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Citation: Gaetz, Christian and Yibo Gao, "Self-dual intervals in the Bruhat order." Selecta Mathematica 26, 5 (November 2020): 77 ©2020 Authors

As Published: https://dx.doi.org/10.1007/s00029-020-00608-z

Publisher: Springer International Publishing

Persistent URL: https://hdl.handle.net/1721.1/129422

Version: Author's final manuscript: final author's manuscript post peer review, without publisher's formatting or copy editing

Self-dual intervals in the Bruhat order

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October 29, 2020

Abstract

Björner-Ekedahl [5] prove that general intervals $[e,w]$ in Bruhat order are "top-heavy", with at least as many elements in the i-th corank as the i-th rank. Well-known results of Carrell [7] and of Lakshmibai-Sandhya [9] give the equality case: $[e,w]$ is rank-symmetric if and only if the permutation w avoids the patterns 3412 and 4231 and these are exactly those w such that the Schubert variety X_w is smooth.

In this paper we study the finer structure of rank-symmetric intervals $[e,w]$, beyond their rank functions. In particular, we show that these intervals are still "top-heavy" if one counts cover relations between different ranks. The equality case in this setting occurs when $[e,w]$ is self-dual as a poset; we characterize these w by pattern avoidance and in several other ways.

1 Introduction

We say a complex projective variety X has a *cellular decomposition* if X is covered by the disjoint open sets $\{C_i\}$, each isomorphic to affine space of some dimension, and such that each boundary $\overline{C_j} \setminus C_j$ is a union of some of the $\{C_i\}$. Given a variety with such a decomposition, it is natural, following Stanley [14], to define a partial order Q^X on the $\{C_i\}$ by setting $C_i \leq C_j$ whenever $C_i \subseteq \overline{C_j}$.

*C.G. was partially supported by an NSF Graduate Research Fellowship under grant No. 1122374

When $X = G/B$, the quotient of a complex semisimple algebraic group by a Borel subgroup, the Bruhat decomposition

$$G = \bigsqcup_{w \in W} BwB$$

induces a cellular decomposition $\{BwB/B \mid w \in W\}$ of X, where W is the Weyl group of G. In this case the partial order Q^X on W is the well known *Bruhat order*. For $w \in W$ the closure $X_w = \overline{BwB}/B$ itself has the cellular decomposition $\{BuB/B \mid u \in W,\ u \leq w\}$, and so its poset of cells Q^{X_w} is the interval $[e, w]$ in Bruhat order on W below the element w. The varieties X_w are called *Schubert varieties*.

Much of the structure of the Bruhat order is well-understood combinatorially; see Section 2 for some basic definitions and results. It is graded with the rank of an element w being the length $\ell(w)$ in the Weyl group, it has minimal element e, the identity element of W and maximal element w_0, the longest element of W. A great deal of work has been done on the structure of intervals $[e, w]$ in Bruhat order [3, 6, 15]. Most of this paper will focus on the "type A_{n-1}" case, where the Weyl group W is the symmetric group $\mathfrak{S}_n$.

For $w \in W$ and $k = 0, 1, \ldots, \ell(w)$, let

$$P_k^w := \{u \leq w : \ell(u) = k\}.$$

We call this set the *k-th rank of* $[e, w]$ and call $P^w_{\ell(w)-k}$ the *k-th corank*. When the element w is well understood, we may simplify our notation and just write P_k instead. We have $P_0^w = \{e\}$ and $P^w_{\ell(w)} = \{w\}$. Let Γ_w (resp. Γ^w) denote the bipartite graph on $P_1^w \sqcup P_2^w$ (resp. $P^w_{\ell(w)-1} \sqcup P^w_{\ell(w)-2}$) with edges given by cover relations in Bruhat order (see Figure 2 for an example).

Theorem 1 (Björner and Ekedahl [5]). *Bruhat intervals are "top-heavy", that is, for all* $0 \leq k \leq \ell(w)/2$,

$$|P_k^w| \leq |P^w_{\ell(w)-k}|.$$

Given a permutation $\pi \in \mathfrak{S}_m$, we say $w \in \mathfrak{S}_n$ *avoids* π if there are no indices $1 \leq i_1 < \cdots < i_m \leq n$ such that $w(i_1), \ldots, w(i_m)$ are in the same relative order as $\pi(1), \ldots, \pi(m)$.

Theorem 2 (Carrell; Lakshmibai and Sandhya [7, 9]). *The following are equivalent for* $w \in \mathfrak{S}_n$*:*

- *S.1 the interval* $[e, w]$ *is rank-symmetric, that is,* $|P_k^w| = |P^w_{\ell(w)-k}|$ *for all* $0 \leq k \leq \ell(w)/2$*;*
- *S.2* w *avoids* 3412 *and* 4231*;*
- *S.3 the Schubert variety* X_w *is smooth.*

Permutations satisfying the equivalent conditions of Theorem 2 are called *smooth permutations.*

Theorem 3 shows that, even when $[e, w]$ is rank-symmetric, so that Theorem 1 does not give an asymmetry between ranks and coranks, the interval is still "top heavy" if we also consider cover relations. For $u \in [e, w]$ we write $\mathrm{udeg}_w(u)$ for the number of $v \in [e, w]$ covering u, and $\mathrm{ddeg}_w(u)$ for the number covered by u. A poset is called *self-dual* if it is isomorphic to its dual poset, which has the same elements with the order relation reversed.

Theorem 3. *Let $w \in \mathfrak{S}_n$ be a smooth permutation, then*

$$\max_{u \in P_1^w} \mathrm{udeg}_w(u) \leq \max_{u \in P_{\ell(w)-1}^w} \mathrm{ddeg}_w(u),$$

with equality if and only if $[e, w]$ is self-dual.

Stanley wondered [14] if the posets Q^X for X smooth are always self-dual (they are rank-symmetric by the Hard Lefschetz Theorem); although this is the case for many small examples, it is not true for the smooth Schubert variety X_{34521} (see Figure 2). Theorem 4 below characterizes self-dual intervals in Bruhat order on the symmetric group.

Theorem 4. *The following are equivalent for $w \in \mathfrak{S}_n$:*

SD.1 the bipartite graphs Γ_w and Γ^w are isomorphic;

SD.2 w avoids the smooth patterns 3412 *and* 4231 *from (S.2) as well as* 34521*,* 45321*,* 54123*, and* 54312*;*

SD.3 w is polished (see Definition 9);

SD.4 the interval $[e, w]$ in Bruhat order is self-dual.

Remark 5. In Section 3.3 we prove that (SD.3)⇒(SD.4) in general finite Coxeter groups, however in Section 4 we give counterexamples to the other implications in general Coxeter groups.

The equivalence of (SD.1) and (SD.4) is notable because it implies that self-duality of $[e, w]$ may demonstrated by comparing only two pairs of ranks and coranks. This is in contrast to the case of rank-symmetry, where Billey and Postnikov [1] conjecture that one must check that $|P_i^w| = |P_{\ell(w)-i}^w|$ for around the first r pairs of ranks and coranks, where r is the rank of the Weyl group. In particular, (SD.1) gives a new sufficient (but not necessary) condition for the smoothness of X_w which may be checked by comparing only two pairs of ranks and coranks. See [11] for discussion of a similar problem in certain infinite Coxeter groups.

The remainder of the paper is organized as follows. In Section 2 we recall background on Bruhat order and give the definition of polished elements. Section

3 gives the proof of Theorem 4 and Theorem 3, with each implication in Theorem 4 (SD.1)⇒(SD.2), (SD.2)⇒(SD.3), and (SD.3)⇒(SD.4) occupying a subsection and the proof of Theorem 3 occupying the last subsection. Finally, Section 4 shows that Theorem 4 does not extend to other finite Coxeter groups.

2 Background and definitions

Let (W, S) be a finite Coxeter system; we write Δ_S for the associated Dynkin diagram (see Björner and Brenti [4] for basic results and definitions). For $w \in W$, the *length* $\ell(w)$ is the shortest possible length for an expression $w = s_1 \cdots s_\ell$ with the $s_i \in S$; such an expression for w of minimal length is called a *reduced expression* or *reduced decomposition*. The *parabolic subgroup* W_J for $J \subseteq S$ is the subgroup generated by J, and (W_J, J) is a Coxeter system. The unique element of maximum length in W_J is denoted $w_0(J)$. Each left coset wW_J (resp. right coset $W_J w$) of W_J in W has a unique representative w^J (resp. ${}^J w$) of minimal length, and the set of these representatives is the *parabolic quotient* W^J (resp. ${}^J W$). Given $J \subseteq S$, each element $w \in W$ may be uniquely written $w = w^J w_J$ with $w^J \in W^J$ and $w_J \in W_J$ (resp. $w = {}_J w^J w$ with ${}^J w$ in ${}^J W$ and ${}_J w$ in W_J) with J and this decomposition satisfies $\ell(w) = \ell(w^J) + \ell(w_J)$; whenever we write an element w as a product of two elements whose lengths sum to $\ell(w)$, we say this product is *length-additive*. The *support* $\mathrm{Supp}(w)$ is the set of $s \in S$ appearing in a given reduced expression for w (it is known that the support does not depend on the reduced expression).

The elements of $T = \{wsw^{-1} \mid w \in W, s \in S\}$ are called *reflections*. For $w \in W$ and $t \in T$, we write $w \leq wt$ whenever $\ell(wt) > \ell(w)$; the *Bruhat order* on W is the transitive closure of this relation. The Bruhat order is graded, with rank function given by ℓ, has unique minimal element e and unique maximal element $w_0 = w_0(S)$. If above we instead require that $t \in S$, the resulting partial order is called the *right weak order*, denoted $\leq_R$ (if we require that $t \in S$ and multiply on the left, we obtain the *left weak order* $\leq_L$ on W). We write $[u, w]$ for the interval between u and w in Bruhat order, and $[u, w]_L$ and $[u, w]_R$ for intervals in left and right weak orders, respectively; we also write $[u, w]^J$ for $[u, w] \cap W^J$.

Proposition 6 (See, e.g. [4])**.** *The map $u \mapsto u^J$ from $W \to W^J$ preserves Bruhat order.*

The *right inversion set* $T_R(w)$ of $w \in W$ is $\{t \in T \mid \ell(wt) < \ell(w)\}$; the *right descent set* is $D_R(w) = T_R(w) \cap S$. We similarly define left inversions and descents by multiplying by t on the left. It is not hard to check that

$$W^J = \{w \in W \mid D_R(w) \subseteq S \setminus J\}$$

and that $D_R(w_0(J)) = D_L(w_0(J)) = J$. It is well known that $s \in D_R(w)$ (resp. $s \in D_L(w)$) if and only if w has a reduced expression ending with s (resp. beginning with s).

The following characterization of Bruhat order is well known.

Proposition 7. *Let $u, w \in W$, then $u \leq w$ if and only if for some (equivalently, for any) reduced expression $w = s_1 \cdots s_\ell$ there is a substring $s_{i_1} \cdots s_{i_k}$ with $i_1 < \cdots < i_k$ which is a reduced expression for u.*

2.1 Billey-Postnikov decompositions

Let $w \in (W, S)$ and $J \subseteq S$, we say the parabolic decomposition $w = w^J w_J$ is a *Billey-Postnikov decomposition* (or *BP-decomposition*) if

$$\mathrm{Supp}(w^J) \cap J \subseteq D_L(w_J).$$

For any $u \in W$ and any $J \subseteq S$, it was shown in [2] that

$$[e, u] \cap W_J = [e, m(u, J)]$$

for some element $m(u, J) \in W$, and we take this as the definition of $m(u, J)$.

Proposition 8 (Richmond and Slofstra [12])**.** *If the parabolic decomposition $u = u^J u_J$ is a BP-decomposition, then $u_J = m(u, J)$.*

2.2 The symmetric group as a Coxeter group

Much of the paper will focus on the case of the symmetric group $\mathfrak{S}_n$, the Coxeter group of type A_{n-1}. We make the conventions for the symmetric group that the simple generators are $S = \{s_1, ..., s_{n-1}\}$ where s_i is the adjacent transposition $(i\, i+1)$. It is not hard to see that the reflections T are exactly the transpositions (ij), for which we sometimes write t_{ij}.

In this case descents and inversions correspond to the familiar notions by the same name which appear in the combinatorics of permutations. Namely, for $w = w(1) \dots w(n)$ in one-line notation, (ij), $i < j$ is a right inversion of w if $w(i) > w(j)$ and a right descent if this is true and $j = i+1$. The length $\ell(w)$ is the number of inversions of w, and the longest element w_0 is the reversed permutation with one-line notation $n\, n-1 \cdots 2\, 1$.

2.3 Polished elements

We now define the *polished elements* appearing in the statement of Theorem 4.

Definition 9. Let (W, S) be a finite Coxeter system, we say that $w \in W$ is *polished* if there exist pairwise disjoint subsets $S_1, ..., S_k \subseteq S$ such that each S_i is a connected subset of the Dynkin diagram and coverings $S_i = J_i \cup J_i'$ for $i = 1, ..., k$ with $J_i \cap J_i'$ totally disconnected so that

$$w = \prod_{i=1}^{k} w_0(J_i) w_0(J_i \cap J_i') w_0(J_i')$$

where the product is taken from left to right as $i = 1, 2, ..., k$ (if the S_j are reordered, we obtain a possibly different polished element).

In light of Theorem 4, the word "polished" is meant to indicate that these elements are even nicer than smooth elements.

Example 10. The following element (shown in Figure 1) with $k = 2$, $J_1 = \{s_8\}$, $J_1' = \emptyset$, $J_2 = \{s_2, s_3, s_4, s_6, s_7\}$, $J_2' = \{s_4, s_5, s_6\}$, and multiplication in the order of

$$\begin{aligned} w =& w_0(J_1)w_0(J_2)s_4 s_6 w_0(J_2') \\ =& 123456798 \cdot 154328769 \cdot 123546789 \cdot 123457689 \cdot 123765489 \\ =& 154973268 \end{aligned}$$

is a polished element. Notice that $J_2 \cap J_2' = \{s_4, s_6\}$ is totally disconnected.

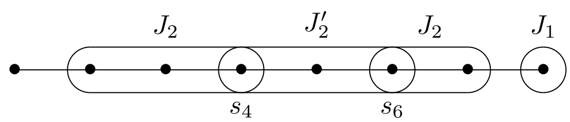

Figure 1: A polished element 154963287 in $\mathfrak{S}_9$.

The permutation $34521 \in \mathfrak{S}_5$, whose graphs Γ_{34521} and Γ^{34521} are shown in Figure 2, is *not* polished. This can be checked directly or seen to follow from Theorem 4, since $\Gamma_{34521} \ncong \Gamma^{34521}$.

3 Proof of Theorem 4

It is clear that (SD.4)⇒(SD.1), as any antiautomorphism of $[e, w]$ induces an isomorphism $\Gamma_w \cong \Gamma^w$. We are going to show that (SD.1)⇒(SD.2), (SD.2)⇒(SD.3) and (SD.3)⇒(SD.4) in the following sections.

3.1 Proof of direction (SD.1)⇒(SD.2)

For $w \in \mathfrak{S}_n$, let $\mathrm{bl}(w)$ be the largest $b \geq 1$ such that $[n] := \{1, 2, \ldots, n\}$ can be partitioned into consecutive intervals $J_1 \sqcup J_2 \sqcup \cdots \sqcup J_b$ such that $w \cdot J_i = J_i$ for all $i = 1, \ldots, b$. We write $w = w^{(1)} \oplus \cdots \oplus w^{(b)}$ where $w^{(i)} \in \mathfrak{S}_{|J_i|}$ and say that w has $\mathrm{bl}(w)$ *blocks*. Equivalently, $\mathrm{bl}(w)$ is the cardinality of $S \setminus \mathrm{Supp}(w)$, thus we see that $\mathrm{bl}(w) = n - |P_1^w|$.

Definition 11. We say that an inversion (i, j) of w is *minimal* if $i < j$, $w(i) > w(j)$ and there does not exist k such that $i < k < j$ and $w(i) > w(k) > w(j)$.

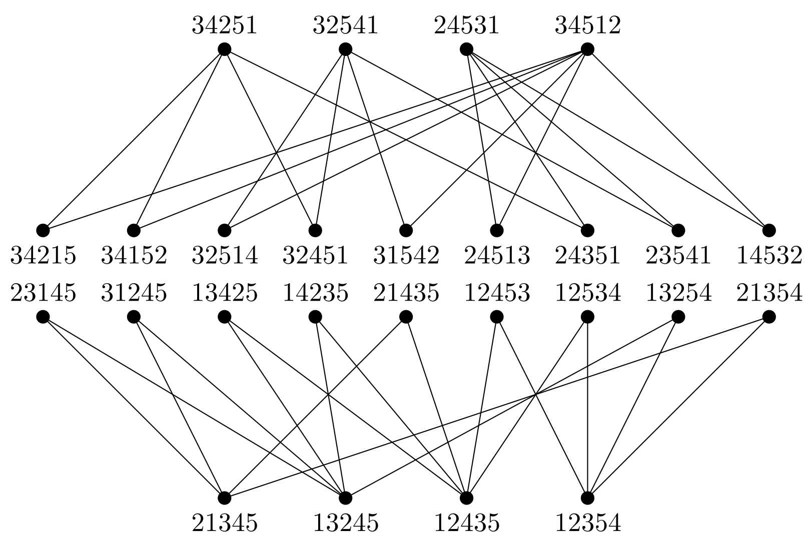

Figure 2: The bipartite graphs Γ^{34521} (top) and Γ_{34521} (bottom). Note that the graphs are not isomorphic.

In other words, (i,j) is a minimal inversion of w if and only if wt_{ij} is covered by w is in the strong Bruhat order. So the minimal inversions of w are in bijection with $P^w_{\ell(w)-1}$. We generalize this definition to minimal pattern containment.

Definition 12. We say that $w \in \mathfrak{S}_n$ *contains* pattern $\pi \in \mathfrak{S}_k$ at indices $a_1 < \cdots < a_k$ if $w(a_i) < w(a_j)$ if and only if $\pi(i) < \pi(j)$ for all $1 \le i < j \le n$. We say that this occurrence of π is *minimal* if there does not exist an occurrence of the pattern π at different indices $a'_1 < \cdots < a'_k$ such that $a'_1 \ge a_1$, $a'_k \le a_k$, $\min_i w(a'_i) \ge \min_i w(a_i)$, $\max_i w(a'_i) \le \max_i w(a_i)$ and at least one of these four inequalities is strict.

Example 13. The permutation 45321 contains the pattern 3421 at indices 1,2,4,5 but this containment is not minimal since 45321 also contains 3421 at indices 1,2,3,4.

Notice that if $w \in \mathfrak{S}_n$ contains $\pi \in \mathfrak{S}_k$, then w must have some minimal occurrence of π.

Lemma 14. *For $w \in \mathfrak{S}_n$, we always have $|P^w_{\ell(w)-1}| \ge |P^w_1|$ and if w contains the pattern 4231, then $|P^w_{\ell(w)-1}| > |P^w_1|$.*

Remark 15. The inequality $|P^w_{\ell(w)-1}| \ge |P^w_1|$ follows directly from Theorem A of [5]. We will still give the full proof here as the idea will also be useful later on.

Proof. Use induction on n. Let $a = \mathrm{bl}(w)$ and $w = w^{(1)} \oplus \cdots \oplus w^{(a)}$. Then $|P^w_{\ell(w)-1}| = \sum_{i=1}^{a} |P^{w^{(i)}}_{\ell(w^{(i)})-1}|$ and $|P^w_1| = \sum_{i=1}^{a} |P^{w^{(i)}}_1|$. As $\mathrm{bl}(4231) = 1$, w contains 4231 if and only if one of $w^{(i)}$ contains 4231. Therefore we can assume without loss of generality that $a = 1$. Consequently, P^w_1 consists of all simple transpositions s_i for $i = 1, \ldots, n-1$ so $|P^w_1| = n - 1$.

Let $u \in \mathfrak{S}_{n-1}$ be the permutation obtained from w by restricting to the relative ordering of $w(2), \ldots, w(n)$. Let $b = \mathrm{bl}(u)$ and $u = u^{(1)} \oplus \cdots \oplus u^{(b)}$ with $u^{(i)}$ being a permutation on $J_i \subset \{2, \ldots, n\}$. An example is shown in Figure 3. Since $\mathrm{bl}(w) = 1$, we necessarily have that $w(1)$ is greater than the smallest entry

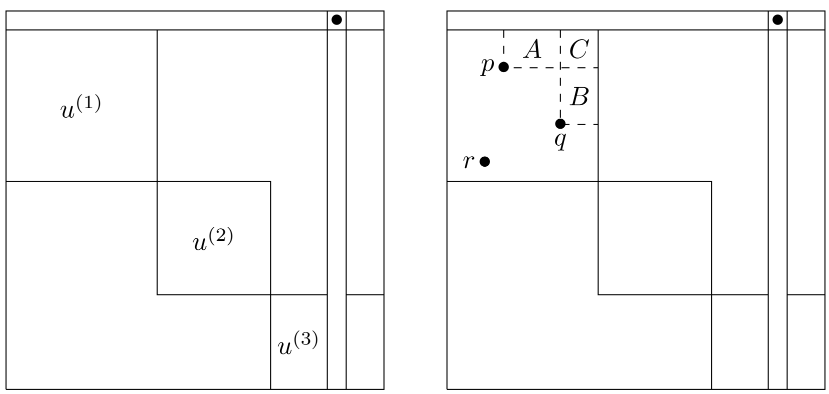

Figure 3: The decomposition of w with the first entry deleted. The permutation diagrams in Figures 3-9 use matrix coordinates; there is a dot in position (i, j) whenever $w(i) = j$.

in J_b. The minimal inversions of w contain all minimal inversions in $u^{(i)}$'s and minimal inversions of the form $(1, k)$. By the induction hypothesis, the number of minimal inversions in $u^{(i)}$ is at least $|J_i| - 1$. And for the minimal inversions in the form of $(1, k)$, we can take $k = w^{-1}(\max J_i - 1)$, for $i = 1, \ldots, b-1$ (the right most element in each block $u^{(i)}$) and $w^{-1}(w(1) - 1)$ (the right most element in the left part of $u^{(b)}$). Together, we obtain $|P^w_{\ell(w)-1}| \geq n - 1$ as desired. Moreover, by the induction hypothesis, if any $u^{(i)}$ contains 4231, then the above inequality is strict as well. Thus, we may assume that none of the $u^{(i)}$'s contain 4231.

We now assume that w contains 4231 and all of the 4231's inside w involve the entry $(1, w(1))$. Among all 4231 patterns at indices $1, p, q, r$, choose one where p is minimal and among those, choose one where $w(q)$ is maximal. Since the pattern 231 satisfies $\mathrm{bl}(231) = 1$, the entries at p, q, r belong to the same block J_i (see

Figure 3). Consider regions A, B, C defined as follows:

$$\begin{aligned} A &= \{k \in J_i : k < p, w(p) < w(k) < w(q)\}, \\ B &= \{k \in J_i : p < k < q, w(q) < w(k) \le |J_1| + \cdots + |J_i|\}, \\ C &= \{k \in J_i : k < p, w(q) < w(k) \le |J_1| + \cdots + |J_i|\}. \end{aligned}$$

By minimality of p, A must be empty and by maximality of $w(q)$, B must be empty. As $u^{(i)}$ avoids 4231, C must be empty. As a result, $A = B = C = \emptyset$. This means that both $(1, p)$ and $(1, q)$ are minimal inversions of w. As w has strictly more than 1 minimal inversions of the form $(1, k)$ for $k \in J_i$, the inequality $|P^w_{\ell(w)-1}| \ge n - 1$ is strict, so we are done. □

Lemma 16. *If $w \in \mathfrak{S}_n$ avoids 4231 and has minimal inversions at (p, q) and (q, r), then both wt_{pq} and wt_{qr} cover $wt_{pq}t_{qr}$ and $wt_{qr}t_{pq}$ in the Bruhat interval $[e, w]$.*

Proof. We have that $p < q < r$ and $w(p) > w(q) > w(r)$. Since (p, q) and (q, r) are minimal inversions, the sets

$$\{(a, w(a)) \mid p < a < q, w(q) < w(a) < w(p)\}$$

and

$$\{(a, w(a)) \mid q < a < r, w(r) < w(a) < w(q)\}$$

must be empty. Moreover, since w avoids 4231,

$$\{(a, w(a)) \mid p < a < q, w(r) < w(a) < w(q)\}$$

and

$$\{(a, w(a)) \mid q < a < r, w(q) < w(a) < w(p)\}$$

must be empty as well. As a result,

$$\{(a, w(a)) \mid p < a < r, w(r) < w(a) < w(p)\} = \{(q, w(q))\}.$$

A useful visualization can be seen in Figure 4.

It is now clear that both (q, r) and (p, r) are minimal inversions of wt_{pq}. So wt_{pq} covers $wt_{pq}t_{qr}$ and $wt_{pq}t_{pr} = wt_{qr}t_{pq}$. Similarly, wt_{qr} also covers $wt_{pq}t_{qr}$ and $wt_{qr}t_{pq}$ as desired. □

Lemma 17. *For $w \in \mathfrak{S}_n$ avoiding 4231, if w satisfies (SD.1) then w avoids 34521, 45321, 54123, 54312 and 3412.*

Proof. All four patterns mentioned in this lemma have one block, so we can again without loss of generality assume that $\mathrm{bl}(w) = 1$ and therefore that $P^w_1 = \{s_1, \ldots, s_{n-1}\}$. Assume that w avoids 4231 and it satisfies condition (SD.1). Thus there exists some graph isomorphism $\Gamma^w \cong \Gamma_w$ identifying $P^w_{\ell(w)-1}$, which is in bijection with minimal inversions, and P_1^w, which is the set of simple transpositions. We will label all minimal inversions by $\{1, 2, \ldots, n-1\}$ corresponding to their associated simple transpositions.

The following fact is going to be very useful. Assume w satisfies (SD.1) and w avoids 4231. Then if w has minimal inversions at (p, q) and (q, r) with labels i and j respectively, then i and j must differ by one (see Figure 4).

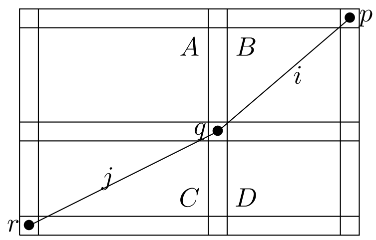

Figure 4: Adjacent labels

To see this fact, we use Lemma 16. The graph isomorphism $\Gamma_w \cong \Gamma^w$ implies that there exists two elements in P_2^w that cover both s_i and s_j in the strong Bruhat order. As a result, $|i-j| = 1$ since otherwise, there exists only one element $s_i s_j = s_j s_i \in P_2^w$ that covers both s_i and s_j.

We first deal with the patterns 34521, 45321, 54123, 54312 of size five. If w contains 45321, take a minimal pattern at indices $a_1 < a_2 < a_3 < a_4 < a_5$ and consider the 16 regions indicated in Figure 5. Since w avoids 4231, we know that $A_{11}, A_{12}, A_{21}, A_{22}, A_{31}, A_{33}, A_{34}, A_{42}, A_{43}, A_{44}$ are all empty. If A_{41} is non empty and contains some $(a', w(a'))$, then w contains a pattern 45321 at indices $a_1 < a_2 < a_3 < a_4 < a'$, contradicting the minimality of $a_1 < a_2 < a_3 < a_4 < a_5$. Similarly, the rest of the regions $A_{13}, A_{14}, A_{23}, A_{24}, A_{32}$ are all empty by the minimality. As

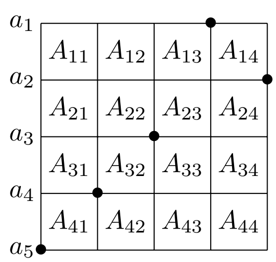

Figure 5: A minimal 45321.

a result, we now have minimal inversions at (a_1, a_3), (a_2, a_3), (a_3, a_4) and (a_4, a_5) and let their labels be i_1, i_2, i_3, i_4 respectively. By the fact regarding adjacent labels above, we know that i_3 is simultaneously adjacent to i_1, i_2 and i_4. This yields a contradiction. We will have the same contradiction if w contains 54312, the inverse of 45321.

So we assume further that w avoids 54312 and 45321. If w contains 34521, we similarly take a minimal 34521 at indices $a_1 < \cdots < a_5$, and consider the regions shown in Figure 6 (left) as before. The cases are slightly more complicated here.

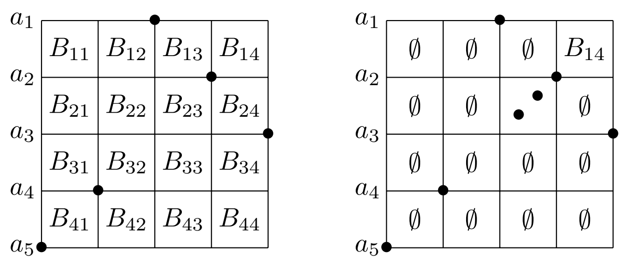

Figure 6: A minimal 34521.

Since w avoids 4231, $B_{11}, B_{21}, B_{31}, B_{42}, B_{43}, B_{44}$ are empty. Since w avoids 45321, B_{22}, B_{33} are empty. Since $a_1 < \cdots < a_5$ is minimal, $B_{41}, B_{32}, B_{12}, B_{13}, B_{24}, B_{34}$ are empty. Thus, among the regions shown in Figure 6, all regions but B_{23} and B_{14} must be empty. Since w avoids 4231, entries in region B_{23} must be decreasing and let them be $(c_1, w(c_1)), \ldots, (c_k, w(c_k))$, $k \geq 0$ where $c_1 < \cdots < c_k$ and $w(c_1) > \cdots > w(c_k)$, shown in Figure 7 (right). By the fact above regarding adjacent labels, we can conclude that the labels of the minimal inversion (a_4, a_5) must be simultaneously adjacent to the labels of (a_1, a_4), (c_k, a_4) and (a_3, a_4) with the convention that $c_0 = a_2$. This yields a contradiction. Elements inside region B_{14} will not affect our argument. The case where w contains 54123 is the same as 54123 is the inverse of 34521.

Finally, we can assume that w avoids 4231, 34521, 45321, 54123 and 54312. Suppose that w contains 3412 and let a minimal 3412 be at indices $a_1 < a_2 < a_3 < a_4$. By minimality, all regions except C_1, C_2, C_3 must be empty, as shown in Figure 7. Since w avoids 4231, elements in C_2 must be decreasing. Then as w

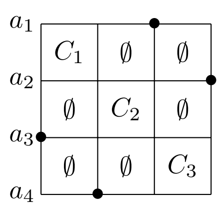

Figure 7: A minimal 3412

avoids 45321 (or 54312), $|C_2| \leq 2$. We divide into cases depending on the value of $|C_2|$.

If $|C_2| = 2$, let it be $(c_1, w(c_1))$ and $(c_2, w(c_2))$ with $c_1 < c_2$ and $w(c_1) > w(c_2)$. As w avoids 4231, C_1 and C_3 must now be empty. The label of the minimal inversion (c_1, c_2) must now be simultaneously adjacent to (a_1, c_1), (a_2, c_1), (c_2, a_3) and (c_2, a_4) and this is clearly impossible. If $|C_1| = 1$, let it be $(c_1, w(c_1))$. Similarly C_1 and C_3 must be empty. Let the labels of the minimal inversions (a_1, c_1), (a_2, c_1), (c_1, a_3) and (c_1, a_4) be i_1, i_2, i_3 and i_4 respectively. Then i_1 is adjacent to i_3, i_1 is adjacent to i_4, i_2 is adjacent to i_3 and i_2 is adjacent to i_4. This is again impossible.

The last remaining case is that C_2 is empty so C_1 and C_3 may not be empty. As w avoids 4231, elements in C_1 and C_3 are decreasing. Now we use the strategy in the proof of Lemma 14 to show that $|P^w_{\ell(w)-1}| > |P^w_1|$, contradicting the fact that w was assumed to satisfy (SD.1). Without of loss generality assume that $\mathrm{bl}(w) = 1$ so that $|P^w_1| = n - 1$. Let u be obtained from w by removing index 1 and let $b = \mathrm{bl}(u)$ with blocks $J_1, \ldots, J_b$. Recall that $|P_{\ell(w)-1}|$ is at least the number of minimal inversions inside each block J_i plus the number of minimal inversions involving index 1 while the number of minimal inversions inside J_i is at least $|J_i| - 1$ by induction and the number of minimal inversions involving 1 and block J_i is at least 1. They sum up to $n - 1$. Now if $a_1 > 1$, since $\mathrm{bl}(3412) = 1$, indices $a_1, \ldots, a_4$ together with C_1 and C_3 must lie in the same block J_i in u. We can then use induction to see that the number of minimal inversions inside J_i is strictly larger than $|J_i| - 1$ and as a result, $|P_{\ell(w)-1}| > n - 1$. The critical case is that $a_1 = 1$. Let C_1 consists of $(c_1, w(c_1)), \ldots, (c_k, w(c_k))$ with $c_1 < \cdots < c_k$ and $w(c_1) > \cdots > w(c_k)$, $k \geq 0$. Again, indices a_2, a_3, a_4 together with C_1 and C_3 all lie in the same block J_i of u. As a result, minimal inversions involving 1 and J_i contain $(1, c_k)$, where $c_0 = a_3$ if $k = 0$, and $(1, a_4)$, contributing at least 2 to the sum. Therefore, we conclude $|P^w_{\ell(w)-1}| > |P^w_1|$ as well. □

Direction (SD.1)⇒(SD.2) follows from Lemma 14 and Lemma 17.

3.2 Proof of direction (SD.2)⇒(SD.3)

Throughout this section, assume that $w \in \mathfrak{S}_n$ is a permutation that avoids 3412, 4231, 34521, 45321, 54123 and 54312. We are going to use the permutation matrix of w, as in Section 3.1, to give a decomposition of w.

We first divide all such permutations w into different "types". Consider the region $C = \{(a, w(a)) \mid 1 \leq a \leq w^{-1}(1), 1 \leq w(a) \leq w(1)\}$ which contains $(1, w(1))$ and $(w^{-1}(1), 1)$ and define $t = t(w) = |C| - 1$ (see Figure 8). If $w(1) = 1$, C contains only $(1, 1)$ and we say that such w is of type n, where n stands for "none". We also observe that entries in C are decreasing, meaning that if $(a_1, w(a_1)), (a_2, w(a_2)) \in C$ with $a_1 < a_2$, then $w(a_1) > w(a_2)$. This is because otherwise, w would contain a pattern 4231 at indices $1, a_1, a_2, w^{-1}$. Assume that C contains $(c_0, w(c_0)), \ldots, (c_t, w(c_t))$ where $1 = c_0 < \cdots < c_t$ and $w(c_0) > \cdots > w(c_t) = 1$.

Then let

$$R = \{(a, w(a)) \mid 1 < a < w^{-1}(1), w(a) > w(1)\}$$

and

$$L = \{(a, w(a)) \mid a > w^{-1}(1), 1 < w(a) < w(1)\}.$$

Since w avoids 3412, at least one of R and L must be empty. Otherwise, say $(a_1, w(a_1)) \in R$ and $(a_2, w(a_2)) \in L$, then automatically $w(1) \neq 1$ and w contains a pattern 3412 at indices $1, a_1, w^{-1}(1), a_2$. It is certainly possible that $L = R = \emptyset$, in which case we say that w is of type n as above. If $L \neq \emptyset$, we say that w is of type l, where l stands for either "left" or "lower" and if $R \neq \emptyset$, we say that w is of type r, where r stands for "right". If w is of type l, then w^{-1} is of type r, so these two cases are completely analogous.

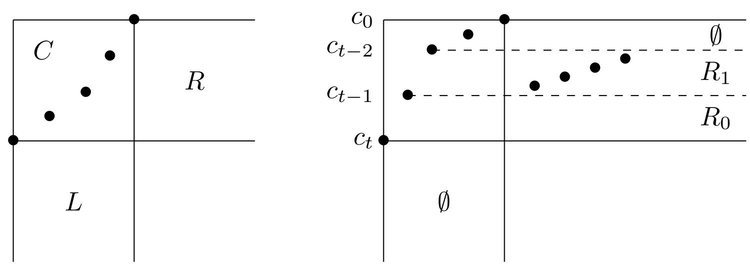

Figure 8: Structure of smooth permutations (left) and structure of permutations avoiding 3412, 4231, 34521, 45321, 54123 and 54312 (right).

So far we have only used the condition that w is smooth, meaning that w avoids 4231 and 3412. The above analysis has also appeared in previous works including [8] and [10].

Now assume that w is of type r so that $L = \emptyset$ and $R \neq \emptyset$. We can further divide R as a disjoint union $R_0 \sqcup R_1 \sqcup R_2$ (see Figure 8) where

$$\begin{aligned} R_0 &= \{(a, w(a)) \mid c_{t-1} < a < c_t\}, \\ R_1 &= \{(a, w(a)) \mid c_{t-2} < a < c_{t-1}\}, \text{ and} \\ R_2 &= \{(a, w(a)) \mid 1 < a < c_{t-2}\}. \end{aligned}$$

As w is of type r, $t \geq 1$. If $t = 1$, $R_1 = R_2 = \emptyset$ and if $t = 2$, $R_2 = \emptyset$ automatically. Regardless, we see that in fact, if $R_2 \neq \emptyset$ and contains $(a, w(a))$, then w would contain a pattern 45321 at indices $1, a, c_{t-2}, c_{t-1}, c_t$. Thus, $R_2 = \emptyset$. Moreover, we see that entries in R_1 must be decreasing: otherwise if $(a, w(a)), (a', (w(a')) \in R_1$ with $a < a'$ and $w(a) < w(a')$, then w would contain a pattern 34521 at indices $1, a, a', c_{t-1}, c_t$, a contradiction. If $R_1 \neq \emptyset$, we further say that w is of type r_1 and if $R_1 = \emptyset$, then $R_0 \neq \emptyset$ and we say that w is of type r_0. Similarly we can define type l_1 and type l_0. Equivalently, we can also say that w is of type l_i if w^{-1} is of type r_i, $i \in \{0, 1\}$.

The following lemma allows us to inductively decompose w. As a piece of notation, if $w \in \mathfrak{S}_n$ satisfies $w(1) = 1, \ldots, w(m) = m$ for some m, then w lies in the parabolic subgroup of $\mathfrak{S}_n$ generated by $J = \{s_{m+1}, \ldots, s_{n-1}\}$. In this case, we will naturally consider $w \in (\mathfrak{S}_n)_J$ as a permutation in $\mathfrak{S}_{n-m}$.

Lemma 18. *Let $w \in \mathfrak{S}_n$ be a permutation that avoids the six patterns in (SD.2). Let $J = \{s_1, \ldots, s_t\} \subset S = \{s_1, \ldots, s_{n-1}\}$ be a connected subset of of the Dynkin diagram of $\mathfrak{S}_n$, where $t = t(w)$ as above.*

- *If w is of type n, $w \cdot w_0(J) = w_0(J) \cdot w \in (\mathfrak{S}_n)_{(S \setminus J) \setminus \{s_{t+1}\}}$ is a permutation of size $n - t - 1$ that avoids the six patterns in (SD.2).*
- *If w is of type r_0, $w_0(J) \cdot w \in (\mathfrak{S}_n)_{S \setminus J}$ is a permutation of size $n - t$ that avoids the six patterns in (SD.2).*
- *If w is of type r_1, $w' = s_t \cdot w_0(J) \cdot w \in (\mathfrak{S}_n)_{(S \setminus J) \cup \{s_t\}}$ is a permutation of size $n - t + 1$ that avoids the six patterns in (SD.2). Considered as a permutation in $\mathfrak{S}_{n-t+1}$, $t(w') = |R_1| + 1$ and w' is not of type r_1. Moreover, if $|R_1| = 1$, w' is not of type l_1 either.*

Proof. First notice the simple fact that if $u \in \mathfrak{S}_n$ contains one of the patterns in (SD.2) and $\{u(1), \ldots, u(m)\} = \{1, \ldots, m\}$, then such a pattern appears either within the first m indices or within the last $n - m$ indices.

If w is of type n, then $w(1) = t + 1, w(2) = t, \ldots, w(t + 1) = 1$. After multiplying by $w_0(J)$ on either side, we obtain $w' = w_0(J)w = ww_0(J)$ satisfying $w'(i) = i$ for $i \leq t+1$ and $w'(i) = w(i)$ for $i > t+1$. Clearly w' avoids the patterns of interest, as w avoids them.

If w is of type r_0, then $w(1) = t + 1, w(2) = t, \ldots, w(t) = 2$ and $w(c_t) = 1$ where $c_t > t + 1$. Let $w' = w_0(J) \cdot w$. We see that $w'(1) = 1, \ldots, w'(t) = t$, $w'(c_t) = t + 1$ and $w'(i) = w(i)$ if $i \notin \{c_0, \ldots, c_t\}$. So we do have $w' \in (\mathfrak{S}_n)_{S \setminus J}$. By our argument above, if w' contains a pattern π mentioned in (SD.2), then none of the indices $1, \ldots, t$ can be involved, and since w avoids π, the index c_t must be involved. Say w' contains pattern π at indices $a_1 < \cdots < a_k$ with $a_i = c_t$. As $a_1 > t$, the relative ordering of the entries does not change after we multiply w by $w_0(J)$ on the left to obtain w', so w must also contain pattern π at the same indices. This yields a contradiction so w' must avoid all six patterns of interest.

The critical case is that w is of type r_1. Let $w' = s_t \cdot w_0(J) \cdot w$ (see Figure 9). We observe that $w'(i) = i$ for $i \leq t-1$, $w'(c_{t-1}) = w(1)$, $w'(c_t) = w(2)$ while w' and w agree on other indices. Thus, w' lies in the parabolic subgroup of $\mathfrak{S}_n$ generated by $s_t, \ldots, s_{n-1}$. We next argue that w' avoids the six patterns of interest. Assume for the sake of contradiction that w' contains one of the patterns in (SD.2) at indices $a_1 < \cdots < a_k$. First, $a_1 > t - 1$ by the argument above. But when restricted to the last $n - t + 1$ indices, w and w' agree by construction, so w must also contain one of the patterns at the same set of indices. This yields a contradiction.

Let $R_1 = \{(t, w(t)), \ldots, (t+m-1, w(t+m-1))\}$ where $|R_1| = m$ with $w(t) > \cdots > w(t+m-1)$. Then $c_{t-1} = t+m$. Let $w'' \in \mathfrak{S}_{n-t+1}$ be the permutation of w' restricted to the last $n-t+1$ indices. In other words, $w''(i) = w'(i+t-1)$. Consider the possible types for w''. It is more convenient to stay with the figure of w'. If w'' were of type r_1, then the set

$$\{(a, w'(a)) \mid t < a < t+m, w'(a) > w'(t)\}$$

cannot be empty, contradicting the fact that entries in R_1 are decreasing. Moreover, if $m = |R_1| = 1$, w'' cannot be of type l_1 because otherwise

$$\{(a, w'(a)) \mid a > c_t, w'(c_t) < w'(a) < w'(c_{t-1})\}$$

cannot be empty, contradicting w being type r. It is also evident that $t(w'') = m+1$, as there are $m+2$ entries weakly inside the rectangle bounded by $(t, w'(t))$ and $(c_t, w'(c_t))$. □

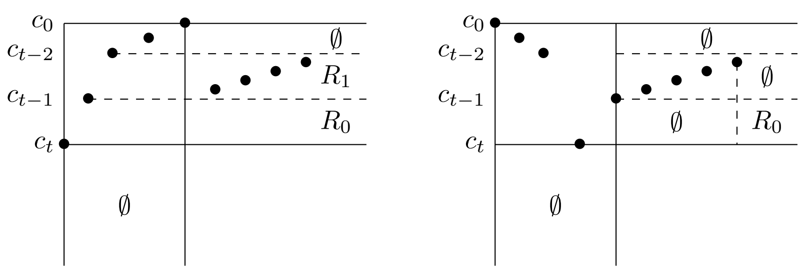

Figure 9: A permutation w of type r_1 (left) and the modified permutation $w' = s_t \cdot w_0(J) \cdot w$ (right).

We are now ready to prove the implication (SD.2)⇒(SD.3) by a repeated application of Lemma 18.

Proof of implication (SD.2)⇒ (SD.3). Given w avoiding the six patterns of interest, with $t = t(w)$ and $J = \{s_1, \ldots, s_t\}$, we can obtain $w' \in (\mathfrak{S}_n)_{S'}$ depending on the type of w listed in Table 1, by Lemma 18.

type of w	w'	S'
n	$w_0(J)w = ww_0(J)$	$\{s_{t+2}, \ldots, s_{n-1}\}$
r_0	$w_0(J)w$	$\{s_{t+1}, \ldots, s_{n-1}\}$
r_1	$s_t w_0(J)w$	$\{s_t, \ldots, s_{n-1}\}$
l_0	$ww_0(J)$	$\{s_{t+1}, \ldots, s_{n-1}\}$
l_1	$ww_0(J)s_t$	$\{s_t, \ldots, s_{n-1}\}$

Table 1: A summary of decomposing w after one step

Continuing with this operation for w' and so on down to the identity, we record each nonempty J as $K^{(1)}, K^{(2)}, \ldots, K^{(m)} \subset \{s_1, \ldots, s_{n-1}\}$ along the way and assume that $w^{(i)}$ is obtained from $w^{(i-1)}$ as w' is obtained from w above, where we start with $w^{(0)} = w$ and end with $w^{(m)} = \mathrm{id}$. Notice that J is empty if and only if $w(1) = 1$, which is equivalent to saying that w is of type n and $t(w) = 0$. When $w(1) = 1$, we will just consider w as living in the parabolic subgroup generated by $\{s_2, \ldots, s_{n-1}\}$. Assume that $K^{(i)} = \{s_{a_i}, \ldots, s_{b_i}\}$, for $a_i \leq b_i$. We label each $K^{(i)}$ by the type of $w^{(i-1)}$. Note that $K^{(m)}$ is of type n.

By Lemma 18, if $K^{(i)}$ is of type n, then $b_i < a_{i+1} - 1$ which is also saying that any two simple transpositions in $K^{(i)}$ and $K^{(i+1)}$ commute; if $K^{(i)}$ is of type r_0 or l_0, then $b_i = a_{i+1} - 1$ and if $K^{(i)}$ if type r_1 or l_1, then $b_i = a_{i+1}$ so $K^{(i)}$ and $K^{(i+1)}$ intersects at exactly one position. Moreover, if $K^{(i)}$ is of type r_1, then $b_i - a_i \geq 1$ and if further $K^{(i+1)}$ is of type l_1, then we necessarily have $b_{i+1} - a_{i+1} \geq 2$ by Lemma 18 so that any simple transposition in $K^{(i)}$ and any simple transposition in $K^{(i+2)}$ commute.

Let $S_1, \ldots, S_k$ be connected components of the Dynkin diagram of $\mathfrak{S}_n$ formed by $K_1, \ldots, K_m$ in this order. We are now going to show that each S_i can be covered by $J_i \cup J'_i$ such that $J_i \cap J'_i$ is totally disconnected and w can be written as the product shown in Definition 9. This is done by induction on k. The base case $k = 0$ and $w = \mathrm{id}$ is trivial. Let $S_1 = K_1 \cup \cdots \cup K_f$. Then $K_1, \ldots, K_{f-1}$ are of types l_1 and r_1 and are alternating between these two. Without loss of generality, let us assume that K_1 is of type r_1, since we can invert everything to go from type l_1 to type r_1. There are the following cases that are almost identical to each other. We will explain the first case in details.

Case 1: $f = 2g - 1$ is odd and K_f is of type r_0. By a repeated application of Lemma 1, we arrive at

$$\begin{aligned} w^{(f)} =& \big(w_0(K_{2g-1})\big)\big(s_{b_{2g-3}} w_0(K_{2g-3})\big) \cdots \big(s_{b_3} w_0(K_3)\big)\big(s_{b_1} w_0(K_1)\big) w \\ & \big(w_0(K_2) s_{b_2}\big)\big(w_0(K_4) s_{b_4}\big) \cdots \big(w_0(K_{2g-2}) s_{b_{2g-2}}\big), \\ w =& \big(w_0(K_1) s_{b_1}\big)\big(w_0(K_3) s_{b_3}\big) \cdots \big(w_0(K_{2g-3}) s_{b_{2g-3}}\big)\big(w_0(K_{2g-1})\big) w^{(f)} \\ & \big(s_{b_{2g-2}} w_0(K_{2g-2})\big) \cdots \big(s_{b_4} w_0(K_4)\big)\big(s_{b_2} w_0(K_2)\big). \end{aligned}$$

Recall that if $j - i \geq 2$, then $a_j - b_i \geq 2$ so any u in the parabolic subgroup generated by K_j would commute with any v in the parabolic subgroup generated by K_i. Inside the above expression for w, $w^{(f)}$ commutes with all the factors on the right hand side so we can move it all the way to the right. We can also move all the $w_0(K_{2i-1})$'s all the way to the left and similarly move all the $w_0(K_{2i})$'s all the way to the right, leaving the s_{b_i}'s in the middle. Let $J = K_1 \cup K_3 \cup \cdots K_{2g-1}$, $J' = K_2 \cup K_4 \cdots K_{2g-2}$ so that $J \cap J' = \{b_1, b_2, \ldots, b_{f-1}\}$ is totally disconnected. We have that $w = w_0(J) w_0(J \cap J') w_0(J') w^{(f)}$.

Case 2: $f = 2g-1$ is odd and K_f is of type l_0. Then

$$w = \big(w_0(K_1)s_{b_1}\big)\big(w_0(K_3)s_{b_3}\big)\cdots\big(w_0(K_{2g-3})s_{b_{2g-3}}\big)w^{(f)}\big(w_0(K_{2g-1})\big)$$
$$\big(s_{b_{2g-2}}w_0(K_{2g-2})\big)\cdots\big(s_{b_4}w_0(K_4)\big)\big(s_{b_2}w_0(K_2)\big).$$

Now we can commute $w^{(f)}$ all the way to the left instead. Also let $J = K_1 \cup K_3 \cup \cdots K_{2g-1}$, $J' = K_2 \cup K_4 \cdots K_{2g-2}$ so that

$$w = w^{(f)}w_0(J)w_0(J \cap J')w_0(J').$$

Case 3: $f = 2g$ is even and K_f is of type r_0. Then

$$w = \big(w_0(K_1)s_{b_1}\big)\big(w_0(K_3)s_{b_3}\big)\cdots\big(w_0(K_{2g-1})s_{b_{2g-1}}\big)\big(w_0(K_{2g})\big)w^{(f)}$$
$$\big(s_{b_{2g-2}}w_0(K_{2g-2})\big)\cdots\big(s_{b_4}w_0(K_4)\big)\big(s_{b_2}w_0(K_2)\big).$$

Let $J = K_1 \cup K_3 \cup \cdots K_{2g-1}$, $J' = K_2 \cup K_4 \cdots K_{2g}$. We have

$$w = w_0(J)w_0(J \cap J')w_0(J')w^{(f)}.$$

Case 4: $f = 2g$ is even and K_f is of type l_0. Then

$$w = \big(w_0(K_1)s_{b_1}\big)\big(w_0(K_3)s_{b_3}\big)\cdots\big(w_0(K_{2g-1})s_{b_{2g-1}}\big)w^{(f)}\big(w_0(K_{2g})\big)$$
$$\big(s_{b_{2g-2}}w_0(K_{2g-2})\big)\cdots\big(s_{b_4}w_0(K_4)\big)\big(s_{b_2}w_0(K_2)\big).$$

Let $J = K_1 \cup K_3 \cup \cdots K_{2g-1}$, $J' = K_2 \cup K_4 \cdots K_{2g}$. We have

$$w = w^{(f)}w_0(J)w_0(J \cap J')w_0(J').$$

The cases where K_f is of type n can be done in the exact same way as either K_f is of type r_0 or l_0. Continuing with the next connected components in $\{K_{f+1}, \ldots, K_m\}$ and so on, we deduce that w has the same form as in Definition 9 so it is polished. ☐

Remark 19. In this section, the purpose of distinguishing between type l and r is to specify the order of multiplying permutations in the decomposition of w. This order can also be seen as governed by the staircase diagram introduced by Richmond and Slofstra [13]. We did not discuss the notion of staircase diagrams since they were not needed in full generality.

3.3 Proof of direction (SD.3)⇒(SD.4)

We now prove the implication (SD.3)⇒(SD.4) for general finite Coxeter groups W. Throughout this section $s_1 \ldots s_n$ is a generic reduced expression; we drop the convention from the previous section that s_i is the specific simple reflection $(i\,i{+}1)$.

Proposition 20. *Suppose that for $w \in W$ we can write $w = uv$ with* $\mathrm{Supp}(u) \cap \mathrm{Supp}(v) = \emptyset$, *then*

$$[e,w] \cong [e,u] \times [e,v].$$

Proof. Let $J = \mathrm{Supp}(v)$; since $D_R(u) \subseteq \mathrm{Supp}(u) \subseteq S \setminus J$, we have $u \in W^J$, so in particular $\ell(w) = \ell(u) + \ell(v)$. Let $u = s_1' \cdots s_m'$ and $v = s_1 \cdots s_n$ be reduced expressions, then

$$w = s_1' \cdots s_m' s_1 \cdots s_n$$

is a reduced expression for w, with all $s_i' \in S \setminus J$ and all $s_j \in J$. By Proposition 7, $[e,w]$ is the set of all reduced subwords of this word ordered by containment as subwords. Any subword σ of $s_1' \cdots s_m' s_1 \cdots s_n$ consists of some elements of $S \setminus J$ followed by some elements of J, and by the above argument σ is reduced if and only if each of these segments is reduced. Thus multiplication gives an isomorphism of posets $[e,u] \times [e,v] \to [e,w]$. □

As products of self-dual posets are clearly self-dual, Proposition 20 implies that it suffices to prove the implication (SD.3)⇒(SD.4) in the case where the polished element w has a single block $S_1 = S$. For the remainder of this section, let $w = w_0(J) \cap w_0(J \cap J') w_0(J')$ with $S = J \cup J'$ and $J \cap J'$ totally disconnected be such a polished element of (W, S).

Lemma 21. *With $w = w_0(J) w_0(J \cap J') w_0(J')$ as above, we have*

$$\begin{aligned} w_{J'} &= w_0(J'), \\ w^{J'} &= w_0(J) w_0(J \cap J'), \end{aligned}$$

and this decomposition $w = w^{J'} w_{J'}$ is a BP-decomposition.

Proof. We know $w_0(J) \geq_L w_0(J \cap J')$ since $w_0(J)$ is the unique maximal element of W_J under weak order, thus we may write

$$w_0(J) = s_1 \cdots s_k w_0(J \cap J')$$

with lengths adding, for some reduced expression $s_1 \cdots s_k$ with each $s_i \in J$. Since $w_0(J \cap J')$ is an involution, we see that

$$w_0(J) w_0(J \cap J') = s_1 \cdots s_k;$$

furthermore, since $s_1 \cdots s_k w_0(J \cap J')$ was length-additive, we know that

$$D_R(s_1 \cdots s_k) \cap (J \cap J') = \emptyset.$$

As $D_R(s_1 \cdots s_k) \subseteq J$, we conclude that $w_0(J) w_0(J \cap J') = s_1 \cdots s_k \in W^{J'}$. Now,

$$w = s_1 \cdots s_k w_0(J')$$

is length-additive, so by uniqueness of parabolic decompositions we conclude $w_{J'} = w_0(J')$ and $w^{J'} = w_0(J)w_0(J \cap J')$. Finally, it is trivially true that

$$(\mathrm{Supp}(w^{J'}) \cap J') \subseteq J' = D_L(w_{J'}),$$

so this is a BP-decomposition. □

Proposition 4.2 of [12] strengthens the following lemma, whose short proof we include for convenience:

Lemma 22. *Let $u \in W$ and $K \subseteq S$ be such that $u = u^K u_K$ is a BP-decomposition, then the multiplication map*

$$[e, u^K]^K \times [e, u_K] \to [e, u]$$

is an order-preserving bijection.

Proof. The map is injective by the uniqueness of parabolic decompositions. To see surjectivity, suppose that $v \in [e, u]$, then by Proposition 6 we have that $v^K \leq u^K$. On the other hand, by Proposition 8, we have $v_K \leq u_K$, since $v_K \leq v \leq u$ and $v_K \in W_K$. Thus $v = v^K v_K$ is in the image. The order-preserving property is immediate from the fact that all products are length-additive and the subword characterization of Bruhat order in Proposition 7. □

Remark 23. A word of caution when reading Lemma 22: except in very special cases it is *not* true that $[e, u^K]^K \times [e, u_K]$ and $[e, u]$ are isomorphic as posets, as $[e, u]$ may contain extra order relations not coming from the product.

We are now ready to prove the implication (SD.3)⇒(SD.4) from Theorem 4.

Proof of implication (SD.3)⇒(SD.4) from Theorem 4. Let w be a polished element of W with

$$w = w_0(J)w_0(J \cap J')w_0(J'),$$

we want to show that the interval $[e, w]$ is self-dual by exhibiting an explicit bijection $[e, w] \to [e, w]$ sending $u \mapsto u^\vee$ such that $u \leq v$ if and only if $v^\vee \leq u^\vee$ (an antiautomorphism).

We observe that

$$w^{J'} = w_0(J)w_0(J \cap J') = w_0(J)^{J \cap J'}.$$

If $u \in [e, w_0(J)^{J \cap J'}]$, then $\mathrm{Supp}(u) \subseteq J$, so $D_R(u) \subseteq J$. Thus if $u \in W^{J \cap J'}$ we have in fact that $u \in W^{J'}$. Thus we have that

$$[e, w_0(J)w_0(J \cap J')]^{J'} = [e, w_0(J)^{J \cap J'}]^{J'} = [e, w_0(J)^{J \cap J'}]^{J \cap J'} = W_J^{J \cap J'}.$$

Clearly we also have $[e, w_0(J')] = W_{J'}$ and so by Lemmas 21 and 22 multiplication is an order preserving bijection

$$W_J^{J \cap J'} \times W_{J'} \to [e, w].$$

It is well known that $W_J^{J\cap J'}$ and $W_{J'}$ are self-dual as posets under Bruhat order with duality maps $u \mapsto w_0(J)uw_0(J\cap J')$ and $u \mapsto uw_0(J')$ respectively (see [4]). This suggests the duality map

$$u \mapsto u^\vee := w_0(J)u^{J'}w_0(J\cap J')\cdot u_{J'}w_0(J')$$

for $[e,w]$. Note that, by Remark 23, we still need to check whether this map is indeed an antiautomorphism of $[e,w]$ (indeed, up to this point we have not needed the assumption that $J\cap J'$ is totally disconnected).

Suppose we have a cover relation $u \lessdot v$ in $[e,w]$; to complete the proof we need to show that $v^\vee \lessdot u^\vee$. Choose reduced decompositions of $v^{J'}$ and $v_{J'}$ to get a reduced decomposition

$$v = v^{J'}v_{J'} = (s_1\cdots s_k)(s'_1\cdots s'_{k'}).$$

By Proposition 7, we know u has a reduced decomposition obtained by omitting one of the simple generators above. If the generator omitted is one of the s'_i, then we have $u^{J'} = v^{J'}$ and $u_{J'} \lessdot v_{J'}$ because $W_{J'}$ is an order ideal under Bruhat order. In this case, the fact that our duality map is known to be an antiautomorphism for $W_J^{J\cap J'}\times W_{J'}$ implies that $v^\vee \lessdot u^\vee$.

The case where the omitted generator is one of the s_i needs another argument, as $W_J^{J\cap J'}$ is not an order ideal (so $u^{J'}$ may not equal $s_1\cdots\widehat{s_i}\cdots s_k$). Suppose we are in this case, with $v^{J'} = s_1\cdots s_k$, $v_{J'} = s'_1\cdot s'_k$, and

$$u = s_1\cdots\widehat{s_i}\cdots s_k s'_1\cdots s'_{k'},$$

and all of these expressions reduced, and let $z = s_1\cdots\widehat{s_i}\cdots s_k$. For convenience, we write x for ${}_{J\cap J'}(v_{J'})$ and y for ${}^{J\cap J'}(v_{J'})$ (so $xy = v_{J'}$ with lengths adding). Then we have length-additive products

$$v = v^{J'}xy \tag{1}$$

$$u = z^{J'}z_{J'}xy. \tag{2}$$

Since $z_{J'}, x$, and y are all in $W_{J'}$, so is their product. And since the above decomposition $u = z^{J'}(z_{J'}xy)$ is length-additive, uniqueness of parabolic decompositions implies that $z^{J'} = u^{J'}$ and $z_{J'}xy = u_{J'}$. Also, because $y \in {}^{J\cap J'}W_{J'}$ has no left descents from $J\cap J'$, we know that $yw_0(J')$ has all elements of $J\cap J'$ as descents, and therefore $y \geq_R w_0(J\cap J')$, so we may write $yw_0(J) = w_0(J\cap J')y'$ for some element y' with $\ell(y) = \ell(w_0(J\cap J')) + \ell(y')$.

Now, we have

$$\begin{aligned}
u^\vee &= w_0(J)u^{J'}w_0(J\cap J')u_{J'}w_0(J')\\
&= w_0(J)u^{J'}w_0(J\cap J')z_{J'}xyw_0(J')\\
&= w_0(J)u^{J'}w_0(J\cap J')z_{J'}xw_0(J\cap J')y'\\
&= w_0(J)u^{J'}z_{J'}xy'
\end{aligned}$$

where in the last step we have used that $z_{J'}x \in W_{J\cap J'}$, which is abelian by our assumption that $J \cap J'$ is totally disconnected, and therefore commutes with $w_0(J \cap J')$. Similarly, we have

$$v^\vee = w_0(J)v^{J'}xy'.$$

In the following computation, we write N_K for $\ell(w_0(K))$ for any subset $K \subseteq S$. Computing lengths, we have

$$\begin{aligned}\ell(u^\vee) &= \ell(w) - \ell(u)\\ &= (N_J + N_{J'} - N_{J\cap J'}) - (\ell(u^{J'}) + \ell(z_{J'}) + \ell(x) + \ell(y))\\ &= (N_J - \ell(u^{J'}) - \ell(z_{J'}) - \ell(x)) + \ell(y')\end{aligned}$$

where in the first step we have used the length-additive decomposition (2) and in the second we have used the fact that $yw_0(J') = w_0(J\cap J')y'$ with the right-hand-side being length-additive, and the left-hand-side having length $N_{J'} - \ell(y)$. This implies that

$$u^\vee = (w_0(J)u^{J'}z_{J'}x) \cdot y'$$

is length-additive. A similar calculation shows that

$$v^\vee = (w_0(J)v^{J'}x) \cdot y'$$

is also length-additive. Thus $v^\vee \lessdot u^\vee$ if and only if

$$w_0(J)v^{J'}x \lessdot w_0(J)u^{J'}z_{J'}x,$$

which, because $w_0(J)$ is an antiautomorphism of Bruhat order on W_J, is true in turn if and only if $u^{J'}z_{J'}x \lessdot v^{J'}x$. These decompositions are length-additive, as they come from parabolic decompositions, thus we need to check that $u^{J'}z_{J'} \lessdot v^{J'}$. Finally we see this is true by recalling that

$$u^{J'}z_{J'} = z^{J'}z_{J'} = z = s_1 \cdots \widehat{s_i} \cdots s_k$$

and $v^{J'} = s_1 \cdots s_k$. This completes the proof of implication (SD.3)⇒(SD.4). □

3.4 Proof of Theorem 3

We obtain Theorem 3 as a corollary of the already established Theorem 4, with technology similar to that of Section 3.1.

Proof of Theorem 3. Let w be smooth so that it avoids 3412 and 4231. We will show that if w contains one of the patterns 34521, 45321, 54123 and 54312, then

$$\max_{u\in P_1^w} \mathrm{udeg}_w(u) < \max_{u\in P_{\ell(w)-1}^w} \mathrm{ddeg}_w(u).$$

On the other hand, we know from Theorem 4 that if w avoids these patterns, then $[e,w]$ in the Bruhat order is self-dual and clearly

$$\max_{u\in P_1^w} \mathrm{udeg}_w(u) = \max_{u\in P^w_{\ell(w)-1}} \mathrm{ddeg}_w(u).$$

Thus, throughout the rest of the proof, assume that w contains one of 34521, 45321, 54123 or 54312.

We use induction on n to show that for any $u \in P_1^w$, $\mathrm{udeg}_w(u) - |P_1^w| \le 1$, and that there exists some $u \in P^w_{\ell(w)-1}$ such that $\mathrm{ddeg}_w(u) - |P_1^w| \ge 2$. This statement suffices for the sake of the theorem.

We first reduce to the case where w does not lie in any proper parabolic subgroup of $\mathfrak{S}_n$, or in other words, $\mathrm{bl}(w) = 1$, with the notation defined in Section 3.1. Let $b = \mathrm{bl}(w) \ge 2$ and $w = w^{(1)} \oplus \cdots \oplus w^{(b)}$. Now the Bruhat interval can be factored as

$$[e,w] \cong [e,w^{(1)}] \times \cdots \times [e,w^{(b)}].$$

Each factor $w^{(i)}$ avoids 3412 and 4231 and is thus smooth, so that $[e,w^{(i)}]$ is rank symmetric. Take $u \in [e,w]$ and write it as $u^{(1)} \oplus \cdots \oplus u^{(b)}$ corresponding to the decomposition of w. If $\ell(u) = 1$, there exists some $j \in \{1,\ldots,b\}$ such that $u^{(i)} = e$ for all $i \ne j$. Then

$$\mathrm{udeg}_w(u) = \sum_{i\ne j} |P_1^{w(i)}| + \mathrm{udeg}_{w^{(j)}}(u^{(j)}) = |P_1^w| + \mathrm{udeg}_{w^{(j)}}(u^{(j)}) - |P_1^{w(j)}|.$$

By the induction hypothesis, $\mathrm{udeg}_{w^{(j)}}(u^{(j)}) - |P_1^{w(j)}| \le 1$ so $\mathrm{udeg}_w(u) - |P_1^w| \le 1$. On the other hand, since all the four patterns of interest do not lie in any proper parabolic subgroup of $\mathfrak{S}_4$, there exists some $w^{(j)}$ containing one of the patterns. By induction hypothesis, there exists some $u^{(j)} \in P^{w^{(j)}}_{\ell(w^{(j)})-1}$ such that $\mathrm{ddeg}_{w^{(j)}}(u^{(j)}) - |P_1^{w^{(j)}}| \ge 2$. Construct $u = u^{(1)} \oplus \cdots \oplus u^{(b)} \in P^w_{\ell(w)-1}$ where $u^{(i)} = w^{(i)}$ for $i \ne j$. Similarly, we see that

$$\begin{aligned}
\mathrm{ddeg}_w(u) =& \sum_{i\ne j} |P^{w^{(i)}}_{\ell(w^{(i)})-1}| + \mathrm{ddeg}_{w^{(j)}}(u^{(j)})\\
\ge& \sum_{i\ne j} |P_1^{w^{(i)}}| + |P_1^{w^{(j)}}| + 2\\
=& |P_1^w| + 2.
\end{aligned}$$

Now we know that w does not lie in any proper parabolic subgroup of $\mathfrak{S}_n$. This means $P_1^w = \{s_1,\ldots,s_{n-1}\}$ contains all simple transpositions. For any s_i, the permutations that cover s_i in P_2^w are contained in

$$\{s_1s_i, s_2s_i, \ldots, s_{i-1}s_i, s_{i+1}s_i, \ldots, s_{n-1}s_i\} \cup \{s_is_{i-1}, s_is_{i+1}\}$$

which has cardinality n if $i \in \{2,\ldots,n-2\}$ and cardinality $n-1$ if $i = 1, n-1$. As a result, $\mathrm{udeg}_w(u) \le n$ for all $u \in P_1^w$. In other words, $\mathrm{udeg}_w(u) - |P_1^w| \le 1$.

Next, we obtain a lower bound of $n+1$ for $\mathrm{ddeg}_w(u)$ for some $u \in P^w_{\ell(w)-1}$. Recall the notion of a *minimal inversion* from Definition 11. The number of minimal inversions of w is exactly $|P^w_{\ell(w)-1}| = |P^w_1| = n-1$. Suppose that (i_1, j_1) and (i_2, j_2) are two minimal inversions of w with $i_1 \leq i_2$, we claim that there exists some $v \in [e, w]$ covered by both $wt_{i_1 j_1}$ and $wt_{i_2 j_2}$ in the Bruhat order. Consider the following cases. If $\{i_1, j_1\}$ and $\{i_2, j_2\}$ are not disjoint, then either $j_1 = j_2$ or $i_1 = i_2$ or $i_2 = j_1$. If $j_1 = j_2$, then $w(i_1) < w(i_2)$ by minimality, and $v = wt_{i_1 j_1} t_{i_2 j_2} = wt_{i_2 j_2} t_{i_1 i_2}$ is covered by both. The case $i_1 = i_2$ is the same. And if $i_2 = j_1$, then by Lemma 16, there are two such v's that serve the purpose. If $\{i_1, j_1\}$ and $\{i_2, j_2\}$ are disjoint, then $t_{i_1 j_1}$ and $t_{i_2 j_2}$ commute. Pictorially, we just need to check that in the permutation diagram, the rectangle formed by $(i_1, w(i_1))$ and $(j_1, w(j_1))$ is disjoint from the rectangle formed by $(i_2, w(i_2))$ and $(j_2, w(j_2))$ so that $v = wt_{i_1 j_1} t_{i_2 j_2}$ is covered by both $wt_{i_1 j_1}$ and $wt_{i_2 j_2}$. These two rectangles overlap precisely when $i_1 < i_2 < j_1 < j_2$ and $w(i_2) < w(i_1) > w(j_2) > w(j_1)$. However, in this case, w contains 3412 at indices i_1, i_2, j_1, j_2, contradicting w being smooth.

Fix a minimal inversion (p, q) of w. For other $n-2$ minimal inversions (i, j), let $V_{(i,j)} = \{v \in P^w_{\ell(w)-2} | v < wt_{pq}, v < wt_{ij}\}$. Since every Bruhat interval of rank 2 is isomorphic to a diamond (see for example [4]), we know that every $v \in P^w_{\ell(w)-2}$ such that $v < wt_{pq}$ belongs to exactly one of $V_{(i,j)}$'s. This means $\mathrm{ddeg}_w(wt_{pq})$ is the sum of $|V_{(i,j)}|$'s. Moreover, we have seen that $|V_{(i,j)}| \geq 1$ for all minimal inversions $(i, j) \neq (p, q)$ from the previous paragraph and that $|V_{(i,j)}| \geq 2$ if $i = q$ or $j = p$ from Lemma 16. As a result, if there are at least three minimal inversions (i, j) of w such that $i = q$ or $j = p$, we know that $\mathrm{ddeg}_w(wt_{pq}) \geq n+1$.

We apply arguments as in the proof of Lemma 17. If w contains 45321, take a minimal pattern 45321 in the sense of Definition 12 at indices $a_1 < a_2 < a_3 < a_4 < a_5$ as in Figure 5 where all the regions $A_{*,*}$'s are empty. Let $(p, q) = (a_3, a_4)$. Since (a_1, a_3), (a_2, a_3) and (a_4, a_5) are all minimal inversions, we know that $\mathrm{ddeg}_w(wt_{pq}) \geq n+1$. The case of 54312, which is the inverse of 45321, is the same. If w avoids 45321 and 54312 but contains 34521, we take a minimal pattern as in Figure 6. With notations in the proof of Lemma 17, we let $(p, q) = (a_4, a_5)$. Since (a_3, a_4), (a_1, a_4) and (c_k, a_4) are all minimal inversions, we also conclude that $\mathrm{ddeg}_w(wt_{pq}) \geq n+1$. The case of 54123, which is the inverse of 34521, is the same. In both cases, $\mathrm{ddeg}_w(wt_{pq}) \geq n+1$ so we are done. □

4 Discussion of other types

Theorem 4 fails in general finite Coxeter groups, in particular we have the following counterexamples for (SD.1)⇒(SD.4) and (SD.4)⇒(SD.3):

- For (W, S) if type B_3 with generators chosen so that $(s_1 s_2)^3 = e$ and $(s_2 s_3)^4 = e$, the element

$$w = s_3 s_2 s_3 s_1 s_2 s_3 s_1 s_2$$

has $\Gamma_w \cong \Gamma^w$, but $[e, w]$ is not self-dual.

- The two elements of length three in W of type B_2 have $[e, w]$ self-dual, but are not polished.

There is a notion of pattern avoidance for general finite Weyl groups (see [1]). This notion was introduced by Billey and Postnikov in order to give a generalization of the Lakshmibai-Sandhya smoothness criterion for Schubert varieties. We do not know whether self-dual Bruhat intervals in types other than A_{n-1} are characterized by pattern avoidance.

Question 24. Is the set of elements w of finite Weyl groups such that $[e, w]$ is self-dual characterized by pattern avoidance in the sense of [1] as in (SD.3)?

Acknowledgements

We are grateful to Sara Billey for suggesting that self-dual intervals may be characterized by pattern avoidance. We also wish to thank Alexander Woo for providing helpful references and Alexander Postnikov and Thomas Lam for their suggestions.

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