STUDIES IN THE THEORY OF ECONOMIC GROWTH AND INCOME DISTRIBUTION

by

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B.A., Amherst College

(1964)

SUBMITTED IN PARTIAL FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

August, 1966

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Dear Professor Brown:

In partial fulfillment of the requirements for the degree of Doctor of Philosophy in Economics, I hereby submit the following thesis entitled

"Studies in the Theory of Economic Growth and Income Distribution".

Respectfully yours,

Joseph E. Stiglitz

ABSTRACT

Title: STUDIES IN THE THEORY OF ECONOMIC GROWTH AND INCOME DISTRIBUTION

Author: Joseph E. Stiglitz

Submitted to the Department of Economics and Social Science on August 16, 1966, in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Economics.

The first chapter treats of the problem of the allocation of heterogeneous capital goods among the sectors of the economy. The central theorem proved is the following: if technological change is Harrod neutral and one sector is more labor intensive than any of the others, all the newest machines go to the most labor intensive sector. Existence and uniqueness of short run equilibrium as well as steady state growth paths are proved, and the comparative dynamics of this economy are investigated.

The second chapter discusses the problems raised by the possibility that one technology is used at two different interest rates, with another technology used at intervening interest rates. Part I. presents a number of conditions under which such "double switching" is impossible. In Part II. explicit expressions for the value of capital per man in terms of the rate of growth, the savings propensities, of workers and capitalists, and the rate of interest, are derived, and it is shown that the value of capital increases with increasing interest rates. Part III. relates the "double switching phenomenon to another phenomenon which has recently received much attention: steady state consumption increasing with increasing interest rates. In Part IV. it is shown that paths of capital accumulation which involve double switches may be efficient, and for some utility functions, optimal. It is also shown that at a switch point the rate of interest is equal to the rate of return.

The third chapter examines the properties of a growth model with a consumption goods sector and a capital goods sector, in which there are two classes, one whose income is derived entirely from capital (the capitalists) and a second which derives its income from both wages and return on savings (the workers). It is shown that there exists at most one balanced growth path with both capitalists and workers present, and conditions for existence of such a path are derived. It is shown that if the consumption goods sector is not more labor intensive than the capital goods sector, or if the sum of the elasticities of substitution of the two sectors is greater than 1, then momentary equilibrium is uniquely determined. The stability properties of the model are analyzed, and it is shown that even if the two class balanced growth path is locally stable, there may exist paths which oscillate around it rather than converge to it.

The fourth chapter investigates the implications of the process of capital accumulation for the distribution of income and wealth among individuals. Both long run and short run changes in the distribution of wealth are described in a series of related models which differ with respect to savings functions, to reproduction behavior, to inheritance policies, and to the homogeneity of the labor force.

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ACKNOWLEDGEMENTS

I owe a particular debtof gratitude to the members of my thesis committee: Professor Robert M. Solow, who served as chairman; Professor Edwin Kuh; and Professor Paul A. Samuelson for their criticism, comments, and suggestions.

I should also like to thank Dr. Frank Hahn of Cambridge, whose comments on an earlier version of the thesis greatly improved both the style and content, and Professor Hirofumi Uzawa of the University of Chicago for his extensive discussions of the first chapter. I also wish to thank the other members of the faculties of Cambridge, Massachusetts Institute of Technology, and Chicago who read earlier versions of some of the chapters.

I am further deeply grateful to the National Science Foundation; the U.S.-U.K. Educational Commission; Gonville and Caius College, Cambridge; the Department of Economics, University of Chicago; and the Department of Economics, Massachusetts Institute of Technology; all of whom gave me generous financial support while this thesis was being written.

Finally, I wish to thank all those who helped with the preparation and typing of the thesis.

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ALLOCATION OF HETEROGENOUS CAPITAL GOODS IN A TWO-SECTOR ECONOMY

One of the most important contributions of the modern theory of economic growth is the demonstration of how important technological progress is in the development of the economy. Solow's classic "Contribution to the Theory of Economic Growth," [5][1] showed that the qualitative facts of growth could not be explained with a very general model that did not include technological change. Then, in 1957 [6], under the assumptions of Hicks neutral technological change and the marginal productivity theory of factor distribution, Solow showed that approximately 87 per cent of the growth in income per capita in the United States since 1890 is due to technological progress, and only 13 per cent to capital accumulation.

It is important then, if we are to understand and perhaps influence the economic growth of a country, that we have a theory of technological progress: how it is produced, introduced, and characterized.

The discussions on the introduction of technological change have centered in recent years primarily around the question of "embodiment"--whether technological change needs new machines to be introduced into the economy, and if so, what implications does this

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[1]Here as elsewhere numbers in brackets refer to references at end of chapter.

have for the growth path of the economy. Obviously, if some machines are better than others, we cannot simply add them together to form an aggregate capital stock, as in the simpler analysis with no technological change; rather, one must now form a "jelly," i.e. a weighted sum, in which the weights are in proportion to the degree, in some sense, of technological progress "embodied" in the machines. Fisher, Samuelson, and Gorman, have shown that the only conditions in which this aggregate jell can be formed are very restrictive conditions. It is thus important for us to formulate our model in such a way as not to require an aggregate capital stock or jelly.

Moreover, the one sector growth models, in which there is a homogeneous commodity, which may either be used for investment or, alternatively, consumed, suffer from the difficulty that, with embodied technological progress, this homogeneous commodity when used for consumption purposes remains unchanged over time, but when used for investment purposes, becomes better and better. This does not make too much sense, and it is therefore important, for the purpose at hand, to have a two sectoral model, one sector producing capital goods, the other consumption goods.

Finally, in the two sectoral growth models advanced thus far, with all capital goods alike, the important and interesting problem of allocating machines between sectors is ignored.[1]

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[1]The only exception to this is M. Kurz's doctoral dissertation, "Patterns of Growth and Valuations in a Two-Sector Model," published in Yale Economic Essays, Volume 2, Number 2, Fall 1962. He makes the assumption of non-transferable capital (what he calls non-shiftable capital); that is, capital in the consumption goods sector cannot be transferred to the capital goods sector, and vice versa. He also makes the assumption--and this does severely restrict the validity of his model--of a Cobb-Douglass production function for both sectors. This

In short then, the kind of model which we are about to set up is important for at least three reasons: (1) it does not require an aggregate capital "jelly;" (2) "embodied" technological change makes most sense in a two sectoral model where consumption and investment goods are not homogeneous; and (3) it is important to know how capital goods of different qualities are allocated between the production of consumption goods and the production of investment goods.

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allows him to form an aggregate capital jelly, as in the Solow one sector Cobb-Douglas model. Unfortunately, as Fisher and Gorman have recently demonstrated, (Review of Economic Studies, [1] and [2]) if technical change is not purely capital augmenting an aggregate capital jelly cannot be formed, and for the Cobb-Douglas case, labor augmenting and capital augmenting change are equivalent. Hence, the assumption of Cobb-Douglas production functions is not only crucial for his results, but also for his methods of arriving at them. We seek to develop techniques here which do not involve the formation of aggregate capital stocks.

1. Production

We shall make the usual neoclassical assumptions of production: constant returns to scale for the two factors of production, capital and labor; i.e. doubling capital and doubling labor doubles the output; diminishing returns to one factor alone; zero output per machine when the labor per machine is zero. In addition, we shall make the assumption of Harrod neutral technological change, (or pure labor augmenting technological progress). This assumption can be modified, and the static analysis can be appropriately modified, but of course there will not then exist a long run steady state path for the economy.

Mathematically, we may express these production conditions as

1. $Y_1 = F_1(K_1(a), aL_1)$

2. $Y_2 = F_2(K_2(a), aL_2)$

where Y_1 is the output of consumption goods, K_1 the capital (of type a) used in the production of consumption goods, and L_1 the labor used in the production of consumption goods. Y_2, similarly, is the output of capital goods, a represents the measure of technological progress (labor augmenting) embodied in the machine of type a.

To see that there are no problems in defining units let us first consider the purely "static" production functions involving only machines of "type 1."

1'. $Y_1 = F_1 \ (K_1(1), L_1)$

2'. $Y_2 = F_2 \ (K_2(1), L_2)$

Now we assume that at times greater than zero, machines of a different "type" are produced, but the number of machines produced by a given

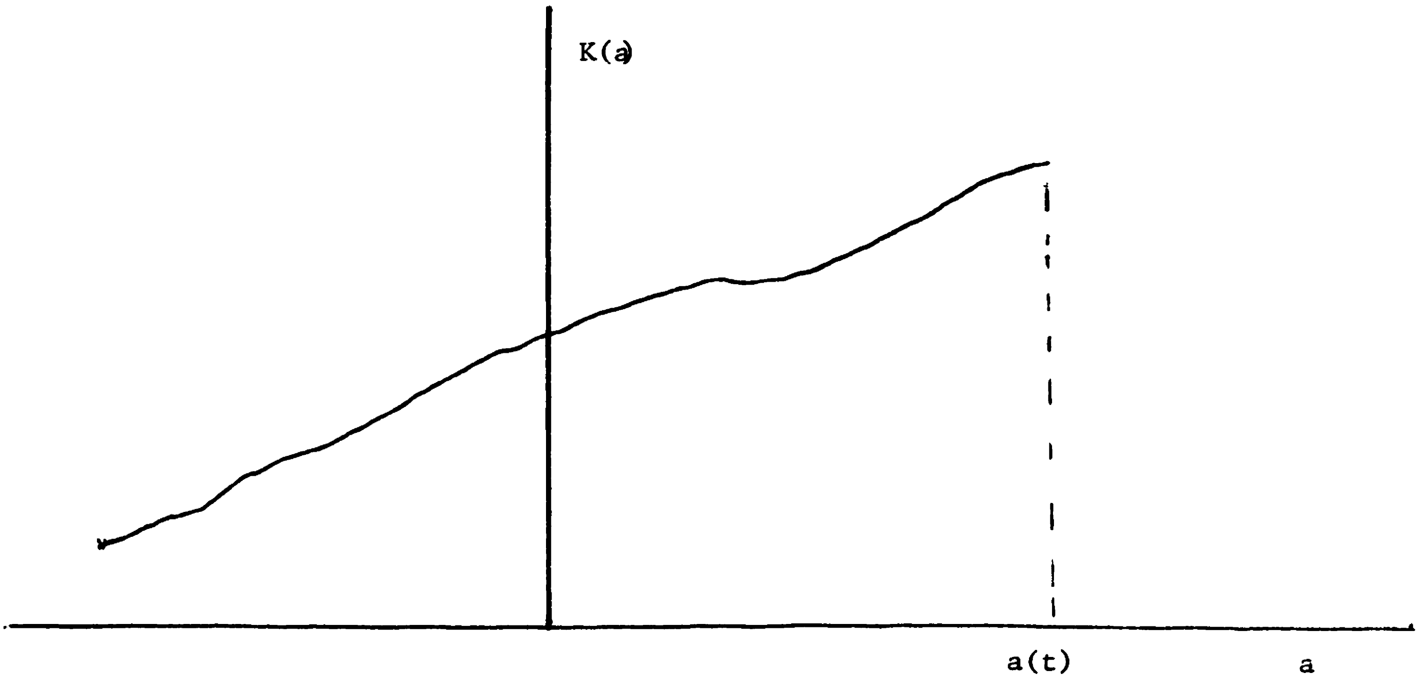

Figure 1

Sample Distribution of Machines

amount of our "old" capital and labor remains unchanged (similarly for times less than zero) (2'' may be considered to define the units of machine goods); we thus have

$$2''. \quad Y_2(a) = F_2(K_2(a), L_2)$$

Finally, we observe that these new types of machines differ from machines built at time zero in a pure labor augmenting way: to produce a given number of consumption goods or machines (as defined above) with a given number of machines of a certain type, we require 1/a the labor on the new machines as on the old (where a is independent of the number of machines or the given output; hence we may speak of a machine as type a). It is clear that it makes no difference for the analysis which type of machine we choose for our "reference" machine (i.e. for which machine we let a = 1.)

A sample distribution of machines is given in figure 1. Note that homogeneity of degree 1 implies that

$$1'''. \quad Y_1 = f_1(x_1(a))K_1$$

$$2'''. \quad Y_2 = f_2(x_2(a))K_2$$

where

$$x_1 = aL_1K_1 \qquad\qquad x_2 = aL_2K_2$$

Thus far, we have imposed the following restrictions on our production functions:

$$3. \quad f_1(0) = 0 \qquad\qquad f_2(0) = 0$$

and

$$4. \quad f_1' > 0 \qquad f_2' > 0 \qquad\qquad 5. \quad f_1'' < 0 \qquad f_2'' < 0$$

In this paper, we shall also impose for analytical convenience the so-called Inada conditions

$$6. \quad f_1'(0) = \infty \qquad\qquad f_2'(0) = \infty$$

the marginal product of labor when the capital labor ratio is zero is infinity. This enables us to ignore the question of whether all machines are used or not, but the analysis is only slightly modified if we drop this restriction.

Total output of consumption goods then is the output on all machines used in the consumption goods sector, and similarly for investment goods:

6. $Y_1 = \int_A f_1(x_1(a))K(a)da$

and

8. $Y_2 = \int_{A'} f_2(x_2(a))K(a)da$

where A is the set of machines used in the consumption goods sector, and A' all other machines. (Because of assumption 6, all machines will be used, since the marginal product of labor gets larger and larger as fewer and fewer laborers are assigned to these machines.)

Our problem then is to determine which machines are used in which sector. It turns out that the answer to this problem is relatively simple: if one sector is always more labor intensive than the other (at least in the relevant range) then all of the newer, and therefore better, machines are used in the labor intensive sector. (One sector is always more labor intensive than the other, if, at any specified wage/rental ratio, the labor capital ratio in one sector is greater than that in the other.) The intuitive reason for this is roughly that the new machines save labor in a multiplicative manner, and it therefore makes some sense to assign newer machines to that sector where it is more important to save labor, the labor intensive sector.

Let us consider an entrepreneur owning a machine of type a. He faces a wage w (in terms of consumption goods numeraire) and a price of new investment goods of p (again in consumption goods numeraire). He is trying to decide whether to allocate the machine to the consumption or the investment sector.

In either case, he will clearly hire labor to the point where the "money" wage is equal to the value of the marginal product. In consumption goods, our numeraire, this is simply

9. $w = af_1'(x_1(a))$

while for the investment sector, this is

10. $w = paf_2'(x_2(a))$

The quasi-rents (per machine) are equal to the output per unit of capital less the wage payments per unit of capital; in the consumption goods industry we have

11. $r_1(a) = f_1(x_1(a)) - \dfrac{wx_1(a)}{a}$

while for investment goods we have

12. $r_2(a) = pf_2(x_2(a) - \dfrac{wx_2(a)}{a}$

which, by 9 and 10, yield

13. $r_1(a) = f_1(x_1(a)) - x_1(a)\ f_1'(x_1a))$

14. $r_2(a) = pf_2(x_2(a)) - px_2(a)\ f_2'(x_2(a))$

A machine will be used in the consumption goods sector if the return to it (the quasi-rent) is greater than in the investment sector. The problem then is to determine the relation between the returns to different kinds of machines in the different sectors, for given w and p. We shall show that if one sector is always more labor intensive than the other, there exists a (unique) â such that all machines

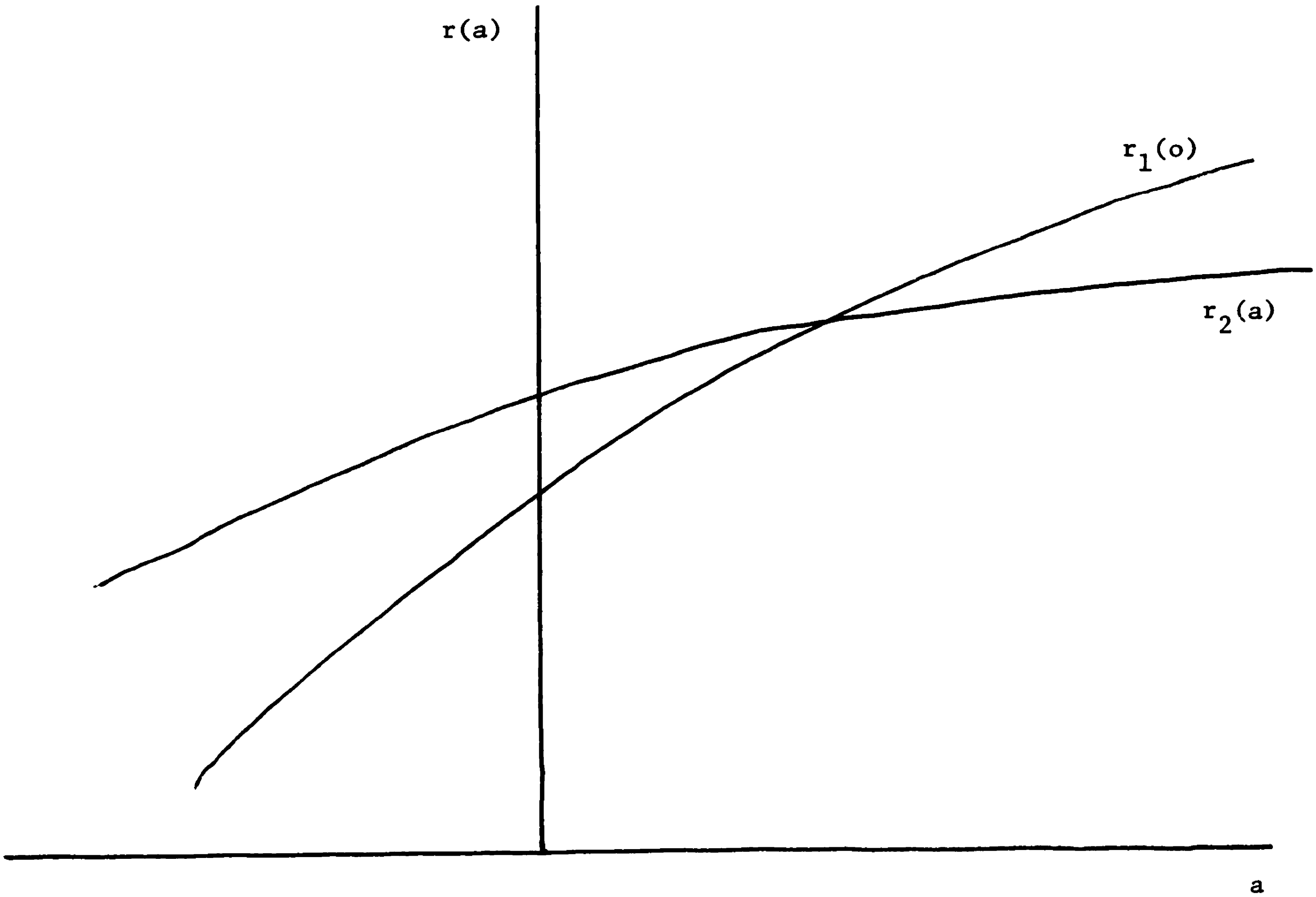

Figure 2

Quasi-rents on Machine of Type a for Fixed w and p "newer" than $\hat{a}$ (i.e. with $a > \hat{a}$) go into the labor intensive sector, while all older machines go into the other sector.

To do this, we first observe that 13 and 14, with 9 and 10, define a relation between r_1 and a, and r_2 and a (for given w and p). We shall show that $\frac{d \ln r_1}{d \ln a} > 0$ and $\frac{d \ln r_2}{d \ln a} > 0$ and for given r $\frac{d \ln r_1}{d \ln a} > \frac{d \ln r_2}{d \ln a}$, and hence there is at most one $\hat{a}$ for which $r_1(\hat{a}) = r_2(\hat{a})$. From 13 and 14, we have

15. $$\frac{dr_1}{da} = \frac{dr_1}{dx_1}\frac{dx_1}{da} = -x_1 f_1'' \frac{dx_1}{da}$$

16. $$\frac{dr_2}{da} = \frac{dr_2}{dx_2}\frac{dx_2}{da} = -px_2 f_2'' \frac{dx_2}{da}$$

For fixed w and p, 9 and 10 define implicit relations between a and x, and a and x_2, from which we easily calculate

17. $$\frac{dx_1}{da} = -\frac{f_1'}{af_1''}$$

18. $$\frac{dx_2}{da} = -\frac{f_2'}{af_2''}$$

Substituting in 15 and 16, and multiplying by a/r, we have

19. $$\frac{d \ln r_1}{d \ln a} = \frac{x_1 f_1'}{r} = \frac{wL_1}{K_1 r_1(a)} > 0$$

20. $$\frac{d \ln r_2}{d \ln a} = \frac{x_2 f_2'}{r} = \frac{wL_2}{K_2 r_2(a)} > 0$$

But for given r (or since w is fixed, for given r/w), if one sector is more labor intensive, the labor-capital ratio will be greater in that sector, and hence $\frac{wL}{rK}$ will be greater. Thus

21. Either $\frac{wL_1}{r_1K_1} > \frac{wL_2}{r_2K_2}$ or $\frac{wL_2}{r_2K_2} > \frac{wL_1}{r_1K_1}$

and hence our theorem is proved.

We shall denote this crucial at at which $r_1(\hat{a}) = r_2(\hat{a})$ as $\hat{m}(w,p)$; it can easily be shown that the elasticity of m to changes in w is 1, since w and a appear symmetrically on both sides of the wage-marginal product equations.

22. $\frac{w}{\hat{m}} \frac{\partial m}{\partial w_2} = 1$

Similarly, it can be shown that

23. $\frac{p}{m} \frac{\partial \hat{m}}{\partial p} < 0$

We have yet to show that there necessarily will exist an intersection, i.e. it is possible for the more labor intensive curve to begin at a high r, i.e. r_j (max) < r_i (min). But using results 22 and 23, we can show that there is some set of wages and prices at which $\hat{m}$ > min a, or in other words, for which, at a feasible a, r_j (max) $\geq$ r (min).

Now that we have established that all new machines will go into the labor intensive sector, and assuming that the investment sector is more labor intensive than the consumption sector, we can rewrite our original production equations 7 and 8:

24. $Y_1 = \int_{-\infty}^{\hat{m}(w,p)} f_1(x_1(a))K(a)da$

25. $Y_2 = \int_{\hat{m}(w,p)}^{a(t)} f_2(x_2(a))K(a)da$

where a(t) is the best machine in existence at time 6, It should be observed that if the elasticity of substitution is greater than 1, as r increases the slope of the logarithm of r as a function of the logarithm of a increases, and hence the r curve is convex, while if the elasticity of substitution is less than 1, as r increases, the slope decreases and hence the r curve is concave. It should also be clear that if one of the sectors has, at every value of r, a higher elasticity of substitution than the other sector has at that r, there can be at most two values of a for which $r_1(a) = r_2(a)$: the very old machines and the very new machines are used in the sector with the greater elasticity of substitution.

2. Employment, Consumption, and Technological Change

Since we know the set of machines which are used in the consumption sector, and the set of machines which are used in the investment sector, we may now also determine the employment in the two sectors:

26. $$L_1 = \int_{-\infty}^{\hat{m}(w,p)} \frac{x_1 K(a)}{a} \, da$$

and

27. $$L_2 = \int_{\hat{m}(w,p)}^{a(t)} \frac{x_2 K(a)}{a} \, da$$

Employment in the two sectors cannot exceed the labor force, and in fact cannot fall short of full employment, by assumption,

28. $$L_1 + l_2 = L(t)$$

The labor force is assumed to be growing exponentially:

29. $$L(t) = L_o \, e^{nt}$$

where n is the growth rate of labor.

Consumption is assumed to be a constant proportion of income, where income is defined as

30. $$Y = Y_1 + pY_2$$

Thus

31. $$pY_2 = sY = sY_2 + spY_2$$

Solving for Y_2, the output of investment goods, we obtain,

32. $$Y_2 = \frac{sY_1}{(1 - s)} \, p$$

For the moment, we assume technological change exogenously determined, (with no labor or capital required), at a constant exponential rate u:

33. $a(t) = a_o e^{ut}$

3. Analytics of the Model.

Our complete model then consists of 2 marginal productivity wage relations, two output equations, two labor demand equations, one labor supply condition, and a consumption function, plus the requirement that if the rate of return in one sector exceeds that in the other, then the machines are used in the sector with the higher return. Using this requirement, and the marginal productivity equation, we have shown that all new machines are used in the labor intensive sector. The critical machine, i.e. the machine with the property that more efficient machines are used in one sector, less efficient machines are used in another, is a function of the wage and price.

Knowing this, we still must solve for x_1, x_2, w, p, Y_1, Y_2, L_1 and L_2, i.e. the amount of consumption goods produced, the amount of investment goods produced, the employment in the investment goods sector, the employment in the consumption goods sector, the wage, the relative price of investment goods to consumption goods, and the effective labor units/capital intensities (for machine of type a used in the sector) in each sector. We can show that there exists one and only one solution, but before we turn to the formal proof, let us briefly suggest the underlying motivation.

For a specified production of investment commodities, there is a maximum amount of consumption goods that can be produced. At that consumption-investment combination, there is a trade off relation between additional consumption, and less investment, and vice versa, which is equal to the relative price ratio, p. As in the usual production possibilities schedule, as investment increases, p increases. But the demand for investment goods is a decreasing function of price. Hence the supply schedule of investment goods as a function of price is upward sloping, while the demand schedule is downward sloping, and a unique equilibrium is thus determined.

4. Proof of Existence of Equilibrium.

More rigorously, we have Theorem II. There will always exist one and only one equilibrium to the set of equations presented above.

We set up the problem as a maximization problem:

34. $\text{Max. } Y_1 + pY_2$

s.t. $K_1(a) + K_2(a) = K(a)$ all a

$L_1 + L_2 = L$

For convenience, we assume that, except possibly for a set of a's of measure zero (e.g. a finite number of a's), if machines of type a are used in sector i, there exist e_1 and e_2, $e_1 e_2 > 0$, such that all a $a + e_1 > a > a - e_2$ are used in sector i. (If the capital intensity hypothesis is satisfied, this condition will clearly be met). This enables us to write

35. $$Y_1 = \sum_i \int_{m_{2i+1}}^{m_{2i+2}} f_1(x_1(a))K(a)da$$

36. $$Y_2 = \sum_i \int_{m_{2i}}^{m_{2i+1}} f_2(x_2(a))K(a)da$$

$$37. \quad L_1 = \sum_i \int_{m_{2i}+1}^{m_{2i}+2} x_1/a\ K(a)da$$

$$L_2 = \sum_i \int_{m_{2i}}^{m_{2i}+1} x_2/a\ K(a)da$$

Taking the partial derivatives of the Lagrangian

$$38. \quad \Psi = \Sigma \int f_1(x_1(a))K(a)da \; + p\ \Sigma \int f_2 x_2(a))K(a)da$$

$$+ w(L - \Sigma \int x_1(a)/a\ K(a)da + \Sigma \int x_2(a)/a\ K(a)da +$$

$$\int r(a)\ \{K(a) - K_1(a) - K_2(a)\}da$$

with respect to m_i, we obtain

$$39. \quad \frac{\partial x}{\partial m_i} = -f_1'\ (x(t-m_1)) + pf_2{}'(x(t-m)) + \frac{wx_1(t-m)}{a} - \frac{pwx_2(t-m)}{a}$$

$$= r_1(x(t-m)) - r_2(x_2(t-m)) = 0$$

i.e. at the "switch points" the quasi rents must be the same in the two sectors. Moreover, if Ψ is to be at a saddle point, it is clear that

$$40. \quad f_1' - \frac{w}{a} = 0$$

and

$$41. \quad pf_2' - \frac{w}{a} = 0$$

40 and 41 are the same as the marginal productivity wage-price relations. The Lagrangean multiplier is to be interpreted as the real wage rate, not unexpectedly.

Finally, if we take the partial derivatives with respect to p and w, we get back our initial resource constraints.[1]

By changing p, we can generate a production possibilities schedule. It can be shown that the schedule is (strictly) concave. The reasoning is clear: consider, for example, all the machines and all the labor used in the production of consumption goods, and they produce c_{max}, and similarly for investment, producing a maximum of i_{max}. Simply transferring any portion of the machines, with the accompanying labor, we can produce any linear combination of c_{max} and i_{max}, without changing any of the labor intensities. But this is not efficient, since if that is done, the value of the marginal product of labor in the two uses will not be the same (except in the case where the two production functions are identical, the famous Solow-Ramsey case). Hence, by changing the intensities, we can obtain a strictly concave production possibilities schedule $Y_1 = H(Y_2)$.

Now, we construct a "utility" function,

42. $U = Y_2^{s} \, Y_1^{1-s}$

This is not a real utility function, since the utility accrues not from the future stream of income, as it logically should, but is immediately consequent to investing. This utility function should be construed solely as a formal mathematical device, the use of which will be clear in a moment.

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[1]Since $\Psi = \Sigma \int \{f_1(x_1(a)) - \frac{wx_1}{a}\} da + \Sigma \int \{pf_2(x_2(a)) - \frac{wx_2}{a}\} da$ ..... if, for all a except possibly a set of measure zero, $f_1(x_1(a)) - wx_1/a$ is not at its maximum, then Ψ will not be at its saddle point; and similarly for $f_2(x_2(a)) - wx_2/a$.

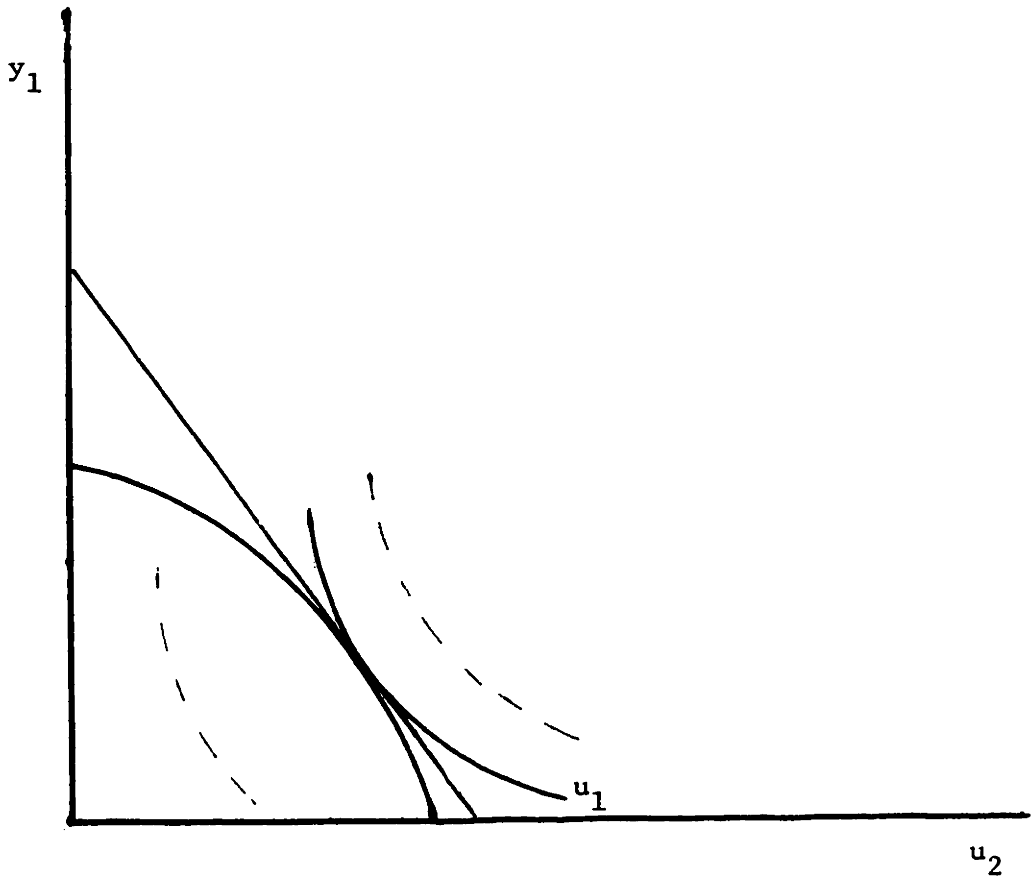

Figure 3
Determination of Short Run Equilibrium

Now, let us

43. Max $U = Y_2^{s} Y_1^{1-s}$

s.t. $Y_2 = H(Y_2)$

Since the production possibilities set is concave, and the utility function is strictly convex, there will be one, and only one solution to the maximization problem. Moreover, the slope of the tangent to the two curves at that point, i.e.

$$44. \; 1/p = \frac{(1-s)(Y_2/Y_1)^{s}}{s(Y_2/Y_1)^{1-s}} = \frac{1-s}{s} \; \frac{Y_2}{Y_1}$$

or

$$44'. \; Y_2 = sY_1/(1-s)p$$

Hence, we have shown that there exists a solution to the system. Moreover, that this solution is unique can easily be shown: consider the solution to the system Y ' and assumed that there existed another solution Y*, with $Y^*_2 > Y_2'$. then Y^*_1 would have to be less than Y'_1 and clearly, $Y_2/Y_1{*} \gtrless Y_2/Y_1'$. But p* must also be greater than p' since more investment goods are being produced. Hence from the demand equation, $Y_2/Y_1{*} < Y_2/Y_1'$, a contradiction. Hence it is impossible for there to be more than one equilibrium.

5. Steady States.

We shall now show that this economy has a unique steady state, and we shall investigate the comparative dynamics of the steady state. Throughout we shall assume that the investment goods industry is more labor intensive than the capital goods industry.

Labour is growing exponentially at the rate n and labour augmenting technical progress is proceeding at the rate u. In a steady state, it is easy to see that Y_1 and Y_2 must be growing at the same rate, equal to n + u. We shall now show that this entails w growing at the rate u, and hence m being constant. If w increases at a rate faster than u, m will be falling and x_2 will be falling, and hence Y_2 will be increasing at a rate less than n + u, and conversely if w increases at a slower rate than u. (Recall that since $Y_1/Y_2 = p(1-s)/s$, if Y_1 and Y_2 are growing at the same rate, p must be constant; and for fixed p, $\frac{d\hat{m}}{dw} = 1$ where $\hat{m}$ is the value of a for the machine which has equal quasi-rents in the two sectors.) Accordingly, if w increases at the rate of technical progress, u, m will be constant.

We shall now show that there is a unique steady state. Since

$$K(a(t)) = K(a(0))e^{(n+u)t} = Y_2(t)$$

we have at time 0

45. $$1 = \int_{-m}^{o} f_2(x_2(a))e^{(n+u)t} \, dt$$

Because of the real wage-marginal product equations, if we are given $x_2(a^*)$ for any a*, when know $x_2(a)$ for all other a; moreover, if any $x_2(a^*)$ increases, they all increase. Hence, if 45 is to be satisfied, m must be a decreasing function of, say, $x_2(a(0))$, since the right hand side increases with m and with x_2 (Figure 4).

From the savings-investment equation, we have

$$Y_1 = \frac{1-s}{s} \, p \, Y_2$$

and substituting in 1, we obtain

46. $$p \frac{(1-s)}{s} = \int_{-\infty}^{-m} f_1(x_1(a(t)))e^{(n+u)t} \, dt$$

For a given m, this means that p is an increasing function of

$x_1(a(0))$ (for instance) (Figure 5). But for a given m, $x_2(a(0))$ is fixed, and hence w/p is fixed, since

47. $w/p = a(0)f_2'(x(a(0)))$

As $x_1(a(0))$ increases, w decreases, since

48. $w = a(0)\dot{f}_1' \ (x_1(a(0)))$ and $f_1'' < 0$

and hence p must decrease. Thus, for a fixed m, there is clearly a unique value of $x_1(a(0))$ and of p. (See Figure 6)

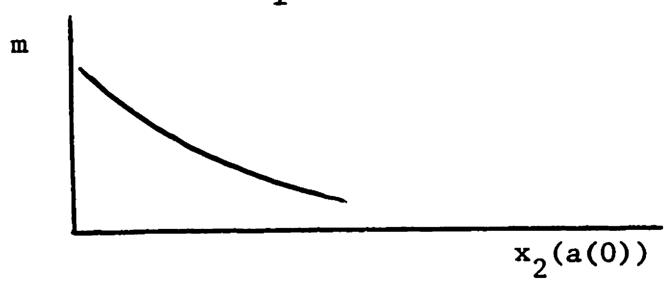

Figure 4

Now let us consider what happens when m is increased. $x_2(a(0))$ is reduced, and hence w/p is increased. For given $x_1(a(0))$ this means that p is decreased. On the other hand, from 46, if m is increased p is decreased for given $x_1(a(0))$. Hence it is unambiguously clear that as m increases, p decreases in equations 45-48.(Figure 9). On the other hand, equation 45 and the allocation of machines equations ensure that as m increases p must increase; assume that $dp/dm \leq 0$; then as m increases, w/p must increase, and hence for any given a, $r_2(a)$ must decrease. There are two cases to consider: w increasing and w decreasing (or remaining the same). The latter must result in m decreasing, and hence is inadmissible. (Figure 7). If w increases, and p remains constant m decreases. Since we have already observed that dm/dp (for fixed w) is positive, it is clear that if p is to decrease, m must decrease, and we again obtain a contradiction. (Figure 8). It is clear then that there is a unique balance growth path (Figure 9).

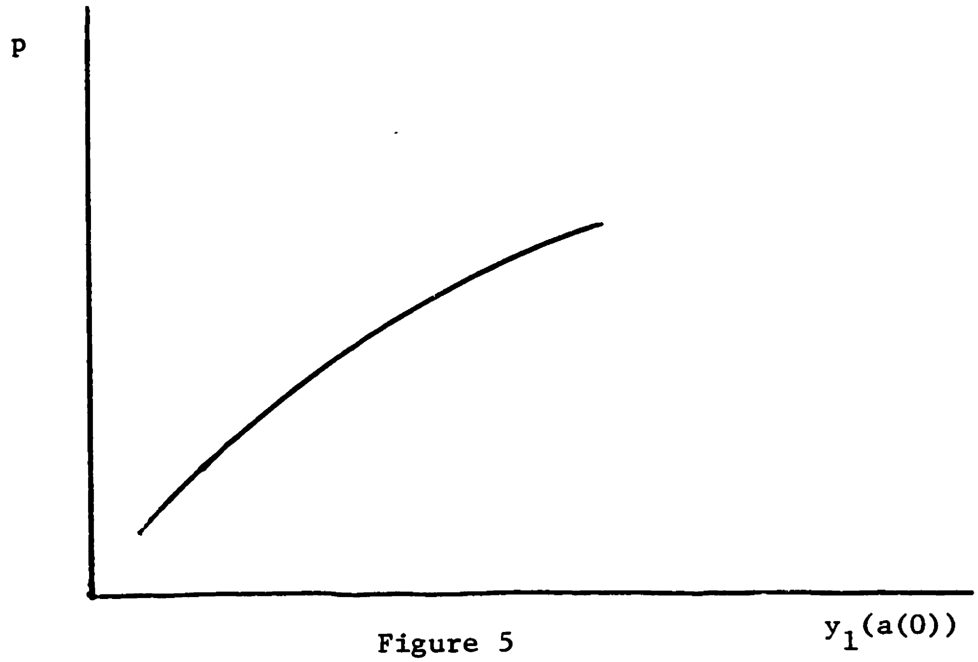

Figure 5

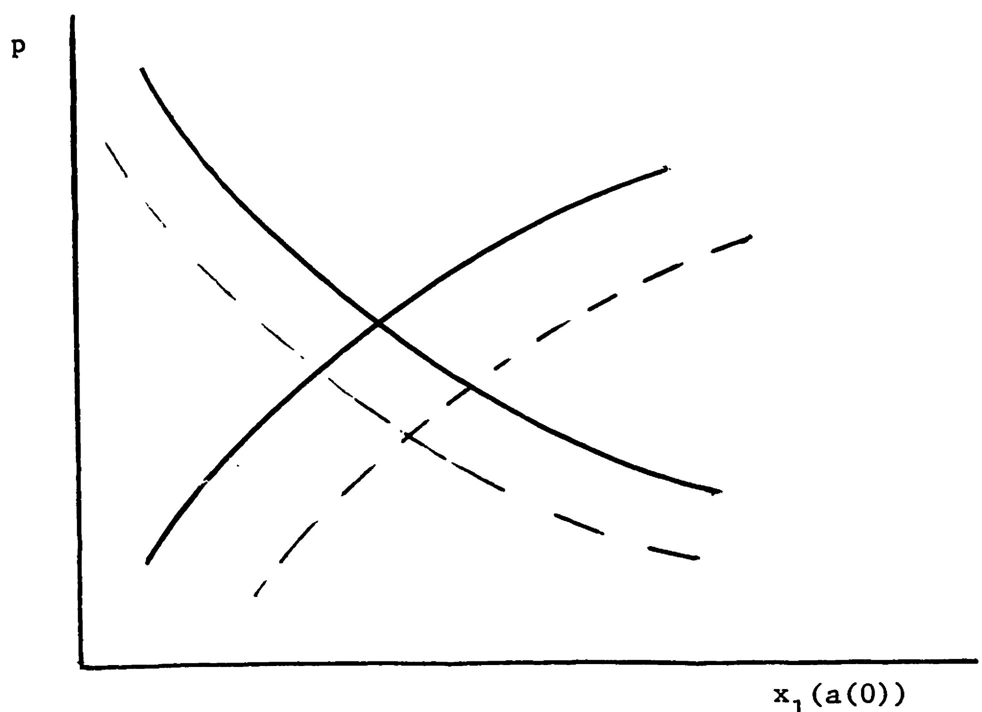

Figure 6

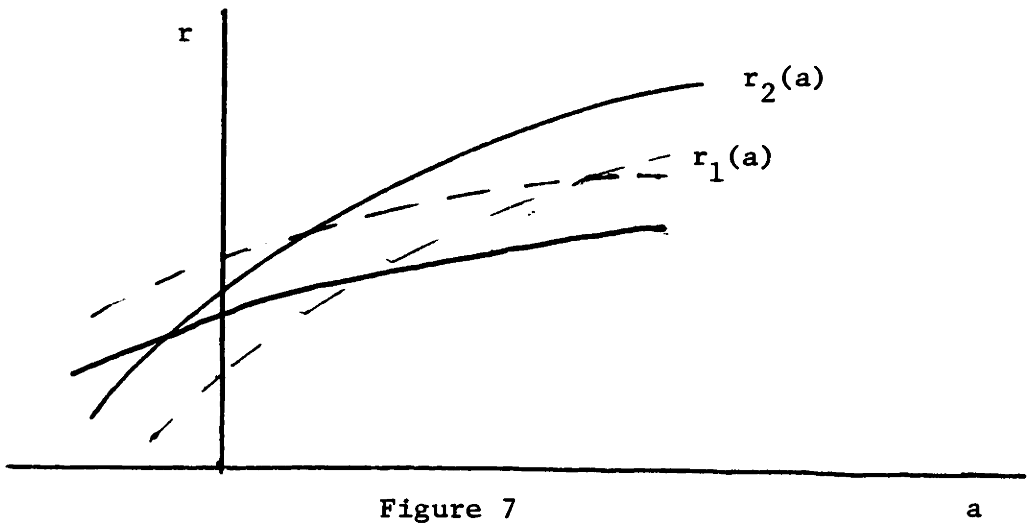

Figure 7

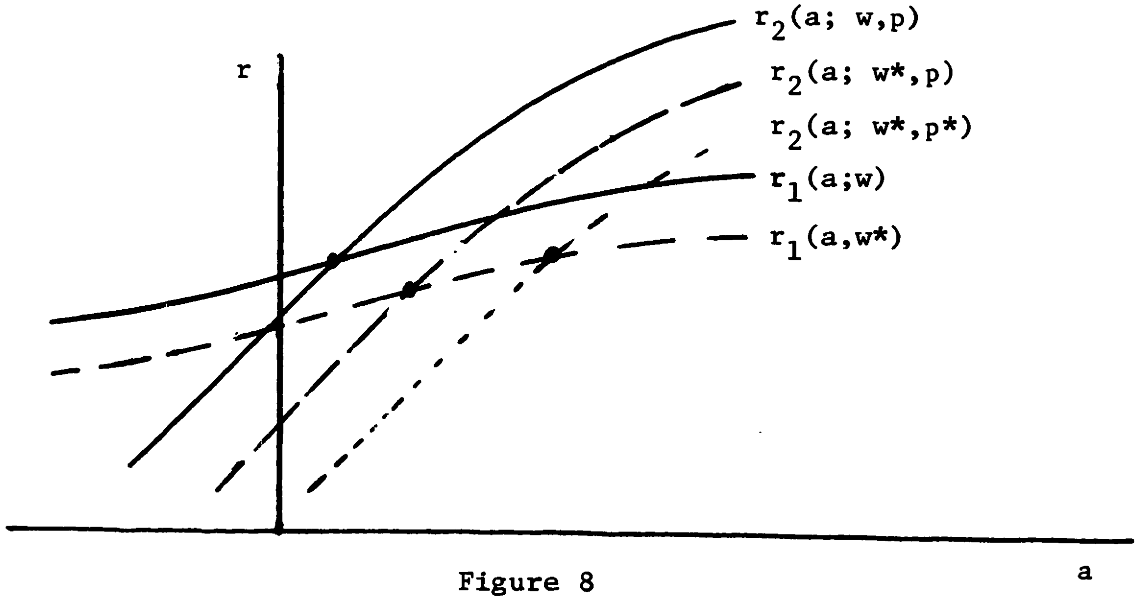

Figure 8

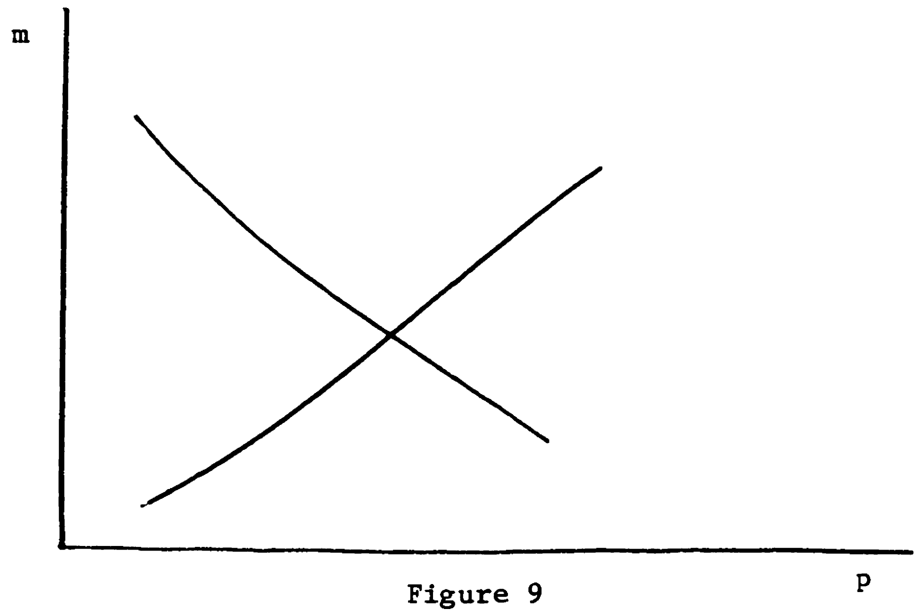

Figure 9

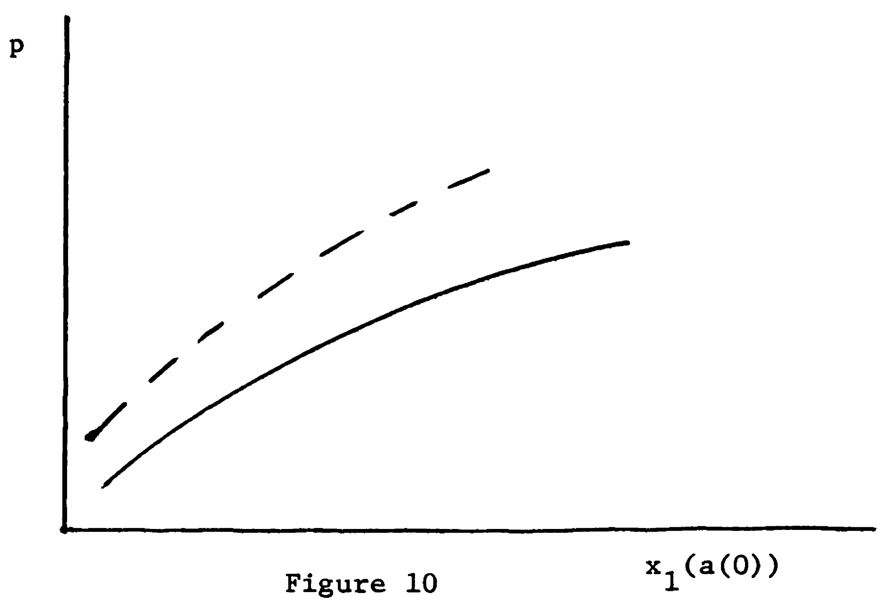

Figure 10

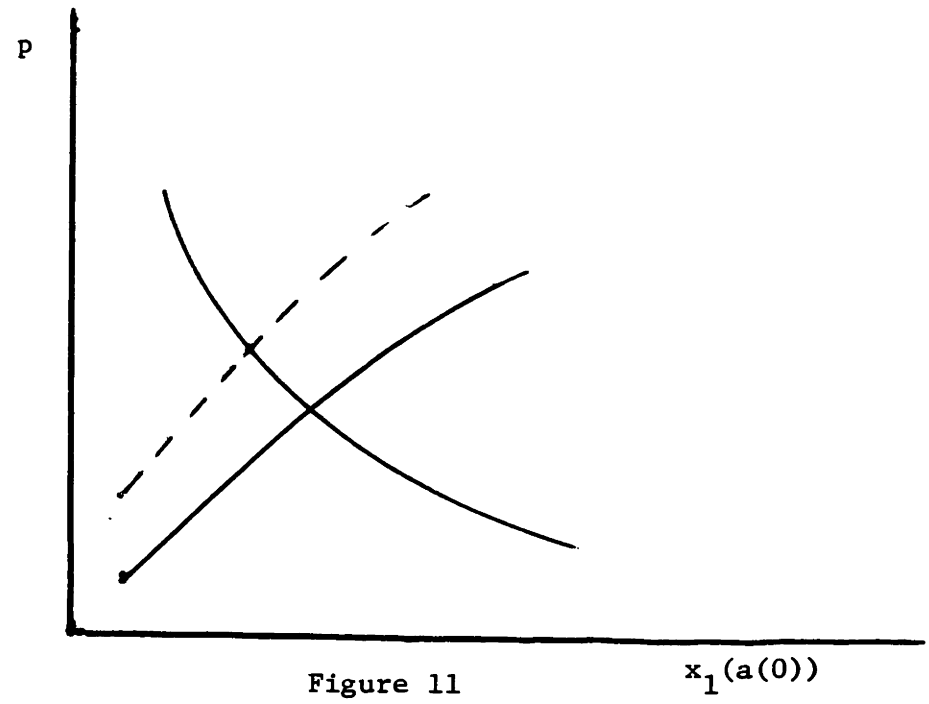

Figure 11

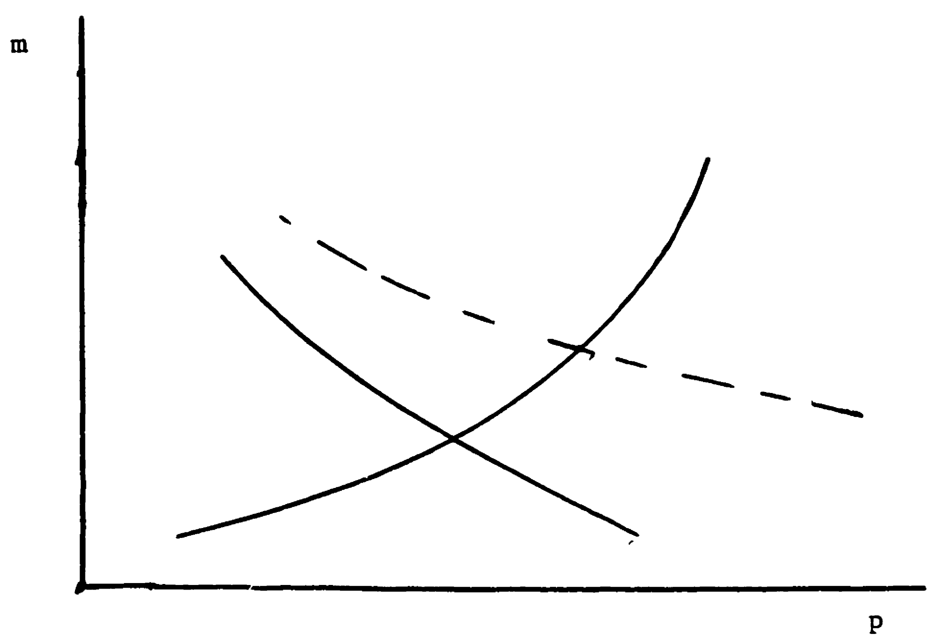

Figure 12

It may be worthwhile to investigate what happens as s increases. Equation 45 remains unchanged, but in equation 46, for any given m and $x_1(a(0))$ p increases (See Figure 10) Hence, the value of p corresponding to any m increases. (11) And from this (see Figure 12) it follows that the value of m increases with increasing s.

The Rate of Return

The private rate of return can be defined in a straightforward manner as that rate of discount which equates the (discounted) sum of quasi-rents to the price of the machine, i.e.

$$p = \int_{\tau}^{\tau+m} e^{-rt}\{pf_2(x_2(a,t)) - wx_2(a,t)/a\}dt + \int_{\tau+m}^{\infty} \{f_1(x_1(a,t)) - wx_1(a,t)/a\}e^{-rt}dt$$

for given $a(\tau)$,

where $x_i(a,t)$ is the effective labor-capital ratio in the i^{th} sector at time t for a machine of type a.

It is easy to see that in a competitive economy the social rate of return is equal to the private rate of return; somewhat heuristically, we observe that the amount of consumption foregone in the present period for a unit of capital goods is given by p (at the margin), while the value of the extra output at the time of this unit of capital good is given by

$p\{f_2\ (x(a,t)) - x_2(a,t)f_2'(x_2(a,t))\}$ while it produces good 2

$f_1\ (x_1(a,t) - x_1(a,t)f_1'(x_1(a,t))$ while it produces good 1

but in a competitive economy, we have observed that

$$f_2'(x_2(a,t)) = w/a \quad \text{and} \quad f_1'(x_1(a,t)) = w/pa$$

Substituting, discounting, and integrating, we obtain identical expression as we obtained for the private rate of return.

It should be observed that in the steady state the rate of return is constant. We have already noticed that in the steady state, the wage rate increases at the rate of labor augmenting technical progress. Hence w/a is constant. Moreover, since p is constant, it also implies that the value of $x_1(a(\tau),t)$ is simply a function of $t - \tau$, and consequently, so is $f_1(x(a(\tau),t)$. Hence we have

$$p = \int_{\tau}^{\tau+m} \{pf_2(x_2(a_2(\tau),t) - \frac{wx_2(a(\tau),t)}{a(\tau)}\} e^{-rt} \quad dt$$

$$+ \int_{\tau+m}^{\infty} f_1(x_1(a(\tau),t)) - \frac{wx_1(a(\tau),t)}{a(\tau)} \quad e^{-rt} \quad dt$$

$$= \int_{o}^{m} \{pf_2(x_2(a(0),t) - \frac{wx_2(a(0),t)}{a(o)}\} e^{-rt} dt + \int_{m}^{\infty} \{f_1(x_1(a(0),t))$$

$$\frac{-wx_1(a(0),t)}{a(o)} \} e^{-rt} dt$$

Finally, we wish to show that as s increases, the rate of return decreases; i.e. along the so-called pseudo-production function of steady states there is always diminishing returns in the Samuelsonian sense.

To see this, we recall that

$dm/ds > 0$

Since dx_2/dm is negative, and dx_1dm is negative, as s increases, both x_1 and x_2 decrease. But since dp/ds is positive, it follows that r, the rate of return must decrease.

Since
$$p = \int_{\tau}^{\tau+m} r_2(a(\tau),t)e^{-rt}dt + \int_{\tau+m}^{\infty} r_1(a(\tau),t)e^{-rt}dt$$

and since at fixed r, the right hand side is unambiguously decreased, while the left hand side is increased, and since the right hand side is a declining function of r, r must decrease as s increases.

VI. Machines not freely transferable between sectors.

So far, we have examined a very general model in which (1) capital and labor are freely substitutable; (2) machines are freely transferable between sectors; and (3) the production functions for machines is different from that for consumption goods.

Because the first two assumptions are open to serious question, it would be worth our while to briefly discuss how the analysis changes when they are dropped. And since much of the literature in this area uses the Ramsey-Solow assumption of the same production function in both sectors, for comparative purposes, we present that case too.

In the case where machines are not freely transferable between sectors (although, within each sector, capital and labor are freely substitutable) there is no problem of the allocation of machines. Output on machines in a given sector is a function only of the labor input into that sector, as we shall shortly show.

Allocating the labor over the machines efficiently, we have the usual condition that the marginal product of labour on all machines must be the same and equal to the real wage:

49. $af_1'(x(a)) = af_1'(x(a)) = w$

50. $af_2'(x(a)) = af_2'f(x\ (a)) \equiv w/p$

For given a, it can easily be shown that x is a strictly decreasing function of the real wage:

51. $dx_1(a)/dw = 1/af_1'' < 0$

52. $dx_2(a)/dw = 1/af_2'' < 0$

Thus we may write x(a) as a function of the real wage

53. $x_1(a) = \Psi_1(w)$

54. $x_2(a) = \Psi_2(w/p)$

Knowing all x(a), we can immediately find L(a), given K(a):

55. $L_1(a) = \dfrac{x_1(a)K_1(a)}{a} = \phi_1(w;\ a,\ K_1(a))$

56. $L_2(a) = \dfrac{x_2(a)K_2(a)}{a} = \phi_2(w/p;\ a,\ K_2(a))$

We can then aggregate immediately to find the total labor demanded as a function of w and p.

57. $L_1 = \int_{-\infty}^{t} \phi_1(w;a,K_1(a))da = L_1(w)$

58. $L_2 = \int_{-\infty}^{t} \phi_2(\underline{w};a,K_1(a))da = L_2(w)$ ρ

Similarly, we can now write the output of consumption goods and capital goods, respectively, as a simple function of w and p.

59. $Y_1 = \int_{-\infty}^{t} f_1(x_1(a))K_1(a)da$

$= \int_{-\infty}^{t} f_1(\psi_1(w))K_1(a)da = Y_1(w)$

60. $Y2 = \int_{-\infty}^{t} f_2(x_2(a))K_2(a)da$

$= \int_{-\infty}^{t} f_2(\psi_2(w/p;a)K_2(a)da = Y_2(w/p)$

and it is clear that output of a commodity is a monotonically decreasing function of the real wage.

The full employment constraint,

61. $L_2 + L_1 = L$

provides an implicit equation between w and p

62. $p = w(w)$

As w increases, L_1 decreases, and to maintain full employment, L_2 must increase, which implies that w/p must decrease, which in turn implies that p must increase (more than w) i.e. $w' > 0$.

And consequently, we also have our production possibilities schedule

$Y_1(w) = P(Y_2(w/p))$ $\qquad\qquad P' < 0$

And because of the always present possibilities of substitution between capital and labor, $P'' < 0$.

The rest of the analysis follows exactly as above, section 4.

There remains the important question, which makes no difference to the static analysis presented thus far, but which is crucial for the dynamic analysis to which we now turn: the allocation of new investment.

In a competitive equilibrium, capital goods must be allocated to the two sectors in such a way that the discounted sum of the stream of quasi-rents must be equal for new capital goods going into each sector (and equal to the price of the capital good.) The rate of interest at which the stream of quasi-rents is to be discounted is a simultaneously determined variable of our system. We shall confine our analysis to steady state paths.[1] Along such paths we have then (and in particular for $\tau = 0$)

63. $p = \int_{\tau}^{\infty} \{pf_2(x_2(a(\tau),t) - wx_2(a(\tau),t)/a(\tau)\}\, e^{-rt}\, dt$

---

[1]By steady state path we mean outputs of consumption and capital goods growing exponentially, price ratio, interest rate, constant, wage rate growing exponentially, etc.

64. $p = \int_{\tau}^{\infty} \{f_1(x_1(a(t),t) - wx_1(a(o),t)/a(t)\} e^{-rt}\, dt$

In addition to these two "capital goods" allocation equations, we have two labour allocation equations (real wage equal marginal production equations).

65. $w/p = a\, f'(x_2(a))$

66. $w = a\, f'(x_1(a))$

If b of the capital goods go into the capital goods sector, and 1-b into the consumption goods sector, then we have, in analogy to equation 45, the following relation

67. $1 = b \int_{-\infty}^{0} f_2(x_2(a(t))e^{(n+u)t}\, dt$

We also have the savings-investment equation

68. $Y_1/Y_2 = \frac{(1-s)p}{a}$

and an equation for the output of consumption goods, which we can write most simply if we divide by the output of capital goods:

69. $Y_1/Y_2 = (1-b) \int_{-\infty}^{0} f_1(x_1(a(t))e^{(n+u)t}\, dt$

To show that there exists at least one set of variables satisfying equations 63-69 is fairly straightforward. We consider a mapping from

$\left(\frac{w}{w+p}, \frac{p}{w+p}\right)$ onto itself defined as follows:

Equation 68 defines a one-to-one continuous mapping from (w/w+p, p/w+p) to $x_2(a)$ for all a, and 67 defines a continuous one-to-one mapping from the set $x_2(a)$ to b. 66, 68 and 69 define a continuous mapping from b, $x_2(a)$, and w/p to w and $x_1(a)$ for all a. (To see this, observe that, as we noted in our earlier analysis, given one $x_1(a^*)$, by 66 we know

$x_1(a)$ for all a, and moreover if $x_1(a^*)$ increases, $x_1(a)$ increases. Consequently, if $x_1(a)$ increases (for any a) for our given b, Y_1/Y_2 increases in 69 and hence p increases (from 68). But from 66 as $x_1(a)$ increases, w decreases, and hence for our given w/p, p decreases.) Since for given $x_1(a)$ and w the left hand side of 64 is a monotonically decreasing function of r, 64 defines a continuous mapping from p, $x_1(a)$, and w to r. Since, for our given w/p, we have already defined a mapping from w/p to w, we immediately have a mapping from w/p to p (defined by $p = w/ (w/p)$). Then 63 defines a continuous mapping from $x_2(a)$, p, w, and r to p'. Finally, we have a continuous mapping from w and p' to $\{(w/w+p)', (p/w+p)'\}$.

Since $w \geq 0$, $p \geq 0$, we have a continuous mapping from a compact space onto itself, and hence we know by the Brouwer fixed point theorem that there exists a fixed point; and the existence of the fixed point ensures that there exists a set of values w^*, p^*, $Y_1/Y_2{}^*$, $x_1(a)^*$ $x_2(a)^*$ (for all a), r^* and p^* satisfying equations 63 to 69. Unfortunately, there may exist more than one solution, as detailed investigation of the properties of the model will readily reveal, or as the Cobb-Douglass case investigated by Kurz serves to illustrate [3]. The kind of general restrictions required in order to ensure this remains an open question. In order to obtain a better understanding of what is involved, we shall now turn to a special case of this model, the fixed coefficients technology.

7. Fixed Coefficients and Number Substitution

This is the most extreme case--and perhaps the most realistic--in which neither substitution between capital and labor nor switching of machines from one sector to another is permitted. In a sense, this case involves the fewest economic decisions. We first investigate the static model. The important point to observe about the fixed coefficients model is that machines actually become economically obsolete. In the previous models, although they are used much less intensively as they grow older, they are never thrown away.

The production functions can now be written as

70. $Q_1(a) = \min \left[\frac{K_1}{c_1}, \frac{a_1 L_1}{e^{-ut}}\right]$

71. $Q_2(a) = \min \left[\frac{K_2}{c_2}, \frac{a_2 L_2}{e^{-ut}}\right]$

If we fix the real wage $\bar{w}$ in the consumption goods industry, and $\bar{w}/\bar{p}$ in the investment goods industry, we immediately determine the return per unit of machine, if a machine is used to capacity:

72. $r_1(t) = 1/c_1[1 - \bar{w}e^{ut}/a_1]$

73. $r_2(t) = 1/c_2[p - \bar{w}e^{ut}/a_2]$

so that if

74. $\bar{w} < e^{ut}/a_1$

or

75. $t > \frac{\ln \bar{w} + \ln a_1}{u} = t_{o_1}(w)$

$K_1(t)$ is used,

and if

76. $\bar{w}/\bar{p} < e^{ut}/a_2$

or

77. $t > \frac{\ln w/p + \ln a_2}{u} = t_{o_2}(w/p)$

then $K_2(t)$ is used.

Following this, output of consumption goods and investment goods may be calculated:

78. $Y_1(t) = \int_{t-t_{o_1}(w)}^{t} 1/c_1 \, K(t)dt$

79. $Y_2(t) = \int_{t-t_{o_2}(w/p)}^{t} 1/c_2 \, K(t)dt$

80. $L_1(t) = \int_{t-t_{o_1}(w)}^{t} \frac{e^{-ut}}{a_1 c_1} K(t)dt$

81. $L_2(t) = \int_{t_{o_2}(w/p)}^{t} \frac{e^{-ut}}{a_2 c_2} K_2(t)dt$

and, as in the previous case, since

82. $dt_{o_1} / dw > 0$

$dt_{o_2} / d(w/p) > 0$

and the integrand in 78-81 is nonnegative.

L_1 and L_2 are both decreasing functions of the real wage, w and w/p, respectively.

And imposing the full employment constraint

83. $L_1(t) + L_2(t) \leq L(t)$

two cases may be distinguished:

(1) where for no non negative value of w and w/p, the equality sign in 83 holds. This implies that the economy is a labor surplus economy, with wage rate equal to zero. In that case, the production possibilities schedule takes on the shape depicted in 13, the same shape that a neoclassical production function with elasticity of substitution equal to zero with homogeneous capital and surplus labor would yield.

(2) where for some non negative values of w and w/p, the equality sign in 83 holds. In that case, if we specify w, to have full employment w/p is specified, and hence p. And in fact, as w increases, w/p must decrease, which implies that p must increase. But as w is increasing, the output of consumption goods is going down, and as w/p is decreasing, the output of investment goods is rising. Hence, we obtain the more usual production possibilities schedule: concave, with decreasing marginal rate of substitution.

It is curious though not surprising that in this, the most restrictive case examined, we obtain exactly the same shape for the production possibilities schedule that we do in the least restrictive case, that examined at the beginning of this paper.

In the dynamic version of this model, we shall again limit ourselves to examining steady states. It can be shown, although we shall not take the time here to do so, that in the steady state the age of obsolescence of the two different kinds of machines remain constant.[1] Our allocation of machines equations, corresponding to 63 and 64 which require that the discounted sum of quasi-rents

[1] Alternatively, we can just assume this as one of the properties of our dynamic equilibrium paths.

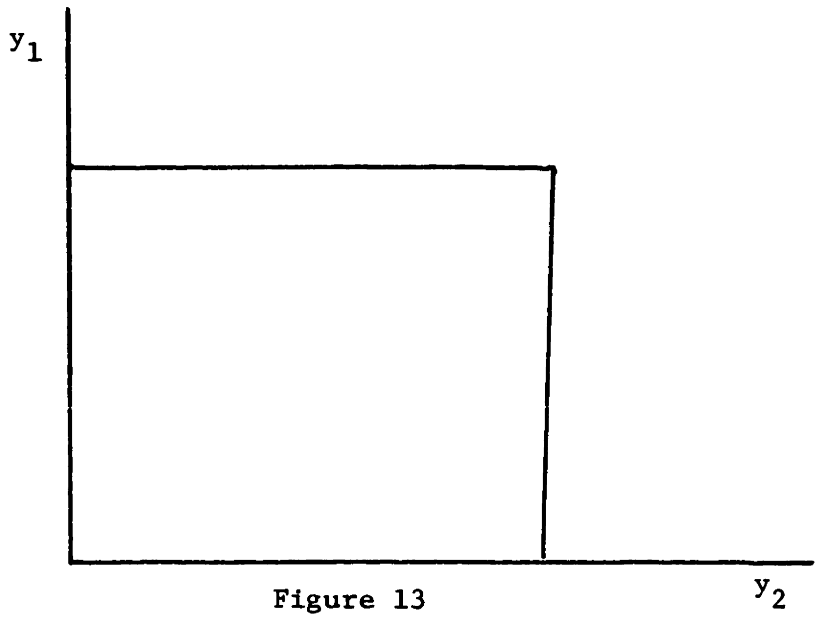

Figure 13

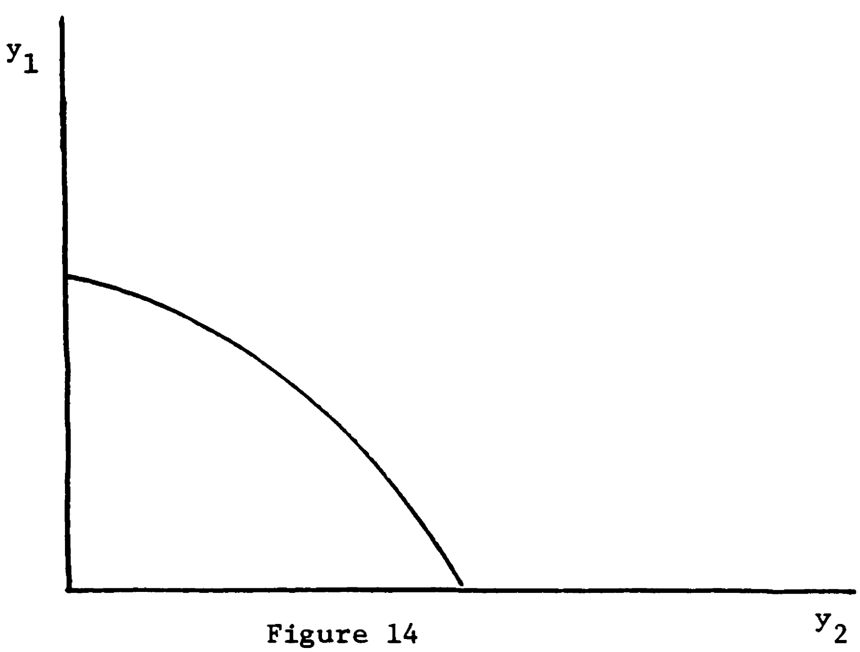

Figure 14

on each of the two kinds of machines equal that price of machines, become (for machines built at time 0, for instance

84. $p = 1/c_2 \int_o^m [p - w_o e^{ut}/a_2] e^{-rt}\, dt = 1/c_2 [\frac{p(1-e^{-rm})}{r} - \frac{w_o(1-e^{um-rm})}{a_2(r-u)}]$

85. $p = 1/c_1 \int_o^q [1 - w_o e^{ut}/a_1] e^{-rt}\, dt = 1/c_1 [\frac{(1-e^{-rq})}{r} - \frac{w_o(1-e^{uq-rq})}{a_1\ (r-u)}]$

Our real wage equal marginal product equations now become (for time 0)

86. $w_o/p = e^{-um}/a_2$

87. $w = e^{uq}/a_1$

In analogy to equation 67, we have

88. $1 = b \int_{-m}^{o} \frac{e^{(n+u)t}}{c_2}\, dt = \frac{b}{c_2\ (n+u)} [1 - e^{-m(n+u)}]$

Our savings-investment equation remains unchanged,

89. $Y_1/Y_2 = (1-s)p/s$

Finally, we have the equation for the output of consumption goods, normalized on the output of investment goods:

90. $Y_1/Y_2 = (1-b)\ \frac{1}{c_1} \int_{-q}^{o} e^{(n+u)t}\, dt = \frac{1 - b}{c_1\ (n+u)}\ 1 - e^{-q(n+u)}$

It is easy to show, along the lines of the proof used in the previous section where the Inada conditions were satisfied, that there exists at least one solution to this set of equations. But unfortunately, multiplicity of solutions cannot be ruled out in general.[1] (Neither, I may add, have I been able to show that there necessarily may exist multiple solutions). In one case, it is possible to show without too much difficulty that there is a unique steady state: that where p = 1. In this case, the model becomes very similar, although not identical, to the Solow, Tobin, Yaari, Von Weizzacker one sector fixed coefficients model, the difference arising from the non-shiftability assumption [7].

8. Identical Production Functions in the Two Sectors

The Ramsey-Solow case, where the production functions in the two sectors are the same, may quickly be disposed of. Again, there are no allocation problems, and the price ratio is equal to 1.

---

[1]The several equations can be reduced to three in the unknowns m, q, and r:

$$\frac{1 - e^{-rm}}{c_2 r} - \frac{e^{-um} - e^{-rm}}{c_2 (r-u)} - 1 = 0$$

$$\frac{a_1}{a_2 c_1 r} \left(e^{-um + uq} - e^{-rq - um + uq}\right) - \frac{1}{a_1 a_2 (r-u)} \left(e^{-um} - e^{-rq - um + uq}\right) - 1 = 0$$

$$\frac{1 - c_2(n+u) - e^{-m(n+u)}}{1 - c^{-mn - mu}} \; \frac{1 - e^{-q(n+u)}}{(n+u)} - \frac{(1-s) a_2 c_1 (e^{-uq + um})}{s a_1} = 0$$

Many of the properties of the equations, particularly the first two, are easily derivable. But the essential difficulty arises in the third equation, in determining dq/dm.

The production functions in this case may be written

91. $Y_1 = F_1(k, aL_1) = F(K, aL)$

$Y_2 = F_2(K, aL_2) = F(K, aL)$

Total output $Y=Y_1 + Y_2$ can be calculated

92. $Y = \int_{-\infty}^{t} f(x(a))K(a)da$

While the labor employment may be simply written as

93. $L = \int_{-\infty}^{t} x(a)/a \quad K(a)da$

We have yet to determine the labor intensity (a) in 92 and 93. This is provided by marginal productivity theory and the full employment constraint:

94. $w = a\, f'(x(a))$

For any given a,

95. $\frac{dx}{dw} = af''(x(a)) < 0$

and from 125

96. $\frac{dL}{dw} = \int_{-\infty}^{t} f'(x(a)) \frac{dx(a)}{dw} K(a)\, dt < 0$

As the wage increases, employment decreases. But the full employment constraint

97. $L = L(t)$

then implies a specified value of the real wage (see Figure 5) and this real wage, in turn, determines x(a) in 94, and this in turn (in 92) determines the level of output.

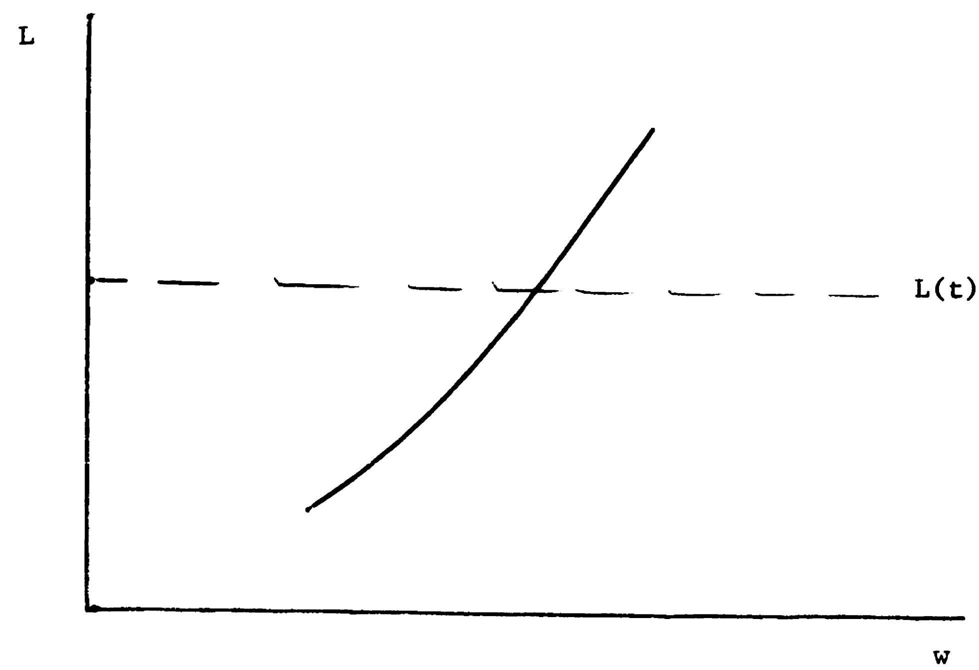

Figure 15

9. Extensions to Three Commodity Markets

The above analysis extends in a straightforward manner to three or more commodities, each produced by a linear homogenous production function

$$Q_i = F_i(K(a), aL) = f_i(x_i(a))K(a)$$

We again impose the restriction that there be no factor intensity reversals. In the case of more than two commodities, this means that if at any given $\frac{w}{r}$, $x_i > x_j$, then $x_i > x_j$ for all $\frac{w}{r}$. And if $x_i = x_j$ for any $\frac{w}{r}$ then $x_u = x_j$ for all $\frac{w}{r}$, and hence we may consider Q_i and Q_j as the same commodity. We can then construct a complete hierarchy of factor intensities. Then, by exactly the same reasoning as in section 2, it is clear that the most labor intensive commodity has a steeper slope (on the r versus a graph) than any other commodity, and hence, for a given w and p, all machines of vintage greater than a' are allocated to the most labor intensive sector. Now eliminate the most labor intensive sector, with its accompanying machines, from the analysis. Then over the range of the remaining machines, and for the remaining commodities, it is clear that the next most labor intensive commodity will have the steepest slope, and hence all machines of vintage greater than a" but less than a' will be allocated to this second most labor intensive sector. And so on.

Thus we have shown that there is a complete hierarchy of the allocation of machines, the newest to the oldest going from the most labor intensive to the least labor intensive commodity.

Of course, for any given w and p, a commodity need not be produced, i.e., if we denote by a: the oldest machine for which $r_i > r_j$ and $r_i > r_k$, a_i may be "newer" than the newest machine in existence.

Similarly, if a_k is the newest machine for which $r_k > r_i$ and $r_k > r_j$, a_k may be below the lowest a in existence. But as in the previous case, by adjusting w, we increase all the a proportionally. In the simpler two commodity case, this means, of course, that we can move a (no subscript needed since there is only one "separation" point) into the feasible range, and both will be produced. But, for given p_i and p_j, we may increase w, and moving, for instance, a_k into the feasible range, but this may move a_i out of the feasible range. But, by adjusting the relative prices, we can bring all (a_i, a_j, a_k) into the feasible range.

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CHAPTER II

ACCUMULATION IN A LEONTIEF-SRAFFA ECONOMY AND THE SWITCHING OF TECHNIQUES

Introduction

We consider an economy with a finite number of commodities, n. Each of the commodities can be produced by any one of a set of processes.[1] Each process is characterized by the inputs of the other commodities and labor required to produce one unit of output.[2]

A given technology of the economy is denoted then by a (1 + n) x n matrix, where each column is a process for producing one of the n commodities.

Corresponding to any technology there is a set of real wages for any given (feasible) rate of interest, i.e. for the technology $\binom{a_o}{A}$, the price vector is simply derived as

$$p = wa_o(1 + r) + pA\,(1 + r) \text{ or } p = a_o(1 + r)(I - (1 + r)A)^{-1},$$

provided that $r < \frac{1 - \lambda^*}{\lambda^*}$ where λ^* is the largest characteristic root of the A matrix.

The Samuelson non-substitution[3] [2,8] theorem states that corresponding to any interest rate, there is a technology which unambiguously maximizes real wages, i.e. for some technology

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[1]For the most part, we shall only consider cases where there are a finite number of such processes, but in section 2.e we shall consider an alternative restriction.

[2]We assume no joint products and only one primary factor (labour): obviously, most of the results will not carry through if either of these assumptions is violated.

[3]Here as elsewhere numbers in brackets refer to references at end of chapter.

$A, P_A \leqslant P_j$ for any $j \neq A$ [1]

In general, one might expect that as the interest rate changed, once we "discarded" a technology, we would never return to that technology; i.e. if A is the "chosen" technology at interest rate r_1, and at some $r_2 > r_1$, B becomes the "chosen" technology, one might not expect that at a still higher interest rate, A again becomes the chosen matrix. Such however is not the case; Mr. Champernowne [1] as early as 1953 presented a counter example. The possibility of such "perversities" was independently discovered by Ruth Cohen [6] (and hence the popular name "Ruth Cohen curiosum") and Piero Sraffa [10].

More recently Mr. Levhari has attempted to show that in fact such a perversity is indeed impossible. The single counterexample of Champernowne was of course enough to disprove his contention. As Dr. Hahn and Dr. Mirrlees have pointed out, three errors, each of which vitiates the proof, led Mr. Levhari to the untrue theorem. [2]

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[1] $\leqslant$ denotes no $p_j^i > P_A^i$ and at least one i, ε $P_A^i < P_j^i$

[2] The errors are the following:

1. In the beginning of the proof he multiplies both sides of an inequality by the matrix $\lambda I - A$; although the inverse is non-negative, it is obvious that the matrix $\lambda I - A$ is not.

2. He asserts by analogy to the Von Neumann model that one can always find a semi-positive vector x, such that (if A and B are non-negative matrices), Ax-Bx is either semi-positive, semi-negative, or zero. Although it is true that one can always find a vector x and a scalar z such that Ax-zBx has the required properties, the statement that for z = 1 there exists a semi-positive vector (or semi-negative) does not follow (and in fact, is not true).

3. Even if one can find the required semi-positive vector x, such that Ax-Bx=0 (identically), one may obtain absolutely no information thereby; consider the case where A-B has some zero columns (i.e. use the same process for some commodities) then letting these components of the vector which correspond to the zero columns be positive and all other components equal to zero, one obtains a vector (A-B)x = 0.

In these notes, I should like to consider the two questions:

1. What can we say about the conditions under which the perversity is possible (or impossible)?
2. What implications does the existence of the Champernowne-Cohen-Sraffa curiousum have for the pattern of capital accumulation and the valuation of capital?

Part I

The Possibility of Multiple Switches

a. Introduction. It will be convenient in this section to introduce the following definitions.

A Direct Switch is a case where the technology changes from some matrix A to some matrix B and back to A (without any other intervening technologies).

In the case of a (General) Switch, of course, there may be any number of intervening technologies.

Most of the results we have obtained are about direct switches, but the theorem presented in the next section and the theorems presented in section 2.e are general.

b. In this section, we shall show that if there is a switch from A back to A with technologies B, C.... intervening, then none of the intervening technologies can have any commodity produced by an unambiguously less capital intensive process than in A.

One process is unambiguously more capital intensive than another process if it requires no less of any commodity than the other process and more of at least one commodity (to be used, the more capital intensive process must require less labor.)

The simplest proof of this proposition is as follows: we know from the constructive proof of Levhari of the non-substitution theorem that if a process reduces the cost of production of one commodity it reduces the price of all.[1]

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[1]If the matrix is not indecomposable, replace the word "reduces" with "does not increase the price of any other commodity and may reduce the price of some other commodities."

Consider the "initial" change in price of the first commodity when a less capital intensive process is used to produce it, leaving all other activities and prices unchanged,

$$\frac{\Delta p_1}{1+r} = \Delta p_1 a_{11}' + (a_{o1} - a_{o1}') + p_1(a_{11}-a_{11}') + p_2(a_{12}-a_{12}') + p_3(a_{13}-a_{13}')\ldots$$

or

$$\Delta p_1 = \frac{[a_{o1} - a_{o1}' + p_1(a_{11}-a_{11}') + p_2(a_{12}-a_{12}') + p_3(a_{13}-a_{13}') \ \ldots]}{\frac{1}{1+r} - a_{11}'}$$

All the terms in the numerator of the right hand side are strictly increasing functions of r, and the denominator of the right hand side is a strictly decreasing function of r; hence there will be only one value of r for which Δp_1 is zero. Since if, and only if, the "initial" change in price is negative will the less capital intensive process be used, below this r, the capital intensive process is used (exclusively), and above this r the labour intensive process is used. (Note that the proof does not depend on the other coefficients of the technology, provided, of course, that they yield a viable technology; the proof depends only on the fact that p is an increasing function of r.)

An immediate corollary of this theorem is that if A switches back to A with intervening technologies B, C..... then there exists no activity levels at which any of the commodities can be produced which lead to any of the intervening technologies unambiguously requiring more capital than technology A.

The force of this theorem is that if we can look at any commodity, or any group of commodities within the economy, and say that in one technology that group of commodities is produced with a more capital intensive process than in some other technology, then the entire technology may be said to be more capital intensive in the sense that it will only appear at lower interest rates. Or, to put it still another way, switching can only occur if there is no commodity which uses an unambiguously more capital intensive process in any one of the intervening matrices. If, as seems realistic, there are some possibilities of substitution (e.g. a "machine" may be worked 8, 9, or 10 hours a day, the other input requirements remaining fixed) then switching cannot occur.

c. Direct switching impossibility theorems.

Theorem c.1. If A-B is non-singular, then direct switching cannot occur. Proof. At a switch point

$$(1+r)(pA + a_o) = p_a = p_b = (1+r)(pB + b_o)$$

or

1. $p_a (A-B) = b_o - a_o$

If A - B is nonsingular, this means that

2. $p_a = (b_o - a_o)(A - B)^{-1}$

There is at most one solution to this equation, since the right hand side of the equation is a constant, and the left hand side is monotonic in r (i.e. every element of the vector p_a is monotonic in r).[1]

An immediate corollary of this result is that if A - B is diagonally dominant, or quasi-diagonally dominant, then direct switching is

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[1]It should be observed that in the usual case, A - B will be singular, since for at least some commodities the same process will be used in the two technologies.

impossible.[1]

A second set of conditions under which direct switching is impossible can be derived as a corollary of the theorem presented above in section 2.b:

Theorem c.2. If A and B are two technologies, such that there exists a set of activity levels, represented by a semi-positive vector x, such that $(A-B)x \geq 0$ or $(A-B)x \leq 0$, then there can be no direct switching.

$(A-B)x \geq 0$ or $(A-B) \leq 0$ says that there is a subset of commodities which can be operated at some activity levels such that the capital requirements of this subset are unambiguously greater (less) in A than in B. But if this is true, then we know that there cannot be any double switching, direct or indirect.[2]

Further insight into what is entailed in a direct switch can be gained by viewing equation 2.1. $P_a(A-B) = b_o - a_o$, as a transformation from n dimensional Euclidean space into itself. If A and B use the same processes for producing m commodities (and the remaining n-m columns of A-B are linearly independent) then the rank of A-B will

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[1]"An n x n matrix A has q.d.d. (a quasi-dominant diagonal") if (1) there exists $d_j > 0$ such that $d_j|a_{jj}| \geq \Sigma_{i \neq j} d_i|a_{ij}|$ $(j=1,\ldots,p)$, and (2) when $a_{ij} = 0$ (given $j \epsilon J$ and $i \notin J$ for some set of indices J), the strict inequality holds for some $j \epsilon J$." A q.d.d. matrix is non-singular. L. McKenzie, "Matrices and Economic Theory," in <u>Mathematical Methods in the Social Sciences</u>, 1959, (Stanford Symposium)

[2]An alternative proof is as follows: Assume that there were two switches, at r and r'; then $p(r)(A-B)=b_o - a_o$ and $-p(r')(A-B) = a_o - b_o$. Adding, we have $\Delta p(A-B)=0$, Δp strictly positive. But for any matrix D, either $xD=0$, $x>>0$, or $Dy \geq 0$ has a solution. Hence, we have a contradiction.

be n-m, and there will be a m dimensional Euclidean subspace, V_m, in the space of p satisfying $p(A-B) = b_o - a_o$. If there is to be double switching, then there must exist two distinct interest rates, r and r', such that $p_a(r)$ and $p_a(r') \varepsilon V_m$.

What does this condition require, and is it likely to be satisfied? Let us take the Sraffa example of an n commodity world in which there is only one technique available for n-1 of the commodities but two (or more) alternative techniques for the first commodity. Then p(r), which is a curve in n-dimensional space, must intersect a particular straight line in the same n-dimensional space at two distinct points. It is misleading to draw (as Sraffa has done) a two dimensional projection of these two lines, for they may very well cross several times, while the actual lines in n-dimensional space do not intersect more than once.

We shall now state and prove a set of explicit conditions on the technologies which must be satisfied if there is to be direct switching between two economies which differ only in the process they use for one commodity:

Theorem c.3. If the matric A-B is of rank n-1, then product of the first row of A-B [less its zero (first) element] and the inverse of the n-1 x n-1 reduced matrix derived from A-B by eliminating its first row and column must be strictly negative if there is to be direct switching.

We have, as before

$$p_a(r)(A-B) = b_o - a_o$$

$$p_1(r)\hat{c} + \tilde{p}\,\tilde{c} = b_o - a_o$$

where $\tilde{p}' = (p_2 \; .... \; p_n)$

$\hat{c}$ is the first row of A-B less its first element, i.e.

$$A_{12}-B_{12}, \; .... \; A_{1n}-B_{1n}$$

$\tilde{c}'$ is the cofactor matrix of $A_{11}-B_{11}$

Hence

$$\tilde{p}\tilde{c} = (b_o-a_o) - p_1(r)\hat{c}$$

$$\tilde{p} = (b_o-a_o)\tilde{c}^{-1} - p\,(r)\hat{c}\tilde{c}^{-1}$$

The left hand side is a monotonic increasing function of r. Every component of the right hand side vector is monotonic. Hence, if the left hand side is to equal the right hand side at more than one r the right hand side must be a monotonically increasing function of r.

$$- p_1'(r)\hat{c}\tilde{c}^{-1} > 0$$

But $p_1'(r) > 0$

Hence $\hat{c}\tilde{c}^{-1} < 0$ [1]

(By assumption A-B is of rank n-1. Hence $\tilde{c}^{-1}$ must exist).

These conditions provide us with what would seem to be fairly severe restrictions on the A-B coefficients, especially for larger matrices. For a two by two matrix (an economy with two commodities),

$$\hat{c} = a_{12} - b_{12} \equiv c_1$$

and $$\tilde{c}^{-1} = \frac{1}{a_{22}-b_{22}} \equiv \frac{1}{c_2}$$

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[1] Analogous conditions can be derived in a straightforward manner for cases where more than one industry uses the same process.

Hence, a necessary condition for a two commodity technology is that one of the matrices be more own-commodity using, while the other matrix be more other-commodity using.

One more condition may be derived for the two commodity case: it can easily be shown that for technology A, the price of one commodity will be rising more rapidly (at all times) than the price of the other commodity.[1]

We require at a switch point

$$p(A-B) = b_o - a_o$$

and if there exists two switch points, then

$$p\ (A-B) = 0$$

Let us choose our units so that at the switch point, with the lower rate of interest, $p_1 = p_2$. Then, except for the singular case

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1

$$p = a_o\ (I\lambda - A)^{-1} = (a_{o1}, a_{o2}) \begin{Bmatrix} \lambda - a_{22} & -a_{21} \\ -a_{12} & \lambda - a_{12} \end{Bmatrix} \Big/ |\lambda I - A|$$

$$p_1 = \frac{a_{o1}\lambda - a_{22} + a_{o2}a_{12}}{|\lambda I - A|} = \frac{N}{D}$$

$$p_1' = \frac{N'}{D} - \frac{N}{D}\frac{D'}{D}$$

$$\frac{p_1'}{p_1} - \frac{p_2'}{p_2} = \frac{N_1'}{N_1} - \frac{N_2'}{N_2}$$

Since $D > 0$, for $\lambda < \lambda^*$ (where $\lambda^* = \min[\lambda_1^*, \lambda_2^*]$, where λ_1^* and λ_2^* are the Frobenius roots of the technologies), for p_i to be positive, N_i must be positive. Hence

$$\frac{a_{o1}}{a_{o1}\ -a_{22} + a_{o2}a_{12}} - \frac{a_{o2}}{a_{o2}\lambda - a_{11} + a_{o1}a_{21}} \gtrless 0 \text{ as } -a_{11}a_{o1} + a_{o1}^2 a_{21} + a_{12}a_{22} - a_{o2}^2 a_{12} \gtrless 0$$

independent of λ.

where the two commodities have essentially the same production function, either commodity one or commodity two will have its price rise at a faster rate with increasing r. For convenience, we will call the commodity with the faster rate of increase of price commodity one. Then, it is clear that $\Delta p_1 > \Delta p_2$. But from what we have already shown, if there are to be two switching points

$$\Delta p_1 (a_{11} - b_{11}) + \Delta p_2 (a_{12} - b_{12}) = 0$$

and

$$\Delta p_1 (a_{21} - b_{21}) + p_2 (a_{22} - b_{22}) = 0$$

so that

$$\frac{|a_{11} - b_{11}|}{|a_{12} - b_{12}|} = \frac{\Delta p_1}{\Delta p_2} > 1$$

$$\frac{|a_{21} - b_{21}|}{|a_{22} - b_{22}|} = \frac{\Delta p_1}{\Delta p_2} > 1$$

In other words, if commodity 1's price rises faster than commodity 2's in, say, the A technology (a property which depends only on the A technology), then the difference in the production of commodity 1, between the A technology's use of commodity 1 and B's use of it, must be greater, in absolute value terms, than the difference between A technology's use of commodity 2 and B's use of it in absolute value and conversely for the production of commodity 2.

d. The construction of examples

In spite of all the restrictions which we have seen have to be satisfied, it is not difficult to construct examples. Let A be an arbitrary Leontief-Sraffa matrix. Then our problem is to find a matrix B, which uses the same processes for the (first) m commodities, which satisfies

$$p_a(r_1)(C) = c_o \qquad c_o = b_o - a_o$$

$$p_a(r_2)(C) = c_o \qquad C = B - A$$

We have 2 (n-m) equations in our $(n-m)^2$ unknown C_{ij} and n-m unknown labour requirement differences, and hence it should not be hard to find a solution in general.[1] (Since $(n-m)^2 + n-m \geq 2n-m$ if $n-m \geq 1$)

Let our technology A be characterized by

$$\begin{matrix} .2 & .4 \\ .3 & .5 \end{matrix} \quad \text{Then} \quad \lambda I - A = \begin{matrix} \lambda-.2 & -.4 \\ -.3 & \lambda-.5 \end{matrix}$$

$$p = (1,2) \begin{matrix} \lambda-.5 & .4 \\ .3 & \lambda-.2 \end{matrix} \Big/ (\lambda-2)(\lambda-4) - .12$$

At $\lambda = .9$ $\qquad\qquad$ $\lambda = .8$

$$p = (1,2) \begin{matrix} .4 & .4 \\ .3 & .7 \end{matrix} /.16 \qquad p = (1,2) \begin{matrix} .3 & .4 \\ .3 & .6 \end{matrix} \Big/ .06$$

$$= \left(\frac{1}{.16}, \frac{1.8}{.16}\right) \qquad \left(\frac{.9}{.06}, \frac{1.6}{.06}\right)$$

Then

$$\frac{1}{.16} c_1 + \frac{1.8}{.16} c_2 = \frac{.9}{.06} c_1 + \frac{1.6}{.06} c_2$$

$$-\frac{c_1}{c_2} = \frac{37}{21}$$

$$\begin{matrix} 1 & .05 \\ .2 & .67 \\ .3 & .29 \end{matrix} = \left(\frac{b_o}{B}\right)$$

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[1]The proviso in general must be inserted because it is possible that $p_a(r_1) = zp_a(r_2)$, i.e. corresponding to two interest rates, the commodity price ratios are the same; and in that case, there will be no solution.

Hence, we have constructed an example of a switch point at

$$\lambda = .9 \ (R = 1/9) \text{ and } \quad \lambda = .8 \ (r = .25)$$

Thus, we have provided a methodology for constructing perverse examples.

e. Differentiable Production Functions

Thus far, we have restricted ourselves to cases where each industry chooses the process with which it produces from a finite set. Now we wish to consider a case where there is an infinity of processes for producing each commodity, and the alternative techniques may be "related" to each other by the following type of functional relationship

$$1 = F_i(a_{oi}, a_{1i}, \ldots\ldots\ldots a_{ni})$$

where F_i is a well behaved differentiable function.[1]

A switch implies that corresponding to a given state of the technology A there are at least two distinct interest rates. Hence, if we can show that, under the above conditions, there is a unique r corresponding to a given technology, then we will have shown that switching (direct or indirect) is impossible.

Theorem e.1. If the (diagonal of the) matrix A is non-hollow, there is a unique r corresponding to the technology A.

This follows immediately from the fact that the marginal product of good i in producing itself must be equal to 1 + r, if good i is used in producing itself.

This theorem states that if any commodity is used in producing itself, there cannot be switching.

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[1]As should be clear as the analysis proceeds, it is not necessary that every industry be able to choose from such a set of processes for many of our results to go through.

We shall now prove a stronger theorem:

Theorem e.2. If the production functions of the economy are well-behaved, differentiable functions, then there is only one r corresponding to a given technology.

If the technology has an indecomposable matrix, then we can, for any two commodities, i and j, find a set of k's such that a_{ik_1}, $a_{k_1k_2}$, ... a_{k_zj} are all greater than zero, and conversely, we can find a set of m's such that a_{jm_1}, $a_{m_1m_2}$ .... a_{m_vi} are all greater than zero.

If $a_{ij} \neq 0$, we know that

$$P_i/P_j(1+r) = \text{marginal product of good i in producing good } j = M^i_j$$

Hence, the existence of such chains implies that for any two commodities, we can find

$$P_i/P_j(1+r)^z = M^i_{k_1}\ M^{k_1}_{k_2}\ .....\ M^{k_z}_j$$

and

$$P_j/P_i(1+r)^v = M^j_{m_1}\ M^{m_1}_{m_2}\ ......\ M^{m_v}_i$$

The right hand side of each of these equations is independent of r. Assume that the matrix were used at two different interest rates, r and r'. Then

$$\frac{P_i(r)/P_j(r)}{P_i(r')/P_j(r')}\quad\frac{(1+r)^z}{(1+r')^z} = \frac{P_j(r)/P_i(r)}{P_j(r')/P_i(r')}\quad\frac{(1+r)^v}{(1+r')^v} = 1$$

But this implies that

$$(1+r)^{z+v} = (1+r')^{z+v}$$

which in turn, implies that

$$r = r'$$

If the matrix is decomposable, we can partition it so that

$A_{11} B_{12} \cdots\cdots$

$0 \quad A_{22} \cdots\cdots$ $\qquad A_{ii}$ indecomposable.

$0 \quad 0$

If A_{11} has more than one element, it follows that there will be only one r consistent with it (by the same reasoning by which we showed that if the whole matrix is indecomposable, there is only one r consistent with it). If A_{11} has only one member, and it is non zero, theorem e.1 applies. If A_{11} has only one member, and it is zero, then (since "nothing will come of nothing"),[1] a_{o1} is positive from which it follows that the wage is equal to the marginal product of labour in producing commodity 1 (which is independent of r, for the given technology) times the price:

$$1 = \text{a number} \times a_{o1}(1 + r)$$

Hence there can only be one r, and our theorem is complete.

Accordingly, we have shown that if there are "enough" different processes switching cannot occur.

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[1] _Lear_, I, ii.

Part II

Value of Capital

For a long time, the possibility of capital per unit of labor moving in the same direction to the rate of interest has been well known,(by, e.g. Wicksell). This meant that when an economist in attempting for instance to explain the higher wage in one economy in comparison to the other, made a statement like, "the higher wage economy has more capital per unit of labor" he did not mean it in a value sense, but in a real sense. And accordingly, the K in the neoclassical production function referred to the real capital stock.

The force of the switching possibility is, of course, that it is impossible in a real sense to say that one technology has a higher wage rate because it has a more capital intensive technique.

But note that switching can only occur in those cases where by looking at the technologies, one cannot say that one technology is more capital intensive than another--or any set of commodities is produced by a more capital intensive process.

If such switching possibilities were of any importance in the real world, one would lose a very neat way of talking about the world, a very convenient shorthand--although in a wide class of cases one could, with caution, still use the old "mode of thinking." In general, however, one would have to enumerate, for instance, the various kinds of "capital goods" used in the production of commodities.

But Mr. Pasinetti has taken this opportunity to discuss the general problem of capital valuation. He asserts that

> " . . . we could easily construct cases with as many techniques as we please, in which . . . an increase in the rate of profit entails sometimes a switch to a lower total value of capital per man and sometimes a switch to a higher total value of capital per man."

In the remainder of this section, we shall attempt to ascertain more precisely how the value of capital (per man) changes with different interest rates.

For convenience, we express everything in per capita terms; then full employment ensures that

$$a_o x(t) \quad = \quad 1$$

The value of output at time tx may now be simply written as (letting the money wage equal 1)

$$p(t)x(t) = (1{+}r)\ (p(t)Ax(t) + 1)$$

The value of net profits is

$$r(1 + pAx)$$

and the value of net output is

$$1 + r(1 + pAx)$$

Net savings must equal net investment,

$$S(t) = g(p(t)Ax(t) + 1)$$

We assume that a certain fraction of workers' income is saved, s_w, and a certain fraction of capitalists' income is saved, s_c, so that

$$s_w + s_c r\ (1 + pAx) = \quad g(pAx + 1)$$

or

$$(1 + pAx)(g - s_c r) = s_w$$

since $1 + pAx > 0$, this means that if $s_w = 0$, $g = s_c r$, but otherwise, $g > s_c r$. If $g = s_c r$, to find the value of capital we shall have recourse to an alternative method of solving for the value of capital per man which will be described below, in the discussion of a model with wages paid at the end of the period. (See page). If $g \neq s_c r$, we can easily solve for the value of capital per man

$$pAx = \frac{s_w - g + s_c r}{g - s_c r} = \frac{sw}{g - s_c r} - 1$$

and the value of output per man

$$px = \frac{(1+r)s_w}{g - s_c r}$$

It is easy to see that, for fixed s_c and r, the value of capital per man increases with increasing s_w (as one might expect) since

$$\frac{\partial\, pAx}{\partial\, s_w} = \frac{1}{g-s_c r}$$

while for fixed s_w and r, the value of capital per man increases with increasing s_c (also as one might expect), since

$$\frac{\partial\, pAx}{\partial\, s_c} = \frac{s_w r}{(g-s_c r)^2}$$

But what is somewhat unexpected is that for fixed s_w and s_c, the value of capital per man increases with increasing r, since

$$\frac{\partial\, pAx}{\partial\, s_c} = \frac{s_w s_c}{(g-s_c r)^2}$$

We shall now investigate the case where wages are paid at the end of the period of production.

The value of gross output is now (in balanced growth) in per capita terms

$$px = (1+r)(pAx) + 1$$

The value of net output is now

$1 + rpAx$

Setting savings equal to investment, we now have

$s_c rpAx + s_w = gpAx$

or

$pAx\ (g - s_c r) = s_w$

Since $pAx > 0$, this means that if $s_w = 0$, $g = s_c r$, but otherwise $g > s_c r$. In the latter case,

$$pAx = \frac{s_w}{g-s_c r}$$

which is exactly one greater than the value of capital per man derived in the previous case. Accordingly, all the partial derivatives remain unchanged and all the qualitative properties are identical.

The value of output per man can easily be derived by substituting the derived value of capital per man expression into the value of output per man equation:

$$\frac{(1+r)s_w}{g-s_c r} + 1$$

We now turn to the problems which arise when $s_w = 0$. One method of solving for the value of capital per man in this case is to introduce explicit demand conditions.

Since we are interested in balanced growth paths, we can confine ourselves to straight line Engels curves (logarithmic utility functions.) For workers, this means that

$$p_i(t)l_i(p(t)) = m_i(p(t))$$

where l_i is the consumption of good i at time t, given prices p(t) (Income for workers being fixed at 1.) $m_i(p(t))$ is then the proportion of income spent by workers on the ith commodity, given prices p(t). m(p) is homogenous of degree 0. Moreover, since we assume that all wages are consumed, we have

$$p\ 1(p) = 1 = \Sigma mi(p)$$

(1(p) is the column vector whose ith element is $1_i(p)$)

Similarly, we have for capitalists

$$p_i(t)n_i(p(t)) = q_i(p(t)) \times \text{profits}$$

where n_i is the consumption of the ith good given prices p(t) and q_i is the proportion of profits (as a function of the prices) spent on commodity i. We observe that

$$p\ q = 1 - s_c$$

where q is the column vector whose ith element is $q_i(p)$

If we define the following matrix

$$Z = A + \frac{1\ (p) a_o}{1 + g} + \frac{q(p) rpA}{1 + g}$$

it is easy to show that[1]

$$p = (1+r)pA + a_o = (1+g)pZ$$

$$x = (1+g)Ax + 1(p) + q(p)rpAx = (1+g)Zx$$

have a solution x*, p*, g*, $r^*=g^*/s_c$, and a value of capital

$$p^*Ax^* = \frac{p^*x^* - 1}{1 + r}$$

As we move from one technology to another, what happens to the value of capital depends on what happens to p*x*, and there is no

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[1]Cf. M. Morishima, "Economic Expansion and the Interest Rate in Generalized Von Neumann Models," Econometrica, 1960, pp. 352-363.

a priori basis on which to believe that p*x* of higher interest rate technologies is greater or less than p*x* of lower interest rate technologies, as we shall see more explicitly in the following section.[1]

This discussion also serves to illustrate two basic difficulties in comparing values of capital at different interest rates:

1. For any given technology, and given consumption functions, r is uniquely determined. Moreover, if we admit several technologies, keeping s_c and our consumption functions fixed, there are only a discrete number of interest rates for which the value of capital can be compared.

2. The commodity bundle of outputs is different in equilibrium for different r. (This is also true in general even if there exist differentiable production functions.)

It may be of interest to note what happens to the value of capital per unit of output of a given industry, as the interest rate changes. We shall now show that the value of capital per unit of output (measured in wage units) increases (decreases) at a switch point for good i as the labor requirement for good i is greater (less) in the first technology than in the second.

The value of capital per unit of output measured in wage units in the different industries is given by the vector

$$pA$$

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[1]We could work out similar results for the Leontief model presented above, in which wages are paid at the beginning of the period. Note that if we are given demand functions, r will be uniquely determined, and our comparative equilibrium dynamics statements, about, for instance, the change in the value of capital with a change in the interest rate, for fixed s_w and s_c, will no longer be meaningful.

and the change in the value when the technology changes to B is

$$pB-pA$$

But from section w, we know that

$$p(B-A) = a_o-b_o.$$

Hence the ith component of p(B-A), which is the change in the value of capital per unit of output in the ith industry as the economy switches from B to A, is equal to $b_o^i-a_o^i$, the difference in labor requirements.

Part III.

Double Switching and the Steady State Consumption Per-Man Perversity

In the recent discussions on the "switching of techniques" two different, but related, problems have been raised:

A. Double Switching Perversity: the use of a given technology at two distinct interest rates with at least one other technology used as some interest rate in between.

B. The Steady State Consumption Per Man Perversity: a higher consumption per man in a steady state with a higher interest rate.

Much disputation has arisen over the confusion of these two problems. Professor Robinson, for instance, in her note "The Badly Behaved Production Function" [7] asserts that it is the second perversity "that there has been all the fuss about." Yet Levhari's paper [2] over which the recent controversy has arisen, is solely about the first problem.

The perversities are, of course, related. The existence of the first perversity is, except in a singular case,[1] a sufficient condition for the existence of the second perversity. In this section I investigate the conditions under which the first perversity is a necessary condition for the second; for unless it is a necessary condition, theorems showing conditions under which double switching is impossible have no direct bearing on the consumption per man perversity.

1. I shall first show that in an economy in which there are only two technologies, e.g. that examined by Sraffa in Part III of Production

[1] It is possible that output per man be the same in the different technologies; see below for a detailed discussion.

of Commodities by Means of Commodities, the two perversities in a stationary state are exactly equivalent, i.e. the double switching perversity is a necessary and sufficient condition for the consumption per man perversity.

For simplicity, we will initially assume that there is only one consumption good (denoted by subscript 1), and no growth, so that if we let x be a column vector representing the gross outputs of the various commodities, A be the Leontief Matrix for the economy, a_o the vector of labor requirements per unit of output, c the level of the one consumption good and e be the column vector

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ . \\ . \\ . \end{pmatrix}$$

we can immediately write the output equals consumption plus investment equation as

$$x = Ax + ce$$

or

$$x = c(I-A)^{-1}e$$

i.e. x is equal to a constant times the elements of the first column of the inverse of I-A.

But denoting the labor force by 1,

$$a_o x = 1$$

or

$$c = \frac{1}{a_o(I-A)^{-1}e}$$

Recall that the pricing equation is

$$p(r) = (1+r)a_o(I-(1+r)A)^{-1}$$

where

we follow the convention that the money wage rate is equal to 1.

Hence

$p(0) = a_o(I-A)^{-1}$

and in particular

$p_1(0) = a_o(I-A)^{-1}e$

Thus we have shown that

$c = 1/p_1(0)$ = real wage

the level of consumption for any given technology is equal to the reciprocal of the price of the consumption good at zero interest rate (for a stationary state); i.e. the real wage.

We know from the nonsubstitution theorem that at any interest rate, the technology which is used is that which minimizes (all) prices, so that in particular, at the zero interest rate, of the two technologies, we use the one with the lower p_1, i.e. a higher c. At the first switch point, then, the economy must go to a lower consumption per head; if and only if there exists another switch, from the second technology back to the first, can consumption per head rise.

More generally, let the growth rate be g, and let b denote the market basket of consumption goods in which we are interested, and c denote the number of such market baskets.

Then

$$c = \frac{1}{a_o(I-(1+g)A)^{-1}{}_o}$$

But since

$p(g) = a_o(I-(1+g)A)^{-1}$

$p(g)b = a_o(I-(1+g)A)^{-1}b$

so that

$c = 1/p(g)b$

the level of output per man (in number of market baskets) is equal to the reciprocal of the cost of one basket at an interest rate equal to the rate of growth. But at g, every price in one technology exceeds the corresponding price in the other, so that

$p_A b > p_B b$ (or <)

independent of the particular composition of the market basket.[1]

But now, the first switch point at an r higher than g must entail a lower consumption per head, and it is only on the "double switch" that we obtain an output per man perversity at an <u>r greater than g</u>. But at the first switch at an r less than g, consumption per head must be lowered, i.e., we have the consumption per man perversity. This of course is not unexpected: it is just the Golden Rule.

Hence, although the double switching perversity is independent of the rate of growth and the market basket, the presence of the consumption per man perversity depends on the rate of growth, although it too is independent of the particular market basket chosen. The following diagrams will help illustrate what is meant. In figure 1, we have the "usual" technology: one switch only, which occurs at an interest rate of r*. Then, if g < r*, there is no consumption per man perversity, since consumption per man falls as we go from the low interest rate technology to the high interest rate technology; if g > r*, consumption per man changes perversely at the switch point; and if g = r*, at the

[1]Note what all this implies for the singular case referred to on page 1; the two technologies have the same consumption per man if and only if there is a switching point at the rate of interest equal to the rate of growth.

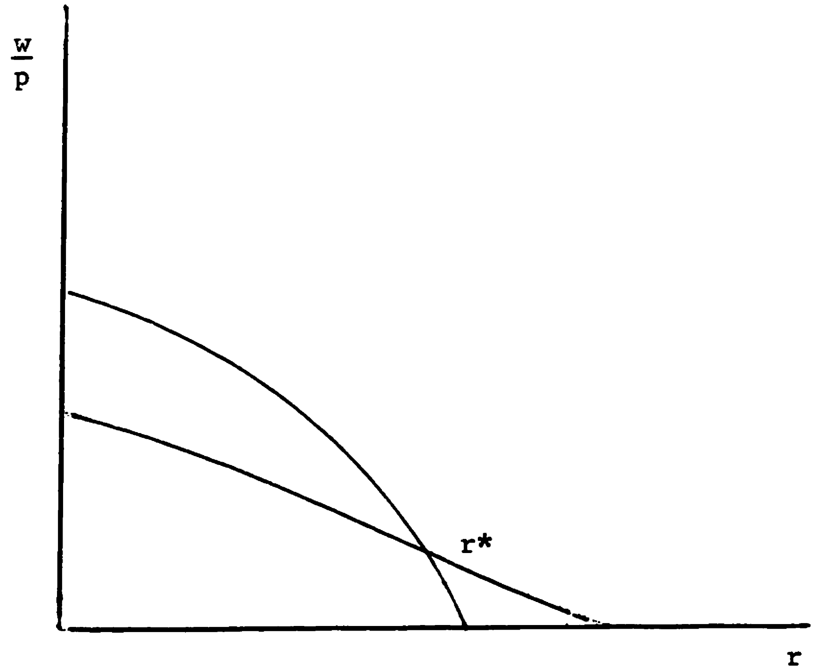

Figure 1

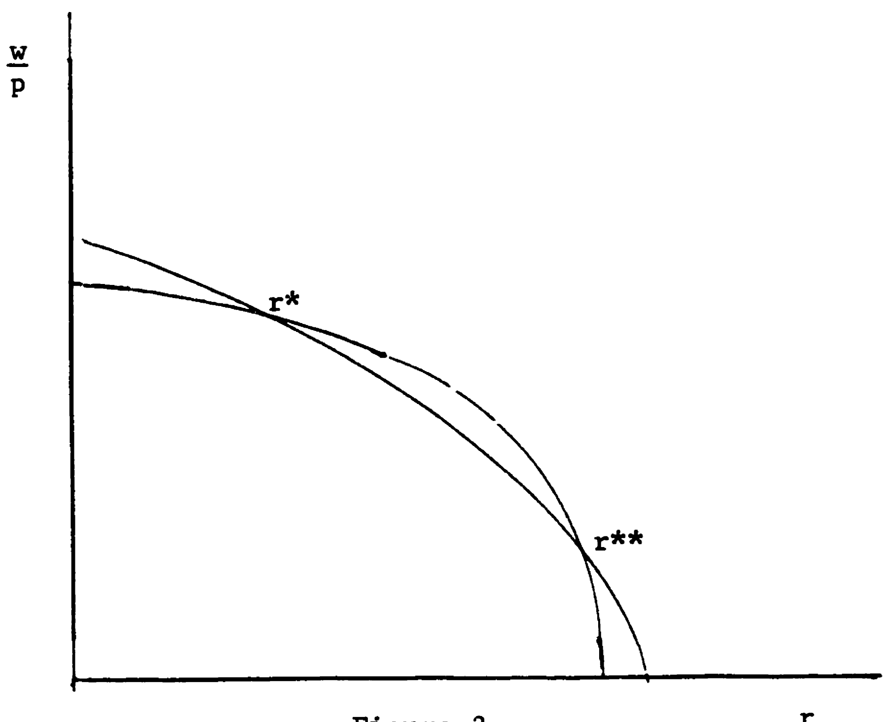

Figure 2

switch point output per man remains unchanged.

In figure 2, we have a technology with a double switch, at r^* and r^{**}. If $g < r^*$, the switch at r^* is normal, that at r^{**} perverse. If $r^*<g<r^{**}$ the switch at r^* is perverse, that at r^{**} normal. If $g>r^{**}$ the switch at r^{**} is perverse, that at r^* is normal; if $g = r^{**}$ or $g = r^*$, consumption per man remains constant.

2. Unfortunately, many of these results do not carry over to economies with more than two technologies. It is still true that the consumption per man is equal to the cost of the particular market basket of goods at prices corresponding to an r equal to the rate of growth. Moreover, it is also still true that the technique actually in use at that r unambiguously minimizes the cost of any market basket of goods, and hence of all technologies has the highest consumption per man (The "Golden Rule") and hence it is still true that the first switch at a higher i interest rate must be normal, and the first switch at a lower interest rate must be "perverse."[1]

But, except for these two "particular" switches (the immediate adjoining ones) whether or not a given switch is perverse depends on the particular market basket of goods in question, i.e. on demand functions. This is because for any two technologies (except the one actually used) it is not true in general that at $r = g$[2].

$p_A(g) > p_B(g)$ or $p_B(g) < p_A(g)$ Hence, because the non-substitution theorem only tells us that the prices for

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[1]If such switches exist.

[2]If the switch between A and B occurs at $r = g$, then $p_A(g) = p_B(g)$ and if A is used at $r = g$, $p_A(g) \leqslant p_B(g)$

the technology actually used are unambiguously lower than for all other technologies, it is in general possible to find two market baskets of goods, b* and b**, such that the cost of b* in A prices is greater than in B prices, and conversely for b**, i.e.

$$p_A(g)b^* > p_B(g)b^* \quad \text{and} \quad p_A(g)b^{**} < p_B(g)b^{**}$$

and accordingly, for b* a switch from A to B entails a reduction in consumption per man, while for b** a switch from A to B entails an increase in consumption per man. Thus, in this more general case, double switching is not necessary for the consumption per man perversity; and moreover, for economies with the same set of technologies the per man perversity may or may not occur depending on the market basket of goods (i.e. on "demand").

Assuming that there is only one consumption good assumes away, of course, this fundamental difficulty; it is as if we have already specified the demand function since we are explicitly specifying the market basket. In this case, we can order the technologies in order of consumption per man, which will be the inverse order of the price of the one consumption good at an interest rate equal to the rate of growth. Consider two technologies. A and B which switch at an interest rate of r'; let us assume that the switch in the consumption per man sense is "perverse." Then, if r' is greater than g, at some interest rate between r' and g, the price of the consumption good given by the two technologies must be the same. This, of course, is a much weaker requirement than that of a double switch.

3. At a switch point, what happens to the value of capital per man? Values in terms of our market basket of goods, b, capital per man is simply

$$\frac{pAx}{pb}$$

Writing the national income identity in two ways, as the sum of investment plus consumption, and as the sum of factor payments, we have

$$px = (1+g)pAx + cpb$$

$$px = (1+r)(1+pAx)$$

so that

$$pAx = \frac{pbc - (1+r)}{r - g}$$

so that at a switch point

$$\frac{pAx - pBy}{pb} = \frac{c_A - c_B}{r - g}$$

Hence, in the usual case of $r > g$ the value of capital per man (at constant prices) increases or decreases when switching from a technology which dominates at a lower interest rate to one which dominates at the higher interest rate as the output per man increases or decreases, and either is possible. This may be termed the Real-Wicksell effect.

Part IV

Patterns of Capital Accumulation

The exitence of the two types of perversity raises serious questions for the "optimality" of competitive equilibrium. If we have an economy with a double switch denoted by ABA, do we, for instance, skip the intervening technology if we wish eventually to be in A? And if so, how can the competitive market perform this feat? Mr. Champernowne, in his 1953 [1] article where he originally showed the possibility of double switching, suggested that in fact the intervening technology would be skipped and that in the transition two price systems would exist side by side. If this is correct, can competitive behavior be reconciled to it? If not, what does happen?

1. <u>An Example</u>

Let us begin with an example, by which we can illustrate all the basic propositions in which we are interested:

a. There exist efficient paths which go through B.

b. There exist optimal paths which go through B.

c. There exist efficient paths which do not go through B.

d. There exist optimal paths which do not go through B.

e. There exist efficient and optimal paths which continually oscillate between A and B.

f. A competitive economy can skip technologies.

g. In general, only one of the balanced growth paths is efficient.

To illustrate these propositions, we consider the following elementary example proposed by Champernowne. We consider an economy in which there exists two processes for producing drink, one of which requires chemicals and no labor and the other of which requires drink and labor. The Leontief matrices for these two processes may be represented as follows:

$$A \quad \begin{matrix} 0 & 1 \\ 0 & 0 \\ 1/60 & 0 \end{matrix} \qquad\qquad B \quad \begin{matrix} 1/98 \\ 40/98 \\ 0 \end{matrix}$$

where the first row are the labor requirements per unit of output, the second row the drink requirements, and the third row the chemical requirements. The first column of A represents the production of drink, the second column in the A technology represents the reduction of chemicals.

In other words, the first (A) technology, with a unit of labor in one period, makes 1 unit of chemical, and then with that unit of chemical makes 60 units of drink in the following period. B takes 4o units of drink with 1 unit of labor to produce 98 units of drink.

We consider an economy which initially has 1 unit of chemical and 9 units of labor available in every period. Neither drink nor chemicals can be stored. In Table 1 we set forth three alternative "histories" of this economy. We wish to show that all three paths are efficient. By definition, an efficient path (along which consumption at time t is given by c^t,) is one such that there does not exist another path, starting with the same initial conditions (along which consumption at time t is given by c'^t,) such that

$$c^t < c'^t \qquad \text{for any} \quad t < T$$

and for which $c^t = c'^t \qquad t \geqslant T$

In other words, if consumption is increased in any period, it must be decreased in some other period. Consider the first path, denoted C_A. Increase consumption in any period to 61 (say), to do this, we must use not only the chemical-process (A), since its maximum production is 60, but also the drink-process (B). But to produce with the drink process, we must use drink from the previous period as an input, i.e. consumption of drink in the previous period must be less than 60. Hence C_A is efficient.

To show that C_{ABA} is efficient, we observe that if we increase the consumption the first period, to say 21, we will only have 39 units

ALTERNATIVE PATHS FOR ECONOMY WITH TWO PROCESSES

Path	C_A				C_{ABA}					C_{ABABA}					
Period	0	1	2	3	0	1	2	3	4	0	1	2	3	4	
L_A	1	1	1	1		1		1	1		1		1	1	
L_B							1					1			
Ch	1	1	1	1	1	1		1	1		1		1	1	
D_A		60	60	60		60	60	0	60		60	60		60	
D_B							98	0				98			
Cons.		60	60	60		20	158	0	60		20	158	0	20	158 0 ....

L_A labor used in A process

L_B labor used in B process

Ch output of chemicals

D_A output of drink by A process (i.e. by chemicals)

D_B output of drink by B process

Cons consumption

of drink available the next period to use in the drink-process production, and hence the maximum output of drink (and the maximum consumption) in the following period is less than 156. There is no way in which production (and consumption) of drink can exceed 158. If consumption is to exceed the 0 level of the third period, either some chemical must be produced the preceding period or some drink must not be consumed. In the former case, some labor will not be available for use in the drink process, so consumption will have to be less than in the previous period (158), while in the latter case, although output is not reduced, consumption clearly will be.

The path ABABA, since it consists simply of repetitions of the first part of the ABA path can similarly be shown to be efficient.

To prove that these paths may be optimal, all we need to do is show that there exist utility functions for which the three paths are, respectively, optimal. For path C_A take $U(C,t) = e^{-1/3t}\, U(c)$ where $U'' < o$. Then C_A with r equal to 1/3 is the unique path satisfying the necessary conditions for the optimal path: $-\frac{\dot{u}''t}{u't} = r - 1/3$. It is not difficult to show that C_A is in fact optimal.

For path C_{ABA} let $U(c,t)$ be $U(c)|\sin(\pi\ \tau/2)|\ t \leqslant 4$

$U(c)e^{-1/3t}$ $\quad t > 4$ and for path C_{ABABA} let $U(c,t)$ be $e^{-\delta t}U(c)|\sin(\pi\ t/2|$

Hence, propositions b, d, and e are confirmed.

The following example demonstrates that these paths can also be optimal for utility functions with positive time preference. Consider the following utility function

$U(c,1) = C$

$U(c,2) = .99C$

$U(c,3) = o1\ C$

$U(c,t) = .0001e^{-1/3c}t \geq 4$

Since almost all the weight is on the first two periods, we attempt to maximize the sum of consumption over those two periods (approximately) and this clearly is given by ABA.

A utility function of the following form can similarly be shown to be optimal for a path of the type ABABA

$U(c,3n+1) = .0001^n\ C$

$U(c,3n+2) = (.0001 - e)^n C$

$U(c,3n+3) = .01 \times .0001^n C$

To see how a competitive market can "skip" the intervening technology we must introduce Walrasian markets, in which contracts for production and consumption are made for all periods in the future. (Since no change, either with respect to the technology or to utility functions will occur, all decisions may as well be made at one time as sequentially.) The "Auctioneer" calls out the price of drink in successive periods; since we are not interested at present in the stability of the market, let us simply have the auctioneer call off all p_{it} where p_{it} is the price of the ith commodity at time t, $p_{it} > 0$. As usual, producers and consumers then announce their production and consumption plans. Let us assume that $U(c,t) = U(c)\ e^{-1/6t}$, and, as before, the economy at t=0 is in a balanced growth path using the chemical process to produce drink, at an interest rate of 30%.[1]

Then it is easy to see that when the Auctioneer calls off the prices corresponding to an interest rate of 1/6 the consumers and producers are in equilibrium; we have skipped technology B.

The phenomenon of efficient paths skipping technologies is not confined merely to economies with only two technologies, as we can illustrate by the following example.

In our economy for producing drink, let us introduce a new, third technique, which we will call technique Z. It takes 44 4/9 units of drink, and one unit of labor, to produce 103 4/9 units of drink. The Leontief matrix for this technology Z can be written as (keeping the same row designations as earlier)

$$\frac{1}{103\ 4/9}$$

$$\frac{1}{44\ 4/9}$$

$$0$$

Hence in steady state it produces 59 units of drink. The price equation for this technology is

$$p(r) = \frac{(1+r)}{59 - 44\frac{4}{9}\,r}$$

If we call this technology Z, the chemical process for producing drink A, and the "old" drink process B, then we obtain the following interest rates where the pairs of technologies have the same real wage.

[1]The reader may think of the following situation. Until time 0, the utility function had been $U(c)e^{-.3t}$. Then for some reason the rate of pure time preference changes. We must now plan for the future.

EFFICIENT PATHS IN A THREE-PROCESS ECONOMY

BƵA

Period	0	1	2	3	4
L_A				1	1
L_B	1	1			
L_Z			1		
Ch				1	1
D_A					60
D_B	98	98			
D_Z			$103\frac{4}{9}$		
Cons.	58	$52\frac{5}{9}$	$103\frac{4}{9}$	0	60

BA

Period	0	1	2	3	4
L_A				1	1
L_B	0	0	0		
L_Z					
Ch				1	1
D_A					60
D_B	98	98	98		
D_Z					
Cons.	58	58	98	0	60

AAA

Period	0	1	2	3
L_A		1	1	1
L_B				
L_Z				
Ch	1	1	1	1
D_A		60	60	60
D_B				
D_Z				
Cons.		60	60	60

ABA

Period	0	1	2	3	4
L_A		1		1	1
L_B			1		
L_Z					
Ch	1	1		1	1
D_A		60	60		60
D_B			98		
D_Z					
Cons.		20	158	0	60

ABƵA

Period	0	1	2	3	4	5
L_A		1			1	1
L_B			1			
L_Z				1		
Ch		1	1		1	1
D_A		60	60			60
D_B			98			
D_Z				$103\frac{4}{9}$		
Cons.		20	$113\frac{5}{9}$	$103\frac{4}{9}$	0	60

AƵA

Period	0	1	2	3	4
L_A		1		1	1
L_B					
L_Z			1		
Ch	1	1		1	1
D_A		60	60		60
D_B					
D_Z			$103\frac{4}{9}$		
Cons.		$15\frac{5}{9}$	$163\frac{4}{9}$	0	60

AZBA

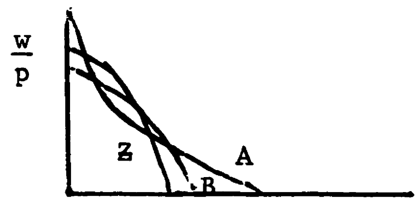
Period	0	1	2	3	4	5
L_A		1			1	1
L_B				1		
L_Z			1			
Ch	1	1			1	1
D_A		60	60			60
D_B				98		
D_Z			$103\frac{4}{9}$			
Cons.		$15\frac{5}{9}$	$123\frac{4}{9}$	98	0	60

A - B at r=.20 and r=.25 (A preferred to B r <.20 and r > .25)

A - Z at r=.229 and r=.10 (A preferred to Z r < .229 and r > .10)

B - Z at r=.225 (B preferred to Z r > .225)

so that we can construct the frontier as follows:



where A is the "chosen" technology for r ≥ .25, B for .25 ≥r≥.225 Z for .225 ≥r≥.10 and A for r ≤.10.

To show that these paths are efficient, we reason exactly as we did above for the two process economy. Consider, for example, the path BCA. To increase consumption in period 1, we must decrease it in period 0 and/or decrease it in period 2. In the former case we increase our output in the first period by using process Z, which requires a larger input of drink; and in the latter case we decrease our "savings" of drink by using process Z in period 2 less intensively or by using process B in period 2. To increase our consumption in period 2, we must use both the A process and the Z process in that period, (or the A process and the B process). Either case requires us to reduce the output and consumption produced by the B process in period 1. (To view this another way, let us keep consumption in period 1 fixed, increase output and consumption in period 2 by 1 unit using A process; this requires us to reduce output by B process in period 1 by 1/6o, and hence output produced by the Z process in period 2 is reduced by $\frac{103\ 4/9}{44\ 4/9} \times \frac{98}{6o}$ which is clearly greater than 1.)

Finally, if we increase consumption in the third period, we must either have "saved" some drink for the previous period, and thus reduced its consumption directly, or we must have some chemical from the previous period, and thus reduced the output of drink the previous period, and hence the level of consumption.

2. Generalizations

Let us now generalize these results: the questions which we are interested in are the following:

1. Does there exist at least one efficient path of the type ABA?[1]

2. Can a competitive[2] economy generate a path of type ABA?

3. Is the rate of return equal to the market rate of interest along an efficient path?

4. If we have a pure time discount utility function, is a path of type ABA ever optimal?

To answer these four questions, we must first notice that the linear technology (with or without double switching) satisfies the Malinvaud technological assumptions. [3]

It is trivial to show that the additivity, divisibility, and convexity assumptions are satisfied (Malinvaud assumptions 1, 2, 3.) Moreover, assumption 4, that if it is possible to obtain an output of y from an input vector (including labor) of x, then it is also possible to obtain it from any vector x', such that $x' \geq x$, is satisfied.

---

[1]Here as elsewhere, a path of type ABA is a path which begins in A technology, passes into the B technology, and then passes back into the A technology.

[2]Throughout this section, we shall consider only a competitive economy with complete futures markets (and complete certainty), so that all contracts for future production and consumption are made today. (In particular, it should be noted that the savings rate is not predetermined.)

Thus all of Malinvaud's technological assumptions are fulfilled. In the analysis we shall also need the fact that there is some decentralization of production that is efficient, since all firms have the same production function.[1]

Under these assumptions, Malinvaud has shown that associated with any efficient path there is at least one sequence of non-negative vectors (interpretable as discounted prices and wages) such that, for all T

$$\sum_{c=o}^{c=T} (-p^t \bar{c}^t + w^t)$$

is minimal for all paths for which

$$c^t = \bar{c}^t \quad t > T$$

Moreover, provided the discounted value of capital tends to zero when T tends to infinity, these prices are those that would be generated by a competitive system.

Since we are only interested at present in paths which, after sufficiently long time, are in equilibrium in technology A, we can assume that $x_{T+1} = \bar{x}$ [2,3]

---

[1]We are primarily concerned with efficient paths, rather than optimal paths. To use Mlinvaud's results on optimal paths, we need to make his two assumptions about the preference ordering: if X is the set of all consumption paths that are preferred to a particular consumption path c*, then X is convex, and if it contains a path c', it contains all paths c such that $c \geqslant c'$. Moreover, if c is contained in X, then there is an $e > 0$ such that if $|c_{it}' - c_{it}| < e$ for all i and t, then c' is contained in X.

Malinvaud's assumption 7 is satisfied since the labor supply is fixed in every period.

[2]We consider only ABA paths such that, when B processes are no longer used $c_t = \bar{c}$, steady state consumption in A; such paths are clearly feasible, as Professor Solow has shown in [9]. Since we are only interested in showing that there exists an efficient path of type ABA, it is sufficient to consider only this case. The more general

2a. Efficiency

We shall now make use of these results. First, we must convert competitive prices, under a convenient normalisation rule, into discounted prices. Let the wage in each period be one. Then using primed p's for discounted prices, unprimed for undiscounted (competitive) prices, and superscripts for time, we have

$$p^t = (1+r^{t-1})\ (p^{t-1}A^t \qquad + \quad a_o^t)$$

(where A^t and $a_o{}^t$ denote the technologies used at time t.)

and

$$p' = p'^{t-1}A^t + w'^t a_o{}^t$$

Consider any path generated by a competitive economy, with its associated prices.[1] We have

$$c_t = -A^{t+1}x^{t+1} + x^t$$

$$p'^{t+1} = p'^t A^{t+1} + a_o^t w'^t$$

---

case can be handled in a similar manner.

Malinvaud's theorem requires of an efficient path that for all $t', \Sigma p^t c^t - w^t$ from $t = 0$ to $t = T$ be minimal at the efficient path for all paths which are identical to it for $t > t'$.

In the subsequent analysis we shall only compare the discounted value of two programs, one of which remains in A, the other of which begins in A, goes to B, and finally returns to A. For convenience, we refer to any path of the latter type as a path of type ABA. Moreover, we shall only compare their discounted values at the time T. The reason for this is that when we compare a particular path of type ABA with any other path which is identical to it after some time $t < T$ (note that $t \neq T$), we are comparing it with another path of the same type ABA.

3 (from preceding page)

We are only considering the case where steady state consumption is higher in A than in B, which, provided that there are only two technologies, in our economy, will always be true.

[1]Equality will hold for any goods which are produced, and since these are the only ones we are interested in, we ignore inequalities.

Multiplying the first equation by p'^{t} and the second equation by x^{t+1}, and adding them together we have

$$p'^{t+1}x^{t+1} - p'^{t}x^{t} = w'^{t} - p'^{t}c^{t}$$

Summing over $0 \leq t \leq T$, we have

$$p'^{T+1}x^{T+1} - p'^{o}x^{o} = \sum_{t=o}^{t=T} (w'^{t} - p'^{t}c^{t})$$

In particular, let us consider an economy with x^{o} the balanced growth proportions for technology A (for some specified demand functions.) Let us take as our vector $(p'^{t}, w'^{t}) = (p^{t}, w^{t})$, $t = 0$, the equilibrium prices generated by a rate of interest $r*$, where $r*$ is the rate of interest at which A and B "switch" along the factor price frontier, and let us consider the competitive discounted and undiscounted prices generated if $r_t = r*$ (where as before we have used as our "normalisation" rule $w_t = 1$). For the switch point equilibrium prices,[1] $p*A + a_o = p*B + b_o$, and accordingly, if $p_o^o = p*$ and $r_t = r*$, if we use B techniques instead of A, p_t remains unchanged. Consider any path which enters B along which

$$c_t = -A^{t+1}x_{t+1} + x_t$$

Since there is no joint production (and no durable capital) this can clearly be satisfied.

Then, for such a path

$$\sum_{t=0}^{t=T} (w'^{t} - p'^{t}c^{t}) = p'^{T+1}x^{T+1} - p'^{o}x^{o}$$

and we now wish to show that this is less than (or equal to)

$$\sum_{t=o}^{t=T} (w'^{t} - p'^{t}c^{*t})$$

---

[1]Those satisfying the equation

$$p* = (1+r*)(p*A + a_o)$$

where c^* is any other consumption program such that for $t > T+1$, $c^{t*} = \bar{c}$, the steady state consumption in A. But for any other program, it is true that

$$x^{*t} > A^{*t+1} x^{*t+1} + c^{*t}$$

$$p'^{t+1} \leq p'^{t} A^{*t+1} + a_o^{*t} w'^{t}$$

so that for any other program c^*

$$\sum_{o}^{T} (w'^{t} - p'^{t} c^{*t}) \gtrsim p'^{T+1} x^{*T+1} - p'^{o} x^{*o}$$

But since $x^{*o} = x^{o}$ and $x^{*T+1} = x^{T+1}$

$$\Sigma\, w'^{t} - p'^{t} c^{*t} \gtrsim \Sigma\, w'^{t} - p'^{t} c^{t}$$

Thus we have shown that there does in fact exist at least one efficient path of the type ABA.

2.b. Rates of Return

Professor Solow in his Dobb festschrift [9] paper has shown that along any efficient path from B, along which interest and prices are constant, the rate of interest is equal to the rate of return. We shall now provide an alternative proof, making use of the results we have already obtained. The rate of return is defined as that rate of interest which, for two given efficient consumption paths, makes the present value of their consumption stream the same. At a switch point, as we have already noticed, we make take

$$p^{t} = p(r=r^*) = p^*, \text{ all } t.$$

and it is clear that

$$p'^{t} = \frac{p^{t}}{(1+r^*)^{t}}$$

The rate of return is that value of R for which

$$p^*c^o + \frac{p^*c^1}{(1+R)} + \ldots\ldots + \frac{p^*c^t}{(1+R)^t} \ldots\ldots = p^*\bar{c}^o + \frac{p^*\bar{c}^1}{(1+R)} \ldots\ldots \frac{p^*\bar{c}^1}{(1+R)^t} \ldots\ldots$$

In particular, we are interested in two paths, one of which has

c^t = steady state consumption per head in technology A.

The other path has after some T , $\bar{c}^t$ = steady state consumption per head in technology B.

The value of the consumption program c until time T is

$$-p'^{T+1}x^{T+1} + p'^{o}x^{o} + \Sigma\, w'^{t}$$

While the value of the consumption program $\bar{c}$ until time T is

$$-\bar{p}'^{T+1}\bar{x}^{T+1} + \bar{p}'^{o}\bar{x}^{o} + \Sigma\, \bar{w}'^{t}$$

$$w' = \bar{w}'$$

Hence

$$\Sigma p'^{t}c^{t} - \Sigma\bar{p}'^{t}\bar{c}^{t} = p'^{o}x^{o} - \bar{p}'^{o}\bar{x}^{o} - p'^{T+1}x^{T+1} + \bar{p}'^{T+1}\bar{x}^{T+1}$$

But since

$p'^{t} = \bar{p}'^{t}$ and $x^{o} = \bar{x}^{o}$, and

$$\lim_{T \to \infty} p'^{T+1} = \lim_{T \to \infty} \bar{p}'^{T+1} \to 0$$

and x is bounded above, for T sufficiently large

$$\left| \sum_{t=o}^{t=T} (p'^{t}c^{t} - p'^{t}c^{t}) \right| < e, \text{ e arbitrarily small.}$$

But recalling the definition of $p'^{t} = p^{t}/(1+r^*)^{t}$ it is immediately clear that $R = r^*$ is at least one solution to the rate of return equation.

2.c. Optimality

We have one final question to answer: if the utility function is of the form

$$U(c,t) = u(c) \quad (1+\delta)^{-\tau} \qquad \delta > r^{**} \text{ or } \delta < r^{*}$$

where U(c) is concave

(where r^{**} and r^{*} are the upper and lower switch point rates of interests, respectively,) is it optimal to go through B? In other words, if we are at a point on the factor price frontier below given by X and we wish to go (eventually) to a point Y (or vice versa) do we use B techniques in the transition? (The reason for wanting to go from X to Y may, for instance, be an unexpected change in the pure rate of time preference.)

The answer is clearly no, as may be shown as follows.[1]

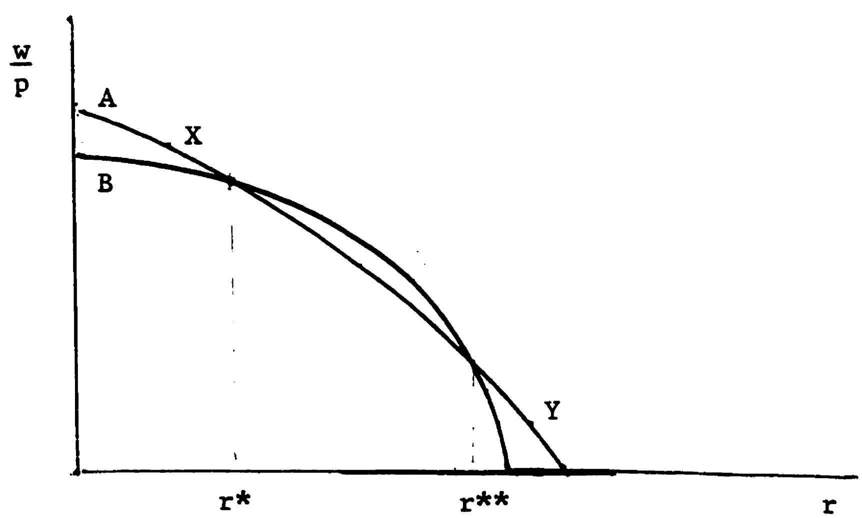


Figure 4

[1] As an aside, it may be worth observing that if it did pay to go through B once, it would pay to go through again, (and again. . .) since the time pattern of consumption would be the same from the beginning of the first passage through B to the end as it would be at successive passages.

If c^* is the A.A.A path, and c is any ABA path $\Sigma u(c^*,t) - \Sigma u(c,t)$ $\geq \Sigma u'(c)(1+\delta)^{-t}(c_t^* - c_t)$

But we already know that

$$\Sigma \frac{c^*t}{(1+r)^t} > \Sigma \frac{c_t}{(1+r)^t} \qquad \begin{array}{l} \text{if } r < r^* \\ \text{or } r > r^{**} \end{array}$$

Hence

$$\Sigma u'(c)(1+\delta)^{-t}(c_t^* - c_t) > 0 \qquad \underline{\text{q.e.d.}}^{1}$$

[1]After this chapter had been written, unpublished papers duplicating some of the results of this chapter by M. Bruno, E. Burmeister, and E. Sheshinsky, "The Nature and Implications of the Reswitching of Techniques," and M. Bruno, "Optimal Accumulation in Discrete Capital Models," were drawn to my attention.

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CHAPTER III

A TWO-SECTOR TWO CLASS MODEL OF ECONOMIC GROWTH

Introduction

This note examines the properties of a growth model with a consumption goods sector and a capital goods sector, in which there are two classes, one whose income is derived entirely from capital (the capitalists) and a second which derives its income from both wages and return on savings (the workers).

In a one sector version of this model, Meade (4) and Samuelson and Modigliani (6) have recently shown that Pasinetti's proposition (5) that in such an economy the rate of profit is equal to the rate of growth divided by the savings propensity of capitalists is true only under certain restrictive conditions.[1]

1. This is not to say that these conditions may (or may not) be satisfied in some actual economies.

I. The Model

The differential equations of capital accumulation are derived as in (6):

1. $\dot{k}_c/k_c = s_c r - n$

2. $\dot{k}_w/k_w = s_w r - n + \frac{s_w(y - rk)}{k_w}$

where s_c is the propensity to save of capitalists, s_w that of workers, r the rate of profit, n the rate of growth of the labor force, k the aggregate capital per worker ratio, k_c the ratio of capital owned by the capitalists to the size of the labor force, k_w the ratio of capital owned by the workers to the size of the labor force, and y the output per man measured in machine goods numeraire. If we let λ be the proportion of the labor force in the capital goods industry (which we denote by a subscript 1), it is clear that

$$y = \lambda y_1 + (1 - \lambda) p y_2 = w + rk$$

where y_i is the output of machines per worker in the i^{th} industry, p the price ratio and w the wage rate.

We may write the savings equal investment equilibrium condition as

3. $\lambda y_1 = s_w y + (s_c - s_w) rk_c$.

Finally, we have the usual production relations of a two sector neoclassical model: factor payments equal marginal products and full employment of labor and capital. Letting $y_i = f_i(k_i)$, $i = 1,2$, where k_i is the capital per worker in the ith industry, and $\underline{w}$ be the wage rentals ratio, we have

4. $\underline{w} = f_i/f_i' - k_i$, $f_i' > 0$, $f_i'' < 0$ $i = 1,2$

5. $\lambda k_1 + (1 - \lambda) k_2 = k$

II. Uniqueness of Momentary Equilibrium

Given k_c and k_w (and hence k), at any point of time, we wish to know whether the path (i.e. $\dot{k}_c$ and $\dot{k}_w$) is determinate; or in other words, does specifying k_c and k_w uniquely specify k_1 and k_2. If we define

$$\overline{s_c} = s_w + (s_c - s_w)k_c/k > s_w$$

our equations 3-5 become identical in form to the corresponding equations in the usual two sector growth models, with the savings-investment equilibrium condition

$$\lambda\, y_1 = (s_w \underline{w} + \overline{s}_e k)r$$

for which it is known that the following are sufficient conditions for uniqueness of momentary equilibrium: 1. The sum of the elasticity of substitutions be greater than or equal to 1; and 2. the capital intensity of the capital goods sector be less than or equal to that of the consumption goods sector.

III. Existence of Balanced Growth Paths

If there is no capitalist class, the model presented above becomes simply the usual neo-classical model with a constant fraction of income (s_w) saved. Hence any balanced growth path of the Uzawa constant savings proportion model is a balanced growth path of our two class model.

But the question we are really interested in is the existence (and uniqueness) of balanced growth paths for which $k_c \neq 0$. It is easy to see that if there exists a balanced growth path with both classes present, it is unique. A balanced growth path is one in which $\dot{k}_c = \dot{k}_w = 0$. Hence

6. $r = n/s_c$

7. $s_w y = s_w r(k - k_w) + nk_w = nk - nk_c + s_w rk_c$, or

7' $s_w y = nk + k_c(s_w r - n)$

From 6, r is uniquely determined; hence from the marginal productivity equal factor price relations k_1 and k_2 are uniquely determined, and hence y_1 and y_2 are fixed. Substituting 7' into the savings-investment equation, we have

8. $\lambda\, y_1 = nk + k_c(s_w r - n) + s_c k_c r - s_w k_c r = nk + k_c(s_c r - n) = nk$

Substituting into the full employment condition, we have

9. $\lambda = -k_2/(k_1 - k_2 - y_1/n)$

from which it is clear that λ is fixed, and since $k = \lambda\, y_1/n$, k is uniquely determined. Thus we have shown that there is at most one balanced growth path with both classes present.

There will exist such a balanced growth path if and only if the values of $\lambda*$, y_1*, y_2*, k_1*, k_2*, k_2^*, and $r*$ (where the * variables are the solutions to equations 3-9) are consistent with a positive value of k_c; i.e. from 9',

10. $s_w y* < nk* = s_c r* k*$

or

$s_w/s_c < r*k*/y*$

or using 8 we have

11. $\lambda\, y_1^* > s_w y*$

$$\frac{I*/L}{Y*/L} > s_w \qquad \text{(where I is investment)}$$

Condition 10 states that the ratio of the savings propensity of workers to that of capitalists must be less than the share of capital (in balanced growth) while condition 11 states that the investment output ratio must be greater than the savings propensity of workers.

What these restrictions imply may be seen in the following way. From 7, letting $k_c = 0$, we have

12. $\psi\,(k_1, k) = nk - s_w(w + rk)$

but since, for this model with $k_c = 0$, it is known[1] that $k = h(k_1)$, and that

---

1. Provided, for instance, $k_1 < k_2$ or the sum of the elasticities is greater than 1.

13. $\emptyset(k_1) = \psi(k_1, h(k_1)) = 0$

at only one point (and $\emptyset'(k_1)_{\emptyset=0}$ 0). The questinn then is whether the solution to 13 gives k_1 greater or less than the solution to

14. $s_c f_1'(k_1) - n = 0$

If the situation is as depicted below, then there exists an equilibrium with two classes, because at A, $s_w y < nk$. (See below for a detailed discussion of stability). (See Figure 1).

On the other hand, if the situation is as depicted below, then there can not be a balanced growth path with both classes, because at A, $s_w y > nk$. (See Figure 2).

The important point to observe is that one cannot look at merely the values of, say, the savings propensity of workers and the investment-output ratio today to tell whether there exists a balanced growth path with both classes; note that this would be true even if the capital output ratio of each sector were fixed (but different between the two sectors.)

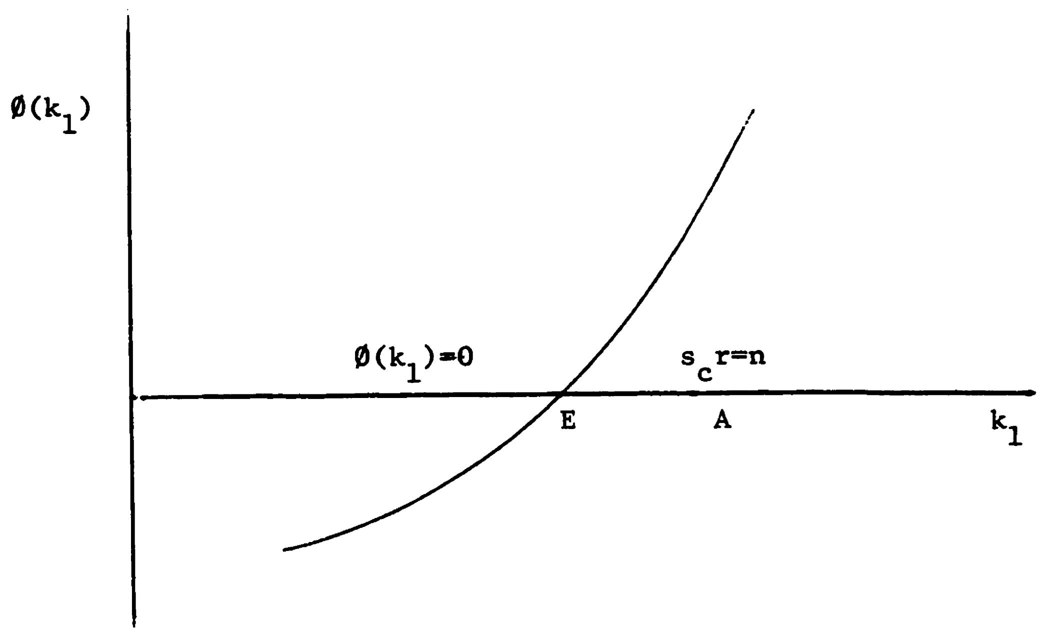

FIGURE 1

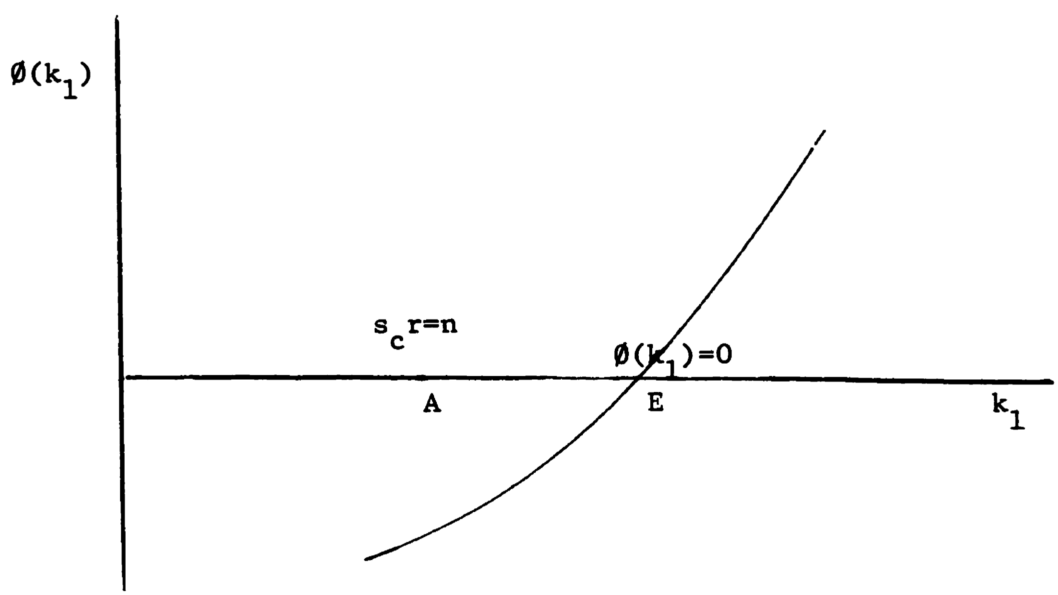

FIGURE 2

IV. Stability

If the only balanced growth path is the one-class balanced growth path (hereafter referred to as the OCBGP) then, of course, the stability properties of the model are exactly as Uzawa and Inada have analyzed them.

The interesting questions arise when there exists both a two class and a one class balanced growth path. In that case, we can ask two questions concerning the stability of the OCBGP:

(a) If k_c is constrained to be zero, is the OCBGP stable?

(b) In general, is the OCBGP stable?

The answer to the first question again is the same as provided by Uzawa and Inada. What we shall now show is that the answer to the second question is no: if k_c is ever position, then there exists a balanced growth path with $k_c > 0$, then k_c will always be positive. But on the other hand, it cannot be shown that the economy will converge to the two-class balanced growth path (TCBGP) it may oscillate around it.

First, it should be observed that under the capital intensity hypothesis the system is always locally stable.[1]

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1. Since $\dot{k}_w = s_w(f_1 - k_1 f_1') + s_w k_w f_1' - nk_w$

and $k_c = s_c k_c f_1' - nk_c$

to show local stability we must show that the characteristic roots of the Jacobian $\dfrac{\partial(k_w, k_c)}{\partial(k_w - k_c)}$ evaluated at $\dot{k}_c = \dot{k}_w = 0$ be negative.

Footnote continued.

The Jacobian, when evaluated at $k_c = k_w = 0$ is found to be

$$\begin{Bmatrix} s_w f_1' - n + s_w f_1'' \dfrac{dk_1}{dk_w}(k_w - k_1) & s_w f_1'' \dfrac{dk_1}{dk_c}(k_w - k_1) \\ s_c f_1'' \dfrac{dk_1}{dk_w} k_c & s_c f_1'' \dfrac{dk_1}{dk_c} k_c \end{Bmatrix}$$

The characteristic equation of the Jacobian is

$$\eta^2 + \left\{ -s_c f_1'' \frac{dk_1}{dk_c} k_c - s_w f_1'' \frac{dk_1}{dk_w}(k - k_c - k_1) - s_w f_1' + n \right\} \eta$$

$$+ (s_w s_c f_1''^2 k_c (k_w - k_1) \frac{dk_1}{dk_c} \frac{dk_1}{dk_w} - s_w s_c f_1''^2 \frac{dk_1}{dk_c} \frac{dk_1}{dk_w} k_c (k_w - k_1)$$

$$+ s_c f_1'' \frac{dk_1}{dk_c} k_c (s_w f_1' - n) = 0$$

Since $s_c > s_w$, $s_w f_1' - n < 0$, and since $k_1 < k_2$, $k_1 < k$. We need to show that $dk_1/dk_w > 0$ and $dk_1/dk_c > 0$. Observing that the savings-investment equality may be considered a function of k_c, k_w and k_1, we write

$H = \lambda y_1 - r[s_w(w + k) + (s_c - s_w)k_c] = 0$; then $\dfrac{dk_1}{dk_w} = \dfrac{-\partial H/\partial k_w}{\partial H/\partial k_1}$ and

$$\frac{dk_1}{dk_c} = \frac{-\partial H/\partial k_c}{\partial H/\partial k_1}$$

Footnote continued.

Since $\lambda = \frac{k-k_2}{k_1-k_2} =$, $\frac{\partial \lambda y_\omega}{\partial k_1} = \frac{y_1[(dk_2/dk_1)(\lambda-1)-\lambda]}{k_1-k_2} + \lambda r > 0$ if $k_1 < k_2$

$$\frac{\partial H}{\partial k_1} = \lambda r + y_1 \frac{\partial \lambda}{\partial k_1} - (\lambda - s_w)\bar{y}_1 \frac{d \ln r}{d k_1} > 0 \quad \text{if } k_1 < k_2$$

while $\frac{\partial H}{\partial k_w} = -s_w f_1' + \frac{f_1}{k_1-k_2} < 0$ if $k_1 < k_2$

and $\frac{\partial H}{\partial k_c} = -s_c f_1' + \frac{f_1}{k_1-k_2} < 0$ if $k_1 < k_2$

Hence all coefficients of the characteristic equation are positive; a necessary and sufficient condition for the modulus of the characteristic roots to be negative is that all coefficients of the characteristic equation be positive, and our theorem is thus proved.

Observe, however, that since we are presently only interested in local stability, all we require is that, near the TCBGP, the consumption goods sector be more capital intensive than the capital goods sector. This is, of course, a much weaker condition than the restriction that this hold for all wage-rentals ratios. What happens at interest rates other than those arbitrarily close to $r=n/s_c$ is completely irrelevant for local stability.

But the fact that the system is locally stable does not guarantee that there are not cyclical paths oscillating around the fixed point; this can only be deduced from a complete analysis of the phase diagram, and it appears that such cyclical paths are a distinct possibility. But what it does exclude is the possibility of there being cyclical paths near the fixed point, i.e. there exists a region around the fixed point in which all paths converge to the fixed point. This is what is meant by local stability; and how big this region is cannot be said without detailed information on the functions.

Secondly, it should be observed that if the OCBGP is unique and stable if k_c is constrained to equal zero, it is globally unstable if k_c is not so constrained (provided there exists a TCBGP). This follows from the fact that under the conditions stated, r must be greater than n/s_c[1], and hence k_c must be positive near the TCBGP. Moreover, since the minimum value of $\dot{k}_c=-nk_c$, if k_c is over positive, it will never become negative; hence if k_c is ever positive, k_c will

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1. See diagram 1

always be positive when the economy enters a small neighborhood of the OCBGP (if it enters it). And, clearly, once in the neighborhood, k_c increases until (k_w, k_c) is no longer arbitrarily close to the OCBGP.

We shall now prove a theorem about the global stability of the system when there exists a two class balanced growth path.[1]

1. If the capital intensity of the consumption goods sector is not less than that of the capital goods sector, then the set of limit points (as $t \to \infty$) in (k_c, k_w) phase space for a path is either the TCBGP point or a limit cycle around it, or the capital labor ratio of the path is unbounded. The Inada derivative conditions are sufficient to rule out the possibility of divergence.

2. If the capital intensity of the capital goods sector is greater than that of the consumptions good sector, all of the above results obtain provided that there exists a unique OCBGP which, in the absence of capitalists, is stable. Moreover, if the Inada derivative conditions are satisfied, the TCBGP point cannot be a saddle-point.

3. If, in addition to the conditions of 2, the TCBGP is unstable, then all paths oscillate around the TCBGP and at least one of the paths is perfectly periodic.

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1. We ignore the case where k_c is ever zero, since our model is then identical to that investigated by Uzawa and Inada.

Proof. By a theorem of Poincare and Bendixson, we know that the limiting set can only be a stable limit-cycle, a stable node, a stable focus, or a saddle point (for a non-diverging path.) Since we have only two fixed points, the OCBGP and the TCBGP points, all we need to establish for the first part of 1 and 2 is that the OCBGP is not a limit set and all limit-cycles must be around the TCBGP. The latter is easy to show, making use of Poincare index numbers. Consider a path which has a closed curve as its limit set. Take as a closed Jordan curve any path which, starting from $(k_c(t), k_w(t))$ for t arbitrarily large, goes around a full orbit to $(k_c(t+z), k_w(t+z))$ where $k_c(t+z) = k_c(t)$ and the short segment of the k_c = constant line required to join $k_w(t+z)$ to $k_w(t)$. For t sufficiently large, the initial and final points must be arbitrarily close to a point on the closed curve which constitutes the limit of the trajectory, and accordingly they must be arbitrarily close to each other. Since all our functions are differentiable, the vector $(\dot{k}_c, \dot{k}_w)$ defined at all points along the line joining the initial and final points must be approximately the same, and accordingly the index number of the Jordan curve so defined must be 1, which implies that it contains a fixed point. Moreover, it cannot contain the OCBGP, since negative values of k_c are inadmissable.

The OCBGP point is not a limiting set, since, from what we have already observed, the OCBGP point is a saddle point, with the k_w axis as separatrix; and since we have ruled out $k_c = 0$, the OCBGP cannot be a limit set.

To show that the Inada conditions are sufficient for non-divergence, we show that at all points along the lines (see figure 3), (0,D)(D,D) and (D,D)

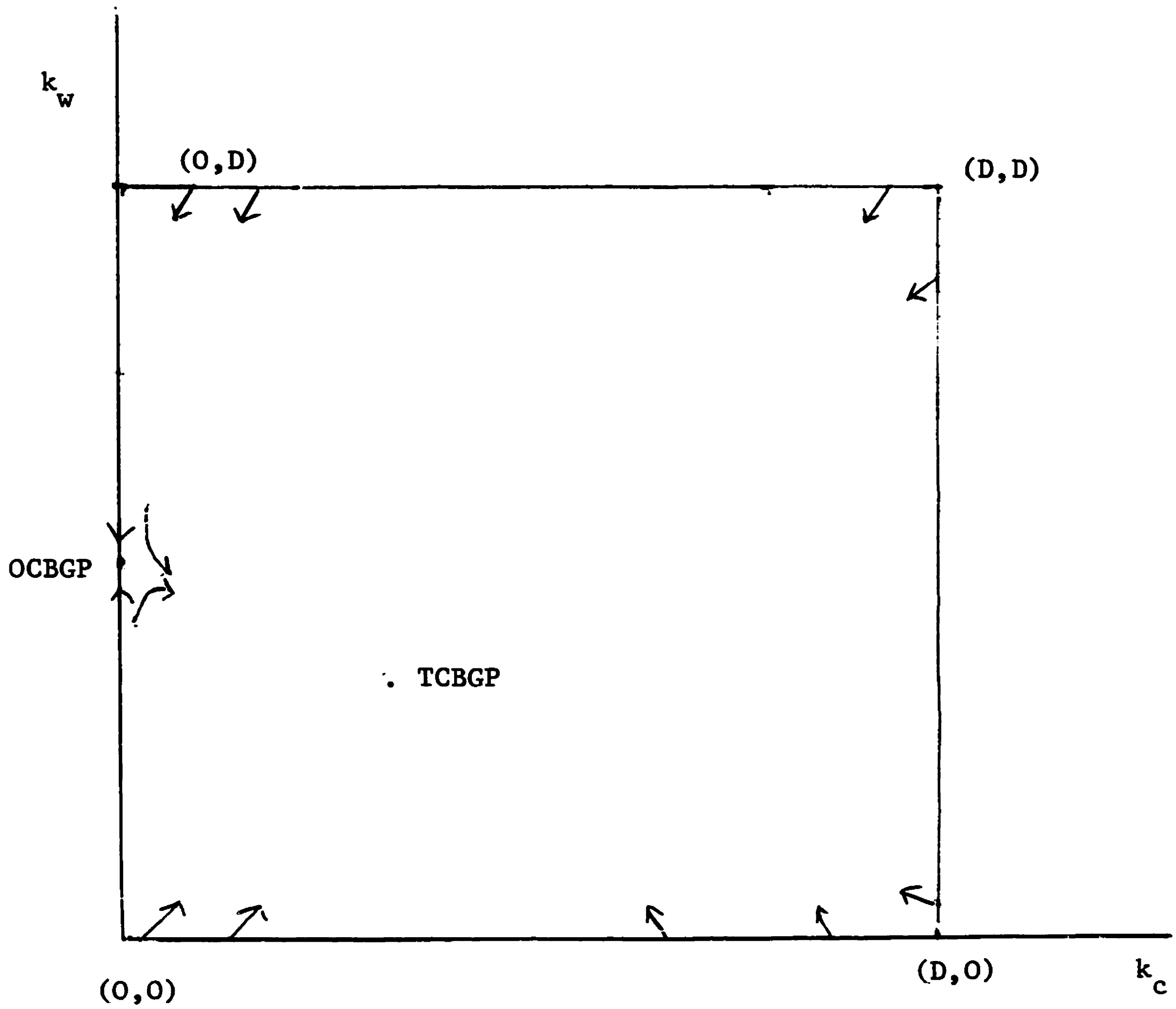

FIGURE 3

(D,O) D sufficiently large in (k_c, k_w) phase space, $\dot{k}_c, \dot{k}_w$ point inward. First, we consider $k_1 < k_2$. For $k_w=0$, it is as if we had an Inada model, with s_c of capital income saved and s_w of wage income. For sufficiently large k, r is arbitrarily small. Moreover, we have already shown that $dk_1/dk_w > 0$. Hence, everywhere along (D,D)(D,O) $\dot{k}_c/k_c = s_c r - n$ is negative. On the other hand, $\dot{k}_w/k_w = s_w w/k_w + s_w r - n < \frac{2s_w w}{k_1} + s_w r - n = 2s_w f_1' \frac{(f_1/k_1 - 1)}{f_1'} + s_w r - n$. For $k_c = 0$, we have a two sector model with $s_w = s_c$, the same fraction of wage and profit income saved, and again we know that for sufficiently large k, k_1 is arbitrarily large, and $dk_1/dk_c > 0$. And by the Inada conditions, we also know that the marginal product of capital must equal the average product as k_1 becomes arbitrarily large. Hence $\dot{k}_w/k_w < s_w r-n < 0$ along (O,D)(D,D).

If $k_1 > k_2$, (since $k_1 > k$), then, for sufficiently large D, $\dot{k}_c < 0$ along (D,D)(D,O) $k_w/k_w \frac{2s_w w}{k_2} + s_w r - n = 2s_w f_1' \frac{(f_2/k_2 - 1)}{f_2'} + s_w r - n < 0$ for D sufficiently large.

To show that the TCBGP cannot be a saddle point (under the stated conditions), we take the boundary of the square (O,O)(O,D)(D,D)(D,O) as a closed Jordan curve. From what we have already said, it is easy to calculate the Poincare index number, which turns out to be +1, and hence the (only) enclosed fixed point, the TCBGP, cannot be a saddle point.

Finally, we wish to show that if the TCBGP point is locally unstable, all paths oscillate around it and at least one path is periodic, except for the trivial cases of paths which remain forever at the TCBGP point and paths which have $k_c = 0$ forever. If we take a sufficiently small neighborhood in (k_w, k_c) phase space around the TCBGP point, and define our region of interest as the intersection of (0,0)(0,D)(D,D)(D,0) and the complement of the neighborhood, it is clear that this new region is compact,[1] and all paths[2] must enter this region and any path which enters it never leaves it. Thus the conditions of the Poincare-Bendixson theorem are satisfied: all paths have a closed curve as the set of their limit points (i.e. are limit-cycles) and at least one path is periodic.[3]

We shall now investigate the properties of these cycles. First, any periodic trajectory must have a time-average value of the interest rate equal to the rate of growth divided by the savings propensity of capitalist.[4] Moreover, the average return on capitalists' capital is

---

1. Since any neighborhood of a point is an open set, and the complement of an open set is closed and the intersection of two closed sets is closed.

2. With the trivial exceptions already noted.

3. Perhaps the difference between a general limit cycle and a periodic trajectcry (as a special case) should be emphasized. A periodic orbit has itself as its limit; while a limit cycle has a closed curve (which in general is not identical to the orbit) as the set of its limit points.

4. $$\lim_{h \to \infty} \int_{2\pi h/z}^{2\pi(h+z)/z} d\ln k_c = s_c \int_{2\pi h/z}^{2\pi(h+z)/z} r\,dt - \frac{2\pi}{z} \to 0 \Rightarrow$$

$$\lim_{h \to \infty} \frac{1}{2\pi/z} \int_{2\pi h/z}^{2\pi(h+z)/z} r\,dt \to n/s_c$$

equal to n/s_c.[1] Secondly, let IC by the closed curve representing the set of limit points for some path. Then any trajectory starting (or coming) sufficiently close to IC spirals around IC, in the sense that it is met an infinite number of times by a line through any point of IC, provided that the direction of the line is not the direction of the vector (k_w, k_c).

Thirdly, since no two trajectories can intersect (because of uniqueness of momentary equilibrium) successive limit cycles must be of unambiguously increasing amplitude.

Fourthly, if C_{i+1} is a periodic trajectory, and C_i is the next smallest in amplitude, then at most one of them can be stable (under positive time).

Fifthly, there are a finite number of limit cycles.

These cycles do not constitute a business cycle, since full employment is an emplicit assumption of the model. But the other properties of the business cycle can be observed: fluctuations in the distribution

---

1. It should be clear that the first does not automatically imply the second. For instance, it cannot be shown that the average return on workers capital is equal to n/s_c. To prove that for large h

$$\int_{2\pi h/z}^{2\pi(h+1)/z} rk_c\,dt \Bigg/ \int_{2\pi h/z}^{2\pi(h+1)/z} k_c\,dt \to n/s_c$$ observe that

$$\int_{2\pi h/z}^{2\pi(h+1)/z} s_c r k_c\,dt \qquad m \int_{2\pi h/z}^{2\pi(h+1)/z} dt = \int_{2\pi h/z}^{2\pi(h+1)/z} d\dot{k}_c \to 0$$

of income, fluctuations in output per man, fluctuations in the rate of growth of output, fluctuations in the investment-output ratio, and constant average rates of profit for varying amplitudes of the cycle. In fact, the "story" of this cycle sounds very much like some of the capital intensity versions found in the business cycle literature (minus, of course, the fluctuations in employment). But note that this cycle is generated without any rigidities or lags in the system.

A concluding word: in this paper, we have attempted to analyze the equilibrium and non-equilibrium dynamics of a two sector neoclassical model with two classes, and we have found that convergence to one of the fixed points, even under the restrictive Inada conditions, could not be guaranteed: oscillatory paths were found to be a definite possibility. But when we extend the model to more than two classes, although conditions for uniqueness of momentary equilibrium and existence of multi-class balanced growth paths remain essentially unchanged, the stability analysis (even for the one sector model) no longer goes through.

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APPENDIX I

To illustrate some of the propositions we have discussed above, we shall consider an example in which the TCBGP is locally stable even though the capital intensity hypothesis ($k_1 < k_2$) is not satisfied.

Let $y_1 = k_1$

$$y_2 = k_2^{\beta} \qquad \alpha > \beta >$$

Then $\underline{w} = k_1 \left(\frac{1}{\alpha} - 1 \right) = k_2 \left(\frac{1}{\beta} - 1 \right)$

or $k_2 = Nk_1$

$$N = \frac{\frac{1}{\alpha} - 1}{\frac{1}{\beta} - 1} < 1$$

a) Uniqueness of momentary equilibrium:

We must show that for given k_c, k_w the following equation can be solved uniquely for k_1:

$$s_w(1-\alpha)k_1^{\alpha} + k_1^{\alpha-1}) + (s_c - s_w)\, k_1^{\alpha-1} k_c - k_1 \frac{k - k_1 N}{k_1(1 - N)} = 0$$

Divide by $k_1^{\alpha-1}$ and rearrange terms:

$$k_1 = \frac{(1 - bs_w)k_w + (1 - bs_c)k_c}{s_w(1 - \alpha)(1 - N) + N} \qquad b = \alpha(1 - N)$$

b) Existence of two class balanced growth path:

We must show that for some feasible set of s_w, s_c, α and β, $k_c = k - k_w > 0$ when $r = n/s_c$:

$$k_w = \frac{s_w \underline{w}}{r(s_c - s_w)}$$

$$k = \frac{\lambda y_1}{n} = \frac{k_2}{k_2 - k_1 + (y_1/n)} \quad \frac{y_1}{n}$$

$$= \frac{Nk_1}{k_1(N - 1 + c)} \qquad k_1 c \qquad c = \frac{1}{\alpha s_c}$$

$$k - k_w = \frac{N}{N - 1 + c} \qquad ck_1 - \frac{s_w w}{r(s_c - s_w)} > 0 \qquad \text{iff}$$

$$\frac{N c k_1}{N - 1 + c} - \frac{s_w (-)}{\alpha (s_c - s_w)} k_1 > 0 \text{ iff}$$

$$\frac{N}{N - 1 + c} > d \qquad \text{where } d = \frac{s_w s_c (1 - \alpha}{s_c - s_w}$$

Since $c > 1$, $N - 1 + c > 0$, and hence the above will be true iff

$N > d(N-1) + cd$ iff

$N (1 - d) > d(c-1)$ iff

$N > \frac{d}{1 - d} (c-1)$ if $d < 1$

$N < \frac{d}{1 - d} (c-1)$ if $d > 1$

But since $d > 0$, if $d > 1$, $1 - d < 0$ and

$N < \frac{d}{1 - d} (c - 1) < 0$ which is impossible.

Hence, we need only consider the case of $d < 1$.

We must now show that it is possible for

$$1 > N > \frac{d}{d - 1} \quad 1 - c$$

All that this requires is that

$$cd - d < 1 - d$$

or $$cd < 1$$

i.e. $$\frac{s_w}{s_c - s_w} \frac{(1-\alpha)}{\alpha} < 1$$ which is surely possible.

In fact, this condition is satisfied iff $s_w/s_c < \alpha$, since

$$1 = \frac{1-\alpha}{\alpha} \quad \frac{\alpha}{1-\alpha} = \frac{s_w}{s_w/\alpha - s_w} \frac{1-\alpha}{\alpha} > \frac{s_w}{s_c - s_w} \quad \frac{1-\alpha}{\alpha}$$

(Observe that if $s_w/s_c <$ share of capital, $s_w/s_c < \alpha$, since $\beta < \alpha$).

c) Stability: The TCBGP is locally stable.

First, we observe that the constant term in the characteristic equation is positive, since $f_1'' < 0$ and (near the TCBGP) $s_w r - n < 0$. Moreover, $dk_1/dk_c = (1-bs_c)/H > 0$ where $H = s_w(1 - \alpha)(1 - N) + N$ since $N < 1$ and $bs_c < 1$. (This rules out the possibility of the TCBGP being a saddle point.)

To show that the TCBGP is locally stable, we must now show that (near the TCBGP) $- s_c f_1'' \frac{dk_1}{dk_c} k_c - s_w f_1'' \frac{dk_1}{dk_w} k_w - k_1) - s_w f_1' + n >$ 0, i.e.

$$\frac{(1-\alpha)s_w(1-bs_w)}{H} \gtrless s_c - s_w + (1-\alpha)\frac{s_c(1-bs_c)k_c + s_w(1-bs_w)k_w}{(1-bs_c)k_c + (1-bs_w)k_w}$$

a sufficient condition for which is

$$(1 - \alpha)s_w(1-bs_w) < H(s_c - \alpha s_w)$$

But from the requirement that $k_c > k_w$, it can be shown that $H > s_w / \alpha s_c$

Hence

$$H(s_c - \alpha s_w) > \frac{s_w(s_c - \alpha s_w)}{s_c} > s_w(1-\alpha)(1-bs_w)$$

The latter inequality is true iff (rearranging terms)

$$N < \frac{1-\alpha(1-\alpha)}{s_w \alpha^2(1-\alpha)} - \frac{(1-\alpha(1-\alpha)s_c)}{s_c \alpha(1-\alpha)}$$

which it surely will be if the right hand expression is greater than 1. But the right hand expression is equal to

$$1 + \frac{(1-\alpha(1-\alpha))}{s_w \alpha^2(1-\alpha)} - \frac{1}{s_c \alpha(-\alpha)}$$

which is greater than 1 if the sum of the last two expressions is positive;

$$1 - \alpha(1-\alpha) - \alpha s_w/s_c > 1 - \alpha(1-\alpha) - \alpha^2 = 1 - \alpha > 0$$

since $s_w/s_c < \alpha$.

It is worth observing that the Bendixson criterion[1] (which is a sufficient condition for the non-existence of oscillatory paths) is not satisfied here. We have already shown that $\frac{\partial \dot{k}_c}{\partial k_c} + \frac{\partial \dot{k}_w}{\partial k_w} < 0$ near the TCBGP point. We shall now show that there are other points with $k_w > 0$, $k_c > 0$ for which the above sum is positive. Take, for instance, a point where k_c, although positive, is arbitrarily small. Then

1. The Bendixson criterion requires that the sum of the diagonal terms of the Jacobian of the system be one signed. See, for instance, (1).

$$\frac{\partial \dot{k}_c}{\partial k_c} + \frac{\partial \dot{k}_w}{\partial k_w} \approx s_w (1 - \alpha) \; \alpha \frac{k_1^{\alpha}}{H} \quad (k-k_1)(1-bs_w) -$$

$$s_w \alpha k_1^{\alpha - 1} + n = -s_w k_1^{\alpha - 1}[(1-\alpha)(z-1)k \quad +1] + n$$

where $z = (1 - s_w b)/H$.

which, even for fairly small values of k, can clearly be positive.

For instance, if $k=5.13$, $s_w=.05$, $s_c=.95$, $\alpha = .1$, $N = .5$, then the above expression is equal to $n-.0279$, positive for values of n greater than 2.79%. (It is easy to verify that, in this example, there exists a TCBCP).

II.

We now consider an example where the TCBGP is locally unstable. We let there be fixed coefficients in both sectors, with the capital goods sector being more capital intensive than the consumption goods sector. It will be convenient in much of the following discussion to use a numerical example. We let $k_1 = 5$, $y_1 = 1$, $s_c = .550$, $n = .1$, $k_2 = .556$ and $s_w = .366$. We can then easily show that in this model momentary equilibrium is always uniquely determined, since given k_c and k_w we can (from the savings-investment equation) uniquely solve for r:

$$r = \frac{\frac{k - k_2}{k_1 - k_2} y_1 - s_w y_1}{s_w k_w + s_c k_c - s_w k_1}, \qquad r = 0, \qquad r = y_1/k_1$$

$$\text{if } k_1 \geq k_w+k_c \geq k_2 \qquad \text{if } k_w+k_c > k_1 \qquad \text{if } k_w+k_c < k_2$$

we can also easily show that a TCBGP may exist; in our numerical example we have

$$k_w = s_w w / \, r(s_c - s_w) = .9937$$

at $r = n/s_c = .1818$; on the other hand, we have

$$k = \frac{y_1 k_2}{n(k_2 - k_1 + \frac{y_1}{n}} = 1.0000$$

Hence,

$$k_c = k = k_w = .0073$$

showing that a TCBGP may in fact exist.

On the other hand, for stability of the TCBGP we require

$$M = -s_c k_c \frac{dr}{dk_c} - s_w \frac{dr}{dk_w} (k_w - k_1) \qquad - s_w r + n > 0$$

From the savings-investment equation, we calculate

$$\frac{dr}{dk_c} = \frac{-rs_c + y_1(k_1 - k_2)}{s_w(k_w - k_1) + s_c k_c} = -.0855$$

and

$$\frac{dr}{dk_w} = \frac{-rs_w + y_1(k_1 - k_2)}{s_w(k_w - k_1) + s_c k_c} = -.109$$

$$M = .126 < 0$$

and the TCBGP is locally unstable. Note that the fact that $dr/dk_c > 0$ is sufficient to rule out the possibility that the TCBGP be a saddle point (and hence it must either be a center or an unstable focus or node.)

It is easy to show that for this model there exists a unique OCBGP[1] with k identical to its value for the TCBGP (since $\lambda y_1 = nk$ along a balanced growth path, and λ is simply a function of k and the parameters k_1 and k_2); but with

$$r = \frac{s_w y_1 - nk}{s_w(k_1 - k)} < \frac{s_w y_1 - nk}{s_w(k_1 - k) - (s_c - s_w)k_c}$$

1. With $r \gtrless 0$; there exists a (stable) OCBGP with $r = 0$ at $k = s_w y_1/n$ and a trivial OCBGP with $k = 0$.

The OCBGP is a saddle point; to see this, we calculate the Jacobian

$$\frac{\partial(\dot{k}_c, \dot{k}_w)}{\partial(k_c, k_w)} = \begin{Bmatrix} s_c r - n & 0 \\ s_w(k_w - k_1)\dfrac{dr}{dk_w} & s_w r - n + s_w(k_w - k_1)\dfrac{dr}{dk_w} \end{Bmatrix}$$

$$= \begin{Bmatrix} -.0000699 & 0 \\ .160 & .126 \end{Bmatrix}$$

whose characteristic roots are real but of opposite sign. The phase diagram for this model is given in figure 4.

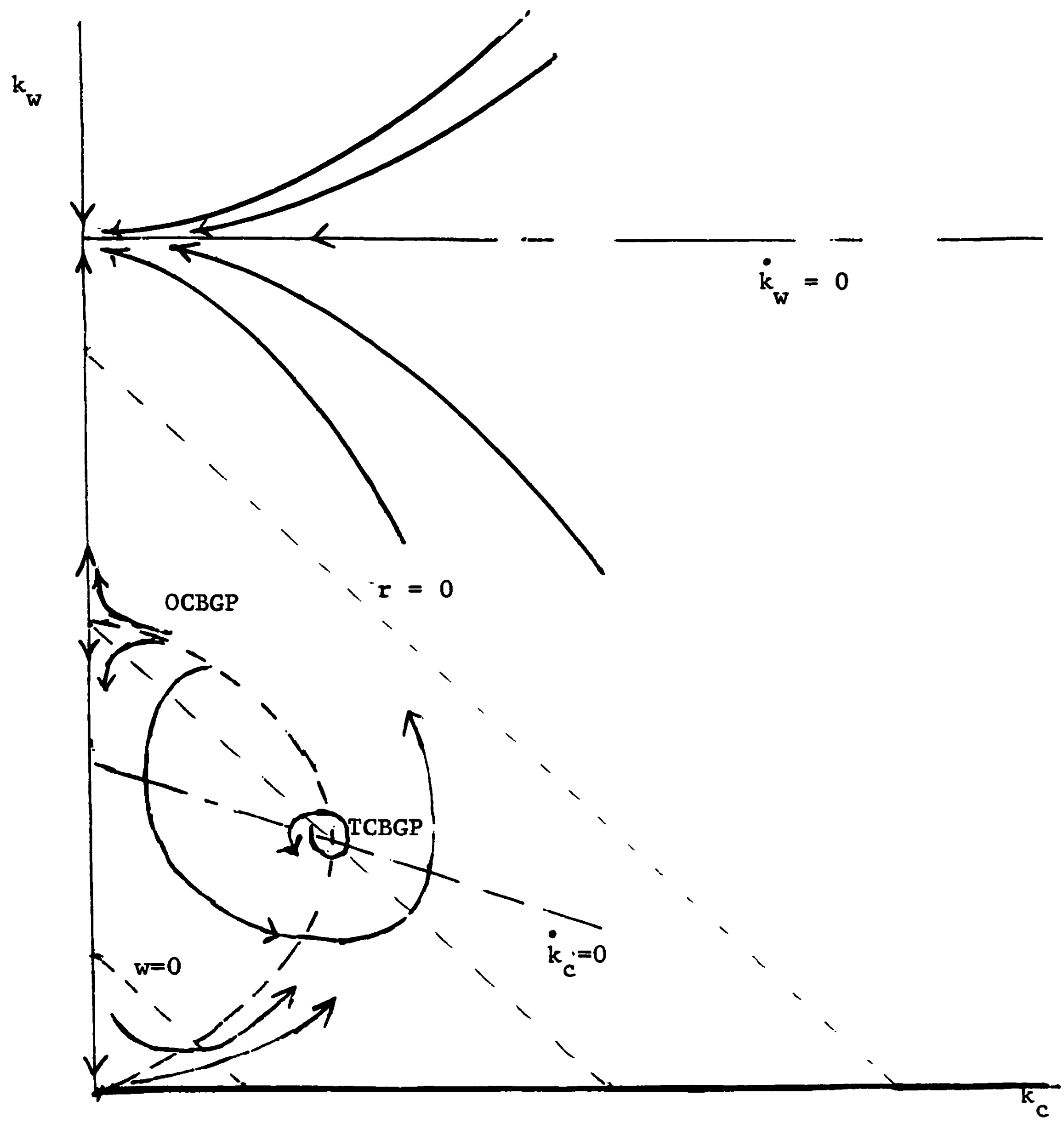

FIGURE 4

Schematic Phase Diagram

CHAPTER IV

THE DISTRIBUTION OF INCOME AND WEALTH AMONG INDIVIDUALS

Introduction

The recent plethora of "alternative theories of distribution" - all profits saved, some profits saved, no wage income saved, some wage income saved, one rate of savings for the working class, another rate of saving for the non-working class, etc. etc. - has been greeted by most (neoclassical) economists with a great deal of skepticism. This paper is not intended as still one more "alternative theory of distribution"; rather, its purpose is to explore in some detail the implications of these growth models for the distribution of income and wealth among individuals, a question which is perhaps of more direct social relevance than the distribution among abstract factors, and which has received little attention in the recent literature. A second purpose of this note is to explore how crucial are the linear (and usual proportional) assumptions made in the usual growth analysis to the results that are obtained.

In this paper, we shall present a number of related models which differ in four important characteristics: (a) Savings behavior; (b) Reproductive behavior; (c) Homogeneity of the labor force; and (d) Inheritance policies. The essential question we are interested in is, given any initial distribution of wealth and income, what is the asymptotic distribution? We also attempt to ascertain, where it is possible to do so, the short run movements in the distribution of wealth.

In section II. of the paper, we consider the Solow one-sector neo-classical growth model [7], with two modifications: wealth is divided equally among one's children and the savings function is linear, but not necessarily proportional. The following two sections consider the same model with alternative savings assumptions: in section III. we investigate non-linear savings functions and in section IV. we consider two cases where there is a wealth term in the savings function. In section V. we revert to the linear savings hypothesis, but let reproduction rates vary with income. In the following section, we investigate what happens if labor is not homogeneous (but the reproduction rate is constant and savings is a linear function of income.) In section VII. we return to the Model of Section II. but introduce primogeniture.

In the following three sections, we investigate what happens if there are different classes who save different proportions of their income, or if individuals save differently out of different sources of income.

In Section XI. we discuss the fiscal policy implications of the analysis and in Section XII. we discuss how the analysis is changed if technological change is introduced into the model. Finally, in order to get a rough idea of the order of magnitude of the numbers involved, we investigate numerically a particularly simple case of the model presented in Section II.

II. The Basic Model

In this section (and throughout most of the paper), it will be convenient to think of society as divided into a number of groups; all the members of any one group have the same wealth; groups differ in their per capita wealth holdings.

We assume that labor is homogeneous; in other words all workers receive the same wage. Thus, all the members of any one group have the same income as well as the same wealth. If y_i is the income per capita of group i, w the wage rate, r the rate of return on capital, and c_i capital per man, 2.1. $y_i = w + r\, c_i$

Savings per capita is assumed to be a linear function of income per capita; hence if s_i is the per capita savings of group i, m the (constant) marginal propensity to save, and b≥ is the per capita savings at zero income,

2.2. $s_i = my_i + b$

Reproduction occurs at a constant rate n, there is no intermarriage between income groups, and wealth is divided equally among one's offspring. These assumptions ensure that the proportion of the population in each group a_i, remains constant.

We can now write down the basic equation of per capita wealth assumulation for group i:

2.3. $\frac{c_i}{c_i} = \frac{s_i}{c_i} - n = \frac{b + mw}{c_i} + mr - n$

We have yet to say how w and r are determined. We assume a well-behaved (concave)neo-classical production function. We also assume that the Inada derivative condition is satisfied, i.e. the marginal product of capital approaches infinity as the capital labor ratio goes to zero. Factors are assumed to be paid their marginal products. (For most of the analysis, all we require is that the interest rate be a declining function of the capital labor ratio and that the wage rate be an increasing function of the capital labor ratio.) If we let k denote the aggregate capital labor ratio, then we have

2.4. $w = w(k)$

and

2.5. $r = r(k)$

Moreover, if we let K_i be the total wealth holdings of group i, and let

2.6. $k_i = K_i/L = a_i c_i$

it is clear that

2.7. $k = \Sigma k_i = \Sigma a_i c_i$

We can now write down the differential equation for aggregate capital accumulation:

2.8. $\dot{k} = \Sigma \dot{k}_i = \Sigma a_i \dot{c}_i = b + mw + rk = nk$

Observe that the aggregate capital accumulation behavior is independent of the distribution of wealth. This is an essential result of the linearity assumption.

In analyzing this model, we shall procede as follows: first we shall discuss the aggregate balanced growth paths and their stability; we shall then discuss the conditions under which a given group is in equilibrium, i.e. has unchanging per capita wealth; next, we shall discuss short and long run movements in the wealth distribution; finally, we investigate what these results imply for movements in the distribution of income.

If the economy is in balanced growth,

2.9. $\dot{k} = 0$

or

2.10 $my = nk - b$

In the case of $b = 0$, a strictly proportional savings function, this is simply the "Solow" equilibrium. If we impose concavity on our production function, it is clear that for $b > 0$, there is a unique value for which $my = nk$, i.e. a unique aggregate balanced growth path. If, on the other hand, $b < 0$, i.e. at a zero income a negative amount is saved, then there will in general exist two balanced growth paths.[1]

If there is only one balanced growth path, it is clear that it is globally stable (See Figure 1), since for capital labor ratios greater than that of the balanced growth path, savings per capita is

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[1]The usual qualifications must be made about the possibility of no equilibrium (in the absence of the Inada condition) and multiplicity of equilibria (in the absence of concavity), even in the case of $b \geq 0$.

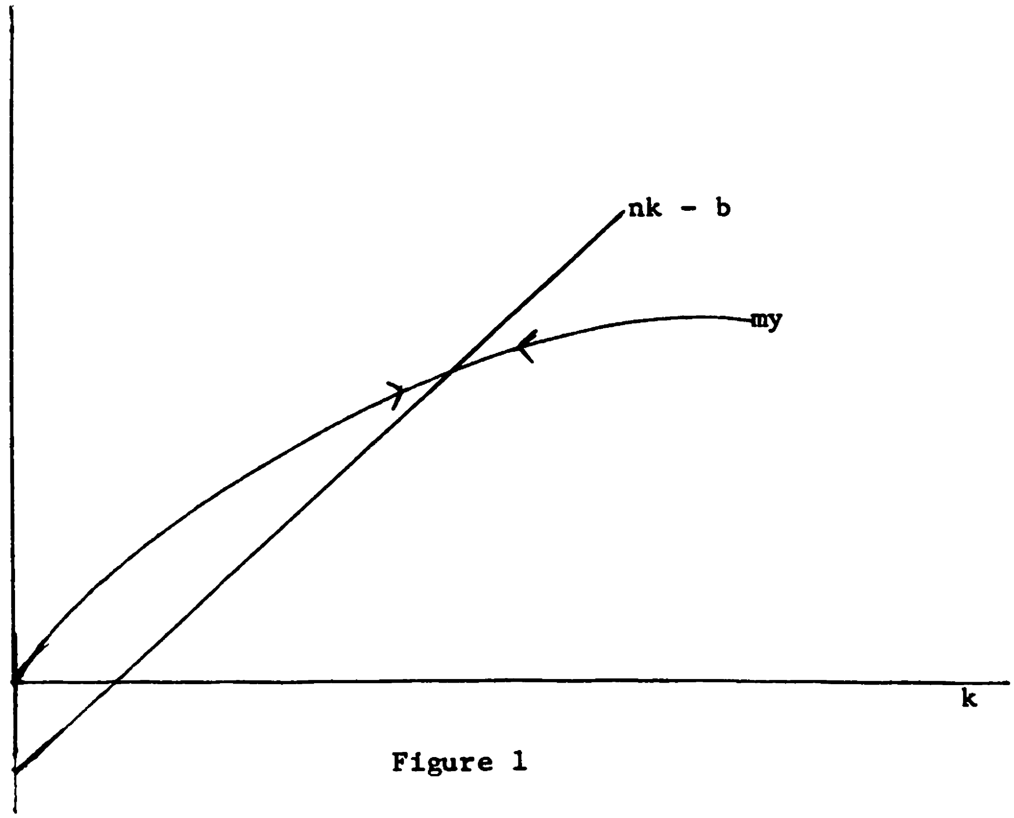

Figure 1

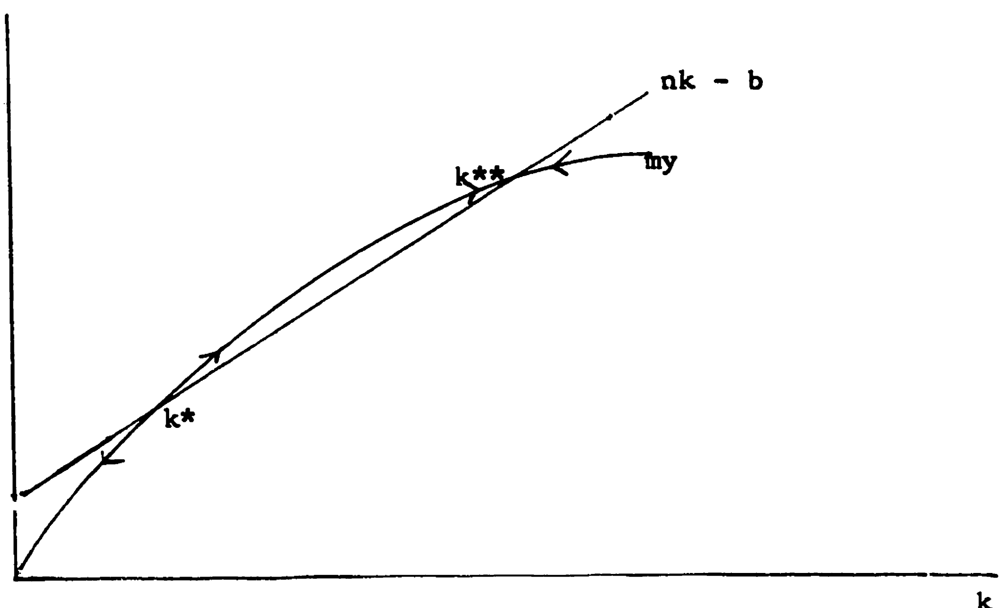

Figure 2

less than that required to maintain the same capital output ratio with population growing at the rate n, and conversely for capital labor ratios less than that of the balanced growth path.

On the other hand, if there are two balanced growth paths (Figure 2) the lower one will be locally unstable, the upper will be locally stable. This can be seen in two ways; graphically, as Figure 2 shows, the nk - b line cuts the my curve from below in the upper equilibrium but from above at the lower equilibrium: analytically, we differentiate the capital accumulation equation 8 with respect to k and evaluate at k = 0, to obtain

$$\frac{\partial \dot{k}}{\partial k} = mr - n$$

The balanced growth path is stable or unstable as $\frac{\partial \dot{k}}{\partial k}$ is less than or greater than zero. mr is the slope of the my curve, and n is the slope of the nk - b curve. Since my is concave, it is clear that the lower intersection must have $mr > n$ and the upper intersection must have $mr < n$.[1]

---

[1]In the singular case of a tangency between the my curve and the nk - b curve, where the upper and lower equilibria merge together, we have a stable - unstable equilibrium: stable with respect to upward deviations, unstable with respect to downward deviations. In this equilibrium $mr = n$, the rate of profit is equal to the rate of growth divided by the marginal propensity to save.

Having analyzed the aggregate properties of the model, we turn now to investigate the behavior of the separate wealth-income groups. First, it should be clear that for any given aggregate capital labor ratio k, there can exist at most only one group, with per capita wealth c^* which is in equilibrium, i.e. only one group whose per capita wealth is neither increasing nor decreasing. We require

$$\dot{c}_i/c_i = 0$$

or

$$c^* = \frac{b + mw(k)}{n - mr(k)}$$

(Observe that c^* is a function of k.)

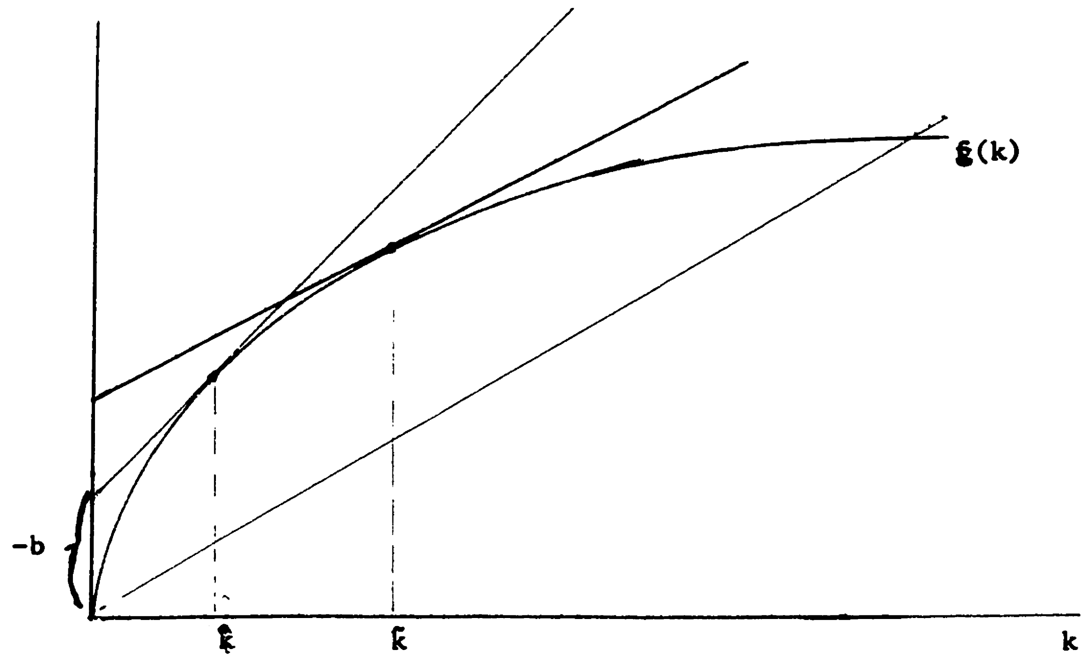

Figure 3

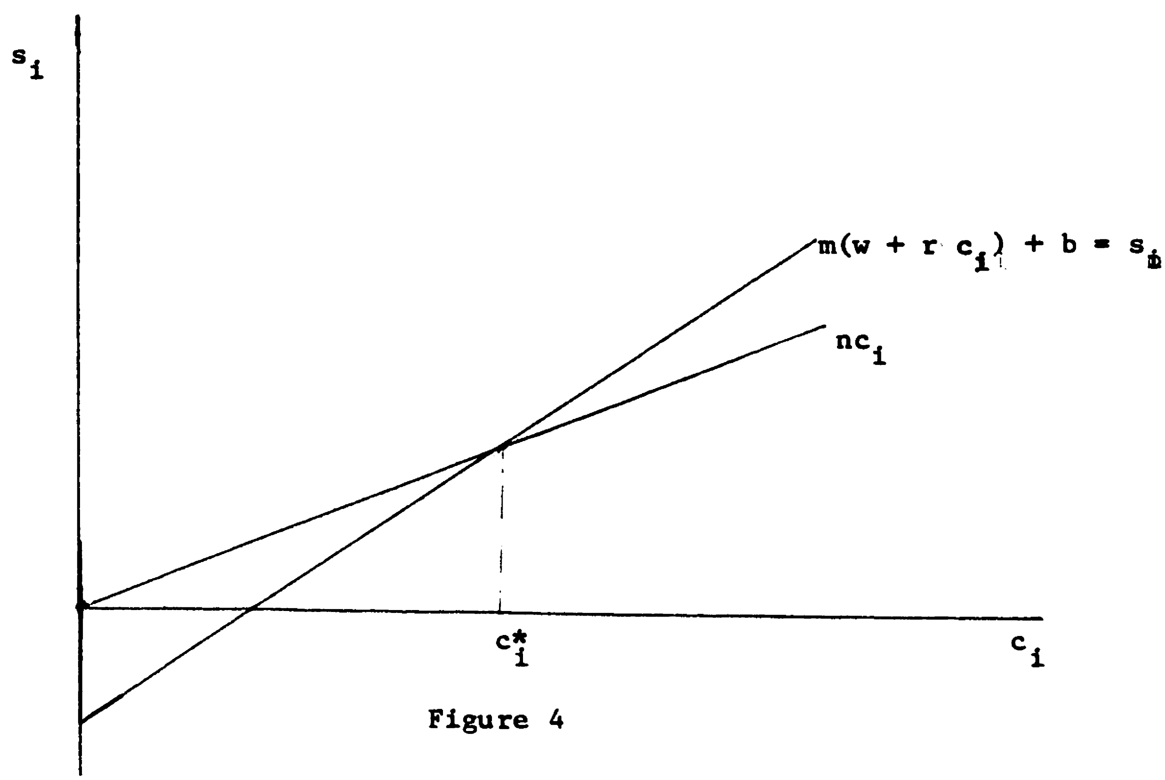

Figure 4

Thus, to the left of the lower equilibrium, savings per man is less than that required to sustain that capital labor ratio, and hence the capital labor ratio falls (continually)[1]; above the lower equilibrium, but below the upper equilibrium (between k^* and k^{**} on Figure 2) the reverse situation holds, so that the economy has an expanding capital labor ratio. Finally, above the upper equilibrium (k^{**}), the economy has a declining capital labor ratio.

This is meaningful only if $c^* > 0$, i.e. only if $(b+mw)(n-mr) > 0$. Using Figure 3, we can see that $b + mw = 0$ at $\hat{k}$, where k is the point of tangency of a straight line beginning at the vertical axis a $-b$ with the my (k) curve (the output per man function, multiplied by the marginal propensity to save). At lower values of k, $b + mw$ is negative, at higher k, $b + mw$ is positive. On the other hand, $n = mr$ at $\tilde{k}$, where $\tilde{k}$ is the point of tangency of the my (k) curve and a line with slope n. For $k > \tilde{k}$, $n - mr > 0$ and conversely for $k < \tilde{k}$. Because of concavity of my, if there exists two balanced growth paths, it is clear that $\hat{k} < \tilde{k}$. Accordingly if $k < \hat{k}$ or $k > \tilde{k}$ (see Figures 4 and 5), then there will exist a unique wealth group which is in equilibrium; but if $\hat{k} < k < \tilde{k}$, (see Figure 6) no wealth group will be in equilibrium.

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[1]What happens when $k = 0$ is a question which we shall postpone for the moment. k can only become negative if there exists foreign countries from whom one can borrow. For a long run savings function. it may well be argued on the basis of econometric evidence that b is zero: we prefer, however, to keep the analysis as general as possible.

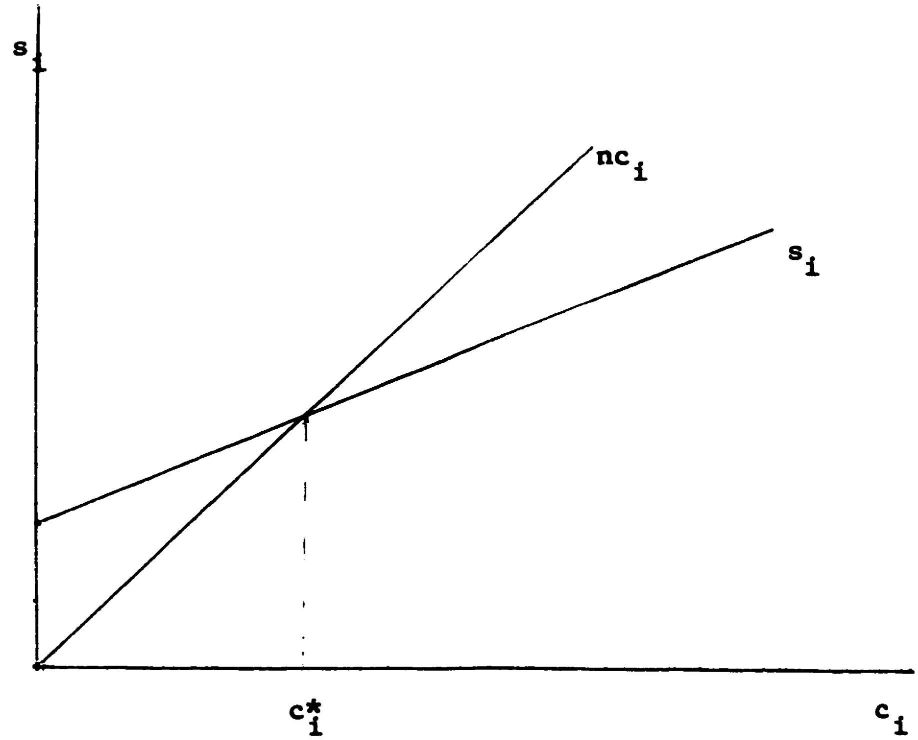

Figure 5

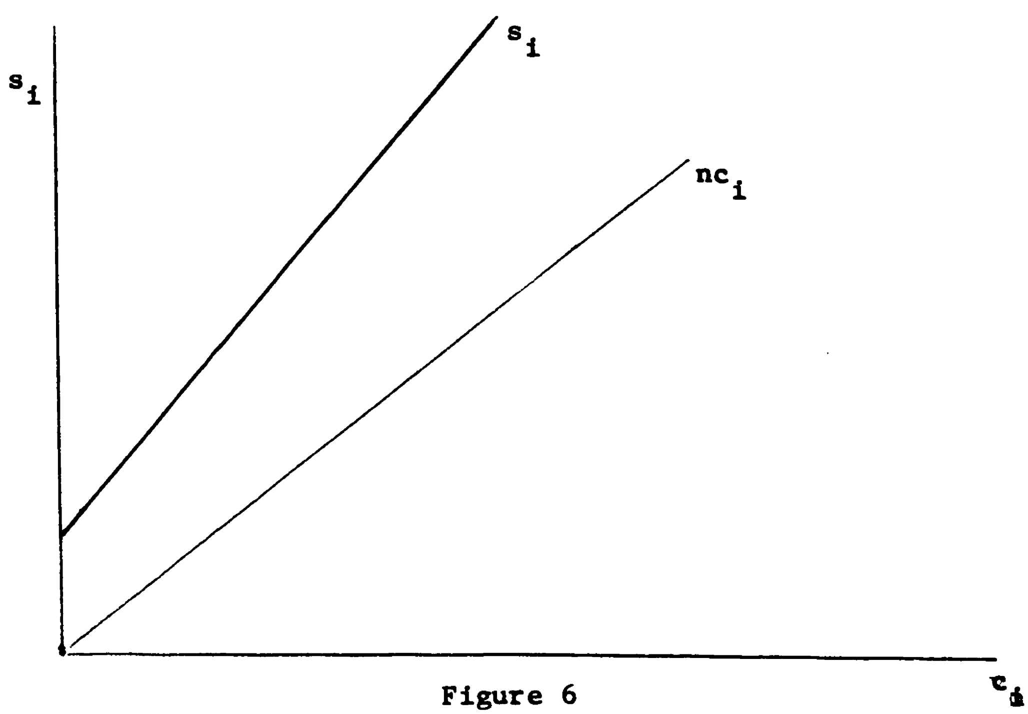

Figure 6

It should be observed, however, that in the first case, with $k < \hat{k}$, groups with per capita wealth less than c* have an increasing per capita wealth. But if $k > \hat{k}$, groups with per capita wealth less than c* have an increasing per capita wealth, and conversely for those with per capita wealth greater than c*. In the intermediate case, all groups have an increasing per capita wealth.

Secondly, we note how the distribution of wealth changes over time. Without loss of generality, we consider the case of two income groups; then we wish to know whether, if $c_1 < c_2$, c_1 is growing faster or slower than c_2; if it is growing faster, then the ownership of wealth (at least in a relative sense) is becoming more "equalitarian"; if it is growing slower, it is becoming less "equalitarian".

But

$$\dot{c}_1/c_1 - \dot{c}_2/c_2 = (b + mw)\ (1/c_1 - 1/c_2)$$

Hence, if $b + mw > 0$, the ownership of wealth becomes (relatively) more equalitarian, while if $b - mw < 0$ it becomes "worse"; if $b = mw$ there is no change in the (relative) ownership of property. Hence, to the left of $\hat{k}$, the ownership of capital is becoming more uneven, to the right, more even.

The economic reasoning behind this result should be clear: if $b + mw$ is equal to zero, increasing per capita wealth by a given percentage, increases savings (mrc) by the same percentage, but increases the savings required to sustain that per capita wealth (nc_i)

by the same percentage, so that whatever c_i happens to be, there it remains. But if b + mw is positive, increasing per capita wealth by a given percentage increases (per capita) savings by a smaller percentage, while the savings required to sustain that per capita wealth ratio goes up in proportion to c_i, and conversely for b + mw less than zero.

These results may be seen still another way by means of Figures 7-9. In each case we plot $\dot{c}_i/c_i$ as a function of c_i. The shape of the curve is, of course, a hyperbola, with asymptote mr-n. Whether it is upward or downward sloping depends on the sign of b + mw. In Figure 7, we take the case of $k < \hat{k}$, so that b + mw < 0 and mr - n > 0. Hence, $\dot{c}_i/c_i$ increases with c_i. In Figure 8 we take the case of $\hat{k} < k < \tilde{k}$, and hence b + mw > 0 and mr - n > 0; hence the curve is downward sloping; $\dot{c}_i/c_i$ is positive for all c_i, but decreases with increasing c_i. Finally, in Figure 9, we take the case of $k > \tilde{k}$, for which mr - n < 0, and b + mw > 0. Accordingly, $\dot{c}_i/c_i$ decreases with increasing c_i, and becomes even negative for sufficiently large values of c_i.

It is clear then at the upper equilibrium in the long run there must be an equalitarian distribution of wealth, since at the upper equilibrium the wealth per man of the poorer groups grows faster than that of the richer groups.

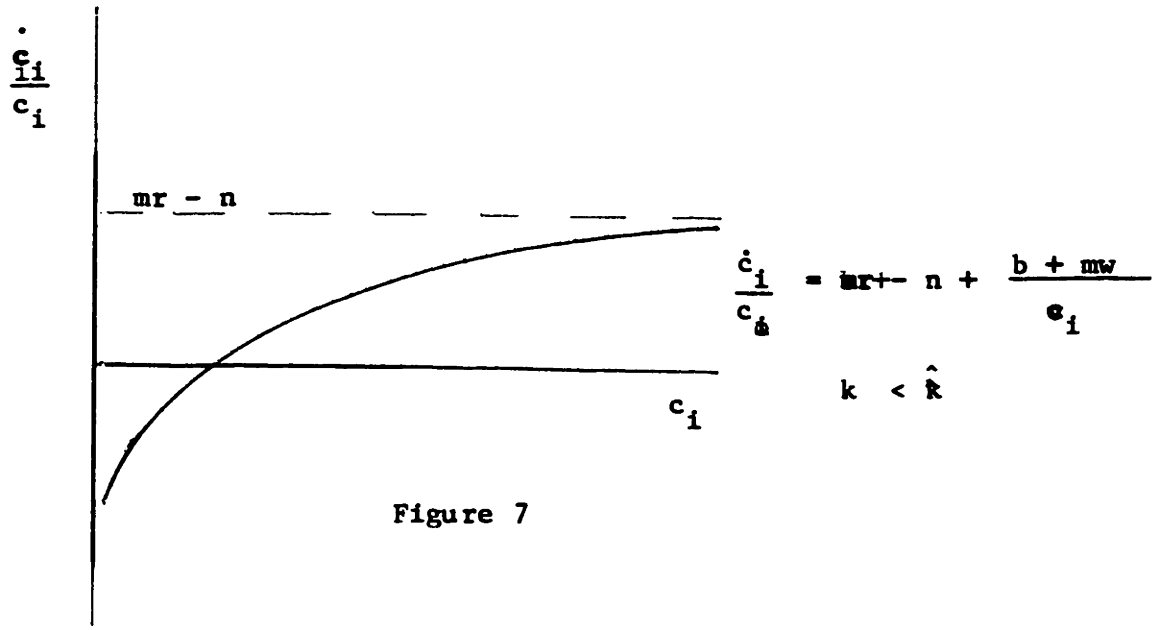

Figure 7

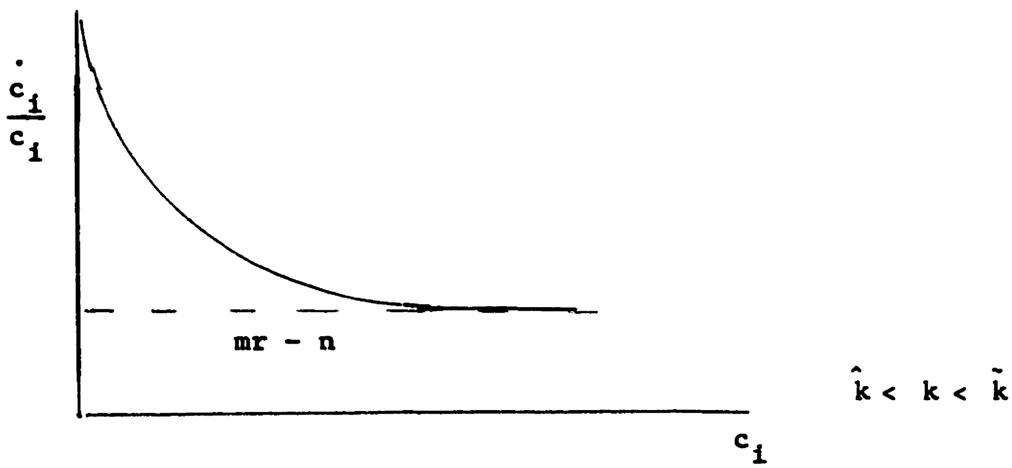

Figure 8

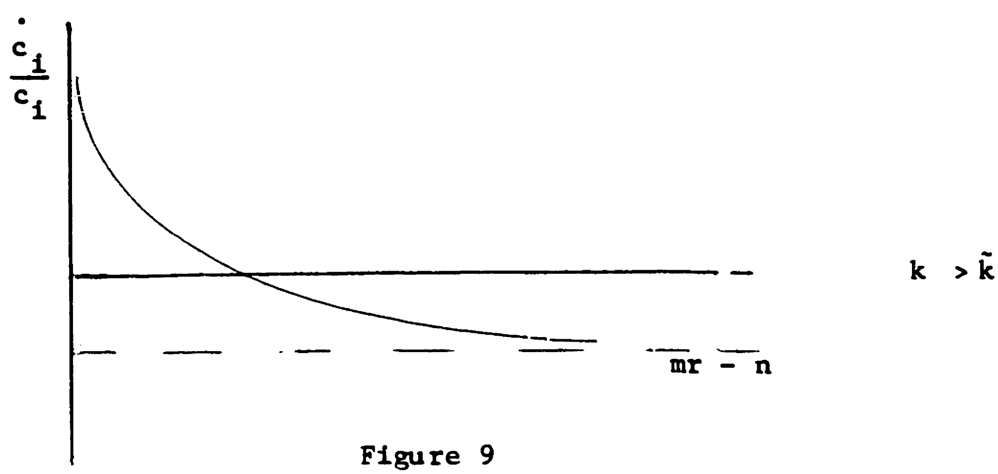

Figure 9

But at the lower equilibrium, those groups with an initial per capital wealth less than the equilibrium will grow continually poorer, while those groups with an initial per capita wealth greater than the equilibrium will grow continually richer; this follows immediately from the fact that those with a lower than average per capita wealth must have a continually declining wealth per man - even when their wealth becomes zero, since

$$\dot{c}_i = mw + b < 0$$

while those with a higher than average per capita wealth must have a continually rising wealth per man

$$\dot{c}_i = my + m(c_i - k)r - nk - n(c_i - k) = (c_i - k)(mr - n) > 0$$

(And of course, those with more initial per capita wealth have a faster rate of growth in per capita wealth.)[1]

---

[1]If there is a lower bound on the amount of capital that one can hold, (an upper bound on indebtedness) then we must modify our savings functions. Assume that the lower bound is zero. Then

$$s_i = b + mw + mrc_i \qquad c_i > 0$$

$$s_i = 0 \qquad c_i = o$$

We assume that there are two groups, a "poor" group with zero wealth and with a of the population, and a "rich" group with 1-a of the population, and all the capital. Then

$$\dot{k} = (1-a)\ (b + mw) + (mr - n)k$$

For a balanced growth, we require $\dot{k} = 0$, or

$$k = \frac{(1-a)(b+mw)}{n - mr}$$

That there exists a unique solution in the region $k > \hat{k}$ to this equation can be shown as follows: the left hand side is clearly an increasing function of k; the right hand side, however, is a decreasing function of k, at $k = (1-a)(b+mw)/(n-mr)$, since

$$\frac{d\ (b+ms)/(n-mr)}{d\ k} = \frac{mw'}{n - mr} + \frac{b + mw}{n - mr}\ \frac{mr'}{n - mr} = \frac{m\ [(1-a)w' + kr']}{(n - mr)(1 - a)}$$

$$= - \frac{aw'm}{(n-mr)(1 - a)} < 0$$

This means that at a balanced growth path, if n - mr is positive $(k > \hat{k})$

$$k - (1-a)\frac{(b+mw)}{n-mr}$$

must be increasing, i.e.,

$$- (1-a)b + nk - (my - amw)$$

must be increasing. Since $nk - (1-a)b$ is a straight line, this means that there can only be one balanced growth path in the region $k > \hat{k}$, where $nk - (1-a)b$ cuts $my - amw$ from above. See Figure 10. Stability properties follow as above for the general model. It does not seem possible to rule out, in general, multiplicity of solutions in the region $k < \hat{k}$.

We can extend these results to the case where the lower limit of per capita wealth, is not zero, but e. Then, in the balanced growth path, we can show that

$$k = ae + (1 - a)\ \frac{(b+mw)}{(n-mr)}$$

We can show, as above, that there can exist at most one solution to this equation for $k > \hat{k}$.

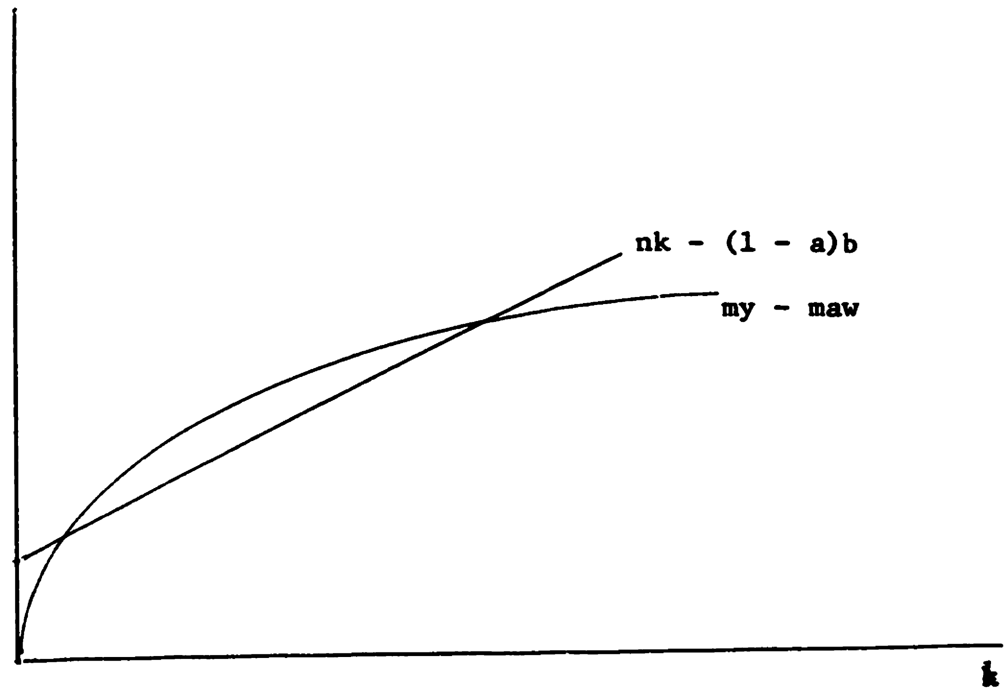

Figure 10

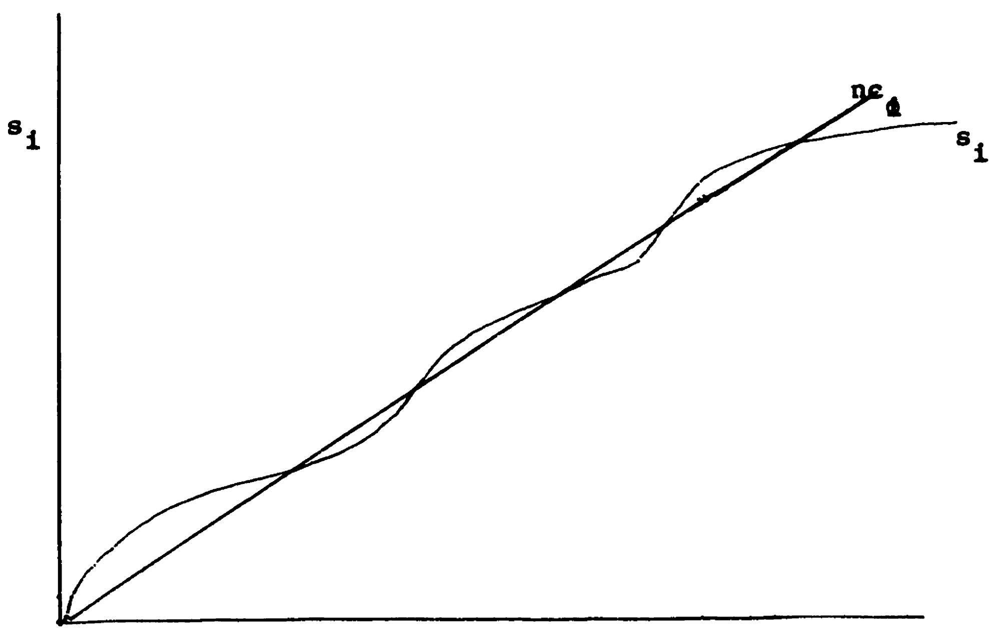

Figure 11

Thus it should be clear that although the fact that each of the individual groups is in equilibrium implies that the aggregate is in equilibrium, the converse is not true: the aggregate can be in equilibrium while the distribution of wealth is changing.

We are finally ready to fully describe movements in the distribution of wealth in our economy:

1. There exist two balanced growth paths, along which the capital labor ratio, output capital ratio, wage rate, etc. are all constant.
2. The one corresponding to the higher capital labor ratio is stable both with respect to the aggregate (locally) and with respect to the component income classes (globally): if the overall capital labor ratio is increased or decreased, (provided it does not fall below $\hat{k}$) the economy returns to the balanced growth path, and if individual income classes are perturbed, the economy eventually returns to the equalitarian state.
3. The one corresponding to the lower capital labor ratio is unstable, both with respect to the aggregate and with respect to the component income classes. If the aggregate k is decreased, it continues to decrease (forever); if it is increased, it continues to increase until it arrives at the upper equilibrium. If individual income classes are perturbed, from the equal distribution position in such a way that the aggregate capital labor ratio remains constant, the classes with per capita wealth greater than the overall capital labor ratio continually increase their per capita wealth, and conversely for those with less wealth than the "average".

4. If the economy is initially within the region between k^* and $\hat{k}$, then the overall capital labor ratio is increasing, the economy eventually arrives in a state with completely equal distribution of income and wealth, but until the overall capital labor ratio becomes equal to $\hat{k}$, the relative distribution of wealth becomes more uneven.[1]

5. For all capital labor ratios greater than k, the distribution of wealth becomes (relatively) more even, eventually reaching complete equality.

The adaptation of these results to movements in the distribution of income is straightforward. If the elasticity of substitution of the production function[2] is equal to one, then the analysis carries over exactly. If the elasticity of substitution is less than 1, for instance

a. In the region $\hat{k} < k < k^{**}$, the decreasing share of capital and the equalization of its ownership both serve to equalize the distribution of income.

---

[1]Perhaps one should not draw morals about the real world from such simple models: if the distribution of wealth appears in the short run to be becoming more uneven, do not lost hope in the capitalist system; eventually, (which may be a long tine) the economy may lead to equalitarian state, by its own accord.

[2]It should be noted that none of the results thus far have depended on the shape (exact concavity and the Inada condition) of the production function.

b. In the region $k > k^{**}$ the increasing share of capital and the equalization of its ownership offset each other; eventually, of course, the equalization tendencies dominate.

c. In the region $k < k < \hat{k}$, the decreasing share of capital and the increasing spread in the ownership of capital offset each other; eventually, the economy moves into the region $\hat{k} < k < k^{**}$.

d. In the region $k < k^{*}$, the increasing share of capital and the increasing spread in its ownership both serve to make the distribution of income more unequal.

Similarly, for elasticities greater than 1.

III. Non-linear Savings Functions

In this section, we make all the assumptions as in the previous, with the exception that the linear savings function is replaced by a non-linear one.

In this case, of course, there may be any number of values for which $s(y_i) = nc_i$, for given k, as Figure 11 illustrates. Accordingly, there may be (for any given balanced growth path) any number of distinct income classes.

But if the savings function is convex, or concave, there can only be (at most) two values of y_i (and hence c_i) for which the wealth per capita is constant; for if the savings function is convex or concave (as a function of y_i) it is convex or concave as a function of c_i:

$$ds/dc_i = s'r$$

$$d^2s/dc_i^2 = s''r \lesseqgtr 0 \quad \text{as} \quad s'' \gtreqless 0$$

See Figures 12, and 13.

But the multiplicity of balanced growth paths is a much more difficult question, for we can no longer simply "add" up the incomes (as we did earlier) and find the savings: in the linear case, savings is independent of the distribution of income, in the non-linear case it is not.

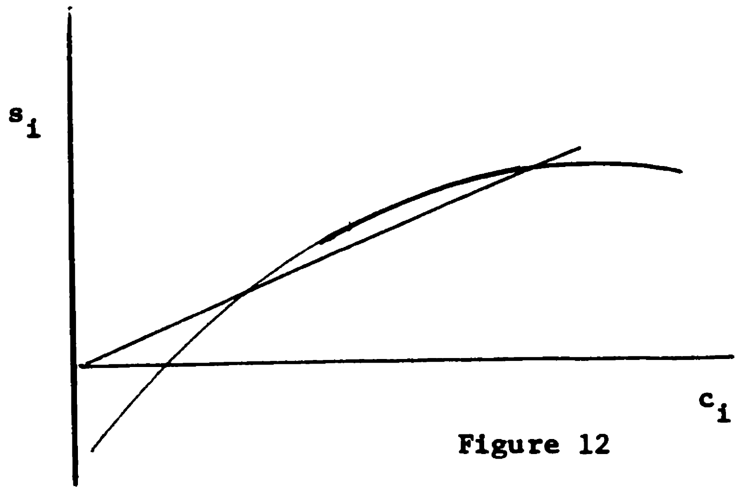

Figure 12

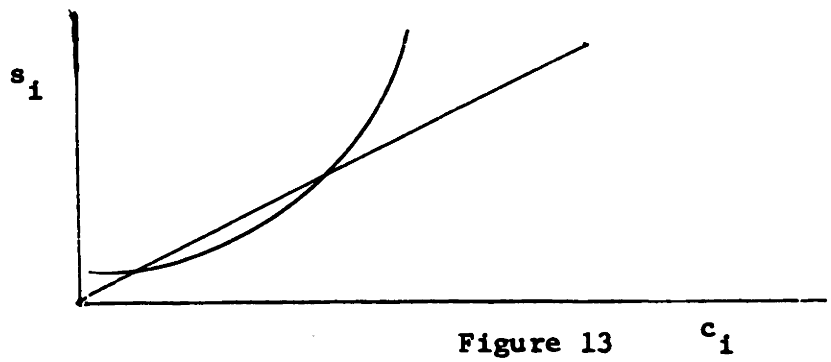

Figure 13

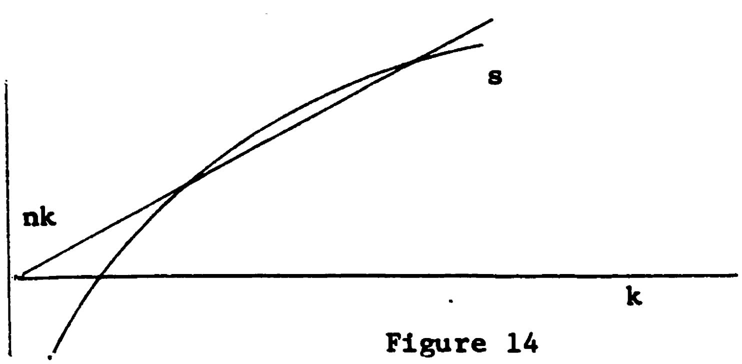

Figure 14

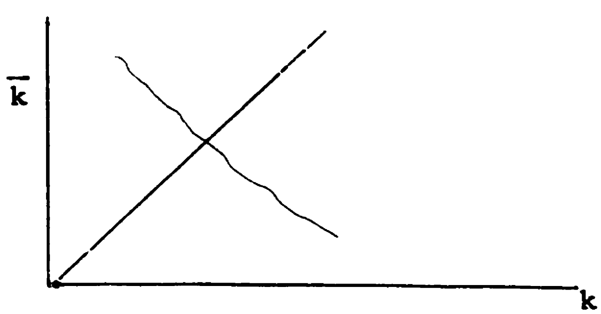

Figure 15

Putting the question mathematically, for how many k is it true that

3.1 $s(w(k) + r(k)c_i) = nc_i$ where $k = \Sigma a_i c_i$

In general, there is an indefinite number, but if the savings function is concave, and the proportion of the population in each of the income groups is fixed, then there can be at most three -- two with only one class present and one with two classes present.

The one class cases require $c_i = k$. Hence, the question becomes, for how many k is

$$s(w + rk) = nk$$

But since $s(k)$ is a concave function of k, and nk a linear function of k, there can be at most two solutions.[1] (See Figure 14).

The two class case is somewhat more difficult to analyze. Let a per cent of the population be in the lower equilibrium, 1-a in the higher. Let

$$\bar{k} = ac_1(k) + (1-a)c_2(k)$$

where $c_1(k)$, $c_2(k)$ are the solutions to equations 3.1 for given k.

$$dc_i/dk = -s'(w' + r'c_i)/(s'r-n)$$

---

[1] $ds/dk = s'y'$; $d^2s/dk^2 = s''y' + s'y''$; if s'', $y'' < 0$, this is clearly negative; if either the savings function or the production function is not concave, $s(k)$ will not have any simple shape (in general).

For the lower class, $c_i < k$, and hence $w' + r'c_i > 0$; if the savings function is concave, $s'r \quad n$ at the lower equilibrium. And conversely for the upper equilibrium. Hence

$$dc_i/dk < 0$$

Hence

$$d\overline{k}/dk < 0$$

There exists at most one k for which $\overline{k} = k$.[1] (Figure 15).

What about the stability properties of these equilibrium, and the "history" of the economy? The two one-class-equilibria have exactly the same stability properties as in the linear case, and nothing more need be said about it here. The two class equilibrium has, as one might expect, properties of both the lower and upper equilibrium one class economies: if a particular subgroup of (or the entire) lower class is disturbed, so that its wealth per capita is less than that in equilibrium, the members of that subgroup become increasingly poorer and if they become slightly richer (in per capita wealth terms) than in equilibrium, they get increasingly rich, until they "merge" with the upper class.

Of course, we have been assuming throughout this process that as individuals shift their class membership the aggregate capital labor ratio changes in the appropriate way; as a larger proportion of the population join the upper class, the aggregate capital labor ratio must rise. But as it rises, it leads all the other members of the lower

---

[1]As in the linear case, there is of course one further possibility existing (provided that at "very large incomes" savings becomes approximately proportional to income) - the poor reducing their capital to a lower bound of say zero, the rich becoming increasingly rich.

class to be out of equilibrium, and since the lower equilibrium is unstable, there is no mechanism for them to reach equilibrium. It is unlikely then that any two class equilibrium situation could ever be maintained for long.

Hence, in this model as in the linear model first examined, there is a tendency (in the long run) for the equalization of wealth and income -- with the possible exception of a group (in an underdeveloped economy perhaps almost the entire economy) whose wealth is at some lower bound (zero, or the upper bound on indebtedness.)

IV. Savings as a Function of Wealth and Income

Recent investigations into savings function have indicated that savings may be a function of wealth as well as income, i.e.

4.1 $$s = b + my + zc$$

or

4.2 $$\dot{c}_i/c_i = \frac{(b+mw)}{c_i} + mr + z - n$$

If $z > n$, of course, c_i increases without bound. In all other cases, the analysis proceeds exactly as in the first section of this paper, with n replaced by n-z. If z is positive, then it is as if the rate of population growth is smaller than it actually is, so that the equilibrium capital labor ratio is higher, r is lower, w is higher, etc. The more reasonable assumption is to make z negative, indicating that the more wealth one has, for any given income, the less one saves (as for instance some of the life cycle stories suggest), then it is as if n is higher, i.e. the equilibrium capital labor ratio will be lower, wages will be lower, and the profit rate will be higher.

An alternative formulation of savings behavior is the following: individuals have a desired wealth-income ratio, given by q^*, and if the wealth-income ratio is less than the desired, they accumulate, if it is greater than the desired, they decumulate. We may write the adjustment process as follows:

4.3 $\dot{c} = h(c^* - c)$

where

4.4 $c^* = q^*y = q^*(w+rc)$

Substituting, we have

4.5 $\dot{c} = h\,[q^*(w=rc) - c] = hq^*w + (q^*rh - h)c$

Since $k = \Sigma\, ac_i$

4.6 $\dot{k} = hq^*w + (rq^*h - h)k = hq^*y - hk$

There is a unique balanced growth path, with $q^* = y/k$, and it is stable. Moreover, for any given aggregate capital labor ratio, there is at most one c for which $\dot{c} = 0$:

4.7 $c = \dfrac{q^*w}{1 - rq^*}$

This is meaningful only if $r(k) < 1/q^*$, i.e. for very low capital labor ratios there exists no positive c for which $\dot{c} = 0$. In all cases, however, the poor accumulate capital faster than the rich, since

4.8 $\dot{c}_1/c_1 - \dot{c}_2/c_2 = hq^*w\,(1/c_1 - 1/c_2)$

and eventually all wealth is evenly distributed.[1]

---

[1] It may be worthwhile to suggest what happens if different groups have different desired wealth-income ratios. Denoting the desired wealth-income ratio of the ith group by q_i^* ,

we have, in analogy to equation 4.5

4.9 $$\dot{c}_i/c_i = \frac{hq_i^* w}{c_i} + q_i^* rh - h$$

which gives us the following aggregate capital accumulation equation:

4.10 $$\dot{k} = h\ (w\ \Sigma q^*a + r\ \Sigma q^*ac - k)$$

If we consider only balanced growth paths for which each group is in equilibrium, i.e. $\dot{c}_i = 0$ for all i, as well as $\dot{k} = 0$, we have

4.11 $$k = w\ \Sigma q_i^*\ a_i + r\ \Sigma \frac{q_i^{*2}\ a_i}{1 - rq_i^*}$$

or $$\frac{rk}{w} = r\ \Sigma\ q_i^*\ a_i + r^2\ \Sigma \frac{q_i^{*2}\ a_i}{1 - rq_i^*}$$

If the elasticity of substitution is greater than or equal to 1, then the left hand side increases with increasing k, while the right hand decreases. Hence, if the elasticity of substitution is equal to or greater than 1, there is a unique balanced growth path with $\dot{c}_i = 0$ for all i; otherwise, there may be more than one balanced growth path.

V. Variable Reproduction Rates

It is clear that different groups in our economy reproduce at different rates, although the relative influence for instance, of income and education is far from settled. In this section, we shall assume that the sole determinant of the reproduction rate of a group is its per capita income. There are various assumptions that one might make about the relation between income and rate of reproduction. Solow, in his classic 1956 paper [7], has investigated one type of reproduction function. We investigate two alternative reproduction functions, one slightly simpler than Solow's, the other somewhat more complicated.[1] We have drawn the two functions in figures 16A and 16B.

In A, the rate of reproduction is an increasing function (but at a decreasing rate) of the level of income; until some maximum is reached, beyond which it is constant (alternatively, n may approach the limit asymptotically). In B, after some level of income the rate of reproduction actually begins to decline.

For simplicity, we revert to the linear savings hypothesis, so that our differential equations become

$$\dot{c}_i = b + mw + mrc_i - n(w+rc_i)c_i$$

For any given capital labor ratio, there can only be two equilibria distribution of income in case A, but in case B, since there is one inflection point, there can be as many as three equilibria levels of income (and rates of reproduction). (See Figures 17a and 17b).

[1] Our reproduction functions are identical to those employed by Professor Meade in a forthcoming book.

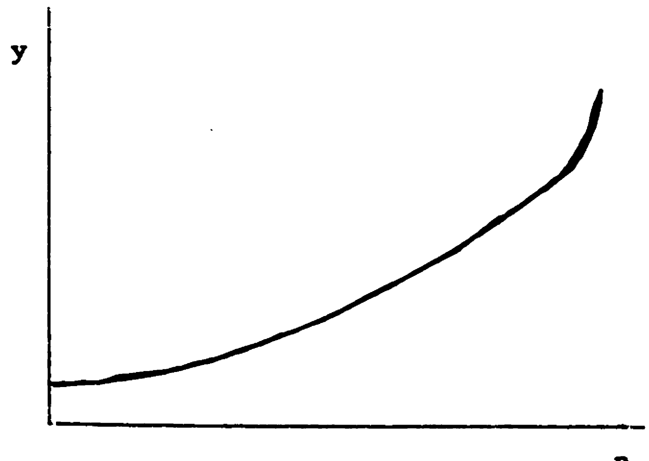

Figure 16 A

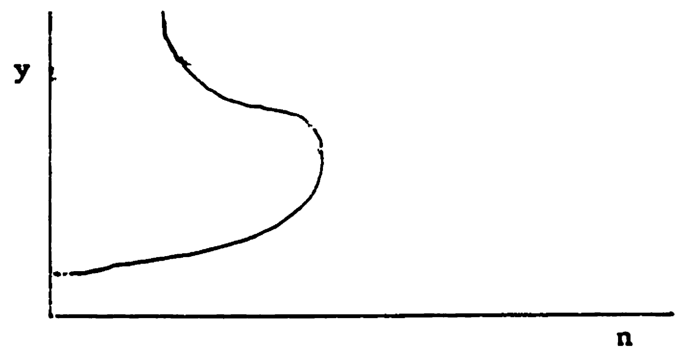

Figure 16 B

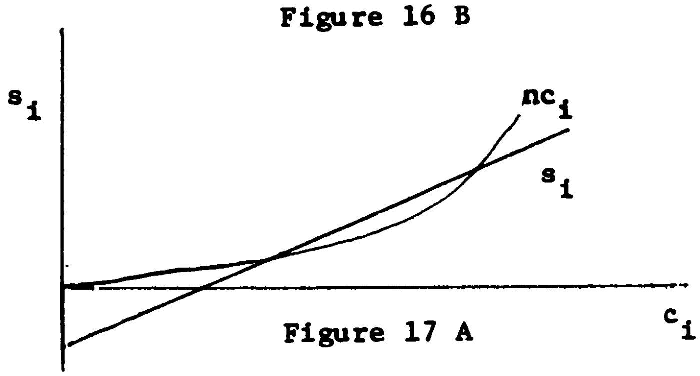

Figure 17 A

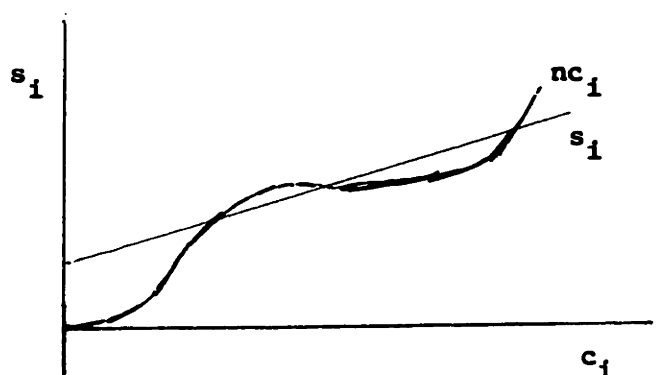

Figure 17 B

In A, groups with slightly more per capita wealth than c** (the upper equilibrium) are increasing their per capita wealth even further, while those with slightly less per capita wealth than c** are decreasing their per capita wealth. On the other hand, those groups with slightly more per capita wealth than c*, the lower equilibrium, have a decreasing per capita wealth, and conversely for those with slightly less per capita wealth than c*. In this sense, we can speak of the upper equilibrium as unstable and the lower equilibrium as stable.

In B, if there exist three equilibria the middle one is surely "stable" (in the sense defined in the previous paragraph) while the other two are unstable.

The analysis of how the aggregate capital labor ratio is determined is beset by the difficulty that if at these different equilibria, there are different rates of reproduction, the group with the highest rate of reproduction "dominates" the entire population. This problem is, however, still worthwhile investigating; the "capitalists" may be small in number but may have a higher per capita income.

Asymptotically then the rate of growth of population, and capital per labor of the entire population approaches that of the income group with the largest growing population. In the very long run, then we have simply

$$sf(k) - nk = sf(c_i) - c_i n(c_i) = 0$$

where n is the rate of growth of the fastest reproducing income group, c_i is that group's capital-labor ratio.

The A situation is easy enough to analyse in these terms: since the upper of the two possible balanced growth paths has the highest growth rate, that is the eventual growth rate. The poor become an insignificantly small proportion of the population. The stability properties of this model are analogous to those of the constant reproduction rate and nothing more will be said about it here. (See Figure 18a)

In the B situation (see Figure 18b) we have as many as four candidates for balanced growth paths. It is clear that of the four, either E" or E"' has the highest rate of reproduction; the rate of reproduction at E" is greater than at E', since we are in the increasing part of the reproduction curve, and E"' is greater than E"", since then we are in the decreasing part of the reproduction curve. We have then only to compare the slope of the lines joining E"and E"' to the origin: the greater slope implies the higher rate of reproduction.

To analyze the stability of the balanced growth paths it will be convenient to make one further assumption: the capital labor ratio in the aggregate is equal to that for each individual. This is what Prof. Meade has called a "propcap" economy [3]. Then, if the economy begins with an initial capital labor ratio less than k', the rate of reproduction exceeds the rate of capital accumulation, and the economy becomes increasingly worse off (the stagnation stage?). If we can get the economy just over the hurdle of k', it takes aff, eventually arriving at k". We are then in a "low-level" equilibrium. In order to get out of this "low level equilibrium trap" the economy must get over the

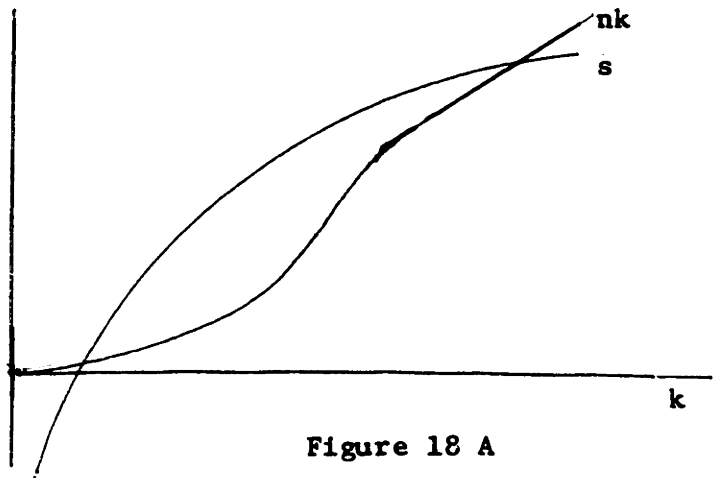

Figure 18 A

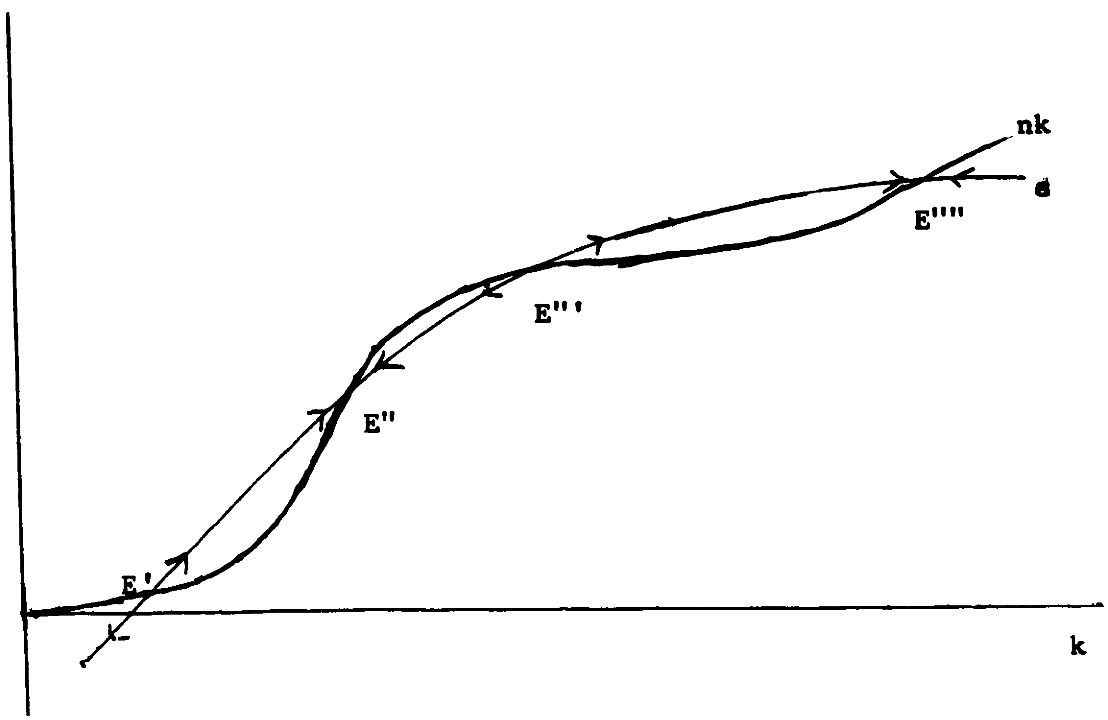

Figure 18 B

new hurdle of E'''; once past E''', which is an unstable equilibrium, it proceeds to E'''', where it can finally come to "rest". Since, as we have noted before, the economy must eventually become (except for in infinitesimal proportion of the population) a one class state, (since the group with the largest rate of growth will--eventually--dominate) the above stability analysis is perfectly general.

But even if, for instance, some rich group is an infinitesimal part of the total population, it does not mean that it has an infinitesimal part of the total wealth of the economy. It is worth calculating then movements in the relative wealth of different groups as well as the movements in their relative per capita wealth. For conveniénce, we assume two groups, the rich denoted by subscript r, and the poor, denoted by subscript p. We have then

$$\dot{c}_r / c_r - \dot{c}_p / c_p = mw + b\ (1/c_r - 1/c_p) - (n_r - n_p)$$

Thus for a society with a "high capital-labor ratio ($k > \hat{k}$, where k is defined in Section II), only if the rich reproduce more slowly than the poor, is it possible for their per capita wealth to increase relative to the poor. (This is, of course, quite possible, depending on the shape of the reproduction function). On the other hand, if $k < \hat{k}$, then $mw+b < 0$, and even if the rich are reproducing faster than the poor, their per capita wealth may be increasing relative to that of the poor.

If K_r is the capital of the rich, and K_p that of the poor, we have

$$\frac{d \ln K_r/K_p}{d\ t} = \frac{2\dot{a}_r}{a_r} + \frac{\dot{c}_r}{c_r} - \frac{\dot{c}_p}{c_p} = (mw + b)\ \frac{1}{c_r} - \frac{1}{c_p} \quad + (1 - 2a)(n_r - n_p)$$

Thus, if the proportion of the population in the rich class is very small (a is small), the only two conditions under which the proportion of capital capital in society which is owned by the rich can increase are (a) the economy is "underdeveloped" with a small capital labor ratio ($k < \hat{k}$), or (b) the rich reproduce much more rapidly than the poor ($n_r > n_p$). If the latter situation is sustained, of course, then eventually the rich will no longer be an infinitesimal proportion of the population; eventually, they will constitute half of the population, and if they continue to grow more rapidly, this will result in their share of ownership of the capital stock of the economy declining.

VI. Heterogeneous Labor Force

So far, we have been following the usual growth model analysis in assuming that all labor is homogeneous. In this section we shall indicate what happens if some labor is paid a higher wage than other labor. We assume that the different kinds of labor are related to each other in a "pure labor augmenting way" so that the ratio of the wage of any two groups is constant. There is no intermarriage between groups. We revert to the linear savings hypothesis and constant rate of growth of population.

The proportion of individuals in each group at time 0 is a_i, and the effective labor force at time 0 is

$$L = (\Sigma a_i p_i)N$$

where p_i is the number of efficiency units incorporated in each member of group i.

Savings of group i per capita is given by

$$s_i = b + mp_i w + rc_i$$

and the rate of growth of per capita wealth is given by

$$\dot{c}_i = b + mp_i w + mc_i - nc_i \quad .$$

At a given capital labor ratio, with w fixed, we have as many equilibria c_i as we have groups.

We now ask, for any given k, are those groups which have a higher equilibrium c_i more or less efficient than those with lower equilibrium c_i? Since

$$c_i = \frac{b + mp_i w}{n-rm}$$

we can easily calculate

$$dc_i / dp_i = mw/(n-mr) \quad .$$

Hence, dc_i/dp_i has the same sign as n-rm, which will be a function of k. For very high rates of interst, n-rm can clearly be negative.[1]

If b = 0, note that c_i is simply proportional to p_i.

How is the general equilibrium determined? Recalling that

$$\frac{K}{L} = \frac{\Sigma a_i c_i}{(\Sigma a_i p_i)N}$$

we have as before

$$my = nk - b$$

Again, we may obtain up to two balanced growth paths.

When we analyze the relative movements in per capita wealth, we have

$$\dot{c}_1/c_1 - c_2/c_2 = b(1/c_1 - 1/c_2) + mw(p_1/c_1 - p_2/c_2) \quad .$$

---

[1]If $(n-mr)(b + mp_i w) < 0$, there is, for group i, no positive value of c_i at which it is in equilibrium for the given k. As before (Section II), we can find $\bar{k}$ such that n-mr = 0 and $\hat{k}_i$ such that $b + mwp_i = 0$. Then $(n-mr)(b + mp_i w) < 0$ for $\min\{\bar{k}, \hat{k}_i\} < k < \max\{\bar{k}, \hat{k}_i\}$.

Even if b + mw is positive, it is now possible that there is no movement towards equalization (but this is not suprising, since there is, as we have noted earlier, no final equalization). What is going on will be somewhat clearer if we observe that the per capita wealth of the first group will be growing faster or slower than that of the second as

$$\frac{c_2}{c_1} \gtrless \frac{b + mwp_2}{b + mwp_1}, \quad \text{if } b + mwp_1 > 0$$

or as

$$\frac{c_2}{c_1} \lessgtr \frac{b + mwp_2}{b + mwp_1}, \quad \text{if } b + mwp_1 < 0 \quad ;$$

i.e., as the present ratio is greater or less than the equilibrium ratio; if $b + mwp_1 = 0$, then $\frac{\dot{c}_2}{c_2} - \frac{\dot{c}_1}{c_1} = \frac{\dot{c}_2}{c_2} \gtrless 0$ as $p_1 \gtrless p_2$.

Further attention ought to be brought to the special case where $b + mp_i w$ and n-rm are of opposite signs. Take the case, for instance, where the economy is in the upper equilibrium, so that n-mr is positive, and let there be two classes, an efficient class whose productivity coefficient we shall denote by 1, and an inefficient class which we shall denote by $p < 1$. In the upper equilibrium, $b + mw > 0$. Then, if p is sufficiently small, i.e.,

$$p \leq - b/mw$$

then all of the capital will be owned by one class. In fact, if

$$p < - b/mw$$

the poorer class actually goes into debt to the richer class, and there exists an equilibrium per capita debt of the poorer class. The rich save enough to lend to the poor and sustain the capita labor ratio.

What happens if $p < b/mw$ but a constraint is imposed on borrowing? (Say, no borrowing is allowed at all). Then we have a two class economy in which the poor consume everything and the rich (the capitalists) consume a (not necessarily constant) proportion of their income. (In the steady state, of course, they would save a constant proportion of their income.) Continuing with our example, we have

$$\dot{k} = a\dot{c}_1 = a(mw + mrc_1 - nc_1 + b)$$

if $\dot{k} = 0$,

$$r = \frac{n-a(b+mw)}{m} = n/m - ab/m - aw .$$

If aw, the wage income of the rich divided by the <u>total</u> population, is small and $b = 0$, we see that

$$r \approx n/m$$

the rate of profit is equal (approximately) to the rate of growth divided by the propensity to save (of the "rich"). If $b \neq 0$, we subtract a constant from the rate of growth divided by the propensity to save to obtain the rate of profit.

Note that in this model we have two classes, differing not in savings functions, but in productivity: the total difference in wealth and income is explained by the greater productivity of the rich. It is of interest (although perhaps not surprising), that the expression we have obtained for the rate of profit in the case when the rich wage income relative to the total population is very small is that of the Cambridge Pasinetti-Robinson theories of distribution.

VII. Primogeniture

So far in this paper we have considered only cases where wealth was divided equally among one's children; without going into a detailed exposition of alternative inheritance programs, let us consider the case perhaps most contrary to that which has been discussed thus far, that of primogeniture, all wealth being left to the first born son. To carry through the analysis we shall need to introduce some further simplifications, and shift the analysis to discrete time.

We consider a period in which the population doubles itself. Each "family" has exactly two sons and two daughters. For simplicity, we shall say that children are born at the end of the period. Everybody lives for only one period, parents dying after giving birth to their quadruplets. We shall examine only equilibrium paths. Then, at the beginning of any period

$1/2$ of the population has zero capital.

Of the remainder, $1/2$ are born to fathers who were first born, $1/2$ to fathers who weren't.

Hence,

$1/2 \cdot 1/2$ of the population has $b + mw$ wealth per capita.

Of the remainder, $1/2$ are born to fathers who were first born, $1/2$ to fathers who weren't.

$1/2 \cdot 1/2 \cdot 1/2$ of the population has $b + m(w+(1+r)(b+mw)$ wealth per capital. And so on.

If we number our groups from the poorest to the richest, then the ith group has
$(1/2)^{i+1}$ of the population where the 0th group has zero wealth and has a per capita wealth of

$$\frac{(b + mw)(m^i(1+r)^i - 1)}{m(1+r) - 1} \qquad i = 1,\ldots n$$

If we compare any two groups, their ratio of per capita wealth is

$$\frac{m^i(1+r)^i - 1}{m^j(1+r)^j - 1} .$$

The total wealth of the economy (per capita) is simply the weighted sum of the per capita wealth, where the weights are the proportion of the population in each group.

In general, we would expect $1/2(m)(1+r)$ to be less than 1; since m is less than 1, for this not to be so, r must be greater than 1, i.e. the rate of interest must be greater than 100%, a situation which would not normally occur.

Hence,

$$k = \frac{1/2(b+mw)}{1-m(1+r)} \quad \left(2 - \frac{m(1+r)}{1-\frac{1}{2}\ m(1-r)}\right)$$

$$= \frac{b + mw}{1-\frac{1}{2}\ m(1+r)}$$

We shall now show that if the elasticity of substitution is equal to or greater than one and $b \geq 0$, then there is a unique solution to the above equation, i.e., a unique balanced growth path; we require

$$\frac{kz - b}{my} = 1 - \frac{\frac{1}{2} r k}{y} \qquad \text{where } z = 1 - (1/2)m \quad 0.$$

The capital output ratio is an increasing function of k, and accordingly, if $b \geq 0$, the left hand side is an increasing function of k. If the elasticity of substitution is greater than or equal to one the right hand side is a non increasing function of k, and hence there is at most one solution. Even in the case when b = 0, if the elasticity condition is not satisfied, it is possible (although not necessary) that there be more than one balanced growth path.

It may be of interest to compare the predictions of this model for a "reasonable" set of parameter values to the actual observed distribution of wealth in the United States. We let m = .2 and r = .1. Then we obtain

% of Spending Units Ranked by Net Worth		% of Net Worth U.S. 1953	Predicted % of Net Worth
Lowest	31	1	0
	23	5	3.6
	35	34	32.0
	11	60	64.4

Source: 1953 Survey of Consumer Finances, from the Federal Reserve Bulletin, 1953, Supplementary Table 5, p. 11

The distribution of wealth does not seem to be very sensitive to moderate changes in m and r. Although it would be wrong to put too much stress on the closeness of our predictions to the actually observed result, it does suggest that further exploration of the implications of inheritance policies for the distribution of wealth under alternative assumptions would be of interest.

VIII. Kaldorian Distribution

Thus far we have assumed that the different amounts of savings of different individuals is explained by differences in income, not by the source of income, nor by any socio-economic characteristics, such as that some individuals are members of the capitalist class, and some are members of the working class, or some workers are skilled while others are unskilled. In this section, we follow Kaldor [1] and assume that a smaller proportion of wage income is saved than of profit income. It will turn out that, provided that a positive proportion there is at least some savings out of wages, eventually all wealth becomes evenly distributed.

We make the same assumption as in Section II. about homogeneity of the labor force, equal division of one's wealth among heirs, and constant reproduction rates. We assume that there is only one possible technique with a capital labor ratio given by k, and we assume that there is some mechanism[1] which assures that the economy maintains an overall capital labor ratio equal to k. If s_w is the savings propensity out of wage income, and so that out of profits the differential equations for the Kaldorian model become

$$s_w w + s_p r c_i - n c_i = \dot{c}_i$$

with

$$s_w w + s_p r k - n k = \dot{k}$$

Since $\dot{k} = 0$, by assumption,

$$s_w w + s_p r k = n k$$

[1] We do not consider what this mechanism is, and whether it is "realistic."

Assume that there existed some income group with $c_i - k \neq 0$. Then

$$\dot{c}_i = a_w w + s_p rk + s_p r(c_i - k) - nk - n(c_i - k)$$

$$= (s_p r - n)\ (c_i - k)$$

But $s_p r-n = -s_w w/k < 0$, provided $s_w \neq 0$, and hence $\dot{c}_i/c_i$ has opposite sign to c_i-k: if $s_w = 0$, whatever the distribution of wealth, that is what it remains, since then $\dot{c}_i/c_i = 0$.

In short, if $s_w \neq 0$, the Kaldorian model leads to the complete equalization of income and wealth among individuals, and in the singular case where $s_w = 0$, the distribution of wealth is a complete historical accident: economic forces cannot change it. Moreover increases in capital by one capitalist must clearly come at the expense of others, since

$$\Sigma ac_i = \bar{k}.$$

IX. A Two Class Model: Workers and Capitalists

We now consider the case where we have two classes, a capitalist class which does not work, and saves s_j of its profits, and a workers class which derives its income from wages plus return on the capital previously saved. Workers save s_i of their income (regardless of its source). Models with this savings behavior have been investigated by Pasinetti [5], Meade [2], Samuelson and Modigliani [6], and Stiglitz [8].

We consider the case where there is a neo-classical aggregate production functions, factors get paid their marginal products; all workers get paid the same wage, capitalists and workers reproduce at the same rate n, and all wealth is divided equally among one's heirs. Then, if c_i is the per capita wealth of the ith group of workers and c_j is the per capita wealth of the jth group of capitalists, our differential equations of per capita wealth accumulation become simply

9.1 $$\dot{c}_i = s_i(w + rc_i) - nc_i$$

9.2 $$\dot{c}_j = s_j rc_j - nc_j \ .$$

If k_i is the total wealth of workers divided by the total labor force (i.e., the total population less the population of capitalists) and k_j the total wealth of capitalists divided by the total labor force, we have

9.3 $\dot{k}_i = s_i(w + rk_i) - nk_i$

9.4 $\dot{k}_j = s_j(w + rk_j) - nk_j$

which are identical to the differential equations derived in [6]; in other words, we find again (because of the linearity of the savings functions) that the aggregate capital accumulation behavior is independent of the distribution of wealth

As has been shown in [6] and [8], there is at most one two class balanced growth path (i.e., a balanced growth path with both capitalists and workers present); along this balanced growth path, $r = n/s_j$, and hence the distribution of wealth among the capitalists is an historic accident, and, as in the Kaldorian case with savings out of wages equal to zero, increases in the capital of one capitalist occur at the expense of other capitalists. All "workers" on the other hand asymptotically have the same wealth and income, since

9.5 $$\dot{c}_i = s_i w + s_i rk_i + s_i r(c_i - k_i) - nk - n(c_i - k_i)$$

$$= (s_i r - n)\ (c_i - k_i)\ .$$

Since $s_i < s_j$, $s_i r - n < 0$, so that if any labor group has per capita wealth greater than the average, k_i, its per capita wealth declines, and conversely for any labor group with less per capita wealth than the average.

There also exists a unique balanced growth path with only workers present (the "dual regime" of [6]). But this case is identical to that investigated above in section II. with b = 0, for which we have already shown that asymptotically all wealth is evenly distributed.

X. A Two Class Model: Skilled and Unskilled Workers

In section VI we took account of the fact that different individuals may in fact have different abilities, and accordingly, their equilibrium per capita wealth may be different. However, we assumed that individuals with different abilities saved according to the same savings function. In this section[1] we investigate the case where individuals of different productivities ("skills") save a different proportion of their income[2], i.e.,

$$\dot{c}_i/c_i = m_i p_i w/c_i + m_i r - n.$$

Provided $m_i r$ is less than n, then, for any given capital labor ratio there is only one possible equilibrium:

$$c_i^*(k) = \frac{m_i p_i w}{n - m_i r} .$$

If we have two classes, the "skilled" and the "unskilled", with a per capita wealth c_i and c_j respectively, then

$$\dot{c}_i/c_i - \dot{c}_j/c_j = w\left(\frac{m_i}{c_1} - \frac{m_j}{c_2}\right) + r(m_i - m_j)$$

$$= \frac{c_i^*}{c_i}(n - m_i r) - \frac{c_j^*}{c_j}(n - m_j r) + r(m_i - m)$$

$$= n(c_i^*/c_i - c_j^*/c_j) + r(m_i(1 - c_i/c_i) - r\ (m_j(1 - c_j^*/c_j))$$

$$\gtrless 0 \text{ as } (c_i^*/c_i)/(c_j^*/c_j) \gtrless \frac{\frac{-r(m_i - m_j)}{c_j^*/c_j} + n - m_j r}{n - m_i r} \quad \text{provided } n - m_i r > 0.$$

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[1] Prof. Meade [4] in an unpublished comment on an earlier version of this paper, investigated a similar model with a Cobb-Douglas production function.

[2] The notation and assumptions are identical to those of Section VI with the following modifications: for simplicity b is assumed to be zero; and m_i is the marginal propensity to save of the ith group.

The implications of this relation can be set forth in a straightforward manner. We shall assume throughout that $m_i > m_j$.

1. If $c_j > c_j^*$, and if c_i/c_i^* is less than or equal to c_j/c_j^* , c_i is growing faster than c_j.

2. If $c_j = c_j^*$, then c_i is growing faster (or slower) than c_j as c_i/c_i^* is less than (or greater than) $\frac{c_j}{c_j^*}$.

3. If $c_j < c_j^*$, and if c_i/c_i^* is greater than or equal to c_j/c_j^*, c_j is growing faster than c_i.

But note that it is possible in this system that c_i/c_i^* be greater than c_j/c_j^* and for c_i still to be growing faster than c_j.[1]

Another way of looking at this problem in terms of distribution of factors has been suggested by Meade [4]. Letting the share of capital owned by members of the ith group be denoted by π; it is easily seen that

$$\dot{c}_i/c_i - \dot{c}_j/c_j \gtreqless 0 \quad \text{as} \quad (m_1-m_2)rk/w \gtreqless \frac{m_j a_j p_j}{j} - \frac{m_i a_i p_i}{i}$$

and in turn

$$\frac{d\pi_i/\pi_j}{dt} \gtreqless 0 \quad \text{as} \quad \dot{c}_i/c_i - \dot{c}_j/c_j \gtreqless 0$$

If rk/w is constant, i.e. we have a Cobb-Douglas production function, it is clear that there is a single stable value to which π_i/π_j converges. To find a more general condition, we must turn to an investigation of the behavior of the aggregate k.

[1] All of this analysis has been done under the assumption of $n-m_i r > 0$ and $n-m_j r > 0$. Since b = 0, it is clear that for some group v in the economy, $n-m_v r > 0$, other groups with larger savings propensities may have $n-m_i r < 0$ and the analysis must accordingly be modified.

$$\dot{k} = w\,\Sigma a_i m_i p_i + r\Sigma\, a_i c_i m_i - nk$$
$$= w\, s_w + rk\Sigma m_i \pi_i - nk\ .$$

Let us assume that we have a balanced growth path. Then along any balanced growth path, it follows that

$m_i \pi_i$ must be constant.

If there are only two classes, it is easy to see that this requires that π be constant, since

$(m_1 - m_2)\pi + m_2$ must be constant;

i.e.,

$$n/r - s_w w/rk = (m_1 - m_2) + m_2$$

If the elasticity of substitution is greater than or equal to one, as k increases, the left hand side unambiguously rises. As k increases, however, it can be shown that (under the convention that $m_1 > m_2$), the right hand side falls.

$$\pi = \frac{m_1 a_1 p_1 w}{(n - m_1 r)\ k}$$

$$d\pi/dk = -kf'' \frac{\pi}{w} + \frac{kf''\ m_1 \pi^2}{w\ m_1 a_1 p_1} - \pi/k\ .$$

The second term is greater in absolute value than the first if

$a_1 p_1 < \pi$

But from the assumption made in the beginning, that the more efficient group have a higher savings rate, it follows that $a_1 p_1 < \pi$; hence $d\pi/dk$ must be less than zero, and hence under the appropriate elasticity assumption, there can be at most one balanced growth path.

Multiplicity of balanced growth paths seems to be a distinct possibility whenever the elasticity condition is violated, or when the number of groups is greater than 2.

XI. Fiscal Policy

In this section, we turn to the implicacions of the analysis presented above for fiscal policy. We concern ourselves only with the redistribution effects of alternative tax policies. For convenience, we will use the simple neo-classical model presented in section II. We will investigate four kinds of taxes: proportional income taxes, progressive income taxes, wealth taxes, and profit taxes. We shall assume that all taxes that are collected are divided equally among the citizens. Hence aggregate disposable income remains unchanged.

1. Proportional Income Tax

If group i's before tax income is $y_i = w + rc_i$, its after tax income is

$$y_i' = (w + rc)(1 - t) + t(w + rk)$$

and hence its per capital wealth accumulation equation is

$$\dot{c}_i = mw + mrc - nc + mrt\ (k - c) + b$$

and, of course, the aggregate capital accumulation behavior of the economy remains unchanged:

$$\dot{k} = b + mw + mrc - nc$$

Let us examine what happens to the relative movements in per capita wealth of two groups:

$$\dot{c}_i/c_i - \dot{c}_j/c_j = (b + mw)(1/c_i - 1/c_j) + mrtk\ (1/c_i - 1/c_j).$$

Again, we observe that for "high" k, the poor increase their per capita wealth relative to the rich, until incomes and wealth are completely equalized, while for "low" k the rich grow richer relative to the poor. But note the two effects of the income tax:

(1) The critical k which determines whether there is income wealth equalization or not is lower, since now the condition is not $b + mw = 0$, but

$$b + mw + mrtk = 0.$$

In fact, at a tax rate greater than

$$1 - n/mr$$

even at the lower balanced growth path, equalization of wealth will occur.

(2) The rate at which equalization occurs is increased (or, if the distribution becomes more uneven, it does so at a slower rate than in the absence of the tax).

2. Progressive Income Tax

For convenience, we assume two income groups, the rich, denoted by a subscript 1, and the poor, denoted by a subscript 2.

We let y_i be the before tax per capita income of group i, y_i' be the after tax per capita disposable income. The tax rate is a function of per capita income, so that somebody with income y_i pays a tax

$$t(y_i)y_i$$

which for convenience we will write as $t_i y_i$. As before, a_i is the proportion of the population which is in group i, so that $a_1 + a_2 = 1$. Then, after tax disposable income may be written as follows:

$$y_1' = y_1\ (1 - t_1) + a_1 t_1 y_1 + a_2 t_2 y_2$$

$$y_2' = y_2\ (1 - t_2) + a_2 t_2 y_2 + a_1 t_1 y_1$$

The per capita capital accumulation equations thus become

$$\dot{c}_1 = b + my_1(1-t_1) + ma_1 t_1 y_1 + ma_2 t_2 y_2 - nc_1$$

$$\dot{c}_2 = b + my_2(1-t_2) + ma_2 t_2 y_2 + ma_1 t_1 y_1 - nc_2$$

and again the aggregate capital accumulation equation remains unchanged

$$\dot{k} = b + mw + mrk - nk$$

As one might expect, the redistributive effects of a progressive tax (for, say, given tax revenue) are greater than for a proportional tax of the same revenue.

$$\dot{c}_1/c_1 - \dot{c}_2/c_2 = b + mw\ (1/c_1 - 1/c_2) + (a_1 - a_2 - 1)(t_1 y_1 - t_2 y_2).$$

3. Profits Tax

If we impose a uniform tax rate on profits, we obtain an after tax income of group i as follows:

$$y_i' = w + rc(1-t) + trk$$

so that the per capita wealth accumulation equation is

$$\dot{c}_i = mw + mrc - nc + mrt(k-c) + b$$

and, again, the aggregate capital accumulation behavior of the economy remains unchanged. Observe that for a given tax rate t, the profits tax has the identical effects that an proportional income tax of exactly the same amount has; alternatively, for the same revenue, the profits tax has much greater redistributive effects.

4. Wealth Tax

For a wealth tax, there are two cases to consider. In the first case we assume that savings is determined solely by current income, so that

$$\dot{c}_i = b + mw + rc_i - nc_i - t(c_i - k)$$

$$\dot{k} = b + mw + rk - nk$$

To achieve the same effects as the profits tax, the wealth tax need be only mr as large (if the savings propensity is .1, and the rate of interest is 10%, the wealth tax rate need only be 1% the profit tax rate for the same effect) and its revenues only need be only m as large. For the same government revenue as the proportional income tax, the wealth tax need be only m times the share of profit as large (if m is 10%, and the share of profit is 25%, then the government revenue need only be 2.5% as large under the wealth tax as under the proportional income tax for the same redistributive effects).

In the second case, savings is assumed to be a function of "Fisherian" income: consumption plus change in wealth. Thus we have as our per capita capital accumulation equation

$$\dot{c}_i = b + mw + mrc_i - mtc_i - nc_i - t(c_i - k)$$

This time the aggregate capital accumulation equation is changed:

$$\dot{k} = b + mw + mrk - mtk - nk$$

The effect of the tax is to lower the capital-labor and output per man of the stable upper equilibrium and raise that of the lower unstable equilibrium, if it exists. The equilibrium capital labor ratio is that that would have obtained in the absence of the tax if population were growing at the rate $n + mt$.

The redistribution effects, on the other hand, are exactly as they were for the wealth tax with the alternative savings function, since

$$\dot{c}_i/c_i - \dot{c}_j/c_j = (b+mw)(1/c_i - 1/c_j) - tk(1/c_i - 1/c_j)$$

XII. Technological Change

So far, we have assumed that there is no technological change; hence in the long run, all incomes (in steady state paths) are constant. If there is technological change, in the case of nonproportional linear savings hypotheses we must make some adjustments in the analysis.

It is clear that the linear savings assumption approaches proportionality as incomes get larger and larger.

In the regular cases, of convex, or concave, savings functions, the savings functions must eventually become proportional, and the long run analysis we revert to the linear model with $b = 0$.

Alternatively--and somewhat more meaningfully within the context of the models examined here--we may assume that the intercept of the savings function shifts upward over time at the same rate as the (Harrod neutral) technological progress; i.e., we have a savings function not in terms of per capita income but in terms of per efficiency unit of labor. Since with Harrod neutral technological change, the rate of profit and the wage--per efficient unit--all are constant, in steady state, we get exactly the same steady state equations as before. If we wish to reconvert back to per capita terms, it may be done in a straightforward manner:

savings per capita = savings per efficiency unit x efficiency units per capita

$$= (b + my) \times e^{zt}$$

XIII. An Example

In this section we shall work through an example to give the rough orders of magnitudes of the time involved. We take Cobb-Douglas production function

$$y = k^{\alpha}$$

The differential equation for the aggregate capital labor ratio is

$$\frac{\dot{k}}{k} = m\, k^{\alpha-1} - n$$

This can easily be solved explicitly if we introduce a change of variables

$$z = k^{\alpha-1}$$

so

$$\frac{\dot{z}}{z} = (\alpha-1)\,\frac{\dot{k}}{k}$$

$$\frac{\dot{z}}{z} = \frac{(\alpha-1)m}{z} - n(\alpha-1)$$

$$\dot{z} = (\alpha-1)m - n(1-\alpha)z$$

$$z = c.e^{(\alpha-1)mt} + \frac{n}{m}$$

At $t = 0$, as $t \to \infty$ $z \to \frac{n}{m}$

$$z(0) = c + \frac{n}{m}$$

since $\alpha < 1$

$$z(t) = (z(0) - z^{*})\ e^{(\alpha-1)mt} + \frac{n}{m}$$

$$k(t) = [z(t)]^{\frac{1}{\alpha-1}}$$

The differential equation for the rate of change of $c_1 - c_2 \equiv g$

$$\frac{\dot{g}}{g} = mr - n = \alpha m z(t) - n$$

$$\ln g(t) = \int_{t_o}^{t} \alpha m z(t)dt - n(t-t_o) + \ln g(0)$$

This can easily be solved explicity if we assume that the economy is in aggregate equilibrium.

Then

$$c_1(t) - c_2(t) = [c_1(0) - c_2(0)]\ e^{(\alpha-1)nt}$$

To get an idea of the numerical values, let us assume that

$\alpha = .25$

$n = .01$

$m = .2$

Let us assume that the economy's capital labor ratio is 80% of its equilibrium value. We can calculate then that it will take (for the given parameters) 5.4 years for the economy's capital labor ratio to increase to 90% of its equilibrium value: or in other words, in 5.4 years its capital labor ratio has increased 12 1/2%. On the other hand, if we assume that the economy is at its equilibrium capital labor ratio, for a reduction of 12 1/2% in the disparity of per capita wealth takes 53.4 years. But note that this latter figure is very sensitive to the rate of growth of the economy, as well as the assumed value of the share of capital, but is not affected at all by the savings propensity. Assume, for instance, that the economy is growing at the rate of 3% and that the share of capital is 1/3; then it will take only 6.67 years for a 12 1/2% reduction in the disparity in per capita wealth. The greater are the share of capital and the rate of growth, the faster the process of equalization of per capita wealth.

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