WAGES AND CAPITAL

by

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49 Irving Street
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August 10, 1966

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Dear Professor Brown:

In partial fulfillment of the requirements for the degree of Doctor of Philosophy in Economics, I hereby submit the following thesis entitled:

"Wages and Capital."

Respectfully yours,

George A. Akerlof

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<u>A B S T R A C T</u>

In the 1950's, as soon as it was clear that the Western economy would weather the postwar period without serious deflation, theoretical economics ended its preoccupation with Keynesian short-run problems and once again turned to answering the fundamental question with which it had begun some eight-score years earlier, namely, what is the cause and what is the source of the wealth of nations.

The new probings into this question have taken new forms, both rich and strange; these forms are called modern growth theory. It is the merit, or perhaps the demerit, of these essays to apply the methods of this growth theory to other areas of economics.

The first essay looks at the relation between short-run and long-run marginal products. The context is the Solow-Johansen putty-clay model with a constant savings rate and a constant rate of growth of the labor force. But we see at the end of this essay that certain problems naturally arise in the distribution of income. The second essay takes off from this point with a changed distribution of income while basically all the other elements of the model remain unchanged.

The wage equation in this system depends on the rate of unemployment, and in this we find the answer to the question whether unemployment is a problem of the long run or a problem of the short run. The answer is that the number of jobs created depends upon the rate of unemployment; for high unemployment rates are associated with low wage rates, and low wage rates induce producers to build machines

of low capital intensity and therefore to create many jobs for a given amount of savings. Thus it is this indirect responsiveness of the number of jobs to the number of unemployed which in the long run keeps unemployment from becoming very high.

But so far these essays have been un-Keynesian. They have imbibed too thoroughly of the spirit of the growth models which were their intellectual progenitors. The next effort is then to try to use the concepts of production functions, etc., to build a Keynesian model whose crucial equation (an investment function) is derived from a production possibilities schedule plus good old profit maximization. With a few slight modifications of our model, we are able to derive a Marginal Efficiency of Investment schedule. The interesting finding of this model is that upward shifts in the wage-bargaining schedule may in fact cause substitution of capital for labor, and hence may increase the level of aggregate demand and also real income.

Finally, this takes us to our last essay. In the Keynesian framework of Chapter III the question of the role of money naturally presents itself. It is the task of the fourth chapter to discuss this role and how the changes in the money supply affect the real variables of the economy.

Thesis Supervisor: Robert M. Solow

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In addition, each chapter has a list of acknowledgments of its own. Chapter I was inspired by Eytan Sheshinski. Joseph Stiglitz, who had some very similar ideas, helped with Chapter II. Chapter III, "Income, Investment and Wages," is a joint product with him. The formulation of Chapter IV, indeed its resurrection from a previous paper, owes much to Giorgio LaMalfa, Eytan Sheshinski, and William Nordhaus.

Chapter III has had many comments which were later embodied in the text. Most notable amongst the suppliers of these were Professors Solow and Kuh, and also Stephen Marglin and Hirofumi Uzuwa, whose reformulations of the model very much clarified our original ideas. In addition Edward Shaw, Zvi Griliches and Oswald Brownlee made helpful criticisms at seminars presented at their respective universities.

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Stability, Marginal Products, Putty and Clay*

by George A. Akerlof

*This paper was inspired by Eytan Sheshinski. The author, who is responsible for all errors, would also like to thank Robert M. Solow for valuable comments. The writing of this paper was supported by the National Science Foundation.

This note is a proof of the stability of the putty-clay model without technical change. As such, it could be an addendum to several papers: Johansen [1959], Solow [1962], Phelps [1963], Inada [1964].

But there is something more important at stake here than a stability exercise. For we have a model founded on the principle that in the short run labor is paid its short-run marginal product; the natural question arises whether in the long run labor will be paid its long-run marginal product. (The role of stability is that both the long-run wage and the long-run marginal product are well-defined concepts.) The answer to the question is a conditional "yes": in the long run labor is paid its long-run marginal product if producers respond to a given wage by building the profit-maximizing machine for that wage rate. Thus, perhaps not unsurprisingly, the chain between short-run marginal products and long-run marginal products depends upon the construction of the proper machines. This appears to be a more tenuous link than in the usual putty-putty models, where the choice of technique is not made irrevocably at the time of construction of the capital good, where the capital-labor ratio can be costlessly changed at the whim of the entrepreneur and where, consequently, long-run marginal product and short-run marginal product are one and the same.

At the risk of repetition, but for the sake of specificity and clarity, we briefly spell out the model. Labor is growing at a constant rate λ. Output can be divided costlessly between machines and consumption goods. There are two production functions for this output: an ex-ante production function and an ex-post production function. The ex-ante production function F is neoclassical:

$$Q = F(K,L); \quad \frac{\partial F}{\partial K} \geq 0, \quad \frac{\partial F}{\partial L} > 0; \quad \frac{\partial^2 F}{\partial K^2} < 0, \quad \frac{\partial^2 F}{\partial L^2} < 0$$

and F is homogeneous of degree one. Q is output, K is capital and L is labor. Once machines have been produced, it is no longer possible to substitute capital for labor. An old machine has fixed labor requirements for the production of output.

New capital is produced by savings. Savings is assumed to be a constant share of output. The wage rate is the average product of labor on the least efficient machine used; labor is allocated first to the most efficient machines, then to successively less efficient machines until labor is exhausted. There is a qualification to this process, however. Labor will not accept a sage less than $\bar{w} \geq 0$. Therefore machines which produce less than $\bar{w}$ with one unit of labor will never be used; and labor will be unemployed if there is not a large enough number of sufficiently productive machines.

Entrepreneurs have stationary expectations; accordingly, they choose the current capital-labor (for newly produced capital) such that $\frac{\partial F}{\partial L}(K,L) = w(t)$, where $w(t)$ is the wage rate at time t. (F was introduced above.)

Machines are only retired because their use would be inefficient; and old machines never die or fade away; they live forever. We add one further assumption which is not necessarily indigenous to putty clay models. This assumption could be stated (imprecisely) as follows: call $k_{\bar{w}}$ the minimum capital intensity used when the wage is $\bar{w}$ (or $f(k_{\bar{w}}) = \bar{w}$). Suppose that at time 0 there is a machine of capital-intensity $\hat{k}$, then according to our assumption "there is a machine" of all intensities between $k_{\bar{w}}$ and $\hat{k}$. Furthermore, there is some $\epsilon > 0$ such that "there are more than ϵ machines" of each intensity between $k_{\bar{w}}$ and $\hat{k}$. This last statement is not quite precise because we are dealing with a whole continuum of machines; therefore the statement "there is a machine" needs some further exposition. We repeat our assumption in more precise form: let $g_o(k)$ be the distribution function of machines of type k. If $dg_o(k) > 0$, or if k is an atom, then either x is an atom or $dg_o(x) > \epsilon_k > 0$ for all $x \epsilon [k_w, k]$.

At this point let us introduce the cast of leading actors in our drama: $k(t)$ is the capital-labor ratio of capital built at time t; $k_{min}(t)$ is the capital-labor ratio of the minimum capital intensity at time t; $S(t)$ is the economy's total savings at time t; s is the constant proportion of income which is saved; λ is the rate of growth of labor; $f(\cdot) = F(.,1)$ where F was introduced above; it is assumed that if F, s, and λ were the corresponding data for a Solow-1956 model, there would be a unique nonzero stable equilibrium k^*. k^* is then the positive root of the equation: $sf(k) = \lambda k$. For feasibility we require that the minimum wage $\bar{w}$ meet the requirement: $\bar{w} < f(k^*) - k^*f'(k^*)$--i.e., the marginal product of labor with the maximum sustainable capital-labor ratio is greater than the minimal wage. We also mention $g_t(k)$ which is the "distribution function" of machines of intensity k at time t; and $k_{\bar{w}}$ which has the property $f(k_{\bar{w}}) = \bar{w}$. By "distribution function" we mean $g_t(k)$ is the number of machines of capital-intensity greater than or equal to k at time t, where the unit of machines is the number of workers employed.

We list these symbols for easy future reference:

$k(t)$	capital-labor ratio built at time t
$S(t)$	total savings at time t
$f(k)$	average product of labor with capital-labor ratio k
λ	rate of growth of labor $L(t)$
$\dot{L}(t)$	change in labor supply at time t
k^*	maximum sustainable capital-labor ratio: $sf(k^*) = \lambda k^* > 0$
s	constant rate of savings
$w(t)$	the wage at time t
$\bar{w}$	minimum wage
$g_t(k)$	distribution function of machines of type k
$k_{\bar{w}}$	capital intensity with the property $f(k_{\bar{w}}) = \bar{w}$.

Our object is to prove the following theorem: under the conditions outlined above $\lim_{t \to \infty} k(t) = k^*$. Rather than immediately plunge into a morass of algebraic detail, we present first an outline of our strategy of proof.

The basic relation of our system is the inequation between new jobs, new labor, and capital-intensity. If new jobs exceed new laborers, the wage will not fall--because old equipment will be retired; however, if new jobs fall short of new laborers, old equipment will be returned to service (if possible) and wages will not rise. (Proposition I)

Suppose that the economy is producing capital-labor ratios below k^*. These low capital-labor ratios foster enough savings to support their own maintenance for an exponentially growing labor force; or, to be more technical and more precise, suppose that all the machines were of the same capital intensity $x < k^*$, and the whole labor force is working on such capital. Then the number of new jobs created by using the economy's savings to produce new capital goods of this intensity x, would exceed the new laborers. The result of this is that if the initial capital intensity produced at time 0 is less than k^*, eventually $k(t)$ must rise. The reader is reminded that $k(t)$ is the capital intensity produced at time t. (Proposition II)

Analogously, if $k(0) > k^*$, eventually $k(0)$ must fall.

(Proposition III)

Propositions IV and V are based on the same logic as Propositions II and III. The results are surprisingly strong. Suppose that at time t the number of jobs created just matches the new entrants to the labor force. $k(t)$, the capital-intensity of capital newly produced will fall (not rise) or rise (not fall) (within a small neighborhood of t) depending upon whether $k(t) \lesseqgtr k^*$. The reason for this behavior is the "savings" created by the extra output from the new machines themselves.

Summarizing propositions II-V, either $k(t)$ will cross k^* an infinite number of times, or $k(t)$ will be monotonic after some time t.

Proposition VI duly limits the number of times that $k(t)$ can cross k^*. $k(t)$ can only cross k^* once in an upward direction: the reason is that savings generated by producing capital of intensity greater than k^* is enough to supply new workers with jobs on capital of intensity k^*, even though no higher capital-intensity can be permanently maintained.

Propositions VII and VIII show that $k(t)$ does not approach an asymptote $\hat{k} \neq k^*$. This is natural, since for such a case, savings gets closer and closer to the savings of a Solow-1956 economy. And in this Solow-1956 analogue the capital-labor ratio approaches k^*.

Again summarizing our results to date: $k(t)$ is monotonic beyond a certain point. $k(t)$ does not approach an asymptote not equal to $k*$. $k(t)$ does not approach ∞, $k(t)$ must approach $k*$.

With this battle plan in mind, we begin our proof. Before any further presentation it is necessary to deal with a technicality. $k(t)$ need not be differentiable, although we shall prove that it must be continuous. At time t, $k(t)$ may have the property that within some neighborhood of t, $k(t+\Delta t) - k(t) \geqq 0$, for $\Delta t > 0$. We shall denote this property $\Delta k(t) \geqq 0$. It is important to note that $\Delta k(t)$ is not a number but a property of the function k at time t. $\Delta k(t) \leqq 0$, $\Delta k(t) > 0$, $\Delta k(t) < 0$, however, have natural and analogous interpretations.

For our purposes, $\Delta k(t)$ replaces the time derivative of $k(t)$ at time t. For the reader who does not worry about technicalities, the sign of $\Delta k(t)$ could be interpreted as the sign of $\dot{k}(t)$, if that animal should exist.

We begin our proof with the following basic proposition:

Proposition I. $$\frac{S(t)}{k(t)} - \hat{L}(t) > 0 \longrightarrow \Delta k(t) \geqq 0 \qquad (1)$$

$$\frac{S(t)}{k(t)} - \hat{L}(t) < 0 \longrightarrow \Delta k(t) \leqq 0 \qquad (2)$$

$$\Delta k(t) > 0 \rightarrow \frac{S(t)}{k(t)} - \dot{L}(t) > 0 \qquad (3)$$

$$\Delta k(t) < 0 \rightarrow \frac{S(t)}{k(t)} - \dot{L}(t) < 0 \qquad (4)$$

The reader is referred to a mathematical appendix for a rigorous proof of this proposition. Below we present the economics of the situation. (Note: numbers ( ) in "proof" refer to relations (1) - (4) above.)

"Proof" (1) $\frac{S(t)}{k(t)}$ is the number of new jobs.

Suppose $\frac{S(t)}{k(t)} - \dot{L}(t) > 0$, then the number of new jobs exceeds the number of new laborers. Therefore old capital equipment must be retired; and the wage will not fall; correspondingly, k(t) will not fall. (2) is analogous.

(3) If $\Delta k(t) > 0$, then old equipment is being retired because w(t) is rising. Therefore $\frac{S(t)}{k(t)} - \dot{L}(t) > 0$. (4) is analogous.

This first proposition enables us to prove our theorem with minimal effort; we need only worry about three interrelationships-- S(t), k(t), and $\dot{L}(t)$. Through the judicious use of inequalities it is unnecessary to maintain a complex catalogue of all the machines on hand.

Proposition II. If $k(t) < k^*$ for some time T, eventually $\frac{S(t)}{k(t)} - \dot{L}'(t) \geqq 0$ for some time $t \geqq T$.

Proof. Suppose the contrary. Suppose $\frac{S(t)}{k(t)} - \dot{L}'(t) < 0$ for all $t \geqq T$.

In this case $\dot{S}(t) \geqq \frac{sf(k(t))}{k(t)} S(t)$ (5)

since the RHS of (1) is the savings from new labor placed on new machines; in addition, output may increase by the use of some machines which were not previously in service.

(6) But $\frac{sf(k(t))}{k(t)} \geqq \min\left(\frac{sf(k(T))}{k(T)}, \quad \frac{sf(\bar{k})}{\bar{k}}\right) > \lambda+\epsilon$, for some $\epsilon > 0$;

since $k < k^*$, and where $\bar{k}$ is the capital-intensity produced when $w(t) = \bar{w}$. [$\bar{k}$ has the property $f(\bar{k}) - \bar{k}\, f'(\bar{k}) = \bar{w}$.]

Relation (6) is true because $k(t) \leqq k(T)$ for all $t \geqq T$ by Proposition I and the assumption that $\frac{S(t)}{k(t)} - \dot{L}(t) < 0$.

Combinging (5) and (6)

$$\dot{S}(t) > (\lambda+\epsilon)\, S(t) \qquad \text{or}$$

$$S(t) > S(T)e^{(\lambda+\epsilon)t}$$

Eventually

$$\frac{S(t)}{k(t)} - \lambda L_o\, e^{\lambda t} \geqq \frac{S(t)}{k(T)} - \lambda L_o\, e^{\lambda t} > 0$$

Hence $\Delta k(t) \geqq 0$ for some $t \geqq T$.

III. Analogously, if $k(T) > k*$, for some time $t \geqq T$, $\frac{S(t)}{k(t)} - \dot{L}(t) \leqq 0$.

Proof. Suppose the contrary. Then

$$\frac{S(t)}{k(t)} - \dot{L}\,(T) > 0 \quad \text{and } \Delta k(t) \geqq 0 \text{ for all } t \geqq T$$

Then $\dot{S}(t) \leqq sf(k(t))$ $\frac{S(t)}{k(t)} \leq \frac{sf(k(T))S(t)}{k(T)} \leqq (\lambda-\epsilon)S(t)$ for some $\epsilon > 0$.

$$S(t) \leqq S(T)e^{(\lambda-\epsilon)t}$$

Eventually $\frac{S(t)}{k(t)} - \dot{L}(t) \leqq \frac{S(t)}{k(T)} - \lambda L_o e^{\lambda t} < 0$.

$\therefore$ there is some $t \geqq T$ such that $\frac{S(t)}{k(t)} - \dot{L}(t) \leqq 0$.

IV. If $\frac{S(t)}{\dot{L}(t)} - k(t) = 0$ and $k(t) < k*$,

$$\Delta k(t) \geqq 0 .$$

Proof. We show that $k(t+\Delta t) - k(t) \geqq 0$ for sufficiently small $\Delta t > 0$. We do so by estimating

$$\Delta \left(\frac{S(t)}{L'(t)} - k(t) \right) \text{ and showing that this is greater than 0.}$$

[Δ here means the same thing as when applied to $k(t)$].

Assume $k(t + \Delta t) < k(t)$. We shall show that this is not true. We estimate

$$\left[\frac{S(t + \Delta t)}{L'(t + \Delta t)} - \frac{S(t)}{L'(t)} \right] - \left[k(t + \Delta t) - k(t) \right] \tag{8}$$

If $k(t + \Delta t) < k(t)$, then $k_{min}(t + \Delta t) < k_{min}(t)$. Also, $S(t + \Delta t) \geqq S(t) + \dot{L}(t)\Delta t \, sf(k(t)) + o(\Delta t^2)$ $\geqq$ here because in this case old machines have been brought back into service.

Also $\dot{L}(t + \Delta t) = \Delta t \lambda \dot{L}(t) + \dot{L}(t) + o(\Delta t^2)$.

Hence

$$\frac{\dot{L}(t)S(t + \Delta t) - S(t)\dot{L}(t + \Delta t)}{L'(t)L'(t + \Delta t)} \geqq \frac{(sf(k)\dot{L}(t)^2 - \lambda S(t)\dot{L}(t))\Delta t + \text{to } (\Delta t^2)}{L'(t)L'(t + \Delta t)}$$

$$= \frac{(sf(k) - \lambda k)(\dot{L}(t))^2 \Delta t + o(\Delta t^2)}{L'(t)L'(t + \Delta t)} > 0$$

for Δt sufficiently small.

But $k(t+\Delta t) < k(t)$. Hence

$$\left[\frac{S(t+\Delta t)}{\dot{L}(t+\Delta t)} - \frac{S(t)}{\dot{L}(t)}\right] - [k(t+\Delta t) - k(t)] > 0$$

Hence

$$\frac{S(t+\Delta t)}{\dot{L}(t+\Delta t)} - k(t+\Delta t) > 0$$

Therefore, by Proposition I

$$k(t+\Delta t) > k(t). \quad \text{Contradiction}$$

Hence $$k(t+\Delta t) \geqq k(t) .$$

V. Similarly for $k(t) > k^*$, if $\frac{S(t)}{k(t)} - \dot{L}(t) = 0$,

$$\Delta k(t) \leqq 0 .$$

The proof is exactly analogous to the proof of IV.

The implications of IV and V are that k(t) is monotonic on one side of k*, once $\frac{S(t)}{k(t)} - \dot{L}(t) \geqq 0$. For if $\frac{S(t)}{\dot{L}(t)} - k(t) = 0$,

$\Delta k(t) \geqq 0$ and $\Delta\left(\frac{S(t)}{\dot{L}(t)} - k(t)\right) > 0$, where Δ has the same meaning applied to the expression $\left(\frac{S(t)}{\dot{L}(t)} - k(t)\right)$ that it did applied to k(t). Therefore $\frac{S(t)}{\dot{L}(t)} - k(t)$ cannot cross 0. And therefore k(t) cannot decrease (for $k < k^*$) or increase for $k > k^*$.

VI. Suppose $k(T) = k^*$ at some time $T \geqq 0$.

Suppose $\Delta k(T) > 0$,

then $k(t) \geqq k^*$ for all $t \geqq T$.

<u>Proof.</u> Let $\hat{T}$ be the time of first return to k* after T. We show that $\Delta k(\hat{T}) \geqq 0$.

$$\frac{S(\hat{T})}{k^*} = \frac{S(T)}{k^*} + \int_T^{\hat{T}} \frac{sf(k(t))}{k^*} (\dot{L}(t) + R(t))\, dt \tag{9}$$

where R(t) is the (algebraic) number of workers retired from old machines, to work on (newer), more capital-intensive machines at time t.

$\int_T^{\hat{T}} R(t)dt = 0$, since the net number of workers retired must be 0; all machines in use at time T are in use at time $\hat{T}$, since the wage is exactly the same at T as at $\hat{T}$. [Note: if $g_T(\ ^{-1}(k^*))$ is an atom, the proof continues slightly modified.]

From (9) and the fact that $k(t) \geqq k^*$ for all t: $T \leqq t \leqq \hat{T}$

$$\frac{S(\hat{T})}{k^*} \geqq \frac{S(T)}{k^*} + \int_T^{\hat{T}} \lambda(\dot{L}(t) + R(t))dt$$

$$> \dot{L}(T) + \lambda(L(\hat{T}) - \dot{L}(T) + \lambda \int_T^{\hat{T}} R(t)dt$$

$$= \dot{L}(T) + \dot{L}(\hat{T}) - \dot{L}(T) = \dot{L}(\hat{T})$$

$\Delta k(\hat{T}) > 0$ actually, since $\hat{T}$ was the moment of first return; if $\Delta k(T) > 0$, then $\Delta k(\hat{T}) > 0$ also.

VII. k either crosses k* or approaches an asymptote, by Propositions II, III and IV.

We shall show that k(t) does not approach an asymptote $\hat{k} < k^*$. Suppose the contrary, that $k \to \hat{k} < k^*$.

$$S(T) \sim \int_{-\infty}^{T} sf(\hat{k})\dot{L}(t)dt \qquad (A \sim B \text{ means } \frac{A}{B} \rightarrow 1)$$

$$\frac{S(T)}{\hat{k}} \sim \int_{-\infty}^{T} \frac{sf(\hat{k})}{\hat{k}} \dot{L}(t)dt$$

$$> (\lambda+\epsilon)\dot{L}(T). \quad \text{some } \epsilon > 0 .$$

$$\therefore \frac{S(T)}{\hat{k}} > (\lambda+\epsilon)L(T) \qquad \text{for all } T \geqq T_o \text{ for some time } T_o.$$

But this implies that eventually all workers are working on capital of capital intensity greater than or equal to $\hat{k}$. Furthermore, there is a surplus of new jobs over new workers. But then $w(T) = f\hat{k} - \epsilon$ does not approach $\hat{k}$ eventually for any $\epsilon > 0$. But for ϵ sufficiently small, $k(T) > \hat{k}$.

$\therefore$ k does not approach $k*$.

Suppose $k \rightarrow \hat{k} > k*$.

$$\frac{S(T)}{\hat{k}} \sim \int_{-\infty}^{T} \frac{sf(\hat{k})}{\hat{k}} \dot{L}(T)$$

$$< \quad (\lambda-\epsilon)\ L(T) \quad \text{for some } \epsilon > 0.$$

This implies that there will eventually be unemployment and therefore k does not approach $\hat{k}$.

V - VIII imply that $\lim_{t \to \infty} k = k^*$.

1. k does not approach an asymptote $k \neq k^*$.
2. k does not approach ∞.
3. k cannot cross k^* more than two times by VI.
4. Yet after being on one side of k^* for more than a time T, k is monotonic, by II, III, IV and V.

   Therefore k has no choice: it must approach k^*.

Q.E.D.

It is clear from the nature of our proof and our logic that a stronger and more general theorem is in fact true. Given an ex-ante production function $Q/L = f(k)$ and an ex-post production function $Q/L = g(k;\bar{k})$ where $\bar{k}$ is the capital-labor ratio originally chosen for optimal production; and where k is the ex-post capital-labor ratio; and furthermore where f is the envelope of the g's; then if labor is paid its short-run marginal product, and $\bar{k}(t)$ is chosen such that $f(\bar{k}(t)) - \bar{k}\, f'(\bar{k}(t)) = w(t)$, this model will also have a stable capital intensity k^*. The specific assumption of fixed Leontief coefficients was by no means necessary for the completion of our appointed task. The reader can quickly verify that Propositions I - VIII will hold with this more general "envelope" model.

Before these results are passed on, either as a bequest or as a burden to future generations, some criticism is well deserved. It is a remarkable fact that in the long run almost all workers will be working on capital stock with capital of intensity within some ϵ-neighborhood of k^*, while an ancient machine whose average product is almost $f(k^*) - k^*f'(k^*)$ determines the wage rate of labor. If all machines die after a fixed period of time, havoc breaks loose in our previously well-ordered model. As Ken-Ichi Inada has shown, it is not possible to have a steady state in such a system. But, at least, we can say something about the difference between Inada's model and the Sheshinski-Kemp contradiction. In the Inada system, if machines of intensity k^* were the only machines produced, there would soon be no machine of average product $f(k^*) - k^*f'(k^*)$. But it is necessary to have at least some machines of this type to maintain a wage w^*, which in turn will induce producers to make machines of intensity k^*.

We are faced with one of those rare situations in economic theory where the difference between ϵ and 0 qualitatively changes the results. Another example of such a predicament is the decomposability of a von Neumann matrix; according to a famous but fallacious argument, a decomposable matrix is "almost" indecomposable, for tiny ϵ's could be fed into the appropriate slots to replace 0's.

The reason for the irregularity, in the case at hand, is that the model we have chosen has certain singular assumptions. If we had isoquants which were not quite fixed-coefficient, and if labor were paid its short-run marginal product on the existing capital, it is no longer obvious that the death of old machines would be disastrous for the stability of $k(t)$.

As promised in the beginning, the stability of our system has something to say about marginal products. In the long run we know that the wage will be approximately w^* and all the capital will be of intensity close to k^*. (Thus the long-run marginal product of labor is well defined.) Since $w^* = f(k^*) - k^*f'(k^*)$, we know that the long-run wage is equal to the long-run marginal product.

Suppose, however, that producers had a different response to a given wage: $k(t) = \phi(w(t))$, where $\phi'(w) > 0$. Then, if k^* is a feasible solution (i.e., if there is a $w^* > w_{min}$ such that $\phi(w^*) = k^*$), it will be a stable solution independent of ϕ. But in general, $\phi^{-1}(w^*) \neq f(k^*) - k^*f'(k^*)$. This makes precise what we said in the beginning--producers must build "the proper machines" in order that the long-run wage be equal to the long-run marginal product.

MATHEMATICAL APPENDIX

We wish to prove Proposition I.

By a basic relation of our system

$$L(t) = \int_{k_{min}(t)}^{\infty} d\, g_t(x) \tag{1A}$$

or

$$k_{min}(t) = k_{\bar{w}} \text{ and } L(t) \geqq \int_{k_{min}(t)}^{\infty} g_t(x)dx$$

where $dg_t(x)$ is the measure associated with the distribution function of machines of intensity $\geqq x$, and where $k_{min}(t)$ is the index of the least productive machine used at time t (or the lim inf of these numbers).

It is immediately clear that the continuity, differentiability and direction of $k_{min}(t)$ are equivalent to the continuity, differentiability and direction of $k(t)$; for the wage is the average product of capital of type $k_{min}(t)$--or $w(t) = f(k_{min}(t))$--f is continuous, differentiable and monotonic; and $k(t)$ is a continuous, differentiable monotonic function of the wage.

To show that $k_{min}(t)$ is continuous, we calculate its change in value in a neighborhood of time t; we show that this value is small if the neighborhood itself is small.

But certain problems present themselves. Namely, to show the continuity of $k_{min}(t)$ it is necessary to show that $dg_t(k_{min})$ is positive for $k_{min}(t+\Delta t)$ within a Δt-neighborhood of time t. A backward glance at the equation in (1A) will show why this last statement is true.

So our argument must be more complex than originally planned. We must simultaneously show that $dg_t(k_{min}) > 0$ and $k(t)$ is continuous. To do this we proceed by induction. We divide time into short intervals of length $\overline{\Delta t}$. Later in our argument we place upper limits on $\overline{\Delta t}$. We show that $k(t)$ is continuous during the period $n\overline{\Delta t} \leqq t \leqq (n+1)\overline{\Delta t}$ if $k(t)$ was continuous up to that time.

Lemma 1. $k(t)$ is continuous in the period $0 \leqq t \leqq \overline{\Delta t}$.

1. Our induction property can be shown to hold in the period $0 \leqq t \leqq \overline{\Delta t}$, (1) by our initial conditions (remember the assumption about $dg_o(x)$) and (2) by the logic which shows that if $k(t)$ is continuous in period $n\overline{\Delta t} \leqq t \leqq (n+1)\overline{\Delta t}$ if $k(t)$ was continuous up to time $n\overline{\Delta t}$. We do not try the reader's patience with such an exercise, though he may later wish to check to see that the claim made here is in fact correct.

Lemma 2. $k(t)$ is continuous in the interval $n\overline{\Delta t} \leqq t \leqq (n+1)\overline{\Delta t}$, and $k(t)$ is continuous for $0 \leqq t \leqq n\overline{\Delta t}$.

Proof: (1) We show that $dg_t(x) > 0$ for $k_{\overline{w}} < x < \max\,(k_{min}(t+\Delta t), k_{min}(t))$. Suppose that this is not true. Then $k(\tau) < \max\,(k_{min}(t+\Delta t),\ k_{min}(t))$ for $\tau \leqq n\overline{\Delta t}$. The reason is that up to time $n\overline{\Delta t}$, $k(t)$ has been continuous.

$$g_t(k) = g_o(k) + \int_{t\,|\,k(t) \geqq k} \frac{S(t)}{k(t)}\,dt$$

i.e., the distribution jobs on machine of type k is the original distribution plus the jobs created at times when the capital intensity of machines built was greater than k. Since $k(t)$ was continuous up to $n\overline{\Delta t}$, and $S(t)$ was greater than $S(0)$, if $k_{min}(t) = k(\tau)$ for some time τ before $n\overline{\Delta t}$ or if $g_o(k_{min}) > 0$, then $g_t(x) > 0$ for $k_{\overline{w}} \leqq x \leqq k_{min}(t)$. Therefore $g_{n\overline{\Delta t}}(x) = 0$ for all $x \geqq \max\,(k_{min}(t),\ k_{min}(t+\Delta t))$ if $dg_t(\max(k_{min}(t),\ k_{min}(t+\Delta t)) = 0$.

This indicates that a massive amount of capital-building has taken place since $n\overline{\Delta t}$. For all workers now have jobs on machines of capital intensity greater than or equal to $\max\,(k_{min}(t),\ k_{min}(t+\Delta t))$.

Thus $$\int_{\substack{n\overline{\Delta t} \\ k(t)\ \text{in use}}}^{t+\Delta t} \frac{S(t)}{k(t)}\,dt = L(t+\Delta t) \qquad (2A)$$

We make three comments on (2A): (1) we shall hereafter consider $k_{min}(t+\Delta t)$ to be max $(k_{min}(t), k_{min}(t+\Delta t))$. This will simplify notation. (2) The integral in (2A) is subscripted by the phrase "k(t) in use," since some capital built since time $n\overline{\Delta t}$ may not be in use. (3) (2A) is an equality since $g_o(k_{\overline{W}}) > 0$.

But (2A) places an impossible strain on the system to produce capital in a short period. $S(n\overline{\Delta t}) \leqq sf(k_{min}(t+\Delta t))L(n\overline{\Delta t})$ since all machines at time $n\overline{\Delta t}$ have capital intensity less than $k_{min}(t+\Delta t)$. And $k(t)$ in use $\geqq k(t)$ for $0 < t < n\Delta t$. Therefore for $\overline{\Delta t}$ sufficiently small (2A) cannot hold; and therefore $g_o(\max(k_{min}(t), k_{min}(t+\Delta t)) > 0$ or $k(\tau) = \max(k_{min}(t+\Delta t), k_{min}(t))$ for some time $0 \leqq \tau \leqq n\overline{\Delta t}$.

It is important to note that $\overline{\Delta t}$ was chosen sufficiently small, independent of $k_{min}(t)$ or t itself. A closer look at (2A) will indicate why. For a proof of this proposition, see Appendix B.

We know now that $dg_t(x) > 0$ for $k_{\overline{W}} \leqq x \leqq \max(k_{min}(t), k_{min}(t+\Delta t))$.

2. We now show that $k_{min}(t)$ is continuous. There are four cases to consider. We show that $(k_{min}(t) - k_{min}(t+\Delta t)$ is small in each of these cases.

Case I.

$$L(t) = \int_{k_{min}(t)}^{\infty} dg_t(k) \quad (3A); \quad L(t+\Delta t) = \int_{k_{min}(t+\Delta t)}^{\infty} dg_{t+\Delta t}(k) \quad (4A)$$

Case II.

$$L(t) = \int_{k_{min}(t)}^{\infty} dg_t(k) \quad (5A); \qquad L(t+\Delta t) < \int_{k_{\overline{w}}}^{\infty} dg_{t+\Delta t}(k) \quad (6a)$$

Case III.

$$L(t) < \int_{k_{\overline{w}}}^{\infty} dg_t(k) \quad (7A); \qquad L(t+\Delta t) = \int_{k_{min}(t+\Delta t)}^{\infty} dg_{t+\Delta t}(k) \quad (8A)$$

Case IV.

$$L(t) < \int_{k_{\overline{w}}}^{\infty} dg_t(k) \quad (9A); \qquad L(t+\Delta t) < \int_{k_{\overline{w}}}^{\infty} dg_{t+\Delta t}(k) \qquad (10A)$$

Case IV is easy. $k_{min}(t) = k_{min}(t+\Delta t) = k_{\overline{w}}$.

We proceed with Case I. Subtracting (3A) from (4A)

$$L(t+\Delta t) - L(t) = \int_{k_{min}(t+\Delta t)}^{\infty} dg_{t+\Delta t}(k) - \int_{k_{min(t)}}^{\infty} dg_t(k)$$

$$= \int_{\substack{t \\ k(t) \text{ in use}}}^{t+\Delta t} \frac{sQ(t)}{k(t)}\, dt + \int_{k_{min}(t+\Delta t)}^{k_{min}(t)} dg_t(k)$$

$$L(t+\Delta t) - L(t) \quad - \int_{t}^{t+\Delta t} \frac{sQ(t)}{k(t)}\, dt = \int_{k_{min}(t+\Delta t)}^{k_{min}(t)} dg_t(k)$$

Since $dg_t(k) > 0$ in the relevant range, it is clear that for Δt sufficiently small

$$k_{min}(t+\Delta t) - k_{min}(t) \text{ must be sufficiently small.}$$

Case II. If $k_{min}(t) \neq k_{\bar{w}}$, it is possible to choose a neighborhood of t such that (6A) does not hold since $dg_t(k)$ varies slowly as does $L(t)$. Therefore $k_{min}(t) = k_{\bar{w}}$; $k_{min}(t+\Delta t) = k_{\bar{w}}$ and $k_{min}(t+\Delta t) - k_{min}(t) = 0$ in a sufficiently small neighborhood.

Case III. Likewise, if $k_{min}(t) = k_{\bar{w}}$ and (7A) is satisfied, it is possible to choose a neighborhood such that

$$L(t+\Delta t) < \int_{k_{\bar{w}}}^{\infty} g_{t+\Delta t}(k)\, dk \,.$$

Summarizing these cases, $k(t)$ is continuous for $n\overline{\Delta t} \leqq t \leqq (n+1)\overline{\Delta t}$, given that $k(t)$ was continuous for $0 \leqq t \leqq n\overline{\Delta t}$.

We can now show the truth of Proposition I, using the following formula

$$L(t+\Delta t) - L(t) = \int_{t}^{t+\Delta t} \frac{sQ(t)}{k(t)}\, dt + \int_{k_{min}(t)}^{k_{min}(t+\Delta t)} d\, g_t(k) \qquad (11A).$$

(11A) should make Proposition I obvious.

APPENDIX B

To show that $\overline{\Delta t}$ can be fixed independently of $k_{max} = \max_{t \leq n\overline{\Delta t}} k(t)$ is the second part of this appendix.

The question at hand is what is the minimum time $\overline{\Delta t}$ beyond time t_o so that workers can all be working on machines newly built since t_o and with intensity greater than or equal to k_{max}.

First, it is clear that this is an optimization problem: to create more than $L(t_o)$ jobs on machines of intensity greater than k_{max} in the minimum time.

Second, it is also clear that this time is reduced (or not increased) if we assume that the stock of capital at time t_o contains an infinite supply of capital of intensity k_{max}. It is also logically clear that workers will work only with this intensity.

Then
$$\dot{S}(t) = \frac{S(t)}{k(t)} [sf(k(t)) - sf(k_{max})]$$

for $\frac{S(t)}{k(t)}$ represents the new jobs created at time t; $f(k(t))$ represents the output per worker on new jobs at time t; but these workers must be transferred from machines with average output $f(k_{max})$.

We wish to minimize the time Δt such that

$$\int_{t_o}^{t_o+\Delta t} \frac{S(t)}{k(t)}\, dt \geqq L(t_o) \qquad \text{subject to}$$

$$S(t_o) = L(t_o)\, sf(k_{max})$$

$$\dot{S}(t) = \frac{S(t)}{k(t)} (sf(k(t)) - sf(k_{max}))$$

$$k(t) \geqq k_{max} \, .$$

A third transformation of our problem makes it still simpler.

$$\dot{S}(t) \geqq \frac{S(t)}{k(t)}\, sf(k(t)).$$

Therefore, if we consider the minimum time to create $L(t_o)$ jobs of intensity greater than k_{max}, where the rule of motion of the system is that

$$\dot{S}(t) = \frac{S(t)}{k(t)}\, sf(k_{max}) \qquad \text{and where}$$

$S(t_o) = sf(k_{max})\, L(t_o)$, again we would find a lower bound for $\overline{\Delta t}$.

More formally, given

$$\dot{S}(t) = \frac{S(t)}{k(t)}\, sf(k(t)), \qquad S(t_o) = sf(k_{max})\, L(t_o)$$

we minimize $\widetilde{\Delta t}$ such that

$$\int_{t_o}^{t_o+\widetilde{\Delta t}} \frac{S(t)}{k(t)}\, dt \geqq L(t_o) \text{ provided } k(t) \geqq k_{max} .$$

Note: $\widetilde{\Delta t}$ will be a function of k_{max} alone. Thus we write $\widetilde{\Delta t}(k_{max})$.

Building machines of type k_{max} increases $S(t)$ by the maximal rate in this system. At the same time it creates the maximum possible number of new jobs, given $k(t) \geqq k_{max}$.

Hence $k(t) = k_{max}$ is the optimal control, and

$$S(t) = S(t_o)\, sf(k_{max}) \exp \frac{sf(k_{max})}{k_{max}}\, t$$

We see that $\widetilde{\Delta t}(k_{max})$ is a rising function of k_{max}--since $\frac{sf(k_{max})}{k_{max}}$ is a decreasing function of k_{max}. But $k_{max} \geqq k_{\overline{w}}$. Therefore, $\widetilde{\Delta t}(k_{max}) \geqq \widetilde{\Delta t}(k_{\overline{w}})$. Therefore, there is a nonzero lower bound for $\overline{\Delta t}$.

WAGES, CAPITAL AND UNEMPLOYMENT

One of the best things about putty-clay models is that they restore to economic theory the concept of a job. One pictures a worker at work on a particular machine; there are only so many machines in existence and therefore there are only so many man-hours of employment possible, ergo, only so many jobs. But, while one reads in the newspaper or in government reports that so many jobs are needed, or so many jobs must be filled, this is a rarely mentioned topic in economic theory. This is not a small matter--for unemployment, a job-related concept, is hard to explain in the absence of a job-related theory.

This model is, accordingly, an attempt to relate the amount of unemployment to the number of jobs. But it is unambitious: it is a long-run theory and does not account for unemployment due to insufficient aggregate demand; rather this is closer to a structural unemployment: not enough machines are created to keep the whole labor force employed.

Consider the real wage rate as dependent on the rate of unemployment with low unemployment rates corresponding to high wage rates, and with high unemployment rates corresponding to low wage rates. Suppose the unemployment rate is low, then wages are high; the result is that capitalists will build capital-intensive machines. But very capital-intensive machines provide few jobs--not enough jobs for the steady flow of entrants into the labor force. Therefore, unemployment rises (the wage falls).

Correspondingly, if unemployment were high, wages would be low; capital intensity of newly built machines would be low; a lot of new jobs would be created. And unemployment would fall.

Between this high rate of unemployment there is some steady-state rate of unemployment. That is what our model is all about.

To begin our description of the model, we have the putty-clay technology of Chapter I. <u>Ex-ante</u> output is produced under neoclassical conditions:

$$Q = F(K,L) \ .$$

F is homogeneous of degree one.

$$\frac{\partial F}{\partial K} > 0. \quad \frac{\partial F}{\partial L} > 0. \quad \frac{\partial^2 F}{\partial K^2} < 0. \quad \frac{\partial^2 F}{\partial L^2} < 0.$$

But once machines are built, the capital hardens into hard-baked clay and the capital intensity is fixed forever after. And labor must work with capital of a given type in fixed proportions.

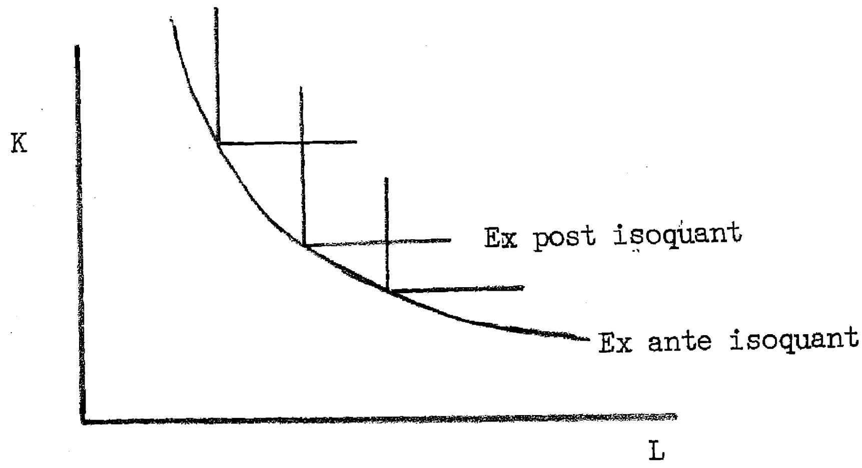

Figure 1

Figure 1 gives a simple depiction of an ex-ante isoquant as an envelope of ex-post isoquants.

The supply of factors to be fed into this production function is traditional: the labor force is growing at a constant rate λ. The new investment in the economy is a constant fraction s of output. Thus $S(t) = sQ(t)$, where $S(t)$ is savings at time t and $Q(t)$ is output at that time.

But we assume that there are hiring and firing costs involved, and that these costs increase per worker (though not per unit output) with the capital intensity of the machines used. Thus a worker on a more capital-intensive machine can demand a higher wage than a worker on a less capital-intensive machine. The reason is that the worker can threaten to leave and the capitalist must undergo the cost of replacing the lost worker. But once replaced, the new worker will in expected value be as litigious and obstinate as the old: he too can bargain. Thus there is some area for wage-bargaining between workers and employers.

Perhaps hiring and firing costs are not the reason for this bargaining; unions may demand high wages, or there may be some sense of a "just" wage, and in particular it may be felt that a rich employer (i.e., with a lot of capital) should pay a higher wage than a poor employer. The important property for our model is that the same worker, if "lucky," will get a high paying job, and if "unlucky" will get a low paying job. What is important is a nonuniformity in the wage rate--for this is what induces a worker to stay in the unemployment pool to search for some job paying more than some minimum wage.

To formalize what we have said earlier, the wage in this model depends upon bargaining. The results of this bargaining depend on three things: first is the capital intensity of the machine used. The second is the average wage in the economy as a whole, and the third is the unemployment rate. For the higher is the average wage , the more believable is the threat that the worker will leave his job to seek a better opportunity; likewise, the lower the unemployment rate the lower is the expected duration of the worker's stay in the unemployment pool; thus the more believable is the threat that the worker will seek another job.

In functional notation, then

$$w_k = w(k,\bar{w},u)\text{--}$$

or the wage paid a worker on capital of intensity k is a function of that capital intensity, the average wage and the unemployment rate.

Of course,

$$\frac{\partial w_k}{\partial \bar{w}} > 0, \qquad \frac{\partial w_k}{\partial u} < 0 .$$

We can write down also the equation for the average wage:

$$\bar{w}(t) = \frac{\int_{k_{min(t)}}^{\infty} w(k(t),\ u(t),\ \bar{w}(t))\ dg_t(k)}{\int_{k_{min(t)}}^{\infty} dg_t(k)} \tag{1}$$

where $g_t(k)$ = number of jobs on machines of capital intensity $\geq$ k at time t,

and where $k_{min(t)}$ is the lowest capital intensity in use at time t.

The wage paid to the workers working on the least productive machine should be exactly equal to the average product on those machines. The reason is that no additional bargaining is possible, but workers demand at least this minimum wage. Thus $w_{min} = f(k_{min})$, where f(x) is the average product of labor on machines of intensity x. The reader should note that $f(x) = F(x,1)$, where F is the ex-ante production function.

But it is also possible to write a construction of the minimal acceptable wage.

Suppose that a job is expected to last n years (units of time).

Suppose that the unemployment rate is constant, and that all labor in the unemployment pool is homogeneous. Furthermore, suppose that all new entrants into the labor force land in the unemployment pool, and in addition, suppose that the unemployment rate is constant. Then the flow into the pool is $\lambda L + \frac{1}{n} L$. The size of the pool is uL.

Dorfman's bathtub theorem indicates that the expected duration in the pool is the size of the pool divided by the outflow, or $\frac{u}{\lambda + 1/n}$.

The worker at time t has a choice of accepting a job paying w_{min}; or he can wait an expected length of time $\frac{u}{\lambda + 1/n}$ and receive an expected wage $\bar{w}$. This job lasts a time n. The marginal gain from waiting and accepting a higher paying job has approximately expected value

$$(\bar{w} - w_{min})n \ .$$

(We should discount this for the fact that this gain is later; but this complicates the mathematics and obfuscates the principles at work.)

Also the expected value is not quite $(\bar{w} - w_{min})n$, but this is an approximation. At the margin, the loss from waiting and not being paid w_{min} should be equal to the expected value from waiting and receiving $\bar{w}$.

Thus

$$(\bar{w} - w_{min})n = \frac{u}{\lambda + 1/n} \quad w_{min}$$

Solving,
$$w_{min} = \frac{n\bar{w}}{n + \frac{u}{\lambda + 1/n}}$$

or
$$w_{min} = g(u,\bar{w})$$

$$\frac{\partial w_{min}}{\partial u} < 0; \qquad \frac{\partial w_{min}}{\partial \bar{w}} > 0 \ .$$

Since $w_{min} = f(k_{min})$ we can write:

$$k_{min} = f^{-1}\ (g(u,\bar{w}))\ .$$

Thus, rewriting (1)

$$\bar{w} = \frac{\int_{f^{-1}\ (g(u,w))}^{\infty} w(k,\bar{w},u)\ dg_t(k)}{\int_{f^{-1}(g(u,\bar{w}))}^{\infty} dg_t(k)}$$

The last relation needed to complete our model is a description of the choice of technique. The capital-labor ratio of new machines is chosen in such a way that profits are maximized for a given unemployment rate and for a given average wage. It is assumed that expectations are static.

Thus if $T(k)$ are training costs for labor on machines of intensity k, we find that entrepreneurs choose k such that

$$\frac{\partial}{\partial k}\left[\frac{f(k) - w(k,u,\bar{w})}{k + T(k)}\right] = 0$$

Thus $$k = k(u,\bar{w})$$

This finishes the description of the model; but several natural questions present themselves. The first of these is whether the model has a steady state, and if it does have a steady state, is there unemployment in that steady state.

At time t the number of new jobs created is $\frac{S(t)}{k(t)}$; the number of new entrants to the labor force is $\dot{L}(t)$. Suppose that there is only one type of capital which has ever been produced in the past--and that this capital intensity is $k*$. Then $S(t) = (1-u(t))L(t)sf(k*)$. For $(1-u(t))L(t)$ is the number of employed workers. Each worker produces an average product $f(k*)$. Savings $S(t)$ is a constant fraction of this.

If the unemployment rate is constant--as is necessary for a steady state--$\dot{u}(t) = 0$.

$$u(t) = 1 - \frac{N(t)}{L(t)} ,$$

where $N(t)$ is the number of employed workers. Thus

$$\dot{u}(t) = \frac{-\dot{N}(t)}{L(t)} + \frac{\dot{L}(t)\ N(t)}{L(t)^2} = \frac{-\dot{N}(t)}{L(t)} + \lambda(1-u(t))$$

But $\dot{N}(t)$ is the number of new jobs. Hence

$$\dot{N}(t) = \frac{S(t)}{k(t)} = \frac{(1-u(t))\ L(t)\ sf(k*)}{k*}$$

If $\dot{u}(t) = 0$, then

$$-(1-u(t))\ \frac{sf(k*)}{k*} + \lambda(1-u(t)) = 0$$

or

$$sf(k*) = \lambda k* .$$

Therefore $k*$ is the steady-state capital-labor ratio of Solow's 1956 model of economic growth--and a very old friend.

Since $w^* = w_k^* = \bar{w}$ in our steady-state, knowing k^* we can infer w^*, for $k^* = k(\bar{w})$. Given w^* we know that

$$w_k^* = w(k^*, u^*, w_k^*) .$$

This equation determines u^*--the steady-state unemployment rate. Thus we have our steady-state; there should be no question that this is one.

The next question we ask is a more difficult one to answer: is the model stable at k^*, u^*? Although there is reason to believe that under certain general conditions this will be the case, we cannot prove it.

But we can prove something slightly weaker: that k^*, u^* may be a stable point--but in any case the path of $(k(t), u(t))$ must oscillate around this, in some sense.

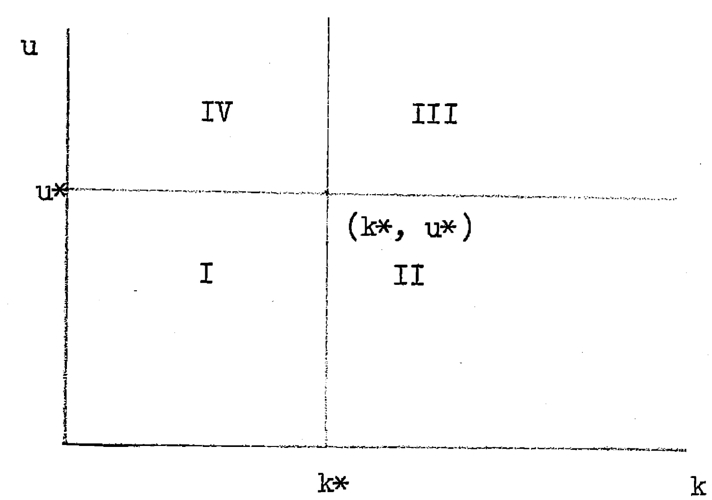

Figure 2

Suppose that $k(t)$ should be less than $k^* - \epsilon$ for an indefinite period of time. Then savings would be very great, u would fall and w would rise. Hence k would rise above $k^* - \epsilon$. Correspondingly, should k remain above $k^* + \epsilon$ for a long period of time, u would rise and hence w would fall, hence $k(t)$ would fall.

Correspondingly, suppose that u remains below $u^* - \epsilon$, then k must rise above $k^* + \epsilon$. But this cannot be maintained indefinitely for then u rises without bound and w falls. A similar log persists if u remains above $u^* + \epsilon$.

Hence we have reason to believe that k^*, u^* is in some sense an equilibrium point with some stability.

In conclusion, we note that this model may have some meaning for macroeconomics: for it answers a fundamental question. Namely, it indicates one intrinsic reason why unemployment is a short-run (and not a long run) problem in a modern capitalist state. The answer, which may not be surprising, is that the number of jobs created depends upon the wage rate; and the wage rate in turn depends on the unemployment rate. And second, we see why unemployment in some countries shies away from both very low levels and very high levels.

INCOME, INVESTMENT, AND WAGES

I.

Three commonly held doctrines of modern economics seemingly conflict, namely, (1) depressions are caused by insufficient investment demand, (2) high wages induce substitution of capital for labor, and (3) if only money wages were less sticky than money prices, producers could be induced to hire all the labor available. For it is reasonable to argue that lower real wages induce less substitution of capital for labor, therefore induce a smaller investment demand, and correspondingly a lower income and a lower level of employment. In truth a subtler logic prevails: lower real wages may at times raise real income by increasing investment demand but at other times they may reduce real income because producers are unwilling to hire additional labor at the higher wages demanded. It is the purpose of this paper to set forth a model in which both kinds of behavior can occur and to juxtapose these two conflicting points of view.

Different models have been explored,[1] but all have roughly the same results, the same analysis and this same deep-rooted and prevailing ambiguity. Basically, desired consumption and investment determine an equilibrium level of output. But investment is determined by

[1] Only one of these models (the basic one) is presented here.

the following considerations: for next period producers have a choice of techniques to produce output: they may either introduce new machines, which saves labor, or they may use old machines in combination with a larger input of labor. Using simple expectations of the wage, income, and interest, investment is determined so that the marginal cost of output is the same for both techniques. This consideration determines the level of desired investment.

Thus our model derives Keynes's marginal efficiency of investment schedule from real considerations. What is new or what is surprising is that this MEI schedule depends upon three variables: expected output, interest, and the real wage.

Our model has a second important feature which differs from the <u>General Theory</u>. The price level is sticky in a downward direction at the level p_0 prevailing at the beginning of the period. Above this level, however, the price is a constant multiple γ of the marginal costs of producing an additional unit of output. Such a pricing system demarcates two regimes; in one of these regimes aggregate supply is independent of the real wage and producers will supply whatever level of output is demanded; in the other regime, aggregate supply depends solely on the real wage. The merit of such an assumption is its focus on the role of aggregate supply in a Keynesian model. In particular, the sole dependence of aggregate supply on the real wage seems both important and strange in this light. But the obvious demerit of this pricing system is that its pointed singularity does not conform to reality.

In the first regime, aggregate supply does not depend on the real wage. An upward shift in workers' demands for money wages does not provoke a retaliatory price rise. Therefore the upward shift in the demand for money wages is in fact an upward shift in the demand for real wages. As a result, aggregate demand will increase because of entrepreneurs' desire to substitute capital for labor. And aggregate supply will just match this demand; so output increases. But in the second (traditional) regime, a rise in money-wage demands will cause a rise in the price level which, in the absence of a resultant insufficiency of transactions balances, would leave all real quantities unchanged. Each of these regimes represents one of the two possible effects of wages on income and investment.

II.

The key to this model is three separate technological relations: for this reason they are presented first.

The first relation tells how output is produced this period: it can only be produced on the machines already existing at the beginning of the period--as we assume that it takes a full period for new machines to be built and brought into production. The collection of machines in existence at the beginning of the period need not be homogeneous. For present purposes the only thing that matters is that these machines with N workers can produce an output Q. Increasing the number of workers increases the output, although in general at a decreasing rate. The justification for this is twofold: (1) the most efficient machines are used first; but as output increases, successively less efficient machines are brought into production; (2) given machines may be used more intensively, but with diminishing returns. We may summarize all of this mathematically in the equation

$$Q = G(N_o) \qquad G'(N_o) > 0,\ G''(N_o) < 0\ . \tag{1}$$

The subscript "o" denotes "old" machines.

One possible picture of $G(N_o)$ is shown in Figure 1.

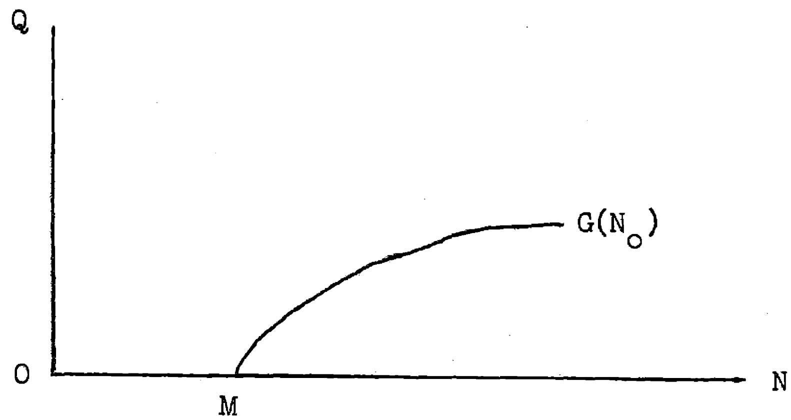

Figure 1

OM represents overhead labor. (It does not contradict the later argument to say that OM = 0 or even that OM is negative.)

The second technological relation is analogous to $G(N_o)$. It relates the output possible in this period's machines on the next period. In the absence of the physical deterioration of machines, this would be exactly $G(N_o)$. But whatever the case, we still generally write $Q = H(N_o)$, which represents next period's output on machines existing at the beginning of this period with an input of labor N_o. Since we assume that machines neither ripen nor mature nor improve with age, we can write $G(N_o) \geqq H(N_o)$. And as in the last paragraph we assumed that $G'(N_o) > 0$ and $G''(N_o) < 0$, similarly and for exactly the same reasons $H'(N_o) > 0$ and $H''(N_o) < 0$.

The third technological relation might be called an Investment Possibilities Schedule. Next period new machines will produce a specified output when combined with a specified number of laborers. Symbolically, we present this in equation (2).

$$(2) \qquad Q_{\text{new machines}} = F(I, N_{nm}) \quad \text{with}$$

$$\frac{\partial F}{\partial I} > 0, \quad \frac{\partial F}{\partial N_{nm}} > 0; \quad \frac{\partial^2 F}{\partial I^2} < 0, \quad \frac{\partial^2 F}{\partial I \partial N_{nm}} > 0, \quad \frac{\partial^2 F}{\partial N_{nm}^2} < 0$$

Again, but more formally, equation (2) reads that the output produced on the new machines next period is a function of the amount of output devoted to investment and the number of laborers working with the new capital goods.

We consider two assumptions about F. One is the usual assumption that F is homogeneous of degree one.

There are strong reasons to suggest that a better assumption is that F has returns to scale less than one, and accordingly this case is also considered. One justification for such an assumption is the existence of limited entrepreneurial ability to manage investment enterprises. The second reason is limited resources of various sorts; the most important of these is limited information about investment projects and prospects, but locational considerations and limited natural resources should not be excluded from this list.

It is worth noting, for the sake of perspective as well as for clarity, that G and H represent ex-post production functions, and F represents the ex-ante production function. The technological specification of our model is only a minor modification of Johansen (1959) and Solow (1962). The difference lies only in our nonspecification of the physical deterioration of machines. This, as will later become clear, is allowable because our model is short-run and is only concerned with a single time period.

III.

Given the stock of machines which will be in existence next period and given the investment possibilities schedule, the problem for the entrepreneur with given expected sales is to minimize cost; the entrepreneur must decide how many new machines should be built and how production will be allocated next period between new and old machines.

For the sake of simplicity we assume that producers have static expectations: they expect wages, the interest rate, and output to be the same this period as next period. Actually, these assumptions are not necessary for the completion of the appointed task (which, for the moment, is to derive an investment function). For instance, we could assume that wages and output are increasing at some trend rate of increase; or any of a number of more complicated assumptions. One variant is the assumption that entrepreneurs have Markovian expectations about the Gross National Product. Thus "high" GNP is expected to continue into the future, but there is the expectation that GNP will in expected value converge toward some normal level. (For a longer run model, one might have GNP deflated by its normal rate of growth.) A similar expectation might apply to the wage rate.

Basically, we say that investment depends on the expectation of future periods' aggregate demand and wage levels. These two variables are macroeconomic in character and thus are not ones which the firm itself is likely to affect; so to the entrepreneur in making his investment decisions, simple expectations are necessary; and in a model such as ours, what expectations are more natural than static ones? But also, as long as output and wages are thought to be Markovian, and high output (wages) this period are assumed to be correlated with high output (wages) next period, it is still true that the important variables in the investment equation will be this period's wages and this period's output, although the specific investment equation will differ somewhat from the one derived below.

The term structure of interest rates is consistent with this kind of Markovian expectations (see Gordon Pye in The Quarterly Journal of Economics). Our concern here is the term structure of wage rates. For a capitalist, in buying a putty-clay machine, must piece together future short-term wage payments, while his return is a certain and fixed amount of output. Similarly, in the bond market, an arbitrageur has the option of buying a long-term bond and receiving a constant fixed payment or of piecing together the short-term interest rates. The term structure of wage rates and the term structure of interest rates seem to be sufficiently analogous that, as a first approximation, it is reasonable to assume the same type of expectational behavior in both markets.

The requirement that old and new techniques must have the same marginal costs for producing output gives two equations, which jointly determine investment and employment on new machines.

(3)
$$\frac{w}{\frac{dH}{dN_0}} = \frac{(\lambda + \delta + r)}{\frac{\partial F(I,N_{nm})}{(\partial I)}}$$

(4)
$$\frac{w}{\frac{dH}{dN_0}} = \frac{w}{\frac{\partial F(I,N_{nm})}{\partial N_{nm}}}$$

The left-hand side of (3) is the marginal cost of output by adding labor to old machines (w is the real wage). The RHS of (3) is the marginal cost of output by investing in new techniques: $1/\partial F/\partial I$ represents the marginal product of new machines. $(\lambda + \delta + r)$ represents the carrying cost of capital per period, where λ is the rate of depreciation due to obsolescence, δ, depreciation due to physical wear and tear, and r the bank rate of interest.

Equation (4) has a similar interpretation. The marginal labor costs (for the last unit of output) must be equal on new and old machines. Or, in other words, equation (4) states that the marginal cost of output by adding labor to new machines must equal the marginal cost of output by adding labor to old machines. Cancelling the w from both sides of the equation and inverting we obtain the perhaps more obvious condition.

$$(4') \qquad \frac{dH}{dN_o} \triangleq \frac{\partial F}{\partial N_{nm}}$$

The marginal product of labor on all machines, both new and old, must be the same.

One further observation is worth noting: Businessmen often say that technical change is what causes them to invest, but obsolescence causes them to demand higher returns. Both of these elements are present in equation (3). λ represents the increased return to investment demanded because of obsolescence. F, on the other hand, represents (in part) the technical change which causes investment.

Equations (3) and (4) may be derived more formally from minimization of costs subject to the condition of producing at least a given output. Symbolically, define costs, E, as follows:

$$(5) \qquad E = (\lambda + \delta + r)\, I + w^e N$$

N represents total employment; $N = N_0 + N_{nm}$. w^e represents the expected real wage next period. We shall always be careful to differentiate the real wage from the money wage by the following convention: the real wage will be w unsubscripted; the money wage will be represented by w_m with the subscript "m". The problem then is to

$$\text{(6)} \qquad \text{Min } E \quad \text{s.t. } H(N_0) + F(I, N_{nm}) \geq Q_0^e$$

where superscript "e" represents "expected"

The Lagrangean is

$$\text{(7)} \qquad L = w^e N + (\lambda + \delta + r)\, I + w^e N_{nm} - \beta(G(N_0) + F(I, N_{nm}) - Q_0)$$

Formally, we obtain the three equations (8), (9), and (10) as necessary conditions for a minimum

$$\text{(8)} \qquad w^e - \beta \frac{dH}{dN_0} = 0$$

$$\text{(9)} \qquad w^e - \beta \frac{\partial F(I, N_{nm})}{\partial N_{nm}} = 0$$

$$\text{(10)} \qquad (\lambda + \delta + r) - \beta \frac{\partial F(I, N_{nm})}{\partial I} = 0$$

$$\text{(11)} \qquad H(N_0) + F(I, N_{nm}) - Q_0^e = 0$$

Eliminating β, (8), (9) and (10) reduce to two equations--exactly the same as (3) and (4)--as expected. The assumption that $H(N_0)$ and $F(I,N_{nm})$ have diminishing returns and returns to scale less than or equal to one is a sufficient condition for a minimum.

The fact that producers use the cheapest available techniques, or at least what are expected to be the cheapest available techniques, to produce next period's output, is equivalent to the determination of a demand for new investment, given the expected wage rate and expected sales. Thus (3), (4) and (11) plus our assumptions about static expectations give an investment demand equation:

$$I = I^d(w,Q)$$

This equation expresses the investment demand given sales Q and a wage w.

Another question naturally arises: in some sense, will this investment be profitable? By our cost-minimization procedure, the marginal cost of producing an additional unit of output is the same by adding labor as by adding additional units of capital. But our pricing mechanism (see below for details) assumes more than a constant percentage mark-up over marginal costs. Therefore, the last unit of output is produced at a profit (and this last unit of output is produced at minimum cost). Therefore, in some sense this investment is expected to be profitable.

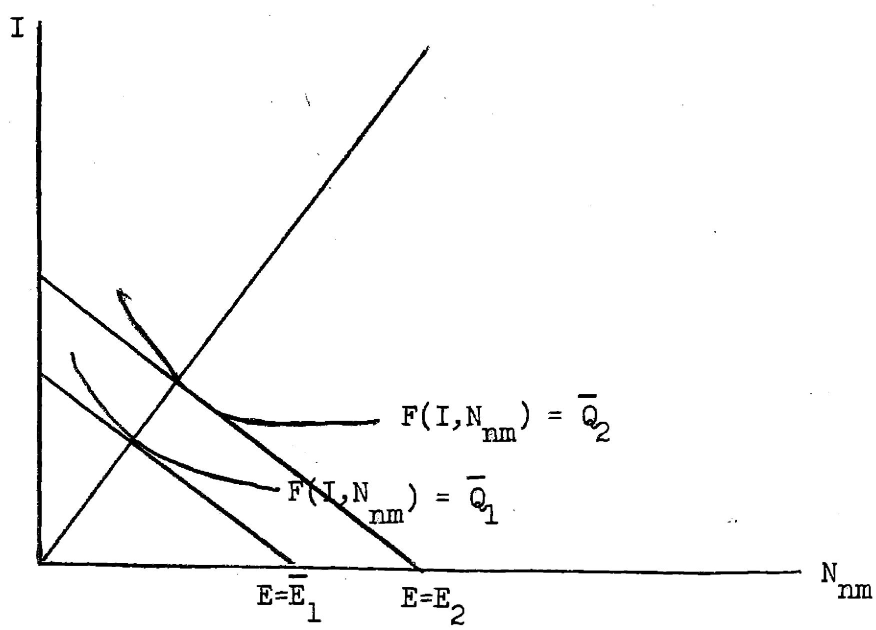

Figure 2

F has constant returns to scale

For given w and given $(\lambda + \delta + r)$, it is clear that if F has constant returns to scale, the following program will minimize producers' costs.

It is clear that there is a minimum cost per unit output on production with new machines for given w and $(\lambda + \delta + r)$. This minimum cost per unit output will be independent of the volume of output with new machines. Call this cost $e_{min}(w)$. Output is produced with old machines up to the point where

$$\frac{w}{\frac{dH}{dN_O}} = e_{min}(w)$$

Beyond that point all added output is produced with new machines and labor working with those new machines.

Thus if one were to trace an investment schedule as a function of output for a given wage, $I^d(\bar{w},Q)$, it must have the following shape:

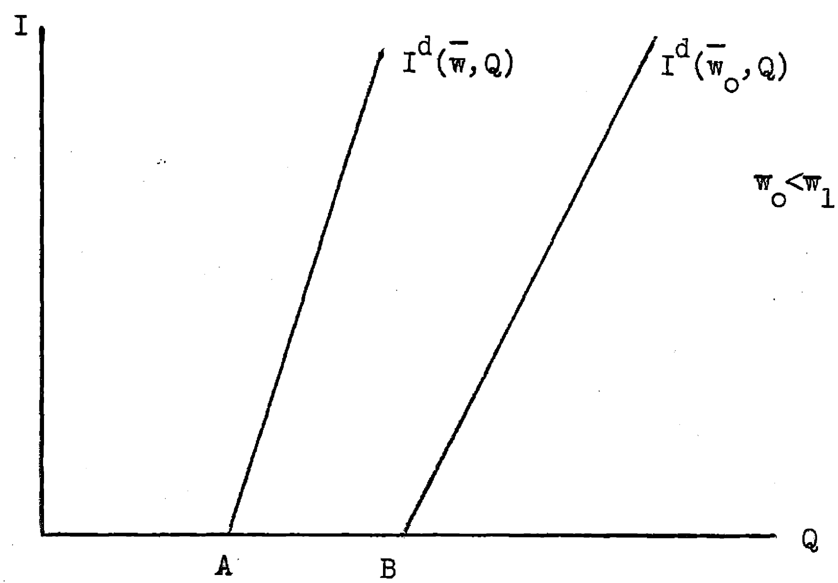

Figure 3

In Figure 3 at A, $\dfrac{\bar{w}_1}{\frac{dH}{dN_0}} = e_{min}(\bar{w}_1)$;

at B, $\dfrac{\bar{w}_o}{\frac{dH}{dN_o}} = e_{min}(\bar{w}_o)$.

In Figure 3 at the higher wage rate $\bar{w}_1$ the investment demand schedule should be steeper than at the lower wage rate $\bar{w}_o$.

IV.

Two of the most important aspects of the economy are yet to be explained: consumption and wage behavior.

Consumption is determined simplybby income and the interest rate. But we assume that the latter is a parameter set, for instance, by William McChesney Martin, and hence we may write simply

$$C = C(Q) = C(C + I) \qquad \text{with}$$

$$1 > C'(Q) > 0, \quad C''(Q) < 0, \; C(0) > 0 \; .$$

This is equivalent to

$$S = S(Q) \quad \text{with } S'(Q) > 0 \quad \text{and } S''(Q) > 0; \quad S(0) < 0 \; .$$

In equilibrium,

$$I^d = S(Q) \; .$$

If the level of investment is fixed, the level of saving is determined and thus the level of income. And a change in the level of investment results in a much larger change in the level of income by the multiplier.

V.

Keynes assumed that workers and management bargain for a money wage. Such behavior should be represented by a wage-bargaining equation. But in The General Theory, real wages are determined only by the level of aggregate demand: for whatever the level of the money wage, the price level will adjust in such a way that aggregate demand will be equal to aggregate supply. In the Keynesian model, admittedly an upward shift of this wage bargaining equation will be associated with a lower level of aggregate demand--for if all real variables remained unchanged, higher levels of the wage bargaining equation would correspond to a higher price level; in turn, a higher price level would correspond to a higher demand for transactions balances. Therefore, the demand for transactions balances would exceed the supply of transactions balances (if all real variables remained unchanged). Hence the interest rate must rise with a corresponding decrease in the level of investment and aggregate demand. But in the last paragraph we swept this effect under the rug by assuming that the Central Bank uses the market rate of interest rather than the supply of money as the measure of tightness in money markets. Thus we have assumed implicitly that an upward shift in the wage bargaining equation will also be accompanied by a sufficient upward shift in the supply of money, to keep the interest rate constant; for now this is convenient but later we deal with the case where the money supply itself is the monetary parameter.

In a manner, now classical, we assume that the money wage equation can be represented as a function of the rate of unemployment, output per capita, and money wages last period. It is of considerable importance to note that we can add the price level to this equation without changing the analysis which follows--so long as a given percentage change in the price level will always be associated with a smaller percentage change in the money wage equation. We shall not add this to our analysis, since the reader already has a weighty load of equations, qualifications and constraints to keep in his mind at the same time.

In any case, we write our Phillips Curve as $w_m^t = w_m(u, Q/L, w_m^{t-1})$, where t represents the time period and where L represents the total population (or labor force) and u the unemployment rate. Since we are considering a strictly one-period model and since $u = 1 - N/L$, we can rewrite our Phillips Curve as

$$w_m = w_m(N,Q) \tag{13}$$

The justification for this curve is that it represents a wage bargaining equation; u, the unemployment rate represents the strength of the workers; for the lower the unemployment rate, the easier it is for a worker to find another job, and therefore the stronger is the bargaining position of the workers.

Similarly, the higher is output per capita the greater is the amount which the employer can afford and is willing to pay. And last period's wages set a standard (or an initial point) for this period's wages. Newspaper articles about labor contracts always discuss the increase or rate of increase of wages in these agreements. There is, presumably, some correspondence between the usual representation of the bargains and the bargaining process itself. As empirical evidence for such a wage bargaining equation, we can only cite the results of Phillips, Lipsey, Perry, Kuh and others. But the reader himself should judge the weight of the conflicting evidence for and against the Phillips Curve.

VI.

A money wage equation should naturally be accompanied by a system of pricing. The system of pricing which we assume in this model is the second area in which we differ markedly from the Keynesian model presented by Mr. Hicks in his classic Mr. Keynes and the Classics. First, we assume that the pricing system is oligopolistic. By this we mean first of all that there is a kink in the demand curve of each representative oligopolist, and this kink occurs at the price level p_o prevailing at the beginning of the period. Thus we say that prices are rigid in a downward direction. But we make no such assumption about the upward direction; we assume that in the upward direction the price level (provided that it is greater than p_o) will be a multiple γ of the marginal cost in money terms. This multiple γ is the standard measure for the degree of imperfect competition.

We can only justify these assumptions as an ad hocery which permits the introduction of markets which are to some degree noncompetitive. This pricing system is the Achilles heel of this essay; but it is left to a future attempt to bring into a Keynesian analysis a fuller array of market structure; in particular, to assume that the firms in these markets are imperfect Chamberlinian competitors, and to try to derive pricing behavior, wage behavior, and a modified Keynesian model from there.

VII.

Before attempting an analysis of our model, we should put together its components; this gives some perspective. Most important, we must remember that our model has two regimes. In one regime the price level is rigid at the level prevailing at the beginning of the period p_o.

In this regime, investment demand can be derived from the three equations (3), (4) and (11):

$$I^d = I^d\ (w, Q)$$

and labor demanded can be derived from (1)

$$N^d = G^{-1}\ (Q)$$

Investment supplied (saving equation) and labor supplied (wage bargaining equation) are represented:

$$I = S = S(Q)$$

$$w = \frac{w_m(N, Q)}{p_o}$$

In the second regime, in which prices are no longer sticky, the real wage equation is modified: Rather,

$$w = \gamma / \frac{dG}{dN_0} \quad . \tag{12}$$

VIII.

A PREVIEW OF RESULTS

In the first regime, an upward shift in the wage-bargaining schedule increases equilibrium income. For such an upward shift tends to raise the real wage. A rise in the real wage induces substitution of capital for labor and therefore raises investment demand. In turn, this raises real income.

Meanwhile, the price level has not responded to the rise in the wage-bargaining schedule; for in this first regime prices stick at p_0. This aspect of the model is critical--for producers are willing to supply a changed level of aggregate demand at an unchanged price level. Thus aggregate supply (in this range) is independent of the real wage.

However, in the second regime an increase in the labor-bargaining schedule is accompanied by a corresponding increase in the price level which leaves real wages unchanged. In this regime aggregate supply depends solely on the real wage. The equilibrium level of output is determined so that aggregate demand is equal to aggregate supply; at this output, and with the employment associated with this output, the money wage is determined by (13); and the price level associated with this output adjusts so the real wage is a multiple γ of the marginal product of labor. And so in this second regime aggregate supply depends solely upon the real wage rate; and this dependence or independence of aggregate supply from the real wage rate demarcates the two regimes.

IX.

As our model is divided into two regimes, so is our analysis. Accordingly, in the next four sections the price level is assumed to be sticky at the level p_o.

The next step of the analysis is to explain several reduced form equations and to try to catalogue their properties. The first of these is a labor-market equilibrium equation.

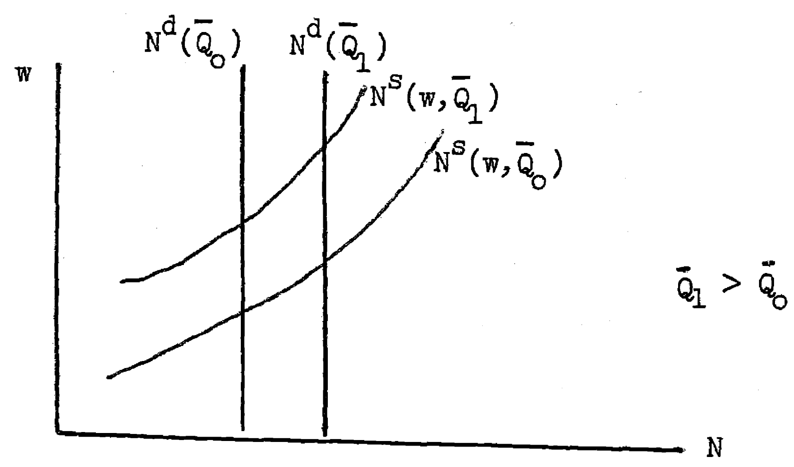

Figure 5

Labor demanded N^d depends upon Q alone, since $N^d = G^{-1}(Q)$. For given Q, Figure 5 pictures labor demanded. Similarly, with price level p_o, labor supplied is a function of the wage rate for some given Q, from equation (13), $w = w_m \frac{(N,Q)}{p_o}$. (This is equivalent for given $\bar{Q}$ to $N^s = N^s(w,\bar{Q})$.) Here we are interpreting the Phillips Curve as a "labor supply equation"; since the Phillips Curve, for a given level of output, relates the wage demanded for a given supply of labor. But no additional connotations should be attached to the word "supply."

Figure 5 shows that the intersection of the labor supply with labor demanded is higher with higher levels of output. The reason is clear, for labor supplied falls with higher levels of output, while labor demanded rises. Thus the equilibrium level of wages w is a rising function of output Q.

Thus we can trace graphically a labor market equilibrium equation LL.

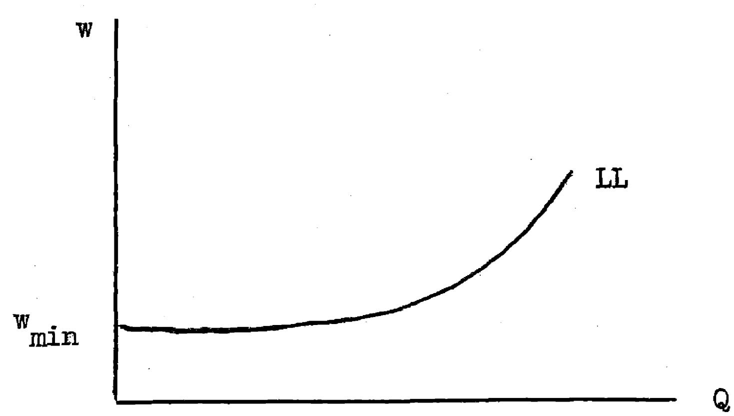

Figure 6

LL is bounded above some $w_{min} > 0$ if the wage demanded by workers (unions) is always greater than some minimum wage w_{min}. This is made an assumption of the model.

$$w(N,Q) \geqq w_{min} > 0$$

The equation LL, mathematically, represents the set of points such that $w = w(N^d(w,G),Q)$, where N^d is the labor that producers demand with output Q.

X.

Remembering that we are still assuming that the price level is rigid at the level p_o, we derive the next reduced form.

For each w on the LL curve there is a unique level of output Q which maintains labor market equilibrium.

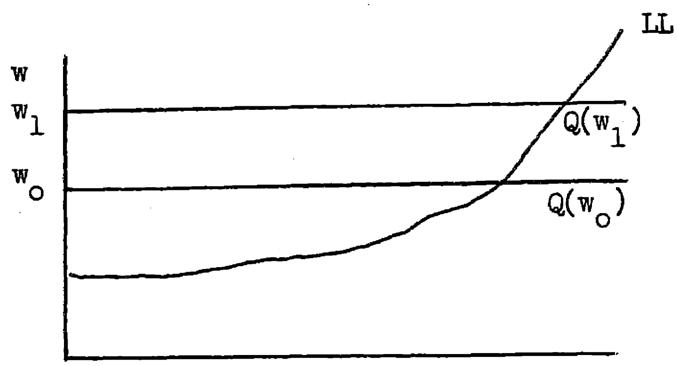

Figure 7

For such Q and such w there will be a level of investment demanded (given by equations (3), (4) and (11)) as producers choose the best techniques. This can be represented as

(16) $$I^d = I^d\ (w, Q_{LL}(w))$$

Also, there is a savings function (12); and $S(Q) = I^d(w, Q_{LL}(w))$ is the equilibrium condition that desired savings = desired investment. This allows us to trace out an IS curve. We denote it an IS-LL curve because Q in equation (16) depends on LL. Figure 8 gives a picture of how this equilibrium looks.

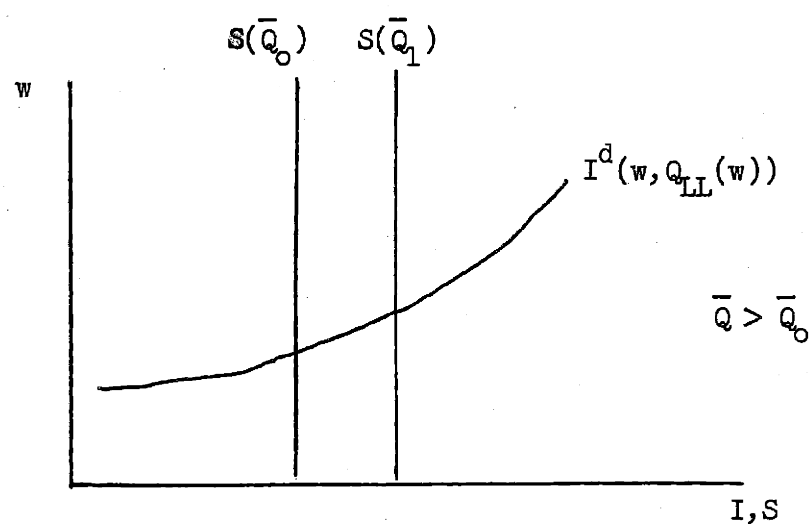

Figure 8

It is clear that the higher Q must be associated with higher equilibrium w in Figure 8. (I^d is independent of Q, but upward sloping. S(Q) is independent of w but moves right with higher Q.)

Thus we get an IS-LL curve and an LL curve in Figure 9.

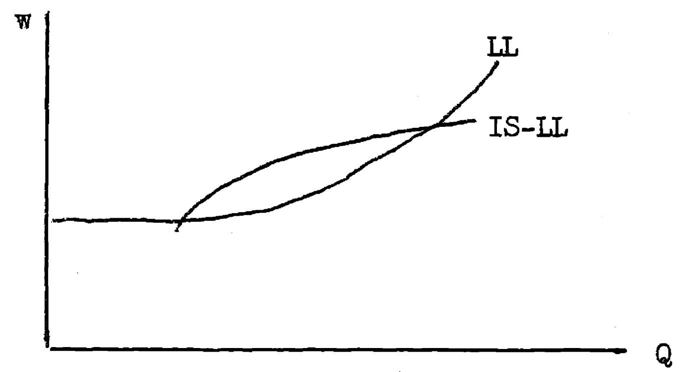

Figure 9

The next question is, what is the reason for showing two intersections of the IS-LL curve and the LL curve? This needs some mathematical argument.

The IS-LL curve is defined by the following relation:

$$S(Q_{IS\text{-}LL}(w)) = I^d(w, Q_{LL}(w)) \tag{17}$$

Differentiating (17) with respect to w, we get

$$\frac{dS}{dQ}\ \frac{dQ_{IS\text{-}LL}}{dw} = \frac{\partial I^d}{\partial w} + \frac{\partial I^d}{\partial Q_{LL}}\ \frac{dQ_{LL}}{dw} \tag{18}$$

All the terms on the right hand side of (18) are positive. For very large Q and higher w

$$\frac{\partial I^d}{\partial Q_{LL}} \cdot \frac{1}{\frac{dS}{dQ}}$$

is assumed to be greater than one. The justification for this is that the added investment necessary to produce added increments of output becomes very large as both investment and labor run into diminishing returns (in the case where F has returns to scale less than one). Where F has CRS higher w induces sufficiently high investment ratios that

$$\frac{\partial I^d}{\partial Q_{LL}}$$

will become large. Also $\frac{1}{\frac{dS}{dQ}}$ is increasing with Q.

Therefore, for high Q

$$\frac{dQ_{IS\text{-}LL}}{dw} > \frac{dQ_{LL}}{dw} .$$

Conversely, for low levels of output, investment in new machines will be low but nonnegative. But $S(0)$ is negative. Therefore for all wage rates Q_{IS-LL} is greater than a level of output Q_{min} at which $S(Q_{min}) = 0$. (See section IV.) And for low levels of output $Q_{IS-LL} < Q_{LL}$, as pictured in Figure 9.

There is no particular reason why our graph should necessarily show two intersections, however. At most we have shown that there are an even number of such intersections; and this even number may be 0 (or a tangency with a "double" intersection).

Further assumptions could show that there are either 0 intersections or two. For instance, one might assume an almost ⌋-shaped $w = w(N,Q)$-schedule; as employment approaches full employment there is a sharp rise in our static Phillips Curve.

XI.

Once again we delay the time when we shall unite the two regimes of our model: the Keynesian regime, in which the real wage is proportional to the marginal cost, and the regime in which the price level is sticky in a downward direction. We derive a third reduced form, again with the assumption that the price level is rigid at the level p_o.

The IS-LL curve was convenient because it showed the occurrence of equilibria in pairs, when our model was seasoned with some "normal" assumptions about the shape of the wage demand schedule, the investment schedule, and savings function. But the IS-LL curve is inconvenient for comparative statics since a shift in the wage demand schedule shifts both the IS-LL curve and the LL curve. We would prefer to develop an IS curve which would shift independently of the LL curve. Accordingly, such an IS curve is derived, and we try to catalogue its relationship with the IS-LL curve of the last section.

The IS schedule we have in mind satisfies the simultaneous relation that $S(Q_{IS}) = I^d(w, Q_{IS})$. If F has constant returns to scale (see section III) we already know that we can represent the I^d schedules for each given wage, as in Figure 10 below.

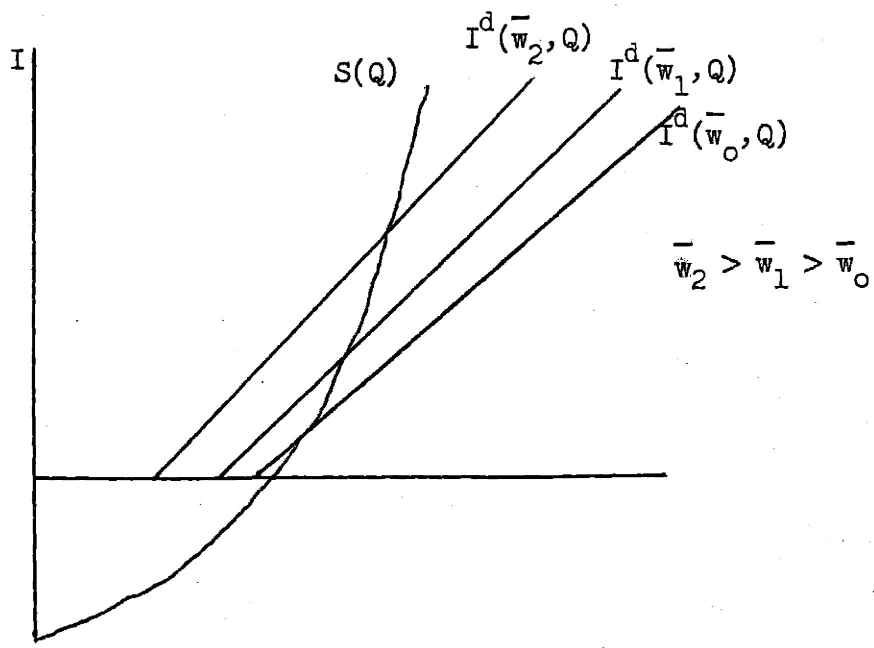

Figure 10

The intersection of the savings schedule and the investment schedules gives at least one level of equilibrium output for each wage level in Figure 10. However, there may be three such levels of equilibria, for sufficiently low wage rates. The middle such equilibria, however, must be unstable (in the sense that $S'(Q) > \frac{\partial I}{\partial Q}$).

Thus we show an IS schedule made up of the two stable branches in Figure 11.

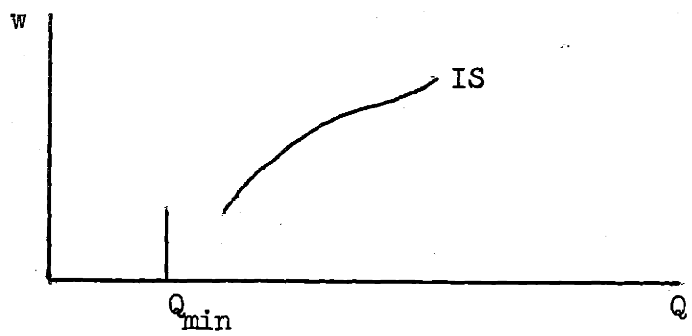

Figure 11

In Figure 11, Q_{min} represents the level of output at which savings is zero: i.e., $S(Q_{min}) = 0$; this is the minimum equilibrium level of output because investment is greater than or equal to zero.

But it is worth noting that only one branch of the IS curve can intersect LL at the same wage. For suppose that both branches intersect LL. Then

$$S(Q_1) = I^d(\bar{w}, Q_1) \quad \text{and} \quad Q_1 = Q_{LL}(\bar{w})$$

Thus $Q_1 = S^{-1}(I^d(\bar{w}, Q_{LL}(\bar{w})))$, which defines Q_1 uniquely.

Thus we derive in Figure 12, two possible pictures of our IS curves.

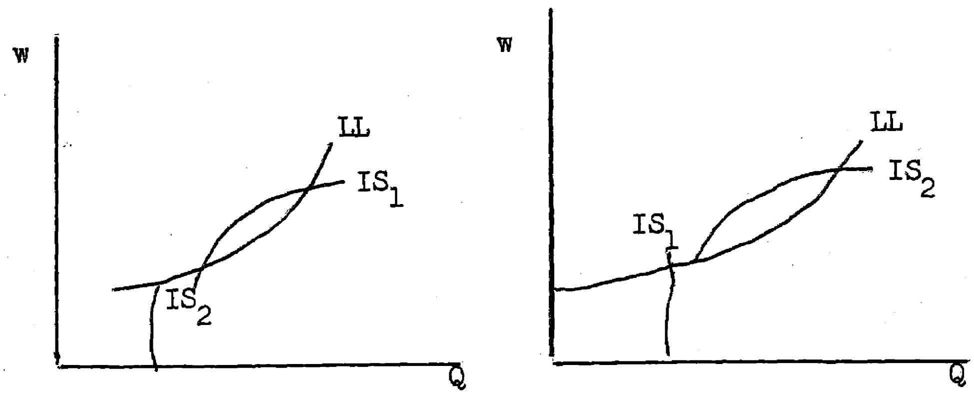

Figure 12A Figure 12B

IS_1 and IS_2 represent the two (possible) branches of the IS curve.

And so we can add to Figure 9 an IS curve pictured below.

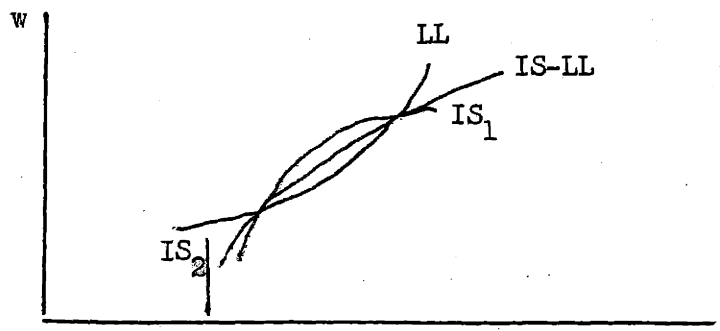

Figure 13

Intersections of IS-LL and LL are also intersections of IS and LL; for IS-LL meets the relation (17)

$$S(Q_{IS-LL}(w)) = I^d(w, Q_{LL}(w)) \ .$$

If $Q_{IS-LL}(w) = Q_{LL}(w)$ then $S(Q_{IS-LL}(w)) = I^d(w, Q_{IS-LL}(w))$. But this is the defining relation for $Q_{IS}(w)$. Therefore, at such points of intersection

$$Q_{IS}(w) = Q_{IS-LL}(w) = Q_{LL}(w) \ .$$

In the original drafts of this paper we did not picture an investment schedule quite so singular as those shown in Figures 3 and 10. Furthermore, diminishing rather than constant returns to scale were assumed. If F has diminishing returns to scale, one can draw investment functions like those in Figure 14.

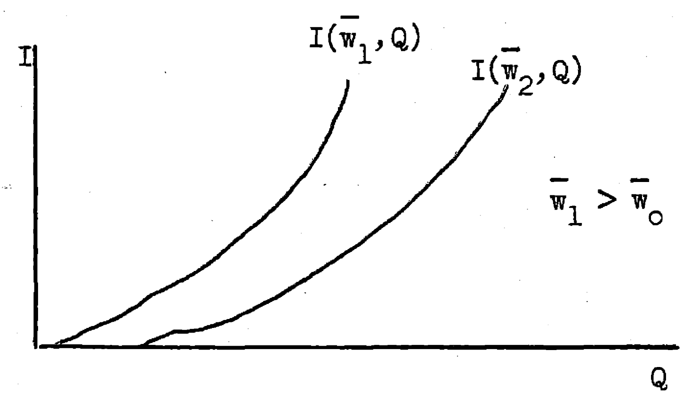

Figure 14

And thus an IS curve can be derived as in Figure 15.

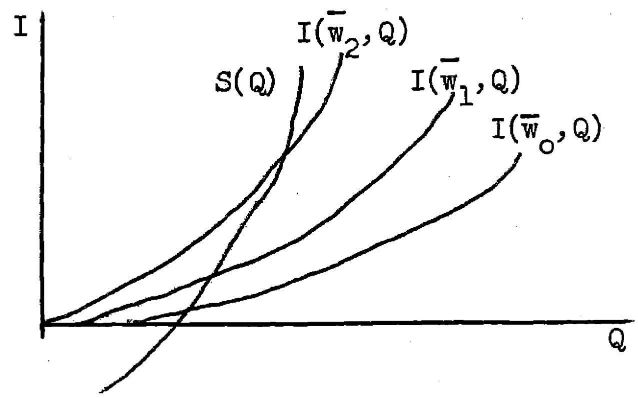

Figure 15

XII.

The stability of any system clearly depends on the adjustment mechanism postulated. Accordingly, either upper equilibria or lower equilibria may be stable. We make a crucial assumption. We assume that at any moment firms are in equilibrium in using the cheapest techniques of that moment. Accordingly, either equation (12) or equation (13) may not be satisfied; but equations (3), (4) and (11) always are satisfied.

With this in mind one can, with caution, make some comments. The underlying mechanism of the LL equation is the wage-bargaining. Therefore, above LL one can assume wages will be falling. Below LL wages will be rising.

The equilibrator of the IS schedule is output produced. To the left of the IS schedule output will rise; to the right of the IS schedule output will fall. One finds that the lower equilibria are stable and the upper equilibria are unstable.

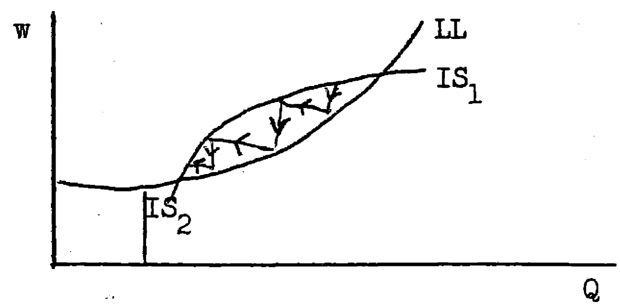

Figure 16

As a footnote to stability, one might propose a business cycle theory with two equilibria; each of them alternately stable and unstable during different phases of the cycle. The w_{min} keeps output from falling too low to the left. The marginal product of labor constraint (12) keeps equilibrium from moving too far to the right.

XII.

Finally, we have reached a stage where it is opportune to re-introduce the second regime of our model. Luckily this can be done in a quite natural manner in the w-Q space of Figures 9, 11, 12, 13 and 16.

Returning to the wage bargaining equation, we see that $w_m = w_m(N(Q),Q)$. Therefore the money wage can be represented as dependent on the level of Q alone. Similarly, from the form of $G(N_o)$ the marginal product of labor depends only on Q, and declines as output goes up.

The price level satisfies the equation

$$p = \gamma \frac{w_m(Q)}{G'(N(Q))} \qquad \text{for} \qquad \gamma\frac{w_m(Q)}{G'(N(Q))} \geqq p_o$$

$$p = p_o \qquad \text{for} \qquad \frac{w_m(Q)}{G'(N(Q))} \leqq p_o$$

$$\frac{d}{dQ}\left[\frac{w_m(Q)}{G'(N(Q))}\right] = \frac{w_m'(Q)}{G'(N(Q))} - \frac{w_m(Q)G''(N(Q))N'(Q)}{[G'(N(Q)]^2}$$

Hence there is a level of output Q_o below which $p = p_o$. Above that level the real wage is a fraction of the marginal product of labor. This can be represented in our Q-w space in the following manner: for levels of output below Q_o the intersection of IS and LL represent equilibrium points of our system. For levels of output above Q_o the intersection of IS and SS represents points of equilibrium. SS is the curve in w-Q space: $G'(N(Q)) = \gamma w$.

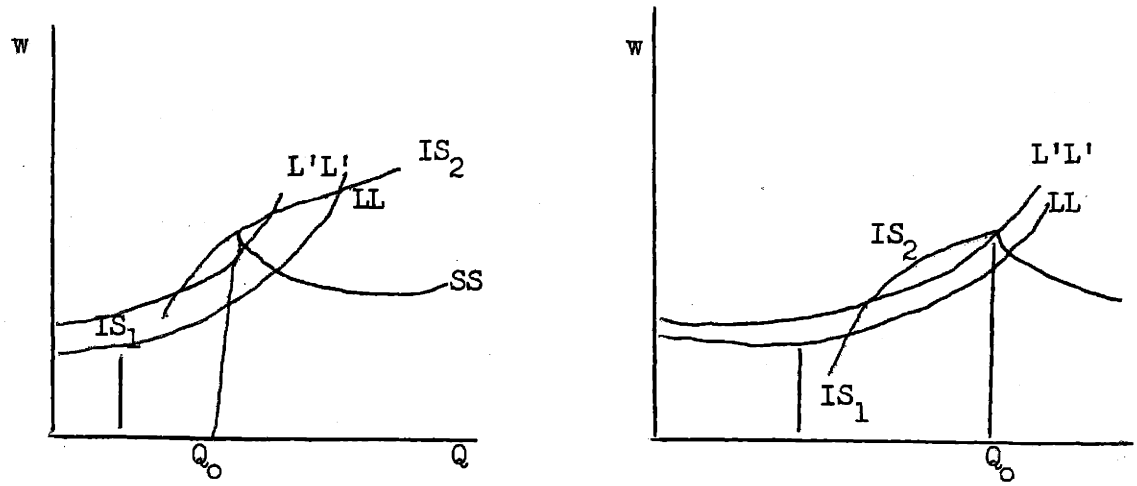

Figure 17

Upward shifts in the wage-bargaining equation will correspond to upward shifts in the LL schedule, which for lower level equilibria below Q_o will correspond to increases in equilibrium output. In Figure 17 this is represented by the upward shift of LL to L'L'. The reason for this upward shift in income is an increased aggregate demand because capitalists respond to the increased wage demands by substituting capital for labor. Aggregate supply has responded positively.

However, for equilibrium levels of output above Q_o, a shift in LL will have no effect on equilibrium output. Graphically, this is represented by the fact that equilibrium occurs where IS intersects SS: both of these curves are independent of the money-wage bargaining equation. In economic terms this result occurs because aggregate supply (which must equal aggregate demand) depends solely on the real wage. Thus increases in money wage demands are correspondingly matched by a rise in the price level which maintains the real part of the system at its former equilibrium.

It is important to remember that these results depend on our previous assumption that the interest rate was the proper monetary parameter of the system. Suppose, however, that M, the money supply, were the parameter of the system. In regime one it is clear that an upward shift in the money wage equation would have no effect on the demand for money if equilibrium output remained constant. Suppose that income goes down. The price level is sticky at p_o and so (at the previous equilibrium interest rate) there will be an overabundance of transactions balances. This would cause the interest rate to fall, which would raise the level of investment demand, and hence would be a second cause for income to rise. For aggregate demand increases both because of an increased real wage and a decreased interest rate. This contradicts the possibility that income falls; so income must rise. But the rise in real income does increase the demand for transactions balances, and therefore equilibrium output changes by less if the money supply is the constant parameter than if the Federal Reserve maintains a constant interest rate.

In the second regime, where prices adjust to changes in the bargaining position of labor, an upward shift in the money-wage equation will depress the level of national income if the money supply is kept constant. The reason is that if output remained the same with an increase in the money-wage equation, the price level would rise to match. The result is a rise in the transactions demand for money; this in turn tends to bid up the interest rate and therefore to depress the level of aggregate demand and national income.

XIV.

In conclusion, we have done what we set out to do at the very beginning. We have set forth two regimes. In these two regimes, shifts in the money wage equation will have different and opposite effects on the levels of employment, income and prices. The old conclusions, about shifts in the wage bargaining equation, stated so firmly in our macroeconomic textbooks (see Ackley) are not necessarily valid. Rather, these conclusions depend crucially upon which regime we live in.

What separates the two regimes is whether the aggregate supply of output is dependent or independent of the real wage rate of labor. If the aggregate supply of output does depend solely on this real wage, it is reasonable to expect the real wage to be correlated with the change in the unemployment rate, for the _ex post_ production functions shift slowly (and hence are fairly stable) while unemployment conditions seem to shift much more rapidly.

To test this theory we tried to see whether there was a correlation between the real wage and the change in the unemployment rate. The notion behind this test was that in a strictly Keynesian model (a la Hicks) increases in unemployment should be accompanied by greater than average increases in the real wage.

Therefore, we regressed the percentage change in wages on changes in the unemployment rate. To account for a possible asymmetry between rises and declines in the unemployment rate, we divided the unemployment variable into two parts.

$$\Delta u^{+} = \max\,(u_t - u_{t-1}, 0)$$

$$\Delta u^{-} = \max\,(u_{t-1} - u_t, 0).$$

Wages in manufacturing were deflated by the wholesale price index. We used data for the years 1913-1957.[2] The test was run in the following form.

$$\ln\left(\frac{w_t}{w_{t-1}}\right) = a + b\Delta u_t^{+} + c\Delta u_t^{-} + \epsilon_t \;.$$

Here a represents the trend rate of change in wages. It should be roughly associated with the rate of growth of productivity.

Results, however, are hardly conclusive:

$$\ln \frac{w_t}{w_{t-1}} = \underset{(.5358)}{.0258} - \underset{(.5358)}{.1297 \Delta u_t^{+}} + \underset{(.7119)}{.5525 \Delta u_t^{-}} \quad R^2 = .0245$$

Durbin Watson = 1.82.

Other tests were taken (various war and depression years were eliminated) but in all, the various combinations and permutations of eliminations the R-squared's similarly confirmed the independence of the real wage and changes in the unemployment rate.

These results reject a simple theory in which aggregate supply is solely dependent on the real wage. A complicated theory (as in Keynes's answer to Tarshis and Dunlop) may be correct; a simple theory is empirically unfounded. But this independence should give us more confidence in the belief that aggregate supply is somewhat independent of the real wage, and hence more confidence in the possibility that an increase in money wage demands will increase real wages, and consequently producers' desire to substitute capital for labor.

AN ESSAY ON THE THEORY OF MONEY

The theory of money is divided into two parts, the first of which is inhabited by fierce loanable-funds theorists; the second of which is inhabited by equally fierce monetary-demand theorists. This essay is an attempt to conquer each; and, the pacification having been made, to build a bridge across the gulf between these formerly warring nations.

PART I

One of the basic quandries in the theory of money is the continuity in the rate of interest. This fact is noted by Keynes: "In normal circumstances the banking system is in fact always able to purchase (or sell) bonds in exchange for cash by bidding the price of bonds up (or down) in the market by a *modest* amount." (italics mine). Of course, some of this money-market continuity must be spurious, for at least in the United States the Federal Reserve takes pains to lean against the prevailing winds, and the Federal Open Market Committee takes pains to prevent the existence of "disorderly markets." Still, in agreement with Keynes, it is surprising that medium-run changes in short-term interest rates are not more violent, as the Federal Reserve and other (exogenous) sources vary both the supply and demand for short-run securities. The reason for this surprise is that if both supply and demand are relatively inelastic one gets a picture, as below.

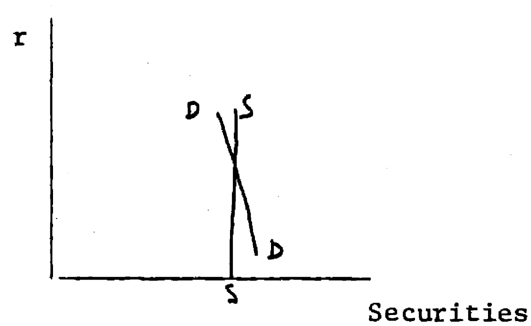

Figure One

Small shifts in SS should change the price "considerably."

Thus it appears that within the range of variation of Federal Reserve operation there is considerable elasticity of demand for short-term credit. Who or what are the institutions that are willing to adjust their holdings dependent upon whether the bill rate is 3 per cent or 3 1/2 per cent? The answer must be the banks and other financial intermediaries, both of which are careful portfolio maximizers. The composition of bill holdings confirms this intuition, for 20 per cent are held by banks, 18 per cent are held by other large financial intermediaries, and 62 per cent are held by corporations and individuals.

Thus, in tracing the effects of monetary action, one is pushed to the next step -- to ask to whom or to what institutions do the banks, etc. lend, and how does the policy of banks affect the level of activity of the economy as a whole.

First, we are going to assume that all bank loans can be easily divided into two categories: long-term and short-term loans. We assume that short-term loans are made for the purpose of current transactions; and we are also going to make the following important assumption -- that within a wide range the volume of income-generating transactions is independent of the availability of short-term loans. The reason for this second assumed inelasticity is that one-month loans at 6 to 12 per cent (per annum) have a 1/2 to 1 per cent interest charge, which seems negligible when compared with normal profit rates on sales (probably over

3 per cent). The reason for making this comparison is that short-run loans are believed to constitute the marginal extra working capital needed for sales. The question at hand is whether businessmen forego sales or pay the extra interest payments when bank rates go up. We believe that the interest payments for the extra working capital are small, and therefore that almost universally businessmen are willing to pay these extra costs. The one area where sizeable adjustments in current transactions may occur because of monetary stringency is in the holding of inventory, however.

A second reason why transactions volume should be little affected by the volume of credit is that in a complex economy, when money is in short supply there are undoubtedly multifarious ways of increasing the velocity of money: through increases of trade credit, and a host of informal arrangements. The conclusions of two empirical studies support this point. Thus, in a study of the distribution of bank deposits by size of firm in the U.S., Allen Meltzer concludes that: "the reduction of cash balances by liquid firms helps to explain the increases in the income velocity of money during the recent tight money period." Similarly Lipsey and Brechling write about trade credit in the U.K.: "there is substantial evidence that firms do feel the quantity effect of monetary squeezes and that they react to them by (on the whole) lengthening their credit periods."

Therefore, for two reasons, the comparison between usual rates of profits on transactions and the bank rates on loans necessary to carry out these transactions, and also a basic faith in the ingenuity of business to deal for short periods of time with less money in the bank because business can make a host of informal arrangements, we are impelled to construct a theory in which the supply of credit only affects the volume of transactions through the supply of long-term loans, and, as a consequence, through the volume of new investment expenditure. We should keep in mind, however, that the desired volume of inventories could be an important exception to this rule.

Thus we are led to the formulation of the following stylized model. There are four securities: long-term loans and long-term bonds, short-term loans and short-term bonds. Banks have a portfolio demand for each of these four types of assets; this demand is derived from a maximization of some utility function of risk and expected value. Furthermore, we assume that this demand function is upward sloping: i.e., the higher the rate of return on each type of security, the higher will be the demand for that security. This is in accordance with the observation that increased quantities of bonds offered by the Federal Open Market Committee usually reduce interest rates.

In a similar vein we assume that there is a nonbank demand for short and for long-term loans. We will consider each of these demands as a supply of long and short-term assets, and make the (realistic) assumption that each of these supplies depends negatively on the rate of interest in its own market. To be precise, in mathematical symbols, the two curves will be represented as $S_L(r_L)$ and $S_S(r_S)$ -- where subscripts L and S refer to the long and short interest rates, respectively.

A well-warranted assumption buys results. We assume that with the addition or subtraction of a risk premium long-term bonds and long-term loans (and, similarly short-term bonds and short-term loans) can be viewed as the same commodity. The reason is that the major variation in the value of these assets can be assumed to result from a common cause: variations in the long (and short) rates of interest.

Demand for each asset should equal supply. The total demand by banks for long and short-term assets should be approximately some fraction of the money supply. Thus we write equations (1) and (2).

$$D^L(r_L,r_S,\alpha_L + \alpha_S) - S_L(r_L) = \alpha_L \qquad (1A)$$

$$(1)$$

$$D^S(r_L,r_S,\alpha_L + \alpha_S) - S_S(r_S) = \alpha_S \qquad (1B)$$

$$D^L(r_L,r_S,\alpha_L + \alpha_S) + D^S(r_L,r_S,\alpha_L + \alpha_S) = \theta\bar{M} + \theta(\alpha_L + \alpha_S) \qquad (2)$$

which reduces to

$$D^L(r_L,r_S,\alpha_L + \alpha_S) = S_L(r_L) + \alpha_L \qquad (3A)$$

$$(3)$$

$$\theta(\bar{M} + \alpha_L + \alpha_S) - D^L(r_L,r_S,\alpha_L + \alpha_S) = S_S(r_S + \alpha_S \qquad (3B)$$

where D^L and D^S represent the bank demand for long and short-term securities; and where α_L and α_S represent the Federal Reserve's parametrically varying supplies of these two types of assets. D^L and D^S depend on α_L and α_S because total bank assets vary with $\alpha_L + \alpha_S$.

To catalogue the assumptions previously made about derivatives:

$$D_2^S > 0 \qquad D_1^L > 0 \qquad S_L' < 0$$

$$D_1^S < 0 \qquad D_2^S < 0 \qquad S_S' < 0$$

From (3) we calculate:

$$\frac{\partial r_L}{\partial \alpha_L} = \frac{\begin{vmatrix} +1 - D_3^L & D_2^L \\ -\theta + D_3^L & -D_2^L \qquad -S_S' \end{vmatrix}}{\Delta}$$

$$\frac{\partial r_L}{\partial \alpha_S} = \frac{\begin{vmatrix} -D_3^L & D_2^L \\ (-\theta + D_3^L) + 1 & -D_2^L - S_S' \end{vmatrix}}{\Delta}$$

$$\frac{\partial r_L}{\partial \alpha_S} - \frac{\partial r_L}{\partial \alpha_S} = \frac{-D_2^L - S_S' + D_2^L}{\Delta} = -\frac{S_S'}{\Delta}$$

where $\Delta = \begin{vmatrix} D_1^L - S_L' & D_2^L \\ -D_1^L & -D_2^L - S_S' \end{vmatrix} > 0$

If one assumes a stable equilibrium of demand and supply in these markets (there does not seem to be a good reason for not assuming this), Δ is greater than zero, and one can make the following statement:

If the Federal Reserve increases the supply of short-term bonds and long-term bonds by a given amount, the change in the long-term rate will be the greater if the Federal Reserve deals in long-term bonds. Returning to the earlier argument, this can be read to say that an offering of α dollars of long-term bonds will have a greater effect on the volume of real transactions than α dollars of short-term bills.

A simplistic demand for the two types of assets will perhpas illustrate why the concern here is real. Suppose that banks hold a constant ratio of long-term assets to short-term assets, say in a ratio of one to two. Then a Federal Reserve purchase of one billion dollars in short-term assets will create five billion dollars in new bank loans (at a 20 per cent reserve requirement). The net result is an increase in short-term loans of 3 2/3 billion; and an increase in long-term loans of 1 1/3 billion; on the other hand, an open-market purchase of one billion dollars of long-term securities would increase the volume of long-term loans by 2 1/3 billion and short-term loans by 2 2/3 billion. The result in this oversimplified case is that open-market policy in the long-term market is considerably more effective than in the short-term market. Remember that the effect on the volume of long-term loans was thought to be the measure of the effectiveness of monetary policy.

Some other results of our line of argument are worth pursuing. First, there is an asymmetry between changing the money supply by changing reserve requirements and changing the money supply through open market operations. First, changing reserve requirements is "neutral" in its portfolio effects -- i.e., there is no change in supply of short or long-term assets; whereas open market policy can have the added leverage of affecting the supplies in the "right" part of the market. There is a second asymmetry in the relation between the volume of bank assets and the money supply. Suppose reserve requirements are M. Then bank loans and bonds L are approximately $L = (1 - \gamma)M$ where M is the money supply. To change the supply of bank loans and bonds by 100y per cent through open market policy, there is a change of M dollars in the money supply. Whereas a change of bank loans by 100y per cent through changing reserve requirements results in a change of $(y - y\gamma)M$ in the money supply. This asymmetry develops because there are two different effects from monetary policy, (1) from changing the volume of loanable funds and (2) from changing the money supply.

Present theory should take a look at the Gurley-Shaw hypothesis. On first glance one would suppose that the larger were financial intermediaries, which arbitrage between long and short-term interest rates, the greater would be the effect on long-term rates of changes in the volume of short-term bills.* The reason for such a guess is that the financial intermediaries transmit the short-term effects to the long-term market. Such may, in fact, be the case empirically, but it need not be true in general, for a subtler logic prevails.

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* i.e. In later terminology, $\frac{\partial}{\partial \gamma} \left(\frac{\partial r_L}{\partial \alpha_S}\right) > 0$.

For, if the banks were in the market alone, what seems to the financial intermediaries as an overadjustment of the long-term rate relative to the short may occur; and the financial intermediaries would then counteract this tendency. The determinants of a system similar to systems (1) - (2) tell us this.

But another result is clear -- that is, that the larger are the financial intermediaries the more difficult it is for the Federal Reserve to twist the structure of interest rates. For the "natural" tendencies of the market are then stronger.

To "prove" this point we construct a system similar to (1) - (2). We pretend that there are only two groups of institutions in the economy that are willing to trade long-term assets for short-term assets and that these institutions are banks and financial intermediaries.

Let $D_L^B\ (r_L,\ r_S,\ \alpha_L + \alpha_S)$ and $D_S^B\ (r_L,\ r_S,\ \alpha_L + \alpha_S)$ be the long and short-term demands for securities by banks. Let $\gamma D_L^F\ (r_L,\ r_S)$ be the demand for long-term securities by financial intermediaries, where γ is a parameter. As previously,

$$D_S^B(r_L,\ r_S,\ \alpha_L + \alpha_S) = \theta(\overline{M} + \alpha_L + \alpha_S) - D_L^B\ (r_L, r_S, \alpha_L + \alpha_S)$$

and similarly,

$$\gamma D_L^F\ (r_L,\ r_S) = \mathbf{A} - \gamma D_S^F\ (r_L,\ r_S)$$

where γA represents the total assets of financial intermediaries, and where γD_S^F represents the demand by financial intermediaries for short-term assets.

Thus we get

$$\theta(\overline{M} + \alpha_S + \alpha_L) - D_L^B\ (r_L,\ r_S,\ \alpha_L + \alpha_S) + \gamma A - \gamma D_L^F\ (r_L,\ r_S) = \alpha_S + S_S(r_S)$$

$$D_L^B\ (r_L,\ r_S,\ \alpha_L + \alpha_S) + \gamma D_L^F\ (r_L,\ r_S) = \alpha_L + S_L\ (r_L)$$

whence

$$\frac{\partial r_L}{\partial \alpha_L} = \frac{\begin{vmatrix} -\theta + D_{L3}^B & -D_{L2}^B - \gamma D_{L2}^F - S_S' \\ -1 - D_{L3}^B & D_{L2}^B + \gamma D_{L2}^F \end{vmatrix}}{\Delta}$$

$$\frac{\partial r_L}{\partial \alpha_S} = \frac{\begin{vmatrix} 1 - \theta + D_{L3}^B & -D_{L2}^B - \gamma D_{L2}^F - S_S' \\ -D_{L3}^B & D_{L2}^B + \gamma D_{L2}^F \end{vmatrix}}{\Delta}$$

where $\Delta = \begin{vmatrix} -D_{L1}^B - \gamma D_{L1}^F & -D_{L2}^B - \gamma D_{L2}^F - S_S' \\ D_{L1}^B + \gamma D_{L1}^F - S_L' & D_{L2}^B + \gamma D_{L2}^F \end{vmatrix}$

Similar to our previous discussion, we assume Δ to be less than zero.

Subtracting, we find

$$\frac{\partial r_L}{\partial \alpha_L} - \frac{\partial r_L}{\partial \alpha_S} = - \frac{S_S'}{\Delta} < 0$$

Computing,

$$\frac{\partial}{\partial \gamma}\left(\frac{\partial r_L}{\partial \alpha_L} - \frac{\partial r_L}{\partial \alpha_S}\right) = + \frac{S_S'}{\Delta^2} \frac{\partial \Delta}{\partial}$$

$$\frac{\partial \Delta}{\partial \gamma} = + D_{L1}^F \quad S_S' - D_{L2}^F \; S_L'$$

$\frac{\partial \Delta}{\partial \gamma} < 0$ if we assume that $D_{L1}^F > 0$ (or that a rise in the long-term interest rate makes it more attractive to hold additional long-term assets).

Hence

$$\frac{\partial}{\partial \gamma}\left(\frac{\partial r_L}{\partial \alpha_L} - \frac{\partial r_L}{\partial \alpha_S}\right) > 0, \text{ or}$$

$$\frac{\partial}{\partial \gamma}\left(- \left(\frac{\partial r_L}{\partial \alpha_L} - \frac{\partial r_L}{\partial \alpha_S}\right)\right) < 0$$

Thus, perhaps there is some reason for alarm in the growth of financial intermediaries. For, while it is possible that financial intermediaries may counteract the transmission of changes in short-term interest rates to the long-term market, it is also true that financial intermediaries make it more difficult for open market policy to control the term structure.

By "more difficult" we mean, precisely, that the shift of a given number of dollars of bond sales between long and short-term bonds will have a smaller effect on the difference between the long and the short-term interest rates. Perhaps this is a poor definition of the word "difficult," but it is the only possible one around.

Also, we can compare the relative strength of monetary policy in depression and in boom -- and test the doctrine that in using monetary policy in depression "the Fed is trying to push on a string." To set up a mathematical structure which illustrates this point, we borrow an investment equation from Chapter III (Income, Investment and Wages).

$$\begin{aligned} I^D &= b(r_L)\,(Y - Y_0(r_L)) & \quad Y &\geq Y_0\,(r_L) \\ I^D &= 0 & \quad Y &< Y_0\,(r_L) \end{aligned} \tag{7}$$

The notation of equation (7) should be self-explanatory: I^D is investment demanded; Y_0 is a minimum level of income below which investment will be zero; it depends positively on r_L; Y is income.

The <u>raison d'être</u> for (7) is that at some point -- $(Y_0(r_L))$ -- which is functionally dependent on r_L, there is a desire for new capacity which rises linearly with output. The rate at which the investment demand increases with income depends negatively on the interest rate, however, because the rate of interest is the determinant of technique. The reader should consult Chapter III for details.

Since $S^S(r_S)$ is a demand for transactions loans, it can be assumed that this is proportional to transactions, which in turn are proportional to income. Thus we write

$$S^S(r_S) = Y\,\gamma(r_S) \;. \qquad \gamma'(r_S) < 0$$

It should be apparent from our formulation that for income sufficiently low -- below Y_0, that is -- more than marginal decreases in the interest rate must be made to coax any additional investment. However, the interest rate may continue to decline with additional purchases of bonds by the Fed -- but "sterile" transactions demand will take up all the additional loans.

The problem with comparing the strength of monetary policy for two income points both greater than Y_0, however, is that there is no natural comparison. Should we compare the investment elasticity at the same short-term rates of interest or at the same long-term rates of interest? One assumption -- emphatically avoided earlier because we were trying to twist the interest rate structure -- will help us here. We shall assume, but only for the sake of comparison, that the short-term interest rate is an increasing function of the long-term interest rate, or $r_S = \xi(r_L)$, $\xi'(r_L) > 0$.

Since the volume of loans is fixed parametrically by the Federal Reserve, we can write that the supply of loans α(a parameter) must equal the demand for loans, or:

$$b(r_L)\,(Y - Y_0(r_L)) + Y\gamma(\xi(r_L)) = \alpha$$

$$\frac{\partial r_L}{\partial \alpha} = \frac{1}{b'\ (r_L)\ (Y - Y_0(r_L)\ ) - Y_0'\ (r_L)\ b(r_L) + Y\gamma'\ (\xi(r_L)\ )\ \xi'(r_L)}$$

(The differentiation is partial because Y is considered fixed.)

From (7) and the consideration that $Y > Y_0(r_L)$, we derive:

$$\frac{\partial I}{\partial r_L} = b'(r_L)\ (Y - Y_0\ (r_L) - b(r_L)\ Y_0'(r_L)$$

Hence, since

$$\left.\frac{\partial I}{\partial \alpha_L}\right|_Y = \left.\frac{\partial I}{\partial r_L}\ \frac{\partial r_L}{\partial \alpha}\right|_Y$$

we derive

$$\frac{\partial I}{\partial \alpha} = \frac{b'(r_L)\ (Y - Y_0\ (r_L)\ ) - b(r_L)\ Y_0'\ (r_L) \qquad \alpha}{b'(r_L)\ (Y - Y_0\ (r_L)\ ) - b(r_L)\ Y_0'\ (r_L) + Y\gamma'\ (\xi(r_L)\ )\xi'(T_L)}$$

and hence

$$\frac{\partial}{\partial Y}\ (\frac{\partial I}{\partial \alpha}) = \frac{b'(r_L)}{b'(r_L)\ (Y - Y_0\ (r_L)\ ) - b(r_L)\ Y_0'\ (rL) + Y\gamma'(\xi(r_L)\xi'\ (r_L)} \ X$$

$$\left[1 - \frac{b'(Y - Y_0) - b\ Y_0'}{b'(Y - Y_0) + b\ Y_0' + Y\gamma'\xi'} \right]$$

$$- \frac{\gamma'(\xi(r_L)\ )\ \xi'(r_L)}{[b'(r_L)\ (Y - Y_0(r_L)\ ) - b(r_L)\ Y_0'(r_L) + Y\gamma\ '(\xi(r_L)\ )\ \xi'(r_L)]^2}$$

$$> - \frac{\gamma'(\xi(r_L)\ )\ \xi'(r_L)}{[b'(r_L)\ (Y - Y_0(r_L)\ ) - b(r_L)\ Y_0'\ (r_L) + Y\gamma\ '(\xi(r_L)\ )\xi'(r_L)]^2} > 0$$

The sign of $\frac{\partial}{\partial Y}\left(\frac{\partial I}{\partial \alpha}\right)$ indicates that the same amount of monetary policy has increasing effectiveness with increasing income. Hence with this specific type of investment accelerator, we find the mathematical structure of "pushing on a string."

We wish now to return to the main theme of the argument. In this first part we have dealt with the worry -- perhaps significant -- that the short-term market is so "thick" that open market operations in the short-term market will not be transmitted to the long-term market. I think that this is an important consideration; I am worried by the question whether increases in the supply of money will not be mainly compensated by reductions in the issuance of certificates of deposit, short-term credit, etc. This does not seem to be an unreal concern; for the Federal Reserve itself deals in short-term securities because the market there is so large and so orderly. Suppose, however, that these considerations are correct; then there is still some hope for open market policy if it can twist the interest-rate structure by dealing in long-term securities; thus the example of the fractional division of assets, fixed-coefficient as this example was, gives some indication of what was in mind. Thus we are worried by the growth of financial intermediaries, which tend to untwist the interest rate structure.

But there are effects of monetary expansion and contraction other than the change in the supply of loans; this comes through the supply and demand for money itself. It is this aspect of monetary policy to which we devote the second part of this essay.

PART II

A.

We have explored briefly some of the mechanics of an increase in the money supply; how this is associated with an increased volume of loans, and how these loans may affect (or may not affect) business activity. This second part continues to trace through the economy the effect of a Federal Reserve open market operation; how the individual holders of money find themselves unknowingly holding an increased (decreased) volume of money <u>in toto</u>; and how these individual agents tend to disburse these extra dollars which they themselves did not even know they held.

To trace the flows of money through the economy the next step is to construct a demand for money. What follows is based on the following notion: that an individual's receipts and expenditures are stochastic. The individual has a probability measure over a path space of future receipts and expenditures of money income. The alternative to holding money is to hold a portfolio asset which pays the individual an interest rate r. [This interest rate may be subjective and vary from individual to individual. It should, however, be correlated with the market interest rate.] However, there is a transactions cost of a + bx for a transaction of value x which rearranges a portfolio between money and interest-bearing assets. [The <u>a</u> and the <u>b</u> may also vary from individual to individual. Let it also be noted that this transactions cost may include the individual's labor and psychic disutility in making such transactions, in addition to the charges made by the banker or broker.] Finally, should the individual's bank account reach 0, he is required to engage in a transaction to increase his money balances.

In this Tobin-Baumol framework, individuals maximize the discounted value of receipts and expenditures. In a nonstochastic case, individuals choose a stream of receipts and expenditures to maximize

$$\int_0^\infty e^{-rt}\, dE(t), \text{ where}$$

$$E(t) = \int_0^t rA(t)dt - \sum_{\substack{\text{transfers made}\\ 0 \leq \tau \leq t}} [a + bx(\tau)]$$

$A(t)$ is the quantity of assets held at time t; these assets bear an interest rate r.

Probabilistically we assume that the individual maximizes the expected value of receipts and expenditures, or

$$\int_\Omega \left[\int_0^\infty e^{-rt}\, dE(t;\, p)\right] dp$$

where the receipts and expendituees vary with the path itself; and where dp represents the measure over the path space Ω.

Furthermore, it is assumed that there are diminishing returns to the holding of money. Successive increments in the money held by an individual with a given past history of receipts and expenditures are assumed to be associated with decreasing increments in the expected value of the individual's future interest earnings net of his payments for transactions. This assumption is based upon the following intuition: the reason for holding extra money now is to avoid future transactions payments. Additions of equal amounts to money balances subtract equal amounts of interest earnings. But suppose that our system of paths has the property that the probability of receiving a given amount of money is independent of past history, then the addition of one unit of money adds to the expected duration of time before a transaction must be made (i.e. when the bank account reaches zero balances) by a length of time d with a probability p(d). A second dollar adds the same amount of time to this duration, but the transactions costs avoided should be weighted by less, since they are placed farther into the future and therefore have a lower discounted value. Thus there is a constant loss in earnings from adding equal increments to money balances, but correspondingly there are decreasing gains. We do not want to make the strong assumption that the future money paths are independent of the past; but there does seem to be good cause to take some middle ground between this assumption and complete generality in assuming that at a given point in time it looks as if additional dollars will add decreasingly to the expected value of transactions costs avoided.

Two results of this present model will be derived:

1. Each individual has a "four-number policy" at every point in time; if the individual's money supply were reduced to 0 at time t he would increase the level of his bank account by $\underline{s}(t)$; at a point $\overline{S}(t)$ the individual would reduce his holdings to $\overline{s}(t)$. In short, each individual at time t has an $\overline{S}(t) - \overline{s}(t) - \underline{s}(t) - 0$ policy of money holding.

2. In due time the random walk of transactions causes a stable system displaced from equilibrium by some Federal Reserve operation to return to a new equilibrium, which is itself independent of the initial loanable-funds displacement described in Part I. This equilibrium will only depend on the demand for money.

B.

Consider a man at time t who receives an exogenous increase in his money supply. Let "exogenous" here mean that there is no effect on the individual's expectations of future receipts and expenditures. This man must decide whether to decrease his money holdings or increase them. Provided that appropriate assumptions are made, several behavioral results are true:

1. The individual never increases the money in his bank account unless his money holdings have reached 0.
2. Any change that the individual makes in his bank account, at a cost of a + bx, will be finite and nonzero.
3. If an individual who holds x decides to reduce his holdings, he would also reduce his holdings if he held q+x, $q \geq 0$.

4. An individual who reduces his holdings will always reduce his holdings to the same level.

5. S(t) is a decreasing function of r and proportional to the price level, if the appropriate assumptions are made.

1. Consider an individual who increases the amount of money which he holds at time t -- even though his bank account contains $z > 0$. At the present moment the man pays a + bx to increase his money holdings to z + x. The man could have done better if he had not increased his holdings. For, take the exact same strategy at z, that the man takes at z+x -- only at time T, when by either pure randomness or an intentional reduction of money holdings, the lower path starting from z hits, or would hit 0, spend a sum (a + bx) to unite the two paths. The man who did not increase his money holdings is richer by interest on x for the period (T - t), and also he has paid transfer costs at a later date. Figures 2a and 2b illustrate the two cases.

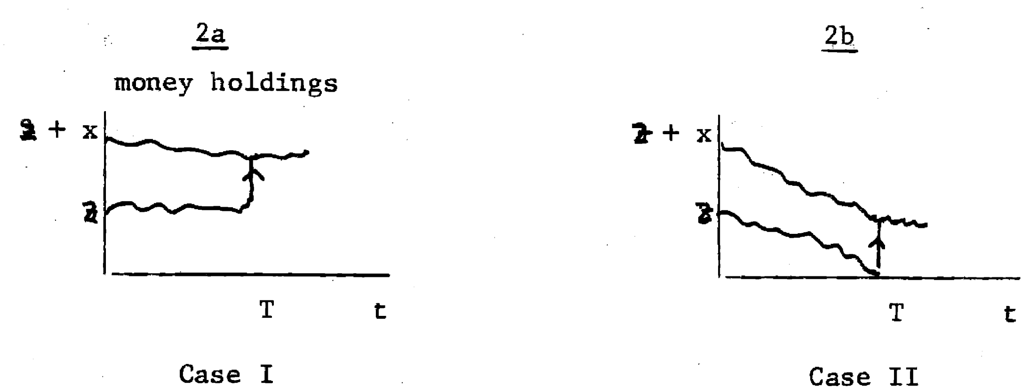

Figure 2

It seems unrealistic to assume that people's bank accounts actually reach 0. Of course the reason for this behavior in our model is that we have failed to associate some "discomfort" with reaching (or going below) zero; or, indeed that checks are paid randomly and are cashed with a variable lag and a variable period of float. But, for the discussion which follows it will not matter whether there is some minimum value $\underline{S}(t)$ below which the individual adjusts his bank account, or whether the individual actually allows his bank account to wander until it reaches the very bottom.

2. When an individual makes a transaction it always pays to move some finite nonzero distance. This is obvious.

3. The third proposition is the most difficult to prove; the most cogent reason for this is that the proposition is not, in general, true. It is necessary to make some assumption about the nature of the individual's path space (and the optimal strategies on those paths) and the gains from holding money. Accordingly, such an assumption is made; it is hoped that the assumption is reasonable.

Before making any assumption, first let us demonstrate the kind of pathology which would render the proposition false. Suppose that the expected paths all look very much like f(t) pictured in Figure 3. Furthermore, all possible paths starting at x are very close to f(t).

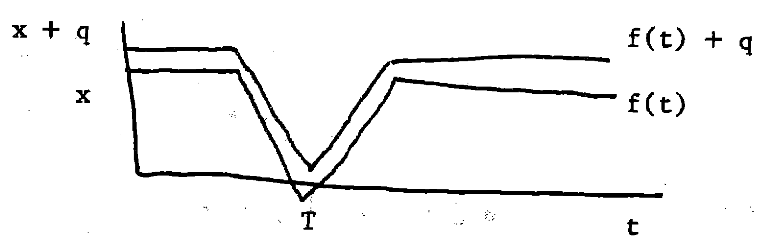

Figure 3

Then a numerical example can be constructed where it would pay to reduce the holdings in the bank account to a near-zero level. The individual will receive interest for a time close to $(T - t_o)$ on a sum which is close to x. This would pay for the transactions cost which the individual must undergo to make such a change. However, if the individual holds $x + q$ initially, he saves two transfers by not decreasing his money holdings at t_o. Therefore, it is conceivable that it will pay to reduce one's money holdings from x to $\bar{s}$, but not from $q > x$ to $\bar{s}$.

We make an assumption about the holding of money:

$$\frac{y - x}{z - x}\left[E(z,\varepsilon) - E(x,\varepsilon)\right] \geqq E(y,\varepsilon) - E(x,\varepsilon) \qquad \text{for } y \geqq z \geqq x, \text{ or } E \text{ is convex.}$$

where $E(q,\varepsilon)$ is the expected value of interest received and transfers paid out, given that the individual holds an amount of money q and makes no deliberate change in his money holdings for a short period ε; beyond that point he is free to pursue his optimal strategy. The raison d'être for such an axiom is that the farther away one is from the origin, the smaller is the likelihood that an increment in money holding will aid the individual in the process of postponing and avoiding transfer costs; and interest earnings, of course, are proportional to the sum on which interest is collected.

Suppose that a man with x dollars in the bank reduces his holdings to $\bar{s}$. Then $E(x,\varepsilon) \leqq E(\bar{s},\varepsilon) + a + b\ (x-\bar{s}) + \frac{x-\bar{s}}{r}$.

Suppose the man held $q + x$.

$$E(q+x) - E(\bar{s},\varepsilon) \leqq \frac{q+x-\bar{s}}{x-\bar{s}}\left(E(x,\varepsilon) - E(\bar{s},\varepsilon)\right) \leqq a + b\ (q + x - \bar{s}) + \frac{q+x-\bar{s}}{r}$$

Therefore it also pays the man to reduce his holdings from $q + x$.

4. If the individual reduces his holdings from $\overline{S}$ to $\overline{s}$, he will reduce his holdings from $q + \overline{S}$ to the same $\overline{s}$. Suppose that he reduced his holdings to $z > \overline{S}$. At z (by point 3) it would pay the individual to reduce his holdings; therefore this would not be an optimal policy. Suppose, on the other hand, that the individual reduces his holdings to some z which is less than $\overline{S}$. Then it would also pay to reduce his holdings from $\overline{S}$ to z, for the advantages and disadvantages are exactly the same irrespective of the initial money-holdings.

5. Suppose that the reigning price level is expected to continue into the far future. Also suppose that the individual's receipts and expenditure flows are expected to be proportional to that price level. It is clear that in this case, if transactions costs are also proportional to the price level p, then the individual's four numbers will be proportional to that price level. Furthermore, it is true that for higher interest rates the individual will have a lower $\overline{S}(t)$; for the gains from reducing his bank account are greater, whereas the losses from increased transactions costs are the same. Very specific assumptions are necessary, however, before it is possible to give $(\overline{S}, \overline{s}, \underline{s}, 0)$ as a function of the rate of interest, given that the individual has experienced a particular past history of receipts and expenditures.

C.

If we assume that what we observed in cross-section in Part IIB can be taken as a snapshot of the behavior of money-holdings over time, then we have constructed a "demand for money." This demand is not simply a number; it is somewhat more complicated, but luckily it does not defy easy description; and, hopefully, from a picture of it we can in turn derive an idea of the dynamics of monetary policy.

Our picture is as follows: there is a "money-box" in n-dimensions where n is the number of bank accounts in the economy. Each of these bank accounts has a 4-S policy: that is, there is an upper limit above which the individual reduces his holdings, and a lower limit below which the individual increases his holdings. If one pictures a state of the money supply as a vector $(m_1, \ldots, m_n)$ where m_i is the holding of the i^{th} individual for $i = 1, \ldots, n$, then we have a restriction $\underline{S}_i \leqq m_i \leqq \overline{S}_i$. There is a further macroeconomic restriction:

$$\sum_{i=1}^{n} m_i = M$$

The set $B = \{m_1, \ldots, m_n \mid S_i \leqq m_i \leqq S_i\}$ forms an n-dimensional box. The set $\Sigma m_i = M$ forms a plane P_M. The vector $(m_1, \ldots, m_n)$ must follow some random walk on the plane-segment $B \cap P_m$.

We pause a moment for repetition and reflection: for all that follows (and the relevance of what precedes) depends crucially upon this concept of the "box." The "box" is the n-dimensional rectangular space given by $\underline{S}_i \leq m_i \leq \overline{S}_i$.

This represents all states of the distribution of money holdings which are permissible, given the policy of the i^{th} individual of reducing his holding above S if they reach $\bar{S}_i$ and increasing his holdings if they fall to $\underline{S}_i$. Figure IV below gives a depiction of this box with three individuals in three dimensions. What we call "the money plane" is the set of points which embody the added restriction that all the money in the system must be divided amongst all n bank accounts: that is, $\sum_{i-1}^{n} m_i = \bar{M}$.

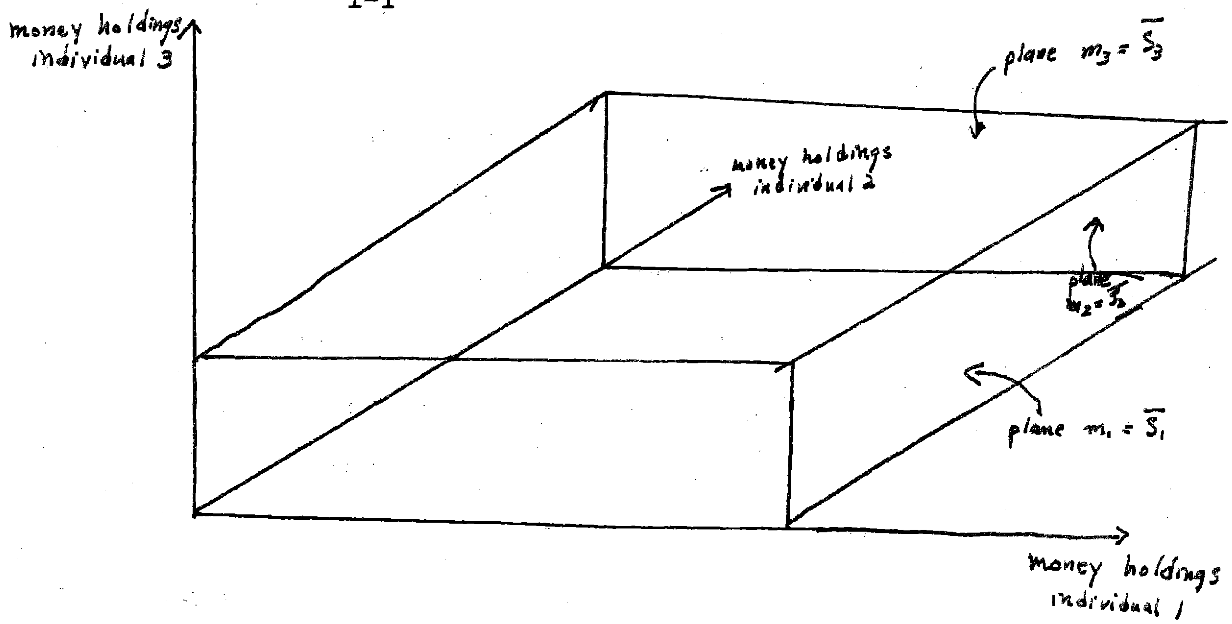

Figure 4

To return to the end of Part I, at that point we have seen that money has been injected into the system via a purchase of government bonds by the Federal Open Market Committee. This money has in turn been distributed to two sources: those with a demand for short-term and those with a demand for long-term loans.

In order to sell this money, however, the price of borrowing money has been bid down (or the interest rate on money loans has been bid down). Corresponding to the language and framework of Part II, we could say that in order for the banks to sell this money the money box B must be inflated (this corresponds to short-term loans) or the volume of long-term loans has been increased. Actually, of course, both of these events occur simultaneously.

But an interesting fact appears: suppose that in the total volume of loans, long-term loans are relatively unimportant. The major demand and supply of loans (i.e., the "thick" markets) lie in transactions demand. Then the interest rate will overadjust; for it is necessary to expand the money box B initially so that in a short space of time more bonds will be offered than would have been at the old interest rate. This corresponds to a short-run substitutability of money for bonds, which is less elastic than the long-run substitutability of money for bonds.

However, the more important are long-term loans, correspondingly the less will be the overadjustment (and very possibly there will be an under-adjustment) of the interest rate.

In studying the lags in monetary policy it is worth noting that the interest rate may have -- in fact, probably ~~have~~ has -- a very rapid adjustment, and probably an overadjustment to its equilibrium level.

The reason for such a supposed over-adjustment is that the Fed, in trying to buy a large quantity of bonds in a short period of time must "look hard" for people to hold the extra money -- where the difficulty of this "search" is measured by the decrease in the interest rate for a given purchase. In terms of the "box" this means that in an open market operation in which the Fed buys bonds the interest rate must fall so much that individuals who would have reached their upper barriers no longer do so, or if they do reach their upper barriers they cut their holdings of money less than if the interest rate had remained the same; and also a greater number of people reach their lower barriers, and those who do increase their money holdings by a greater amount than if the interest rate had remained fixed. It is clear that the shorter the time of duration of the Fed's purchase of these bonds the greater will be the decrease in the interest rate -- for the fewer are the candidates whose money holdings can be altered by any change in the interest rate. But the interest rate, just necessary to induce the system to hold the additional money in the long-run, is that interest rate which would prevail if the Fed tried to purchase these bonds over an infinite period of time. Using the previous logic, which indicates that the interest rate decrease entailed in the pruchase of the bonds diminishes with the time allotted for this sale, it becomes clear that the initial interest rate adjustment was an over-adjustment.

Thus the lag in the application of monetary policy from the point of view of loanable funds is the lag between long-term loan and the use of this money in actual construction. The length of this lag depends crucially upon (1) whether banks have on file (or in process) a long pipeline of loan requests; and (2) whether the plans on file at the bank are ready for execution. A study of this is of vital importance; perhaps Professor Ando's study of Morgan Guarantee Trust Company will provide some clues.

But from the viewpoint of monetary demand, adjustment may be much slower. To recount the steps: (1) the size of the money box is increased by the initial purchase of bonds. (2) However, at the same time the money plane is raised by the increased volume of money (from P_M to $P_{M+\Delta M}$). (3) From this point on the normal transactions demand for money, via the bombardment of the sides of the money box, returns (or tends to return) the system to an equilibrium position where the volume of money demanded equals the supply of money.

The mechanism of adjustment is that if there is "too much" money in the system, the money plane is "too high" for the size of the money box. This means that on the average too many people want to disburse of their money holdings and too few people want to increase their money holdings. The net result is that the interest rate is bid down, (or if people buy commodities rather than bonds, income is bid up.) This increases the size of the box; and furthermore, this force continues until the size of the box has increased so that, at this new (lower) interest rate, the quantity of money which people want to disburse will equal the quantity of money which people want to buy.

It is instructive to see how an increase in the amount of money in the system with no change in the size of the box will create such forces; accordingly, we compare two situations: in the first the amount of money is M, in the second, the amount of money is $M + \Delta M$ and ΔM is greater than zero. We compare two random walks caused by exactly the same income-induced transactions; each of these random walks are exactly the same except that each starts as a positive displacement of the other, and one will reach a first barrier before the other. But the random walk in the system with more money will hit an upper barrier before its twin with less money in the system; similarly it will hit its first lower barrier after its twin with less money in the system. The result is that the system with more money has a greater propensity to disburse of money holdings than the system with less money, and similarly a smaller propensity to increase its money holdings. Supposing that in the system with less money the desire to disburse of money is equal to the desire to increase money holdings on the average, then we can say that the system with more money will initially be out of equilibrium. In the money market, there will be a downward pressure on the interest rate; this has two effects: first, there will be an increase in the volume of bans, which in use will be randomly placed in the n-bank accounts of the transactors; second, there is an induced increase in the size of the "box."

In a Patinkin world, however, where real income or the real interest rate cannot change, it is clear that the random walk mechanism will ensure a quantity theory -- i.e., the equilibrium price level will be proportional to the quantity of money. However, in a less rigid framework, the interest rate and real income may change as well -- <u>thus</u> affecting the size of the box.

One note might also be worthwhile; suppose that the demand for long-term loans were very elastic (i.e., increased volume of bonds could be purchased by the Fed without substantially affecting the size of the box). Then a large increase in the money supply would lead to a very rapid adjustment -- for the intersection of the money plane and the box is small (perhaps nought); a small increase in the money supply, however, has a less than proportional change in the area of $BnP_{M+\Delta M}$ (and therefore in the permissible area on the money plane for the random walk). Thus the quantity theory is more potent for large increases in the money supply than for small increases.

The purpose of this essay was to explain how a loanable funds theory of monetary policy fits in with a monetary demand theory of the rate of interest. The basic device we have used is the box. Initially, money is sold to loan-demanders. In turn money is diffused through the economy's many bank accounts; if interest rates are above or below equilibrium there will be pressures toward adjustment caused by a greater (smaller) desired disbursal of money than the desired increase in money holdings. We have seen the two part process which takes place in monetary policy: first the sale of the bonds, then the multiple expansion of loans -- with interest rates falling -- and then the longer-run adjustment mechanism as the loan-money is spent randomly in transactions and creates either an excess demand for money at the given interest rates or an excess supply of money at these interest rates. This somewhat complicated picture seems to picture how the money market operates.

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