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# Exact solutions of a quantum system placed in a Kratzer potential and under a uniform magnetic field

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# Exact solutions of a quantum system placed in a Kratzer potential and under a uniform magnetic field

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**Abstract.** We propose the Whittaker function approach as a theoretical method for finding exact solutions of a quantum mechanical system placed in the Kratzer potential. We then show that the effect of an external uniform magnetic field on this system can be satisfactorily determined using variational method. By following the one-step treatment suggested in this study, we increase the reliability and the accuracy of the solutions of Schrödinger equation for a quantum mechanical system placed in potential energy and perturbed by a uniform magnetic field that proves to be useful in modelling physical phenomena. We find that the achieved numerical and analytical results agree very well with those already published and those calculated using the Numerov method.

**Keywords.** Kratzer potential; variational method; Numerov method; energy eigenvalues; Landau problem; Whittaker functions.

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## 1. Introduction

The fundamental equation of quantum mechanics is the Schrödinger equation [1]. The solution for this equation is the wave function, which contains all the information that can be known about a quantum system. Some of the simplest systems that have been exactly solved are: the hydrogen atom, infinite square well, harmonic oscillator and quantum box. Solutions become difficult when the system consists of two or more particles and the number of exact solutions is very limited. To solve this problem, there are some approximate approaches that can analytically provide accepted solutions such as the WKB, the variational method (VM) and the perturbation theory. Moreover, many numerical methods, such as the Airy function approach, the asymptotic iteration, the Numerov method (NM) and the finite element method [2–11] have been suggested as solutions. Finding exact solutions to the Schrödinger equation for potentials that prove useful in the modelling of physical phenomena is a very important challenge for a deep understanding of the structures and interactions in such systems. In quantum chemistry and molecular physics, Kratzer's potential is often considered to explain interactions in a molecular system. This potential is exactly solvable [12], and numerically solved by many methods, such as the asymptotic iteration method (AIM) [13–22]. This potential is widely used. For example, it was presented in the study of anharmonic oscillatory systems having potential energy of Kratzer type [23,24]. Relativistic and non-relativistic treatment of Hulthen–Kratzer potential and the Kratzer potential plus a dipole in 2D systems was discussed in [25–27]. The solution of two-dimensional Schrödinger equation for the Kratzer potential for low vibrational and rotational energy levels without and with a constant magnetic field having arbitrary Larmor frequencies have been studied in [28]. The bound state solutions to the radial Schrödinger equation, in the case of the pseudoharmonic and Kratzer potentials, are obtained in three-dimensional space using the series expansion method within the framework of a general interaction potential [29]. Under Kratzer potential, in the presence of an external magnetic field, the two-dimensional Dirac equation for a fermion was solved [30]. By considering the non-central modified Kratzer potential, Sadeghi [31] tried to connect the corresponding Schrödinger equation to the associated Laguerre and Jacobi equations. A canonical transformation has been used to obtain energy eigenvalues and eigenfunctions of the generalised Morse potential from those of the Kratzer oscillator [32], and this transformation was used

to construct energy-raising and lowering operators for the Morse potential system from the Infeld–Hull ladder operators of the Kratzer oscillator system. In fractional dimensions, the spectrum of Schrödinger Hamiltonian operator with singular inverted complex and Kratzer's molecular potentials was presented by El-nabulsi [33]. Effects of a Landau-type system on a neutral particle subjected to the Kratzer potential in a rotating frame was studied by Oliveira and Bakke [34]. In quantum mechanics, the VM is one way of finding approximations to the lowest energy eigenstate or ground state, and some excited states, but the AIM is applied to obtain highly accurate eigenvalues of the Schrödinger equation for several potential' energy systems [6, 21, 22, 27, 35–44]. Singular inverse square potential in coordinate space with a minimal length was studied, regularised and renormalised with a generalised uncertainty principle, characterised by the existence of a minimal length [45–47]. The Schrödinger equation solutions for numerous quantum systems are nothing but the Whittaker functions [48–54], and the solutions are connected to the hypergeometric series [55–57]. The Whittaker functions are, moreover, among the most important special functions and they are solutions of Whittaker's differential equation which is very close to the Schrödinger equation for some quantum systems. To our knowledge, the study of the quantum system placed in Kratzer potential and under uniform magnetic field by the asymptotic iteration and variational methods is missing in the literature.

In this paper, we investigate a quantum system placed in the Kratzer potential by using the advantage of the Whittaker special function. We give in §2 a brief reminder of both Schrödinger and Whittaker equations. Then, in §3, we achieve exact eigenvalues for the Kratzer potential energy and we focus on the external uniform magnetic field effects on the system's Hamiltonian. In §4 we use VM and the numerical NM to find eigenvalues of the quantum system placed in Kratzer potential and under a uniform magnetic field. Results and discussions are given in §5. Finally, we conclude the paper in §6 with a summary.

## 2. Problem formulation

For many quantum systems, principally the Whittaker function has been considered as the solution for the Schrödinger equation. In more detail, in the case of central potential, the three-dimensional Schrödinger equation is $H\psi(\vec{r}) = E\psi(\vec{r})$ and in the spherical system of coordinates $(r, \theta, \phi)$ the solution is written as

$$\psi(\vec{r}) = R_{n,l}(r)Y_{l,m}(\theta, \phi)$$

where

$$Y_{l,m}(\theta, \phi) = (-1)^m \sqrt{\left(\frac{2l+1}{4\pi}\right)\left(\frac{(l-m)!}{(l+m)!}\right)} P_l^m(\cos(\theta))\mathrm{e}^{im\phi}$$

are the normalised spherical harmonics, $m \geq 0$ and $P_l^m(\cos(\theta))$ are the associated Legendre functions. However, the radial wave function $R_{n,l}$ satisfies the one-dimensional radial equation

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{R_{n,l}(r)}{\mathrm{d}r}\right) - \frac{2\mu}{\hbar^2}r^2(V(r) - E)R_{n,l}(r) = l(l+1)R_{n,l}(r). \tag{1}$$

For suitability, we assume that $P_{n,l}(r) = rR_{n,l}(r)$. The previous equation, therefore, can be rewritten as

$$-\frac{\hbar^2}{2\mu}\left(\frac{\mathrm{d}^2P_{n,l}(r)}{\mathrm{d}r^2}\right) + \left(V(r) + \frac{\hbar^2}{2\mu}\frac{l(l+1)}{2r^2} - E\right)P_{n,l}(r) = 0. \tag{2}$$

The domain of the solution for this equation is the set $\mathbb{R}_+[0, \infty]$, the function $P_{n,l}(r)$ satisfies the condition $4\pi\int_0^\infty |P_{n,l}(r)|^2\,\mathrm{d}r = 1$, and it vanishes as $r \to 0$ and as $r \to \infty$. The Whittaker equation $W(x)$, in comparison, can be written as [36]

$$\frac{\mathrm{d}^2W(x)}{\mathrm{d}x^2} + \left(-\frac{1}{4} + \frac{k}{x} + \frac{\frac{1}{4} - m^2}{x^2}\right)W(x) = 0. \tag{3}$$

The solution for this equation is given by

$$W(k, m; x) = \mathrm{e}^{-x/2}x^{m+\frac{1}{2}}F\left(m + \frac{1}{2} - k, 1 + 2m; x\right), \tag{4}$$

where $F(m + \frac{1}{2} - k, 1 + 2m; x)$ is the confluent hypergeometric function, defined as

$$F(\alpha, \gamma; x) = 1 + \frac{\alpha x}{\gamma 1!} + \frac{\alpha(\alpha+1)x^2}{\gamma(\gamma+1)2!} + \cdots.$$

The Whittaker equation is a second-order differential equation. From the symmetry of $m$, which appears as $m^2$ in the differential equation, there must exist a second independent solution, and it is readily obtained as $W(k, -m)(x)$. The two solutions are designed by the Whittaker functions *WhittakerM* $(k, m, x)$ and *WhittakerW* $(k, m, x)$. These solutions can be expressed as a hypergeometric function, which is a power series that reminder the power series solution of Schrödinger equation for many quantum systems such as harmonic oscillator, Coulomb and Morse potential energies. This infinite power series solution should terminate at a particular term, to guarantee its convergence. This requires that $\alpha$ is zero or negative integer $\alpha \in \mathbb{Z}_-$ which represents the termination condition. Hence, $F(\alpha, \gamma; x)$ is reduced to a polynomial function of degree $|\alpha|$ in $x$.

## 3. The non-perturbed system in Kratzer potential energy

By using the Whittaker equation, we now seek to obtain the non-relativistic analytical solutions for the case of a quantum system placed in the Kratzer potential energy. This potential is central and can be represented in the form $V(r) = a/r + b/r^2 + c$, where $a = -2D_e r_e$ and $b = D_e r_e^2$, with $D_e$ is the dissociation energy, $r_e$ is the equilibrium internuclear separation and $r$ the internuclear molecular distance. The Hamiltonian is

$$H_0 = -\frac{\hbar^2}{2\mu}\overrightarrow{\nabla}^2 + \frac{\hbar^2}{2\mu}\frac{l(l+1)}{r^2} + V(r).$$

According to eq. (2), Schrödinger equation $H_0\psi(\vec{r}) = E\psi(\vec{r})$, can be simplified as (the atomic unit's system is habitually adopted, i.e., $\hbar = \mu = q = 1$)

$$-\frac{1}{2}\frac{\mathrm{d}^2 P_{n,l}(r)}{\mathrm{d}r^2} + \left(\frac{a}{r} + \frac{b}{r^2} + c + \frac{1}{2}\frac{l(l+1)}{r^2} - E\right)P_{n,l}(r) = 0. \tag{5}$$

The Whittaker equation shows that the first solution for the previous Schrödinger equation can be expressed as

$$W\left(\frac{-a}{\sqrt{2(E-c)}}, \sqrt{\left(l+\frac{1}{2}\right)^2 + 2b}; 2\sqrt{2(E-c)}r\right). \tag{6}$$

As the *WhittakerW* function flows up, which means it is not normalisable, the accepted solution is the *WhittakerM* function. The energy eigenvalues, therefore, can be determined by setting

$$k = \frac{-a}{\sqrt{2(E-c)}}$$

and

$$m = \sqrt{\left(l+\frac{1}{2}\right)^2 + 2b}.$$

The termination condition leads to the solution $m - k + 1/2 = -n$ where $n \in \mathbb{N}$; consequently, the energy eigenvalue is given by

$$E_n = c - \frac{2a^2}{(2m+1+2n)^2}.$$

Using the formula for $m$ above, and in the absence of uniform magnetic field, one finds that the discrete energy expression can be written as

$$E^0_{n,l} = c - \frac{2a^2}{(2\sqrt{(l+\frac{1}{2})^2 + 2b} + 1 + 2n)^2}. \tag{7}$$

## 4. The uniform magnetic field effects

The Hamiltonian for a charged particle in an electromagnetic field is

$$H = \frac{1}{2\mu}\left(\vec{p} - \frac{q}{c}\vec{A}\right)^2 + q\Phi, \tag{8}$$

where $\vec{A}$ and $\Phi$ are respectively the vector and scalar potentials, $\vec{p}$ is the linear momentum, $c$ is the velocity of light and $q = -e$ is the particle electric charge. The time-independent Schrödinger equation is given by [14, 28, 58–63]

$$\frac{1}{2\mu}\left(i\hbar\vec{\nabla} + \frac{q}{c}\vec{A}\right)^2\psi(\vec{r}) + q\Phi(\vec{r})\psi(\vec{r}) = E\psi(\vec{r}). \tag{9}$$

The Schrödinger equation written above can be extended to

$$\begin{aligned}&\frac{-\hbar^2}{2\mu}\vec{\nabla}^2 + \frac{i\hbar q}{\mu c}(\vec{A}\cdot\vec{\nabla}) + \frac{q^2}{2\mu c^2}\vec{A}^2 + \\ &q\Phi)\psi(\vec{r}) + \frac{i\hbar q}{2\mu c}\psi(\vec{r})(\vec{\nabla}\cdot\vec{A}) = E\psi(\vec{r}).\end{aligned} \tag{10}$$

At this point, the equation can be simplified by choosing a gauge. Given any $\vec{A}$ and $\phi$, we can perform a gauge transformation, such that the resultant $\vec{A}$ and $\phi$ satisfy $\vec{\nabla}\cdot\vec{A} = 0$ and $\phi = 0$, that is the Coulomb gauge conditions. Thus, the Schrödinger equation in the Coulomb gauge changes to the following form:

$$\frac{-\hbar^2}{2\mu}\vec{\nabla}^2\psi(\vec{r}) + \frac{i\hbar q}{\mu c}\vec{A}\cdot\vec{\nabla}\psi(\vec{r}) + \frac{q^2}{2\mu c^2}\vec{A}^2\psi(\vec{r}) = E\psi(\vec{r}). \tag{11}$$

In the case of a uniform in space and time-independent magnetic field $\vec{B}$, we may choose: $\vec{A} = -\frac{1}{2}(\vec{r}\times\vec{B})$ and $\Phi = 0$. It follows that the electric field is null ($\vec{E} = \vec{0}$) and ($\vec{B} = \vec{\nabla}\times\vec{A}$). Furthermore, we note that $\vec{\nabla}\cdot\vec{A} = -\frac{1}{2}\vec{\nabla}\cdot(\vec{r}\times\vec{B}) = 0$, we have indeed chosen the Coulomb gauge. Hence, the time-independent Schrödinger equation becomes

$$\begin{aligned}&\frac{-\hbar^2}{2\mu}\vec{\nabla}^2\psi(\vec{r}) - \frac{i\hbar q}{2\mu c}(\vec{r}\times\vec{B})\cdot\vec{\nabla}\psi(\vec{r}) + \\ &\frac{q^2}{8\mu c^2}(\vec{r}\times\vec{B})^2\psi(\vec{r}) = E\psi(\vec{r}).\end{aligned} \tag{12}$$

Using the vectors' identities $(\vec{r}\times\vec{B})\cdot\vec{\nabla}\psi(\vec{r}) = -\vec{B}\cdot(\vec{r}\times\vec{\nabla}\psi(\vec{r}))$, $(\vec{r}\times\vec{B})^2 = r^2B^2 - (\vec{r}\cdot\vec{B})^2$ and the canonical angular momentum operator, $\vec{L} \equiv \vec{r}\times\frac{\hbar}{i}\vec{\nabla}$, we can write

$$-\frac{i\hbar q}{2\mu c}(\vec{r}\times\vec{B})\cdot\vec{\nabla}\psi(\vec{r}) = -\frac{q}{2\mu c}(\vec{B}\cdot\vec{L})\psi(\vec{r})$$

and simplify the time-independent Schrödinger equation for a charged particle of charge $q$ in an external uniform magnetic field $\vec{B}$ to

$$\frac{-\hbar^2}{2\mu}\vec{\nabla}^2\psi(\vec{r}) - \frac{q}{2\mu c}(\vec{B}\cdot\vec{L})\psi(\vec{r}) + \frac{q^2}{8\mu c^2}(r^2B^2 - (\vec{r}\cdot\vec{B})^2)\psi(\vec{r}) = E\psi(\vec{r}). \tag{13}$$

If the quantum system is placed in a Kratzer scalar potential

$$V(r) = \frac{a}{r} + \frac{b}{r^2} + c$$

and perturbed by the field $\vec{B} = \beta\vec{k}$, where $\beta$ is a real constant, the simplified Schrödinger equation for such a quantum system can be written as in refs [28,58]:

$$\frac{-\hbar^2}{2\mu}\left(\vec{\nabla}^2 - \frac{l(l+1)}{r^2}\right) - \frac{q}{2\mu c}(\vec{B}\cdot\vec{L}) + \frac{q^2}{8\mu c^2}(r^2B^2 - (\vec{r}\cdot\vec{B})^2) + V(r)\psi(\vec{r}) = E\psi(\vec{r}). \tag{14}$$

The previous equation leads to the Schrödinger equation [14,28,58,59]:

$$\left(\frac{-\hbar^2}{2\mu}\vec{\nabla}^2 + \omega_L L_z + \frac{\mu}{2}\omega_L^2 r^2\sin^2(\theta) + \frac{\hbar^2}{2\mu}\frac{l(l+1)}{r^2} + V(r)\right)\psi(\vec{r}) = E\psi(\vec{r}), \tag{15}$$

where $\omega_L$ is the Larmor frequency given by $\omega_L = e\beta/2\mu c$, $\theta$ is the angle defined by $\theta = \widehat{(\vec{B},\vec{r})}$, angle limited by the directions of the magnetic field and the system of diatomic molecule and $m_l$ is the magnetic quantum number ($|m_l| \leqslant l$).

If one can ignore the quadratic term $\frac{\mu}{2}\omega_L^2 r^2\sin^2(\theta)$, eq. (15) reduces to

$$H\psi(\vec{r}) = (H_0 + \omega_L L_z)\psi(\vec{r}) = E\psi(\vec{r}). \tag{16}$$

As $H_0$ commutes with $L_z$, the operators $H$, $L_z$ and $H_0$ mutually commute. Hence, they possess a set of common eigenfunctions: $\psi(\vec{r})$. The eigenvalues $E_{n,l,m_l}$ are expressed in atomic units system

$$E_{n,l,m_l} = \langle\psi(\vec{r})|H|\psi(\vec{r})\rangle = \langle\psi(\vec{r})|H_0|\psi(\vec{r})\rangle + \omega_L\langle\psi(\vec{r})|L_z|\psi(\vec{r})\rangle = E^0_{n,l} + m_l\omega_L. \tag{17}$$

Otherwise, for fixed angle $\theta$, we adopt the Hamiltonian in two dimensions (polar coordinates ($r$,$\varphi$)). The suggestion is $\psi(\vec{r}) = \frac{1}{r}P_{n,l}(r)\Phi_l(\varphi)$, and here the separation of variables is between the two variables $r$ and $\varphi$, and then eq. (15) leads to

$$\frac{\mathrm{d}^2P_{n,l}(r)}{\mathrm{d}r^2} + \left(-\frac{2\mu}{\hbar}\omega_L m_l - \frac{\mu^2}{\hbar^2}r^2\omega_L^2\sin^2(\theta) - \frac{l(l+1)}{r^2} - \frac{2\mu}{\hbar^2}\left(\frac{a}{r} + \frac{b}{r^2} + c - E\right)\right)P_{n,l} = 0. \tag{18}$$

Using the concept of effective potential energy, eq. (18) may be written as

$$\frac{\mathrm{d}^2P_{n,l}(r)}{\mathrm{d}r^2} + \frac{2\mu}{\hbar^2}(E - V_{\mathrm{eff}}(r)P_{n,l}(r)), \tag{19}$$

where

$$V_{\mathrm{eff}}(r) = m_l\hbar\omega_L + \frac{\mu}{2}r^2\omega_L^2\sin^2(\theta) + \frac{\hbar^2}{2\mu}\frac{l(l+1)}{r^2} + \left(\frac{a}{r} + \frac{b}{r^2} + c\right).$$

Effective potential energy is based on four terms:

$V_1 = m_l\hbar\omega_L$: This term is constant and depends on the Larmor frequency $\omega_L$ and the magnetic quantum number $m_l$. It is due to the interaction between the magnetic field and angular momentum.

$V_2 = \frac{\mu}{2}r^2\omega_L^2\sin^2(\theta)$: This term appears as a harmonic part and depends on the Larmor frequency $\omega_L$ and the angle $\theta$. Its origin is the not null potential vector $\vec{A}$. It is independent of the magnetic quantum number $m_l$.

$V_3 = \frac{\hbar^2}{2\mu}\frac{l(l+1)}{r^2}$: This term arises upon acting the kinetic energy operator on the wave function. It is the centrifugal term and it tends to throw the particle outward.

$V_4 = \frac{a}{r} + \frac{b}{r^2} + c$: This term constitutes a Kratzer potential.

To solve the problem, we suggest using the variational method. For that, the acceptable asymptotic solution at the origin $r = 0$ is proportional to the power of $r$. Furthermore, the rational and logical trial wave function is

$$P_{n,l}(r) = r^\alpha W\left(\frac{-a}{\sqrt{2(E-c)}}, \sqrt{\left(l+\frac{1}{2}\right)^2 + 2b}; 2\sqrt{2(E-c)}r\right),$$

where $\alpha$ is treated as the variational parameter which is determined by minimising the expectation value of the Hamiltonian for each state. After having $\alpha$ value, it was used in the wave function to obtain energy eigenvalues known as

$$E_{n,l,m_l} = \frac{\langle P_{n,l}(r)|H|P_{n,l}(r)\rangle}{\langle P_{n,l}(r)|P_{n,l}(r)\rangle}, \tag{20}$$

where

$$H = -\frac{\hbar^2}{2\mu}\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \hbar\omega_L m_l + \frac{\mu}{2}r^2\omega_L^2\sin^2(\theta) + \frac{\hbar^2}{2\mu}\frac{l(l+1)}{r^2} + \left(\frac{a}{r} + \frac{b}{r^2} + c\right).$$

In the next paragraph, we shall discuss and check the validity of our approach by comparing the obtained results with those already published. To be sure that our results are in the acceptable range, we compare these results, also, with numerical founded ones.

**Table 1**. Comparison of the energy eigenvalues obtained by using the variational and the Numerov methods with those of ref [58], calculated for $a = -20, b = 10, c = 0, \omega_L = 0.1$ and $\theta = \pi/6$.

| $n$ | $l$ | $m_l$ | $E^{(\mathrm{VM})}_{n,l,m_l}$ | $E^{(\mathrm{NM})}_{n,l,m_l}$ | $E^{(\mathrm{Ref})}_{n,l,m_l}$ |
|---|---|---|---|---|---|
| 0 | 0 | 0 | −7.997422 | −7.997420 | 7.99751 |
| 1 | 0 | 0 | −5.548762 | −5.548755 | 5.54914 |
| 1 | 1 | 0 | −5.166786 | −5.16678 | 5.16726 |
| 1 | 1 | 1 | −5.066786 | −5.06678 | 5.06726 |
| 2 | 0 | 0 | −4.067494 | −4.067439 | 4.06853 |
| 2 | 1 | 0 | −3.8241203 | −3.82404 | 3.82536 |
| 2 | 1 | 1 | −3.7241203 | −3.72404 | 3.72536 |
| 2 | 2 | 0 | −3.422179 | −3.422017 | 3.42388 |
| 2 | 2 | 1 | −3.322179 | −3.322017 | 3.32388 |
| 2 | 2 | 2 | −3.222179 | −3.222017 | 3.22388 |
| 3 | 0 | 0 | −3.099398 | −3.099082 | 3.10157 |
| 3 | 1 | 0 | −2.934049 | −2.933591 | 2.93656 |
| 3 | 1 | 1 | −2.834049 | −2.833591 | 2.83656 |
| 3 | 2 | 0 | −2.656366 | −2.655423 | 2.65955 |
| 3 | 2 | 1 | −2.556366 | −2.555423 | 2.55955 |
| 3 | 2 | 2 | −2.456366 | −2.455423 | 2.45955 |

## 5. Discussion of the results

In the following, we verify the results of our approach for a quantum system placed in a Kratzer potential and a possible magnetic field. For a null magnetic field, the exact solutions are given by the Whittaker equation. Rightly, our result for a quantum system placed in a Kratzer potential without a magnetic field is

$$E^0_{n,l} = c - \frac{2a^2}{(2\sqrt{(l+\frac{1}{2})^2 + 2b} + 1 + 2n)^2}.$$

This result is the same as the results achieved in refs [28, 64, 65].

Next, we solve the same problem when the quantum system is placed in a uniform magnetic field. For this case, we recall the VM and the NM. To test our results in the case of a quantum system placed in a Kratzer potential and under a uniform magnetic field, we choose the values of the parameters in expression (20):

$$E_{n,l,m_l} = \frac{\langle P_{n,l}(r)|H|P_{n,l}(r)\rangle}{\langle P_{n,l}(r)|P_{n,l}(r)\rangle},$$

suggested in ref. [58]. The values of these parameter are: $a = -20, b = 10, c = 0, \omega_L = 0.1$ and $\theta = \pi/6, \pi/4, \pi/3$ and $\pi/2$.

**Table 2**. Comparison of the energy eigenvalues obtained by using the VM and NM with those of ref. [58] , calculated for $a = -20, b = 10, c = 0, \omega_L = 0.1$ and $\theta = \pi/4$.

| $n$ | $l$ | $m_l$ | $E^{(\mathrm{VM})}_{n,l,m_l}$ | $E^{(\mathrm{NM})}_{n,l,m_l}$ | $E^{(\mathrm{Ref})}_{n,l,m_l}$ |
|---|---|---|---|---|---|
| 0 | 0 | 0 | −7.994846 | −7.994843 | 7.99509 |
| 1 | 0 | 0 | −5.541995 | −5.541978 | 5.54301 |
| 1 | 1 | 0 | −5.159097 | −5.159067 | 5.16036 |
| 1 | 1 | 1 | −5.059097 | −5.059068 | 5.06036 |
| 2 | 0 | 0 | −4.053552 | −4.053336 | 4.05623 |
| 2 | 1 | 0 | −3.808634 | −3.808315 | 3.81181 |
| 2 | 1 | 1 | −3.708634 | −3.708315 | 3.71181 |
| 2 | 2 | 0 | −3.403542 | −3.402926 | 3.40779 |
| 2 | 2 | 1 | −3.3035420 | −3.302926 | 3.30779 |
| 2 | 2 | 2 | −3.2035420 | −3.202926 | 3.20779 |
| 3 | 0 | 0 | −3.0746754 | −3.073535 | 3.07990 |
| 3 | 1 | 0 | −2.9071331 | −2.905548 | 2.91304 |
| 3 | 1 | 1 | −2.8071331 | −2.805548 | 2.81304 |
| 3 | 2 | 0 | −2.6251884 | −2.622307 | 2.63235 |
| 3 | 2 | 1 | −2.5251884 | −2.522307 | 2.53235 |
| 3 | 2 | 2 | −2.4251884 | −2.422307 | 2.43235 |

Tables 1–4 show the founded energy values by using the VM and NM, for various values of $n$, $l$ and $m_l$. We note, firstly, that the energy level values increase with different quantum numbers. Secondly, these values progress also with the angle $\theta = \widehat{(\vec{B}, \vec{r})}$ and obviously with the Larmor frequency $\omega_L$. Thirdly, these results are very closed, and those founded by our proposed VM are localised between those obtained by the NM and those accomplished by using the perturbation method in the frame of AIM [58]. The recorded results are slightly smaller than those attained by the NM and slightly larger than those of ref. [58]. This confirms once again that our choice of the trial wave function is really logical and rational. The chosen wave function is the product of the founded wave function of the free quantum system placed in a Kratzer potential, and the acceptable term of the asymptotic solution of the Schrödinger equation at the vicinity of the axis origin. We observe that for the same values of $n$ and $l$, energy level values are equally spaced. This equidistant splitting of the levels is due to the first term of potential

**Table 3**. Comparison of the energy eigenvalues obtained by using the VM and NM with those of ref. [58], calculated for $a = -20, b = 10, c = 0, \omega_L = 0.1$ and $\theta = \pi/3$.

| $n$ | $l$ | $m_l$ | $E^{(\mathrm{VM})}_{n,l,m_l}$ | $E^{(\mathrm{NM})}_{n,l,m_l}$ | $E^{(\mathrm{Ref})}_{n,l,m_l}$ |
|---|---|---|---|---|---|
| 0 | 0 | 0 | −7.992271 | −7.992281 | 7.99271 |
| 1 | 0 | 0 | −5.5352543 | −5.535216 | 5.53706 |
| 1 | 1 | 0 | −5.151445568 | −5.151385 | 5.15368 |
| 1 | 1 | 1 | −5.051445567 | −5.051385 | 5.05336 |
| 2 | 0 | 0 | −4.039796907 | −4.039319 | 4.04439 |
| 2 | 1 | 0 | −3.793404638 | −3.792712 | 3.79880 |
| 2 | 1 | 1 | −3.69340464 | −3.692712 | 3.69880 |
| 2 | 2 | 0 | −3.3853538 | −3.384023 | 3.39244 |
| 2 | 2 | 1 | −3.2853538 | −3.284023 | 3.29244 |
| 2 | 2 | 2 | −3.1853538 | −3.184023 | 3.19244 |
| 3 | 0 | 0 | −3.0506973 | −3.048328 | 3.05917 |
| 3 | 1 | 0 | −2.8811871 | −2.877925 | 2.89060 |
| 3 | 1 | 1 | −2.7811871 | −2.777925 | 2.79060 |
| 3 | 2 | 0 | −2.5955433 | −2.58983 | 2.60656 |
| 3 | 2 | 1 | −2.4955433 | −2.48983 | 2.50656 |
| 3 | 2 | 2 | −2.3955433 | −2.38983 | 2.40656 |

**Table 4**. Comparison of the energy eigenvalues obtained by using the VM and NM with those of ref. [58], calculated for $a = -20, b = 10, c = 0, \omega_L = 0.1$ and $\theta = \pi/2$.

| $n$ | $l$ | $m_l$ | $E^{(\mathrm{VM})}_{n,l,m_l}$ | $E^{(\mathrm{NM})}_{n,l,m_l}$ |
|---|---|---|---|---|
| 0 | 0 | 0 | −7.989697 | −7.989705 |
| 1 | 0 | 0 | −5.528538 | −5.52847 |
| 1 | 1 | 0 | −5.143830 | −5.143724 |
| 1 | 1 | 1 | −5.043830 | −5.043724 |
| 2 | 0 | 0 | −4.026216 | −4.025395 |
| 2 | 1 | 0 | −3.778415 | −3.777222 |
| 2 | 1 | 1 | −3.678414 | −3.677221 |
| 2 | 2 | 0 | −3.367578 | −3.365308 |
| 2 | 2 | 1 | −3.267578 | −3.265308 |
| 2 | 2 | 2 | −3.1675786 | −3.165308 |
| 3 | 0 | 0 | −3.0273729 | −3.023442 |
| 3 | 1 | 0 | −2.8560809 | −2.85071 |
| 3 | 1 | 1 | −2.7560809 | −2.75071 |
| 3 | 2 | 0 | −2.5671890 | −2.557943 |
| 3 | 2 | 1 | −2.4671890 | −2.457943 |
| 3 | 2 | 2 | −2.3671889 | −2.357943 |

energy: $V_1 = m_l \hbar \omega_L$. For each triplet $(n, l, m_l)$, energy level values reduce with the angle $\theta$, and this is due to the second term of potential energy

$$V_2 = \frac{\mu}{2} r^2 \omega_L^2 \sin^2(\theta).$$

## 6. Conclusion

In summary, we have utilised the Whittaker functions approach for finding the exact vibrational levels of energy and the associated wave functions in a quantum system placed in the Kratzer potential. By using the VM, the effect of an external magnetic field on such a system has been studied, and the eigenvalues of energy have been founded. To check the validity of our results, we also recall the NM and compare the results provided by our approach with the results calculated by this method. We find that all numerical and analytical results agree very well, and they agree with the results given in ref. [58]. We believe that the treatment followed in this study will be useful for investigating exact solutions of Schrödinger' equation of a quantum mechanical system subjected to potentials that are useful in modelling physical phenomena.

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