default periods. This effect is seen in Table 3 where consumers in the first economy with very low values of μ choose one-period contracts. They give up the default option because it is worth very little to them in expectation. The curvature of their utility function is low enough that they are practically risk-neutral, and thus, they maximize the expected sum of consumption in each period, discounted by β.

Changes in various parameters affect each economy in similar ways. As mentioned above one reason why the average term length of contracts has increased steadily since the mid-70's may be that the depreciation rate of automobiles has declined. A second reason may be that real personal income for at least one segment of the population has not grown as quickly as the real value of automobiles. The editors of *Automotive News* suggest this as a reason why the contract length varied from 1979 to 1980. Each of these effects can be captured in this simple model. For the depreciation rate one simply reduces the value of α, while for the second, one increases the value of $\overline{K}$. Each of these exercises result in more consumers choosing two-period contracts as is evidenced in Figures 2 and 3. These figures show the percentage of each group choosing two-period contracts in the economy where consumers differ in μ as well. For increases in α the return to lenders on a default increases, thereby reducing R_{B_2} in equilibrium. More consumers, especially (N), choose two-period contracts as the relative cost is reduced. For increases in $\overline{K}$, more consumers have a stronger smoothing motive as paying off the entire loan in period 1 becomes more burdensome, more so than the increased damage from paying the loan off in state $2m$ for defaulters or in states $2l$ and $2m$ for non-defaulters. The relative mix of people who sign two-period contracts improves as more members of group (N) move to two-period contracts than members of group (D). This also decreases R_{B_2}, amplifying the effect until equilibrium is reached. The changes in the equilibria for both static exercises are similar in the end though the mechanisms are different.

3. Microeconomic Evidence

This section is intended to test whether the risk premium and the length of the loan contract are related in a microeconomic data set at one point in time. The data set is the Consumer Expenditure Survey in the Fourth Quarter of 1985, which contains data on personal characteristics and financing data for those who recently bought motor vehicles. I restrict the sample to those who bought vehicles in the last twelve months and who did not have missing data on a subset of variables that are of *a priori* importance. This leaves 1,561 observations, 786 financed their car, while 775 paid in full. For those who financed their car, the amount the consumer borrowed, the amount of the downpayment, the length of the contract, both when the payments start and the number of monthly payments, and the amount of the monthly payments are available.

To calculate the risk premium for the loan contract of individual i, let P_i denote the principal of the loan signed at time t_i, and let p_i denote the monthly payments. Every contract in the sample called for monthly payments. Let n_i denote the number of months before consumer i has to begin payments, and let N_i equal the total number of payments. Let $\varphi(t,s)$ denote the price as of time t of a discount bond that pays \$1 at time s. These prices are calculated from McCulloch (1990) and are explained more fully in the Data Appendix.

If there were no risk premium, the discounted stream of future payments would equal the principal, that is, the value of the portfolio of p_i units of discount bonds for each period the payments are made would equal P_i. Any difference between the discounted stream of payments and the principal, is related to default risk, since to a first approximation this valuation method adjusts for inflationary expectations and real term structure effects. Since the period the contracts are signed are all within a twelve month period anyway, these considerations do not differ greatly across observations. The risk premium is calculated by

$$R_i^* = \left(p \sum_{s=t+n_i}^{t+n_i+N_i} \varphi(t,s) - P_i\right) / P_i. \tag{5}$$

Note this method defines the risk premium atemporally; it does not produce any correlation endogenously. It values the future payment stream in terms of today's prices and then asks how much large this value is than the amount the lender pays to buy it, the principal. The contract length is simply $n_i + N_i$.

The simplest analysis is to consider the correlation between individual risk premia and the contract lengths. This correlation over 786 observations is 0.227. A slightly more advanced analysis is to take those consumers who finance their purchases and regress the risk premium on the contract length, the downpayment, and other risk factors such as the ratio of income to the loan value. As is explained in Maddala (1983) this introduces a bias in the coefficient estimates. The reason is simply that it ignores the effects of the decision to finance in the first place. The correct method is to use a tobit estimator that takes into account the censoring of the sample in the first step.

Traditional tobit analysis assumes that the factors involved in censoring are the same that explains the relationship conditional on observation. In this case, however, the financing outcome may be influenced by several factors other than time horizon which are included in the second stage. One method of estimation is to use a two-step procedure as advanced by Heckman (1979) and amended by Lee *et al.* (1980). The procedure is to first model the finance/not finance outcome with a typical maximum likelihood probit estimator. The predicted values are then transformed and included as a regressor in the second stage where least squares is applied. Let $\hat{y}_i$ denote the predicted values from the first stage, ϕ the density of a standard normal and Φ the cumulative normal, the transformation is simply $\phi(\hat{y}_i)/\Phi(\hat{y}_i)$. This variable is included as a regressor because otherwise the expectation of the regression error in the second-stage conditional on the other right-hand side variables is not zero. The non-zero expectation leads to a classical omitted variables bias; including the Heckman correction factor, ϕ/Φ, clears up the problem.

The analysis, however, leaves out the decision to buy a car in the first place. If borrowing constraints are important, that is, some consumers are denied credit because of observable characteristics or other factors, then some may want to buy a car but cannot. This prior step in the decision process should also be included in the estimation. It is excluded because not enough data is available to explain the purchase decision in the first place. Most importantly, the data on the characteristics of the vehicle people owned previous to the one they just bought is missing. This presumably is the most important element in the decision to buy a new one. Therefore, the subsequent analysis, especially the distributional assumptions, are all conditioned on buying a car in the first place. Care must be given in interpreting the estimated coefficients.

The results from estimating the system are reported in Table 4a. The first half reports the results from the first-stage probit analysis. The right hand side variables include a dummy for whether the auto is used or new ($\text{Used}_i = 1$), a dummy for whether the consumer owns a car which was financed in the past

($\text{Borrowed}_i = 1$ if so), the ratio of income to the purchase price (Income_i), the ratio of existing liquid assets, that is checking accounts, savings accounts, stocks and bonds, to the purchase price (Liquid_i), a dummy for whether the consumer is unemployed ($\text{Unemp}_i = 1$), a dummy for whether the family has multiple wage earners ($\text{Multearn}_i = 1$), a dummy for whether the consumer attended college ($\text{College}_i=1$), a dummy for whether the consumer is retired ($\text{Retired}_i = 1$), a dummy for whether the consumer is white ($\text{White}_i = 1$), a dummy for whether the consumer lives in a rural area ($\text{Rural}_i = 1$), a dummy for whether the consumer is a female ($\text{Female}_i = 1$), the number of members in the family (Fmlsize_i), and a constant (Constant_i).

The table suggests that most of these factors explain the finance outcome. Used cars are less likely to be financed than new cars, consistent with the evidence in the *Automotive News*. Individuals with more financial assets or higher income are less likely to finance their purchases (Income & Liquid). The retired also are less likely to finance. They presumably cannot earn a higher riskless return on their assets as the loan rate, and they have the ability to pay for the motor vehicle in full. Paying in cash saves them money. On the other hand, the unemployed also are less likely to finance, which makes sense in some credit rationing stories. Since these people do not have an expected high stream of income in the future to pay off their debt, they are more likely to default. Suitably modifying the model of the previous section by increasing the probability of the worst state would deliver this prediction. Lenders do not raise the interest rate to reflect this probability but instead refuse to lend at all. Multiple earner families are more likely to finance their purchases, presumably because they more likely can absorb some bad news yet still make their car payments.

Three other factors are also significant or close to traditional levels. Being white comes in negative, which is surprising given traditional redlining stories. The white dummy may pick-up other wealth levels measured poorly by the variables above. Owning a car that one financed in the past is also negative. Again a positive number is expected, but since this variable is one only if the family owns more than one vehicle, it too may be picking up wealth effects not captured elsewhere. Finally, family size is significantly positive. This may be due to the smoothing effects described above, or it may simply reflect younger families that do not have the resources to pay in full at the time of the future. Having attended college, living in a rural area or being female are insignificant.

Table 4b shows that 74.8% of the finance outcomes are correctly predicted. This number comes from adding the number of families who did not finance their automobiles and had an estimated probability less than .5 and the number who did finance and had an estimated probability greater than .5 and then dividing by the total number of observations, 1,561. The pseudo R^2 (Maddala, 1983), another goodness-of-fit measure that behaves like the traditional OLS R^2, equals .345.[7]

As for the second stage two equations are of interest. The first equation considers the determination of the risk premium. In most models the risk premium depends on the factors that affect the probability of default and affect the value

[7] Its formula is $\text{Var}(\hat{y}_i)/(E(\hat{y}_i) - E^2(\hat{y}_i))$.

of the collateral in case of default. Variables such as Income_i, Multearn_i and Ownhome_i are taken as indicators of default risk, where Ownhome_i is a dummy for whether the consumer owns their home. The higher the first two, the more likely the consumer will be able to make their payments. The last may also reflect wealth that can be used to offset negative income shocks, and it also may reflect a good credit rating, *i.e.* a smaller propensity to default. Variables such as Downpayment_i and Used_i are related to the value of the automobile in case of default. Downpayment_i is defines as one minus the ratio of the principal of the loan to the purchase price of the automobile. The higher the downpayment, the lower is the principal to the value of the car. At any point in time, the lower the principal–to–value ratio, the less the lender loses in case of default. If the automobile is used at any time in the future, its depreciation may be worse than the remaining principal due.

This analysis leads to the following specification

$$\begin{aligned} R_i^* = \alpha_0 &+ \alpha_1 \text{ContractLength}_i + \alpha_2 \text{Downpayment}_i + \alpha_3 \text{Used}_i \\ &+ \alpha_4 \text{Income}_i + \alpha_5 \text{Multearn}_i + \alpha_6 \text{Ownhome}_i \qquad (6) \\ &+ \alpha_7 \text{Othrcrd}_i + \alpha_8 \phi(\hat{y}_i)/\Phi(\hat{y}_i) + \epsilon_i^r. \end{aligned}$$

The second to last variable Othrcrd_i is a dummy for whether the consumer obtained financing from some source other than an auto dealer or a financial institution. One example of such a source is relatives who may not charge market interest rates. The last variable is the Heckman correction. The contract length is included to test if after controlling for other observable risk factors, the contract length matters for the length of the risk premium, and more specifically, if it is positive.

Another test is to consider the determinants of the contract length itself. The model presented in the previous section suggests that those consumers who have higher smoothing and prudence considerations will choose longer contract lengths. Smoothing motives are related to the expected slope of future income. Variables such as Age_i and Retired_i are observable indicators. The younger the consumer is, the more likely the income profile is upward sloped, and hence, the more likely the consumer will choose a longer contract. If the consumer is retired, the income profile is flat, and hence, the less likely the consumer will choose a longer contract. The variable Famsize_i can be one measure of both smoothing and prudence motives as the larger the family size, the more commitments the consumer has. The family has other important expenses that have to be made each period or that may arise stochastically in the future. This means the consumer is more likely to be squeezed in the future, and thus, the consumer wants to limit the value of future payments. This leads to longer contracts. In addition, the lower income is compared to the purchases price, the more burdensome the monthly payment, keeping the contract length constant is. These horizon factors may even be important in a model where signaling default probabilities through the contract length is not important.

The suppliers of credit may also have some preferences on contract length. As before Othrcrd_i is included. In addition a dummy variable for whether the supplier of credit is an automobile company, $\text{Autocrd}_i = 1$, if so. The evidence discussed earlier from GMAC annual reports suggests automobile companies care about the contract length. The model above may explain this preference, and it may not be different from the preferences of finance companies. If both finance companies and auto financing arms have the same preferences, then the slope coefficient on this variable will be zero. One cannot rule out *a priori*, however, that it is non-zero.

This discussion suggests the following regression:

$$\begin{aligned}\text{ContractLength}_i = \beta_0 + \beta_1 R_i^* + \beta_2 \text{Used}_i + \beta_3 \text{Income}_i \\ + \beta_4 \text{Age}_i + \beta_5 \text{Retired}_i + \beta_6 \text{Famsize}_i \\ + \beta_7 \text{Othrcrd}_i + \beta_8 \text{Autocrd}_i + \beta_9 \phi(\hat{y}_i)/\Phi(\hat{y}_i) + \epsilon_i^c.\end{aligned} \tag{7}$$

The risk premium is included to test whether the two are correlated after controlling for these factors. It may be positive because the higher R_i^* is, the higher the monthly payments are. *Ceteris paribus* the consumer will still want to smooth the payments over time. It may be negative if the smoothing motive is already controlled for and therefore weak, and the consumer wants to limit the financial loss from paying a high premium. Most importantly, however, if there are other determinants of the length that are missing because they are unobservable, the correlation will still last. The model of the previous section explicitly has an unobservable variable, ψ_i, which helps determine the contract length. Leaving variables such as ψ_i out of the regression equation leads to an omitted variables bias, which will preserve the positive relationship. The standard errors for α in equation (6) and β in (7) are corrected for the inclusion of an estimated regressor, as is described in Lee *et al.* and Lee and Trost (1978).

Table 5 shows the results from the two second-stage regressions where neither regression is instrumented; the first half reports the results for (6), while the second reports the results from (7). The corrected standard errors are large, so firm conclusions are hard to draw. Still, it is clear that the risk premium and contract length are positively correlated. Each enters positively and significantly in the other's regression. The downpayment may have some effects on the risk premium, and financing the vehicle through some other place than a financial institution or the auto dealer significantly lowers the risk premium. The standard errors on the other variables are so large that no other coefficients is significantly different from zero. That the horizon variables are insignificant in the second regression equation (7) is disappointing.

Having shown suggestive evidence that the contract length and the risk premium are positively correlated, one may want to estimate a more structural relationship. Running both sets of regressions suggest that one may want to run instrumental variables. In the first equation instrumenting for the contract length should still give a positive α if the equilibrium signaling conditions are important. In the second equation, instrumenting for the risk premium also does not alter the implications for a positive correlation. Lee *et al.* (1980) shows how to use two-stage estimation techniques in this environment.

Table 6 reports the results from this IV estimation. It uses the exclusion restrictions implied by the simultaneous equations through two-stage least squares to identify the parameters. As before the corrected standard errors blow-up so that one cannot make strong conclusions. In the first regression the contract length enters negatively and insignificantly. Nothing else comes in important.

In the second regression the coefficient on the risk premium is positive but insignificant. It is also the only variable that is close to being important. That the other coefficients are insignificant, as they are in Table 5, suggests identification is problematical.

4. Macroeconomic Evidence

Credit markets are an important element in the consumer automobile market. In 1989 70% of automobiles bought were at least partially financed. Though this figure fluctuates somewhat from year-to-year, in 1990 it was only 62%, it has remained between 60% and 70% since the 1950's.

Most of the empirical and theoretical analysis considers the relationship between the quantity of loans issued and its price, either the gross loan rate or net of some risk-free rate. Figure 4 plots the average loan rate and the analogous Treasury Note rate. Before 1983 the average rate reported is the annualized rate for a three year loan, while after the 1983 the average rate is for a four year loan. The Treasury Note rates before 1983 are three year rates, while after 1983 they are an average of three and five year rates. The figure illustrates that while the loan rates trace the general movements in the Treasury rates the year-to-year movements in the loan rates are dampened relative to the Treasury rates.

Figure 5 illustrates this effect also. It plots both the spread between the loan and Treasury rates and the Treasury rate itself. A regression of the changes in the spread on changes in the Treasury rate yield a coefficient of $-.917$ with a t-statistic of -15.129. This strong correlation suggests that variation in the difference between the two rates does not arise solely in changes in default risk. While there are several explanations why institutional considerations make different financial assets less than perfectly substitutable (Kashyap, Stein and Wilcox, 1991) and therefore helps explain the correlation, one explanation why auto loan rates do not move closely with the cost of funds is that non-price factors may help to clear the market.

One such non-price factor, considered above, is the length of the loan contract. A few observations on the average length stand out. First, Figure 6 shows a strong upward trend in the average length of loan contracts for both new and used cars. In 1974 the average contract on a new car lasted 35.7 months; in 1990 the average contract lasted 54.6 months. An interesting side question discussed a bit earlier is the reason for the upward trend. Weber (1984) suggests automobiles hold their value better, and therefore, creditors are better protected. Holding everything else equal, loans can be longer because the collateral is more valuable. The model above confirms this effect. I digress in order to suggest that the trend in contract lengths is immediate evidence that some of the factors discussed above are at work. Note that this trend in the contract length is not offset by increases in the average downpayment (Figure 7). In fact a simple regression of the downpayment on a constant and a trend yields a significant, negative coefficient, though only 18.5% of the variation is explained, which is small.

The above model and discussion suggests the following empirical specification. As in the microeconomic section consider two equations, one for the aggregate risk premium, one for the average contract length. For the risk premium, riskless Treasury securities and consumer loans may be imperfect investment substitutes as discussed in Kashyap *et al.* (1991). Thus, the riskless rate R_{ft} should be included. Second, there may be variation over time in the default probabilities of individuals due to aggregate conditions. As a proxy for expected default probabilities in the future, I use the expected growth rate in consumption over the next three or four years (ConsmpGrowth_t). Current conditions may also matter, and thus, I include a recession dummy, Recess_t. Finally, movements in the average downpayment also may effect the risk premium.

Let M_t denote the maturity of the loan, R_t^* the risk-premium, that is the average auto loan rate minus the appropriate Treasury Note rate (three years before 1983:1 and the average between three and five years after 1983:1) and Dum83_t a dummy variable equal to one if $t \geq 1983{:}1$, zero otherwise. The equation for the risk premium is:

$$R_t^* = \alpha_0 + \alpha_1 M_t + \alpha_2 \text{Downpymt}_t + \alpha_3 \text{ConsmpGrowth}_t + \alpha_4 R_{f\,t} + \alpha_5 \text{Recess}_t + \alpha_6 \text{Dum83}_t + \eta_t^r. \tag{8}$$

The average maturity M_t is included to test whether it is important after controlling for these other factors.

For the average maturity, some of the same variables used above are included in the second equation. Business cycle conditions may effect the choice of maturities. If the future is expected to be good compared to current conditions, consumers may want to push the burden of loan payments in the future by choosing long contracts. Current conditions may also affect the maturity choice for the same reasons. To account for these smoothing and risk effects, the expected consumption growth rate, a recession dummy and the ratio of the value of the loan to per-capita income P_t/Y_t are included. In addition, as is obvious from an examination of Figure 6, the average contract length has increased steadily. Weber (1984) suggests increases in the value of collateral help explain this trend. A trend term is included to account for these effects.

The equation for the average maturity is

$$\begin{aligned} M_t = \beta_0 + \beta_1 R_t^* + +\beta_2 \text{ConsmpGrowth}_t + \beta_3 t + \beta_4 \text{Recess}_t \\ + \beta_5 P_t/Y_t + \eta_t^m. \end{aligned} \tag{9}$$

As before the risk premium is included in order to test its relevance after other factors are accounted for.

Using a rational expectations argument, one can include the realized real interest rate and the realized growth rate instead of the expected rates as long as one instruments for the expectation error in the realized rates. Any variable dated at t or earlier is appropriate. Thus, the variables used to instrument for consumption growth are the contemporaneous and past six, one-month growth rates in consumption. The instruments used for the expectation error in realized inflation, are the current, appropriate nominal Treasury Note rate and the contemporaneous and past six one-month inflation rates in the consumption deflator.

Because of the serial correlation implied in the use of overlapping data (Hansen and Hodrick, 1980), a GLS estimation procedure combined with the optimal instruments technique of Hansen (1982) is used. The procedure takes two steps. First, the coefficients are estimated using OLS. The asymptotic covariance matrix of the residuals with the instruments is calculated using the technique of Newey and West (1987) to ensure positive definiteness. 48 lags are used; more lags do not change the coefficient estimates but do improve the standard errors. The reported results usually have the smallest t-statistics in absolute value. This asymptotic covariance matrix is then used as part of a GLS estimator in the second step.

Table 7 reports the estimation results from (8) and (9), when the maturity and the risk premium are not instrumented; *i.e.* the implied simultaneity of (8) and (9) is not accounted for. In the first equation the maturity is positive and significant. Raising the length by one month raises the premium by 28 basis points after controlling for other factors. I interpret this as evidence for the effects discussed above. The real interest rate is significant and negative. Increasing the real interest rate by one percentage point decreases the risk premium by about a half, suggesting the supply of credit is still important. Note that this estimated effect is smaller in absolute value than simply regressing R_t^* on the nominal T-Note rate. The expected growth rate in consumption is positive and significant. While the above story suggests this effect should be negative, it is important to note that this procedure does not identify demand and supply curves. The positive relationship may reflect credit supply factors not full captured in the real interest rate. The coefficient on the recession dummy is positive and significant, suggesting a higher risk premium by 73 basis points.

The downpayment is also positive and significant. If all of the other effects were controlled, one would expect this coefficient would be negative. A higher downpayment, however, may be indicative of higher risk factors not controlled for elsewhere and not fully offset by higher downpayments. In other words, when default risk is high, the equilibrium risk premium and the downpayment may be jointly higher. One such risk factor not fully accounted for elsewhere is compositional effects. The number of credit worthy individuals who ask for and receive loans may vary over time. Such variation may be reflected both in the risk premium and in the downpayment.

The second equation suggests some of the same effects. The risk premium enters positively and significant, though its effect is economically small. As before the growth rate in consumption has the opposite sign expected, as does P_t/Y_t. These may also be due to compositional effects. The recession dummy is positive, as expected, but it is insignificant. The trend term is of course very important.

Table 8 reports the results from estimating the same equations (8) and (9), except this time the maturity and risk premium are instrumented. The exclusion restrictions provide identification. The first equation, when the risk premium is the dependent variable, is hardly affected by instrumenting. The coefficient on the maturity length increases by only 4 basis points. Presumably, this is because the risk premium has such a small effect on the maturity length in the second equation. All of the other variables are not qualitatively or quantitatively altered.

The second equation, when the maturity length is the dependent variable, on the other hand, is somewhat affected by instrumenting. The coefficient on the risk premium becomes negative, and nearly significant at conventional levels. The economic importance of the effect, however, is still small. Increasing the risk premium by a full percentage point, decreases the average contract length by only .06 months. The coefficient on ConsmpGrowth$_t$ is a bit closer to zero, while the coefficient on P_t/Y_t is even more negative. Both are still significant. The coefficient on the recession dummy is zero for all practical purposes.

5. Conclusion

The paper suggests that in an environment where borrowing constraints for nondurable consumption is important, consumers will have preferences over non-price terms in their auto loan contracts. Attitudes towards risk, intertemporal smoothing and default affect the choice of contracts. In equilibrium lenders take these preferences in account, building the implied set of default rates in the offered term structure of loan rates. Heterogeneous agents separate themselves by choosing different combinations of interest rates and contract lengths. Some microeconomic and some macroeconomic evidence is presented that non-price terms influence the equilibrium contracts offered and accepted. One non-price term emphasized in this paper and neglected elsewhere is the contract length. The paper offers that the fact that the contract length matters has implications that reach farther than durable credit markets: consumers have to take costly measures to smooth nondurable consumption.

The next step of the analysis is to test what aspects of individual loans move to produce movements in aggregate series. One hypothesis is that the terms of most contracts move over time because of aggregate risk factors or the supply of credit. Some evidence in the macroeconomic section such as the recession dummy suggests that aggregate variables that affect all individuals are important. A different hypothesis is that individual contracts do not change as much but that the mix of individuals requesting and receiving loans does move due to aggregate circumstances. Such a hypothesis may explain the otherwise anomalous results on some of the macroeconomic variables such as the consumption growth rate and the average downpayment. An analysis of the terms of individual loan contracts over time should shed light on the amount of information that lies in aggregate series such as average contract lengths and average interest rates that are presented in Section 4. The relationship among movements in default rates, contract lengths and interest rates are of primary interest.

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Table 2: Specific Consumption Patterns

	Defaulters						Non-Defaulters					
$\mu \in [.001, 2.0]$	C_1	C_{2h}	C_{2m}	C_{2l}	A	*Matr.*	C_1	C_{2h}	C_{2m}	C_{2l}	A	*Matr.*
$\mu =$ .001	1.223	5.000	1.500	.500	.000	1	1.223	5.000	1.500	.500	.000	1
$\mu =$ .002	1.720	4.495	.995	.500	.000	2	1.223	5.000	1.500	.500	.000	1
$\mu =$ .011	1.720	4.495	.995	.500	.000	2	1.223	5.000	1.500	.500	.000	1
$\mu =$ .012	1.720	4.495	.995	.500	.000	2	1.712	4.503	1.003	.003	.008	2
$\mu =$ 1.644	1.378	4.840	1.340	.845	.342	2	1.263	4.957	1.457	.457	.458	2
$\mu =$ 1.645	1.378	4.840	1.340	.845	.342	2	1.223	5.000	1.500	.500	.000	1
$\mu =$ 3.000	1.321	4.898	1.398	.903	.399	2	1.146	5.075	1.575	.575	.075	1
$y_l \in [-.5000, .0000]$	C_1	C_{2h}	C_{2m}	C_{2l}	A	*Matr.*	C_1	C_{2h}	C_{2m}	C_{2l}	A	*Matr.*
$y_l =$ -.5000	1.438	4.779	1.279	.785	.282	2	1.351	4.867	1.367	.367	.368	2
$y_l =$ -.0693	1.438	4.779	1.279	.785	.350	2	1.351	4.867	1.367	.367	.437	2
$y_l =$ -.0692	1.438	4.779	1.279	.785	.350	2	1.292	4.931	1.431	.431	.000	1
$y_l =$ .0000	1.438	4.779	1.279	.785	.777	2	1.353	4.868	1.368	.368	.365	1
$s \in [-1.500, 1.500]$	C_1	C_{2h}	C_{2m}	C_{2l}	A	*Matr.*	C_1	C_{2h}	C_{2m}	C_{2l}	A	*Matr.*
$s =$ -1.500	1.554	3.764	1.539	.668	.167	2	1.447	3.871	1.646	.271	.273	2
$s =$ 1.086	1.341	5.530	1.107	.883	.379	2	1.269	5.602	1.179	.451	.451	2
$s =$ 1.087	1.341	5.530	1.106	.883	.379	2	1.223	5.652	1.228	.500	.000	1
$s =$ 1.500	1.302	5.818	1.043	.992	.418	2	1.223	5.900	1.125	.500	.000	1

Notes:

1. Other parameters values are: $\alpha = .8$, $\beta = .99$, $R_f = 1/\beta$, $\overline{K} = .992$, $\delta = .5$, $\lambda = .25$, $y_0 = 0.0$, $p_h = .25$, $p_m = .60$, and the base cases for income are $y_1 = 2.225$, $y_{2h} = 5.00$, $y_{2m} = 1.50$, $y_{2l} = .50$. *Matr.* is the number of periods in which the loan is paid.

Table 3: Equilibrium Results

		$\mu \in [.001, 3.000]$	$v \in [-.5000, .0000]$	$s \in [-1.500, 1.500]$
Defaulters	2-period if	$\mu \in [.002, 3.000]$	$v \in [-.5000, .0000]$	$s \in [-1.500, 1.500]$
	ψ_d	.095	.409	.407
Non-Defaulters	2-period if	$\mu \in [.012, 1.644]$	$v \in [-.0692, .0000]$	$s \in [-1.500, 1.086]$
	ψ_n	1.194	.879	.965
	R_{B_2}	1.023	1.024	1.023

Notes:

1. Other parameters values are: $\alpha = .8$, $\beta = .99$, $R_f = 1/\beta$, $\bar{K} = .992$, $\delta = .5$, $\lambda = .25$, $y_0 = 0.0$, $p_h = .25$, $p_m = .60$, and the base cases for income are $y_1 = 2.225$, $y_{2h} = 5.00$, $y_{2m} = 1.50$, $y_{2l} = .50$.

Table 4a: First-Stage (Probit) Results

	Coeff.	*Std. Err.*	*t-stat*
Used	-.857	.084	-10.170
Borrowed	-.366	.078	-4.711
Income	-.048	.004	-11.221
Liquid	-.069	.024	-2.832
Multearn	.395	.097	4.089
Unemp	-.123	.161	-.763
College	.041	.078	.523
Retired	-.514	.149	-3.448
White	-.209	.131	-1.592
Rural	.023	.098	.233
Famsize	.086	.030	2.899
Female	.101	.107	.947
Constant	.989	.175	5.653

Table 4b: First-Stage (Probit) Results

	pr. $< .5$	*pr.* $\geq .5$	*total*
Paid in Cash	556	219	775
Borrowed	174	612	786
Total	730	831	1561

Notes:

1. Dependent variable equlās Finance=1 if car was financed.

Table 5: Second-Stage Results

	Dependent Variable = Risk Premium		
	Coeff.	*Std. Err.*	*t-stat*
Cntrlngth	.188	.045	4.150
Downpymt	.018	.021	.866
Constant	-.026	.100	-.261
Used	.040	.058	.697
Income	.001	.605	.001
Multearn	-.004	.078	-.046
Ownhome	.002	.075	.028
Othrcrd	-.034	.012	-2.948
Heckman	.007	.061	.117
	Dependent Variable = Contract Length		
	Coeff.	*Std. Err.*	*t-stat*
Riskprm	.521	.207	2.518
Constant	.531	3.040	.175
Used	-.104	1.783	-.058
Age	-.000	127.553	-.000
Retired	.003	.594	.005
Famlsize	.003	10.454	.000
Income	-.006	18.747	-.000
Othrcrd	-.024	.186	-.128
Autocrd	.016	.815	.020
Heckman	.221	1.271	.174

Notes:

1. Standard errors are corrected for the estimated Heckman correction, as described in Lee *et al.* (1980).

Table 6: Second-Stage Results — Instrumented

	Dependent Variable = Risk Premium		
	Coeff.	*Std. Err.*	*t-stat*
Cntrlngth	-.132	.512	-.257
Downpymt	-.012	.177	-.065
Constant	.152	1.052	.144
Used	.009	.611	.014
Income	-.001	6.414	-.000
Ownhome	.007	.791	.009
Multearn	.002	.827	.002
Othrcrd	-.049	.065	-.743
Heckman	.076	.441	.171
	Dependent Variable = Contract Length		
	Coeff.	*Std. Err.*	*t-stat*
Riskprm	1.847	2.213	.835
Constant	.443	2.699	.164
Used	-.126	1.581	-.080
Age	-.000	113.070	-.000
Retired	.025	.528	.047
Famlsize	.002	9.267	.000
Income	-.005	16.618	-.000
Othrcrd	.035	.192	.181
Autocrd	.019	.722	.026
Heckman	.196	1.128	.174

Notes:

1. Standard errors are corrected for the estimated Heckman correction, as described in Lee *et al.* (1980).
2. The instrument technique is two-stage least squares applied to the the risk premium and the contract length.

Table 7: Aggregate Time Series Regressions

Dependent Variable = Risk Premium

	Coeff.	*Std. Err.*	*t-stat*
Contract Length	.276	.032	8.711
Down Payment	.283	.048	5.910
Consmp Growth	.249	.055	4.515
R_f	-.520	.033	-15.764
Recession	.732	.275	2.661
Dum83	-.920	.534	-1.724
Constant	15.266	4.510	3.385

Dependent Variable = Average Maturity of New Cars

	Coeff.	*Std. Err.*	*t-stat*
Risk Premium	.078	.029	2.706
Consmp Growth	-.264	.050	-5.302
Trend	.111	.002	50.540
Recession	.290	.239	1.213
P_t/Y_t	-3.725	2.234	-1.667
Constant	37.732	1.578	23.907

Notes:

1. Each regression uses the right-hand side regressors as instruments except for R_f and Consmp Growth. Instruments for R_f are the nominal 3-month Treasury Note rate and the contemporary and the last six one-month inflation rates in car prices. Instruments for Consmp Growth are the contemporary and last six one-month growth rates in consumption.
2. The estimation procedure uses Hansen's (1982) estimation procedure though ammended given the results of Newey-West (1987) with 48 lags. The time length is 1973:8 1987:11.
3. A test of the overidentifying restrictions yields a $\chi^2_{14} = 4.236$ for the first regression with a p-value of .994 while $\chi^2_6 = 2.630$ for the second regression with a p-value of .854.

Table 7: Aggregate Time Series Regressions – Instrumented

Dependent Variable = Risk Premium

	Coeff.	*Std. Err.*	*t-stat*
Contract Length	.323	.016	19.616
Down Payment	.301	.048	6.303
Consmp Growth	.280	.036	7.674
Dum83	-1.253	.281	-4.464
R_f	-.561	.022	-26.040
Recession	.714	.138	5.176
Constant	14.782	4.816	3.531

Dependent Variable = Average Maturity of New Cars

	Coeff.	*Std. Err.*	*t-stat*
Risk Premium	-.060	.037	-1.631
Consmp Growth	-.193	.016	-11.867
Trend	.108	.002	65.380
Recession	.044	.119	.372
P_t/Y_t	-5.199	1.372	-3.788
Constant	39.073	.994	39.306

Notes:

1. Each regression uses as instruments a constant, a trend, Dum83, the nominal 3-month Treasury Note rate, the contemporary and the last six one-month inflation rates in car prices, the contemporary and last six one-month growth rates in consumption, the recession dummy, and P_t/Y_t. In addition the downpayment is included as an instrument when the risk premium is the dependent variable.
2. The estimation procedure uses Hansen's (1982) estimation procedure though ammended given the results of Newey-West (1987) with 48 lags. The time length is 1972:8 1987:11.
3. A test of the overidentifying restrictions yields a $\chi^2_{14} = 4.248$ for the first regression with a p-value of .994 while $\chi^2_{14} = 3.835$ for the second regression with a p-value of .996.

Fig. 1: Arrival of Information and Timing of Decisions

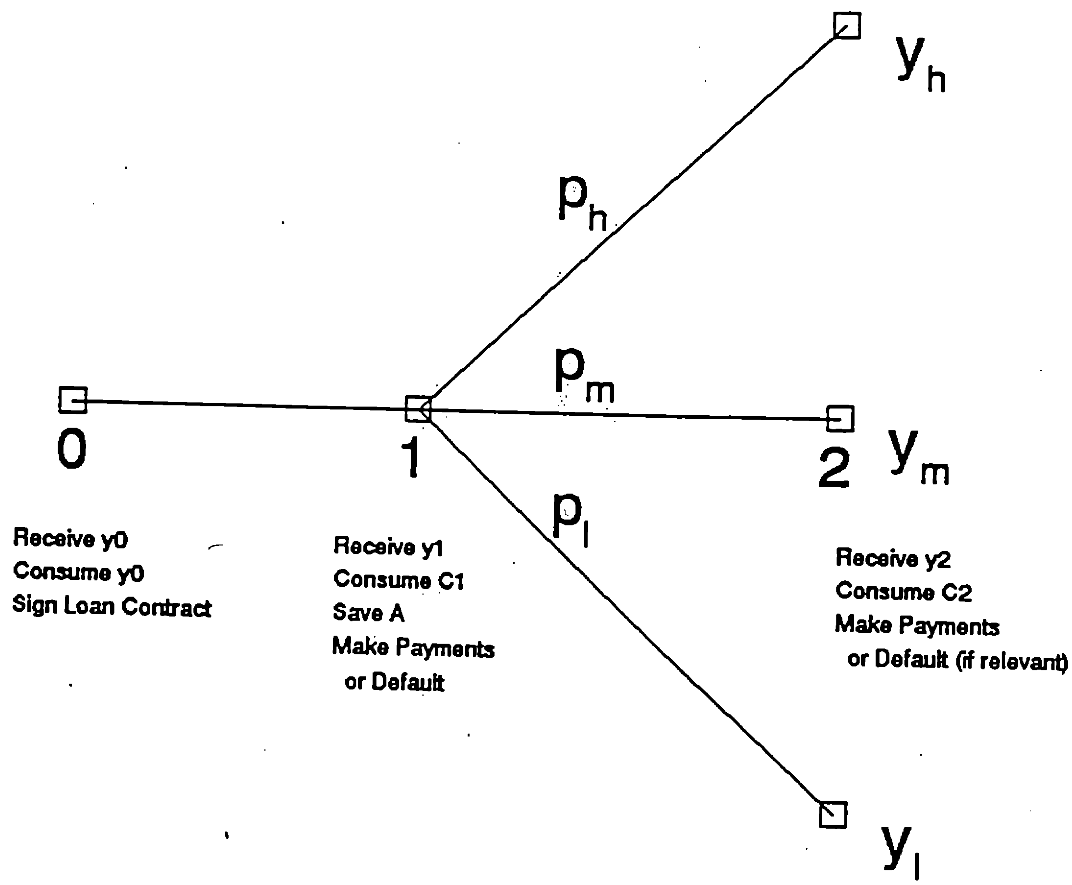


Fig. 2: Effect on Maturity by Chg. Alpha

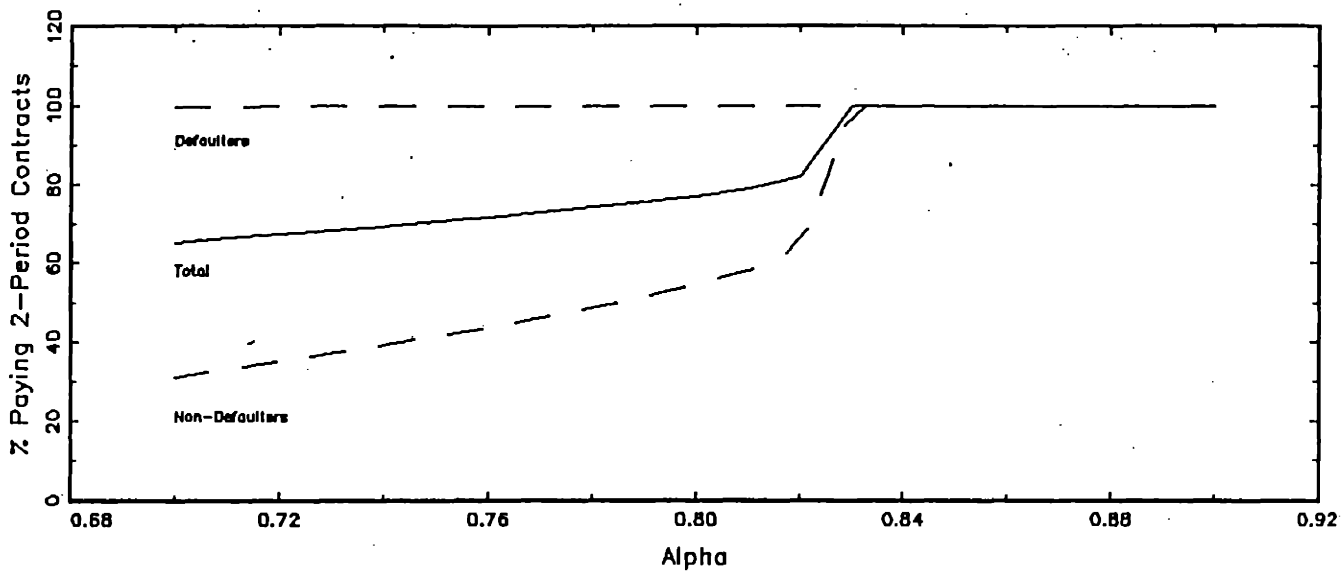


Fig. 3: Effect on Maturity by Chg. K

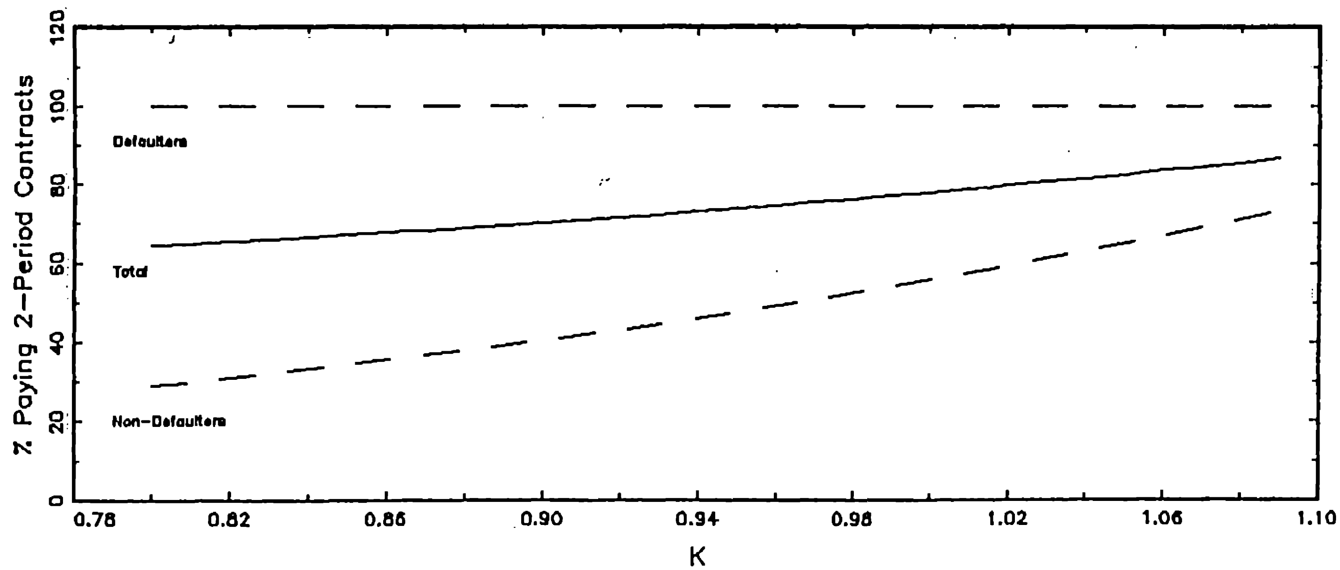


Fig. 4: Autoloan and Treasury Note Rates

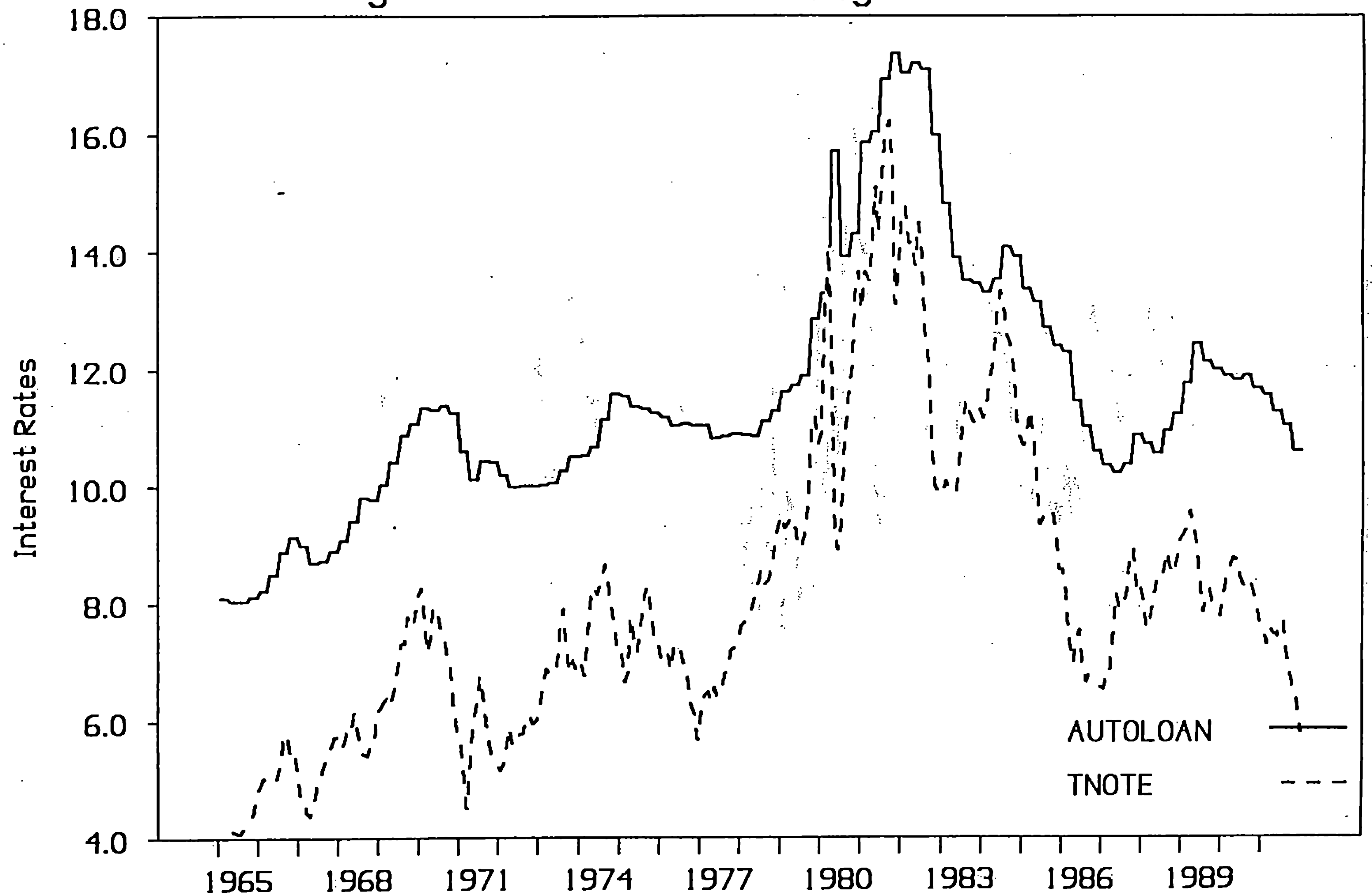


Fig. 5: Autoloan and Treasury Note Spread

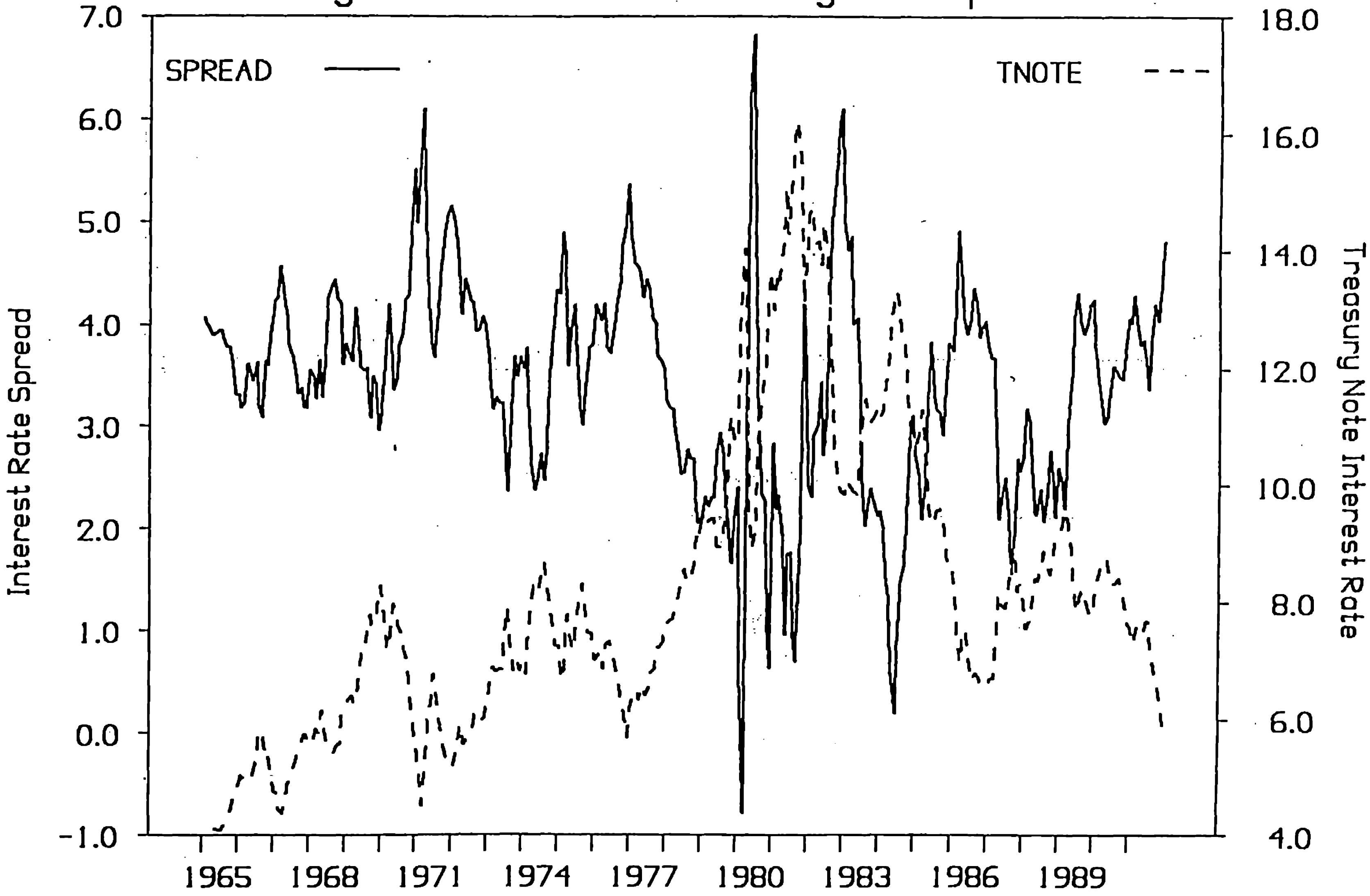


Fig 6: Maturities of New Auto Loans

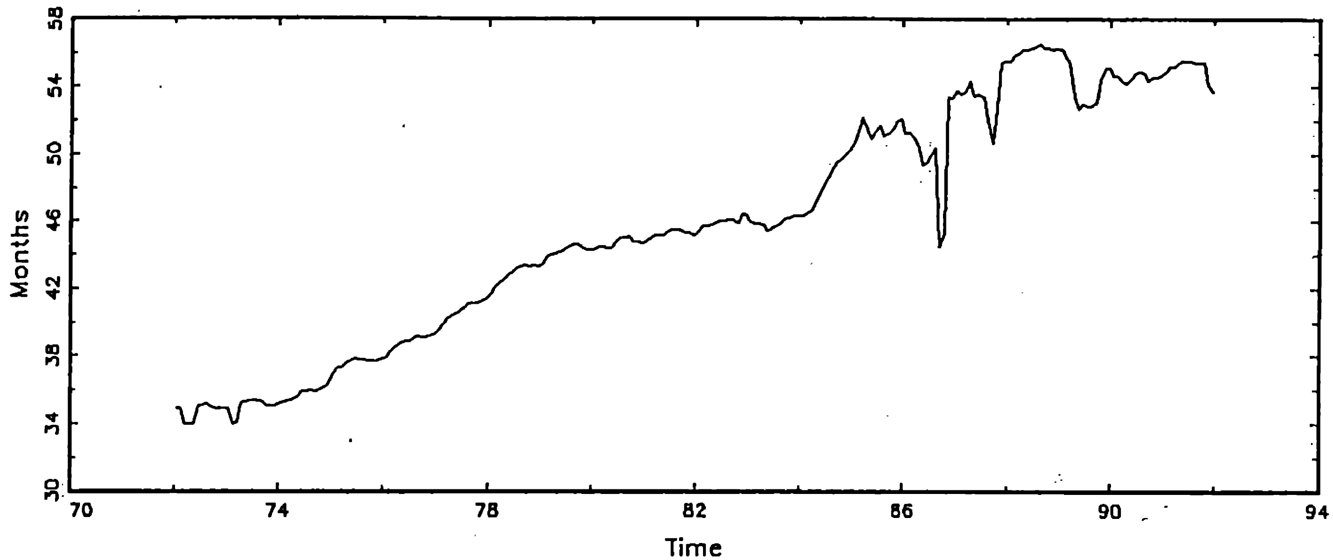
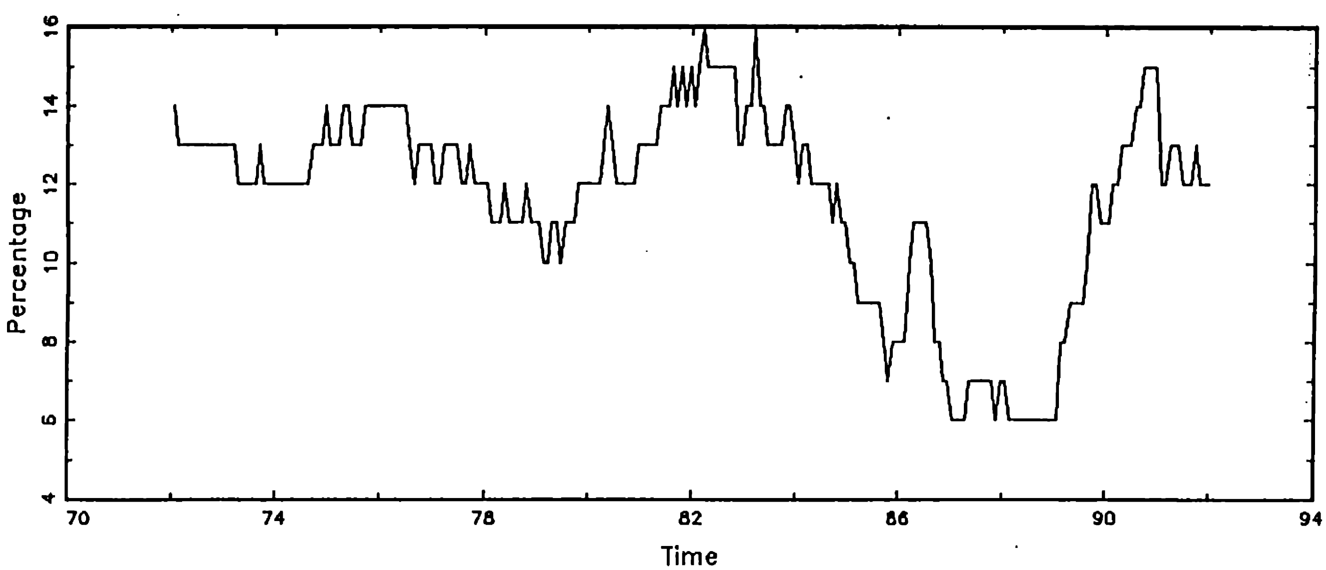


Fig 7: Downpayments of New Auto Loans



APPENDIX A: EQUILIBRIUM CONDITIONS

To calculate and ensure the equilibrium described above, one must verify that certain conditions are met. There are seven:

1. Find the savings rules of consumers in period 1. Let A_d denote the savings of consumers in period 1 who have two-period contracts and will default in $2l$; let A_n denote the savings of consumers who have two-period contracts and do not default; let A_1 denote the savings of people who sign one-period contracts and do not default; and let A_1^* denote the savings of people who default in period 1. These values solve the following first-order conditions:

$$U'(y_1 - R_{B_1}K - A_1) = \beta R_f\left[p_h U'(y_p + R_f A_1) + p_m U'(y_q + R_f A_1) + p_l U'(y_l + R_f A_1)\right]$$

$$U'(y_1 - A_1^*) = \beta R_f\left[p_h U'(y_h + R_f A_1^*) + p_m U'(y_m + R_f A_1^*) + p_l U'(y_l + R_f A_1^*)\right]$$

$$U'(y_1 - A_n - P) = \beta R_f\left[p_h U'(y_h + R_f A_n - P) + p_m U'(y_m + R_f A_n - P) + p_l U'(y_l + R_f A_n - P)\right]$$

$$U'(y_1 - A_d - P) = \beta R_f\left[p_h U'(y_h + R_f A_d - P) + p_m U'(y_m + R_f A_d - P) + p_l U'(y_l + R_f A_d)\right].$$

2. Find the conditions on ψ to ensure some default in state $2l$ while others do not, and that none default in state $2m$:

$$U_d^k(0) + U(y_l + R_f A_d) > U_d^k(\alpha^2 K) + U(y_l + R_f A_d - P)$$

$$U_n^k(0) + U(y_l + R_f A_n) < U_n^k(\alpha^2 K) + U(y_l + R_f A_n - P)$$

$$U_{d,n}^k(0) + U(y_m + R_f A_P) < U_{d,n}^k(\alpha^2 K) + U(y_m + R_f A_x - P),$$

where $A_x = A_n$ or A_d.

3. Find the conditions on ψ so that none default in 2:

$$(1+\beta)U^k(0) + U(y_1 - A_1^*) + \beta EU(y_2 + R_f A_1^*)$$

$$< U^k(\alpha K) + \beta U^k(\alpha^2 K) + U(y_1 - R_{B_1} K - A_1)$$

$$+ \beta EU(y_3 + R_f A_1)$$

$$< U^k(\alpha K) + \beta U^k(\alpha^2 K) + U(y_1 - P - A_x)$$

$$+ \beta EU(y_2 + R_f A_x - P \cdot 1_{[P_2]}).$$

4. Find the conditions such that consumers choose to pay in 1 or in 1 and 2 for defaulters and non-defaulters. They pay the entire loan off in 1 if:

$$U(y_1 - R_{B_1} K - A_1) + \beta EU(y_2 + R_f A_1) \geq U(y_1 - P - A_x)$$

$$+ \beta EU(y_2 + R_f A_x - P \cdot 1_{[P_2]}).$$

5. Find the conditions on y_0 so that consumers do not make any downpayments:

$$U'(y_0) > \beta R_{B_1} U'(y_1 - R_{B_1} K - A_1)$$

$$U'(y_0) > \tfrac{1}{2} \beta R_{B_2} U'(y_1 - P - A_x).$$

6. Check to make sure selling isn't better than defaulting in period $2l$:

$$P/\overline{K} > \alpha^2(1-\lambda)$$

7. Find R_{B_2} consistent with ϕ and γ as given in (2) and (4).

Computationally, to find the equilibrium, one first picks an arbitrary R_{B_2}, $\widehat{R}_{B_2}$ and calculates the optimizing behavior of individuals taking the loan rates as given. This induces default probabilities and one calculates what the loan rates should be given consumer behavior. This leads to a different rate, $\widetilde{R}_{B_2}$. Then one sets $\widehat{R}_{B_2} = \widetilde{R}_{B_2}$ and proceeds in this fashion until convergence is achieved, that is until $\widetilde{R}_{B_2} = \widehat{R}_{B_2}$, which is rapid.

Appendix B: Data Sources

The microeconomic data is from the Consumer Expenditure Survey, 1985, and it is described in an appendix to the second chapter on data or in the text to this chapter.

The prices of discount bonds are calculated from McCulloch (1990). McCulloch reports interest rates for 1–6 month bonds, 9 month bonds, and 1–5 year bonds. Interest rates for interim months are linear interpolations from the two closest interest rates on either side. Let $r(t,s)$ denote the interest rate on a bond at time t that pays 1$ at time s that McCulloch reports, and let $\varphi(t,s)$ denote the price of that bond. Then $\varphi_{t,s} = e^{-r(t,s)(s-t)/1200}$. The interest rate is multiplied by $(s-t)/1200$ because McCulloch lists annualized rates in percentage terms.

As for the macroeconomic data, the auto loan rate data is from Data Resources Incorporated. Three and five year Treasury note rates, real and nominal aggregate consumption, new car price deflators, and per-capita income come from Citibase. Per-capita income is calculated by dividing disposable personal income (GMYD) measured in thousands of dollars, seasonally adjusted, divided by the resident population (POPRES), also measured in thousands. Aggregate contract lengths, downpayments and nominal values of loans are averages for auto financing companies and are from the Federal Reserve Board. Contemporaneous values are reported in the *Bulletin*. Real loan values are deflated by the new car price deflator.